A Collection of Essays on Electric Grid Operations: Optimizing Energy Storage and Enhancing the Effect of Social Comparisons

by

Santhosh Suresh

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Doctoral Committee:

Assistant Professor Owen Wu, Co-Chair
Professor Roman Kapuscinski, Co-Chair
Professor Xiuli Chao
Assistant Professor Xun Wu

For my parents, advisors, family and friends. Dedicated to my childhood hero and mathematician Srinivasa Ramanujam

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"If I have seen further it is by standing on the shoulders of giants."

-Isaac Newton

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

LIST OF TABLES

Table

ABSTRACT

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Every year, a significant portion of the energy the U.S. electric grid consumes is wasted through transmission, heat losses, and inefficient technology, translating into significant costs to individual consumers and businesses. Improving the efficiency of the electric grid is one of the easiest and most cost effective ways to combat climate change, clean the air we breathe, improve the competitiveness of our businesses and reduce energy costs for consumers. Governments across the world are focusing on ways to achieve it. This dissertation explores various ways of improving the grid operations related to energy storage. Storage allows for smoothing production and hence avoiding the costly peaking plants. However, scalable energy storage technology, such as Lead Acid Batteries or Pumped Storage units are expensive and have significant conversion losses in both the storing and withdrawing processes. The first two essays in the dissertation consider energy storage.

The first essay considers the problem of locating storage facilities in an electric grid. Due to transmission losses, congestion and reliability issues, the location of storage technology on the grid affects its economic value. We consider a trade-off between locating storage closer to the generation unit, increasing flexibility of the storage unit, or closer to the demand hub, reducing transmission losses and study how changes in system parameters will affect this trade-off. We derive a structure for the optimal operating policy and conclude that

current system parameters greatly favour locating of Energy Storage investments closer to the demand hub, due to lesser transmission losses, and minimal benefit of pooling.

The second essay considers the choice of storage technology. There exist a number of storage technologies differing in investment costs per unit capacity and conversion efficiencies. One of the critical trade-offs is between investing in technology with greater conversion efficiency, incurring greater fixed costs or investing in technology with low efficiency, incurring lower fixed cost but greater variable costs. We study the system parameters under which different types of technology are most suitable. We also show that under some situations investing in a portfolio of technologies may perform better than choosing a single technology. However, we observe that the benefit of divesting in multiple technologies is decreasing in the number of technologies used.

The third essay deals with behavioural energy efficiency. Opower, a technology company has recently shown that comparing energy consumptions of households to their peers can motivate consumers to significantly reduce their energy usage through peer pressure. Their Home Energy Reports (HER) program has been documented to reduce up to 3% in energy consumption of consumers across the spectrum and this has resulted in a growing field of energy conservation known as behavioural energy efficiency. We model the effects of these social incentives against more standard investment incentives. We observe that, including both types of incentives dramatically reduces demand consumption when compared to the sum of the reduction by applying each of the incentives separately, in some situations.

CHAPTER I

Introduction

Operating the North American electric grid is a complex business involving tens of thousands of nodes spread across the entire continent. Each instant, operators have to make thousands of trade-offs deciding the location, quantity and quality of power that is being generated, transmitted and consumed, subject to several physical and economic constraints. The annual revenue of the electric grid runs in billions of dollars, making efficient operation of the grid, of paramount economic importance. In this dissertation, we discuss some ways to improve the operation of the electric grid, both at the supply and consumption end of the grid.

At the supply side of the grid, use of Energy storage technologies can substantially reduce the cost of power generation. This is due to the convexity of the energy generation costs, i.e., during high demand periods, the marginal cost of energy is higher, owing to the use of more expensive but flexible peaking units. Hence, energy storage allows for smoothing of power generation, storing energy during inexpensive low demand periods for later use. Chapters II and III discuss the problems related to Energy Storage Investment and Operation. In chapter II, we discuss the problem of locating Energy Storage Technologies on the grid, specifically, the trade-off between locating storage closer to the consumer end or closer to the central generation end of the grid. In chapter III, we consider the issue of choosing the right portfolio of storage technologies, given the many options available, such as Lead Acid Batteries, Pumped Storage Units, Nickel Hydride batteries etc.

At the consumption end of the grid, Chapter IV explores the potential of motivating consumers to consume less energy because of social comparisons and by incentivizing them to invest in more energy efficient equipment. We compare the potential of both these types of incentives. In particular, we consider whether the two treatments are complements or substitutes, in terms of reducing energy consumption.

We conclude this dissertation in Chapter V by summarizing our contributions and discussing future research opportunities in this area.

CHAPTER II

Operations and Investment of Energy Storage in the Presence of Transmission Losses

2.1. Introduction

It is well known in OM practice and theory, that inventory provides a wide range of benefits. It helps meet uncertain demand as well as smooth predictable cyclical patterns in demand, allows for batching where economies of scale in procurement exist, allows to take advantage of quantity discounts and protects against long lead times. Not unlike other industries, in the omnipresent electricity industry, storage technology also has several benefits, including enhancement of sustainability, reliability and utilization of generation and transmission assets (Rastler and Kamath 2005). The value of electric storage has increased due to significant technical improvements (de Morsella 2011, Rastler 2010) and the potential of energy storage has been established by several independent studies (Eyer and Corey 2010, Chu and Majumdar 2012, Greenberger 2011). Makansi (2004) argues that the benefit of storage will likely increase because of increasing use of intermittent renewable sources of energy.

Despite the high potential benefits of energy storage (Akhil et al. 2013), only about 3% of the energy served in the US is cycled through energy storage, as opposed to 10-15% in Europe and Japan (Gyuk 2003). To improve energy storage usage, the American Recovery and Reinvestment Act allocated nearly $650 million to energy storage technologies and related smart-grid technologies. These projects will naturally raise questions about the best use of storage technologies and their locational benefits. Due to transmission losses (Energy

Information Administration 2009) and congestion issues (due to limited capacities, see Rivier and Perez-Arriaga 1993), the allocation of storage on the grid is a non-trivial problem. Hoffman et al. (2010) point out the need for modeling tools to answer this question. Sbiliris and Dedoussis (2013) and Gayme and Topcu (2013) use examples to show that intelligent location of storage can significantly impact the value of storage. Nourai (2007) discuss and numerically evaluate the investment projects of American Electric Power (AEP), where they consider whether to localize storage investments for each community or centralize storage investments at the grid level. We address exactly the same question, but their analysis is computational, while we model this problem analytically.

While the problem of optimal location is fairly complicated, we consider the fundamental trade-off between locating storage closer to the generating station (centrally) or closer to the the usage point (locally). Pooling of storage capacity is beneficial when there is uncertainty in the destination of stored energy, which arises from variability in demands across nodes in the grid. Hence, pooling reduces transmission losses on average. Interestingly, centrally located storage capacity must also "store" the transmission losses, which means that local storage achieves greater *efficacy* of storage capacity. In a parsimonious model, we consider the trade-off of pooling vs. localizing.[1] Using stochastic dynamic programming, we identify the structure of the optimal policy of operating the grid during each period. We then compare investment strategies, given the optimal operating policy and identify the features of the system that favor pooling versus localizing of storage capacity.

In a traditional operations sense, the greater the variability, the greater the benefits of centralized storage capacity. In our study, however, we observe that increasing variability increases the value of storage, which causes higher *efficacy* of the localized invested capacity to smooth production. We investigate how the increasing penetration of the intermittent renewable sources affects the storage location problem. The high wind generation during low demand periods results in temporary surplus of energy, increasing the benefit of storage capacity. Furthermore, the presence of local generation (due to distributed sources of energy, such as wind farms) increases the benefit of localizing storage by enabling a cycle of locally

[1]Throughout the paper, the words pooling and centralizing of storage capacity are used interchangeably. Similarly, the words localizing and decentralizing are used interchangeably.

storing and using energy.

We discuss how the optimal pooling and localizing strategies are affected by system parameters such as the cost of storage capacity, storage conversion efficiency, transmission efficiency, and demand distribution parameters. We find that an increase in the cost of storage increases the tendency towards localizing. Also, the benefits of localizing increase when the average load increases. The impact of storage efficiency depends, however on where the energy is generated. Under greater occurrence of local generation, increased storage efficiency implies more localizing.

Contrary to the current practice, we find that, under current system parameters, it is overwhelmingly more beneficial to localize storage capacity investments, mostly due to the following two reasons. Firstly, the minimum of demand at a local node tends to be high enough, so that local storage can primarily be used to satisfy local demand, decreasing potential for remote transmission. Secondly, the greater *efficacy* of localizing is beneficial due to the high price of storage capacity today ($1000-2000/KWh).

2.2. Literature Review

Since the objective of this paper is to analyze the value of storage siting and sizing in energy markets, a few streams of literature are relevant. We first discuss literature pertaining to evaluation of energy storage without transmission constraints or losses, i.e., single-node energy storage systems. Then, we discuss energy storage literature in the context of grid operations, i.e., with transmission factors. We then discuss a related resource-allocation problem: Distributed Generation (DG). Finally, we compare energy storage to the traditional OM context of capacitated systems.

2.2.1 Single-Node Energy Storage Systems

Several papers provide insights on the evaluation of a single-storage facility, co-located with renewable energy and/or directly serving customer demand. We wish to extend these insights by considering an electric network with more nodes and transmission losses, and focusing on storage location.

The evaluation of the economic value of energy storage is of extreme importance and single-node is the natural step in this direction. See Mokrian and Stephen (2006) for a comprehensive review of methodology related to evaluation of the economic benefit of energy storage due to arbitrage, when operating in a market with exogenous prices. Several more recent papers extend this stream of literature of revenue maximizing storage owner. Korpaas et al. (2003), Castronuovo and Lopes (2004), Brunetto and Tina (2007), attempt this problem as a deterministic optimization problem given a particular sample path over a finite horizon and then averaging the results over the sample paths. Bitar et al. (2010), Bitar et al. (2011), Kim and Powell (2011) consider a stochastic generalization of this problem. They derive closed form expressions for the value of storage under certain special cases of the energy price and wind distributions, to help evaluate storage investments.

Harsha and Dahleh (2011), Granado et al. (2012), Van De Ven et al. (2011) extend the literature by considering the objective of players who are obligated to serve the consumer demand, while maximizing revenue from a wind farm combined with storage. These papers are different from the previous group in that, here, they are obligated to satisfy demand. However, all of the above papers have an exogenous price process, while we endogenize the price to the generation costs. Brown et al. (2008) consider a version of the storage investment problem with additionally, a conventional generation unit, with deterministic production cost in one-node setting. Our focus is different, we concentrate on the trade-off between locating storage at central and local sites, which requires multi-node setting. We borrow and generalize some assumptions from the single node literature.

2.2.2 Operation of Storage on the Electric Grid

In order to understand where to locate resources on the grid, it is necessary to first understand how they would be optimally operated given locations are already chosen.

The traditional approach to solve grid operations is to apply Optimal Power Flow (OPF) models, which include all the decisions to be made for each node of the grid, keeping in mind the energy flow constraints, generation constraints, and transmission constraints (See Cain et al. (2012) for a full summary on OPF). The presence of energy storage facilities on the grid provides additional benefits for operations, but makes solving the problem more difficult, since energy can be generated and stored to potentially serve future demand. Due to difficulty of solving a system with storage units, Kraning et al. (2011) show that the optimization of a portfolio of storage technologies in order to minimize total system investment and operating costs in the grid can be approximated to a convex optimization problem. Similar convexifications of Power Flow problems can be seen in Low (2014), Sojoudi and Lavaei (2013).

Our focus is understanding the trade-offs and generating insights. We therefore consider a stylized model with three nodes (two demand hubs and one generation node), to demonstrate the trade-offs between localizing and centralizing of storage capacity. Several papers model operation of grids with energy storage on related specialized structures of grids: Single-bus systems (Su and El Gamal 2011, Zhou et al. 2011, Chandy et al. 2010, Denholm and Sioshansi 2009), are common. We add to this literature by proving structural properties of the parameter-dependent grid operation policy in our stylized model with two demand hubs. Other papers consider lossless 1-D and 2-D grids (Kanoria et al. 2011) and IEEE benchmark network systems (Gayme and Topcu 2013) with deterministic demand to show that storage flattens the generation profile. We use stochastic demands in our stylized model with transmission losses and extend some of these patterns in the operating policy. Our insights additionally extend to the investment problem.

2.2.3 Location of Storage on the Electric Grid

The problem of locating energy storage on the grid is complicated due to multiple locations, as well as losses and limited capacities along transmission lines. A general computational framework to optimize energy storage on the grid has been designed recently and studied through simulations in Sjodin et al. (2012), Bose et al. (2012) using tools similar to the Optimal Power Flow. Sjodin et al. (2012) use the approximate Direct Current OPF (Purchala et al. 2005, Pandya and Joshi 2008), while Bose et al. (2012) use the relaxation of the Alternating Current OPF (Bai et al. 2008, Bai and Wei 2009) on IEEE benchmark systems based on semi-definite programming (Boyd and Vandenberghe 2004, Wolcowicz et al. 2000). In contrast to these papers, our focus is on understanding the trade-offs in a stylized network model to derive insights on the investment policy, rather than exact solutions.

Some papers consider the location of storage in single bus systems (Denholm and Sioshansi 2009, Zhou et al. 2011). Denholm and Sioshansi (2009) compare the benefits of deploying storage capacity at the wind farm or the load center connected via a transmission line. They show that locating storage technology close to the wind farms is beneficial and results in utilization of transmission capacities. Zhou et al. (2011) consider the problem of operating a storage facility on a wind farm connected through a transmission line to a load center. Both of these papers assume a single storage facility and an exogenous price process, which is independent of the energy stored. In an independent but related work, Thrampoulidis et al. (2013) consider a generic network model to derive insights on storage investment in a network. They prove that it is never optimal to co-locate storage capacity with a generator which is connected to the grid via a sole transmission line. These results are consistent with our model, but our focus is on multiple demand nodes. Also, we model the impact of storage on energy prices implicitly, through convex costs, instead of exogenous prices. Our model shows that the intuition of locating storage closer to the point of "variability", i.e., the wind farm or demand hubs in the models in these papers, continues to hold in many settings.

Recent implementation of energy storage projects considered at American Electric Power (AEP), Nourai (2007) have similar lessons. They concluded that localized storage ('Community Energy Storage') has greater benefits to the grid, then central storage. We confirm their observations based on theoretical framework, that is different from other streams of work. We formally compare localizing with pooling. Our model suggests that localizing of storage increases the *efficacy* of storage assets by making better use of storage.

2.2.4 Location of Distributed Generation Resources on the Grid

A related field of resource allocation on the grid is that Distributed Grid (DG) location. DG resource allocation problems present similar trade-offs as storage location in electrical networks with transmission issues, as the objective is to invest in resources that reduce generation costs, while still minimizing energy losses. In their seminal paper, Ackermann et al. (2001) define Distributed Generation as electric power generation within distribution networks or on the customer side of the network. Building on Ackermann et al. (2001), El Khattam et al. (2004) investigate optimal investment in DG resources across the grid. Madarshahian et al. (2009) consider the trade-off between customer interruption cost and distributed generation cost and provide some heuristics for optimal DG investment. Rodriguez et al. (2010) review the literature of DG optimization problems accounting for the multiple objectives that DG systems often carry. This research is geared towards providing computational methodologies for the industry, that can solve the distributed storage investment problem in the grid. Additionally, Wang and Nehrir (2004), Acharya et al. (2006) etc. compare several analytical approaches to finding the optimal DG investment in simplified bus networks. Additionally, Atwa et al. (2010) show that Mixed Integer Linear Programs can be used to find an investment solution with huge performance improvement. While these papers focus on comparing performance of methodologies, we focus on the insights of the optimal investment policy, for the storage problem. We observe a unifying trend between the investment of Distributed Generation capacities and decentralized Storage capacities: They both tend to achieve greater value when invested away from the conventional generation units.

2.2.5 Related work in Traditional Operations Management

Our model includes the elements of storage capacity and losses, as well as transmission of energy, topics studied in traditional OM concepts of Inventory Control, Warehousing and Transhipment.

The electric energy storage optimization problem is similar to a classical multi-period inventory problem with stochastic demand. However, most of the inventory literature assumes linear production costs, whereas in energy markets production costs are highly convex. A treatment of the traditional inventory model with convex production costs can be found in Karlin (1960). The paper however does not consider storage capacity or possibility of multiple nodes. Several elements of energy storage models are different from OM models: production cost is non-linear, all demand has to be satisfied and sequence of demand/production tends to be different. Also, the nature of storage costs is significantly different, with conversion losses. Our paper provides another inventory control model, with convex generation costs and non-linear holding costs and provides a structure to the optimal operating policy, contributing to the Inventory Control literature.

Our research is also related to the warehousing and the commodity trading literature. Cahn (1948) introduces the problem that aims to identify optimal purchasing, storage and sales decisions of a stock given a warehouse with a fixed capacity. Bellman (1956) formulates this as a dynamic program, and Dreyfus (1957) provides analytic solutions for deterministic prices and shows that the optimal solutions at each stage are to either fill-up the warehouse, or empty it, or do nothing. Charnes et al. (1966) show that for the case of positive stochastic prices the optimal trading decisions remain the same. Rempala (1994) and Secomandi (2010) extend this work to incorporate a limit to the injection and withdrawal and more recently, Zhou et al. (2012) to the case of negative prices as well. Faghih et al. (2011) address this problem in the energy storage context where storage is used for arbitrage. For independently distributed prices in each period and a storage with a perfect round-trip efficiency, they derive explicit formulae for the optimal thresholds (closed form or recursive). Our paper treats prices as endogenous by modelling the convex production costs, and we observe that in this system, the threshold structure of 'fill up' or 'empty' does not apply, as storing energy affects the marginal cost of energy in each period. In a related context, Netessine (2006) considers the problem of endogenously pricing inventory with limited storage capacity constraint. While they provide insights on the pricing decision, prices are set by the production costs in our model and we have multiple storage units, while they have just one.

Another related traditional OM concept is transhipment. A more traditional analysis of inventory management under transshipment can be seen in Hu et al. (2008) where they discuss the co-ordination transhipment prices of a two location production/inventory model. In energy markets, the transmission losses impose a different cost structure, correspondingly, operating policy is different. The impact of transmission constraints on energy prices is analyzed in Lee et al. (2011), where congestion pricing plays an important role in modulating demand. We complement this research by considering storage and transmission in a centralized setting.

2.3. Model

2.3.1 Storage and Transmission

We consider a storage investment problem on a simple tree network that consists of one central generation facility (node G) and two demand centers (leaf nodes A and B). Transmission lines G-A and G-B connect the generation facility and the demand centers. Let $\mathcal{T} \stackrel{\text{def}}{=} \{1, 2, \dots, T\}$ denote an operating horizon. In a period preceding the operating horizon (labeled as period 0), storage investment is decided at each node and storage facilities are built. Once built, the storage size is fixed throughout the operating horizon $\mathcal{T}$. An operating period $t \in \mathcal{T}$ represents a few hours, while period 0 is much longer.

Assumption II.1 (Storage)**.** *(i) Each storage facility can be filled up or emptied within one operating period. (ii) Energy loss during storage operation is linear in the amount of energy stored or released.*

Part (i) is not restrictive, as many storage technologies allow fully charging or discharging within a few hours (Denholm et al. 2010, KEMA 2012). Part (ii) approximates the reality well (Ibrahim et al. 2008) and is a standard assumption in most engineering and OM literature (Denholm and Sioshansi 2009, Zhou et al. 2012). Most storage losses occur during energy conversion processes (storing or releasing); keeping energy in the storage for one period has negligible losses.

We denote storage size by $\mathbf{S} = (S^A, S^B, S^G)$, where $S^i \geq 0$ is the storage size at node i. Denote by $\mathbf{s}_t = (s_t^A, s_t^B, s_t^G)$ the storage level (also referred to as inventory level) at the beginning of period t, where $\mathbf{s}_t \in \mathcal{A} \stackrel{\text{def}}{=} \{\mathbf{s} : \mathbf{0} \leq \mathbf{s} \leq \mathbf{S}\}$, $t \in \mathcal{T}$.

