Online Supplement of

Dynamic Pricing and Inventory Management with Dual Suppliers of Different Leadtimes and Disruption Risks

by

Xiting Gong, Xiuli Chao, and Shaohui Zheng

In this online supplement, we provide the missing proofs in the paper. We first prove Lemmas 2 and 3. For easy reference, we state the results again.

Lemma 2. *Suppose $\phi_i(x,\theta) : \Re \times [0,1] \to \Re$ is submodular in (x,θ), $i = 1,2$; $\phi_1(x,\theta)$ is concave in x; and $\phi_1(x,\theta) - \phi_2(x,\theta)$ is decreasing in x, then*

$$\psi(x,\theta) := \max_{y \geq x} \{\theta\phi_1(y,\theta) + (1-\theta)\phi_2(x,\theta)\}$$

is also submodular in (x,θ).

Proof. We prove the submodularity of $\psi(x,\theta)$ by definition, i.e., for any $x_1 < x_2$ and $0 \leq \theta_1 < \theta_2 \leq 1$, verify that

$$\psi(x_1,\theta_1) + \psi(x_2,\theta_2) \leq \psi(x_1,\theta_2) + \psi(x_2,\theta_1). \tag{23}$$

Denote $x^*(\theta) = \arg\max_x \phi_1(x,\theta)$. Then, $\psi(x,\theta) = \theta\phi_1(\max\{x, x^*(\theta)\},\theta) + (1-\theta)\phi_2(x,\theta)$. Since $\phi_1(x,\theta)$ is a submodular function, $x^*(\theta)$ is decreasing in θ. Thus, $x^*(\theta_1) \geq x^*(\theta_2)$. In what follows, we divide the analysis into three cases and separately verify (23) holds under all cases.

Case 1. $x_1 \geq x^*(\theta_1)$. In this case, $x_2 \geq x_1 \geq x^*(\theta_1) \geq x^*(\theta_2)$. Then, $\psi(x_i,\theta_j) = \theta_j\phi_1(x_i,\theta_j) + (1-\theta_j)\phi_2(x_i,\theta_j)$, $i,j = 1,2$. Then,

$$\begin{aligned}
\psi(x_2,\theta_2) - \psi(x_1,\theta_2) &= \theta_2(\phi_1(x_2,\theta_2) - \phi_1(x_1,\theta_2)) + (1-\theta_2)(\phi_2(x_2,\theta_2) - \phi_2(x_1,\theta_2)) \\
&\leq \theta_2(\phi_1(x_2,\theta_1) - \phi_1(x_1,\theta_1)) + (1-\theta_2)(\phi_2(x_2,\theta_1) - \phi_2(x_1,\theta_1)) \\
&\leq \theta_1(\phi_1(x_2,\theta_1) - \phi_1(x_1,\theta_1)) + (1-\theta_1)(\phi_2(x_2,\theta_1) - \phi_2(x_1,\theta_1)) \\
&= \psi(x_2,\theta_1) - \psi(x_1,\theta_1),
\end{aligned}$$

where the first inequality is from the submodularity of $\phi_1(x,\theta)$ and $\phi_2(x,\theta)$, and the second one holds since $\theta_1 < \theta_2$ and $\phi_1(x,\theta) - \phi_2(x,\theta)$ is decreasing in x. Thus, (23) holds in Case 1.

Case 2. $x_2 \le x^*(\theta_1)$. In this case, $x^*(\theta_1) \ge x_2 > x_1$. Then, $\psi(x_i, \theta_1) = \theta_1\phi_1(x^*(\theta_1), \theta_1) + (1 - \theta_1)\phi_2(x_i, \theta_1)$, $i = 1, 2$. Since $\phi_1(x, \theta)$ is concave in x and $\phi_1(x, \theta_1) - \phi_2(x, \theta_1)$ is decreasing in x, we have $\phi_1(x_2, \theta_1) \ge \phi_1(x_1, \theta_1)$ and $\phi_2(x_2, \theta_1) \ge \phi_2(x_1, \theta_1)$. Then,

$$\begin{aligned}\psi(x_2, \theta_2) - \psi(x_1, \theta_2) &= (1 - \theta_2)(\phi_2(x_2, \theta_2) - \phi_2(x_1, \theta_2)) + \theta_2\Big(\max_{y \ge x_2} \phi_1(y, \theta_2) - \max_{y \ge x_1} \phi_1(y, \theta_2)\Big) \\ &\le (1 - \theta_2)(\phi_2(x_2, \theta_2) - \phi_2(x_1, \theta_2)) \\ &\le (1 - \theta_1)(\phi_2(x_2, \theta_1) - \phi_2(x_1, \theta_1)) \\ &= \psi(x_2, \theta_1) - \psi(x_1, \theta_1),\end{aligned}$$

