Ideals Generated by Principal Minors

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Symbols are listed approximately in order of appearance. Many symbols used throughout have alternate, shorthand notation.

Symbol	Explanation Alternate Notation
$X_{r\times s}$	$r \times s$ matrix of indeterminates $(x_{ij})_{1\leq i\leq r, 1\leq j\leq s}$ X
K	arbitrary algebraically closed field
$K[X]$	polynomial ring over K whose variables are the entries of X
$\mathfrak{P}_t(X)$	ideal in $K[X]$ generated by the principal t-minors of X, when X is a square matrix $\mathfrak{P}_t$
$\det X$	determinant of X, when X is square Δ
$\mathrm{GL}(r, K)$	general linear group of order r over K
$\mathrm{Grass}(r, n)$	Grassmannian, or Grassmann variety, of r-dimensional subspaces of K^n $\mathcal{G}$, $\mathrm{Grass}(r, K^n)$
$\mathcal{V}(I)$	algebraic set defined by the ideal I
$\mathbb{A}^r$	affine r-space $\mathbb{A}^r_K$
$\mathrm{rad}(I)$	radical of an ideal I $\mathrm{rad}\ I$
$\wedge^r A$	rth exterior power of the matrix A (basis induced by the standard basis)
$\mathbb{P}^r$	projective r-space (over K)
Ⓢ	Segre product
A^{T}	transpose of the matrix A
$K^\times$	multiplicative group of K
$\mathrm{ht}(I)$	height (or codimension) of the ideal I $\mathrm{ht}\ I$

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Symbol	Explanation Alternate Notation
$\dim R$	Krull dimension of the ring R
$\mathrm{Pf}_t(X)$	ideal in $K[X]$ generated by the size t Pfaffians of X, when X is alternating Pf_t, Pf_{2h}
$\mathrm{I}_t(X)$	ideal in $K[X]$ generated by all t-minors of X I_t
$\mathcal{Z}_{n,r}$	set of $n \times n$ matrices of rank exactly r
$\mathcal{Y}_{n,r,t}$	set of $n \times n$ matrices of rank exactly r, whose size t principal minors vanish
$\bar{\mathcal{Y}}$	Zariski closure of the algebraic set $\mathcal{Y}$
$\mathfrak{Q}_{n-1}$	defining ideal for $\bar{\mathcal{Y}}_{n,n,n-1}$ $\mathfrak{Q}_{n-1}(X)$
$I :_R J$	ideal quotient, $\{r \in R \mid rJ \subseteq I\}$ $\mathrm{I} : \mathrm{J}$
$\mathrm{Sp}(r, K)$	sympletic group of order r over K
R^G	ring of invariants in the ring R under the action of the group G
$\mathrm{SL}(r, K)$	special linear group of order r over K
$\underline{\mathrm{i}}$	$\{i_1, \ldots, i_r\}$, an indexing r-tuple where $i_1 < \cdots < i_r \in \{1, \ldots, n\}$ for $r \leq n$
$x_{\underline{\mathrm{i}}}$	Plücker coordinate indexed by $\underline{\mathrm{i}}$
$\ker \varphi$	kernel of the map φ
$\mathrm{Proj}(R)$	set of all homogeneous prime ideals of the graded ring R which do not contain the irrelevant ideal
R_d	degree d summand of a graded ring R
$\mathfrak{m}$	(homogeneous) maximal ideal of a (graded) ring
$\mathrm{Ass}_R\, I$	set of associated primes of the R-module R/I $\mathrm{Ass}(I)$, $\mathrm{Ass}\, I$
R_P	localization of the ring R at the prime ideal P
$\mathrm{depth}(I)$	depth of the R-module R/I in $\mathfrak{m}$, where $(R, \mathfrak{m})$ is (graded) local $\mathrm{depth}\, I$
$A_{\underline{\mathrm{i}},\underline{\mathrm{j}}}$	$(\underline{\mathrm{i}}, \underline{\mathrm{j}})$th cofactor of the matrix A A_{ij} (when $\underline{\mathrm{i}} = \{i\}$, $\underline{\mathrm{j}} = \{j\}$)
$A(\underline{\mathrm{i}}; \underline{\mathrm{j}})$	submatrix of A given by its $\underline{\mathrm{i}}$-indexed rows and $\underline{\mathrm{j}}$-indexed columns
$\mathrm{cof}(A)$	cofactor matrix for the matrix A $\mathrm{cof}\, A$

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Symbol	Explanation Alternate Notation
adj A	classical adjoint of the matrix A
$\mathbf{I}_r$	identity matrix of size r
D_f	distinguished open set of points which do not vanish on the polynomial f
Spec R	set of primes in the ring R
row A	row space of the matrix A
col A	column space of the matrix A
rank A	rank of the matrix A
$\mathbf{e}_i$	standard basis vector where 1 is in the ith position
$\mathfrak{I}(\mathcal{Y})$	defining ideal for the algebraic set $\mathcal{Y}$
$\mathcal{O}(\mathcal{Y})$	affine coordinate ring for $\mathcal{Y}$
ω_R	canonical module of the ring R
type R	type of the ring R
$\dim_K(M)$	K-vector space dimension of the module M
$\mathrm{Ext}^i_R(M,N)$	ith right derived functor for $\mathrm{Hom}_R(M,N)$
$\mu(M)$	minimal number of generators for the module M
$\Delta(\underline{\mathrm{i}};\underline{\mathrm{j}})$	determinant of $X(\underline{\mathrm{i}};\underline{\mathrm{j}})$
$I : J^\infty$	$\bigcup_t I : J^t$
$\underline{\mathrm{I}}$	set of Plücker indices
$\mathcal{V}(\underline{\mathrm{I}})$	algebraic set defined by the Plücker variables indexed by elements in $\underline{\mathrm{I}}$
Graph($\mathbf{g}$)	graph associated to a point $\mathbf{g}$ in a Grassmannian
Graph(A)	graph associated to the column, or row, depending on context, space of the matrix A
Graph($\mathcal{S}$)	graph associated to a permissible subvariety $\mathcal{S}$ of a Grassmannian
G_{triv}	set of isolated or dominating vertices in a permissible graph G
$(E,\mathcal{I})$	a finite set E with a collection $\mathcal{I}$ of subsets of E
$\mathcal{P}(E)$	power set of a finite set E
$\mathcal{V}(E,\mathcal{I})$	algebraic set associated to a matroid given by $(E,\mathcal{I})$

ABSTRACT

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Chair: Mel Hochster

Let X be a square matrix of indeterminates. Let $K[X]$ denote the polynomial ring in those indeterminates over an algebraically closed field, K. A minor is *principal* means it is defined by the same row and column indices. We prove various statements about ideals $\mathfrak{P}_t \subset K[X]$ generated by principal minors of a fixed size t. When $t = 2$ the resulting quotient ring is a normal complete intersection domain. When $t > 2$ we break the problem into cases by intersecting $\mathcal{V}(\mathfrak{P}_t)$ with the locally closed variety of rank r matrices. We show when $r = n$ for any t, there is a K-automorphism of $K[X][\frac{1}{\det X}]$ that maps $\mathfrak{P}_t\, K[X][\frac{1}{\det X}]$ to $\mathfrak{P}_{n-t}\, K[X][\frac{1}{\det X}]$, inducing an isomorphism on the respectively defined schemes. When $t = r$ we factor $A \in \mathcal{V}(\mathfrak{P}_t)$ as the product of its row space matrix, a matrix in $\mathrm{GL}(t, K)$, and its column space matrix. We show that for the analysis of components it is enough to consider irreducible algebraic sets in the product of Grassmannians, $\mathrm{Grass}(t, n) \times \mathrm{Grass}(t, n)$. For $t = r$ we also observe the connection between such decompositions and matroid theory.

CHAPTER 1

Introduction: Systems of Polynomial Equations

This thesis lies squarely in the realm of pure math, yet its motivation comes from throughout mathematics, industry, and government. At issue is the following persistent question: What is the geometry of the solution set to a system of polynomial equations?

Example 1. The polynomial $f(x, y) = y - x^2$, when set to 0, gives a system of one equation. The coefficients of f are in $\mathbb{R}$, the field of real numbers. The solution set is all points (a, b) in the xy-plane that satisfy $b - a^2 = 0$. We say the *zero-locus* of f in $\mathbb{R}^2$ is a parabola. See Figure 1.1 for a plot of the equation $y - x^2 = 0$ in $\mathbb{R}^2$.

Sometimes the problem is simple enough to include on an algebra homework assignment, like Example 1. Unfortunately many other times the problem is far from trivial. Regardless of its difficulty, however, the problem of solving systems of polynomial equations remains a timeless one. To emphasize this point the author excitedly quotes the following thesis introduction from 1961, which could easily have been lifted from this thesis (among many others published since then):

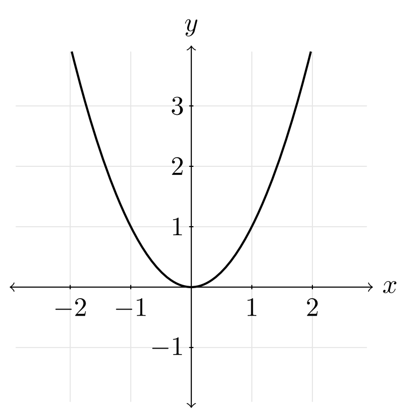

Figure 1.1: The parabola is the zero-locus of $f(x, y) = y - x^2$ in $\mathbb{R}^2$.

> The problem solved in this dissertation is a special case of a very general question in the dimension theory of ideals in a Noetherian ring... [I]n algebro-geometric terms, [given a set of generators of an ideal,] what is the dimension of the corresponding variety, and what can one say about the dimension of each irreducible component of that variety? [10]

To translate to the non-expert: an *ideal* is the set of all polynomials that equal zero as a consequence of a given system of equations. We call the equations defining the system *generators* for the ideal. A *variety* refers to the notion of a "basic" shape cut out by an equation or equations; for example, a parabola is a variety (see Example 1 and Figure 1.1).

A system of equations may contain any number of variables, with coefficients over any ring (such as the integers, $\mathbb{Z}$). In this dissertation we shall only consider systems over an arbitrary algebraically closed field (such as the complex numbers, $\mathbb{C}$), and we shall only work with finitely many variables. These restrictions happen to simplify a lot; over an algebraically closed field, for example, every polynomial in one variable completely factors. Let $S = K[x_1, \ldots, x_N]$ denote the polynomial ring in N variables over K. One advantage to using only finitely many variables is that S satisfies the *Noetherian* property, implying any ideal I in S has finitely many generators. Suppose the polynomials $f_1, \ldots, f_h \in S$ generate I. Let R denote the quotient ring S/I. We let $\mathcal{V}(I)$ denote the solution set to the system of equations $\{f_j(x_1, \ldots, x_N) = 0 \,|\, j = 1, \ldots, h\}$ in the *affine space* $\mathbb{A}^N = \mathbb{A}^N_K = K^N$. In order to gain insight into the geometry of $\mathcal{V}(I)$, commutative algebraists like to answer questions such as the following:

- Is I prime, or even reduced? If not, then Hilbert's *Nullstellensatz* says there is a "better" set of polynomials that define $\mathcal{V}(I)$, in the sense that the ideal they generate, denoted rad I, includes all polynomials vanishing on $\mathcal{V}(I)$ and is the unique reduced ideal that does so. In general, finding such generators is hard.
- What is the *primary decomposition* for I? A unique decomposition always exists, because of the Noetherian property. Its minimal components are in bijection with the irreducible components of $\mathcal{V}(I)$.
- Is R *Cohen-Macaulay*? If so, what is its type? (See Section 3.3.2 for the notion of *type*.) In particular, is R *Gorenstein*?

For the remainder of the thesis we take for granted the reader's knowledge of introductory commutative ring theory with algebraic geometry, though we attempt to black-box some of the more technical terminology and results throughout. Chapter 2 also provides relevant background. For additional references, we prefer [2, 11] for basic commutative ring theory and [3] specifically for Cohen-Macaulay ring theory, [14, 15, 35] for an introduction to algebraic geometry, and Chapter 11 of [6] for a multilinear algebra refresher.

1.1 Ideals Generated by Principal Minors

We now introduce the thesis topic, ideals generated by principal minors. Let X denote a generic matrix, which we write as

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = X_{n\times n}.$$

Let $K[X]$ denote the polynomial ring over an arbitrary algebraically closed field K, with the entries of X as its variables. In any square matrix, the minors whose row and column indices are the same are called *principal*. We focus on polynomials given by principal minors of a fixed size, t, where $1 \leq t \leq n$. Let $\mathfrak{P}_t = \mathfrak{P}_t(X) \subseteq K[X]$ denote the ideal generated by the size t principal minors of X.

Question. What geometric properties do algebraic sets defined by principal minors satisfy?

We include in this section some minor observations.

1.1.1 Initial Observations

Our first observation follows from basic multilinear algebra:

Proposition 1. *The principal t-minors of a square matrix A vanish if and only if the diagonal entries of the exterior power matrix $\wedge^t A$ (using the basis induced by the standard basis on K^N) vanish.* □

Next, for all n, we immediately see $K[X]/\mathfrak{P}_1$ is isomorphic to a polynomial ring in $n^2 - n$ variables over K, since the generators for $\mathfrak{P}_1$ are just the diagonal entries of X. We also recognize, for $X = X_{2\times 2}$, $K[X]/\mathfrak{P}_2$ as the homogeneous coordinate ring of the image of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ under the Segre embedding:

$$K\left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right] \to K[y_0, y_1] \otimes K[z_0, z_1]$$

$$x_{11},\ x_{12},\ x_{21},\ x_{22} \mapsto y_0z_0,\ y_1z_0,\ y_0z_1,\ y_1z_1,$$

respectively, yields the isomorphism

$$\frac{K[X]}{(x_{11}x_{22} - x_{12}x_{21})} \cong K[y_0, y_1] \textcircled{s} K[z_0, z_1] \subset K[y_0, y_1, z_0, z_1],$$

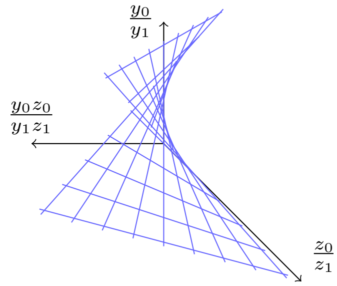

Figure 1.2: (Affine open set of) the quadric ruled surface in $\mathbb{P}^3$.

where Ⓢ denotes the Segre product, and

$$K[y_0, y_1, z_0, z_1] \cong K[y_0, y_1] \otimes K[z_0, z_1],$$

a homogeneous coordinate ring for $\mathbb{P}^3$. The corresponding algebraic set is a quadric ruled surface. A projection of it to affine space is shown in Figure 1.2.

1.1.2 Group Actions on $\mathfrak{P}_t$

We note the group actions on $\mathfrak{P}_t$ which leave it invariant. The most obvious action is $\mathbb{Z}/2\mathbb{Z}$, given by $X \mapsto X^{\mathrm{T}}$, the transpose of X. Another useful action is the symmetric group, S_n, of degree n. Suppose τ is a permutation matrix representing an element in S_n, with τ^{T} its transpose and inverse. The action $\tau : X \mapsto \tau \cdot X \cdot \tau^{\mathrm{T}}$ performs the same permutation on the rows of X as it does the columns, and thus preserves $\mathfrak{P}_t$. We also observe $\mathfrak{P}_t$ is unaffected by non-zero scalars, i.e., the actions of $\mathrm{GL}(1, K) \cong K^{\times}$ on each row and each column of X.

1.1.3 Bounds on Codimension

Let ht I denote the height of an ideal I. In our case (as in any polynomial ring), $\mathrm{ht}(\mathfrak{P}_t) = n^2 - \dim(K[X]/\mathfrak{P}_t)$. We may interchangeably use the term *codimension* of $\mathfrak{P}_t$, in which case we mean the codimension of $\mathcal{V}(\mathfrak{P}_t)$ in $\mathbb{A}^{n^2}$.

Proposition 2. $\mathrm{ht}(\mathfrak{P}_t) \leq \min\{\binom{n}{t}, (n-t+1)^2\}$.

Proof. By Krull's Height Theorem the number of generators of an ideal is always an upper bound for its height, and $\mathfrak{P}_t$ has $\binom{n}{t}$ generators. The determinantal ideal (see Section 2.1.1) I_t contains $\mathfrak{P}_t$ and has height $(n-t+1)^2$, which gives the other upper bound. □

Corollary 1. *If* $\binom{n}{t} > (n-t+1)^2$*, then* $\mathfrak{P}_t \subseteq K[X]$ *is not a complete intersection.* □

We can obtain another bound by comparing the ideals $\mathfrak{P}_t(X)$ to the corresponding Pfaffian ideals $\mathrm{Pf}_t(X)$, when we impose the alternating condition on X (see Section 2.1.2) .

Proposition 3. *If t is odd, then* $\mathrm{ht}(\mathfrak{P}_t) \leq \binom{n+1}{2}$.

Proof. Let $\mathfrak{A} \in K[X]$ denote the ideal defining the alternating condition, i.e.,

$$\mathfrak{A} = (x_{ii} \,|1 \leq i \leq n) + (x_{ij} + x_{ji} \,|1 \leq i \neq j \leq n)\,.$$

It is straightforward to see $\mathfrak{A}$ is a complete intersection, so its height is $\binom{n+1}{2}$. The generators for $\mathfrak{P}_t$, modulo $\mathfrak{A}$, are exactly the squares of the generators for Pf_t (in other words, $\mathrm{rad}\,\mathfrak{P}_t\,/\mathfrak{A} = \mathrm{Pf}_t$). Size t Pfaffians vanish when t is odd, in which case the image $\mathfrak{P}_t\,/\mathfrak{A}$ is zero. Therefore $\mathrm{ht}\,(\mathfrak{P}_t) \leq \binom{n+1}{2}$. ☐

The bound in Proposition 3 becomes relevant when $t \geq n + 1 - \sqrt{\binom{n+1}{2}}$. The first such case is when $n = 8$ and $t = 3$. We can also compare $\mathfrak{P}_t(X)$ to $\mathrm{I}_t(X)$ when we require X to be symmetric.

Proposition 4. $\mathrm{ht}(\mathfrak{P}_t) \geq (n - t + 1)^2 - \binom{n}{2}$.

Proof. We have

$$\mathfrak{P}_t + (x_{ij} - x_{ji} \,|\, i \neq j) \supseteq \mathrm{I}_t\,.$$

There are $\binom{n}{2}$ distinct linear forms which give the symmetric condition, so each contributes 1 to the codimension, and I_t has height $(n - t + 1)^2$. ☐

1.2 Statement of Results

We collect and summarize the original results in this thesis.

Theorem (Theorems 1 and 2, Corollary 2; see Section 3.1)**.** *For all n, $\mathfrak{P}_2$ is prime, normal, a complete intersection, and toric. Hence, $\mathfrak{P}_2$ is strongly F-regular and Gorenstein. Its codimension is $\binom{n}{2}$.*

See Chapter 4 of [36] for information about toric ideals. The strategies in proving Theorems 1 and 2 heavily exploit the fact that the generators for $\mathfrak{P}_2$ are binomial. The $t > 2$ cases require a different approach. It turns out for any t, the irreducible components of $\mathcal{V}(\mathfrak{P}_t)$ may be classified according to the rank of a generic element.