Let $\alpha \in (0, 1]$ denote the one-way efficiency of the storage; the round-trip efficiency is α^2. That is, reducing s_t^i by one unit releases α units of energy, and raising s_t^i by one unit requires α^{-1} units of energy. Thus, the energy flow associated with an inventory change of $\Delta s_t^i = s_{t+1}^i - s_t^i$ is

$$\psi_\alpha(\Delta s_t^i) \stackrel{\text{def}}{=} \begin{cases} \alpha^{-1}\Delta s_t^i, & \text{if } \Delta s_t^i \geq 0, \\ \alpha \Delta s_t^i, & \text{if } \Delta s_t^i < 0, \end{cases} \tag{2.1}$$

where $\psi_\alpha(\Delta s_t^i) > 0$ is the energy inflow into storage, and $\psi_\alpha(\Delta s_t^i) < 0$ is the energy outflow.

The assumption of the same efficiency in both ways brings notational and analytical convenience, but does not cause any loss of generality, because if the storing efficiency α_1 differs from the releasing efficiency α_2, we can use $\alpha = \sqrt{\alpha_1\alpha_2}$ and scale the storage size and storage levels accordingly. We do not consider constraints on the speed of charging and discharging energy storage in this paper. See Wu et al. (2012) for a recent discussion on the impact of charging and discharging rates on the value of energy storage.

Our analysis focuses on storage operations in the presence of transmission losses. To clearly analyze the tradeoffs, we assume that transmission capacity is not constrained. Linear transmission loss is a good approximation for most transmission lines with moderate utilization, under destressed conditions (Gomez-Exposito et al. 2000).

Assumption II.2 (Transmission)**.** *(i) The transmission lines have sufficient capacity, i.e., capacity constraints are non-binding in all periods. (ii) Energy loss during transmission is linear in the amount of energy transmitted.*

Let $\beta \in (0, 1]$ denote the efficiency of the transmission lines G-A and G-B in either direction, i.e., transmitting one unit of energy leads to $1 - \beta$ units of transmission loss. Let u_t^i denote the transmitted energy measured at leaf node $i \in \{A, B\}$ in period t. To indicate the direction of transmission, we use $u_t^i > 0$ for the energy transmitted from G to leaf node i and $u_t^i < 0$ for the reverse flow (from leaf node i to G). The corresponding transmission flow measured at G can be written as

$$\psi_\beta(u_t^i) \stackrel{\text{def}}{=} \begin{cases} \beta^{-1}u_t^i, & \text{if } u_t^i \geq 0, \\ \beta u_t^i, & \text{if } u_t^i < 0. \end{cases} \tag{2.2}$$

2.3.2 Balancing Demand and Supply

Let $\mathbf{d}_t = (d_t^A, d_t^B) \geq 0$ denote the demand for electricity in period t, where d_t^i is the demand at leaf node i. We assume the demand process $\{\mathbf{d}_t : t \in \mathcal{T}\}$ is Markovian. We assume $\mathbf{d}_t$ is realized at the beginning of period t and must be satisfied in period t.

We refer to the tree network with generation and storage facilities as an electricity system. A system operator makes production and inventory decisions in every period. At the beginning of period t, the system operator observes the period-starting storage level $\mathbf{s}_t$ and demand $\mathbf{d}_t$, and decides the period-ending storage level $\mathbf{s}_{t+1}$, which is the next period's starting storage level. The inventory change and demand determine the transmission flows and central generation, as detailed below and illustrated in Figure 2.1.

At the leaf nodes, the flow balance constraint is

$$u_t^i \;=\; d_t^i + \psi_\alpha(\Delta s_t^i), \qquad i = A, B. \tag{2.3}$$

Then, (2.2) and (2.3) imply together that the transmitted energy measured at G is $\sum_{i=A,B} \psi_\beta(u_t^i) = \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(\Delta s_t^i)\big)$. Flow balance at G implies that the central facility's production quantity, denoted as q_t, is a function of $\mathbf{d}_t$ and inventory change $\Delta\mathbf{s}_t = \mathbf{s}_{t+1} - \mathbf{s}_t$:

$$q_t \;=\; q(\Delta\mathbf{s}_t, \mathbf{d}_t) \;\stackrel{\text{def}}{=}\; \psi_\alpha(\Delta s_t^G) + \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(\Delta s_t^i)\big). \tag{2.4}$$

Because $\psi_\alpha(\cdot)$ and $\psi_\beta(\cdot)$ are convex and increasing functions, $q(\Delta\mathbf{s}_t, \mathbf{d}_t)$ is convex and increasing in $\Delta\mathbf{s}_t$. As the central facility produces $q(\Delta\mathbf{s}_t, \mathbf{d}_t)$ to balance the energy flows at node G, the system operator must choose storage level $\mathbf{s}_{t+1}$ such that $q(\Delta\mathbf{s}_t, \mathbf{d}_t) \geq 0$.

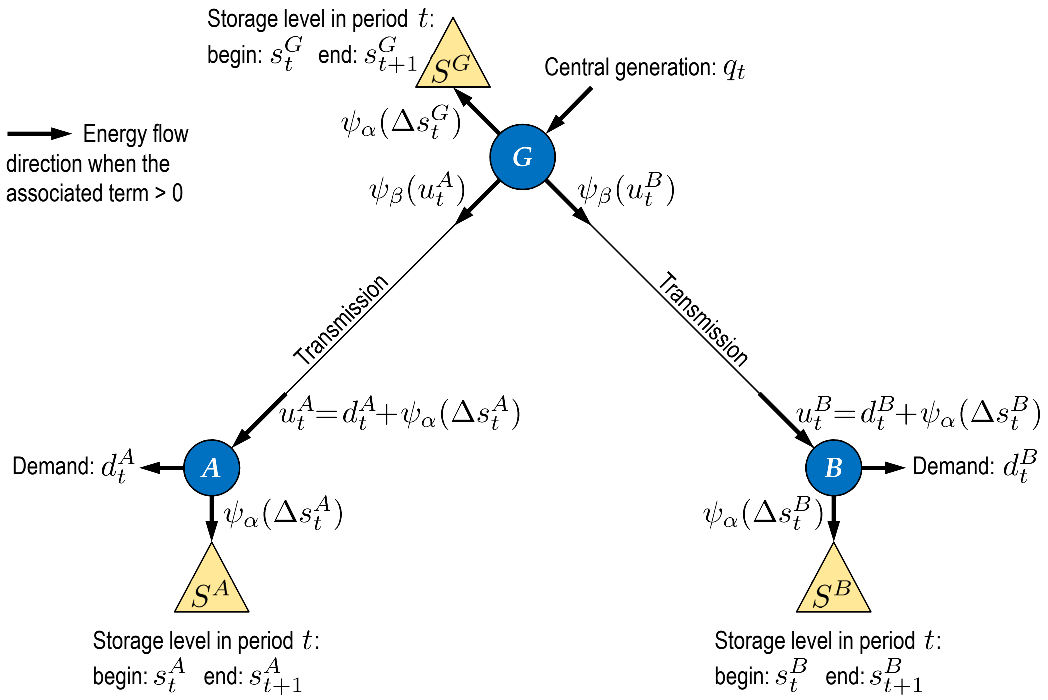
Let $c(q_t)$ denote the cost of producing q_t in period t at the central facility at G. The production satisfies the following assumption.

Assumption II.3 (Production). *(i) $c(q_t)$ is convex and increasing in q_t for $q_t \geq 0$, and $c(0) = 0$; (ii) In every period t, central production q_t can be adjusted to any non-negative level at negligible adjustment cost.*

2.3.3 Problem Formulation

We aim to decide the storage investment strategy and the corresponding operating policy that satisfies the demand at the minimum cost. To evaluate a storage investment decision $\mathbf{S}$, let $V_t(\mathbf{s}_t, \mathbf{d}_t; \mathbf{S})$, $t \geq 1$, denote the minimum expected discounted operating cost from period t onward when the state is $(\mathbf{s}_t, \mathbf{d}_t)$, and let $\gamma \in (0, 1]$ be the discount factor. The optimal operating policy for given storage $\mathbf{S}$ is determined by the following stochastic dynamic

Figure 2.1: Network Model and Key Variables



program:

$$V_t(\mathbf{s}_t, \mathbf{d}_t; \mathbf{S}) = \min_{\mathbf{s}_{t+1}} \left\{ c\big(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)\big) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1}; \mathbf{S})\big] \right\}, \quad t \in \mathcal{T}, \tag{2.5}$$

$$\text{s.t. } \mathbf{s}_{t+1} \in \mathcal{A}, \quad q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \geq 0, \tag{2.6}$$

where E_t denotes the expectation with respect to future demand $\mathbf{d}_{t+1}$, conditioned on $\mathbf{d}_t$. The terminal condition is $V_{T+1}(\mathbf{s}_{T+1}, .; \mathbf{S}) = 0$.

The storage facilities are installed in period 0; no additional investment or divestment can be made during the operating horizon. The storage investment decision trades off between the upfront investment cost and the ongoing operating cost. The investment of $|\mathbf{S}| \overset{\text{def}}{=} S^A + S^B + S^G$ units of storage capacity incurs an upfront investment cost of $p\,|\mathbf{S}|$, charged at the end of period 0, where p is the investment cost per unit of storage capacity (we consider a single storage technology in this paper). We assume storage facilities are full after installation: $\mathbf{s}_1 = \mathbf{S}$. Thus, at the end of period 0, the investment and operating cost combined is $p\,|\mathbf{S}| + V_1(\mathbf{S}, \mathbf{d}_1; \mathbf{S})$. Because the investment decision $\mathbf{S}$ is made at the *beginning* of period 0, we define $V(\mathbf{S}) \overset{\text{def}}{=} \mathsf{E}_0 V_1(\mathbf{S}, \mathbf{d}_1; \mathbf{S})$ and $B(\mathbf{S}) \overset{\text{def}}{=} V(\mathbf{0}) - V(\mathbf{S}) - p\,|\mathbf{S}|$, which is the net benefit of $\mathbf{S}$, i.e., the operating cost reduction brought by the storage net the investment cost. Then, the optimal investment is determined by

$$\max_{\mathbf{S} \geq 0} \; B(\mathbf{S}). \tag{2.7}$$

This problem is to make the storage siting and sizing decisions on a simple tree network, taking into account the storage and transmission losses and the convex production cost. Our model can be extended to include a fixed cost or other general cost structures, but we believe that focusing on the linear investment cost helps us understand the basic tradeoffs in storage investment decisions.

2.4. Optimal Operating Policy for Given Storage Investment

This section derives the structural properties of the optimal operating policy under given storage size $\mathbf{S} \geq 0$. Because $\mathbf{S}$ is fixed for $t \in \mathcal{T}$, we write $V_t(\mathbf{s}_t, \mathbf{d}_t; \mathbf{S})$ as $V_t(\mathbf{s}_t, \mathbf{d}_t)$ and write the optimal decision for (2.5)-(2.6) as $\mathbf{s}^*_{t+1}(\mathbf{s}_t, \mathbf{d}_t)$ in this section.

2.4.1 Basic Properties and Problem Decomposition

We first derive some properties of the operating cost function $V_t(\mathbf{s}_t, \mathbf{d}_t)$. Intuitively, stored energy has an operating-cost reduction effect. Lemma II.4 confirms this intuition and further shows that this effect declines when the storage level increases. (Throughout this paper, monotone and convex properties are not in strict sense, unless otherwise noted.)

Lemma II.4. *$V_t(\mathbf{s}_t, \mathbf{d}_t)$ is decreasing and convex in $\mathbf{s}_t$ for any $\mathbf{d}_t$ and $t \in \mathcal{T}$.*

Note that constraint (2.6) defines a non-convex feasible region for $\mathbf{s}_{t+1}$, and thus, the proof of Lemma II.4 is not obvious. All proofs are included in the online supplement.

The next lemma shows that energy from any storage facility would not be withdrawn only to store it in another location. Intuitively, because transmission capacity is non-binding (Assumption II.2), there is no benefit from moving stored energy only to incur transmission and storage losses.

Lemma II.5. *(i) Let* $\mathbf{s}_t, \widetilde{\mathbf{s}}_t \in \mathcal{A}$*, and* $\mathbf{s}_t - \widetilde{\mathbf{s}}_t = (-\delta,\, \beta^2\delta,\, 0)$ *or* $(\beta^2\delta,\, -\delta,\, 0)$ *for some* $\delta > 0$*. Then,* $V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq V_t(\mathbf{s}_t, \mathbf{d}_t)$ *for any* $\mathbf{d}_t$ *and* $t \in \mathcal{T}$*.*
(ii) Let $\mathbf{s}_t, \widetilde{\mathbf{s}}_t \in \mathcal{A}$*, and* $\mathbf{s}_t - \widetilde{\mathbf{s}}_t = (-\delta,\, 0,\, \beta\delta)$ *or* $(0,\, -\delta,\, \beta\delta)$ *or* $(\beta\delta,\, 0,\, -\delta)$ *or* $(0,\, \beta\delta,\, -\delta)$ *for some* $\delta > 0$*. Then,* $V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq V_t(\mathbf{s}_t, \mathbf{d}_t)$ *for any* $\mathbf{d}_t$ *and* $t \in \mathcal{T}$*.*

To analyze the structures of the optimal policy, we decompose the problem in (2.5)-(2.6) into a master problem and a subproblem.

- The <u>master</u> problem decides the production level q_t,

$$V_t(\mathbf{s}_t, \mathbf{d}_t) = \min_{q_t} \big\{ c(q_t) + \gamma W_t(q_t, \mathbf{s}_t, \mathbf{d}_t) : q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t) \big\}. \tag{2.8}$$

- While the <u>subproblem</u> finds the optimal use of q_t by deciding the inventory levels:

$$W_t(q_t, \mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1}} \big\{ \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big] : \mathbf{s}_{t+1} \in \mathcal{A}(q_t) \big\}, \quad q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t), \tag{2.9}$$

where $\mathbf{s}_{t+1}$ is chosen from an *iso-production surface* $\mathcal{A}(q_t)$, defined as

$$\mathcal{A}(q_t) \overset{\text{def}}{=} \big\{ \mathbf{s}_{t+1} \in \mathcal{A} : q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) = q_t \big\}, \tag{2.10}$$

and q_t is chosen from $\mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t) \overset{\text{def}}{=} \big[\underline{\underline{\underline{q}}}_t, \overline{\overline{\overline{q}}}_t\big]$, where $\overline{\overline{\overline{q}}}_t = q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t)$ is the maximum production in period t, which satisfies the demand and fills up the storage at all three nodes, and $\underline{\underline{\underline{q}}}_t = \big(q(-\mathbf{s}_t, \mathbf{d}_t)\big)^+$ is the minimum production, which satisfies the remaining demand after inventory from all three nodes is used to meet as much demand as possible. (Throughout the paper, $x^+ = \max\{x, 0\}$.) For brevity of notations, we do not explicitly express the dependence of $\mathcal{A}(q_t)$, $\overline{\overline{\overline{q}}}_t$, and $\underline{\underline{\underline{q}}}_t$ on $(\mathbf{s}_t, \mathbf{d}_t)$.

Let $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ denote an optimal solution to the subproblem (2.9). Solving (2.9) leads to the minimum expected cost-to-go function $W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$ that is well-behaved.

Lemma II.6. $W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$ *is convex and decreasing in* q_t *for any given* $(\mathbf{s}_t, \mathbf{d}_t)$ *and* $t \in \mathcal{T}$*.*

The master problem (2.8) decides the optimal production, trading off between the production cost $c(q_t)$ and the minimum expected cost-to-go function $W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$. Because

$W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$ and $c(q_t)$ are convex in q_t (Lemma II.6 and Assumption II.3), the master problem (2.8) is a one-dimensional convex optimization problem. Therefore, the rest of the analysis in this section is devoted to describing the structures of $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$, the solution to the subproblem (2.9).

2.4.2 Structures of the Optimal Inventory Policy

The key construct for describing the optimal solution $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ is a *balance curve*, denoted as $\{\mathbf{b}(z; s^G, \mathbf{d}_t) : z \in [0, S^A + S^B]\}$. A balance curve assumes the inventory in node G is fixed at $s^G \in [0, S^G]$. For each total leaf inventory level $z \in [0, S^A + S^B]$, $\mathbf{b}(z; s^G, \mathbf{d}_t)$ prescribes an allocation of z among the leaf nodes A and B so that the future expected operating cost is minimized:

$$\mathbf{b}(z; s^G, \mathbf{d}_t) \in \underset{\mathbf{s}_{t+1} \in \mathcal{A}}{\arg\min} \big\{ \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big] : s^A_{t+1} + s^B_{t+1} = z,\ s^G_{t+1} = s^G \big\}. \tag{2.11}$$

We characterize the structural properties of the optimal solution $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ using the balance curve, with explicit formulae when possible. We will show that for $\alpha \le \beta$, $s^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ lies on the balance curve if (loosely speaking) the production level q_t allows $s^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ to reach the balance curve; otherwise, $s^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ will be as close to the balance curve as possible. These structural properties bring computational benefits and also reveal insights on storage management. To clearly illustrate these structures, we first present a special case with symmetric demand across the leaf nodes and $S^G = 0$ (Theorem II.8). We then relax the constraint $S^G = 0$. In Theorem II.9 we further relax the symmetry and show that the policy structure remains the same. In Theorem II.11 we consider the case with $\beta < \alpha$.

For a given leaf node, we refer to the storage at this leaf node as local (L) storage, and the storage at the other leaf node as remote (R) storage. The storage at node G is referred to as central storage.

2.4.2.1 Special Case with Symmetric Leaf Nodes and $\boldsymbol{\alpha \le \beta}$

The leaf nodes are said to be symmetric in period t if they have the same storage size, $S^A = S^B$, and the demand distributions for future periods satisfy

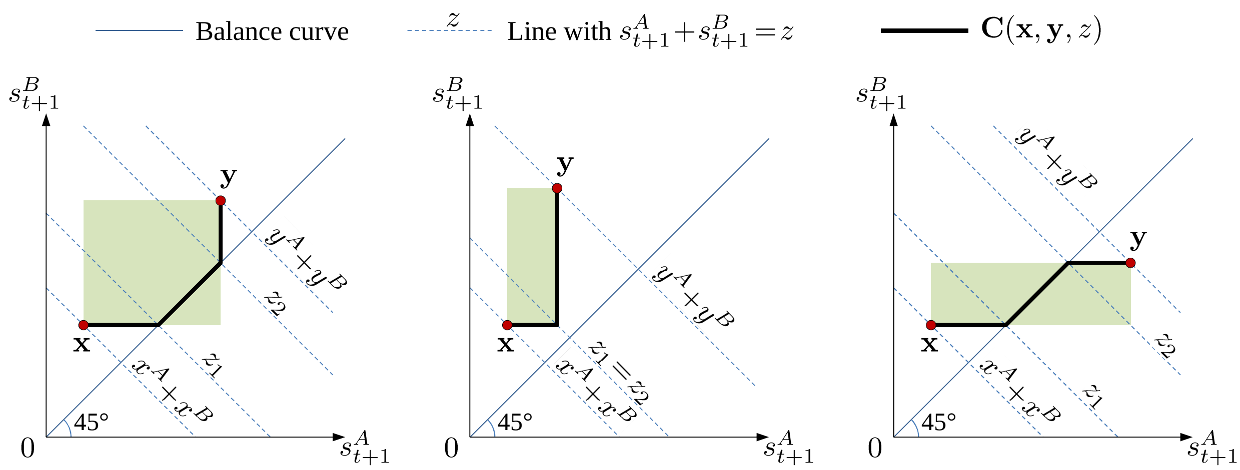
$$\mathsf{P}\{d_\tau^A \le x,\, d_\tau^B \le y \mid \mathbf{d}_t\} = \mathsf{P}\{d_\tau^A \le y,\, d_\tau^B \le x \mid \mathbf{d}_t\}, \quad \forall\, \tau = t+1, \ldots, T,\ \forall\, x,\ \forall\, y. \tag{2.12}$$

When the leaf nodes are symmetric in period t, allocating inventory evenly across the leaf nodes minimizes the future expected operating cost in (2.11). Lemma II.7 confirms this intuition.