where the first inequality holds since $\max_{y \ge x} \phi_1(y, \theta_2)$ is decreasing in x and the second inequality holds since when $\phi_2(x_2, \theta_2) \le \phi_2(x_1, \theta_2)$, $(1-\theta_2)(\phi_2(x_2, \theta_2) - \phi_2(x_1, \theta_2)) \le 0 \le (1-\theta_1)(\phi_2(x_2, \theta_1) - \phi_2(x_1, \theta_1))$; and when $\phi_2(x_2, \theta_2) > \phi_2(x_1, \theta_2)$, $(1-\theta_2)(\phi_2(x_2, \theta_2) - \phi_2(x_1, \theta_2)) \le (1-\theta_2)(\phi_2(x_2, \theta_1) - \phi_2(x_1, \theta_1)) \le (1 - \theta_1)(\phi_2(x_2, \theta_1) - \phi_2(x_1, \theta_1))$ due to the submodularity of $\phi_2(x, \theta)$ and $0 \le \theta_1 < \theta_2 \le 1$. Thus, (23) holds in Case 2.

Case 3. $x_1 < x^*(\theta_1) < x_2$. In this case,

$$\psi(x_2, \theta_2) - \psi(x_2, \theta_1) \le \psi(x^*(\theta_1), \theta_2) - \psi(x^*(\theta_1), \theta_1) \le \psi(x_1, \theta_2) - \psi(x_1, \theta_1),$$

where the first inequality follows since (23) holds under Case 1 and the second inequality follows since (23) holds under Case 2. Thus, (23) holds in Case 3.

In summary, since we have shown (23) holds under all possible cases, $\psi(x, \theta)$ is submodular in (x, θ). The proof is complete. □

Lemma 3. *Suppose $\phi_1(x, q) : \Re \times \Re^+ \to \Re$ is concave and submodular in (x, q) and $\phi_2(x) : \Re \to \Re$ is concave in x. In addition, suppose $\max_{(x,q) \in \Re \times \Re^+} \phi_1(x, q) = \max_{x \in \Re} \phi_1(x, 0)$ and $\phi_1(x, 0) - \phi_2(x)$ is increasing in x. Then,*

$$\psi(x) := \max_{q \ge 0} \phi_1(x, q) - \max_{q \ge 0} \phi_2(x + q)$$

is also increasing in x.

Proof. Define $x_1^* = \arg\max_{x \in \Re} \phi_1(x, 0)$ and $x_2^* = \arg\max_{x \in \Re} \phi_2(x)$. Then, it follows that

$$\phi_1(x_1^*, 0) - \phi_2(x_1^*) \ge \phi_1(x_2^*, 0) - \phi_2(x_2^*).$$

Since $\phi_1(x, 0) - \phi_2(x)$ is increasing in x, we have $x_1^* \ge x_2^*$.

We first prove $\psi(x)$ is increasing in x when $x \le x_1^*$. By the definition of x_1^* and the assumption that $\max_{x\in\Re} \phi_1(x,0) = \max_{(x,q)\in\Re\times\Re^+} \phi_1(x,q)$, we have

$$\max_{x\in\Re}\Big\{\max_{q\ge 0} \phi_1(x,q)\Big\} = \max_{x\in\Re} \phi_1(x,0) = \phi_1(x_1^*,0) \le \max_{q\ge 0} \phi_1(x_1^*,q). \tag{24}$$

Thus, $\max_{q\ge 0} \phi_1(x,q)$ achieves its maximization when $x = x_1^*$. Since $\phi_1(x,q)$ is concave in (x,q), $\max_{q\ge 0} \phi_1(x,q)$ is also concave in x, and thus, it is increasing in x when $x \le x_1^*$. Since $\max_{q\ge 0} \phi_2(x+q) = \max_{y\ge x} \phi_2(y)$ is decreasing in x, it follows that $\psi(x)$ is increasing in x when $x \le x_1^*$.

We next prove $\psi(x)$ is also increasing in x when $x \ge x_1^*$. Define $q_1^*(x) = \arg\max_{q\ge 0} \phi_1(x,q)$. Since $\phi_1(x,q)$ is submodular in (x,q), then $q_1^*(x)$ is decreasing in x. Note from (24) that $\phi_1(x_1^*,0) = \max_{q\ge 0} \phi_1(x_1^*,q)$. Then, $q_1^*(x_1^*) = 0$, and thus, $q_1^*(x) = 0$ for $x \ge x_1^*$. Since $\phi_2(x)$ is concave in x and by the definition of x_2^*, $\max_{q\ge 0} \phi_2(x+q) = \phi_2(\max\{x, x_2^*\})$. Thus, when $x \ge x_1^*$,

$$\psi(x) = \phi_1(x, q_1^*(x)) - \phi_2(\max\{x_2^*, x\}) = \phi_1(x,0) - \phi_2(x).$$

As $\phi_1(x,0) - \phi_2(x)$ is increasing in x, it follows that $\psi(x)$ is increasing in x when $x \ge x_1^*$. □

Proof of Theorem 2

We prove (a) by induction on t; and (b) will be proved simultaneously. Since $V_{T+1}(x,i,j) = 0$, (a) is obviously true for $t = T+1$. Now assume inductively that (a) holds for $t+1$ and we shall prove the theorem holds for t. In what follows, we will only prove the theorem holds for t when $k = 1$, and a similar approach can prove the results when $k = 2$.