Let $\mathcal{Z}_{n,r}$ denote the set of all $n \times n$ matrices of rank r. This is a locally closed set whose closure is defined by the ideal $\mathrm{I}_{r+1}(X)$ of size $(r + 1)$-minors of X. As a tool in studying the components of $\mathcal{V}(\mathfrak{P}_t)$, we study the components of the locally closed sets $\mathcal{Y}_{n,r,t} = \mathcal{Z}_{n,r} \cap \mathcal{V}(\mathfrak{P}_t)$. The set $\mathcal{V}(\mathfrak{P}_t)$ is covered by the $\mathcal{Y}_{n,r,t}$. Hence, it is covered by the closures of the irreducible components of the $\mathcal{Y}_{n,r,t}$. Whenever a closed set is a finite union of closed varieties, we can find its irreducible components by making the union irredundant. Thus, the components of $\mathcal{V}(\mathfrak{P}_t)$ can be found by first finding the closures of all the components of the $\mathcal{Y}_{n,r,t}$ and then omitting the ones that are not maximal in the family.

Theorem (3, Section 3.2.1)**.** *In the localized ring $K[X][\frac{1}{\det X}]$, the K-algebra automorphism $X \mapsto X^{-1}$ induces an isomorphism $\mathcal{Y}_{n,n,t} \cong \mathcal{Y}_{n,n,n-t}$.*

Theorem (4, Section 3.3)**.** *For $n \geq 4$, $\mathcal{V}(\mathfrak{P}_{n-1})$ has two components. One is defined by the determinantal ideal $\mathrm{I}_{n-1}(X)$. The other is the Zariski closure of the locally closed set $\mathcal{Y}_{n,n,n-1}$, and has codimension n.*

Theorem 4 improves the bound in Proposition 3, including removal of the requirement t be odd.

Corollary (8, Section 3.3)**.** *For $n \neq 3$, $\mathrm{ht}(\mathfrak{P}_t) \leq \binom{n+1}{2} - \binom{t+2}{2} + 4$.*

We conjecture $\mathfrak{P}_{n-1}$ is reduced and we prove it for $n = 4$. It follows that $\mathrm{I}_3 = \mathrm{I}_3(X_{4\times 4})$ and $\mathfrak{Q}_3 = \mathfrak{Q}_3(X_{4\times 4})$, the defining ideal for the Zariski closed set $\bar{\mathcal{Y}}_{4,4,3}$, are *algebraically linked*. In Section 3.3.2 we discuss these consequences.

Theorem (Corollaries 9 in Section 3.3.1 and 10, 11, 12 in Section 3.3.2, Theorem 5 in Section 3.3.1)**.** *Suppose $n = 4$. Then $\mathfrak{P}_3$ is reduced. Consequently, I_3 and $\mathfrak{Q}_3$ are algebraically linked and hence $\mathfrak{Q}_3$ is Cohen-Macaulay with 5 generators. Furthermore, $\mathfrak{Q}_3 = \mathfrak{P}_3 :_{K[X]} \Delta$, where $\Delta = \det X$.*

The above results completely describe the geometry of all schemes defined by principal minors of a fixed size for matrices of size ≤ 4. For $n > 4$, the problem is very difficult, so we focus instead on the stratification of $\mathcal{V}(\mathfrak{P}_t)$ by the locally closed sets $\mathcal{Y}_{n,r,t}$. We begin with the locally closed sets $\mathcal{Y}_{n,n-2,n-2}$.

In studying the components of $\mathcal{Y}_{n,n-2,n-2}$ we define a surjection

$$\Theta : \mathcal{Z}_{n,n-2} \to \mathrm{Grass}(n-2,n) \times \mathrm{Grass}(n-2,n),$$

which we show has a fibre isomorphic to $\mathrm{GL}(n-2,K)$, and reduce the problem to studying pairs of sets in $\mathrm{Grass}(n-2,n)$. Given a point in the Grassmannian, we encode exactly which of its Plücker coordinates do and do not vanish in a graph. Such graphs are called *permissible* (see Section 3.4.1). We then define the notion of a permissible subvariety of the Grassmannian, along with its corresponding graph. Using the properties of permissible graphs we compute the dimension of $\mathcal{Y}_{n,n-2,n-2}$.

Theorem (8, Section 3.4)**.** dim $\mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n$.

The structure of $\mathcal{Y}_{n,t,t}$, i.e., when $r = t$, turns out to be of great interest in its own right. In fact, it leads to the study of questions that are NP-hard in general (see Ford's paper ([12]) on matroid varieties and the result he cites from [34]). In Chapter 4 we prove a few lemmas about the vanishing of particular minors in a generic matrix.

Finally, in Section 4.3.2 we show the connection between matroid varieties and subvarieties of a Grassmannian. Positroid varieties are a particularly well-behaved type of matroid variety; they are normal, Cohen-Macaulay, have rational singularities, and their defining ideals are given exactly by Plücker coordinates ([26]).

Theorem (9, Section 4.3.2)**.** *If a subset of Plücker coordinates for* $\mathrm{Grass}(n-2, n)$ *defines an irreducible algebraic set, then it is positroidal. Every irreducible component of every matroid scheme in* $\mathrm{Grass}(n-2, n)$ *is of this form, and so is positroidal.*

CHAPTER 2

Motivation and Background

In this chapter, we develop some motivation for the study of principal minors, beginning with generic matrices. We then record a handful of relevant commutative algebra lemmas. K shall always denote an arbitrary algebraically closed field.

2.1 Generic Matrices and Invariant Theory

Ideals with generators defined in terms of generic matrices arise naturally in invariant theory. Over the past half century or so, commutative algebraists have produced numerous results and applications concerning them; see, for example, [4, 7–10, 19, 23–25, 27, 32, 33]. Therefore it already seems to make sense why studying ideals generated by principal minors falls right into this tradition.

By *generic* matrix, we mean an $r \times s$ matrix $X = X_{r\times s}$ of indeterminates. When we impose certain conditions on X, such as matrix rank, we obtain systems of equations in the entries. Although there are many ways to define an ideal in a polynomial ring using generic matrices, and these matrices may be over a more general commutative ring than a field, in this chapter we only mention the three most relevant examples to the thesis: determinantal ideals, Pfaffian ideals, and defining ideals for Grassmann varieties.

2.1.1 Determinantal Ideals, I_t

Perhaps the most classical work involving generic matrices is the study of *determinantal ideals*, I_t. Let $Y = (y_{ij})_{1\le i\le r, 1\le j\le t-1}$ a generic $r \times (t-1)$ matrix and similarly, $Z = (z_{ij})$ is a generic $(t-1)\times s$ matrix. Let $T = K[Y, Z]$ denote the polynomial ring over K generated by the entries of Y and Z. Let $G = \mathrm{GL}(t-1, K)$, the general linear group of degree $t-1$ over K, and finally, suppose K is infinite. Each element $g \in G$ can be identified with an invertible $(t-1)\times(t-1)$ matrix M_g, using the standard basis on the vector space K^{t-1}.

We get an action of G on T given by, for each $g \in G$,

$$\begin{aligned} g : YZ &\mapsto YM_g^{-1}M_gZ \\ (y_{ij} &\mapsto (i,j)\text{th entry of } YM_g^{-1}, \\ z_{ij} &\mapsto (i,j)\text{th entry of } M_gZ)\,. \end{aligned}$$

The elements of T which remain invariant under the action of G comprise the subring, $T^G = K[YZ] \subseteq T$, called the *ring of invariants* in T under the action of G. De Concini and Procesi proved this, completely independent of characteristic, in 1976 ([5]). The *Second Fundamental Problem of Invariant Theory* (from [38]) is to find a defining ideal for T^G. We give a K-algebra surjection $S \twoheadrightarrow T^G$, where S is some polynomial ring over K, whose kernel is the defining ideal. Let $X = (x_{ij})$ denote a generic $r \times s$ matrix and put $S = K[X]$. Define the K-algebra map

$$\begin{aligned} \varphi : X &\mapsto YZ \\ (x_{ij} &\mapsto (i,j)\text{th entry of } YZ)\,. \end{aligned}$$

By construction, φ is surjective. The kernel is exactly $\mathrm{I}_t = \mathrm{I}_t(X)$, the ideal generated by the size t-minors of X. Hochster and Roberts famously proved, in 1974, the quotient rings $K[X]/\,\mathrm{I}_t$, along with other rings of invariants of reductive groups acting on regular rings (see [19]), are Cohen-Macaulay.

Many results about determinantal ideals include additional hypotheses about X where the entries are not as general. However, as long as the ideals have the same height as in the generic case, the results about determinantal ideals in the latter case apply ([9]). In his thesis ([10], quoted in the introduction), Eagon bounded the height of $\mathrm{I}_t(X)$, where the entries of X are any elements in a fixed commutative Noetherian ring with 1. He also proved unmixedness when $t = r$, provided I_t has maximal depth, $(r-t+1)(s-t+1) = s-r+1$. In 1971 Hochster and Eagon ([9]) showed the ideals I_t are *generically perfect*, a notion developed in [8], and they showed quotients $K[X]/I_t$ are normal. Svanes ([37] 1974) showed determinantal ideals $\mathrm{I}_t(X)$ are Gorenstein if and only if X is square.

2.1.2 Pfaffian Ideals, Pf_{2h}

The techniques from [9] can be directly applied to the family of *Pfaffian ideals* and in fact, Kleppe and Laksov ([24]) did this in 1980. Let X be an $n \times n$ alternating matrix, in which case $K[X]$ only consists of $\binom{n}{2}$ indeterminates. (To avoid any characteristic 2 caveats, we say *alternating*, in the tradition of [1], to describe a skew symmetric matrix whose main diagonal vanishes.) The Pfaffian ideals $\mathrm{Pf}_t = \mathrm{Pf}_t(X)$ are then generated by the $t = 2h$ square roots of each of the symmetrically placed $2h$-minors of X. Pfaffian ideals are defined as zero when t is odd. The *Pfaffian rings* are the quotients $K[X]/\mathrm{Pf}_t$.

Pfaffian rings also appear in invariant theory; let $Y = (y_{ij})$ denote a generic $(2h-2) \times n$ matrix and let $G = \mathrm{Sp}(2h-2, K)$ denote the symplectic group over K^{2h-2}. We let G act on $T = K[Y]$, again via matrix multiplication

$$M_g : y_{ij} \mapsto (i,j)\text{th entry of } M_g Y$$

for each $g \in G$, where M_g is the matrix realization (using the standard basis for K^{2h-2}) of the group element g. The ring of invariants T^G is K-generated by the skew products $\langle \mathbf{y}_i, \mathbf{y}_j \rangle$, where $1 \leq i, j \leq n$ and $\mathbf{y}_i, \mathbf{y}_j$ denote the column vectors of the matrix Y indexed by their respective subscripts. Let X denote a generic size n alternating matrix. Map $K[X] \to T^G$ via

$$\varphi : x_{ij} \mapsto \langle \mathbf{y}_i, \mathbf{y}_j \rangle$$

for each $i, j = 1, \dots n$. The kernel of φ is $\mathrm{Pf}_{2h}(X)$. See [5] for a characteristic-free proof of these facts.

Along with Cohen-Macaulayness, Kleppe and Laksov in [24] showed, characteristic-freely, that Pfaffian ideals are normal and Gorenstein. Kleppe ([25]) showed in 1978 that for any size alternating matrix Pf_{2h} is reduced, irreducible, and has singular locus Pf_{2h-2}. He also included a new computation of Room's result, $\mathrm{ht}\,\mathrm{Pf}_{2h} = \binom{n-2h+2}{2}$. Finally, we mention a famous result of Buchsbaum and Eisenbud ([4] 1979): Any Gorenstein ideal of height 3 in a regular local ring arises by depth-preserving base change from a generic Pfaffian example. In other words, a Gorenstein ideal of height 3 in a regular local ring has a realization as $\mathrm{Pf}_{2h}(X'_{(2h+1)\times(2h+1)})$, where X' is some alternating matrix such that ideals in $K[X']$ have with the same height over the generic $(2h+1)\times(2h+1)$ alternating matrix of indeterminates.

2.1.3 Grassmannians

Let $r \leq n$. The *Grassmann variety*, also called the *Grassmannian*, denoted $\mathrm{Grass}(r, K^n)$ or, when context is clear, $\mathrm{Grass}(r, n)$, is the set of r-dimensional vector subspaces of K^n. See Example 10.31 of [22] for more information on the following construction: Let $X = X_{r\times n}$ denote a generic matrix, with $r \leq n$, and let $T = K[X]$. Let $G = \mathrm{SL}(r, K)$, the special linear group of degree r over K. An element $g \in G$ can be identified with an $r \times r$ matrix M_g whose determinant is 1. Let G act on T via

$$M_g : x_{ij} \mapsto (i,j)\text{th entry of } M_g X.$$

In all characteristics ([5]), the K-algebra generators for the ring of invariants T^G are exactly the $\binom{n}{r}$ size r minors of X.

To describe the defining ideal for T^G, we first introduce some notation. Let

$$\underline{\mathrm{i}} = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\},$$

where we assert $i_1 < \cdots < i_r$. Let $\Delta_{\underline{\mathrm{i}}}$ denote the r-minor given by the $\underline{\mathrm{i}}$th columns of X. Put $S = K\left[y_{\underline{\mathrm{j}}} \,|\, \underline{\mathrm{j}} = \{j_1, \ldots, j_{n-r}\} \subseteq \{1, \ldots, n\}\right]$. We define a K-algebra map

$$\begin{aligned} \varphi : S &\twoheadrightarrow T^G \\ y_{\underline{\mathrm{j}}} &\mapsto \Delta_{\{1,\ldots,n\}\setminus \underline{\mathrm{j}}}. \end{aligned}$$

The generators for the kernel of φ, also called the *Plücker relations* on the size r minors of X, are quadratic and they give the defining ideal for $T^G \cong K[\wedge^r X]$.

As a projective variety we identify $\mathrm{Grass}(r, n)$ with $\mathrm{Proj}(S/\ker\varphi) \subseteq \mathbb{P}^{\binom{n}{r}-1}$. When we consider a vector space $V \in \mathrm{Grass}(r, n)$, we may identify it with a point $\mathbf{g} = \mathbf{g}_V = \left[\cdots : g_{\underline{\mathrm{i}}} : \cdots\right]$ in $\mathbb{P}^{\binom{n}{r}-1}$, under the *Plücker embedding*. The entries $g_{\underline{\mathrm{i}}}$ are called *Plücker coordinates* for $\mathbf{g}$.

2.2 The Geometry of an Ideal

Much of the material we present in this section is standard, but we provide it for the sake of context. When we use an ideal to define an algebraic set $\mathcal{V}(I)$, various qualities of I allow us to describe the geometry of $\mathcal{V}(I)$. All rings in this section we shall assert are commutative, Noetherian, and with 1.

2.2.1 Local vs. Graded Local Rings

Graded rings are a generalized notion of polynomial rings. Formally, a ring R is *graded* means it can be writen

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

where the direct summands are $\mathbb{Z}$-modules, and for all $i, j \in \mathbb{Z}$, $R_i R_j \subset R_{i+j}$. The polynomial rings $K[X]$ are $\mathbb{N}$-graded, meaning we may write

$$R = K[X] = K \oplus R_1 \oplus R_2 \oplus \ldots$$

with $R_0 = K$ and for each $d > 0$, R_d is generated, as a $\mathbb{Z}$-module and in fact as a K-vector space, by the monomials of degree d. Quotients of $K[X]$ by homogeneous ideals are also $\mathbb{N}$-graded and collectively they fall into the category of *standard graded* K-algebras.

Graded rings appear as homogeneous coordinate rings of projective varieties, with homogeneous ideals defining algebraic sets. If R is standard graded as a K-algebra, then the generators of R_1 also generate the unique homogeneous maximal ideal, $\mathfrak{m}$, of R. $\mathrm{Proj}(R)$, by definition, has no point corresponding to $\mathfrak{m}$, which is why we sometimes refer to $\mathfrak{m}$ as the *irrelevant ideal*. The analogous relationship between projective and affine varieties explains a very useful property of graded rings: Many statements about local rings remain true when we replace "local" with "graded local", meaning the ring has a unique homogeneous maximal ideal, and we require all ideals and modules in the statements to be homogeneous as well. Chapter 1.5 of [3] develops the validity of this claim. Unless stated otherwise, when we state a result about local rings, it is implicit the result applies to graded local rings as well.

2.2.2 Primary Decomposition

We, like many authors, use the term *variety* to mean reduced and irreducible scheme. Thus, for an ideal I, the algebraic set $\mathcal{V}(I)$ is a variety if and only if its radical, $\mathrm{rad}\, I$, is prime. The Noetherian property implies every ideal I has a *primary decomposition* (sometimes called a *Noether-Lasker decomposition*). The *associated primes* of I are the radicals of the components in its primary decomposition. I is reduced if and only if its components are prime. The minimal primes, with respect to containment in the set $\mathrm{Ass}(I)$ of associated primes for I, are unique, but any *embedded primes* are not. The following lemma permits one of the main premises we use in proving Theorem 5:

Lemma 1. *Suppose the ideal $I \in R$ has no embedded primes. R/I is reduced if and only if the localization R_P/IR_P is reduced for all minimal primes P of I. In particular, this is the case when R/I is Cohen-Macaulay.*

Proof. Suppose the minimal primes of I are $P_1, \dots, P_h$, and put $W = R \setminus (P_1 \cup \cdots \cup P_h)$, the set of non-zero-divisors for R/I. Then

$$R/I \hookrightarrow W^{-1}R/I \cong \frac{R_{P_1}}{IR_{P_1}} \times \cdots \times \frac{R_{P_h}}{IR_{P_h}}.$$

□

CHAPTER 3

Ideals Generated by Principal Minors

For the remainder of this thesis let $X = X_{n\times n} = (x_{ij})_{1\le i,j\le n}$ denote a size n square matrix of indeterminates and let $K[X]$ denote the polynomial ring over the algebraically closed field K, generated by the entries of X. Let $\mathfrak{P}_t = \mathfrak{P}_t(X) \subseteq K[X]$ denote the ideal generated by the size t principal minors of X.

3.1 Principal 2-Minors Case

As we saw in Section 1.1.1, the characterization of ideals $\mathfrak{P}_1$, for any n, is trivial. The next case we consider is when $t = 2$, fixing n. We will first show $R = K[X]/\mathfrak{P}_2$ is a complete intersection, which implies Gorenstein. We then show $\mathfrak{P}_2$ is toric, hence is prime. We then show R is normal. Hochster showed in [16] (1972) that normal quotients of toric ideals are direct summands of Laurent polynomial rings, a fact that implies $K[X]/\mathfrak{P}_2$ is F-regular ([17]). Furthermore, since R is Gorenstein all notions for R of F-regularity, strong F-regularity, and weak F-regularity are equivalent ([18]).

Theorem 1 (–)**.** *For all n, $K[X]/\mathfrak{P}_2$ is a complete intersection domain.*

A key in the proof of Theorem 1 is that $\mathfrak{P}_2$ is *toric*, meaning the quotient $K[X]/\mathfrak{P}_2$ is isomorphic to a ring generated by monomials. We also exploit the fact that complete intersections are Cohen-Macaulay.