Lemma II.7. *If $S^A = S^B$ and (2.12) holds in period t, then the balance curve is $\mathbf{b}(z; s^G, \mathbf{d}_t) = \big(z/2, z/2, s^G\big)$ for all $z \in [0, S^A + S^B]$.*

Next, we characterize the optimal decision $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$. A component of the optimal decision is represented by a curve $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$, where $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $x^A \le y^A$, $x^B \le y^B$, and $x^G = y^G$. The curve $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ connects $\mathbf{x}$ and $\mathbf{y}$ within the rectangle formed by $\mathbf{x}$ and $\mathbf{y}$, staying as close to the balance curve $\mathbf{b}(z; x^G, \mathbf{d}_t)$ as possible. Figure 2.2 illustrates the curve $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$. The curve's parameter $z \in [x^A + x^B,\ y^A + y^B]$ is the total leaf storage level.

Figure 2.2: Examples of $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ under symmetric leaf nodes



Formally, for the case of symmetric leaf nodes, $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ is defined as follows:

$$\mathfrak{B}(\mathbf{x},\mathbf{y},z) \stackrel{\text{def}}{=} \begin{cases} \mathbf{1}_{\{x^A\le x^B\}}\big(z-x^B,\, x^B,\, x^G\big)+\mathbf{1}_{\{x^A>x^B\}}\big(x^A,\, z-x^A,\, x^G\big), & \text{if } z\in[x^A{+}x^B,\, z_1),\\ \big(z/2, z/2, x^G\big), & \text{if } z\in[z_1,z_2],\\ \mathbf{1}_{\{y^A\ge y^B\}}\big(z-y^B,\, y^B,\, x^G\big)+\mathbf{1}_{\{y^A<y^B\}}(y^A,\, z-y^A,\, x^G\big), & \text{if } z\in(z_2, y^A{+}y^B], \end{cases} \tag{2.13}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $z_1 = (x^A \vee x^B)\wedge y^A + (x^A \vee x^B)\wedge y^B$ and $z_2 = (y^A \wedge y^B)\vee x^A + (y^A \wedge y^B)\vee x^B$, where $a\vee b = \min\{a,b\}$ and $a\wedge b = \max\{a,b\}$.

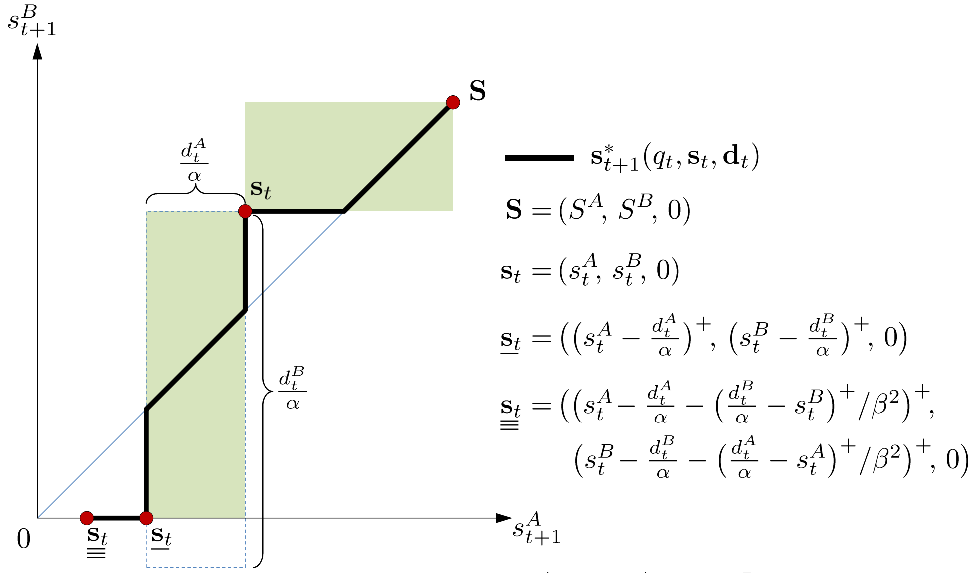
We focus on the case of $\alpha \le \beta$ for now, i.e., storing energy incurs more energy loss than transmitting energy from the central to the leaf nodes. Thus, intuitively, it is preferred to use the central generation q_t to meet as much demand as possible. If q_t is insufficient to cover the entire demand, stored energy is then released; if q_t is more than enough to meet the demand, the excess energy is stored. The following analysis formalizes this intuition and prescribes the optimal inventory policy.

Theorem II.8 below focuses on the case of $S^G = 0$. We first give a graphical representation of the theorem in Figure 2.3. As q_t increases from $\underline{\underline{\underline{q}}}_t$ to $\overline{\overline{\overline{q}}}_t$, the optimal period-ending inventory $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ moves from $\underline{\underline{\underline{\mathbf{s}}}}_t$ to $\mathbf{S}$ along the bold curve. The following structures are true for $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ for any q_t, and thus also true for $\mathbf{s}^*_{t+1}(\mathbf{s}_t, \mathbf{d}_t)$, the solution for (2.5)-(2.6):

- Either store energy in both leaf nodes and end up (weakly) above the period-starting inventory $\mathbf{s}_t$, or withdraw energy from both leaf nodes and end up (weakly) below $\mathbf{s}_t$.
- When storing energy, keep the inventory levels as close to the balance curve as possible, i.e., the segment of the curve between $\mathbf{s}_t$ and $\mathbf{S}$ can be described by $\mathfrak{B}(\mathbf{x},\mathbf{y},z)$.
- When withdrawing energy to meet the demand, withdraw from local storage first and stay as close to the balance curve as possible (i.e., part of the curve below $\mathbf{s}_t$ can also be described by $\mathfrak{B}(\mathbf{x},\mathbf{y},z)$), and then withdraw energy from the remote storage if necessary.

To precisely describe the above structure, we define some critical production and inventory levels. Let $q^o_t = (d^A_t + d^B_t)/\beta$ denote the production that meets the demand in period t without changing inventory levels. For a given state $(\mathbf{s}_t, \mathbf{d}_t)$, $(d^i_t - s^i_t\alpha)^+$ is the remaining demand at leaf node i after being served by local storage, and thus, the production needed to serve this remaining demand is $\underline{q}_t = \sum_{i=A,B} (d^i_t - s^i_t\alpha)^+/\beta$, where the single under-bar represents

Figure 2.3: Optimal inventory $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$: Case of $\alpha \leq \beta$, symmetric leaf nodes ($S^A = S^B$ and (2.12) holds), and $S^G = 0$



The graph illustrates the situation when $d^A_t < \alpha s^A_t$ and $d^B_t > \alpha s^B_t$.

that only local storage is used to satisfy demand. The corresponding remaining storage level is $\underline{\mathbf{s}}_t = \big((s^A_t - d^A_t/\alpha)^+,\, (s^B_t - d^B_t/\alpha)^+,\, s^G_t\big)$.

We denote by $\underline{\underline{\underline{\mathbf{s}}}}_t$ the inventory level after the demand is served by the stored energy, in the sequence of local, central, and remote location (hence the triple under-bars). For the case of $S^G = 0$, we have $\underline{\underline{\underline{\mathbf{s}}}}_t = \Big(\big(s^A_t - \frac{d^A_t}{\alpha} - \big(\frac{d^B_t}{\alpha} - s^B_t\big)^+/\beta^2\big)^+,\ \big(s^B_t - \frac{d^B_t}{\alpha} - \big(\frac{d^A_t}{\alpha} - s^A_t\big)^+/\beta^2\big)^+,\ 0\Big)$; the expression for the general case will be given later. After the demand is served by the stored energy, the remaining demand is served by production $\underline{\underline{\underline{q}}}_t$, which is exactly the minimum production defined after (2.10).

Theorem II.8. *When $\alpha \leq \beta$, $S^G = 0$, and leaf nodes are symmetric ($S^A = S^B$ and (2.12) holds), for given state $(\mathbf{s}_t, \mathbf{d}_t)$ and feasible production quantity $q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t)$, an optimal* *inventory decision* $\mathbf{s}^*_{t+1}$ *is as follows:*

$$\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t) = \begin{cases} \textbf{[withdraw L fully, R partially]:} & \\ \quad \underline{\mathbf{s}}_t - (\underline{\mathbf{s}}_t - \underline{\underline{\underline{\mathbf{s}}}}_t)(\underline{q}_t - q_t)/(\underline{q}_t - \underline{\underline{\underline{q}}}_t), & \textit{if } \underline{\underline{\underline{q}}}_t \leq q_t \leq \underline{q}_t, \quad (2.14a) \\ \textbf{[withdraw partially from L]:} & \\ \quad \mathfrak{B}\big(\underline{\mathbf{s}}_t, \mathbf{s}_t, s_t^A + s_t^B - (q_t^o - q_t)\beta/\alpha\big), & \textit{if } \underline{q}_t < q_t \leq q_t^o, \quad (2.14b) \\ \textbf{[store at both leaf nodes]:} & \\ \quad \mathfrak{B}\big(\mathbf{s}_t, \mathbf{S}_t, s_t^A + s_t^B + (q_t - q_t^o)\beta\alpha\big), & \textit{if } q_t^o < q_t \leq \overline{\overline{\overline{q}}}_t. \quad (2.14c) \end{cases}$$

Theorem II.8 prescribes that when $q_t < q_t^o$, it is optimal to use q_t to serve as much as demand as possible and serve the remaining demand by withdrawing stored energy. In (2.14a), "withdraw fully from L" refers to withdrawing down to $\underline{\mathbf{s}}_t$, i.e., using local storage to serve as much demand as possible. "Withdraw partially from L" in (2.14b) refers to withdrawing to a level higher than $\underline{\mathbf{s}}_t$. When $q_t > q_t^o$, it is optimal to use q_t to meet the demand entirely and store the excess energy according to (2.14c). In all cases, we try to use the current supply, q_t, to satisfy the demand, and resolve the supply-demand mismatch by using storage.

We now extend the structures of the optimal policy to the case of $S^G > 0$, as illustrated in Figure 2.4. Same as the case of $S^G = 0$, the optimal period-ending inventory $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ is either (weakly) above or below the period-starting inventory $\mathbf{s}_t$. The sequence in which energy is stored or withdrawn now involves the central storage:

- When storing energy, first store at the central storage until full (moving from $\mathbf{s}_t$ to $\overline{\mathbf{s}}_t = (s_t^A, s_t^B, S^G)$ in Figure 2.4), and then store in the leaf nodes, keeping inventory as close to the balance curve as possible (following $\mathfrak{B}(\overline{\mathbf{s}}_t, \mathbf{S}, z)$, the curve connecting $\overline{\mathbf{s}}_t$ and $\mathbf{S}$ in Figure 2.4). The production quantities corresponding to $\overline{\mathbf{s}}_t$ and $\mathbf{S}$ are $\overline{q}_t = q_t^o + (S^G - s_t^G)/\alpha$ and $\overline{\overline{\overline{q}}}_t$, respectively.
- When withdrawing energy, withdraw first from local storage (following $\mathfrak{B}(\underline{\mathbf{s}}_t, \mathbf{s}_t, z)$), then from central storage (moving from $\underline{\mathbf{s}}_t$ to $\underline{\underline{\mathbf{s}}}_t$), and finally from remote node (moving from

$\underline{\mathbf{s}}_t$ to $\underline{\underline{\underline{\mathbf{s}}}}_t$), where the critical inventory levels are expressed below:

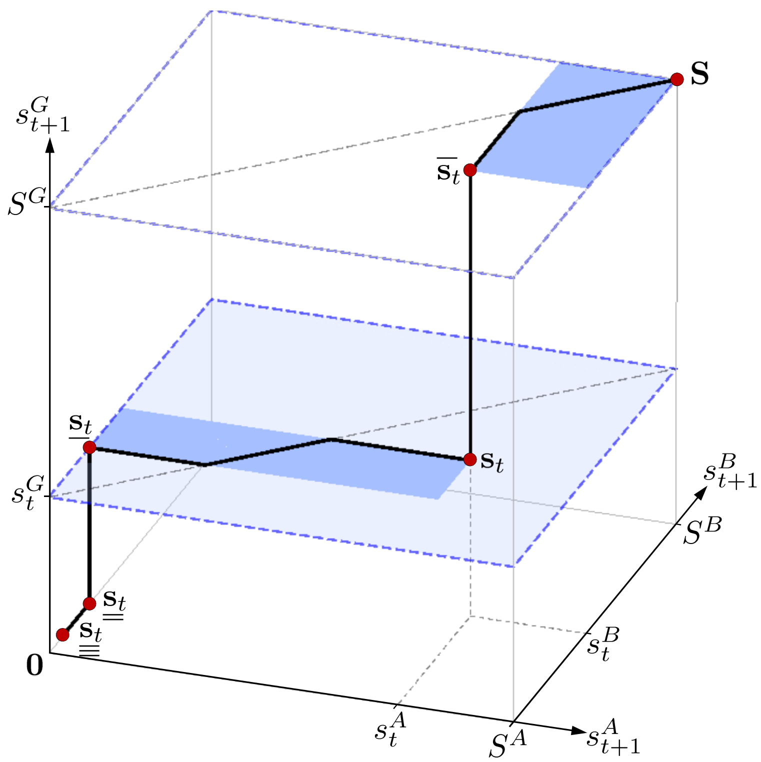
$$\underline{\mathbf{s}}_t = \Big(\big(s_t^A - \tfrac{d_t^A}{\alpha}\big)^+,\ \big(s_t^B - \tfrac{d_t^B}{\alpha}\big)^+,\ s_t^G \Big),$$

$$\underline{\underline{\mathbf{s}}}_t = \Big(\big(s_t^A - \tfrac{d_t^A}{\alpha}\big)^+,\ \big(s_t^B - \tfrac{d_t^B}{\alpha}\big)^+,\ \big[s_t^G - \big(\tfrac{d_t^A}{\alpha} - s_t^A\big)^+/\beta - \big(\tfrac{d_t^B}{\alpha} - s_t^B\big)^+/\beta\big]^+ \Big),$$

$$\underline{\underline{\underline{\mathbf{s}}}}_t = \begin{cases} \big(0,\ 0,\ \big[s_t^G - \big(\tfrac{d_t^A}{\alpha} - s_t^A + \tfrac{d_t^B}{\alpha} - s_t^B\big)/\beta\big]^+\big), & \text{if } s_t^A \le \tfrac{d_t^A}{\alpha},\ s_t^B \le \tfrac{d_t^B}{\alpha}, \\ \big(0,\ \big[s_t^B - \tfrac{d_t^B}{\alpha} - \big(\tfrac{d_t^A}{\alpha} - s_t^A - s_t^G\beta\big)^+/\beta^2\big]^+,\ \big[s_t^G - \big(\tfrac{d_t^A}{\alpha} - s_t^A\big)/\beta\big]^+\big), & \text{if } s_t^A \le \tfrac{d_t^A}{\alpha},\ s_t^B > \tfrac{d_t^B}{\alpha}, \\ \big(\big[s_t^A - \tfrac{d_t^A}{\alpha} - \big(\tfrac{d_t^B}{\alpha} - s_t^B - s_t^G\beta\big)^+/\beta^2\big]^+,\ 0,\ \big[s_t^G - \big(\tfrac{d_t^B}{\alpha} - s_t^B\big)/\beta\big]^+\big), & \text{if } s_t^A > \tfrac{d_t^A}{\alpha},\ s_t^B \le \tfrac{d_t^B}{\alpha}, \\ \big(s_t^A - \tfrac{d_t^A}{\alpha},\ s_t^B - \tfrac{d_t^B}{\alpha},\ s^G\big), & \text{if } s_t^A > \tfrac{d_t^A}{\alpha},\ s_t^B > \tfrac{d_t^B}{\alpha}. \end{cases}$$

The corresponding production quantities are $\underline{q}_t$, $\underline{\underline{q}}_t = \big(\underline{q}_t - s_t^G\alpha\big)^+$, and $\underline{\underline{\underline{q}}}_t$, respectively.

Figure 2.4: Example of $\mathbf{s}_{t+1}^*(q_t; \mathbf{s}_t, \mathbf{d}_t)$: Case of $\alpha \le \beta$, symmetric leaf nodes, and $S^G > 0$



The structures illustrated in Figure 2.4 for symmetric leaf nodes turn out to be optimal in general as long as $\alpha \le \beta$. This generalization is detailed next and the structures will be formally stated in Theorem II.9. The above critical inventory levels, $\underline{\mathbf{s}}_t$, $\underline{\underline{\mathbf{s}}}_t$, and $\underline{\underline{\underline{\mathbf{s}}}}_t$, remain the same in the general case.

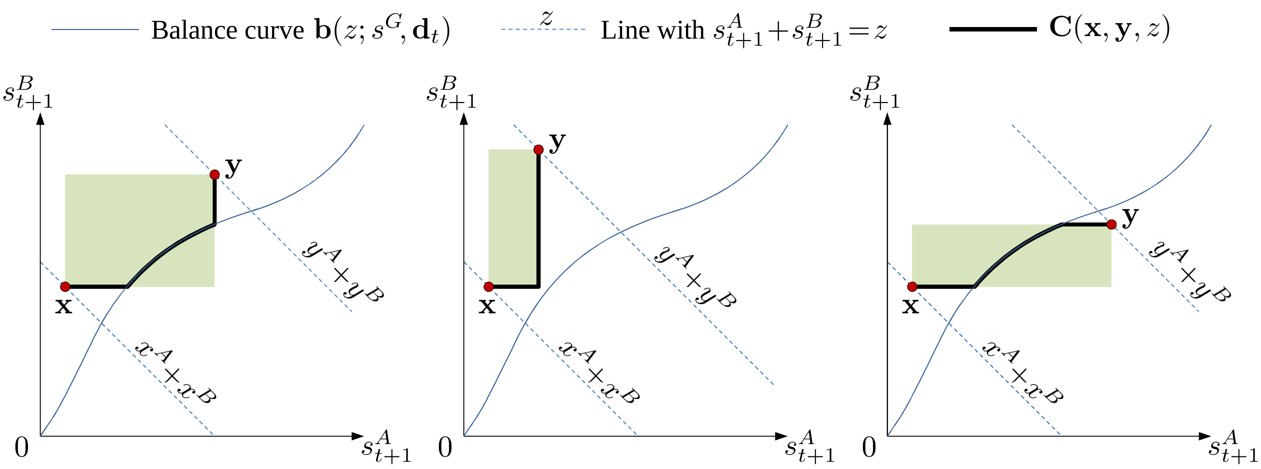
2.4.2.2 Case with General Demand Distribution and $\alpha \le \beta$

For general demand distribution, the balance curve $\mathbf{b}(z; s^G_{t+1}, \mathbf{d}_t)$ defined in (2.11) may be non-linear. Same as before, for $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ with $x^A \le y^A$, $x^B \le y^B$, and $x^G = y^G$, the curve $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ connects $\mathbf{x}$ and $\mathbf{y}$ within the rectangle formed by $\mathbf{x}$ and $\mathbf{y}$, staying as close to the balance curve $\mathbf{b}(z; x^G, \mathbf{d}_t)$ as possible. Formally,

$$\mathfrak{B}(\mathbf{x}, \mathbf{y}, z) \overset{\text{def}}{=} \mathbf{b}(z; x^G, \mathbf{d}_t) + (1, -1, 0)\, l(z), \qquad z \in [x^A + x^B, y^A + y^B], \tag{2.15}$$

where $l(z) = \arg\min_{\ell}\{\,|\,\ell\,| : \mathbf{x} \le \mathbf{b}(z; x^G, \mathbf{d}_t) + (\ell, -\ell, 0) \le \mathbf{y}\}$. In particular, if $\mathbf{x} \le \mathbf{b}(z; x^G, \mathbf{d}_t) \le \mathbf{y}$, then $l(z) = 0$. If $\mathbf{b}(z; s^G, \mathbf{d}_t) = \big(z/2, z/2, s^G\big)$, then (2.15) becomes (2.13). Figure 2.5 illustrates $\mathbf{b}(z; s^G_{t+1}, \mathbf{d}_t)$ and $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$.

Figure 2.5: Examples of $\mathbf{b}(z; s^G, \mathbf{d}_t)$ and $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$



The next theorem extends the structures illustrated in Figure 2.4 to the general case, using the general definition of $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ in (2.15).