We first prove $V_t(x,i,j|\, c_1)$ is decreasing in c_1. Since $V_{t+1}(x,i,j|\, c_1)$ is decreasing in c_1 by the inductive assumption, it follows from (9) and (10) that $L_t(x,i,j|\, c_1)$ and $W_t(x,y,i,j|\, c_1) + c_1 x$ are both decreasing in c_1. Thus, for any $q_1 \ge 0$, one can easily verify that the maximand in (15) is decreasing in c_1, and consequently, $V_t(x,i,j|\, c_1)$ is decreasing in c_1.

We next prove $V_t(x,i,j|\, c_1)$ is supermodular in (x, c_1) and part (b) when $k = 1$. By replacing q_1, q_2, and d with $\tilde{q}_1 = -q_1$, $\tilde{q}_2 = x + q_2 - d$ and $\tilde{d} = x - d$, the optimality equation (15) can be rewritten as

$$V_t(x,i,j|\, c_1) = \max_{\substack{\tilde{q}_1 \le 0, \tilde{q}_2 \ge \tilde{d} \\ \underline{d}_t \le x - \tilde{d} \le \bar{d}_t}} \Big\{ R_t(x-\tilde{d}) + \gamma^{1,i}[c_1\tilde{d} + W_t(\tilde{d}-\tilde{q}_1, \tilde{q}_2-\tilde{q}_1, 1, j|\, c_1)] + (1-\gamma^{1,i})[c_1\tilde{d} + W_t(\tilde{d}, \tilde{q}_2, 0, j|\, c_1)]\Big\}. \tag{25}$$

By the convexity of $G_t(\cdot)$ and the inductively assumption on $V_{t+1}(x,i,j|\, c_1)$, it can be easily verified from (9) and Proposition 1 that $L_t(x,i,j|\, c_1)$ is concave in x and supermodular in (x,c_1) while $L_t(x,i,j|\, c_1) - c_1x$ is submodular in (x,c_1). Thus, it follows from (10) that $c_1\tilde{d} + W_t(\tilde{d},\tilde{q}_2,0,j|\, c_1)$ is supermodular in $(\tilde{d},\tilde{q}_2,c_1)$. In addition, from (10), we have

$$\begin{aligned}&c_1\tilde{d} + W_t(\tilde{d}-\tilde{q}_1,\tilde{q}_2-\tilde{q}_1,1,j|\, c_1)\\ =&\alpha c_2(\tilde{d}-\tilde{q}_1) + \gamma^{2,j}\left[c_1\tilde{q}_1 + L_t(\tilde{q}_2-\tilde{q}_1,i,1|\, c_1)\right] + (1-\gamma^{2,j})[c_1\tilde{q}_1 + L_t(\tilde{d}-\tilde{q}_1,i,0|\, c_1)].\end{aligned}$$

Then, it can be easily verified from the above expression that $c_1\tilde{d} + W_t(\tilde{d}-\tilde{q}_1,\tilde{q}_2-\tilde{q}_1,1,j|\, c_1)$ is supermodular in $(\tilde{d},\tilde{q}_2,\tilde{q}_2,c_1)$. Since $R_t(x-\tilde{d})$ is supermodular in $(x,\tilde{d})$ from the concavity of $R_t(\cdot)$, the maximand in (25) is a supermodular function in $(x,\tilde{q}_1,\tilde{q}_2,\tilde{d},c_1)$. Notice that the constraint in (25) is a lattice. Thus, by applying Lemma 1, $V_t(x,i,j|\, c_1)$ is supermodular in (x,c_1), and $\tilde{q}_1^*(x,i,j|\, c_1)$, $\tilde{q}_2^*(x,i,j|\, c_1)$ and $\tilde{d}^*(x,i,j|\, c_1)$ are all increasing in c_1.

Note that $\tilde{q}_1^*(x,i,j|\, c_1) = -q_{t,1}^*(x,i,j|\, c_1)$, $\tilde{q}_2^*(x,i,j|\, c_1) = x + q_{t,2}^*(x,i,j|\, c_1) - d_t^*(x,i,j|\, c_1)$ and $\tilde{d}^*(x,i,j|\, c_1) = x - d_t^*(x,i,j|\, c_1)$. It follows that $q_{t,1}^*(x,i,j|\, c_1)$ and $d_t^*(x,i,j|\, c_1)$ are decreasing in c_1 while $q_{t,2}^*(x,i,j|\, c_1) - d_t^*(x,i,j|\, c_1)$ is increasing in c_1. Since $p_t(\cdot)$ is a decreasing function, $p_t^*(x,i,j|\, c_1) = p_t(d_t^*(x,i,j|\, c_1))$ is increasing in c_1. In addition, since $q_{t,1}^*(x,i,j|\, c_1)$ is decreasing in x from Theorem 1, $\xi_{1t}(i,j|\, c_1) = \sup\{x|\, q_{t,1}^*(x,i,j|\, c_1) > 0\}$ is also decreasing in c_1. Thus, (b) holds for t when $k=1$.