Proof. We saw in Section 1.1.1 that the $n = 2$ case gives the homogeneous coordinate ring for $\mathbb{P}^1 \times \mathbb{P}^1$, and that it is a complete intersection domain. We proceed by induction on n: let $X' = (u_{ij})_{1\le i,j\le n-1}$ denote a size $n-1$ matrix of indeterminates and suppose $\mathfrak{P}_2(X')$ satisfies the theorem. Append to the bottom of X' the row $(x_1 \ \cdots \ x_{n-1})$, then to the far right the column $(y_1, \dots, y_{n-1}, z)$. Let X denote the resulting size n matrix. The ideal generated by the principal 2-minors is

$$\mathfrak{P}_2(X) = \mathfrak{P}_2(X')K[X] + (zu_{ii} - x_iy_i \,|\, i = 1, \dots, n-1).$$

Put $A = K[X']/\mathfrak{P}_2(X')$, and

$$R = \frac{A[x_1, \ldots, x_{n-1}, z, y_1, \ldots, y_{n-1}]}{(zu_{ii} - x_i y_i)} \cong \frac{K[X]}{\mathfrak{P}_2(X)}.$$

First, we show R is a complete intersection. By the induction hypothesis, A is a complete intersection domain, hence so is the polynomial ring $\tilde{A} = A[x_1, \ldots, x_{n-1}, z, y_1, \ldots, y_{n-1}]$. In particular, $\tilde{A}$ is Cohen-Macaulay. It follows that if $zu_{ii} - x_i y_i$, for $i = 1, \ldots, n-1$, form a regular sequence on $\tilde{A}$, then R is also a complete intersection. Our strategy is to show $x_i y_i$ form a regular sequence in $\tilde{A}/(z)$. All polynomials we consider are homogeneous, so $zu_{ii} - x_i y_i$ form a regular sequence if and only if

$$zu_{11} - x_1 y_1, \ldots, zu_{n-1,n-1} - x_{n-1} y_{n-1}, z$$

form a regular sequence, if and only if

$$z, zu_{11} - x_1 y_1, \ldots, zu_{n-1,n-1} - x_{n-1} y_{n-1}$$

form a regular sequence. The strategy works because clearly z is not a zero-divisor in the domain $\tilde{A}$. Working now in $\tilde{A}/(z)$, there are 2^{n-1} minimal primes for the ideal $I = (x_i y_i \mid i = 1, \ldots, n-1)$, each generated by picking one variable from each pair $\{x_i, y_i\}$. Therefore I has (pure) height $n-1$. Height and depth of an ideal are equal in a Cohen-Macaulay ring, so the generators for I must form a regular sequence, as desired.

We now show R is a domain, by showing it is isomorphic to a *semigroup ring* (see Chapter 7 of [28]). We first claim the variables x_i, y_i, for $i = 1, \ldots, n-1$, are not zero-divisors on R. Since R is Cohen-Macaulay it suffices to show each is a homogeneous parameter. Fix i and suppose we kill a minimal prime, P, of (x_i). The minor $zu_{ii} - x_i y_i$ is in $(x_i) \subset P$, so P must also contain either u_{ii} or z. If $z \in P$ then we already know the dimension drops, since z is a parameter. On the other hand, suppose $u_{ii} \in P$. Then the relations $u_{ii}u_{jj} - u_{ij}u_{ji} = 0$ imply, for each $j \neq i$, either u_{1j} or u_{j1} is in P. By the induction hypothesis none of these variables are zero-divisors, so again, the dimension drops and we are done.

Having shown x_i, y_i are not zero-divisors on R, we next observe R injects into its localization at any subset of the variables $\{x_i,\, y_i \mid i = 1, \ldots, n-1\}$. Fixing i, if we invert either x_i or y_i then we can use the principal 2-minor relations to solve for the other. The same arguments held for the smaller matrix X', so we may invert, say, all of the entries below the diagonal of X, then solve for the ones above the diagonal. The resulting K-algebra, isomorphic to R, is generated by the $\binom{n+1}{2}$ indeterminates on or below the diagonal of X, along with monomials of the form $x_{ii}x_{jj}x_{ji}^{-1}$ for $i < j$. Therefore R is a semigroup ring. $\square$

Example 2. We illustrate the proof of Theorem 1 in an example. Suppose $n = 3$. If $X' = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$, then we saw $K[X']/\mathfrak{P}_2(X')$ satisfies the theorem. Put

$$X = \begin{pmatrix} X' & \begin{matrix} y_1 \\ y_2 \end{matrix} \\ \begin{matrix} x_1 & x_2 \end{matrix} & z \end{pmatrix}.$$

Then

$$\mathfrak{P}_2(X) = (u_{11}u_{22} - u_{12}u_{21})K[X] + (zu_{11} - x_1y_1, zu_{22} - x_2y_2).$$

Let $R = K[X]/\mathfrak{P}_2(X)$. We can invert the entries u_{21}, x_1, x_2, which are below the main diagonal of X, then use the equations in $\mathfrak{P}_2$ to get an expression for the variables above the diagonal. Therefore,

$$R \cong K\left[u_{11}, u_{22}, z, u_{21}, x_1, x_2, \frac{u_{11}u_{22}}{u_{21}}, \frac{u_{11}z}{x_1}, \frac{u_{22}z}{x_2}\right].$$

Having proved $K[X]/\mathfrak{P}_2$ is a domain, we may apply Serre's criteria for normality (see [3], Chapter 2.2). The following lemma describes the condition, which we use in proving Theorem 2.

Lemma 2. *Let R denote a domain with fraction field $\mathcal{K}$.*
(a) Suppose $\frac{a}{b} \in \mathcal{K} \setminus R$, for some $a, b \in R$. Then the ideal $I = \{r \in R \,|\, r\frac{a}{b} \in R\}$ has depth 1.
(b) Let x, y be a regular sequence in R such that $R[\frac{1}{x}]$ and $R[\frac{1}{y}]$ are regular. Then R is normal.

Proof. I kills the image of a in R/bR because $b \in I$. Therefore, a kills I/bR and thus I has depth 1, proving (a). To show (b), suppose $\frac{a}{b} \in \mathcal{K} \setminus R$ is integral over R. Then there exist N, N' such that $x^N\frac{a}{b} \in R$ and $y^{N'}\frac{a}{b} \in R$. Let I denote the ideal from (a), so that $x^N, y^{N'} \in I$. Since x, y form a regular sequence, so do $x^N, y^{N'}$. But this contradicts depth $I = 1$. We therefore conclude $\frac{a}{b} \in R$. □

Theorem 2 (–)**.** *For all n, $K[X]/\mathfrak{P}_2$ is normal.*

Proof. Let J denote the defining ideal of the singular locus for $R = K[X]/\mathfrak{P}_2$. Serre's condition says if J has depth at least 2, then R is normal. Theorem 1 showed R is Cohen-Macaulay, so it is enough to show J has height at least 2.

From the proof of Theorem 1, we saw the product of the entries below the diagonal of X is in J, as is the product of the entries above the diagonal. More generally, let J' denote the ideal generated by the degree n monomials whose factors consist of exactly one variable from each pair $\{x_{ij}, x_{ji}\}$, $1 \le i \neq j \le n$. Then $J' \subseteq J$, since for all such monomials μ, $R[\frac{1}{\mu}]$ is a localized polynomial ring. We will show each of the minimal primes of J' contains some height 2 ideal (x_{ij}, x_{ji}).

Let P be a minimal prime of J'. If for each pair $\{x_{ij}, x_{ji}\}$ we can choose one not in P, multiply these choices together to get an element $u \in J' \setminus P$, a contradiction. Therefore there exists some pair $x_{ij}, x_{ji} \in P$. Now suppose we kill that pair. The principal 2-minor relation implies either x_{ii} or x_{jj} must also vanish; without loss of generality, say x_{ii}. Then, for any $k \neq i, j$, the relation $x_{ii}x_{kk} - x_{ik}x_{ki}$ implies either x_{ik} or x_{ki} must vanish. Thus the dimension drops by at least $2 + 1 + (n-2) = n+1$. There are $\binom{n-1}{2}$ principal 2-minors not involving variables with i in the index. Thus the dimension goes down to $n^2 - (n+1) - \binom{n-1}{2} = \binom{n+1}{2} - 2$, as desired. $\square$

Corollary 2 (–)**.** *For all n, $K[X]/\mathfrak{P}_2$ is strongly F-regular, and hence, F-regular.*

Proof. Since a normal ring generated by monomials is a direct summand of a regular ring, this follows from [16] and [17]. $\square$

3.2 Using Matrix Rank to Find Minimal Primes of Principal Minor Ideals

Characterizing the ideals $\mathfrak{P}_2$ was a unique endeavor because the generators, the principal 2-minors, are binomials. This fact made it easy to solve for entries in X and construct a ring, isomorphic to $K[X]/\mathfrak{P}_2$, that was easier to describe. Unfortunately, once $t > 2$, the generators for $\mathfrak{P}_t$ are not binomial and another strategy is required. It turns out for any t, the components of $\mathcal{V}(\mathfrak{P}_t)$ may be classified according to the rank, r, of a generic element. For fixed n, r, t with $1 \le t, r \le n$, let $\mathcal{Y}_{n,r,t} \subset \mathcal{V}(\mathfrak{P}_t)$ denote the locally closed subset of matrices with rank exactly r.

For any scheme, $\mathcal{Y}$, the closures of the irreducible components of a non-empty open set $\mathcal{U}$ are irreducible components of $\mathcal{Y}$: Say $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_h$ are the components. Then

$$\mathcal{U}_0 = \mathcal{U} \setminus (\mathcal{U}_2 \cup \cdots \cup \mathcal{U}_h) \neq \emptyset$$

is open because $\mathcal{U}_2 \cup \cdots \cup \mathcal{U}_h$ is closed in $\mathcal{U}$, so is open in $\mathcal{U}_1$. Let $\mathrm{cl}_{\mathcal{Y}}(\mathcal{U}_1)$ denote the closure of $\mathcal{U}_1$ in $\mathcal{Y}$. Then $\mathrm{cl}_{\mathcal{Y}}(\mathcal{U}_1) = \mathrm{cl}_{\mathcal{Y}}(\mathcal{U}_0)$ is irreducible, closed, and contains $\mathcal{U}_0$, which is open in

$\mathcal{Y}$. We have

$$\begin{aligned}\bar{\mathcal{U}}_0 &= \bigcup_{i=1}^{h'} \bar{\mathcal{U}}_0 \cap \mathcal{Y}_i \\ &\subseteq \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_{h'} = \mathcal{Y},\end{aligned}$$

where $\mathcal{Y}_1, \ldots, \mathcal{Y}_{h'}$ are the components of $\mathcal{Y}$. This implies $\bar{\mathcal{U}}_0 \subseteq \mathcal{Y}_i$ for some i. $\mathcal{U}_0$ is open in $\mathcal{Y}_i$ and $\mathcal{Y}_i$ is irreducible, hence $\mathcal{U}_0$ is dense in $\mathcal{Y}_i$. Therefore $\bar{\mathcal{U}}_0 = \bar{\mathcal{U}}_1 = \mathcal{Y}_i$ is a component of $\mathcal{Y}$.

We study the components of $\mathcal{Y}_{n,r,t}$ and take their Zariski closures, $\bar{\mathcal{Y}}_{n,r,t}$, in $\mathcal{V}(\mathfrak{P}_t)$. The components of $\mathcal{V}(\mathfrak{P}_t)$ will be among those closures as we vary r, since they are irreducible closed sets whose union is $\mathcal{V}(\mathfrak{P}_t)$. The issue is which ones are maximal.

3.2.1 Rank $r = n$

We have a convenient way to simplify the study of components of $\mathcal{Y}_{n,n,t}$, which relies on the following classical theorem, stated and proved in Sir Thomas Muir's 1882 text, *A Treatise on the Theory of Determinants.*

Theorem (Muir, [29] §96)**.** *If the determinant adjugate to a given determinant be formed, any minor of it of the tth order is equal to the product obtained by multiplying the cofactor of the corresponding minor in the original determinant by the $(t-1)$th power of the original determinant.* □

To frame Muir's theorem in modern terms, we first recall some definitions: For any $n \times n$ matrix A, suppose $\underline{\mathrm{i}}, \underline{\mathrm{j}} \subset \{1, \ldots, n\}$ are indexing sets of cardinality t:

$$\begin{aligned}\underline{\mathrm{i}} &= \{i_1, \ldots, i_t\} \\ \underline{\mathrm{j}} &= \{j_1, \ldots, j_t\}\end{aligned}$$

The $(\underline{i}, \underline{j})$*th cofactor* of A is

$$A_{\underline{\mathrm{i}},\underline{\mathrm{j}}} = (-1)^{\sigma} \det(A'),$$

where $A' = A(\{1, \ldots, n\} \setminus \underline{\mathrm{i}}; \{1, \ldots, n\} \setminus \underline{\mathrm{j}})$ is the submatrix of A given by the rows indexed by the complement $\{1, \ldots, n\} \setminus \underline{\mathrm{i}}$ and the columns indexed by $\{1, \ldots n\} \setminus \underline{\mathrm{j}}$, and

$$\sigma = i_1 + \cdots + i_t + j_1 + \cdots + j_t.$$

The *cofactor matrix* of A, denoted $\mathrm{cof}(A)$, is the matrix whose (i, j)th entry is the cofactor $A_{\{i\},\{j\}}$. We shall often abuse notation and write $A_{ij} = A_{\{i\},\{j\}}$. The *classical adjoint* of A, denoted $\mathrm{adj}(A)$, is the transpose, $\mathrm{cof}(A)^{\mathrm{T}}$, of the cofactor matrix. We have the identity

$$A \cdot (\mathrm{adj}\ A) = (\mathrm{adj}\ A) \cdot A = (\det\ A) \cdot \mathbf{I}_n, \tag{3.1}$$

which also gives the better known formula, $A^{-1} = \frac{1}{\det\ A}\,\mathrm{adj}\ A$, provided $\det\ A \neq 0$.

Restating Muir's theorem, let A be an $n \times n$ matrix. Suppose μ is a size t minor of adj A, indexed by the rows $\underline{i}$ and columns $\underline{j}$ of adj A. Then

$$\mu = (\det\ A)^{t-1} \cdot A_{\underline{i},\underline{j}}. \tag{3.2}$$

Corollary 3. *Suppose A is an $n \times n$ matrix and* $\det\ A = 0$*. Then* adj A *has rank at most one.*

Proof. By Equation (3.2), $\det\ A = 0$ implies every minor of adj A of size $t \geq 2$ vanishes. □

Corollary 4. *Suppose A is an invertible $n \times n$ matrix. For $1 \leq t < n$, the principal $(n-t)$-minors of A vanish if and only if the principal t-minors of A^{-1} vanish (if and only if the principal t-minors of* adj A *vanish).*

Proof. Principal minors are exactly those with symmetric indices. Therefore Equation (3.2) gives the bijection between the principal t-minors of adj $A = (\det\ A)A^{-1}$ and the principal $(n-t)$-minors of A. □

In fact, we can deduce a much stronger statement than that of Corollary 4. Let S denote the polynomial ring $K[X]$. Put $\Delta = \det\ X$, and let $S_\Delta = S[\frac{1}{\Delta}]$. Finally, though we only use this notation after stating and proving Theorem 3, let $D_\Delta \subset \mathbb{A}^{n^2} = \mathrm{Spec}(S)$ denote the distinguished open set of $n \times n$ matrices with non-vanishing determinant.

Theorem 3 (–)**.** $\mathcal{Y}_{n,n,t} \cong \mathcal{Y}_{n,n,n-t}$.

Proof. $\mathcal{Y}_{n,n,t}$ is the subscheme of $\mathrm{Spec}(S_\Delta) \cong \mathrm{GL}(n,K)$, defined by the vanishing of the ideal $\mathfrak{P}_t\, S_\Delta$. $\mathrm{GL}(n,K)$ has an automorphism, sending $g \mapsto g^{-1}$ for each $g \in \mathrm{GL}(n,K)$, which induces an algebra automorphism, $\Phi : S_\Delta \to S_\Delta$, sending the entries of X to the respectively indexed entries of X^{-1}:

$$\Phi : x_{ij} \mapsto (-1)^{i+j}\frac{1}{\Delta}X_{ji}$$

It is clear Φ is its own inverse. So by Muir's theorem (Equation (3.2)), each $t \times t$ minor of X is mapped to the complementarily indexed $(n-t) \times (n-t)$ minor of X^{-1}, multiplied by Δ^{1-t}. Hence $\mathfrak{P}_t\, S_\Delta \leftrightarrow \mathfrak{P}_{n-t}\, S_\Delta$. And,

$$\frac{S_\Delta}{\mathfrak{P}_t\, S_\Delta} \cong \frac{S_\Delta}{\mathfrak{P}_{n-t}\, S_\Delta}$$

induces the isomorphism on the respective schemes. □

Corollary 5 (–). *$\mathcal{Y}_{n,n,n-1}$ has one component, and its codimension is n (where $n \geq 2$).*

Proof. Equation (3.1) implies, when $\Delta \neq 0$, that the principal $(n-1)$-minors of X vanish if and only if the diagonal entries of X^{-1} vanish. Put $S' = S/\mathfrak{P}_1$, a domain since it is isomorphic to a polynomial ring with $n^2 - n$ variables, and redefine $\Phi = \phi' \circ \phi$ as the composition map

$$S_\Delta \xrightarrow{\phi} S_\Delta \xrightarrow{\phi'} S'_\Delta \tag{3.3}$$

which takes X to the image of its inverse in S'_Δ, the coordinate ring for the invertible matrices with zeros on the main diagonal. By construction, Φ is surjective. The Krull dimension of S_Δ is n^2, and likewise, the Krull dimension of S'_Δ does not change upon localization, so is $n^2 - n$. The rings S_Δ and S'_Δ both happen to be Cohen-Macaulay, so in particular, are catenary. Therefore, we may conclude $S_\Delta/(\ker \Phi) \cong S'_\Delta$ implies $\operatorname{ht}(\ker \Phi) = n$. □

Corollary 6 (–). *$\mathcal{Y}_{n,n,n-2}$ has one component, and its codimension is $\binom{n}{2}$.*

Proof. The proof is virtually identical to that of Corollary 5, with the following modifications:

- Equation (3.1) implies, when $\Delta \neq 0$, that the principal $(n-2)$-minors of X vanish if and only if the diagonal entries of $\wedge^2 X^{-1}$ vanish.
- Put $S' = S/\mathfrak{P}_2$, a domain by Theorem 1.
- S'_Δ, the coordinate ring for the invertible matrices whose principal 2-minors vanish.
- The Krull dimension of S'_Δ does not change upon localization, so by Theorem 1, is $\binom{n}{2}$.
- $\operatorname{ht}(\ker \Phi) = \binom{n}{2}$.

□

3.2.2 Rank $r = t$

To study the locally closed sets $\mathcal{Y}_{n,t,t}$, we observe and use their relationship with Grassmann varieties. To begin, for any matrix A, let $\operatorname{col} A$ and $\operatorname{row} A$ denote, respectively, the column space and row space of A. Recall, the Grassmann variety, $\operatorname{Grass}(r,n) \subset \mathbb{P}^{\binom{n}{r}-1}$, is the projective variety whose points are in bijection with the r-dimensional vector spaces of K^n. As shorthand, put $\mathcal{G} = \operatorname{Grass}(r,n)$ for fixed $t \leq r \leq n$.

Suppose $\mathbf{g} \in \mathcal{G}$ is the column space of an $n \times r$ matrix B of full rank. Letting $B(\underline{\mathrm{i}})$ denote the submatrix of B consisting of the rows indexed by elements in $\underline{\mathrm{i}} \subset \{1, \ldots, n\}$, there exists such an $\underline{\mathrm{i}} = \{i_1, \ldots, i_r\}$ satisfying $\det B(\underline{\mathrm{i}}) \neq 0$. Thus we may perform row operations on B to get a unique matrix B', such that $\operatorname{col} B' = \mathbf{g}$ and $B'(\underline{\mathrm{i}}) = \mathbf{I}_r$, the $r \times r$ identity matrix. We shall call B' the *normalized* form of B with respect to $\underline{\mathrm{i}}$. Likewise, if $\operatorname{row} C \in \mathcal{G}$ for some $r \times n$ matrix C, then we may define the normalized form C' of C with respect to a set of column indices, $\underline{j} = \{j_1, \ldots, j_r\}$, provided the submatrix of C consisting of the columns indexed by elements in $\underline{j}$, $C(\underline{j})$, is non-zero.