Theorem II.9. *When $\alpha \le \beta$, given state $(\mathbf{s}_t, \mathbf{d}_t)$ and feasible production quantity $q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t)$, an optimal inventory decision $\mathbf{s}^*_{t+1}$ can be expressed as follows:*

$$
\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t) = \begin{cases}
\textbf{[withdraw L,G fully, R partially]:} & \\
\quad \underline{\underline{\mathbf{s}}}_t - (\underline{\underline{\mathbf{s}}}_t - \underline{\underline{\underline{\mathbf{s}}}}_t)(\underline{\underline{q}}_t - q_t)/(\underline{\underline{q}}_t - \underline{\underline{\underline{q}}}_t), & \text{if } \underline{\underline{\underline{q}}}_t \le q_t \le \underline{\underline{q}}_t, \qquad (2.16a) \\
\textbf{[withdraw L fully, G partially]:} & \\
\quad \big((s^A_t - d^A_t/\alpha)^+,\ (s^B_t - d^B_t/\alpha)^+,\ s^G_t - (\underline{q}_t - q_t)/\alpha\big), & \text{if } \underline{\underline{q}}_t < q_t \le \underline{q}_t, \qquad (2.16b) \\
\textbf{[withdraw partially from L]:} & \\
\quad \mathfrak{B}\big(\underline{\mathbf{s}}_t, \mathbf{s}_t, s^A_t + s^B_t - (q^o_t - q_t)\beta/\alpha\big), & \text{if } \underline{q}_t < q_t \le q^o_t, \qquad (2.16c) \\
\textbf{[store partially at G]:} & \\
\quad \big(s^A_t,\ s^B_t,\ s^G_t + \alpha(q - q^o_t)\big), & \text{if } q^o_t < q_t < \overline{q}_t, \qquad (2.16d) \\
\textbf{[store to full at G, partially at leaves]:} & \\
\quad \mathfrak{B}\big(\overline{\mathbf{s}}_t, \mathbf{S}, s^A_t + s^B_t + (q_t - \overline{q}_t)\beta\alpha\big), & \text{if } \overline{q}_t \le q_t \le \overline{\overline{\overline{q}}}_t. \qquad (2.16e)
\end{cases}
$$

Theorem II.9 characterizes the optimal policy as follows:

1) Use current generation q_t to serve the demand.
2) If $q_t > q^o_t$, store the excess energy first at the central storage (expressed in (2.16d)), and then store in the leaf storage, following the balance curve $\mathbf{b}(z; S^G, \mathbf{d}_t)$ as closely as possible (expressed in (2.16e)).
3) If $q_t < q^o_t$, use the stored energy to serve the remaining demand: First withdraw energy stored locally, following the balance curve $\mathbf{b}(z; s^G, \mathbf{d}_t)$ as closely as possible (expressed in (2.16c)), then (if needed) withdraw from the central storage (expressed in (2.16b)), and finally, (if needed) use the remote storage (expressed in (2.16a)).

Intuitively, demand in a period can be met by two types of energy: Stored energy generated in previous periods and current-period generation q_t. Because $\alpha \le \beta$ (i.e., storage is less efficient than transmission), using the current supply q_t to meet demand is preferred to using the stored energy. The mismatch between current supply and demand is resolved by storing or withdrawing energy.

When storing energy, it is optimal to fill up the central storage before storing at the leaf nodes. Intuitively, central inventory provides more operational flexibility with the same storage efficiency. When withdrawing energy to serve the remaining demand, using the closer storage first minimizes the transmission losses and thus is more economical.

2.4.2.3 Case with $\alpha > \beta$

When storage is more efficient than transmission ($\alpha > \beta$), it may not always be beneficial to use current-period generation q_t to satisfy as much demand as possible. As the following example demonstrates, even if q_t can meet the current demand, it may be superior to use some stored energy.

Example II.10. *Consider a system with $\alpha = 1$, $\beta = 2/3$, and sufficiently high storage space. Suppose we are at the beginning of period $T-1$ and observe demand $\mathbf{d}_{T-1} = (2,0)$ and storage level $\mathbf{s}_{T-1} = (2,0,0)$. The final-period demand $\mathbf{d}_T$ is expected to be either $(0,2)$ or $(2,0)$ with equal probabilities. We fix $q_{T-1} = q^o_{T-1} = 3$ in this example.*

Theorem II.9 suggests using q_{T-1} to meet as much demand as possible. Because $q_{T-1}\beta = 2 = d^A_{T-1}$, q_{T-1} exactly meets the demand, and inventory is unchanged: $\mathbf{s}_T = \mathbf{s}_{T-1} = (2,0,0)$. Then, in the final period, if $\mathbf{d}_T = (0,2)$, we must produce a positive amount ($q_T = 5/3$) to meet part of the demand at node B; the remaining demand is met from energy transmitted from remote node A.

Consider an alternative decision in period $T-1$: Store $q_{T-1} = 3$ at the central node G and use the local storage $s^A_{T-1} = 2$ to meet the local demand $d^A_{T-1} = 2$. Consequently, $\mathbf{s}_T = (0,0,3)$. Then, the stored energy s^G_T can serve 2 units of demand in the final period regardless whether demand is $(0,2)$ or $(2,0)$. Hence, production cost is zero in period T, which implies that this alternative decision is optimal for period $T-1$ under the given q_{T-1}. □

The above example demonstrates that the policy in Theorem II.9 may not be optimal when $\alpha > \beta$; the optimal policy may involve serving demand by local storage and simultaneously storing energy at the central node. Such operations are formalized in Theorem II.11 and the formalization requires another critical inventory level: $\overline{\underline{q}}_t \stackrel{\text{def}}{=} \underline{q}_t + (S^G - s^G_t)/\alpha$. Recall $\underline{q}_t$ is the production required to satisfy the demand after the local storage is used to meet as much local demand as possible. If we produce more than $\underline{q}_t$ to fill up the central storage, then the total production is $\overline{\underline{q}}_t$. The resulting inventory level is $\overline{\underline{\mathbf{s}}}_t = \big((s^A_t - d^A_t/\alpha)^+,\ (s^B_t - d^B_t/\alpha)^+,\ S^G\big)$.

Theorem II.11. *When $\alpha > \beta$, given state $(\mathbf{s}_t, \mathbf{d}_t)$ and feasible production quantity $q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t)$, an optimal inventory decision $\mathbf{s}^*_{t+1}$ can be expressed as follows.*

(i) If storage operations are perfectly efficient, $\beta < \alpha = 1$,

$$
\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t) =
\begin{cases}
\textbf{[same as (2.16a) and (2.16b)]} & \textit{if } \underline{\underline{\underline{q}}}_t \le q_t \le \underline{q}_t, \quad (2.17a)\\[2ex]
\textbf{[withdraw fully from L, store partially at G]:} & \\
\quad \big((s^A_t - d^A_t)^+,\ (s^B_t - d^B_t)^+,\ s^G_t + q_t - \underline{q}_t\big), & \textit{if } \underline{q}_t < q_t \le \underline{\overline{q}}_t, \quad (2.17b)\\[2ex]
\textbf{[store to full at G, store or withdraw at leaves]:} & \\
\quad \mathfrak{B}\big(\underline{\overline{\mathbf{s}}}_t,\ \mathbf{S},\ s^A_t + s^B_t + (q_t - \underline{\overline{q}}_t)\beta\big), & \textit{if } \underline{\overline{q}}_t < q_t \le \overline{\overline{\overline{q}}}_t. \quad (2.17c)
\end{cases}
$$

(ii) If storage operations are more efficient than transmission, but not perfectly efficient, $\beta < \alpha < 1$,

$$
\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t) =
\begin{cases}
\textbf{[same as (2.16a) and (2.16b)]} & \textit{if } \underline{\underline{\underline{q}}}_t \le q_t \le \underline{q}_t, \quad (2.18a)\\[2ex]
\textbf{[withdraw partially from L, store partially at G]:} & \\
\quad \mathbf{s}^*_{t+1} \in F_t, & \textit{if } \underline{q}_t < q_t \le \underline{\overline{q}}_t, \quad (2.18b)\\[2ex]
\textbf{[withdraw or store at L, store at G]:} & \\
\quad \mathbf{s}^*_{t+1} \in F_t \cup E_t, & \textit{if } \underline{\overline{q}}_t < q_t \le \overline{q}_t, \quad (2.18c)\\[2ex]
\textbf{[store to full at G, store or withdraw at leaves]:} & \\
\quad \mathbf{s}^*_{t+1} \in E_t, & \textit{if } \overline{q}_t < q_t \le \overline{\overline{\overline{q}}}_t. \quad (2.18d)
\end{cases}
$$

where the face F_t and edges E_t are defined as

$$
\begin{aligned}
F_t &\overset{def}{=} \big\{\mathbf{s}_{t+1} \in \mathcal{A}(q_t) : s^G_{t+1} \ge s^G_t,\ s^i_{t+1} \in \big[(s^i_t - d^i_t/\alpha)^+,\ s^i_t\big],\ i = A, B\big\},\\
E_t &\overset{def}{=} \big\{\mathbf{s}_{t+1} \in \mathcal{A}(q_t) : s^G_{t+1} = S^G,\ s^i_{t+1} \in \big[(s^i_t - d^i_t/\alpha)^+,\ S^i\big],\ i = A, B\big\}.
\end{aligned}
$$

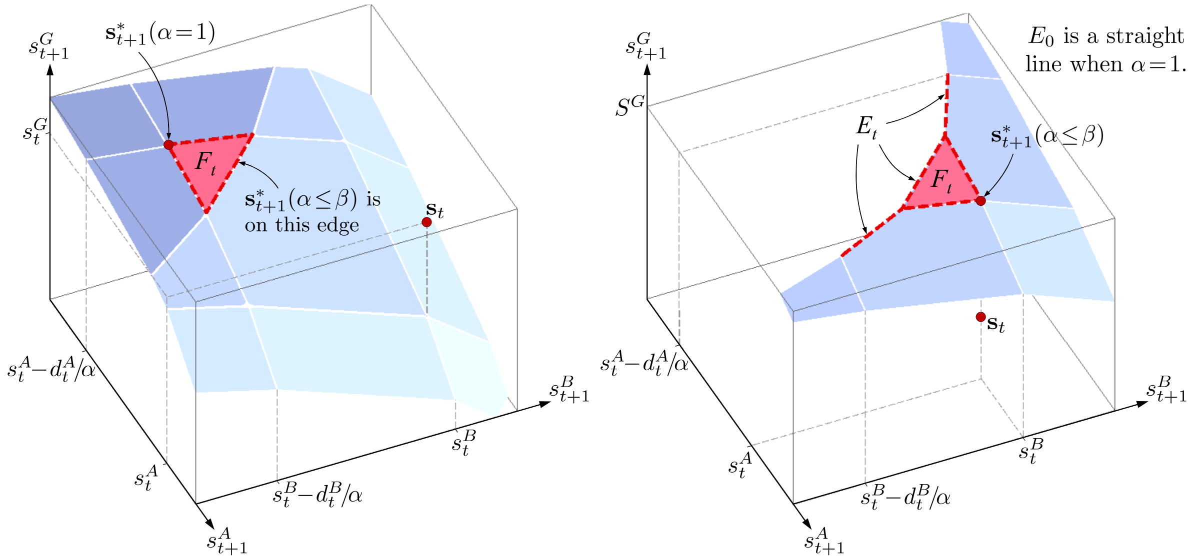
As illustrated in Example II.10 and proved in Theorem II.11, when $\alpha > \beta$ the current-period demand is not always satisfied from current-period generation to the extent possible, which is in contrast with Theorem II.9. This distinction between the cases of $\alpha \le \beta$ and $\alpha > \beta$ is illustrated in Figure 2.6.

For $q_t \in (\underline{q}_t, \underline{\overline{q}}_t)$, when $\alpha = 1$, (2.17b) in Theorem II.11 shows that it is optimal to use local inventory to serve as much local demand as possible and transmit only $\underline{q}_t$ to serve the remaining demand. The excess production $q_t - \underline{q}_t$ is stored at G. This inventory decision is shown as $\mathbf{s}^*_{t+1}(\alpha = 1)$ in Figure 2.6(a). Compared to $\mathbf{s}^*_{t+1}$ in Theorem II.9 (marked as

Figure 2.6: Optimal Policy Structure Examples

(a) $\underline{q}_t < q_t < \min\{q_t^o, \overline{q}_t\}$
Equations (2.16c), (2.17b), (2.18b)

(b) $\max\{q_t^o, \overline{q}_t\} < q_t < \overline{q}_t$
Equations (2.16d), (2.17c), (2.18c)



The piecewise linear surface represents the set of feasible $\mathbf{s}_{t+1}$ for given q_t, i.e., the iso-production surface $\mathcal{A}(q_t)$ defined in (2.10). Each edge of the surface is in one of the following planes: $s^i_{t+1} = s^i_t$, $s^i_{t+1} = s^i_t - d^i_t/\alpha$, $i = A, B$, and $s^G_{t+1} = s^G_t$. In panel (a), $\mathbf{s}_t$ is above $\mathcal{A}(q_t)$ as $q_t < q^o_t$ (production falls short of meeting all demands). In panel (b), $\mathbf{s}_t$ is below $\mathcal{A}(q_t)$ as $q_t > q^o_t$ (there is excess energy after meeting the demand).

$\mathbf{s}^*_{t+1}(\alpha \leq \beta)$ in Figure 2.6(a)), this strategy effectively reduces local inventory and raises the central inventory, providing more operational flexibility for future periods. When $\beta < \alpha < 1$, storage losses become part of the tradeoffs and (2.18b) in Theorem II.11 states that $\mathbf{s}^*_{t+1}$ belongs to F_t, a triangular face shown in Figure 2.6(a). The precise value of $\mathbf{s}^*_{t+1}$ depends on the demand distribution.

For $q_t > \underline{q}_t$, when $\alpha = 1$, (2.17c) shows that the central storage is filled up and the remaining energy is sent to the leaf nodes. The remaining energy can be split with some flexibility, bringing inventory level to the balance curve.[2] When $\beta < \alpha < 1$, storage losses need to be taken into account and $\mathbf{s}^*_{t+1}$ is in $F_t \cup E_t$ shown in Figure 2.6(b), with location depending on the demand distribution. Note that one vertex of the face F_t is exactly the optimal decision in Theorem II.9, shown as $\mathbf{s}^*_{t+1}(\alpha \leq \beta)$ in Figure 2.6(b).

[2]Depending on the future demand distribution, the optimal decision may involve sending no energy to a leaf node, or sending some energy to partially serve the demand, or sending more than the demand and storing the excess energy at a leaf node.

When $q_t > \overline{q}_t$, F_t no longer exists and thus $\mathbf{s}^*_{t+1} \in E_t$. When q_t further increases such that $q_t \geq \overline{\overline{q}}_t \stackrel{\text{def}}{=} \overline{q}_t + \max\{S^A - s^A_t, S^B - s^B_t\}/(\alpha\beta)$, E_t will reduce to only one line segment, which includes only the action of storing energy at both leaf nodes. In such case, the optimal decision falls on the balance curve and coincides with (2.16e) in Theorem II.9.

We summarize the optimal policy structures from Theorems II.9 and II.11 in Table 2.1.

Table 2.1: Optimal Policy Structures Under Given Central Production q

Note: $\underline{\overline{q}}_t \geq q^o_t$ and $\underline{\overline{q}}_t < q^o_t$ are possible.

Range of q	Efficiency: Storage ≤ Transmission ($\alpha \leq \beta$)		Efficiency: Storage > Transmission ($\alpha > \beta$)	
$\overline{\overline{q}}_t$ to $\overline{\overline{\overline{q}}}_t$	Use q to satisfy demand entirely	Store rest of q in central storage (G) until full, and then store in leaf storage	Same as $\alpha \leq \beta$	
$\overline{q}_t$ to $\overline{\overline{q}}_t$	Use q to satisfy demand entirely	Store rest of q in central storage (G) until full, and then store in leaf storage	Use part of q to fill up central storage (G); use rest of q to serve demand and to store in leaf storage	Satisfy remaining demand using local storage (L)
q^o_t to $\overline{q}_t$	Use q to satisfy demand entirely	Store rest of q in central storage (G)	Use part of q to serve demand; store rest of q in central storage (G) first, and then in leaf storage	Satisfy remaining demand using local storage (L)
$\underline{\overline{q}}_t$ to q^o_t	Use q to serve as much demand as possible	Satisfy remaining demand using local storage (L)	Use part of q to serve demand; store rest of q in central storage (G) first, and then in leaf storage	Satisfy remaining demand using local storage (L)
$\underline{q}_t$ to $\underline{\overline{q}}_t$	Use q to serve as much demand as possible	Satisfy remaining demand using local storage (L)	Use part of q to serve demand; store rest of q in central storage (G)	Satisfy remaining demand using local storage (L)
$\underline{\underline{q}}_t$ to $\underline{q}_t$	Use q to serve as much demand as possible	Satisfy remaining demand using local storage (L), and then central storage (G)	Same as $\alpha \leq \beta$	
$\underline{\underline{\underline{q}}}_t$ to $\underline{\underline{q}}_t$	Use q to serve as much demand as possible	Satisfy remaining demand using local storage (L), then central storage (G), and then remote storage (R)	Same as $\alpha \leq \beta$	

In Appendix 2.8, we extend the insights of Theorem II.11 to the case with Distributed Generation. Here, when Distributed Intermittent Generaion exceeds demands at leaf nodes, the net surplus of energy at the local leaf node may be transmitted to the remote leaf node to satisfy demand before withdrawing from any storage, when $\alpha \leq \beta^2$, else, the ordering of the use of sources of energy and locations for storage of energy would depend on the demand distributions. However, the basic intuition of withdrawing from the closest storage source and storing first at the central storage remains in this extension.

2.5. Optimal Investment Decisions

In this section, we consider the storage investment problem in (2.7): $\max_{\mathbf{S}\geq 0} \; B(\mathbf{S}) \equiv V(\mathbf{0}) - V(\mathbf{S}) - p\,|\mathbf{S}|$, where $V(\mathbf{S}) = \mathsf{E}_0 V_1(\mathbf{S}, \mathbf{d}_1; \mathbf{S})$, $p > 0$ is the cost per unit of storage capacity, and

$|\mathbf{S}| = S^A + S^B + S^G$. Our objective is to understand how various factors affect investment decisions, in particular the trade-off between localizing and pooling of storage investment. We analyze the effects of both demand factors (Section 2.5.1) and storage parameters (Section 2.5.2) on investment decisions.

We first present the basic properties of $V(\mathbf{S})$.

Lemma II.12. *$V(\mathbf{S})$ is decreasing and convex in $\mathbf{S}$; $B(\mathbf{S})$ is concave in $\mathbf{S}$.*

Lemma II.13. *(i) Let $\mathbf{S} = (S^A,\, S^B,\, S^G)$ and $\mathbf{S}_g = \big(0,\, 0,\, S^G + \beta^{-1}(S^A + S^B)\big)$. If $d_t^A, d_t^B \geq 0$, $\forall\, t \in \mathcal{T}$, then $V(\mathbf{S}_g) \leq V(\mathbf{S})$.*
(ii) Let $\mathbf{S} = (S^A,\, S^B,\, S^G)$ and $\mathbf{S}_l = (S^A + \beta S^G,\, S^B + \beta S^G,\, 0)$. Then, $V(\mathbf{S}_l) \leq V(\mathbf{S})$.

To compare the benefits of storage investment at different locations, we define a *pooled* investment as investment at node G only, and *localized* or *distributed* investment as investment at the leaf node(s) only. Lemma II.13 shows that, for any investment decision $\mathbf{S}$, there exist a pooled investment decision $\mathbf{S}_g$ and a localized investment decision $\mathbf{S}_l$ that yield a lower expected operating cost but a higher investment cost. (To see why investment cost is higher, note that $|\mathbf{S}| \leq |\mathbf{S}_g|$ because $\beta \in (0, 1)$, and that $|\mathbf{S}| \leq |\mathbf{S}_l|$ if $\beta \geq 1/2$, which is true for most systems.)