We finally prove that $V_t(x,i,j|\, c_1) - c_1x$ is submodular in (x,c_1). By replacing q_1 and q_2 with $\hat{q}_1 = x + q_1$ and $\hat{q}_2 = d - q_2$, (15) can be rewritten as

$$\begin{aligned}V_t(x,i,j|\, -c_1) + c_1x = \max_{\substack{\hat{q}_1\ge x,\hat{q}_2\le d\\ \underline{d}_t\le d\le \bar{d}_t}}\Big\{&R_t(d) + \gamma^{1,i}\left[c_1d + W_t(\hat{q}_1-d,\hat{q}_1-\hat{q}_2,1,j|\, -c_1)\right]\\ &+ (1-\gamma^{1,i})\left[c_1d + W_t(x-d,x-\hat{q}_2,0,j|\, -c_1)\right]\Big\}. \quad (26)\end{aligned}$$

From (10), we have

$$\begin{aligned}&c_1d + W_t(\hat{q}_1-d,\hat{q}_1-\hat{q}_2,1,j|\, -c_1)\\ =&\alpha c_2(\hat{q}_1-d) + \gamma^{2,j}\left[c_1\hat{q}_1 + L_t(\hat{q}_1-\hat{q}_2,1,1|\, -c_1)\right] + (1-\gamma^{2,j})\left[c_1\hat{q}_1 + L_t(\hat{q}_1-d,1,0|\, -c_1)\right].\end{aligned}$$

Since $L_t(x,i,j|\, -c_1)$ is concave in x and $L_t(-x,i,j|\, -c_1)$ and $L_t(x,i,j|\, -c_1) + c_1x$ are both supermodular in (x,c_1), one can easily verify from the above expression that $c_1d + W_t(\hat{q}_1-d,\hat{q}_1-\hat{q}_2,1,j|\, -c_1)$ is supermodular in $(\hat{q}_1,\hat{q}_2,d,c_1)$. Similarly, $c_1d + W_t(x-d,x-\hat{q}_2,0,j|\, -c_1)$ is supermodular in $(x,\hat{q}_2,d,c_1)$. Thus, the maximand in (26) is a supermodular function in $(x,\hat{q}_1,\hat{q}_2,d,c_1)$.

Since the constraint in (26) is a lattice, by applying Lemma 1, $V_t(x,i,j|\,-c_1)+c_1x$ is supermodular in (x,c_1) and thus $V_t(x,i,j|\,c_1)-c_1x$ is submodular in (x,c_1). In summary, (a) holds for t when $k=1$. The proof is complete. □

Proof of Proposition 2

Since Theorem 2 holds for the case when supplier 1 is perfectly reliable, it follows that $V_{t+1}(x,j|\,c_k)$ is supermodular in (x,c_k) while $V_{t+1}(x,j|\,c_k)-c_kx$ is submodular in (x,c_k), $k=1,2$. Then, from (4), $f_t(z,j|\,c_1,c_2)$ is supermodular in (x,c_1) but submodular in (x,c_2). Therefore, by the definition of $z^*_{t,2}$ and by applying Lemma 1, we have $z^*_{t,2}(c_1,c_2)$ is increasing in c_1 while decreasing in c_2.

We next prove the monotonicity results on $z^*_{t,1}(j|\,c_1,c_2)$. Based on Lemma 1, it suffices to prove that $g_t(z,j|\,c_1,c_2)$ is submodular in (x,c_1) while supermodular in (x,c_2). We first prove $g_t(z,j|c_1)$ is submodular in (z,c_1). Notice that (5) can be rewritten as

$$g_t(z,j|c_1)=\alpha c_2 z-G_t(z)+\gamma^j\max_{\tilde q\le 0}\{f_t(z-\tilde q,1|c_1)-c_1z\}+(1-\gamma^j)\left(f(z,0|c_1)-c_1z\right). \qquad (27)$$