The following proposition gives the relationship we need between Grassmann varieties and locally closed sets $\mathcal{Y}_{n,r,t}$, for $1 \leq t \leq r \leq n$.

Proposition 5. *Let $\mathcal{Z}_{n,r} \in \mathbb{A}^{n^2}$ denote the set of $n \times n$ matrices of rank exactly r. Then*

$$\begin{aligned} \Theta :\ \mathcal{Z}_{n,r} &\to \mathcal{G} \times \mathcal{G} \\ A &\mapsto (\operatorname{col} A, \operatorname{row} A)\,. \end{aligned} \tag{3.4}$$

is a bundle map whose fibres are each isomorphic to $\mathrm{GL}(r, K)$.

Proof. The sets where a specified Plücker coordinate does not vanish give an affine open cover of $\mathcal{G}$, and hence of $\mathcal{G} \times \mathcal{G}$. Explicitly, $\mathcal{G}$ is covered by the (open) sets

$$\mathcal{G}_{\underline{i}} = \{\mathbf{g} \in \mathcal{G} \,|\, g_{\underline{i}} = 1\} \cong \mathbb{A}^{r(n-r)}.$$

(See Section 2.1.3 for notation regarding Grassmannians.)

We will show, for each open set $\mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}}$, that the diagram

$$\begin{tikzcd} \Theta^{-1}\left(\mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}}\right) \arrow[r, "\cong"] \arrow[d, "\Theta"'] & \mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}} \times \mathrm{GL}(r, K) \arrow[d, "\pi"] \\ \mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}} \arrow[r, "="] & \mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}} \end{tikzcd}$$

commutes, where π is the product projection. The preimage of $\mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}}$ consists of matrices $A \in \mathcal{Z}_{n,r}$ that factor

$$A = BC = B'A(\underline{i};\underline{j})C', \tag{3.5}$$

where B' is the normalization with respect to $\underline{i}$ of the $n \times r$ matrix B and C' is the normalization with respect to $\underline{j}$ of the $r \times n$ matrix C, and $A(\underline{i};\underline{j})$ is the submatrix of A consisting of its $\underline{i}$-rows and $\underline{j}$-columns. By uniqueness of the normalizations, given fixed $\underline{i}, \underline{j}$, such pairs of matrices (B', C') are in bijection with points in $\mathcal{G}_{\underline{i}} \times \mathcal{G}_{\underline{j}}$. For any fixed pair (B', C'), the set of all possibilities for $A(\underline{i};\underline{j})$ that satisfy Equation (3.5) is in bijection with $\mathrm{GL}(r, K)$. The maps are clearly regular. □

We can use Proposition 5 to get a bound on the codimension of $\mathcal{Y}_{n,r,t}$ in $\mathbb{A}^{n^2}$.

Corollary 7. $\dim \mathcal{Y}_{n,r,t} \leq 2rn - r^2$.

Proof. We have the containment $\mathcal{Y}_{n,r,t} \subset \mathcal{Z}_{n,r}$. Since dim $\mathcal{G} = r(r-n)$, the bundle map Θ (Equation (3.4)) gives

$$\dim\, \mathcal{Z}_{n,r} = r(n-r) + r^2 + r(n-r) = 2rn - r^2 \geq \dim\, \mathcal{Y}_{n,r,t}.$$

$\square$

Example 3. Suppose we factor $A \in \mathcal{Y}_{n,r,t} \subset \mathcal{Z}_{n,r}$ as in Equation (3.5), with

$$\begin{aligned}\underline{\mathrm{i}} &= \{1, \ldots, r\}\\ \underline{\mathrm{j}} &= \{n-r+1, \ldots, n\}.\end{aligned}$$

Points in a Grassmannian are parametrized by the entries of normalized matrices, and in fact, this is one way to show $\dim \mathcal{G} = r(n-r)$. By our hypotheses on A, we have $\Theta(A) \in \mathcal{G}_{\underline{\mathrm{i}}} \times \mathcal{G}_{\underline{\mathrm{j}}}$. We count how many parameters uniquely determine A by writing

$$A = \begin{pmatrix} & \mathbf{I}_r & & \\ b_{r+1,1} & b_{r+1,2} & \cdots & b_{r+1,r} \\ b_{r+2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ b_{n,1} & \cdots & \cdots & b_{n,r} \end{pmatrix} \begin{pmatrix} a_{1,n-r+1} & a_{1,n-r+2} & \cdots & a_{1,n} \\ a_{2,n-r+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{r,n-r+1} & \cdots & \cdots & a_{r,n} \end{pmatrix} \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-r} & \\ c_{2,1} & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \vdots & \mathbf{I}_r \\ c_{r,1} & \cdots & \cdots & c_{n,n-r} & \end{pmatrix}.$$

The matrix in the middle is $A(\underline{\mathrm{i}};\underline{\mathrm{j}})$ and is invertible. Since invertibility is a Zariski open condition, the dimension is not affected (i.e., $\dim(\mathrm{GL}(r,K)) = r^2$). Thus there are $r(n-r)+r^2+r(n-r) = 2rn-r^2$ parameters which uniquely determine A. By the symmetric group action described in Section 1.1.2, analogous arguments may be applied for any $\underline{\mathrm{i}}, \underline{\mathrm{j}}$. We conclude all sets in the cover

$$\bigcup_{\substack{\underline{\mathrm{i}},\underline{\mathrm{j}} \subseteq \{1,\ldots,n\} \\ |\underline{\mathrm{i}}|=|\underline{\mathrm{j}}|=r}} \left(\mathcal{G}_{\underline{\mathrm{i}}} \times \mathcal{G}_{\underline{\mathrm{j}}}\right) \cap \Theta(\mathcal{Y}_{n,r,t}) = (\mathcal{G} \times \mathcal{G}) \cap \Theta(\mathcal{Y}_{n,r,t})$$

have dimension $2r(n-r)$; add the dimension of the fibre, r^2, to get the desired statement of Corollary 7.

Corollary 7 is independent of t. We now describe the conditions on A that are inherent in the fact that A is in $\mathcal{Y}_{n,r,t}$. Suppose we factor A as in Equation (3.5). Then we must have $\underline{\mathrm{i}} \neq \underline{\mathrm{j}}$. Furthermore, requiring the size t principal minors of A to vanish means, equivalently, the diagonal entries of the exterior power matrix $\wedge^t A$ must vanish (Proposition 1). Write

$$\wedge^t A = \left(\wedge^t B\right) \cdot \left(\wedge^t \left(A(\underline{\mathrm{i}};\underline{\mathrm{j}})\right)\right) \cdot \left(\wedge^t C\right).$$

In the special case where $t = r$ each of the factors $\wedge^t B, \wedge^t C$ are, respectively, column and row vectors, while $\wedge^t A(\underline{\mathrm{i}};\underline{\mathrm{j}})$ is a (non-zero) scalar.

Note how, up to sign, the Plücker coordinates of the column space of any $n \times r$ matrix of full rank, $r \le n$, are the coordinates of the wedge of the columns with respect to the basis $\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r} \,|\, 1 \le i_1 < \cdots < i_r \le n\}$, where for $i \in \{1, \ldots, n\}$, $\mathbf{e}_i$ is the standard basis vector in K^n given by the ith row of the identity matrix (and the analogous statement holds for the row space of any $r \times n$ matrix of full rank, $r \le n$). Thus, the principal t-minors of $A \in \mathcal{Z}_{n,t}$ vanish if and only if the component-wise product of $\wedge^t B$ and $\wedge^t C$ is zero.

3.2.3 Other Ranks $r \neq n, t$

When $r < t$ for fixed n, r, t, the components of $\mathcal{Y}_{n,r,t}$ are easy to classify. In this case matrices in $\mathcal{Y}_{n,r,t}$ have rank strictly less than t, so are in $\mathcal{V}(\mathrm{I}_r) \subset \mathcal{V}(\mathrm{I}_t)$, where $\mathrm{I}_r, \mathrm{I}_t$ are the determinantal ideals described in Section 2.1.1.

Proposition 6. *If $r < t$, then any associated prime for the defining ideal of the closure of $\mathcal{Y}_{n,r,t}$ must also contain I_t.* □

For the most part, the components of the sets $\mathcal{Y}_{n,r,t}$ remain a mystery. In the meantime, we apply our current results to $\mathfrak{P}_t$, where $t \neq 1, 2, n$ (as $t = n$ simply gives a hypersurface).

3.3 Principal $(n-1)$-Minors Case

In this section we shall assert $n \geq 4$. This suffices in studying the minimal primes for $\mathfrak{P}_{n-1}$, since $\mathfrak{P}_2(X_{3\times 3})$ and $\mathfrak{P}_1(X_{2\times 2})$ are both prime. It turns out that when $n \geq 4$, the determinantal ideal, I_{n-1}, is a minimal prime for $\mathfrak{P}_{n-1}$ (this is part of the statement of Theorem 4). To see I_{n-1} cannot be the only minimal prime, note the following examples.

Example 4. Say $n = 4$. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{Y}_{4,4,3} \subset \mathcal{V}(\mathfrak{P}_3)$$

is of full rank and its principal 3-minors vanish. Therefore, $\mathcal{V}(\mathrm{I}_3) \subsetneq \mathcal{V}(\mathfrak{P}_3)$ as algebraic sets.

Example 5. In general, given a permutation τ of $\{1, \ldots, n\}$ with no fixed points, its corresponding matrix has zeros on the main diagonal. Hence, the matrix of τ^{-1} has full rank and its $(n-1)$-size principal minors vanish. Since τ and τ^{-1} have the same fixed points, this applies to τ as well.

We shall see the only other minimal prime for $\mathfrak{P}_{n-1}$ is the defining ideal for the closure of $\mathcal{Y}_{n,n,n-1}$. In other words, we claim the contraction, $\mathfrak{Q}_{n-1} = \mathfrak{Q}_{n-1}(X)$, of $\ker \Phi = \mathfrak{P}_{n-1} S_\Delta$ to S as in Corollary 5, is a minimal prime for $\mathfrak{P}_{n-1}$. Before we state and prove Theorem 4, for any subset $\mathcal{Y} \subset \mathbb{A}^{n^2}$, let $\bar{\mathcal{Y}}$ denote its Zariski closure. If $\mathcal{Y}$ is closed, then let $\mathfrak{I}(\mathcal{Y})$ denote its defining ideal in $K[X] = \mathcal{O}(\mathbb{A}^{n^2})$. With this notation, we have

$$\mathfrak{Q}_{n-1} = \mathfrak{I}(\bar{\mathcal{Y}}_{n,n,n-1}).$$

Theorem 4 (–)**.** *For all $n \geq 4$, the minimal primes of $\mathfrak{P}_{n-1}$ are exactly I_{n-1} and $\mathfrak{Q}_{n-1}$.*

Proof. The proof proceeds as follows: First, we show $\mathfrak{Q}_{n-1}$ is indeed a minimal prime for $\mathfrak{P}_{n-1}$. Then, we show $\mathfrak{Q}_{n-1}$ and I_{n-1} are incomparable. By Proposition 6 it remains only to analyze the case where a point $A \in \mathcal{V}(\mathfrak{P}_{n-1})$ has rank $r = n-1$. We will show the components containing A in that case are embedded components in $\mathcal{V}(\mathfrak{P}_{n-1})$. From that, we then conclude I_{n-1} is the only other minimal prime for $\mathfrak{P}_{n-1}$.

Recall the notation introduced in Section 3.2.1; D_Δ denotes the distinguished open set in $\mathbb{A}^{n^2}$ consisting of invertible matrices. We have

$$(\mathcal{V}(\mathfrak{P}_{n-1}) \cap D_\Delta) \subseteq \mathcal{V}(\mathfrak{Q}_{n-1}) \subseteq \mathcal{V}(\mathfrak{P}_{n-1}).$$

Furthermore, by Corollary 5,

$$\mathcal{V}(\mathfrak{Q}_{n-1}) = \overline{\mathcal{V}(\mathfrak{P}_{n-1}) \cap D_\Delta} \cong \overline{\mathcal{V}(\mathfrak{P}_1) \cap D_\Delta}$$

is exactly the closure in $\mathbb{A}^{n^2}$ of the set of invertible matrices whose inverses have all zeros on the diagonal. On the other hand, $\mathcal{V}(\mathfrak{P}_{n-1}) \cap D_\Delta$ is dense in $\mathcal{V}(\mathfrak{P}_{n-1})$, so

$$\dim\left(\mathcal{V}(\mathfrak{P}_{n-1}) \cap D_\Delta\right) = \dim(\mathfrak{P}_{n-1}).$$

Any prime contained in $\mathfrak{Q}_{n-1}$ must have height smaller than ht $\mathfrak{Q}_{n-1} = \mathrm{ht}\,(\mathfrak{P}_{n-1} S_\Delta)$ and so cannot contain $\mathfrak{P}_{n-1}$. Therefore $\mathfrak{Q}_{n-1}$ is a minimal prime for $\mathfrak{P}_{n-1}$.

We now show I_{n-1} and $\mathfrak{Q}_{n-1}$ are incomparable. When $n > 4$ we clearly cannot have $\mathfrak{P}_{n-1} \subseteq \mathrm{I}_{n-1} \subseteq \mathfrak{Q}_{n-1}$, because $\mathfrak{Q}_{n-1}$ is a minimal prime, and $\mathrm{ht}\,\mathrm{I}_{n-1} = 4 \neq n$. The difference in height also shows why we cannot have $\mathfrak{Q}_{n-1} \subseteq \mathrm{I}_{n-1}$ for $n > 4$. When $n = 4$, I_{n-1} and

$\mathfrak{Q}_{n-1}$ each have height 4 and thus since they're prime, containment between them occurs if and only if they are equal. However, Example 4 exhibits a matrix in $\mathcal{V}(\mathfrak{Q}_{n-1}) \setminus \mathcal{V}(\mathrm{I}_{n-1})$, so the algebraic sets cannot be equal.

We look for any additional minimal primes of $\mathfrak{P}_{n-1}$, according to rank. Choose $A \in \mathcal{V}(\mathfrak{P}_{n-1})$. If the rank, $\operatorname{rank} A$, of A is n, then $A \in \mathcal{V}(\mathfrak{Q}_{n-1})$. If $\operatorname{rank} A < n-1$, then the $(n-1)$-minors of A must vanish, so $A \in \mathcal{V}(\mathrm{I}_{n-1})$ (as implied by Proposition 6). It remains to find the components of $\mathcal{V}(\mathfrak{P}_{n-1})$ containing A when $\operatorname{rank} A = n-1$. We claim any such component is not defined by a minimal prime. This will also imply I_{n-1} is minimal, since I_{n-1} and $\mathfrak{Q}_{n-1}$ are incomparable.

Say $\mathcal{V}(\mathfrak{P}_{n-1})$ has a component, $\mathcal{Y} \neq \mathcal{V}(\mathfrak{Q}_{n-1})$, whose dimension is h and suppose $A \in \mathcal{Y}$ has rank $n-1$, so $\mathcal{Y} \neq \mathrm{I}_{n-1}$. Then there exist $\underline{i} \neq \underline{j}$ such that $\det(A(\underline{i};\underline{j})) \neq 0$, an open condition. Thus there exists a non-empty open set $\mathcal{U} \subseteq \mathcal{Y}$, still irreducible, on which $\det(B(\underline{i};\underline{j})) \neq 0$ for all $B \in \mathcal{U}$, and $\dim(\mathcal{U}) = h$. The symmetric action on X described in Section 1.1.2 preserves $\mathfrak{P}_{n-1}$, so we assert, without loss of generality, that $\underline{i} = \{1, \ldots, n-1\}$ and $\underline{j} = \{2, \ldots, n\}$. Then, by the observations from Section 3.2.1 a point $B \in \mathcal{U}$ factors as

$$B = \begin{pmatrix} \mathbf{I}_{n-1} \\ 0 \quad c_2 \quad \cdots \quad c_{n-1} \end{pmatrix} B(\underline{i};\underline{j}) \begin{pmatrix} c'_1 & \\ \vdots & \mathbf{I}_{n-1} \\ c'_{n-2} & \\ 0 & \end{pmatrix}.$$

The remaining $n-2$ principal minor conditions force $n-2$ of the parameters $c_2, \ldots, c_{n-1}$, $c'_1, \ldots, c'_{n-2}$ to vanish. What is left are $n-2$ non-zero parameters, along with the $(n-1)^2$ parameters that give $B(\underline{i};\underline{j})$. We have

$$\dim \mathcal{U} \leq (n-1)^2 + (n-2) = n^2 - (n+1).$$

But $\mathfrak{P}_{n-1}$ has $\binom{n}{n-1} = n < n+1$ generators, so the closure $\bar{\mathcal{U}} = \mathcal{Y}$ cannot be a component. $\square$

Corollary 8 (–)**.** *For $n \neq 3$, $\operatorname{ht} \mathfrak{P}_t \leq \binom{n+1}{2} - \binom{t+2}{2} + 4$.*

Proof. We estimate the height by killing variables in $K[X]$, as in Figure 3.1. If we first kill the last row of X then any principal minor involving that row, and hence, the last column, must vanish. Therefore, if we want the principal minors involving the second-to-last row to vanish, it is enough to kill the first $n-1$ entries. We may continue this argument inductively until we get to the $(t+1)$th row, having killed $n + (n-1) + \cdots + n - (n-t-2) = \binom{n+1}{2} - \binom{t+2}{2}$

$$X \rightsquigarrow \begin{pmatrix} x_{11} & \cdots & x_{1,t+1} & x_{1,t+2} & \cdots & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ x_{t+1,1} & \cdots & x_{t+1,t+1} & x_{t+1,t+2} & \cdots & \cdots & x_{t+1,n} \\ 0 & \cdots & \cdots & 0 & x_{t+2,t+2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Figure 3.1: Estimation of $\operatorname{ht}(\mathfrak{P}_t(X))$ from Corollary 8. It is enough to kill the first t' entries of the t'th row, $t+2 \le t' \le n$, to ensure the principal minors involving that row vanish.

variables so far. When $n \geq 4$, Theorem 4 says the t-minors of the upper left $(t+1)\times(t+1)$ submatrix of X have height 4, and that is independent of the variables we already killed, so we get the desired bound in that case. In the cases where $n = 1, 2$ we may directly compute the height to see it satisfies the desired bound. $\square$

3.3.1 Complete Intersection Case: $\mathfrak{P}_3(X_{4\times 4})$

Corollary 9 (–). *$\mathfrak{P}_3(X_{4\times 4})$ is a complete intersection.*

Proof. From Theorem 4 $\mathfrak{P}_3(X_{4\times 4})$ has pure height 4, and also has 4 generators. $\square$

In general, Theorem 4 gives us $\mathfrak{P}_{n-1} \subseteq \mathrm{I}_{n-1} \cap \mathfrak{Q}_{n-1}$. However, computations in *Macaulay2* ([13]) show, in several prime characteristics, that equality holds for $n = 4$. We shall show, in fact, that $\mathfrak{P}_3(X_{4\times 4})$ is reduced in *all* characteristics. We state this as Theorem 5, whose proof requires the following lemma.