We define the optimal pooled and localized investment decisions as follows:

$$\mathbf{S}_g^* \in \arg\max \big\{B(\mathbf{S}) \,:\, S^A = S^B = 0,\, S^G \geq 0\big\}, \tag{2.19}$$

$$\mathbf{S}_l^* \in \arg\max \big\{B(\mathbf{S}) \,:\, S^A \geq 0,\, S^B \geq 0,\, S^G = 0\big\}. \tag{2.20}$$

The optimal investment $\mathbf{S}^* \in \arg\max \big\{B(\mathbf{S}) : \mathbf{S} \geq 0\big\}$ may coincide with $\mathbf{S}_g^*$ or $\mathbf{S}_l^*$, or may involve investing at both node G and leaf nodes, which we refer to as *mixed* investment.

2.5.1 Impact of Demand Factors

We first identify demand characteristics for which localized investment is advantageous over pooled investment. These characteristics include positively correlated demand across leaf nodes, high minimum demand, and distributed generation that exceeds local demand.

Before going into depth, we introduce a useful way of thinking the economic value of storage in our context. If storage is located right at the demand node, one unit of energy released from the storage can serve one unit of demand, whereas if energy needs to be transmitted to serve the demand, one unit of energy released from the storage can serve β units of demand for central-to-leaf transmission or only β^2 units of demand for leaf-to-leaf transmission. When part of the energy released from the storage is lost during transmission, the economic value of storage is reduced.

Spacial Correlation of Demands

The effect of demand correlation is best visible when demands are perfectly correlated. Under this condition, localized investment decision is optimal.

Theorem II.14. *If $d_t^A = k\, d_t^B$ for some constant $k > 0$ and for all $t \in \mathcal{T}$, then the optimal localized investment $\mathbf{S}_l^*$ is an optimal investment for problem* (2.7)*.*

Note that electricity systems are different from other logistic systems in that energy production and transmission have zero lead-time. Thus, the result in Theorem II.14 is not driven by lead-time related reasons, as in the classic inventory theory. The driving force is the transmission losses, as explained below.

Multiple leaf nodes with perfectly correlated demands can be treated as a single demand node. Obviously, for a system with node G and only one demand node, investing at node G is never optimal, because a smaller investment, βS^G, at the demand node provides the same operational benefit as investing S^G as node G. If investing in storage at node G, $(1-\beta)$ fraction of the energy released from G is lost during transmission to the leaf nodes.

Minimum Demand

When demands are not perfectly correlated, localized investment may still be optimal as long as storing energy locally does not reduce the economic value of the storage.

Localizing storage may reduce its economic value when leaf-to-leaf transmission is part of the optimal operating policy (see Theorems II.9 and II.11), resulting in transmission losses of $1-\beta^2$ fraction of energy. A sufficient condition for leaf-to-leaf transmission not to occur is that the minimum demand is sufficiently high. Specifically, if $d_{\min}^i > \alpha S_l^{i*}$, for $i = A, B$, where $d_{\min}^i$ is the minimum demand at node i and S_l^{i*} is given by (2.20), then locally stored energy can always serve the local demand instead of the remote demand. (Under $\mathbf{S}_l^*$ and

$d^i_{\min} > \alpha S^{i*}_l$, it can be verified that $\underline{q}_t = \underline{\underline{q}}_t = \underline{\underline{\underline{q}}}_t$. Thus, the case of using remote storage in Table 2.1 does not arise.)

The result is formalized in Theorem II.16, whose proof relies on Lemma II.15.

Lemma II.15. *(i) If* $\widetilde{\mathbf{S}} = \mathbf{S} + (\beta\delta,\, 0,\, -\delta)$ *for some* $\delta > 0$*, and* $\widetilde{S}^A < \alpha^{-1} d^A_{\min}$*, then* $V(\mathbf{S}) = V(\widetilde{\mathbf{S}})$*.*
(ii) If $\widetilde{\mathbf{S}} = \mathbf{S} + (\delta,\, -\delta,\, 0)$ *for some* $\delta > 0$*, and* $\widetilde{S}^A < \alpha^{-1} d^A_{\min}$*, then* $V(\widetilde{\mathbf{S}}) \le V(\mathbf{S})$*.*

Theorem II.16. *If the optimal localized investment* $\mathbf{S}^*_l$ *satisfies* $S^{i*}_l < \alpha^{-1} d^i_{\min}$ *for* $i = A, B$*, then it is an optimal investment for the problem* (2.7)*, and any other investment with* $S^G > 0$ *is suboptimal.*

According to Theorem II.16, we may first identify the optimal localized investment, $\mathbf{S}^*_l$, and confirm its global optimality if it satisfies the condition in Theorem II.16.

Distributed Generation

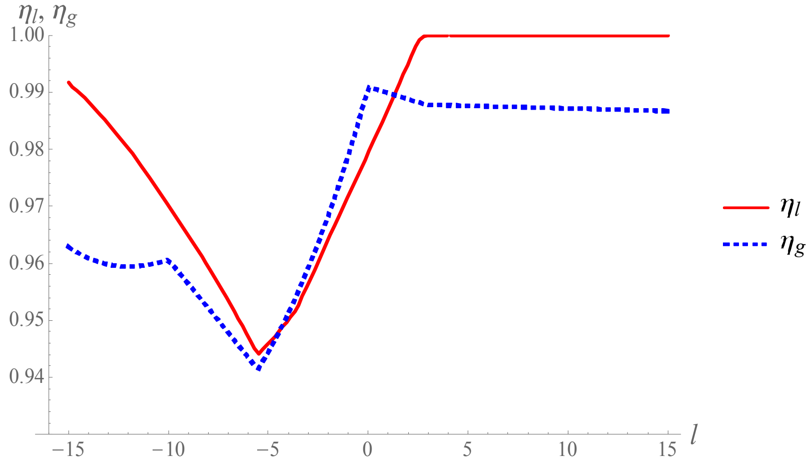
When the amount of local generation exceeds the local demand (i.e., net demand of a leaf node is negative), localized storage investment allows the local generation to be stored and consumed without any transmission. In contrast, with pooled storage, storing local generation at node G and withdrawing energy from node G would incur transmission losses in both ways.

Example II.17 (Effects of minimum demand and distributed generation)**.** *Suppose demand has three levels: l (low), m (medium), and h (high), with $l \le m \le h$. Suppose* $\mathbf{d}_{2k-1} = (m, h)$ *or* (h, m) *with equal probabilities and* $\mathbf{d}_{2k} = (l, l)$*, for* $k = 1, 2, \ldots$*. We consider an infinite operating horizon with discount factor* $\gamma = 0.99$*, and assume a quadratic production cost* $c(q) = q^2$ *and* $\beta \in (0.5,\, 1)$*.*

Because $B(\mathbf{S})$ *is concave in* $\mathbf{S}$ *and demands are symmetric across leaf nodes, we can restrict our attention to symmetric investment decisions:* $S^A = S^B$*. We derive the optimal policy and investment decisions explicitly in Appendix 2.9.*

To facilitate comparing the pooled and localized investment decisions, we define the benefit ratios: $\eta_g \overset{def}{=} B(\mathbf{S}^*_g)/B(\mathbf{S}^*)$ *and* $\eta_l \overset{def}{=} B(\mathbf{S}^*_l)/B(\mathbf{S}^*)$*. Clearly,* $\eta_g, \eta_l \in [0, 1]$*. If* $\eta_g = 1$*, pooled*

Figure 2.7: Impact of minimum demand and distributed generation: $\alpha = \beta = 0.9$, $m = l{+}10$, $h = l + 60$, $p = 400$.



investment is optimal. If $\eta_l = 1$, localized investment is optimal. If $\eta_g < 1$ and $\eta_l < 1$, then a mixed investment is optimal.

In Figure 2.7, we vary the minimum demand $l \in [-15, 15]$ and keep $m = l + 10$ and $h = l + 60$, i.e., we shift all three demand levels in parallel.

When the minimum demand l is positive and not too low, $\eta_l = 1$, i.e., the localized investment is optimal. When l decreases to around zero and slightly negative, localization increasingly deviates away from the optimal investment, and the pooled investment may outperform the localized investment. When l becomes more negative, i.e., distributed generation increases, localized investment returns to be preferred to pooled investment.

This example illustrates the insight that localized investment is a favorable choice under high minimum demand or significant distributed generation. □

The discussions thus far suggest that if the optimal investment involves investing at node G, then demands must not be highly correlated across nodes, minimum demand should be small enough, and distributed generation should not result in frequent, large negative demand.

Next, we discuss the benefits of investing in storage at node G. First, storage at node G reduces the need for leaf-to-leaf transmission. Energy stored at node G can serve demand at either leaf node without incurring leaf-to-leaf transmission losses. Second, storage at node G may avoid investing in storage capacity dedicated to each leaf node. Both are illustrated in the following example.

Example II.18 (Benefits of pooled investment)**.** *We analyze the same setting as in Example II.17 with different parameter values:* $l = m = 0$ *and* $h = 10$*, i.e., demand alternates between* $(0,0)$ *and* $(0,10)$ *or* $(10,0)$*.*

As shown in Figure 2.8(a), for the entire range of p *values considered in this example, the pooled investment is optimal (*$\eta_g = 1$*), but the reason for the optimality is different at different values of* p*, revealed in Figure 2.8(b).*

Figure 2.8: Benefits of pooled investment: $\alpha = \beta = 0.8$, $l = m = 0$, $h = 10$.

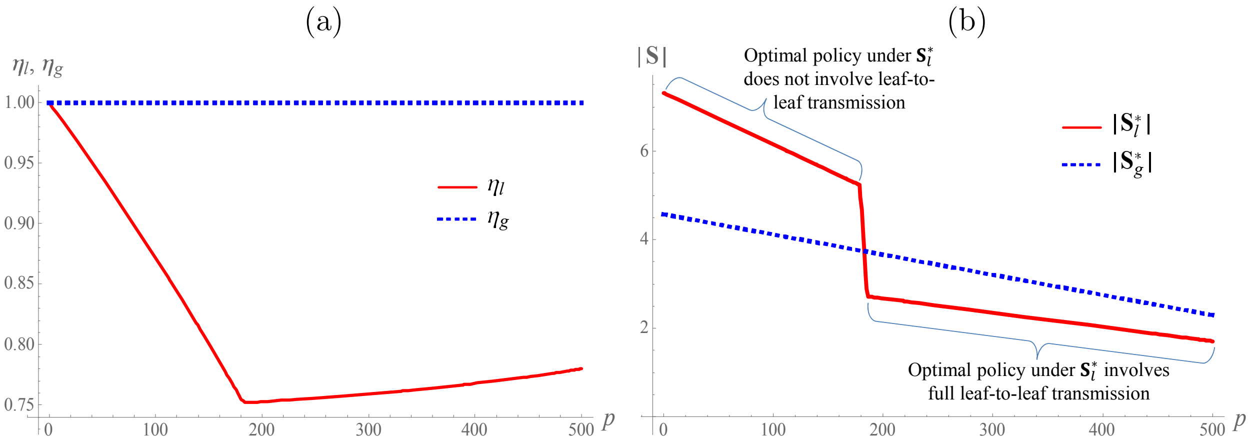


Figure 2.8(b) shows the total capacity invested under the pooled and localized decisions. At high p*, under the localized investment* $\mathbf{S}_l^*$*, the optimal operating policy is to fill the storage when demand is* $(0,0)$*, and empty the storage to meet the demand* $(0,10)$ *or* $(10,0)$*, incurring leaf-to-leaf transmission losses, which significantly reduces the economic value of storage. The pooled investment increases the operational flexibility by transmitting energy from* G *to only the node with high demand and avoiding leaf-to-leaf transmission.*

At low p*, the optimal localized investment is to invest storage capacity dedicated to each leaf node: The optimal operating policy is to fill the storage when demand is* $(0,0)$*, but only withdraw locally stored energy to serve the demand. Dedicated storage investment for each leaf node results in excessive amount of storage capacity, seen in Figure 2.8(b) for the range when* $p < 178$*. Although leaf-to-leaf transmission is avoided, the storage capacity is under-utilized. A pooled storage doubles the storage capacity utilization in this case, and is the optimal investment.* □

In the above example, pooled investment is optimal, but in general when the minimum demand is positive, the next theorem asserts that investment at node G alone cannot be optimal, i.e., mixed investment is optimal.

Theorem II.19. *Let* $\mathbf{S}^*$ *be an optimal solution to* (2.7)*. If* $S^{G*} > 0$*, then we have* $S^{A*} \geq \alpha^{-1} d^A_{\min}$ *and* $S^{B*} \geq \alpha^{-1} d^B_{\min}$*.*

Proof. For given $\mathbf{S}^*$ with $S^{G*} > 0$, suppose $S^{A*} < \alpha^{-1} d^A_{\min}$. Let $\widetilde{\mathbf{S}} = \mathbf{S}^* + (\beta\delta,\, 0,\, -\delta)$, where $\delta = \min\{S^{G*},\, (\alpha^{-1} d^A_{\min} - S^{A*})/2\}$. Note that $\widetilde{S}^A < \alpha^{-1} d^A_{\min}$. Then, by Lemma II.15(i), we have $V(\mathbf{S}^*) = V(\widetilde{\mathbf{S}})$. Because $|\mathbf{S}^*| > |\widetilde{\mathbf{S}}|$, we have $B(\mathbf{S}^*) < B(\widetilde{\mathbf{S}})$, contradicting to the optimality of $\mathbf{S}^*$. $\square$

2.5.2 Impact of Storage Parameters

Storage technologies evolve with improvement in efficiency and reduced cost. In this section, we analyze the effects storage cost and efficiency on the optimal investment.

Impact of the per-unit cost of storage capacity p

In this subsection, we emphasize the dependence on p by writing the optimal investment as $\mathbf{S}^*(p)$. We first observe some basic properties of $\mathbf{S}^*(p)$.

Lemma II.20. *(i) The optimal net benefit* $B(\mathbf{S}^*(p))$ *decreases in* p*; (ii) The optimal total investment* $|\mathbf{S}^*(p)|$ *decreases in* p*.*

Lemma II.20 indicates that as the cost of storage declines, more storage capacity is desirable for the system. The analysis in §2.5.1 suggests that localized investment $\mathbf{S}^*_l(p)$ is optimal in many demand situations. In such case, reduced storage cost will in favor of more investment at the demand nodes.

Even if localized investment is not optimal, Theorem II.21 below reveals that the localized investment $\mathbf{S}^*_l(p)$ is asymptotically optimal as $p \to 0$. This asymptotic optimality also holds for the pooled investment $\mathbf{S}^*_g(p)$ if demand is non-negative at each node.

Theorem II.21. *(i)* $\lim_{p\to 0} \left[B(\mathbf{S}^*(p)) - B(\mathbf{S}^*_l(p))\right] = 0$,
(ii) If $d^A_t, d^B_t \geq 0$, $t \in \mathcal{T}$*, then* $\lim_{p\to 0} \left[B(\mathbf{S}^*(p)) - B(\mathbf{S}^*_g(p))\right] = 0$.

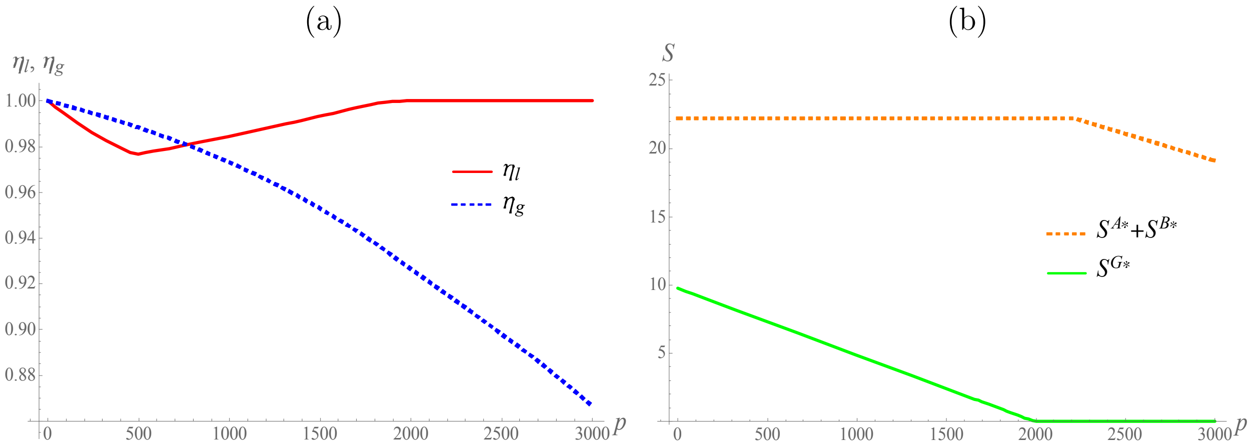
Intuitively, at very low p, we could invest in dedicated local storage capacity that is large enough to eliminate the need for leaf-to-leaf transmission.

When pooling is part of the optimal investment, reduced storage cost also stimulates pooled investment, as illustrated in the following example.

Example II.22. *We set the minimum demand $l = 0$ in Example 2 and vary p in a wide range. The results are shown in Figure 2.9. At high storage cost, it is costly to use part of the storage capacity to store the energy that will eventually be lost during transmission. Hence, localized investment is optimal, and the localized investment increases as storage cost p declines, shown in Figure 2.9(b) for large p.*

As storage cost decreases, the benefits of pooling as explained in §2.5.1 rise and the localized investment is no longer optimal (seen from $\eta_l < 1$ in Figure 2.9(a)). The optimal investment is mixed, as shown in Figure 2.9(b). Note that as the p decreases, η_g increases and S^{G*} also increases in the optimal investment. Finally, as $p \to 0$, both localized and pooled investments converge to be optimal ($\eta_l \to 1$ and $\eta_g \to 1$), consistent with Theorem II.21. □

Figure 2.9: Impact of cost of storage p: $\alpha = \beta = 0.9$, $l = 0$, $m = 10$, $h = 60$



Impact of Storage Efficiency α

We first discuss some monotonicities in the optimal decisions and values with respect to these parameters. We observe that as α increases, $B(\mathbf{S})$ is increasing, for given $\mathbf{S}$ and as β increases, the cost of the system is decreasing.

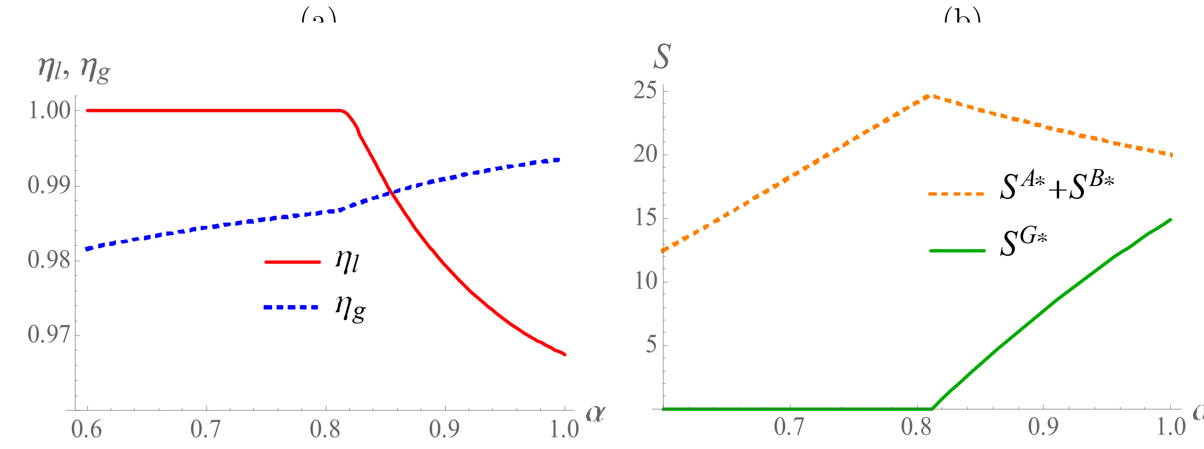
Theorem II.23. *(i) $B(\mathbf{S})$ is increasing in α, for given $\mathbf{S}$. (ii) $V(\mathbf{S})$ is monotonically decreasing in β, for given $\mathbf{S}$. (iii) Given $p > 0$, if $\beta < 1/2$, and $\mathbf{S}^*$ is an optimal solution to (2.7), then $S^{G*} = 0$.*

Next, we consider the effect of α on storage investment. As storage becomes more efficient, on the one hand, the same amount of capacity provides a higher economic value, hence stimulating more investment. On the other hand, with more efficient storage provide better production smoothing, the marginal cost of production at the peak-demand period may reduce significantly, which in turn reduces the need for storage to smooth production. Our numerical experiments find that both of these effects exist and the optimal optimal investment may increase or decrease in storage efficiency.