Since $V_{t+1}(x,j|c_1)$ is supermodular in (x,c_1) and $V_{t+1}(x,j|c_1)-c_1x$ is submodular in (x,c_1), it follows from (4) that $f_t(z,j|c_1)$ is supermodular in (z,c_1) and $f_t(z,j|c_1)-c_1z$ is submodular in (z,c_1). Thus, $f_t(z-\tilde q,1|-c_1)+c_1z$ is supermodular in $(z,\tilde q,c_1)$. Since the constraint $\{(z,\tilde q,c_1)|\tilde q\le 0\}$ is a lattice, by applying Lemma 1, $\max_{\tilde q\le 0}\{f_t(z-\tilde q,1|-c_1)+c_1z\}$ is supermodular in (z,c_1) and thus $\max_{\tilde q\le 0}\{f_t(z-\tilde q,1|c_1)-c_1z\}$ is submodular in (z,c_1). Since $0\le\gamma^{2,j}\le 1$ and $f(z,0|c_1)-c_1z$ is submodular in (z,c_1), it follows from (27) that $g_t(z,j|c_1)$ is submodular in (z,c_1).

We next prove $g_t(z,j|c_2)$ is supermodular in (z,c_2). Note that $f_t(z,0|\,c_2)+\alpha c_2z=\alpha\mathsf{E}V_{t+1}(z-\epsilon_t,j|\,c_2)$ is supermodular in (z,c_2). Thus, from (5), it remains to prove

$$\max_{\bar z\ge z}f_t(\bar z,1|\,c_2)+\alpha c_2z=\max_{\tilde q\le 0}\{f_t(z-\tilde q,1|\,c_2)+\alpha c_2z\}$$

is also supermodular in (z,c_2). From the concavity of f_t and Theorem 2, it is easy to verify that $f_t(z-\tilde q,1|\,c_2)+\alpha c_2z$ is supermodular in $(z,\tilde q,c_2)$. Since the constraint set is a lattice, by applying Lemma 1 we obtain the desired result.

We finally prove $q^*_{t,1}(x,j|\,c_2)$ is increasing in c_2. To this end, we rewrite (7) as

$$q^*_{t,1}(x,j|\,c_2)=\max\{\hat d_t+z^*_{t,1}(j|\,c_2)-x,0\}.$$

Since $z^*_{t,1}(j|\,c_2)$ is increasing in c_2, it follows from the above equation that $q^*_{t,1}(x,j|\,c_2)$ is also increasing in c_2. □

Proof of Theorem 5

We prove (a) by induction; and (b) will be proved simultaneously. Since $V_{T+1}(x,i,j) = V_{T+1}(x,j) = 0$, (a) is trivially true for $t = T+1$. Now assume inductively that (a) holds for $t+1$. To complete the proof, we shall show that the theorem holds for t.

For convenience, we define

$$\phi_t(x,q_1,i,j) = \max_{q_2\geq 0}\Big\{\gamma^{1,i}W_t(x+q_1,x+q_1+q_2,1,j) + (1-\gamma^{1,i})W_t(x,x+q_2,0,j)\Big\}.$$

Then, the optimality equations (6) and (15) can be rewritten as

$$V_t(x,j) = \max_{\underline{d}_t\leq d\leq \bar{d}_t}\Big\{R_t(d) + c_1(x-d) + \max_{q_1\geq 0} g_t(x+q_1-d,j)\Big\}; \tag{28}$$

$$V_t(x,i,j) = \max_{\underline{d}_t\leq d\leq \bar{d}_t}\Big\{R_t(d) + c_1(x-d) + \max_{q_1\geq 0} \phi_t(x-d,q_1,i,j)\Big\}. \tag{29}$$

We first prove $V_t(x,i,j) \leq V_t(x,j)$. From (28) and (29), it suffices to prove $\max_{q_1\geq 0}\phi_t(x,q_1,i,j) \leq \max_{q_1\geq 0} g_t(x+q_1,j)$, or equivalently,

$$\begin{aligned}&\max_{q_2\geq 0}\left\{\gamma^{1,i}\max_{q_1\geq 0} W_t(x+q_1,x+q_1+q_2,1,j) + (1-\gamma^{1,i})W_t(x,x+q_2,0,j)\right\}\\ \leq &\max_{q_2\geq 0}\left\{\max_{q_1\geq 0}\{-(c_1-\alpha c_2)(x+q_1) - G_t(x+q_1) + \gamma^{2,j}f_t(x+q_1+q_2,1) + (1-\gamma^{2,j})f_t(x+q_1,0)\}\right\}.\end{aligned}$$

Note that $\max_{q_1\geq 0} W_t(x+q_1,x+q_1+q_2,1,j) \geq W_t(x,x+q_2,1,j) \geq W_t(x,x+q_2,0,j)$ by Proposition 1. Then, to prove the above inequality, it suffices to show that, for any x, q_1, and q_2,

$$\begin{aligned}&W_t(x+q_1,x+q_1+q_2,1,j)\\ =&-(c_1-\alpha c_2)(x+q_1) - G_t(x+q_1) + \gamma^{2,j}L_t(x+q_1+q_2,1,1) + (1-\gamma^{2,j})L_t(x+q_1,1,0)\\ \leq&-(c_1-\alpha c_2)(x+q_1) - G_t(x+q_1) + \gamma^{2,j}f_t(x+q_1+q_2,1) + (1-\gamma^{2,j})f_t(x+q_1,0).\end{aligned}$$

Since $V_{t+1}(x,i,j) \leq V_{t+1}(x,j)$ by the inductive assumption, by (4) and (9), $L_t(x,1,j) \leq f_t(x,j)$, this shows that the above inequality holds. Hence, $V_t(x,i,j) \leq V_t(x,j)$.