Lemma 3 (–). *For any n, $\mathfrak{Q}_{n-1}$ does not contain any size $r < n-1$ minors of X.*

Proof. Choose an $r \times r$ minor of X, indexed by the rows $\underline{\mathrm{i}} = \{i_1, \ldots, i_r\}$ and columns $\underline{\mathrm{j}} = \{j_1, \ldots, j_r\}$ of X. We shall exhibit an $n \times n$ matrix $A \in \mathcal{V}(\mathfrak{Q}_{n-1})$, whose $(\underline{\mathrm{i}}, \underline{\mathrm{j}})$-minor is non-zero. If we choose A as a permutation matrix of the identity matrix $\mathbf{I}_n$, then since permutation matrices are orthogonal, it shall suffice to construct a matrix $A' = A^{\mathrm{T}}$, the transpose of A, whose main diagonal is all zeros, and whose $(\underline{\mathrm{j}}, \underline{\mathrm{i}})$-minor does not vanish.

Consider the submatrix $A'(\underline{\mathrm{j}}; \underline{\mathrm{i}})$. Set that submatrix equal to a permutation matrix of $\mathbf{I}_r$ such that any entries on the main diagonal of A' are zero. Then in A' put zeros in the remaining entries in the columns $\underline{\mathrm{i}}$. Now complete the standard basis of column vectors, permuting the remaining columns so that the entries on the main diagonal of A' are zero. $\square$

Theorem 5 (–). *Suppose $n = 4$. Then $\mathfrak{P}_{n-1} = \mathfrak{P}_3$ is reduced, and hence $\mathfrak{P}_3 = \mathrm{I}_3 \cap \mathfrak{Q}_3$.*

Proof. Theorem 4 implies $\mathfrak{P}_3(X_{4\times 4})$ is unmixed, so it suffices to show its primary decomposition is exactly $\mathrm{I}_3 \cap \mathfrak{Q}_3$. Also, we have the property $\mathfrak{P}_3$ is reduced if and only if its image at any localization is also reduced. Let $Y_i \in \mathfrak{P}_3$ denote the principal minor obtained by omitting the ith row and column of $X = X_{4\times 4}$. By inverting elements of $S = K[X]$, we first solve for variables using the equations $Y_i = 0$. Then we check the image of $\mathfrak{P}_3$ is reduced.

Invert the minor $\delta_1 = x_{11}x_{22} - x_{12}x_{21}$, which is not in I_3 because I_3 is generated by degree three polynomials, and which is not in $\mathfrak{Q}_3$ by Lemma 3. Put $S_{\delta_1} = S[\frac{1}{\delta_1}]$. We use $Y_4 = 0$ to solve for x_{33}; let $F = Y_4|_{x_{33}=0}$, i.e., the determinant Y_4, evaluated at $x_{33} = 0$. Then $F \equiv -x_{33}\delta_1 \mod Y_4$, and

$$\frac{S_{\delta_1}}{\mathfrak{P}_3\, S_{\delta_1}} \cong \frac{K[\frac{1}{\delta_1}, X|_{x_{33}=-\frac{F}{\delta_1}}]}{\left(Y_1, Y_2, Y_3 \mid x_{33} = -\frac{F}{\delta_1}\right)}.$$

Similarly, put $G = Y_3|_{x_{44}=0}$. Since G is an expression independent of F, we have

$$\frac{S_{\delta_1}}{\mathfrak{P}_3\, S_{\delta_1}} \cong \frac{K[\frac{1}{\delta_1}, X|_{x_{33}=-\frac{F}{\delta_1}, x_{44}=-\frac{G}{\delta_1}}]}{\left(\underbrace{\begin{vmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & -\frac{F}{\delta_1} & x_{34} \\ x_{42} & x_{43} & -\frac{G}{\delta_1} \end{vmatrix}}_{Y_1|_{x_{33}=-\frac{F}{\delta_1}, x_{44}=-\frac{G}{\delta_1}}}, \underbrace{\begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{31} & -\frac{F}{\delta_1} & x_{34} \\ x_{41} & x_{43} & -\frac{G}{\delta_1} \end{vmatrix}}_{Y_2|_{x_{33}=-\frac{F}{\delta_1}, x_{44}=-\frac{G}{\delta_1}}}\right)}.$$

We now solve for a variable not appearing in either polynomial

$$F \equiv x_{31}(x_{12}x_{23} - x_{13}x_{22}) - x_{32}(x_{11}x_{23} - x_{13}x_{21}) \mod Y_4$$
$$G \equiv x_{41}(x_{12}x_{24} - x_{14}x_{22}) - x_{42}(x_{11}x_{24} - x_{14}x_{21}) \mod Y_3,$$

using Y_2. Invert $\delta_2 = x_{11}x_{34} - x_{14}x_{31}$, which, again, is not in I_3 nor $\mathfrak{Q}_3$. Then define $H = Y_2|_{x_{43}=0}$. The image of $\mathfrak{P}_3$ is now principal:

$$\mathfrak{P}_3\, S_{\delta_1,\delta_2} \cong \left(\begin{vmatrix} x_{22} & x_{23} & x_{24} \\ x_{32} & -\frac{F}{\delta_1} & x_{34} \\ x_{42} & \frac{H}{\delta_2} & -\frac{G}{\delta_1} \end{vmatrix}\right) \cdot K\left[\frac{1}{\delta_1}, \frac{1}{\delta_2}, X|_{x_{33}=-\frac{F}{\delta_1}, x_{44}=-\frac{G}{\delta_1}, x_{43}=\frac{H}{\delta_2}}\right]$$

Let γ denote the generator of the image of $\mathfrak{P}_3\, S_{\delta_1,\delta_2}$, upon clearing denominators.

The localized polynomial ring S_{δ_1,δ_2} is a unique factorization domain (UFD). Therefore, it suffices to prove the irreducible factors of $\gamma \in S$ with non-zero image in $\mathrm{I}_3(S/\mathfrak{P}_3)$ or $\mathfrak{Q}_3(S/\mathfrak{P}_3)$ are square-free. We factor γ in the polynomial ring S, using *Macaulay2* ([13]), where we put $K = \mathbb{Z}$ (which will imply the factorization is valid in all characteristics):

$$\gamma = \Delta(\underline{\mathrm{i}}; \underline{\mathrm{j}}) f',$$

where $\Delta(\underline{\mathrm{i}}; \underline{\mathrm{j}}) \in \mathrm{I}_3$ is the minor of X given by the rows and columns indexed by $\underline{\mathrm{i}} = \{1, 2, 3\}$, $\underline{\mathrm{j}} = \{1, 2, 4\}$, respectively, and

$$f' = x_{31}(x_{12}x_{23} - x_{13}x_{22})Y_3 + x_{14}(x_{21}x_{42} - x_{22}x_{41})Y_4 + \delta_1 f,$$

where f is a degree 4 irreducible polynomial (see Equation (3.6)). Modulo the ideal $\mathfrak{P}_3$, we get

$$\gamma \equiv \Delta(\underline{\mathrm{i}}; \underline{\mathrm{j}})\delta_1 f \mod \mathfrak{P}_3,$$

which is square-free, as desired. □

The polynomial f in the proof of Theorem 5 is

$$\begin{aligned} f = -x_{14}x_{21}x_{33}x_{42} + x_{11}x_{23}x_{34}x_{42} + x_{14}x_{22}x_{31}x_{43} \\ - x_{11}x_{22}x_{34}x_{43} - x_{12}x_{23}x_{31}x_{44} + x_{12}x_{21}x_{33}x_{44}. \end{aligned} \tag{3.6}$$

The fact that $\mathfrak{P}_3$ is a reduced complete intersection implies its minimal primes, I_3 and $\mathfrak{Q}_3$, are *algebraically linked*. See [30] for an introduction to algebraic linkage and [21] for a more modern account. In the following section, we state the relevant results about linkage which lead to more conclusions about $\mathfrak{P}_3(X_{4\times 4})$.

3.3.2 Consequences of Algebraic Linkage

In this section, unless stated otherwise, $X = X_{4\times 4}$. Two ideals I, J in a Cohen-Macaulay ring R are *linked* (or *algebraically linked*) means there exists a regular sequence $\mathbf{f} = f_1, \ldots, f_h$ in $I \cap J$ such that $J = (\mathbf{f}) :_R I$ and $I = (\mathbf{f}) :_R J$. Corollary 9 implies

- $K[X]/\mathfrak{P}_3$ is Gorenstein and
- the generators for $\mathfrak{P}_3$, the principal 3-minors, form a regular sequence in $K[X]$.

Therefore I_3 and $\mathfrak{Q}_3$ are linked. We use statements from Proposition 2.5 and Remark 2.7 in [20] to pull some corollaries from Theorem 5. In our context, all results for local rings also hold for *graded local* rings (see Chapter 1.5 of [3] to gain a justification for that statement).

Proposition ([30], 2.5 of [20]). *Let I be an unmixed ideal of height h in a (not necessarily local) Gorenstein ring R, and let $\mathbf{f} = f_1, \ldots, f_h$ be a regular sequence inside I with $(\mathbf{f}) \neq I$, and set $J = (\mathbf{f}) : I$.*
(a) $I = (\mathbf{f}) : J$ (i.e., I and J are linked).
(b) R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay.
(c) Let R be local and let R/I be Cohen-Macaulay. Then $\omega_{R/J} \cong I/(\mathbf{f})$ and $\omega_{R/I} \cong J/(\mathbf{f})$.

□

The modules $\omega_{R/I}, \omega_{R,J}$ are the respective *canonical modules* for $R/I, R/J$ (see Chapter 3.6 of [3]). Given a local Cohen-Macaulay ring R with maximal ideal $\mathfrak{m}$, the *type* of R is

$$\operatorname{type}(R) = \dim_{R/\mathfrak{m}} \left(\operatorname{Ext}^d_R(R/\mathfrak{m}, R)\right),$$

where d is the Krull dimension of R. It can be shown that Gorenstein rings have type 1.

For any finitely generated R-module, M, let $\mu(M)$ denote the minimal number of generators for M. An ideal J is called *minimally linked* to I means I and J are linked with respect to a regular sequence $\mathbf{f}$, such that $\mathbf{f}$ is part of a minimally generating set for I.

Proposition ([20], Remark 2.7). *Let I be an unmixed ideal of height h in a (not necessarily local) Gorenstein ring R, and let $\mathbf{f} = f_1, \ldots, f_h$ be a regular sequence inside I with $(\mathbf{f}) \neq I$, and set $J = (\mathbf{f}) : I$. Let R be local with maximal ideal $\mathfrak{m}$ and let R/I be Cohen-Macaulay. Then*
(a) $\operatorname{type}(R/J) = \mu(I/(\mathbf{f})) = \mu(I) - \dim_{R/\mathfrak{m}}((\mathbf{f} + \mathfrak{m}I)/\mathfrak{m}I) \geq \mu(I) - \operatorname{depth}(I)$.
In particular, if R/J is Gorenstein, then I is an almost complete intersection.
(b) $\operatorname{type}(R/J) = \mu(I) - \operatorname{depth}(I)$ *if and only if J is minimally linked to I.*
In particular if J is minimally linked to I and I is an almost complete intersection, then R/J is Gorenstein.

□

Corollary 10 (–). *$\mathfrak{Q}_3$ is Cohen-Macaulay.*

Proof. For any X, $\mathrm{I}_3(X)$ is known to be Cohen-Macaulay, as described in Section 2.1.1. By algebraic linkage, $\mathfrak{Q}_3 = \mathfrak{Q}_3(X_{4\times4})$ must also be Cohen-Macaulay. □

For the next corollary, we introduce an $\mathbb{N}^{2n}$-multigrading on $K[X]$: a polynomial has degree $(r_1, \ldots, r_n; c_1, \ldots, c_n)$ means under the standard grading, its degree in the variables from the ith row (resp., jth column) of X is r_i (resp. c_j), for all $i = 1, \ldots, n$ (resp., $j = 1, \ldots, n$). Alternatively, deg $x_{ij} = (\mathbf{e}_i; \mathbf{e}_j)$, the entries from the standard basis vectors for K^n. We freely use that the sum, product, intersection, and colon of multigraded ideals is multigraded, as well as all associated primes over a multigraded ideal. Observe how in the standard grading, the degree of any polynomial in the standard grading equals the sums $r_1 + \cdots + r_n = c_1 + \cdots + c_n$.

Corollary 11 (–). *$\mathfrak{Q}_3 = \mathfrak{P}_3 + (f)$, where f is as in Equation (3.6).*

Proof. Because $X = X_{4\times 4}$ is a square matrix, $\mathrm{I}_3(X)$ is Gorenstein ([37]). Algebraic linkage implies the canonical module, $\omega_{K[X]/\mathrm{I}_3} \cong \mathfrak{Q}_3/\mathfrak{P}_3$, is cyclic. The proof of Theorem 5 shows the image of f in $\mathfrak{Q}_3/\mathfrak{P}_3$ is non-zero. It remains to show f actually generates $\omega_{K[X]/\mathrm{I}_3}$.

We saw f has degree 4 in the standard grading, so we show no polynomial of degree strictly less than 4 can generate $\mathfrak{Q}_3/\mathfrak{P}_3$. Assume $g \in \mathfrak{Q}_3$ is such a polynomial. Algebraic linkage says $\mathfrak{P}_3 :_{K[X]} \mathrm{I}_3 = \mathfrak{Q}_3$, so g must multiply every generator $\Delta(\underline{\mathrm{i}};\underline{\mathrm{j}}) \in \mathrm{I}_3$ into $\mathfrak{P}_3$. Suppose g has degree 0 in the ith row. The product of g with any 3-minor not involving the ith row must then be a multiple of $\Delta(\underline{\mathrm{i}};\underline{\mathrm{i}}) \in \mathfrak{P}_3$. If g also has degree 0 in the ith column then either $g \in \mathfrak{P}_3$, a contradiction of the choice of g, or there exists another column j where g has degree 0 as well. Then the product $g\Delta(\underline{\mathrm{i}};\underline{\mathrm{j}})$, where $i \notin \underline{\mathrm{i}}$ and $j \notin \underline{\mathrm{j}}$, must simultaneously divide $\Delta(\underline{\mathrm{i}};\underline{\mathrm{i}})$ and $\Delta(\underline{\mathrm{j}};\underline{\mathrm{j}})$. But this cannot happen, because the product has degree 0 in both the ith row and the jth column. $\square$

Recall, for general n, how we defined $\mathfrak{Q}_{n-1}$ as the contraction of $\ker \Phi$ to $K[X]$ where Φ is the map from Corollary 5. By definition,

$$\mathfrak{Q}_{n-1} = \mathfrak{P}_{n-1} :_{K[X]} \Delta^{\infty},$$

where $\Delta = \det X$.

Corollary 12 (–). *$\mathfrak{Q}_3 = \mathfrak{P}_3 :_{K[X]} \Delta$.*

Proof. From Corollary 11, the only generator for $\mathfrak{Q}_3$ outside of $\mathfrak{P}_3$ is f, and direct computation shows $f\Delta \in \mathfrak{P}_3$. $\square$

We end this section with some conjectures.

Conjecture 1. *For all n, $\mathfrak{P}_{n-1}$ is reduced.*

Conjecture 2. *For all n, $\mathfrak{Q}_{n-1}$ is an* almost complete intersection, *i.e., $\mu(Q_{n-1}) = n + 1$.*

3.4 Principal $(n-2)$-Minors Case

Recall, $\mathcal{G}$ denotes the Grassmann variety $\mathrm{Grass}(r,n)$. The inclusion $\mathcal{Y}_{n,r,t} \hookrightarrow \mathcal{Z}_{n,r}$ induces, via Θ in Proposition 5, a bundle map:

$$\begin{tikzcd} \mathcal{Y}_{n,r,t} \arrow[r, hook] \arrow[d] & \mathcal{Z}_{n,r} \arrow[d, "\Theta"] \\ \Theta(\mathcal{Y}_{n,r,t}) \arrow[r, hook] & \mathcal{G} \times \mathcal{G} \end{tikzcd} \tag{3.7}$$

We use the diagram (3.7) to study components of $\mathcal{Y}_{n,t,t}$, the locally closed sets of $\mathcal{V}(\mathfrak{P}_t)$ where $r = t$. From (3.7), let $\mathcal{H} \subseteq \mathcal{G} \times \mathcal{G}$ denote the closed set consisting of pairs $\mathbf{g} = \left[\cdots : g_{\underline{\mathrm{i}}} : \cdots\right], \mathbf{h} = \left[\cdots : h_{\underline{\mathrm{i}}} : \cdots\right]$ in $\mathcal{G}$ where for each index $\underline{\mathrm{i}}$, either g_i or h_i vanishes. Then $\mathcal{Y}_{n,t,t}$ is the inverse image of $\mathcal{H}$ under Θ, and the components of $\mathcal{Y}_{n,t,t}$ correspond bijectively to the components of $\mathcal{H}$. It follows that to get an irreducible component of $\mathcal{H}$, we must partition the set of indices for the Plücker coordinates into two sets, $\underline{\mathrm{I}}, \underline{\mathrm{J}}$. Let $\mathcal{V}(\underline{\mathrm{I}}), \mathcal{V}(\underline{\mathrm{J}})$ denote the respective closed subsets of $\mathcal{G}$ defined by the vanishing of Plücker coordinates respectively indexed by $\underline{\mathrm{I}}, \underline{\mathrm{J}}$.

Quite generally, for any product of spaces $\mathcal{X} \times \mathcal{Y}$, each component is the product of a component of $\mathcal{X}$ with a component of $\mathcal{Y}$, and all of these are required to cover $\mathcal{X} \times \mathcal{Y}$. Thus each component of $\mathcal{H}$ must be a component of $\mathcal{V}(\underline{\mathrm{I}}) \times \mathcal{V}(\underline{\mathrm{J}})$ for some partition $\underline{\mathrm{I}} \coprod \underline{\mathrm{J}}$ and every component of $\mathcal{V}(\underline{\mathrm{I}}) \times \mathcal{V}(\underline{\mathrm{J}})$ arises as the product of a component of $\mathcal{V}(\underline{\mathrm{I}})$ and a component of $\mathcal{V}(\underline{\mathrm{J}})$. We will look at all such partitions $\underline{\mathrm{I}}, \underline{\mathrm{J}}$, and then for each component $\mathcal{C}$ of $\mathcal{V}(\underline{\mathrm{I}})$ and each component $\mathcal{D}$ of $\mathcal{V}(\underline{\mathrm{J}})$, we shall consider the irreducible set $\mathcal{C} \times \mathcal{D}$. The components of $\mathcal{H}$ are the maximal such sets $\mathcal{C} \times \mathcal{D}$, and their inverse images under the bundle map Θ give the irreducible components of $\mathcal{Y}_{n,t,t}$.

The problem of understanding all the components of the various sets $\mathcal{V}(\underline{\mathrm{I}})$ is known to be extremely hard (as mentioned in [12, 34]). However, we shall be able to understand the situation completely when $r = t = n-2$. Each $n-2$ size minor of an $n \times (n-2)$ matrix is determined by the two rows that are not used. Therefore, $\underline{\mathrm{I}}$ can be described by a set of 2 element subsets of an n element set indexing the rows of the matrix, say $\{1, \ldots, n\}$. This set may be thought of as a graph whose vertices are $\{1, \ldots, n\}$, and whose edges correspond to the pairs of rows omitted from the minors. (There is a similar graph arising from $\underline{\mathrm{J}}$.) We shall see that a component of a closed set of the form $\mathcal{V}(\underline{\mathrm{I}})$ has the same form and give a condition on its graph that is equivalent to irreducibility. We will then classify the minimal pairs of graphs that together cover (all edges of) $\{1, \ldots, n\}$ and such that each graph corresponds to an irreducible closed set in $\mathcal{G}$. Finally, we shall work through an explicit case, $n = 5$.