Furthermore, as storage becomes more efficient, less storage capacity is used to store the energy that eventually becomes lost during transmission. Hence, the benefit of pooled investment increases. This effect is demonstrated in the following example.

Example II.24. *We set the minimum demand $l = 0$ in Example 2 and vary $\alpha \in [0.6, 1]$. Figure 4.9 reveals that as storage becomes more efficient, the optimal investment changes from localized investment to mixed investment, and within the mixture, S^{G*} increases while local investment decreases as α increases. The total storage investment increases.* □

Figure 2.10: Impact of storage efficiency α: $\beta = 0.9$, $l = 0$, $m = 10$, $h = 60$, $p = 400$



In short, we find that declining storage cost definitely encourages storage investment, but improved storage efficiency does not necessarily stimulate storage investment. Storage cost reduction and efficiency improvement also tend to increase the benefit of pooled investment, although in most demand situations localization is still the optimal investment.

2.6. Numerical Analysis

The goal of this section is to investigate the nature of the optimal investment decision under a moderately realistic demand setting, for a three node model. Theorem II.11 part (i) provides an explicit solution for $\mathbf{s}^*_{t+1}$ given q_t, and part (ii) reduces the search space for $\mathbf{s}^*_{t+1}$. These allow for efficient computational study.

2.6.1 Model Parameters and Simulation Details

The round-trip storage efficiency ranges in practice between 80% and 96% (ES-Select 2012). The transmission losses range from about 1% to 15% of energy produced (Energy Information Administration 2009). This implies ranges of $\alpha^2 = 0.80 - 0.96$, and $\beta = 0.85 - 0.99$.

Aggregate production cost of the grid is often approximated as quadratic in the total energy produced in a given period (Bessembinder and Lemmon 2006). We assume the cost function as follows, satisfying Assumption II.3:

$$c(q) = 0.2\,q^2 + 20q \tag{2.21}$$

The evolutions of load and wind power exhibit predictable patterns and random fluctuations. Let $\boldsymbol{L}_t = \left[L_t^A,\, L_t^B\right]$ and $\boldsymbol{l}_t = \left[l_t^A,\, l_t^B\right]$ be the predictable and random components of the load at time t at leaf nodes A and B. Let $\mathbf{W}_t = \left[W_t^A,\, W_t^B\right]$ and $\boldsymbol{w}_t = \left[w_t^A,\, w_t^B\right]$ denote the predictable and random components of wind power. The net demand equals the load net the wind power:

$$\left[d_t^A,\, d_t^B\right] = (\boldsymbol{L}_t + \boldsymbol{l}_t) - (\mathbf{W}_t + \boldsymbol{w}_t).$$

The predictable components $\{\boldsymbol{L}_t\}$ and $\{\mathbf{W}_t\}$ are deterministic processes whose values are known prior to time zero. The stochastic processes $\{\boldsymbol{l}_t\}$ and $\{\boldsymbol{w}_t\}$ represent the deviations from the deterministic levels and evolve according to preset probability distributions.

We consider a cycle of $T = 8$ periods. For given investment in storage capacities, denoted by $\mathbf{S}$, we employ an infinite-horizon average-cost model assuming every cycle faces the same distribution of demands. In the second stage of the optimization, we search for optimal investment capacity $\mathbf{S}$ that minimizes the total investment and operating costs.

We consider a stylized model for demand, to illustrate a realistic pattern of demands. The objective of this model is to demonstrate the pattern of optimal investments under realistic settings. Consider 8 periods per day (of 3 hrs each), with each period representing 3 hours. Predictable components of load and wind are cyclic over these 8 periods.

Table 2.2: Predictable Components of Load and Wind

Time (hour of the day)	0-3	3-6	6-9	9-12	12-15	15-18	18-21	21-24	Average
Net Load $L_t^A - W_t^A (= L_t^B - W_t^B)$ (MWh)	12	14	25	36	40	36	25	14	25

Demand is deterministic, while wind is stochastic $\boldsymbol{w}_t = \xi \boldsymbol{w}_t^0$, where $\boldsymbol{w}_t^0$ is given by,

$$\boldsymbol{w}_t = \begin{cases} (-12, -12), & p = 0.01, \\ (-12, +12), & p = 0.15, \\ (+12, -12), & p = 0.15, \\ (+12, +12), & p = 0.69, \end{cases} \tag{2.22}$$

and ξ is a proxy for the penetration of wind energy into the grid. We vary ξ from 0.5-0.85, corresponding to a penetration of 0-28%. Thus, without using storage, the maximum possible production in a period is 104 MWh, with marginal cost of \$61.6 per Mwh (3.14). The average net demand per period is 50 MWh (excluding $\boldsymbol{w}_t$), with a marginal cost of \$40 per MWh.

We use the same output metrics, η_l, η_g as in Section 2.5.

2.6.2 Computational results

For the system with described parameters, we observe that $\eta_l = 1$ (i.e., localizing is optimal) across the entire range of values of α, β, p considered (Figure 2.11). The primary reason is the high minimum demand during periods of storage withdrawal. However, increasing wind penetration (which acts as a proxy for wind variability) increases η_g, because of increasing variability. We observe this for the $\boldsymbol{w}_t$ per (2.22) and for $\boldsymbol{w}_t$ with same individual node demand distribution, but varying correlation. In other words, the effect is the same for the impact of variability independent of correlation between the variabilities of the two leaf nodes. Similar to other OM literature (Eppen 1979), we observe that increasing correlation reduces the benefits of pooling.

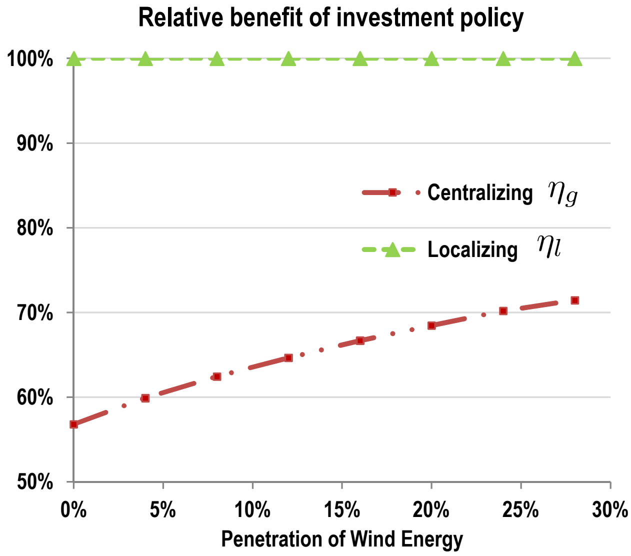


Figure 2.11: Impact of wind penetration on the investment policies, with $p = 1033\$/\text{KWh}$, $\beta = 0.85$, $\alpha = 0.84$.

2.7. Conclusions and Extensions

In this paper, we have derived the structure of the optimal operating policy for an electric grid system with convex generation costs, stochastic demand, finite energy storage capacity, storage and transmission losses. The goal of this paper was to understand the trade-offs between localizing or centralizing storage capacity investments.

Our overarching conclusion is that under current system parameters, localizing of storage investments is preferable primarily because of high minimum demand during withdrawal periods (Theorem II.16), which allows to take full advantage of the greater efficacy of localizing. Nourai (2007) showed that, in AEP's storage experiment, localized storage investments, referred to as Community Energy Storage, created greater value to the grid than centralized storage for a number of reasons. These reasons included reliability, fire risks and easing of congestion in transmission lines. Our model provides a theoretical justification of their observations from their energy storage experiment, in terms of investment costs, storage efficiency and transmission losses.

To further understand the trade-off of localizing vs. centralizing, we studied the impact of changes in various system parameters on the storage location decision. We observe that minimum demand close to zero, or negative correlation of demands can cause pooling. We also note that expensive storage capacity, increased penetration of wind energy increase preference for localizing. The impact of storage efficiency on storage location trade-offs seems to depend on the demand distribution.

While our analysis shows that localizing is often optimal in our 3-node model, we question the scenario when there may be more than 2 leaf nodes in the system, as is often the case in the real grid. Our preliminary analysis suggests that the benefit of pooling does not increase significantly with n unless the minimum demand is low. In future work, we believe the impact of pooling may be considered in larger grid systems.

Finally, we discuss some of the challenges involved in implementing community energy storage investments across the grid. Firstly, there are challenges involved in building the infrastructure to control multiple storage investments across the grid, in real time, reflecting the latest information. Secondly, implementing community storage will influence revenue streams of storage owners and other market players. New market settlement schemes would be required to make sure no market participant prefers to invest storage differently from the co-ordinated investment strategy that improves overall system efficiency.

2.8. Appendix: Extension to Distributed Intermittent Generation

In this section, we extend the analysis to include distributed intermittent generation, such as wind and solar power, at the leaf nodes. Unlike the generation at node G, intermittent generation does not use any fuel and, thus, incurs negligible operating cost. Its output, depends on factors such as wind speed or solar radiation. If the maximum output is always below the demand at the corresponding leaf node, the analysis in the previous section can be directly applied, with d_t^i representing the *net demand* (demand minus the intermittent generation). In this section, we consider a more general case, where intermittent generation may exceed the demand, resulting in a negative net demand $d_t^i < 0$.

2.8.1 Problem Formulation and Optimal Curtailment Decision

Because intermittent generation has negligible operating cost, it is desirable to store the excess generation for future use. However, if total remaining storage space (in local, central and remote nodes) is insufficient to store the excess generation, some intermittent generation must be curtailed (e.g., rotating solar panels or pitching the blades of a wind power generator to reduce its output). Despite negative demand and non-zero probability of curtailment, our formulation remains unchanged, except for the following modifications.

We allow $\mathbf{d}_t$, to include the distributed intermittent generation, i.e., the net demand d_t^i may be negative at a given node i. Definition (3.2) is unchanged: $q(\Delta\mathbf{s}_t, \mathbf{d}_t) = \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(\Delta s_t^i)\big) + \psi_\alpha(\Delta s_t^G)$, but note that when $d_t^i < 0$, then $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) < 0$ is possible, which means storage is filled up but excess energy still exists. We, therefore, extend the definition of the cost function:

$$c(q) \overset{\text{def}}{=} 0, \qquad \text{for } q < 0. \tag{2.23}$$

$q < 0$ corresponds to curtailment, and $q > 0$ is the generation.

Theorem II.25. *$V_t(\mathbf{s}_t, \mathbf{d}_t)$ satisfies the following recursive equation:*

$$V_t(\mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1}\in\mathcal{A}} \big\{c(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + \gamma\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]\big\}, \qquad t = 1, 2, \ldots, T. \tag{2.24}$$

There exists an optimal solution $\mathbf{s}^*_{t+1}$ *to* (2.24) *and the corresponding production, inventory, and curtailment decisions satisfy the following properties:*
(i) If $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) \geq 0$*, then* $q(\mathbf{s}^*_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \geq 0$*, and it is optimal not to curtail distributed generation.*
(ii) If $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) < 0$*, then* $\mathbf{s}^*_{t+1} = \mathbf{S}$*, and it is optimal not to produce at node* G*, curtail distributed generation by* $\beta^{-1}\big|q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t)\big|$*.*

Theorem II.25 establishes a dynamic program for the problem with curtailment and formalizes the intuition that storing distributed generation is preferred to curtailing it.

2.8.2 Optimal Storage and Transmission Operations

The dynamic program in (2.24) is almost identical to (2.5), except that $c(q)$ is extended per (2.23) and $q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)$ can take negative values. With the absence of non-negativity constraint (2.6), we can verify that Lemmas II.4, II.5, and II.6 continue to hold. Furthermore, Theorems II.9 and II.11 continue to apply whenever $d^i_t \geq 0$, $i = A, B$. Theorem II.25 provides the optimal decision if curtailment is necessary (i.e., net local generation exceeds available storage space, or $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) < 0$). Below we analyze the remaining case, when intermittent generation exceeds the demand at one or both leaf nodes, but no curtailment takes place.

When distributed generation exceeds demand at both leaf nodes, intuitively, it is desirable to store the excess generation at the nearest location. We formalize this after Theorem II.26. The interesting case is when the intermittent generation exceeds demand at one leaf node. Without loss of generality, let $d^A_t \geq 0$ and $d^B_t < 0$. $|d^B_t|$ can be stored at B or transmitted to node A to meet d^A_t. In contrast to Theorem II.9, which suggests that leaf-to-leaf transmission is the least desirable option when net demand at both nodes is positive, it may be cost-effective to use $|d^B_t|$ to serve d^A_t even before using local inventory s^A_t.

As an example, consider the case when the storage efficiency is below the leaf-to-leaf transmission efficiency, i.e., $\alpha \leq \beta^2$. Then using $|d^B_t|$ to serve d^A_t is more cost-effective than withdrawing from local storage s^A_t. If $\alpha > \beta^2$, d^A_t may be satisfied by a combination of remote generation $|d^B_t|$, and local inventory s^A_t, and central production, depending on future demand distribution.[3] We divide our example into two cases: when $\alpha \leq \beta$ and $\alpha > \beta$, if

[3]For example, if future demand at B is expected to be high, storing $|d^B_t|$ at node B may be desirable;

$\alpha \leq \beta$, storage efficiency is below leaf to central transmission efficiency, and hence, $|d_t^B|$ is used before withdrawing from s_t^G. Else, the priority between $|d_t^B|$ and s_t^G would depend on the demand distribution.

Theorem II.26 captures the intuition from the above example: while withdrawing energy, if $\alpha \leq \beta^2$, to satisfy demand at A, it is optimal to use the DG at B, local stored energy at A, G, B in that order. If storing energy, DG in B is first stored at B, then excess energy is filled up in G and the leaf nodes, similar to Theorem II.9. If $\beta^2 < \alpha \leq \beta$, the only difference is that DG at B and locally stored energy at A may be both used simultaneously to satisfy demand at A, but DG at B is still used before stored energy at G. Finally, if $\beta < \alpha$, the preference between DG at B and stored energy at G would also depend on the demand distribution.

We now specify some notation relevant for this case: Let $o_t^B = \beta(|d_t^B| - (S^B - s_t^B)/\alpha)^+$ represent the overage or 'spill over' available at G after filling up the storage at B using local generation . We let $r_t^A = \beta(o_t^B - (S^G - s_t^G)/\alpha)^+$ denote the remaining energy after node B's spill over is used to fill up G, measured at node A. Hence, $(r_t^A - d_t^A)^+$ is the energy available at node A from excess spill over of remote generation, netting local demand. Thus, the storage levels at B, G, and A, achieved by storing the spill overs, constrained by capacity are, $\ddot{s}_t^B = \min\{s_t^B + \alpha|d_t^B|, S^B\}$, $\ddot{s}_t^G = \min\{s_t^G + \alpha o_t^B, S^G\}$, and $\ddot{s}_t^A = \min\{s_t^A + \alpha(r_t^A - d_t^A)^+, S^A\}$ respectively. Note that demand at node A is essentially reduced by r_t^A, i.e., $\ddot{d}_t^A = (d_t^A - r_t^A)^+$ is the perceived demand, while $\ddot{d}_t^B = 0$. Let $\ddot{\mathbf{s}}_t = (\ddot{s}_t^A, \ddot{s}_t^B, \ddot{s}_t^G)$ and $\ddot{\mathbf{d}}_t = (\ddot{d}_t^A, \ddot{d}_t^B)$ be the equivalent state achieved.

Further, we redefine the critical production quantities defined in Section 2.4.2 to refer to the state $(\ddot{\mathbf{s}}_t, \ddot{\mathbf{d}}_t)$, after storing spill overs,

$$\overline{q}_t \stackrel{\text{def}}{=} \sum_{i=A,B} \ddot{d}_t^i/\beta + (S^G - \ddot{s}_t^G)/\alpha, \qquad \overline{\overline{q}}_t \stackrel{\text{def}}{=} \overline{q}_t + \max\{S^A - \ddot{s}_t^A, S^B - \ddot{s}_t^B\}/(\alpha\beta). \quad (2.25)$$

We further reinterpret some of the critical production quantities defined earlier and in-

otherwise, using $|d_t^B|$ to meet d_t^A may be cost-effective. In either case, central production q_t is used before $|d_t^B|$ is used.

troduce two more critical quantities:

$$\widetilde{q}_t \stackrel{\text{def}}{=} (d_t^A/\beta - |d_t^B|\beta)^+, \qquad \underline{q}_t \stackrel{\text{def}}{=} \big(\widetilde{q}_t - s_t^A\alpha/\beta\big)^+, \qquad \underline{\underline{q}}_t \stackrel{\text{def}}{=} \big(\underline{q}_t - s_t^G\alpha\big)^+, \tag{2.26}$$

$$q_t^o \stackrel{\text{def}}{=} \big(d_t^A/\beta - o_t^B\big)^+, \qquad \underline{\ddot{q}}_t \stackrel{\text{def}}{=} (\ddot{d}_t^A - s_t^A\alpha)^+/\beta, \qquad \overline{q}_t \stackrel{\text{def}}{=} \underline{\ddot{q}}_t + (S^G - s_t^G)/\alpha. \tag{2.27}$$

$\widetilde{q}_t$ is the minimum production quantity to meet demand without using any storage. $\underline{q}_t$ is the energy required to satisfy demand at A after using distributed generation at B and local storage at A. $\underline{\underline{q}}_t$ is the minimum production quantity required to meet demand at A without using remote storage. q_t^o is the energy needed to satisfy demand at A along with spill over from B. $\underline{\ddot{q}}_t$ is the energy needed to satisfy demand at A after using local storage and remaining spill over r_t^A. Note that $\underline{\underline{q}}_t \leq \underline{q}_t \leq \underline{\ddot{q}}_t, \widetilde{q}_t \leq q_t^o, \underline{q}_t \leq \overline{q}_t \leq \overline{\overline{q}}_t$ as $-d_t^B \geq o_t^B \geq r_t^B$. With the same interpretation as before, the critical states are $\underline{\underline{\mathbf{s}}}_t = \Big(\big[s_t^A - (d_t^A + d_t^B\beta^2)^+/\alpha\big]^+, \ \big[s_t^B - \big(\frac{d_t^A + d_t^B\beta^2}{\alpha} - s_t^A - s_t^G\beta\big)^+/\beta^2\big]^+, \ \big[s_t^G - \big(\frac{d_t^A + d_t^B\beta^2}{\alpha} - s_t^A\big)/\beta\big]^+\Big)$, $\underline{\underline{\mathbf{s}}}_t = \Big(\big[s_t^A - (d_t^A + d_t^B\beta^2)^+/\alpha\big]^+, s_t^B, \ \big[s_t^G - \big(\frac{d_t^A + d_t^B\beta^2}{\alpha} - s_t^A\big)/\beta\big]^+\Big)$, $\overline{\mathbf{s}}_t = (\ddot{s}_t^A, \ddot{s}_t^B, S^G)$.