We next prove $V_t(x,i,j) - V_t(x,j)$ is increasing in x. In what follows, we will first apply Lemma 3 to show that $\max_{q_1\geq 0}\phi_t(x,q_1,i,j) - \max_{q_1\geq 0} g_t(x+q_1,j)$ is increasing in x. To this end, we need to verify that $\phi_t(x,q_1,i,j)$ and $g_t(x,j)$ satisfy the following properties:

(a) $\phi_t(x,q_1,i,j)$ is concave and submodular in (x,q_1), and $g_t(x,j)$ is concave in x;

(b) $\max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j) = \max_{x\in\Re}\phi_t(x,0,i,j)$; and

(c) $\phi_t(x,0,i,j) - g_t(x,j)$ is increasing in x.

Firstly, since $W_t(x,y,i,j)$ is concave and separable in (x,y) by (9), (10), and Proposition 1, it is easily seen that $\phi_1(x,q_1,i,j)$ is concave and submodular in (x,q_1). In addition, $g_t(x,j)$ is clearly concave in x.

We next verify $\max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j) = \max_{x\in\Re}\phi_t(x,0,i,j)$. Denote

$$(x^*,q_1^*,q_2^*) = \arg\max_{(x,q_1,q_2)\in\Re\times\Re^+\times\Re^+}\Big\{\gamma^{1,i}W_t(x+q_1,x+q_1+q_2,1,j) + (1-\gamma^{1,i})W_t(x,x+q_2,0,j)\Big\}.$$

Then, by the definition of $\phi_t(x,q_1,i,j)$, we have

$$\max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j) = \gamma^{1,i}W_t(x^*+q_1^*,x^*+q_1^*+q_2^*,1,j) + (1-\gamma^{1,i})W_t(x^*,x^*+q_2^*,0,j). \quad (30)$$

Since $W_t(x,x+q_2^*,i,j)$ is submodular in (x,i) following from Proposition 1 and by the optimality of q_1^*, we have

$$\begin{aligned}&W_t(x^*+q_1^*,x+q_1^*+q_2^*,0,j) - W_t(x^*,x+q_2^*,0,j)\\ \geq&W_t(x^*+q_1^*,x+q_1^*+q_2^*,1,j) - W_t(x^*,x+q_2^*,1,j) \geq 0.\end{aligned}$$

Then, $W_t(x^*+q_1^*,x+q_1^*+q_2^*,0,j) \geq W_t(x^*,x+q_2^*,0,j)$. According to (30), we have

$$\begin{aligned}&\max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j)\\ \leq&\gamma^{1,i}W_t(x^*+q_1^*,x^*+q_1^*+q_2^*,1,j) + (1-\gamma^{1,i})W_t(x^*+q_1^*,x^*+q_1^*+q_2^*,0,j)\\ \leq&\phi_1(x^*+q_1^*,0,i,j) \leq \max_{x\in\Re}\phi_t(x,0,i,j) \leq \max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j).\end{aligned}$$

Hence, $\max_{(x,q_1)\in\Re\times\Re^+}\phi_t(x,q_1,i,j) = \max_{x\in\Re}\phi_t(x,0,i,j)$.

We lastly verify that $\phi_t(x,0,i,j) - g_t(x,j)$ is increasing in x. By the definition of $\phi_1(x,q_1,i,j)$ and $W_t(x,y,i,j)$, we have

$$\begin{aligned}\phi_1(x,0,i,j) &= \max_{q_2\geq 0}\left\{\gamma^{1,i}W_t(x,x+q_2,1,j) + (1-\gamma^{1,i})W_t(x,x+q_2,0,j)\right\}\\ &= -(c_1-\alpha c_2)x - G_t(x) + (1-\gamma^{2,j})(\gamma^{1,i}L_t(x,1,0) + (1-\gamma^{1,i})L_t(x,0,0))\\ &\quad + \gamma^{2,j}\max_{q_2\geq 0}\left\{\gamma^{1,i}L_t(x+q_2,1,1) + (1-\gamma^{1,i})L_t(x+q_2,0,1)\right\}. \end{aligned} \quad (31)$$