3.4.1 Plücker Coordinates to Graphs

Let G denote a graph. We provide definitions of the graph theory vocabulary we use in this section. For example, G is *size* n means it has n vertices. The *degree* of a vertex in G is the number of edges incident to it, where a *loop*, an edge joining a vertex to itself, counts as two edges. A vertex is *isolated* means it has no edges. G is *simple* means every edge joins exactly two vertices (i.e., there are no *loops*) and any two vertices are joined by at most one edge (i.e., there are no *parallel edges*). All graphs to which we refer from now on we shall assert are simple.

Suppose G is a graph of size n. A vertex in G is *dominating* means it has degree $n-1$, i.e., it is joined to every other vertex in G. G is *complete* means every possible edge is present. A *clique* of size a is a subgraph $G' \subseteq G$ which is complete and not contained in any larger complete subgraph of G.[1] The *complement* H of G is the set of vertices from G, along with the condition any two vertices in H are joined by an edge if and only if they are not joined by an edge in G. A collection of simple graphs which use the same set of n vertices is a *covering* means the union of their edges is complete.

For any point $\mathbf{g} = \left[\cdots : g_{\underline{\mathrm{i}}} : \cdots\right] \in \mathcal{G}$, we encode exactly which of its Plücker coordinates do and do not vanish in a graph, $G = \mathrm{Graph}(\mathbf{g})$. The construction is as follows: Label the vertices of G using $1, \ldots, n$. For any two vertices v and v', draw an edge joining them if and only if the Plücker coordinate $g_{\underline{\mathrm{i}}}$, where $\underline{\mathrm{i}} = \{1, \ldots, n\} \setminus \{v, v'\}$, vanishes. The following definition characterizes the types of graphs we will analyze.

Definition 1 (–)**.** A graph G of size n is *permissible* means the following are satisfied:

(1) G has at most $\binom{n}{n-2} - 1$ edges, i.e., G is not complete.

(2) The subgraph obtained by omitting all dominating vertices is a disjoint union of cliques.

Proposition 7 (–)**.** *Condition (2) in Definition 1 is equivalent to the condition that every vertex in G of degree $d < n-1$ is part of a clique of size d.*

Proof. We show Condition (2) from Definition 1 follows if every vertex in G of degree $d < n-1$ is part of a clique of size d. The reverse implication is even more immediate. Suppose $v \in G$ is a vertex of degree $d < n-1$. By hypothesis, v is part of a size d clique, G'. Upon omitting all dominating vertices of G, G' is still a clique. If any remaining vertex in G' is joined to some non-dominating vertex $v' \notin G'$, then the hypothesis says v' is joined to every other vertex from G'. In that case the degree of v is larger than d, a contradiction. $\square$

If A is a matrix whose column span (resp., row span) is $\mathbf{g} \in \mathcal{G}$, then we write $\mathrm{Graph}(A) = \mathrm{Graph}(\mathrm{col}\ A)$ (resp. $\mathrm{Graph}(A) = \mathrm{Graph}(\mathrm{row}\ A)$).

[1]Some authors use the term *maximal clique* while reserving term *clique* for any complete subgraph in G.

Proposition 8 (–)**.** *For any point* $\mathbf{g} = \left[\cdots : g_{\underline{i}} : \cdots\right] \in \mathcal{G}$*, its associated graph,* $\mathrm{Graph}(\mathbf{g})$ *is permissible.*

Proof. Without loss of generality, we may assert the Plücker coordinate $g_{\{1,\ldots,n-2\}}$ is non-zero. Write

$$A = \begin{pmatrix} & \mathbf{I}_{n-2} & \\ a_{11} & \cdots & a_{1,n-2} \\ a_{21} & \cdots & a_{2,n-2} \end{pmatrix}$$

so that $\operatorname{col} A = \mathbf{g}$. Let $G = \mathrm{Graph}(A)$. By construction, Condition (1) in Definition 1 holds. We will show the alternative condition from Proposition 7 holds as well.

Suppose v is a vertex in G of degree d, where $1 < d < n-1$. If v equals $n-1$ or n, then any vertex joined to it must be one of the first $n-2$. Say $v = n$. Then each vertex v' joined to v indicates the vanishing of the entry $a_{1,v'}$. Any minors involving two such coordinates a_{1,v'_1}, a_{1,v'_2} must also vanish, and those minors correspond exactly to the edges joining v'_1, v'_2. In other words, in this case the set of vertices joined to v must also form a clique.

On the other hand, suppose $v = 1$; the situation will be analogous if we choose v to equal any of the first $n-2$ indexed vertices. There are two subcases to consider. The first subcase is when v is not joined to either of $n-1, n$. This means a collection of 2-minors of the submatrix $A' = A(\{n-1, n\}; \{1, \ldots, n-2\})$, all of which involve the first column, vanish. This can only happen if either

A) $a_{1,1} = a_{2,1} = 0$ and equivalently, v is dominating; or
B) all Plücker coordinates involving the columns indexed by vertices joined to v vanish, in which case these vertices, along with v, form a clique.

For the other subcase, suppose v is joined to one of the last two vertices, say $v' = n$. As before, all vertices joined to v' form a clique, and $a_{1,1} = 0$. If any 2-minor of A' involving a_{11} also vanishes, then either

A) $a_{2,1} = 0$ and v is a dominating vertex; or
B) $a_{1,v'} = 0$ for all v' joined to v, in which case v, along with the vertices v', form a clique.

$\square$

3.4.2 Consequences

Given a permissible graph G, the set of points in $\mathcal{G}$ with that graph is locally closed. Its closure is all points whose graph contains G.

Definition 2 (–)**.** A subvariety $\mathcal{S} \subseteq \mathcal{G}$ is *permissible* means it is the closure of the set of all points with the same fixed permissible graph, which we denote $\mathrm{Graph}(\mathcal{S})$.

Theorem 6 (–)**.** *In $\mathcal{G} = \mathrm{Grass}(n-2, n)$, the irreducible components of $\mathcal{V}(\underline{I})$, where $\underline{I}$ is a set of $(n-2)$-subsets of $\{1, \dots n\}$, are permissible subvarieties of $\mathcal{G}$.*

Proof. The result follows from a known observation about ideals generated by minors of a $2 \times s$ generic matrix. In our case $s = n-2$. Recall, $\mathcal{G}$ is covered by open affine sets $\mathcal{G}_{\underline{\mathrm{i}}}$, where the $\underline{\mathrm{i}}$th Plücker coordinate does not vanish. Fix $\underline{\mathrm{i}} = \{1, \dots, n\} \setminus \{i, j\}$ and let A denote a matrix, normalized with respect to $\underline{\mathrm{i}}$, whose column space is a point in $\mathcal{G}$. The Plücker coordinates for $\mathrm{col}\, A$ are, up to a scalar, the minors of the submatrix $A\left(\{i, j\}; \{1, \dots, n-2\}\right)$. The choice of $\underline{\mathrm{i}}$ was arbitrary; for each class of sets $\underline{\mathrm{I}}$ not containing $\underline{\mathrm{i}}$, we construct $G = \mathrm{Graph}(\mathrm{col}\, A)$, having supposed $\mathrm{col}\, A \in \mathcal{V}(\underline{\mathrm{I}})$. We claim the components of $\mathcal{V}(\underline{\mathrm{I}})$ are in bijection with the minimal possible graphs for A for fixed $\underline{\mathrm{I}}$. We let I denote the ideal in the homogeneous coordinate ring for $\mathcal{G}$, generated by the Plücker variables with indices in $\underline{\mathrm{I}}$.

The first case we consider is the most fundamental point of the proof. Suppose two overlapping 2-minors of A' vanish, so $\underline{\mathrm{I}}$ contains their respective Plücker indices. Consequently, either

A) the third 2-minor in the three involved columns vanishes, or

B) both entries in the overlapping column of the minors vanish.

No other conditions on the Plücker coordinates follow. Case A) is represented as a triangle (3-cycle) in G, whereas Case B) is represented as a dominating vertex. On the other hand, two non-overlapping 2-minors of A' are algebraically independent of each other, because their Plücker indices, together, cannot satisfy those involved in any Plücker relation.

It is possible for 2-minors of A' to vanish when we simply require a collection of its entries to vanish. However, such vanishing is exactly a consequence of the Plücker relations. So supposing the entries in

$$\mathbf{a} = \{a_{ik_1}, \dots, a_{ik_{r_i}}, a_{jk_1}, \dots, a_{jk_{r_j}}\}$$

are zero, and no other entries of A' vanish, any additional generators of the ideal

$$I = \sum_{\substack{\mathbf{a}_k \,=\, k\text{th} \\ \text{element of } \mathbf{a}}} \mathfrak{I}(A \mid \mathrm{col}\ A = \mathbf{g} \in \mathcal{G} \subset \mathbb{P}^{\binom{n}{n-2}-1}, \mathbf{a}_k = 0)$$

must also be Plücker coordinates – and our requirement no other entries of A' vanish implies the additional generators cause 2-minors of A' to vanish. Note, the entries of A' themselves are algebraically independent of each other. If there are no other generators for I, then $\mathcal{V}(\underline{\mathrm{I}})$ is a permissible subvariety. On the other hand, any overlapping 2-minors defined by additional generators either satisfy Case B) above, or, the same row in both minors vanishes, implying Case A).

The final case to consider is when we suppose an entry a of A' and a 2-minor μ of A' vanish. If μ does not involve a, then clearly the two Plücker coordinates are algebraically independent. The other possibility is when a is nested in the submatrix giving the minor μ. Say

$$\mu = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

We lose no generality, as the position of a only determines a name change of the other entries in order to get the same formula, $ad - bc$, for μ. It follows either b or c must also vanish. If c vanishes, then all 2-minors involving the column $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ vanish, but again, that is a condition on Plücker coordinates. As we can see, vanishing of any collection of Plücker coordinates can only cause other Plücker coordinates to vanish. We conclude the minimal primes of $\mathcal{V}(\underline{\mathrm{I}})$ are generated by Plücker coordinates, and thus, each have a unique corresponding graph. $\square$

Suppose $\mathcal{S}$ is a permissible subvariety. It follows from the definition for a permissible graph that $G = \mathrm{Graph}(\mathcal{S})$ contains either isolated vertices, dominating vertices, or neither, but not both. Let G_{triv} denote this set of vertices. We will say G_{triv} is *dominating* to mean its vertices are dominating, or *isolated*, to mean its vertices are isolated. G_{triv} may be empty, and $|G_{triv}| \neq n$. Again, by permissibility of G, the set $G \setminus G_{triv}$ is a disjoint union of c cliques of respective sizes $a_1, \ldots, a_c$.

Theorem 7 (–)**.** *Suppose $\mathcal{S} \subset \mathcal{G}$ is a permissible subvariety. Let G_{triv} denote the (possibly empty) set of either isolated or dominating vertices of $G = \mathrm{Graph}(\mathcal{S})$, and let $a_1, \ldots, a_c$ denote the respective sizes of the cliques in $G \setminus G_{triv}$. Put $m = |G_{triv}|$ and $l = \sum_{j=1}^{c}(a_j - 1)$. Then the codimension of $\mathcal{S}$ in $\mathcal{G}$ is*

$$\operatorname{codim} \mathcal{S} = \begin{cases} n - c + m = 2m + l & \text{if } G_{triv} \neq \emptyset, \text{ dominating} \\ n - c - m = l & \text{otherwise.} \end{cases}$$

Proof. An isolated vertex contributes nothing to the codimension. Say A is an $n \times (n-2)$ matrix, normalized with respect to some set of indices $\underline{\mathrm{i}}$, and such that $\mathrm{Graph}(A) = G$. Let A' denote the complementary submatrix to the size $n-2$ identity submatrix of A. A dominating vertex $v \in G$ indicates that two entries on a column of A' vanish, contributing 2 to the codimension. All other edges joined to v correspond to 2-minors of A' involving that vanishing column and thus contribute nothing to the codimension.

Put $G' = G \setminus G_{triv}$. Columns of A' involved in a given clique of G' are independent of those involved in minors corresponding to edges joined to dominating vertices. There are two cases left to consider. The first case is where a clique of size $a \geq 2$ indicates a collection of $a-1$ entries from the same row of A' vanish, so contributes $a-1$ to the codimension. The other case is when a clique of size $a \leq 2$ indicates all 2-minors involving some set of a columns in A' vanish. The condition is equivalent to the condition that a generic $2 \times a$ matrix have rank 1. It is known (see [9]) that a generic rank 1 matrix of size $2 \times a$ defines an ideal of height $a-1$. $\square$

Given a permissible graph G, let H denote its complement. In understanding components of $\Theta(\mathcal{Y}_{n,n-2,n-2})$ in Equation (3.4), we wish to minimally enlarge H to a permissible graph, $\tilde{H}$, and then take a minimal permissible subgraph $\tilde{G} \subseteq G$ such that together, $\tilde{G}, \tilde{H}$ cover the vertices $\{1, \ldots, n\}$. Specifically, $\tilde{H}$ should not properly contain any permissible subgraph containing H, and $\tilde{G}$ should not contain any permissible subgraph that, with $\tilde{H}$, forms a covering. Upon finding such a pair $(\tilde{G}, \tilde{H})$, we let $(\mathcal{S}, \mathcal{T})$ denote the pair of permissible subvarieties with the respective graphs.

Lemma 4 (–). *A minimal pair, up to permutation, of permissible subvarieties* $(\mathcal{S}, \mathcal{T})$ *whose associated graphs form a covering must satisfy:*
(a) $\mathrm{Graph}(\mathcal{S})$ *consists of a clique of size a with the remaining vertices isolated, and*
(b) $\mathrm{Graph}(\mathcal{T})$ *is its complement, a size n graph with $n-a$ dominating vertices.*
These pairs completely describe the components of $\mathcal{Y}_{n,n-2,n-2}$*. The number of cliques in* $\mathrm{Graph}(\mathcal{S})$ *may take on any value* $2 \leq a \leq n-1$*.*

Proof. We now give an algorithm for producing such a pair from a fixed permissible graph G. Let G_{triv} denote the (possibly empty) set of isolated or dominating vertices of G, let $G' = G \setminus G_{triv}$ and let H' denote the complement of G'. Let $a_1, \ldots, a_c$ denote the sizes of the respective cliques in G'.

Suppose G_{triv} is non-empty and consists of dominating vertices. If H' is permissible and $a_j = 1$ for all $j = 1, \ldots, c$ then let H denote the union of H' and the vertices from G_{triv}, so that G, H give respective graphs for permissible subvarieties $\mathcal{S}, \mathcal{T}$ and we are done. If, on the other hand, H' is permissible but $a_j > 1$ for some j then $a_i = 1$ for all $i \neq j$ and we have two ways to enlarge H':

A) Complete H', then let $\tilde{H}$ denote its union with the vertices from G_{triv}. Then let $\tilde{G} \subseteq G$ denote the complement of $\tilde{H}$. $\tilde{G}$ is permissible because it consists of the edges incident to vertices in G_{triv}.

B) There is at least one isolated vertex in G'. To construct $\tilde{H}$ make the isolated vertices from G' into dominating vertices. Remove their edges from G to get a subgraph $\tilde{G}$.

If H' is not permissible, and G_{triv} is non-empty and consists of dominating vertices, then we can either do A), as above, or we can do the following: choose a clique B from G' of size $a_j \geq 2$. In constructing $\tilde{H}$, make all vertices in $G \setminus B$ dominating. The complement, $\tilde{G}$, of $\tilde{H}$ is a clique of size $a_j + m$, with the remaining vertices isolated.

To finish the proof, now suppose G_{triv} is either empty or consists of isolated vertices. If H' is permissible then adding the vertices from G_{triv} to H' and making them dominating does not change permissibility and we are done. If H' is not permissible then let H denote H', together with the vertices from G_{triv} as dominating vertices. Choose j such that $a_j > 1$ and enlarge H by making all vertices not in that clique, call it B, dominating. The complement $\tilde{G}$ is exactly the clique B. □

Theorem 8 (–)**.** $\dim \mathcal{Y}_{n,n-2,n-2} = n^2 - 4 - n$.

Proof. A matrix $A \in \mathcal{Y}_{n,n-2,n-2}$ has a normalized factorization given by $2(n-2) + (n-2)^2 + 2(n-2) = n^2 - 4$ parameters. Subtract the minimal codimension of $\mathcal{S} \times \mathcal{T}$, as computed in Theorem 7, over all possible pairs as described in Lemma 4. □

3.4.3 Explicit Case: $n = 5$

We explain Theorem 8 by focusing on the first non-trivial case, $n = 5$. A matrix $A \in \mathcal{Y}_{5,3,3}$ if and only if rank $A = 3$ and the size 3 principal minors of A vanish. We have the identification:

$$\begin{aligned} \mathcal{Y}_{5,3,3} &\to \mathrm{Grass}(3,5) \times \mathrm{Grass}(3,5) \\ A &\mapsto (\mathrm{col}\ A, \mathrm{row}\ A) \end{aligned}$$

Without loss of generality, say $\underline{i} = \{1,2,3\}$ and $\underline{j} = \{1,2,4\}$ index the respective Plücker coordinates of $(\mathrm{col}\ A, \mathrm{row}\ A)$ which do not vanish. The factorization

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{pmatrix} \cdot A_{\{1,2,3\};\{1,2,4\}} \cdot \begin{pmatrix} 1 & 0 & c_{13} & 0 & c_{15} \\ 0 & 1 & c_{23} & 0 & c_{25} \\ 0 & 0 & c_{33} & 1 & c_{35} \end{pmatrix} \qquad (3.8)$$

shows $(2\times3)+(3\times3)+(2\times3) = 21 = 25-4$ parameters, not yet considering the requirement that the size 3 principal minors of A vanish. Then the principal 3-minors of A vanish if and only if the diagonal entries of $\wedge^3 A$ vanish, if and only if for each $i = 1, \ldots, 10$, the ith entry of either the column vector $\wedge^3 B$ or the row vector $\wedge^3 C$ vanishes.

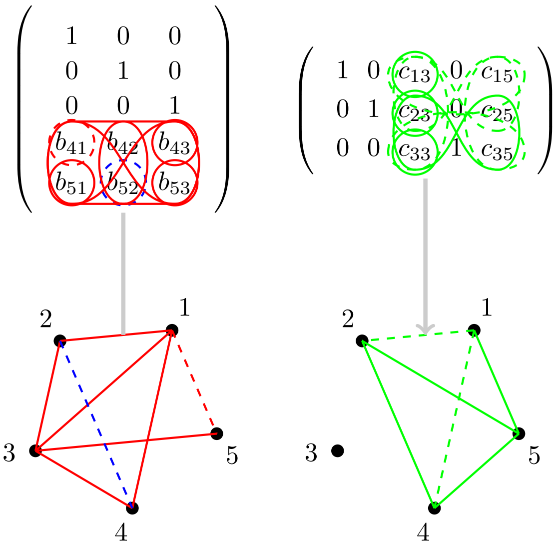

Figure 3.2: The dotted lines indicate Plücker coordinates which vanish as a consequence of the solid ones vanishing. For the matrix on the left, it is enough for either the red dotted line or the blue dotted line to be present. In the graphs, an edge joining vertices v and v' is drawn if and only if the Plücker coordinate with index $\{1, \ldots, 5\} \setminus \{v, v'\}$ vanishes.