Theorem II.26. *Consider state $(\mathbf{s}_t, \mathbf{d}_t)$ such that $d_t^A > 0$ and $d_t^B < 0$. For given feasible production quantity q_t, the optimal inventory decision $\mathbf{s}_{t+1}^*$ are as follows:*[4]

(i) If storage operations are not more efficient than transmission, $\alpha \leq \beta$:

[4]For this theorem, we replace L, R with A, B respectively as the demand is only at node A. The notations are reversed when demand is at node B.

$$
\begin{cases}
\textbf{[withdraw fully from A, DG at B,G, partially from R]:} & \\
\mathbf{s}^*_{t+1} = \big(0, s^B_t - (\underline{\underline{q}}_t - q_t)/\alpha\beta^2,\, 0\big), & \textit{if } \underline{\underline{\underline{q}}}_t \le q_t \le \underline{\underline{q}}_t, \quad (2.28a) \\
\textbf{[withdraw fully from A and DG at B, partially from G]:} & \\
\mathbf{s}^*_{t+1} = \big(0, s^B_t,\, s^G_t - (\underline{q}_t - q_t)/\alpha\beta\big), & \textit{if } \underline{\underline{q}}_t < q_t \le \underline{q}_t, \quad (2.28b) \\
\textit{If } \alpha \le \beta^2, & \\
\textbf{[use DG from B first, then withdraw from A]:} & \\
\quad \mathbf{s}^*_{t+1} = \big(s^A_t - (\widetilde{q}_t - q_t)\beta/\alpha, s^B_t, s^G_t\big), & \textit{if } \underline{q}_t \le q_t \le \widetilde{q}_t, \quad (2.28c) \\
\quad \mathbf{s}^*_{t+1} = \big(s^A_t, s^B_t + (q_t - \widetilde{q}_t)\beta/\alpha, s^G_t\big), & \textit{if } \widetilde{q}_t < q_t \le q^o_t, \quad (2.28d) \\
\textit{Else if, } \beta^2 < \alpha \le \beta^2, & \\
\textbf{[use DG from B partially, withdraw partially from A]:} & \\
\quad \mathbf{s}^*_{t+1} \in L_{AB}(q_t), & \textit{if } \underline{q}_t \le q_t \le q^o_t, \quad (2.28e) \\
\textbf{[store DG at B, store partially at G]:} & \\
\mathbf{s}^*_{t+1} = \big(\ddot{s}^A_t,\, \ddot{s}^B_t,\, S^G - \alpha(\overline{q}_t - q_t)\big), & \textit{if } q^o_t < q_t \le \overline{q}_t, \quad (2.28f) \\
\textbf{[store to full at G, partially at leaf nodes]:} & \\
\mathfrak{B}\big(\overline{\mathbf{s}}_t, \mathbf{S}_t, s^A_t + s^B_t + (q_t - \overline{q}_t)\beta\alpha\big), & \textit{if } \overline{q}_t < q_t \le \overline{\overline{\overline{q}}}_t. \quad (2.28g)
\end{cases}
$$

where

$$
L_{AB}(q_t) \overset{def}{=} \big\{\mathbf{s} \in \mathcal{A}(q) : s^G = s^G_t,\; s^A \in [(s^A_t - d^A_t/\alpha)^+, s^A_t], s^B \in [s^B_t, \ddot{s}^B_t]\big\}.
$$

(ii) If storage operations are more efficient than transmission, $\beta < \alpha \le 1$:

$$
\begin{cases}
\textbf{[same as } (2.28\text{a})\textbf{]} & \textit{if } \underline{\underline{\underline{q_t}}} < q_t \le \underline{\underline{q_t}}, \qquad (2.29\text{a}) \\[1ex]
\textbf{[withdraw fully from A, partially from G and DG at B:]} & \\
\mathbf{s}^*_{t+1} \in L_{BG}(q_t), & \textit{if } \underline{\underline{q_t}} \le q_t \le \underline{q_t}, \qquad (2.29\text{b}) \\[1ex]
\textbf{[withdraw partially from A, G and DG at B:]} & \\
\mathbf{s}^*_{t+1} \in L_{BG}(q_t) \cup L_{AB}(q_t), & \textit{if } \underline{q_t} < q_t \le \underline{\ddot{q}_t}, \qquad (2.29\text{c}) \\[1ex]
\textbf{[store DG at B, store at G:]} & \\
\mathbf{s}^*_{t+1} \in L_{AB}(q_t) \cup L_{AG}(q_t) \cup \overline{L}_{AB}(q_t), & \textit{if } \underline{\ddot{q}_t} < q_t \le \overline{q}_t, \qquad (2.29\text{d}) \\[1ex]
\textbf{[store to full at G, store DG:]} & \\
\mathbf{s}^*_{t+1} \in \ddot{E}_0 & \textit{if } \overline{q}_t < q_t \le \overline{\overline{\overline{q}}}_t. \qquad (2.29\text{e})
\end{cases}
$$

where

$$
\begin{aligned}
L_{BG}(q_t) &\overset{def}{=} \left\{\mathbf{s} \in \mathcal{A}(q_t) : s^A = 0,\ s^B \in [s^B_t, \ddot{s}^B_t], s^G \in [0, s^G_t]\right\}, \\
L_{AG}(q_t) &\overset{def}{=} \left\{\mathbf{s} \in \mathcal{A}(q_t) : s^B = \ddot{s}^B_t,\ s^A \in [(s^A_t - d^A_t/\alpha)^+, s^A_t], s^G \in [s^G_t, S^G]\right\}, \\
\overline{L}_{AB}(q_t) &\overset{def}{=} \left\{\mathbf{s} \in \mathcal{A}(q_t) : s^G = S^G,\ s^A \in [(s^A_t - d^A_t/\alpha)^+, s^A_t], s^B \in [\ddot{s}^B_t, S^B]\right\}, \\
\ddot{E}_0 &\overset{def}{=} \left\{\mathbf{s}_{t+1} \in \mathcal{A}(q_t) : s^G_{t+1} = S^G,\ s^i_{t+1} \in \left[(\ddot{s}^i_t - \ddot{d}^i_t/\alpha)^+,\ S^i\right],\ i = A, B\right\}.
\end{aligned}
$$

When $\alpha = 1$, there is no difference between current period generation and stored energy in terms of costs. Hence, the optimal policy is given directly by Theorem II.11 for the equivalent state $(\ddot{\mathbf{s}}_t, \ddot{\mathbf{d}}_t)$.

When $d^i_t \le 0$, $i = A, B$, note that $q^o_t = \underline{\underline{\underline{q_t}}} = \max\{0, q(-\mathbf{s}_t, \mathbf{d}_t)\} = 0$ is the energy needed to satisfy 'demand' without depleting storage. Recall the intuition that the optimal policy is to store each location's intermittent generation and central production at the closest possible node respectively, moving to the next closest location when storage is full, regardless of the relative values of α and β. Any excess generation is curtailed and no storage releases energy, i.e., $\ddot{\mathbf{s}}_t$ is the optimal policy for $q = q^o_t$. For any production $q \in (q^o_t, \overline{\overline{\overline{q}}}_t]$, the optimal policy is per Theorem II.9 for the equivalent state $(\ddot{\mathbf{s}}_t, \ddot{\mathbf{d}}_t)$ where the critical production quantities are defined per (2.25). To include for spill overs from both nodes, we generalize the notation,

$\ddot{s}_t^i = \min\{s_t^i + \alpha(-d_t^i + r_t^i)^+, S^i\}$, $\ddot{s}_t^G = \min\{s_t^G + \alpha \sum_{i=A,B} o_t^i, S^G\}$, $\ddot{d}_t^i = (d_t^i - r_t^i)^+$ for nodes $i = A, B$ where r_t^B, o_t^A are defined similarly.

In summary, Theorem II.26 describes the structure of the optimal policy for a system with intermittent generation when there's negative demand at one node. Using similar notation, we extend the discussion to the case with negative demand at both nodes. While many of the insights from the previous theorems are retained, we also observe that it may sometimes be optimal to transmit negative demand to the remote node to satisfy demand.

2.9. Appendix: Discussion of Example II.17

Because $B(\mathbf{S})$ is concave in $\mathbf{S}$ and demands are symmetric across leaf nodes, we can restrict our attention to symmetric investment decisions: $\mathbf{S} = (S^L, S^L, S^G)$, where S^L is the storage capacity at each leaf node. Under the optimal investment $\mathbf{S}^*$, the corresponding operating policy has the following properties. First, when $\mathbf{d}_t = (l, l)$, the entire storage capacity $\mathbf{S}^*$ is filled up. Second, when $\mathbf{d}_t = (m, h)$ or (h, m) and $\beta > 0.5$, the entire storage $\mathbf{S}^*$ is emptied to serve the demand. Obviously, if inventory remains in node G or in both leaf nodes, then some storage capacity is never used and the investment cannot be optimal. Furthermore, if the optimal policy empties storage at G and the high-demand node but leaves $s^L > 0$ at the medium-demand node, then an alternative investment $\widetilde{\mathbf{S}}$ with $\widetilde{S}^G = S^G + s^L/\beta$ and $\widetilde{S}^L = S^L - s^L$ has a lower investment cost ($|\mathbf{S}| - |\widetilde{\mathbf{S}}| = s^L(2 - 1/\beta) > 0$ as $\beta > 0.5$) and the same operating cost when storage is emptied to serve demand (m, h) or (h, m).[5]

Hence, we restriction our attention to the policies under which in a high-demand period, the entire stored energy is used to serve demand and the system produces $q_1 = \left(D - \alpha(S^L + \beta S^G + \beta^2 S^L)\right)/\beta$, and in a low-demand period, the system produces $q_2 = 2S^L/(\alpha\beta) + S^G/\alpha$ to fill up all storage. Since the system produces q_1 and q_2 alternately, the long-run discounted production cost is $V(S^L, S^G) \stackrel{\text{def}}{=} \dfrac{c(q_1) + \gamma\, c(q_2)}{1 - \gamma^2}$. The investment problem in (2.7) is equivalent to $\displaystyle\min_{S^L \geq 0,\, S^G \geq 0} p(S^G + 2S^L) + V(S^L, S^G)$.

This convex optimization can be solved using the Karush-Kuhn-Tucker conditions. The solution is characterized by three critical prices: $p_1 > p_2 > p_3 > 0$, where

$$p_1 = \frac{D\alpha(1 + \beta^2)}{\beta^2(1 - \gamma^2)}, \quad p_2 = \frac{4D\gamma\alpha(1 + \beta)}{\beta(1 - \gamma^2)\left(\alpha^4(1 - \beta)(1 + \beta^2) + 4\gamma\right)}, \quad p_3 = \frac{2D\gamma\alpha(1 + \beta)}{\beta(1 - \gamma^2)\left(\alpha^4(1 - \beta) + 2\gamma\right)}.$$

[5] When demand is (m, h) or (h, m), under the alternative investment $\widetilde{\mathbf{S}}$, when demand is (m, h) or (h, m), we empty all storage and the production is $q = -\widetilde{S}^G\alpha + (h - \widetilde{S}^L\alpha)/\beta + \psi_\beta(m - \widetilde{S}^L\alpha) = -S^G\alpha + (h - S^L\alpha)/\beta + \psi_\beta(m - (S^L - s^L)\alpha)$, which is exactly the production under the original policy. Hence, energy generation under the alternative investment remains the same.

The optimal investment decision is

$$\mathbf{S}^* = \begin{cases} \mathbf{0}, & \text{if } p \geq p_1, \\ (A_1(p_1 - p),\, A_1(p_1 - p),\, 0) = \mathbf{S}_l^*, & \text{if } p_2 \leq p < p_1, \\ (A_2(p - p_3),\, A_2(p - p_3),\, A_3(p_2 - p)), & \text{if } p_3 < p < p_2, \\ \big(0,\, 0,\, (p_3 - p)A_4 + A_5\big) = \mathbf{S}_g^*, & \text{if } 0 < p \leq p_3, \end{cases} \tag{2.30}$$

where, $A_1 = \frac{\alpha^2\beta^2(1-\gamma^2)}{\alpha^4(1+\beta^2)^2+4\gamma}$, $A_2 = \frac{\beta(1-\gamma^2)\left(\alpha^4(1-\beta)+2\gamma\right)}{2\alpha^2(1-\beta)(1+\beta)^2\gamma}$, $A_3 = \frac{(1-\gamma^2)\left(\alpha^4(1-\beta)(1+\beta^2)+4\gamma\right)}{2\alpha^2(1-\beta)(1+\beta)^2\gamma}$, $A_4 = \frac{\alpha^2(1-\gamma^2)}{2(\alpha^4+\gamma)}$, $A_5 = \frac{D\alpha^3(1-\beta)}{\beta(\alpha^4(1-\beta)+2\gamma)}$

2.10. Appendix: Proofs and Derivations

Proof of Lemma II.4: The statement of the lemma holds for period T because $V_T(\cdot,\cdot) = 0$. Suppose $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is decreasing and convex in $\mathbf{s}_{t+1}$ for any $\mathbf{d}_{t+1}$.

For any $\mathbf{d}_t$, the objective function in (2.5) is defined on a non-convex set $\{(\mathbf{s}_t, \mathbf{s}_{t+1}) \in \mathcal{A} \times \mathcal{A} : q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \geq 0\}$. We introduce the following auxiliary function, which is an extension of the objective function in (2.5) to a larger convex set:

$$f_t(\mathbf{s}_{t+1}, \mathbf{s}_t, \mathbf{d}_t) \stackrel{\text{def}}{=} c\big([q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)]^+\big) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big], \qquad \text{for } (\mathbf{s}_{t+1}, \mathbf{s}_t) \in \mathcal{A} \times \mathcal{A}.$$

For state $(\mathbf{s}_t, \mathbf{d}_t)$, let $\mathbf{s}_{t+1}^*$ be an optimal decision found by (2.5)-(2.6). Consider any $\widetilde{\mathbf{s}}_t \geq \mathbf{s}_t$. If $\mathbf{s}_{t+1}^*$ is feasible for state $(\widetilde{\mathbf{s}}_t, \mathbf{d}_t)$, i.e., $q(\mathbf{s}_{t+1}^* - \widetilde{\mathbf{s}}_t, \mathbf{d}_t) \geq 0$, then

$$V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq f_t(\mathbf{s}_{t+1}^*, \widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq f_t(\mathbf{s}_{t+1}^*, \mathbf{s}_t, \mathbf{d}_t) = V_t(\mathbf{s}_t, \mathbf{d}_t).$$

If $q(\mathbf{s}_{t+1}^* - \widetilde{\mathbf{s}}_t, \mathbf{d}_t) < 0$ (infeasible), then using $q(\mathbf{S} - \widetilde{\mathbf{s}}_t, \mathbf{d}_t) \geq 0$ and applying the intermediate value theorem, we can find a feasible decision $\widetilde{\mathbf{s}}_{t+1}$ with $\mathbf{s}_{t+1}^* \leq \widetilde{\mathbf{s}}_{t+1} \leq \mathbf{S}$ and $q(\widetilde{\mathbf{s}}_{t+1} - \widetilde{\mathbf{s}}_t, \mathbf{d}_t) = 0$. Thus,

$$V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq \gamma \mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})\big] \leq \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}^*, \mathbf{d}_{t+1})\big] \leq f_t(\mathbf{s}_{t+1}^*, \mathbf{s}_t, \mathbf{d}_t) = V_t(\mathbf{s}_t, \mathbf{d}_t),$$

where the second inequality follows from the induction hypothesis and $\widetilde{\mathbf{s}}_{t+1} \geq \mathbf{s}_{t+1}^*$. Using the intermediate value theorem, we can also show that $\min_{\mathbf{s}_{t+1}\in\mathcal{A}} f_t(\mathbf{s}_{t+1}, \mathbf{s}_t, \mathbf{d}_t) = V_t(\mathbf{s}_t, \mathbf{d}_t)$.

Note that $c\big([q(\Delta\mathbf{s}, \mathbf{d}_t)]^+\big)$ is a composition of convex increasing functions, and thus it is convex in $\Delta\mathbf{s}$. From the induction hypothesis, $\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]$ is convex in $\mathbf{s}_{t+1}$. Therefore, $f_t(\mathbf{s}_{t+1}, \mathbf{s}_t, \mathbf{d}_t)$ is jointly convex in $(\mathbf{s}_{t+1}, \mathbf{s}_t)$ on closed convex set $\mathcal{A} \times \mathcal{A}$. Then, using the theorem on convexity preservation under minimization from Heyman and Sobel (1984, p. 525), we conclude that $V_t(\mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1}\in\mathcal{A}} f_t(\mathbf{s}_{t+1}, \mathbf{s}_t, \mathbf{d}_t)$ is convex in $\mathbf{s}_t$. □

Proof of Lemma II.5: (i) Suppose the statement in part (i) holds for period $t+1$ (it clearly holds for T since $V_T(\cdot,\cdot) = 0$). In this proof, we omit subscript t when no confusion arises.

For period t, we consider states $(\mathbf{s}, \mathbf{d})$ and $(\widetilde{\mathbf{s}}, \mathbf{d})$, with $\mathbf{s} = \widetilde{\mathbf{s}} + (-\delta,\ \beta^2\delta,\ 0)$ for some $\delta > 0$. Let $\mathbf{s}^*_{t+1}$ be the optimal decision for state $(\mathbf{s}, \mathbf{d})$, and denote $\Delta\mathbf{s} = \mathbf{s}^*_{t+1} - \mathbf{s}$ and $q^* = q(\Delta\mathbf{s}, \mathbf{d})$. We now construct a feasible decision for state $(\widetilde{\mathbf{s}}, \mathbf{d})$. Consider three cases:

Case 1: $\widetilde{\mathbf{s}} + \Delta\mathbf{s} \in \mathcal{A}$. In this case, a feasible decision for $(\widetilde{\mathbf{s}}, \mathbf{d})$ is to produce q^* and change inventory to $\widetilde{\mathbf{s}}_{t+1} = \widetilde{\mathbf{s}} + \Delta\mathbf{s}$. Then, $\widetilde{\mathbf{s}}_{t+1} = \mathbf{s}^*_{t+1} - (-\delta,\ \beta^2\delta,\ 0)$, and the induction hypothesis leads to:

$$V_t(\widetilde{\mathbf{s}}, \mathbf{d}) \le c(q^*) + \gamma\mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})\big] \le c(q^*) + \gamma\mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})\big] = V_t(\mathbf{s}, \mathbf{d}).$$

Case 2: $\widetilde{\mathbf{s}} + \Delta\mathbf{s} \notin \mathcal{A}$. This condition means $\widetilde{\mathbf{s}} + \Delta\mathbf{s} = \widetilde{\mathbf{s}} + \mathbf{s}^*_{t+1} - \mathbf{s} = \mathbf{s}^*_{t+1} - (-\delta,\ \beta^2\delta,\ 0) \notin \mathcal{A}$, which can be written as $s^{A*}_{t+1} + \delta > S^A$ or $s^{B*}_{t+1} - \beta^2\delta < 0$ or both inequalities hold.

To identify a feasible inventory decision for $(\widetilde{\mathbf{s}}, \mathbf{d})$, we consider $\widetilde{\mathbf{s}}_{t+1} = \mathbf{s}^*_{t+1} - (-\widetilde{\delta},\ \beta^2\widetilde{\delta},\ 0)$, where $\widetilde{\delta} = \min\big\{S^A - s^{A*}_{t+1},\ s^{B*}_{t+1}/\beta^2\big\}$. Define $\Delta\widetilde{\mathbf{s}} = \widetilde{\mathbf{s}}_{t+1} - \widetilde{\mathbf{s}}$ and $\widetilde{q} = q(\Delta\widetilde{\mathbf{s}}, \mathbf{d})$. There are two subcases: $\widetilde{q} < 0$ and $\widetilde{q} \ge 0$. If $\widetilde{q} < 0$, then applying the intermediate value theorem, we can find a feasible decision $\widetilde{\mathbf{s}}'_{t+1}$ with $\widetilde{\mathbf{s}}_{t+1} \le \widetilde{\mathbf{s}}'_{t+1} \le \mathbf{S}$ and $q(\widetilde{\mathbf{s}}'_{t+1} - \widetilde{\mathbf{s}}, \mathbf{d}) = 0$. Then,

$$V_t(\widetilde{\mathbf{s}}, \mathbf{d}) \le \gamma\mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}'_{t+1}, \mathbf{d}_{t+1})\big] \le \gamma\mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})\big] \le \gamma\mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})\big] \le V_t(\mathbf{s}, \mathbf{d}),$$

where the second inequality uses Lemma II.4 and the third inequality is due to the induction hypothesis.

If $\widetilde{q} \ge 0$, then the feasible decision is $\widetilde{\mathbf{s}}_{t+1} = \mathbf{s}^*_{t+1} - (-\widetilde{\delta},\ \beta^2\widetilde{\delta},\ 0)$ with $\widetilde{\delta}$ defined above.