Then, from (5) and (31), we have

$$\begin{aligned}\phi_t(x,0,i,j) - g_t(x,j) &= (1-\gamma^{2,j})(\gamma^{1,i}L_t(x,1,0) + (1-\gamma^{1,i})L_t(x,0,0) - f_t(x,0))\\ &\quad + \gamma^{2,j}\left(\max_{q_2\geq 0}\left\{\gamma^{1,i}L_t(x+q_2,1,1) + (1-\gamma^{1,i})L_t(x+q_2,0,1)\right\} - \max_{q_2\geq 0}f_t(x+q_2,1)\right).\end{aligned}$$

By the inductive assumption, $V_{t+1}(x,i,j) - V_{t+1}(x,j)$ is increasing in x. Then, it follows from (4) and (9) that $L_t(x,i,j) - f_t(x,j)$ is increasing in x. Then, $\gamma^{1,i}L_t(x,1,j) + (1-\gamma^{1,i})L_t(x,0,j) - f_t(x,j)$ is also increasing in x. Notice that $\gamma^{1,i}L_t(x,1,1) + (1-\gamma^{1,i})L_t(x,0,1)$ and $f_t(x,1)$ are both concave in x. Thus, one can easily verify that

$$\max_{q_2 \geq 0} \left\{\gamma^{1,i}L_t(x+q_2,1,1) + (1-\gamma^{1,i})L_t(x+q_2,0,1)\right\} - \max_{q_2 \geq 0} f_t(x+q_2,1)$$

is increasing in x. Hence, $\phi_t(x,0,i,j) - g_t(x,j)$ is increasing in x.

In summary, since $\phi_t(x,q_1,i,j)$ and $g_t(x,j)$ satisfy the properties (a)-(c), by applying Lemma 3, we conclude that $\max_{q_1 \geq 0} \phi_t(x,q_1,i,j) - \max_{q_1 \geq 0} g_t(x+q_1,j)$ is increasing in x.

Now we prove $V_t(x,i,j) - V_t(x,j)$ is increasing in x and $d_t^*(x,i,j) \leq d_t^*(x,j)$. Define

$$V_t(x,i,j,n) := \max_{\underline{d}_t \leq d \leq \bar{d}_t} \Big\{R_t(d) + c_1(x-d) + \hat{V}_t(x-d,i,j,n)\Big\}, \tag{32}$$

where $\hat{V}_t(x,i,j,1) := \max_{q_1 \geq 0} \phi_t(x,q_1,i,j)$ and $\hat{V}_t(x,i,j,0) := \max_{q_1 \geq 0} g_t(x+q_1,j)$. Then, $\hat{V}_t(x,i,j,n)$ is supermodular in (x,n). By replacing d as $\hat{d} = x - d$, (32) can be rewritten as

$$V_t(x,i,j,n) = \max_{\underline{d}_t \leq x-\hat{d} \leq \bar{d}_t} \Big\{R_t(x-\hat{d}) + c_1\hat{d} + \hat{V}_t(\hat{d},i,j,n)\Big\}.$$

Since $R_t(\cdot)$ is a concave function, the maximand in the above optimization problem is supermodular in $(x,\hat{d},n)$. Since the constraint is a lattice, by applying Lemma 1, $V_t(x,i,j,n)$ is supermodular in (x,n) and $\hat{d}^*(x,i,j,n)$ is increasing in n. By the definition of $V_t(x,i,j,n)$ and $\hat{V}_t(x,i,j,n)$, then $V_t(x,i,j) - V_t(x,j)$ is increasing in x and $d_t^*(x,i,j) \leq d_t^*(x,j)$. Furthermore, $p_t^*(x,i,j) \geq p_t^*(x,j)$. The proof of Theorem 5 thus completed. □

Numerical studies on additive-multiplicative demand

We report some of the results of the numerical studies conducted on models with additive-multiplicative demand. Suppose $T = 4$, $c_1 = 5$, $c_2 = 3$, $\alpha = 0.97$, $h = 1$, $b = 8$, and

$$D_t(p) = w_t(100 - 10p) + \epsilon_t,$$

where w_t, ϵ_t, $1 \leq t \leq T$ are independent random variables with $\Pr(w_t = 0.5) = \Pr(w_t = 1.5) = 0.5$, $\epsilon_t \sim$ Uniform$[-10, 10]$, $\underline{p}_t = 4$, $\bar{p}_t = 8$, $\gamma^{1,0} = \gamma^{1,1} = 1$, $\gamma^{2,0} = 0.3$, and $\gamma^{2,1} = 0.9$. The current period $t = 1$.

In the following, Figure 6 shows the impact of adding a second supplier on the firm's expected total discounted revenue and optimal pricing when the firm originally only sources from supplier 1.

From this figure, it can be seen that both the firm and its customers benefit from dual sourcing, and the benefits decrease when supplier 2 becomes less reliable or when it charges higher unit ordering cost. Figure 7 illustrates the impact of unit ordering cost c_1 on the optimal order quantity from supplier 2, and it demonstrates that supplier 2 may receive less orders when supplier 1 becomes more expensive, implying that supplier 2 may receive more orders when supplier 1 is added into the firm's sourcing system. Finally, Figure 8 shows that supplier 2 may receive less orders when it becomes more reliable.