Example 6. We give a quick example of a possible point $A \in \mathcal{Y}_{5,3,3}$. Put A as in (3.8), where we set the colored expressions from Equation (3.9) equal to 0.

$$\left.\begin{array}{r} 1(c_{33}) \\ b_{43}(1) \\ b_{53}c_{35} \\ -b_{42}c_{23} \\ -b_{52}(c_{23}c_{35} - c_{25}c_{33}) \\ (b_{42}b_{53} - b_{43}b_{52})(-c_{25}) \\ b_{41}(-c_{13}) \\ b_{51}(-c_{13}c_{35} + c_{15}c_{33}) \\ (-b_{41}b_{53} + b_{43}b_{51})c_{15} \\ (-b_{41}b_{52} + b_{42}b_{51})(c_{13}c_{25} - c_{15}c_{23}) \end{array}\right\} = 0. \tag{3.9}$$

The Plücker indices for the chosen expressions comprise $\underline{\mathrm{I}}, \underline{\mathrm{J}}$:

$$\underline{\mathrm{I}} = \{\{1,2,4\}, \{1,2,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}\}$$
$$\underline{\mathrm{J}} = \{\{1,2,3\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}\}$$

The solution is shown as the circled Plücker coordinates in Figure 3.2. Notice how the highlighted solution in (3.9) implies the vanishing of additional Plücker coordinates, as described in the proof of Theorem 6; in particular, we have

$$\left.\begin{array}{r} -b_{41}b_{52}+b_{42}b_{51} \\ b_{51} \end{array}\right\} = 0 \qquad \text{implies either } b_{41}=0 \text{ or } b_{52}=0$$

and

$$\left.\begin{array}{r} c_{33} \\ c_{23} \\ -c_{13} \end{array}\right\} = 0 \qquad \begin{array}{r} \text{implies both } -c_{13}c_{35}+c_{15}c_{23}=0 \\ \text{and } c_{13}c_{25}-c_{15}c_{23}=0. \end{array}$$

In Figure 3.2 the dotted lines indicate other Plücker coordinates which vanish as a consequence, making the respective graphs for col A, row A permissible. The different colored dotted lines in the lefthand matrix and graph reflect the condition that only one of b_{41} or b_{52} is required to vanish.

For any $A \in \mathcal{Y}_{5,3,3}$, we wish to find minimal pairs of permissible subvarieties whose respective graphs cover the vertices $\{1,\ldots,n\}$. Figure 3.3 shows examples of how to construct a pair of permissible graphs, which cover 5 vertices, given an arbitrary partition of the Plücker coordinates, i.e., a graph covering of 5 vertices using a permissible graph and its complement.

Finally, Figure 3.4 shows the types of configurations that give a minimally permissible pair of subvarieties.

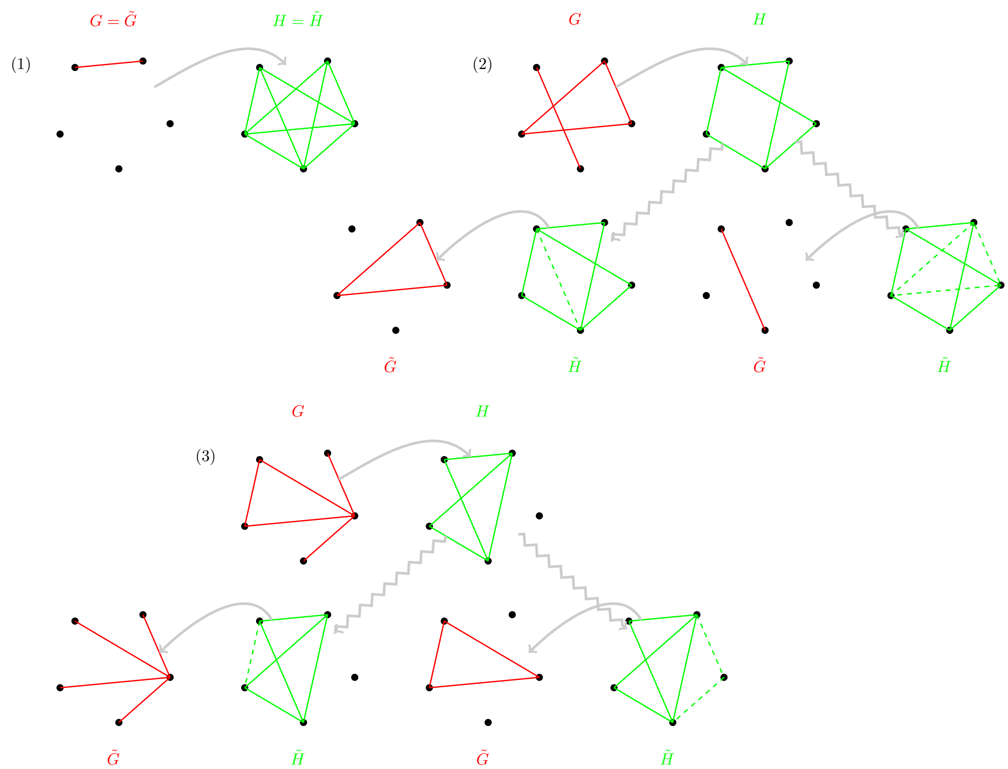

Figure 3.3: What are the minimal pairs of permissible graphs that cover $n = 5$ vertices? We begin with a permissible red graph, G. The green graph to its right is its complement, H. The arrows point to minimal ways to enlarge H to make it permissible; in (1), H is already permissible. After enlarging H to $\tilde{H}$, we remove as many edges from G to obtain $\tilde{G}$, such that $\tilde{G}, \tilde{H}$ still form a covering. It turns out $\tilde{G}$ will always be permissible, and furthermore, will always be the complement of $\tilde{H}$.

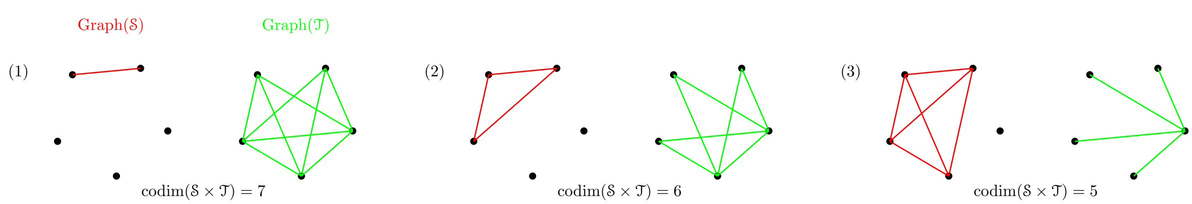

Figure 3.4: Characterization of the permissible subvarieties $\mathcal{S} \times \mathcal{T} \subset \mathrm{Grass}(3, 5) \times \mathrm{Grass}(3, 5)$ that give components of $\mathcal{Y}_{5,3,3}$.

CHAPTER 4

Next: Arbitrary Vanishing of Plücker Coordinates

Describing the minimal primes of $\mathfrak{P}_{n-2}$ remains incomplete until we have analyzed the components of $\mathcal{Y}_{n,n-1,n-2}$. A complete description of $\mathfrak{P}_{n-2}$ would be useful particularly for $n = 5$, because we would have another example of an ideal $\mathfrak{P}_3$. Then by Theorem 3, a natural next step would be to begin analysis of the ideals $\mathfrak{P}_{n-3}$. We anticipate the difficulty will be in studying the locally closed sets $\mathcal{Y}_{n,n-1,n-3}$ and $\mathcal{Y}_{n,n-2,n-3}$. A possible strategy would be to apply the map Θ from Proposition 5, restricted to those sets. The advantage is that studying subvarieties of Grassmannians lends itself to techniques in matroid theory.

4.1 Components of $\mathcal{Y}_{n,n-3,n-3}$

As in the $r = t = n-2$ case a matrix $A \in \mathcal{Y}_{n,n-3,n-3}$ factors so that we may identify A with a pair of points in $\mathrm{Grass}(n-3, n)$. Every subset of the Plücker coordinates corresponds to a simplicial complex that is a union of 2-simplices, by taking complements of indexing sets. There is a notion of permissiblity; permissible 2-complexes are the ones that actually come from a matrix.

Question 1. Given a permissible 2-complex, is the closure of the algebraic set defined by it irreducible?

Question 2. Is every algebraic set defined by vanishing of Plücker coordinates a union of 2-permissible ones which are irreducible?

The problem reduces to finding the conditions for a set of minors of a generic $3 \times (n-3)$ matrix, U, to define a prime ideal in $K[U]$. We used *Macaulay2* ([13]) for $n = 8$ and $K = \mathbb{Z}/101\mathbb{Z}$ to compute the minimal primes for various collections of Plücker coordinates (see Figure 4.1 for an example). One interesting case is when we require a 3×3 minor and

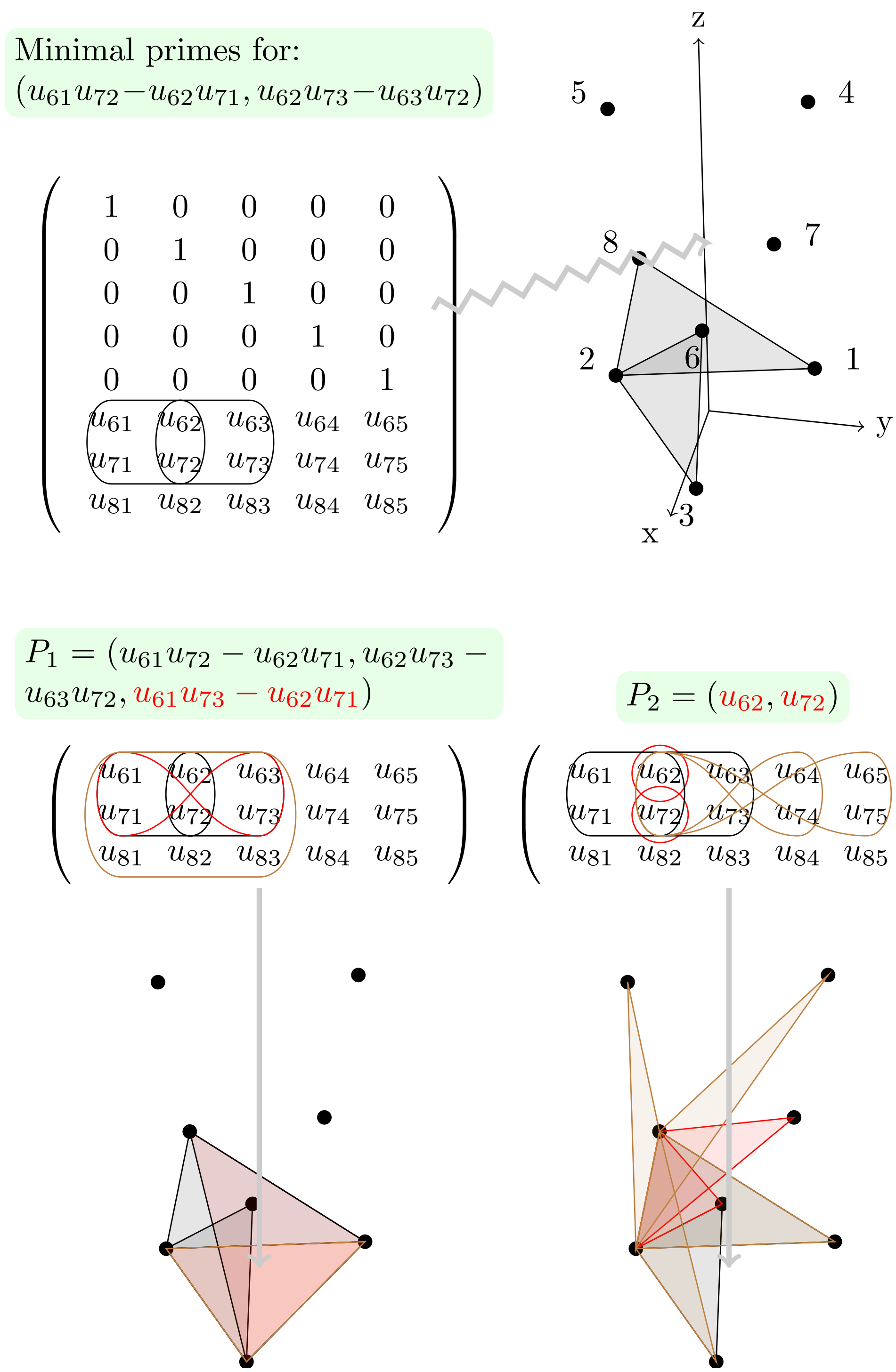

Figure 4.1: Minimal primes for two overlapping 2-minors of a $3 \times (n-3)$ matrix, for $n = 8$ and $K = \mathbb{Z}/101\mathbb{Z}$.

one of its nested 2×2 minors to vanish. We can prove *Macaulay2*'s result directly. In the following lemma, we use an alternate notation for minors to save space; put

$$\begin{aligned} \Delta_{i_1 j_1}^{i_2 j_2} &= \Delta\left(\{i_1, j_1\}; \{i_2, j_2\}\right) = \det X\left(\{i_1, j_1\}; \{i_2, j_2\}\right) \\ \Delta_{ij} &= \Delta\left(\{i, j\}; \{i, j\}\right). \end{aligned}$$

Lemma 5. *Let X denote a size 3 generic square matrix over an algebraically closed field K. Suppose $\Delta = \det X$, along with some size 2 minor $\Delta_{i_1 j_1}^{i_2 j_2}$, generate an ideal I. Then I has two minimal primes:*

$$(\Delta_{i_1 j_1}, \Delta_{i_1 j_1}^{i_1 k_1}, \Delta_{i_1 j_1}^{j_1 k_1}) \quad \text{and} \quad (\Delta_{i_2 j_2}, \Delta_{i_2 k_2}^{i_2 j_2}, \Delta_{j_2 k_2}^{i_2 j_2}).$$

In the notation, $\{i_1, j_1, k_1\} = \{i_2, j_2, k_2\} = \{1, 2, 3\}$ as sets.

Proof. Let $R = K[X]/I$ and write $X = \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{smallmatrix}\right)$. I has two generators, so its height is at most 2. We first show $x_{k_1 j_2}$ is not a zero-divisor on R by showing $J = I + x_{k_1 j_2}$ has height 3 in $K[X]$. In $K[X]/J$,

$$\begin{aligned} 0 &= \Delta \\ &= \pm\, x_{k_1 k_2} \Delta_{i_1 j_1}^{i_2 j_2} \;\pm\; x_{k_1 i_2} \Delta_{i_1 j_1}^{j_2 k_2} \;\pm\; x_{k_1 j_2} \Delta_{i_1 j_1}^{i_2 k_2} \\ &= \pm\, x_{k_1 i_2} \Delta_{i_1 j_1}^{j_2 k_2} \end{aligned}$$

implies we can decompose

$$\begin{aligned} \mathcal{V}(J) &= \mathcal{V}(\Delta_{i_1 j_1}^{i_2 j_2}, x_{k_1 j_2}, x_{k_1 i_2}) \;\cup\; \mathcal{V}(\Delta_{i_1 j_1}^{i_2 j_2}, x_{k_1 j_2}, \Delta_{i_1 j_1}^{j_2 k_2}) \\ &= \mathcal{V}\underbrace{(\Delta_{i_1 j_1}^{i_2 j_2}, x_{k_1 j_2}, x_{k_1 i_2})}_{\text{height} = 3} \cup \mathcal{V}\underbrace{(\Delta_{i_1 j_1}^{i_2 j_2}, \Delta_{i_1 j_1}^{j_2 k_2}, \Delta_{i_1 j_1}^{i_2 k_2}, x_{k_1 j_2})}_{\text{height} = 2+1=3} \cup \mathcal{V}\underbrace{(x_{i_1 j_2}, x_{j_1 j_2}, x_{k_1 j_2})}_{\text{height} = 3}. \end{aligned}$$

Now we localize at $x_{k_1 j_2}$. Over $R' = R_{x_{k_1 j_2}}$ we can clear the remaining entries in row K_1 of X. Let X' denote the resulting matrix. Its entries include

$$\begin{array}{ll} x'_{i_1 k_2} = x_{i_1 k_2} - \frac{x_{k_1 k_2}}{x_{k_1 j_2}} x_{i_1 j_2} & \qquad x'_{i_1 i_2} = x_{i_1 i_2} - \frac{x_{k_1 i_2}}{x_{k_1 j_2}} x_{i_1 j_2} \\ x'_{j_1 k_2} = x_{j_1 k_2} - \frac{x_{k_1 k_2}}{x_{k_1 j_2}} x_{j_1 j_2} & \qquad x'_{j_1 i_2} = x_{j_1 i_2} - \frac{x_{k_1 i_2}}{x_{k_1 j_2}} x_{j_1 j_2} \end{array}.$$

Again, $\det X' = 0$, and so if we expand along the K_1th row we see the size 2 minor $\delta_{i_1 j_2}^{i_1 k_2}$ vanishes, as well as $\delta_{i_1 j_1}^{i_2 j_2} = \Delta_{i_1 j_1}^{i_2 j_2}$. Thus the minimal primes of IR' are

$$(\delta_{i_1 j_1}^{i_2 j_2}, \delta_{i_1 j_1}^{j_2 k_2}, \delta_{i_1 j_1}^{i_2 k_2})R' \quad \text{and} \quad (x'_{i_1 i_2}, x'_{j_1 i_2})R'.$$

Finally, the respective contractions to R are the minimal primes for I.

$$\begin{aligned}
R \cap (\delta_{i_1j_1}^{i_2j_2}, \delta_{i_1j_1}^{j_2k_2}, \delta_{i_1j_1}^{i_2k_2})R' &= \Big(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1j_1}^{j_2k_2},\, x_{k_1j_2}\Delta_{i_1j_1}^{i_2k_2} + x_{k_1k_2}x_{i_1j_2}x_{j_1k_2} \\
&\qquad - x_{k_1i_2}x_{i_1j_2}x_{j_1i_2} - x_{k_1k_2}x_{j_1j_2}x_{i_1i_2} + x_{k_1i_2}x_{j_1j_2}x_{i_1k_2}\Big) : x_{k_1j_2}^{\infty} \\
&= \Big(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1j_1}^{j_2k_2},\, \Delta\Big) : x_{k_1j_2}^{\infty} = \Big(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1j_1}^{j_2k_2},\, \Delta_{i_1j_1}^{i_2k_2}\Big) : x_{k_1j_2}^{\infty} \\
&= \Big(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1j_1}^{j_2k_2},\, \Delta_{i_1j_1}^{i_2k_2}\Big).
\end{aligned}$$

By hypothesis, $\Delta_{i_1j_1}^{i_2j_2} \in (x'_{i_1i_2}, x'_{j_1i_2})R'$. Therefore,

$$\begin{aligned}
R \cap (x'_{i_1i_2}, x'_{j_1i_2})R' &= \left(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1k_1}^{i_2j_2},\, \Delta_{j_1k_1}^{i_2j_2}\right) : x_{k_1i_2}^{\infty} \\
&= \left(\Delta_{i_1j_1}^{i_2j_2},\, \Delta_{i_1k_1}^{i_2j_2},\, \Delta_{j_1k_1}^{i_2j_2}\right).
\end{aligned}$$

□

4.2 More General $\mathcal{Y}_{n,t,t}$

Lemma 5 generalizes immediately:

Lemma 6. *Let X denote a generic $n \times n$ matrix, $n \geq 2$. Let $\Delta = \det X$ and let Δ_{n-1} denote some size $n-1$ minor of X. The ideal $I = (\Delta, \Delta_{n-1})$ has two minimal primes, one of which is generated by all size $n-1$ minors with the same rows as Δ_{n-1}, the other generated by all size $n-1$ minors with the same columns as Δ_{n-1}. Furthermore, I is reduced.*

Proof. Let $X' \subset X$ denote the submatrix consisting of the rows of Δ_{n-1} and let $X'' \subset X$ denote the submatrix consisting of the columns of Δ_{n-1}. We first show

$$\mathcal{V}(I) = \mathcal{V}(\mathrm{I}_{n-1}\, X') \cup \mathcal{V}(\mathrm{I}_{n-1}\, X'').$$

The left-hand inclusion is clear, since a matrix in either component cannot have full rank. Suppose a matrix $A \in \mathcal{V}(I)$. Then the classical adjoint of A has an entry that vanishes, say the (i,j)th entry. Since $\det A = 0$, the classical adjoint of A has rank one. This implies any size 2 minor involving the (i,j)th entry vanishes; this happens if and only if either the ith row or jth column of adj A vanishes. Such entries are exactly the size $n-1$ minors sharing, respectively, the rows of Δ_{n-1} or the columns of Δ_{n-1}.