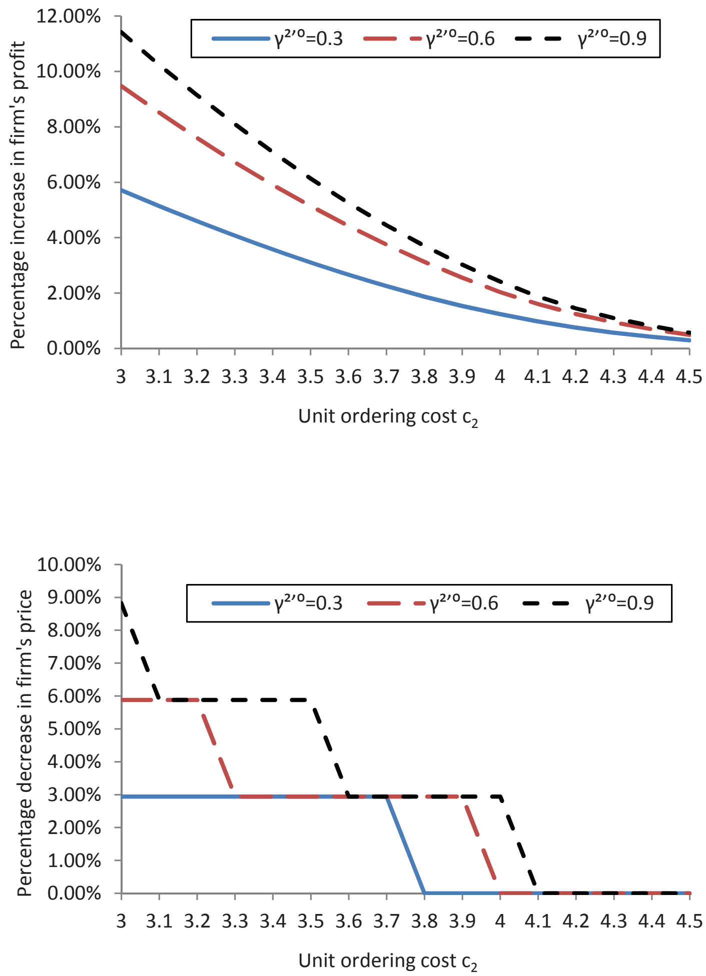

Figure 6: Impact of dual sourcing on firm's revenue and price when $(x, i, j) = (70, 0, 0)$

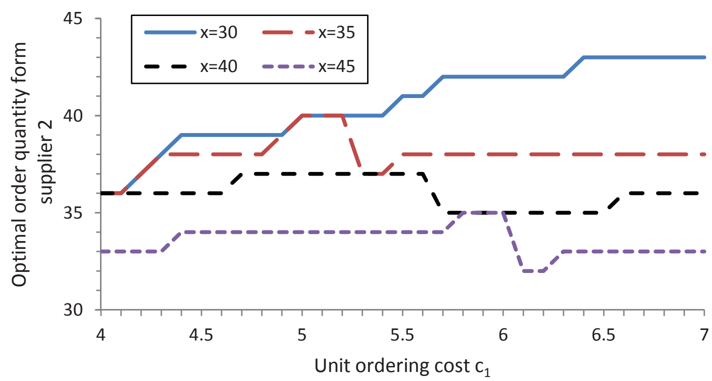

Figure 7: Optimal order quantity from supplier 2 when $(i,j) = (0,1)$

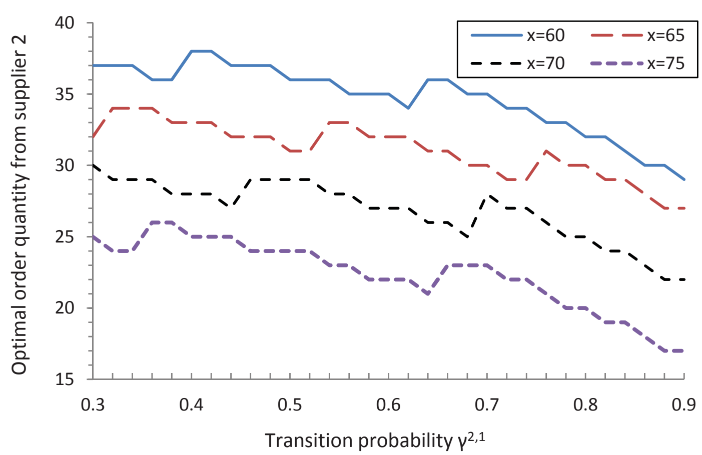

Figure 8: Optimal order quantity from supplier 2 when $\gamma^{1,0} = 0.6$, $\gamma^{1,1} = 0.9$, and $(i,j) = (0,1)$