We show I is reduced by induction on n. Suppose $n = 2$. Modulo I, the determinant of X is a product of two of its entries, and these are the only zero-divisors. Hence I is reduced. Now suppose I is reduced for all $n' = 2, \ldots, n-1$. The minimal primes for I are generated by degree $n-1$ polynomials, so cannot contain any entry of X. In particular, we may invert any entry (i,j) in the submatrix for Δ_{n-1}, then clear all other entries in the ith row and jth column. Such row and column operations do not affect I, and the submatrix obtained by eliminating the ith row and jth column is exactly the $n-1$ case. $\square$

The remaining two lemmas characterize some of the ideal-theoretic consequences of certain combinations of minors vanishing in a generic matrix. Lemma 7 describes the ideal generated by a minor and a nested minor, by reducing to the case of a square matrix whose determinant and one nested minor vanish. Lemma 8 describes the case where two overlapping maximal minors of a generic matrix vanish.

Lemma 7. *Suppose a nested size r minor, Δ_r, of an $n\times n$ generic matrix $X=(x_{ij})$ vanishes, along with $\Delta = \det X$, and suppose $1 \le r < r+2 \le n$. Then the ideal $I=(\Delta,\Delta_r)$ is prime (and a complete intersection).*

Proof. Let X_r denote the submatrix whose determinant is Δ_r and without loss of generality, assert X_r is the lowermost rightmost submatrix of X. First suppose $r=1$. Expanding along the column containing $\Delta_1 = x_{nn}$, since $n \ge r+2$, Δ is linear in at least two variables not equal to x_{nn}. Also, all coefficients of $x_{n1},\dots,x_{nn}$ in the chosen expansion of Δ are relatively prime, and none of those coefficients involve the variable x_{nn}. Thus

$$K[X]/I \cong \left(\frac{K[X]}{(\Delta_1)}\right)/(\Delta)$$

is a domain with Krull dimension n^2-2.

Next, suppose $r=2$. We may assert some entry of X_2 is non-zero;

$$\begin{aligned}\mathrm{ht}\,(\mathrm{I}_1(X_2)) &= \mathrm{ht}\left(\begin{smallmatrix} x_{n-1,n-1} & x_{n-1,n} \\ x_{n-1,n} & x_{n,n}\end{smallmatrix}\right)\\ &= 4 > 2 \text{ (the number of generators of } I)\end{aligned}$$

implies all matrices $A=(a_{ij}) \in \mathcal{V}(I)$ with $a_{n-1,n-1}, a_{n-1,n}, a_{n,n-1}, a_{n,n} = 0$ lie in a subset of too small a dimension to include any components of $\mathcal{V}(I)$. Thus we assert, without loss of generality, $x = x_{nn} \neq 0$. We exhibit a point $A \in \mathcal{V}(I)$ to show x is not in any minimal prime of I:

$$A = \begin{pmatrix} \mathbf{I}_{n-2} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Since we can invert x we may clear the entries of the last row and column of X, and the problem reduces to $r = 1$.

Now, for $r > 2$, we use induction; suppose the claim is true for $r' = 1, \ldots, r-1$. We use the same arguments to show we can invert an entry x_{ij} of X_r. Then, when we clear the entries in the ith row and jth column we reduce to the $r-1$ case. $\square$

Lemma 8. *Let X denote a generic $h \times n$ matrix where $h+2 \leq n \leq 2h-1$. Suppose the two respective minors given by the first h columns and last h columns of X generate the ideal $I \subseteq K[X]$. Then I is a prime complete intersection.*

Proof. The two vanishing minors are relatively prime, so I is a complete intersection. Write $X = \big(U \,|\, Z \,|\, V \big)$, so that $\det(U \,|\, Z) = \det(Z \,|\, V) = 0$. We next show $\mathcal{V}(I)$ is irreducible. Put $c = 2h - n$, the number of columns in Z (thus the hypotheses imply $1 \leq c \leq h-2$). It is enough to map an irreducible set onto the open set

$$\mathcal{V}(I) \cap \left(\mathbb{A}^{h \times n} \setminus \mathcal{V}\left(\mathrm{I}_c(Z)\right)\right),$$

because I is a complete intersection and

$$\dim \mathcal{V}\left(\mathrm{I}_c(Z)\right) = hn - (h - c + 1)$$

is strictly less than $hn - 2$, the dimension of any component of $\mathcal{V}(I)$.

The vanishing of I implies there exist column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{h-c-1}$, $\mathbf{d}_1, \ldots, \mathbf{d}_{h-c-1}$ such that we may decompose

$$\begin{aligned} U &= \Big(\mathbf{b}_1 \mid \cdots \mid \mathbf{b}_{h-c-1} \mid Z \Big) \cdot U' \\ V &= \Big(\mathbf{d}_1 \mid \cdots \mid \mathbf{d}_{h-c-1} \mid Z \Big) \cdot V', \end{aligned}$$

where U' and V' each have size $(h-1) \times (n-h)$. Counting parameters over K, we get a surjection

$$\underbrace{\mathbb{A}^{h \times c} \setminus \mathcal{V}\left(\mathrm{I}_c(Z)\right)}_{Z} \times \underbrace{\mathbb{A}^{2(h-c-1)}}_{\substack{\mathbf{b}_1,\ldots,\mathbf{b}_{h-c-1}, \\ \mathbf{d}_1,\ldots,\mathbf{d}_{h-c-1}}} \times \underbrace{\mathbb{A}^{(h-1)\times(n-h)}}_{U'} \times \underbrace{\mathbb{A}^{(h-1)\times(n-h)}}_{V'} \twoheadrightarrow \mathcal{V}(I) \cap \left(\mathbb{A}^{h \times n} \setminus \mathcal{V}\left(\mathrm{I}_c(Z)\right)\right).$$

To complete the proof, we must show I is reduced, and hence, prime. Each of the minors generating I can be expanded along the same column of Z, call it $(z_1, \ldots, z_h)$. Then I gives a system of equations

$$\begin{aligned} \Delta_1^U z_1 + \cdots + \Delta_h^U z_h &= 0 \\ \Delta_1^V z_1 + \cdots + \Delta_h^V z_h &= 0. \end{aligned} \tag{4.1}$$

The $2 \times h$ matrix, Φ, that expresses the homogeneous system (4.1) has entries consisting of size $h-1$ minors from $(U \,|\, Z)$ in the first row, and size $h-1$ minors from $(Z \,|\, V)$ in the second row. If rank $\Phi = 2$ then we may solve for, wolog, z_1 and z_2 in terms of $z_3, \ldots, z_h$. That case happens if and only if the minor $\delta = \Delta_1^U \Delta_2^V - \Delta_2^U \Delta_1^V$ is not a zero-divisor on I, i.e., $\delta \notin \operatorname{rad} I$. We exhibit a matrix, $A \in \mathcal{V}(I) \setminus \mathcal{V}(\delta)$:

$$\begin{pmatrix} \multicolumn{3}{c}{\mathbf{0}} & 1 \\ \mathbf{I}_{h-c} & \multicolumn{3}{c}{\mathbf{0}} \\ & \mathbf{0} & \mathbf{I}_{h-c-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_c & \multicolumn{2}{c}{\mathbf{0}} \end{pmatrix}$$

We conclude the quotient ring $K[X]/I$ is isomorphic to a polynomial ring in $hn-2$ variables, localized at one element, which is certainly a domain. $\square$

4.3 Connection to Matroid Theory

Matroids are a type of combinatorial data used to describe many seemingly unrelated objects in mathematics, including graphs, transversals, vector spaces, and networks. For more information than given here, we defer the reader to the recently published monograph, *Topics in Matroid Theory* ([31]), by Pitsoulis, particularly Chapters 2 and 3.

4.3.1 The Numerous Equivalent Definitions for a Matroid

Matroid has many equivalent definitions, including a characterization using the Greedy Algorithm. To foster some intuition of what a matroid really is, behind all the combinatorial language, we state several of those definitions here. Let E be a finite set. An *independence system* is a set system $(E, \mathcal{I})$, where $\mathcal{I}$ is a collection of subsets in E, satisfying the following:

(1) $\emptyset \in \mathcal{I}$.

(2) If $S \in \mathcal{I}$ and $T \subseteq S$ then $T \in \mathcal{I}$.

Unless otherwise stated, in this section $(E, \mathcal{I})$ shall always refer to an independence system.

Definition 3 (Independence Definition of a Matroid)**.** An independence system $(E, \mathcal{I})$ is a *matroid* if and only if it satisfies the *independence augmentation axiom*: If $S, T \in \mathcal{I}$ and $|S| > |T|$ then there exists $e \in S \setminus T$ such that $T \cup \{e\} \in \mathcal{I}$.

Members of the collection $\mathcal{I}$ are called *independent*, while members of the the complement $\mathcal{P}(E) \setminus \mathcal{I}$ of $\mathcal{I}$ in the power set $\mathcal{P}(E)$ are called *dependent*.

Definition 4 (Dependent Sets Definition of a Matroid)**.** A collection $\mathcal{D} \subseteq \mathcal{P}(E)$ is the set of dependent sets of a *matroid* if and only if

(1) $\emptyset \notin \mathcal{D}$,
(2) if $S \in \mathcal{D}$ and $T \subseteq S$ then $T \in \mathcal{D}$, and
(3) if $S, T \in \mathcal{D}$ and $S \cap T \notin \mathcal{D}$, then for every $e \in E$, $(S \cup T) \setminus \{e\} \in \mathcal{D}$.

The maximal sets in $\mathcal{I}$ are called *bases*. Let $\mathcal{B}$ denote the collection of bases in $\mathcal{I}$.

Definition 5 (Basis Definition of a Matroid)**.** A collection $\mathcal{B} \subset \mathcal{P}(E)$ is the set of bases of a *matroid* if and only if

(1) $\mathcal{B} \neq \emptyset$ and
(2) (*base exchange axiom*) if $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, then there exists $e' \in B' \setminus B$ such that $(B \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

The *rank function* from the power set of E to the non-negative integers is defined as

$$\begin{aligned} r : \mathcal{P}(E) &\to \mathbb{Z}_+ \\ S &\mapsto \max_{\substack{T \subseteq S \\ T \in \mathcal{I}}} \{|T|\}. \end{aligned}$$

Definition 6 (Rank Definition of a Matroid)**.** A function $r : \mathcal{P}(E) \to \mathbb{Z}_+$ is the rank function of a *matroid* if and only for all $S, T \subseteq E$

(1) $0 \leq r(S) \leq |S|$,
(2) if $T \subseteq S$ then $r(T) \leq r(S)$, and
(3) (*submodularity*) $r(S) + r(T) \geq r(S \cup T) + r(S \cap T)$.

The *closure operator* for $(E, \mathcal{I})$ is defined as

$$\begin{aligned} \mathrm{cl} : \mathcal{P}(E) &\to \mathcal{P}(E) \\ S &\mapsto \{e \in E \,|\, r(S \cup \{e\}) = r(S)\}. \end{aligned}$$

Definition 7 (Closure Definition of a Matroid)**.** A function $\mathrm{cl} : \mathcal{P}(E) \to \mathcal{P}(E)$ is the closure operator of a *matroid* if and only if, for all $S, T \subseteq E$ and for all $e, e' \in E$,

(1) $S \subseteq \mathrm{cl}(S)$,
(2) if $T \subseteq S \subseteq E$ then $\mathrm{cl}(T) \subseteq \mathrm{cl}(S)$,
(3) $\mathrm{cl}\,(\mathrm{cl}(S)) = \mathrm{cl}(S)$, and
(4) (*MacLane-Steinitz exchange property*) if $e' \in \mathrm{cl}\,(S \cup \{e\}) \setminus \mathrm{cl}(S)$ then $e \in \mathrm{cl}\,(S \cup \{e'\})$.

If $\mathrm{cl}(S) = E$ for some $S \subseteq E$, then S is called a *spanning set* of $(E, \mathcal{I})$.

Definition 8 (Spanning Sets Definition of a Matroid)**.** A collection $\mathcal{I} \subseteq \mathcal{P}(E)$ is the set of spanning sets of a *matroid* if and only if
(1) $\mathcal{I} \neq \emptyset$,
(2) if $S \in \mathcal{I}$ and $S \subseteq T$, then $T \in \mathcal{I}$, and
(3) if $S, T \in \mathcal{I}$ and $|S| > |T|$, then there exists $e \in S \setminus T$ such that $S \setminus \{e\} \in \mathcal{I}$.

Example 7. A matroid defined by a K-vector space is called *K-representable*. Let $E = \{1, \ldots, n\}$ index the columns of a matrix A, whose span is a vector space, V. The independent sets, those that comprise $\mathcal{I}$, are the sets of indices of columns of A that are linearly independent. The independence augmentation axiom is a generalized statement that all maximal sets in $\mathcal{I}$ have the same cardinality. The maximal sets in $\mathcal{I}$ index columns that form a basis for V. The collection of these sets, $\mathcal{B}$, satisfies Definition 5.

Let $S \subseteq E$. The closure of S is the vector space span of the correspondingly indexed columns of A. The rank function maps a set of columns in A to the dimension of their span. The spanning sets are the maximal sets of $\mathcal{I}$.

4.3.2 Matroid Subvarieties of a Grassmannian

Fix $r \leq n$. We get a matroid structure (see Example 7) on the finite set $E = \{1, \ldots, n\}$ of columns of an $r \times n$ matrix when we prescribe a subset of Plücker coordinates to vanish. Given a set of Plücker coordinates for $\mathcal{G} = \mathrm{Grass}(r, n)$, let $\mathcal{D}$ denote the set of their indices; $\mathcal{D}$ shall consist of the "dependent sets" as described in Definition 4. Put $\mathcal{I} = \mathcal{P}(E) \setminus \mathcal{D}$. If A is a matrix whose Plücker coordinates with indices in $\mathcal{D}$ vanish, i.e., its matroid is $(E, \mathcal{I})$, then clearly the orbits of A under the action of $\mathrm{GL}(r, K)$ (matrix multiplication from the left) also have the matroid $(E, \mathcal{I})$.

For a fixed matroid $(E, \mathcal{I})$, the *open matroid variety* is the subset of points in $\mathcal{G}$ whose matroid is $(E, \mathcal{I})$. Its closure is called a *matroid variety*, which we shall denote by $\mathcal{V}(E, \mathcal{I})$.

Example 8. Schubert varieties are matroid varieties.

For any Plücker coordinate with index $\underline{\mathrm{i}}$, let $x_{\underline{\mathrm{i}}}$ denote the correspondingly indexed variable in the homogeneous coordinate ring for $\mathcal{G}$. The following example shows we cannot, in general, simply use the indices from $\mathcal{D}$ on the Plücker variables to generate the defining ideal for $\mathcal{V}(E, \mathcal{I})$.

Example 9 (Counterexample 2.6 of [12])**.** Put $r = 3, n = 7$, and

$$\mathcal{D} = \{\{1,2,7\}, \{3,4,7\}, \{5,6,7\}\},$$

the set of indices for Plücker coordinates we require to vanish. We get a matroid $(E, \mathcal{I})$, where $E = \{1, \ldots, 7\}$ and $\mathcal{I} = \mathcal{P}(E) \setminus \mathcal{D}$. One hopes the defining ideal for $\mathcal{V}(E, \mathcal{I})$ is

$$I = (x_{\{1,4,7\}}, x_{\{3,4,7\}}, x_{\{5,6,7\}}).$$

However, the defining ideal is actually

$$J = I + (x_{\{1,2,4\}} x_{\{3,5,6\}} - x_{\{1,2,3\}} x_{\{4,5,6\}}).$$

A particular class of matroid varieties exists, however, where the geometry is better behaved. A *positroid* is a matroid $(E, \mathcal{I})$, such that $E = \{1, \ldots, n\}$ and the matroid is determined by a rank condition on cyclic intervals in E, where a cyclic interval is an ordinary interval or its complement. *Positroid varieties* are the matroid varieties we get from positroids. Positroid varieties are normal, Cohen-Macaulay, have rational singularities, and their defining ideals are given by Plücker variables ([26]).

Before proving the final theorem of the thesis we need the notion of duality for matroids. Given a matroid $(E, \mathcal{B})$, where $\mathcal{B} \subseteq \mathcal{I}$ is the set of bases defining the matroid, the *dual* matroid is defined as $(E, \mathcal{B}^*)$, where

$$\mathcal{B}^* = \{B' \mid B' = E \setminus B \text{ for some } B \in \mathcal{B}\}.$$

From this definition it is clear that a matroid is a positroid if and only if its dual is a positroid.

Theorem 9 (–)**.** *If a subset of Plücker coordinates for* $\mathrm{Grass}(n-2, n)$ *defines an irreducible algebraic set, i.e., a variety, then, after renumbering columns, it is positroidal. Every irreducible component of every matroid scheme in* $\mathrm{Grass}(n-2, n)$ *(one defined by vanishing of a subset of Plücker variables) is of this form, and so, after renumbering, is positroidal.*

Proof. We shall show, equivalently, that the dual matroid variety is positroidal. Once we fix a non-vanishing Plücker coordinate, which we can assume corresponds to $\{1, \ldots, n-2\}$, a point in $\mathrm{Grass}(n-2, n)$ has a unique representation as a size $\mathbf{I}_{n-2}$ identity matrix with a $2 \times (n-2)$ matrix, appended to the bottom. The rows give a matroid structure, whose dual is defined by the columns of $(\mathbf{I}_2 \mid A')$, for some $2 \times (n-2)$ matrix A'. We will show any rank conditions on A' are positroidal.

Fix a set of (indices for) the Plücker coordinates that vanish for A'. If 2×2 minors overlap in one column, we get a decomposition in which either the overlap column vanishes or the third minor vanishes. It follows that in the irreducible components, the columns with no zeros fall into equivalence classes, where two columns are equivalent if and only if the minor they form vanishes.

Let $\mathcal{V}$ denote a fixed irreducible component that contains A'. We construct a positroid that defines a variety isomorphic to $\mathcal{V}$, by reordering the columns of A'. Reorder the columns of A' as follows: Write columns that vanish first. Next, put the columns where the entry in the top row vanishes. There are two cases to consider:

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} = 0.$$

In either case, the vanishing of the minor implies the vanishing of at least one of the starred entries, contradicting our hypotheses. Likewise, we list the columns with zero in the bottom entry and non-zero in the top, which form another equivalence class. Finally, for columns where neither entry vanishes, we list columns in the same equivalence class consecutively. The resulting matrix is

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}.$$

$\square$

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