If we can show $\widetilde{q} \le q^*$, then the feasibility of $\widetilde{\mathbf{s}}_{t+1}$ and the induction hypothesis lead to the intended result:

$$V_t(\widetilde{\mathbf{s}}, \mathbf{d}) \le c(\widetilde{q}) + \gamma \mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})\big] \le c(q^*) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})\big] = V_t(\mathbf{s}, \mathbf{d}). \quad (2.31)$$

The rest of the proof shows $\widetilde{q} \le q^*$. The choice of $\widetilde{\delta}$ gives $\widetilde{s}^A_{t+1} = S^A$ or $\widetilde{s}^B_{t+1} = 0$, which leads to

$$\Delta\widetilde{s}^A = \widetilde{s}^A_{t+1} - \widetilde{s}^A \ge 0 \quad \text{or} \quad \Delta\widetilde{s}^B = \widetilde{s}^B_{t+1} - \widetilde{s}^B \le 0. \quad (2.32)$$

Furthermore, $\widetilde{\delta} < \delta$. Let $\varepsilon = \delta - \widetilde{\delta}$. Then, by definitions, we have

$$\Delta\mathbf{s} - \Delta\widetilde{\mathbf{s}} \;=\; \mathbf{s}^*_{t+1} - \widetilde{\mathbf{s}}_{t+1} - \mathbf{s} + \widetilde{\mathbf{s}} \;=\; (-\widetilde{\delta},\, \beta^2\widetilde{\delta},\, 0) - (-\delta,\, \beta^2\delta,\, 0) \;=\; (\varepsilon,\, -\beta^2\varepsilon,\, 0). \quad (2.33)$$

That is, $\Delta s^A = \Delta\widetilde{s}^A + \varepsilon$, $\Delta s^B = \Delta\widetilde{s}^B - \beta^2\varepsilon$, and $\Delta s^G = \Delta\widetilde{s}^G$. Using the definition in (3.2), we have

$$\begin{aligned} q^* - \widetilde{q} &= \psi_\beta\big(d^A + \psi_\alpha(\Delta s^A)\big) - \psi_\beta\big(d^A + \psi_\alpha(\Delta\widetilde{s}^A)\big) - \Big[\psi_\beta\big(d^B + \psi_\alpha(\Delta\widetilde{s}^B)\big) - \psi_\beta\big(d^B + \psi_\alpha(\Delta s^B)\big)\Big] \\ &\ge \beta\big[\psi_\alpha(\Delta\widetilde{s}^A + \varepsilon) - \psi_\alpha(\Delta\widetilde{s}^A)\big] - \beta^{-1}\big[\psi_\alpha(\Delta\widetilde{s}^B) - \psi_\alpha(\Delta\widetilde{s}^B - \beta^2\varepsilon)\big] \;\equiv\; \Gamma, \end{aligned} \quad (2.34)$$

where the inequality is because $\psi_\beta(u)$ increases in u with a slope of either β or β^{-1}. Now consider the cases under the two conditions derived in (3.28):

- If $\Delta\widetilde{s}^A \ge 0$, then $\Gamma = \beta\alpha^{-1}\varepsilon - \beta^{-1}\big[\psi_\alpha(\Delta\widetilde{s}^B) - \psi_\alpha(\Delta\widetilde{s}^B - \beta^2\varepsilon)\big] \;\ge\; \beta\alpha^{-1}\varepsilon - \beta^{-1}\alpha^{-1}\beta^2\varepsilon \;=\; 0$.
- If $\Delta\widetilde{s}^B \le 0$, then $\Gamma = \beta\big[\psi_\alpha(\Delta\widetilde{s}^A + \varepsilon) - \psi_\alpha(\Delta\widetilde{s}^A)\big] - \beta^{-1}\alpha\beta^2\varepsilon \;\ge\; \beta\alpha\varepsilon - \beta\alpha\varepsilon \;=\; 0$.

Hence, $q^* \ge \widetilde{q}$ and the result in (3.27) holds.

(ii) For the case $\mathbf{s} - \widetilde{\mathbf{s}} = (-\delta,\, 0,\, \beta\delta)$, the proof follows the same lines as in part (i), except that the inventory increases at node G instead of node B. The other cases can be proved similarly. □

Proof of Lemma II.6: Because $q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)$ increases in $\mathbf{s}_{t+1}$ and $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ decreases in $\mathbf{s}_{t+1}$ (Lemma II.4), the subproblem in (2.9) is equivalent to the following problem with an inequality constraint:

$$W_t(q_t, \mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1} \in \mathcal{A}} \big\{ \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big] : q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \le q_t \big\}. \tag{2.35}$$

Because the feasible set in (2.35) expands as q_t increases, $W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$ decreases in q_t.

To prove convexity, note that the set $\mathcal{Y} \stackrel{\text{def}}{=} \{(q_t, \mathbf{s}_{t+1}) : q_t \in \mathcal{Q}(\mathbf{s}_t, \mathbf{d}_t),\ \mathbf{s}_{t+1} \in \mathcal{A},\ q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \le q_t\}$ is a closed convex set. From Lemma II.4, the objective $\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]$ in (2.35) is convex in $\mathbf{s}_{t+1}$, and thus, it is also convex on the set $\mathcal{Y}$. Using the theorem on convexity preservation under minimization from Heyman and Sobel (1984, p. 525), we conclude $W_t(q_t, \mathbf{s}_t, \mathbf{d}_t)$ is convex in q_t. □

Proof of Lemma II.7: Under (2.12) and $S^A = S^B$, the expected cost-to-go function is symmetric with respect to s^A_{t+1} and s^B_{t+1}. That is, $\mathsf{E}_t\big[V_{t+1}\big((s^A_{t+1}, s^B_{t+1}, s^G_{t+1}), \mathbf{d}_{t+1}\big)\big] = \mathsf{E}_t\big[V_{t+1}\big((s^B_{t+1}, s^A_{t+1}, s^G_{t+1}), \mathbf{d}_{t+1}\big)\big]$.

For any $s^A, s^B \in [0, S^A]$ satisfying $s^A + s^B = z$, we have

$$\begin{aligned}\mathsf{E}_t\big[V_{t+1}\big((s^A, s^B, s^G), \mathbf{d}_{t+1}\big)\big] &= \frac{1}{2}\Big(\mathsf{E}_t\big[V_{t+1}\big((s^A, s^B, s^G), \mathbf{d}_{t+1}\big)\big] + \mathsf{E}_t\big[V_{t+1}\big((s^B, s^A, s^G), \mathbf{d}_{t+1}\big)\big]\Big) \\ &\ge \mathsf{E}_t\big[V_{t+1}\big((z/2, z/2, s^G), \mathbf{d}_{t+1}\big)\big],\end{aligned}$$

where the inequality is due to the convexity of $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ with respect to $\mathbf{s}_{t+1}$ (Lemma II.4). Hence, $\big(z/2, z/2, s^G\big)$ is a minimizer to the problem: $\min_{\mathbf{s}_{t+1} \in \mathcal{A}} \big\{ \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big] : s^A_{t+1} + s^B_{t+1} = z,\ s^G_{t+1} = s^G \big\}$. Therefore, $\mathbf{b}(z; s^G, \mathbf{d}_t) = \big(z/2, z/2, s^G\big)$. □

Proof of Theorem II.8: As $S^G = 0$ and $S^A = S^B$, the feasible inventory set $\mathcal{A}$ is the square region shown in Figure 2.12. Also shown in the figure is the piecewise-linear iso-production curve $\mathcal{A}(q_t)$:

$$\mathcal{A}(q_t) = \Big\{ \mathbf{s}_{t+1} \in \mathcal{A} : \sum_{i=A,B} \psi_\beta\big(d^i_t + \psi_\alpha(s^i_{t+1} - s^i_t)\big) = q_t \Big\}. \tag{2.36}$$

In (2.35), for any given q_t, we minimize a convex function over a convex set. Thus, to prove the solution prescribed in Theorem II.8 is optimal for (2.35) and hence for (2.9), we only need to show that it achieves a local minimum on the set $\mathcal{A}(q_t)$. Below, we show that the objective value increases when $\mathbf{s}_{t+1}$ deviates from the prescribed $\mathbf{s}^*_{t+1}$; the deviation is along $\mathcal{A}(q_t)$.

Figure 2.12: Optimal $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$ and the deviation from it along $\mathcal{A}(q_t)$, for various q_t

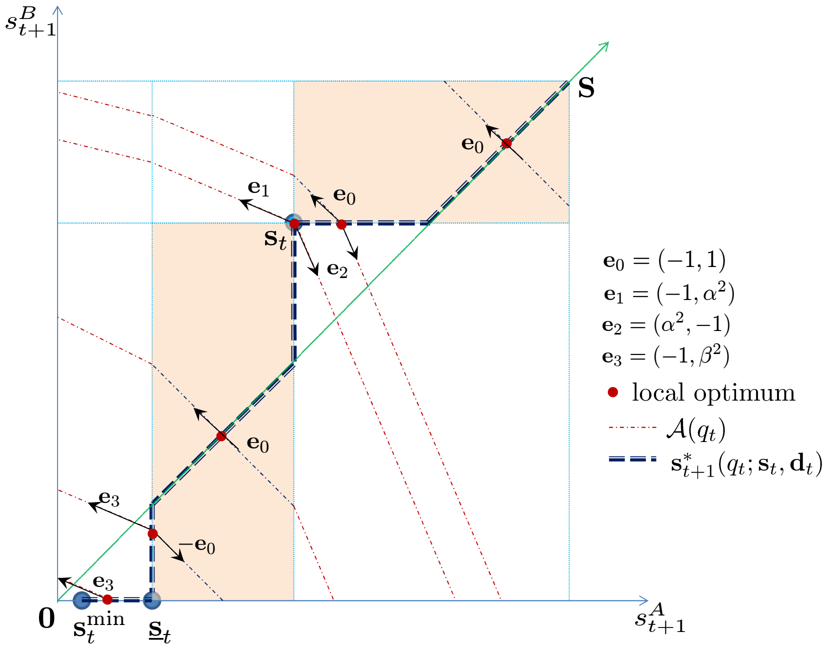


Figure 2.12 shows four possible directions to move along $\mathcal{A}(q_t)$. Moving along $\mathbf{e}_0$ increases the objective value by the definition of the balance curve (2.11). It follows directly from Lemma II.5(i) that the objective increases along $\mathbf{e}_3 = (-1, \beta^2)$. The objective increases along $\mathbf{e}_1$ because

$$V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1}) \leq V_{t+1}(\mathbf{s}_{t+1} + (-\delta, \beta^2\delta, 0), \mathbf{d}_{t+1}) \leq V_{t+1}(\mathbf{s}_{t+1} + (-\delta, \alpha^2\delta, 0), \mathbf{d}_{t+1}), \quad (2.37)$$

where the first inequality is due to Lemma II.5(i) and the second inequality follows from $\alpha \leq \beta$ and the monotonicity in Lemma II.4. The proof for $\mathbf{e}_2$ is similar.

Although the above proof is for one case of the optimal decision $\mathbf{s}^*_{t+1}(q_t, \mathbf{s}_t, \mathbf{d}_t)$, the proofs for other cases are similar. A general proof for the general case with $S^G > 0$ and asymmetric leaf nodes will be presented in the proof for Theorem II.9. □

Proofs of Theorems II.9 and II.11: Overview and Preliminaries

These two theorems provide structural properties for the optimal solution to (2.9), which is equivalent to (2.35), where we minimize a convex function over a convex set. Thus, to prove a solution is optimal for (2.35) and hence for (2.9), we only need to show that it achieves a local minimum in (2.9).

Using the definition from (3.2), the set of feasible actions for (2.9) is

$$\mathcal{A}(q_t) = \Big\{ \mathbf{s}_{t+1} \in \mathcal{A} : \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(s_{t+1}^i - s_t^i)\big) + \psi_\alpha(s_{t+1}^G - s_t^G) = q \Big\}, \tag{2.38}$$

where $\psi_\alpha(s_{t+1}^G - s_t^G)$ is piecewise linear in s_{t+1}^G with slopes α and α^{-1} (slope changes at $s_{t+1}^G = s_t^G$), and $\psi_\beta\big(d_t^i + \psi_\alpha(s_{t+1}^i - s_t^i)\big)$ is piecewise linear in s_{t+1}^i with slopes $\alpha\beta$, $\alpha\beta^{-1}$, and $\alpha^{-1}\beta^{-1}$ (slope changes at $s_{t+1}^i = s_t^i - d_t^i/\alpha$ and $s_{t+1}^i = s_t^i$), for $i = A, B$. If $s_t^i - d_t^i/\alpha \le 0$, the segment with slope $\alpha\beta$ does not exist. The iso-production surface $\mathcal{A}(q_t)$ is thus a piecewise linear surface in $\mathcal{A}$.

To prove local minimum, we show that the objective value $\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]$ in (2.9) increases as $\mathbf{s}_{t+1}$ deviates from the prescribed $\mathbf{s}_{t+1}^*$ (or the set containing $\mathbf{s}_{t+1}^*$). We prove this using two steps:

Step 1. Find all faces of $\mathcal{A}(q_t)$ that intersect the prescribed $\mathbf{s}_{t+1}^*$ (or the set containing $\mathbf{s}_{t+1}^*$), and identify the edges formed by these faces.

Step 2. Prove that the objective value $\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]$ increases when $\mathbf{s}_{t+1}$ moves away from $\mathbf{s}_{t+1}^*$ (or the set containing $\mathbf{s}_{t+1}^*$) in the direction parallel to any of the edges identified in Step 1. (We in fact prove a stronger result that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases for any realization of $\mathbf{d}_{t+1}$.)

Steps 1 and 2 lead to the optimality of the prescribed $\mathbf{s}_{t+1}^*$, because from $\mathbf{s}_{t+1}^*$ (or the set containing $\mathbf{s}_{t+1}^*$) we can reach any $\mathbf{s}_{t+1}$ in any face identified in Step 1 by taking at most two moves parallel to the edges of the face; both moves increase the objective value, as shown in Step 2.

Instead of repeating Step 1 for every case, we first identify all possible faces and edges of $\mathcal{A}(q_t)$. We use k to index the faces of $\mathcal{A}(q_t)$. Face k satisfies (2.38), which can be expressed as

$$\mathbf{a}_k \cdot \mathbf{s}_{t+1} \equiv a_k^A s_{t+1}^A + a_k^B s_{t+1}^B + a_k^G s_{t+1}^G = b_k, \qquad \text{for } \mathbf{s}_{t+1} \in \text{face } k,$$

where a_k^A and a_k^B have three possible values ($\alpha\beta$, $\alpha\beta^{-1}$, and $\alpha^{-1}\beta^{-1}$) and a_k^G is either α or α^{-1}. These values are exactly the slopes discussed after (2.38).

For the part of $\mathcal{A}(q_t)$ with $s_{t+1}^G > s_t^G$ (storing energy at G), we have $a_k^G = \alpha^{-1}$, while a_k^A and a_k^B have 9 combinations, as shown in Figure 2.13(a). Thus, this part of $\mathcal{A}(q_t)$ can have up to 9 faces, which are labeled in clockwise order, with a center face 0. Figure 2.13(a) also illustrates the contours of $\mathcal{A}(q_t)$ when it contains all nine faces (each contour line represents a fixed s_{t+1}^G level, with lower-left being the highest s_{t+1}^G). In general, $\mathcal{A}(q_t)$ contains only a subset of these faces.

The other part of $\mathcal{A}(q_t)$ with $s_{t+1}^G < s_t^G$ consists of faces with $a_k^G = \alpha$. These faces are shown in Figure 2.13(b) and labeled in the same order. Note that the lower-left area cannot be part of $\mathcal{A}(q_t)$ because $s_{t+1}^G < s_t^G$ implies that $\sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(s_{t+1}^i - s_t^i)\big) > 0$ due to (2.38).

The boundary between the above two parts of $\mathcal{A}(q_t)$ has $s_{t+1}^G = s_t^G$ (meaning no storage operations at G), which can be formally written as

$$H(q_t) \stackrel{\text{def}}{=} \Big\{(s_{t+1}^A, s_{t+1}^B, s_t^G) \in \mathcal{A}(q_t) : \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(s_{t+1}^i - s_t^i)\big) = q_t\Big\}. \tag{2.39}$$

When $H(q_t)$ is non-empty, each segment of $H(q_t)$ is an edge formed by faces k and k', for some k. The projection of $H(q_t)$ onto (s_{t+1}^A, s_{t+1}^B) plane is exactly a contour line.

Let $\mathbf{e}_{ij}$ denote a direction parallel to the edge formed by faces i and j, shown as arrows in Figure 2.13. For any two adjacent faces i and j, their coefficient vectors $\mathbf{a}_i$ and $\mathbf{a}_j$ differ in only one element, which gives a simple method to derive $\mathbf{e}_{ij}$. Use $\mathbf{e}_{45}$ as an example. Comparing $\mathbf{a}_4$ and $\mathbf{a}_5$, we see $a_4^B = a_5^B = \alpha\beta$ and $a_4^G = a_5^G = \alpha^{-1}$. Thus, within faces 4 and 5, if we hold s_{t+1}^A constant and reduce s_{t+1}^B by α^{-1}, then s_{t+1}^G must increase by $\alpha\beta$. Thus, $\mathbf{e}_{45} = (0, -\alpha^{-1}, \alpha\beta)$ or scaled to $(0, -1, \alpha^2\beta)$. We scale $\mathbf{e}_{ij}$ such that it contains -1 as an element, which facilitates physical interpretation for the direction.

Having identified all possible faces and edges of $\mathcal{A}(q_t)$, we next prove a lemma on how the value function change along these directions. (Each direction is parallel to an edge; see Figure 2.13.)

Lemma II.27. *For any* $\mathbf{d}_{t+1}$*, we have* *(i)* $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ *increases as* $\mathbf{s}_{t+1}$ *moves along* $\mathbf{e}_{55'}$, $\mathbf{e}_{77'}$.

(ii) $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ *increases as* $\mathbf{s}_{t+1}$ *moves along* $\mathbf{e}_{12}$, $\mathbf{e}_{18}$, $\mathbf{e}_{23}$, $\mathbf{e}_{34}$, $\mathbf{e}_{0'3'}$, $\mathbf{e}_{0'7'}$, $\mathbf{e}_{0'1'}$, $\mathbf{e}_{0'5'}$, $\mathbf{e}_{4'5'}$, $\mathbf{e}_{7'8'}$.

(iii) $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ *increases as* $\mathbf{s}_{t+1}$ *moves along* $\mathbf{e}_{45}$, $\mathbf{e}_{56}$, $\mathbf{e}_{67}$, $\mathbf{e}_{78}$, $\mathbf{e}_{1'2'}$, $\mathbf{e}_{1'8'}$, $\mathbf{e}_{2'3'}$, $\mathbf{e}_{3'4'}$, $\mathbf{e}_{44'}$, $\mathbf{e}_{88'}$.

(iv) If $\alpha \leq \beta$, *then* $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ *increases as* $\mathbf{s}_{t+1}$ *moves along* $\mathbf{e}_{01}$, $\mathbf{e}_{03}$, $\mathbf{e}_{05}$, $\mathbf{e}_{07}$, $\mathbf{e}_{11'}$, $\mathbf{e}_{33'}$.

Proof of Lemma II.27: Parts (i) and (ii) follow directly from Lemma II.5(i) and (ii), respectively. The proofs for parts (iii) and (iv) are similar to (2.37) in the proof for Theorem II.8; the proofs rely on Lemmas II.4 and II.5, and use $\alpha \leq 1$ for proving part (iii) and $\alpha \leq \beta$ for proving part (iv). $\square$

Each case of Theorems II.9 and II.11 involves a subset of the faces and edges from Figure 2.13. We next prove the theorems based on Lemma II.27.

Proof of Theorem II.9: For $q_t \in [\underline{\underline{\underline{q}}}_t, \underline{\underline{q}}_t)$, the theorem states that the remote storage is used. If using s_t^B to serve d_t^A, we must have $s_t^B > \frac{d_t^B}{\alpha}$ and $s_t^A + s_t^G \beta < \frac{d_t^A}{\alpha}$. In this case, the theorem implies that $s_{t+1}^{G*} = s_{t+1}^{A*} = 0$ and $s_{t+1}^{B*} < s_t^B - d_t^B/\alpha$. The solution $\mathbf{s}_{t+1}^*$ is on face $5'$ as illustrated in Figure 2.14(a). Because $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along directions $\mathbf{e}_{55'}$ and $\mathbf{e}_{4'5'}$ (Lemma II.27(i), (ii)), $\mathbf{s}_{t+1}^*$ is locally optimal. The proof for the case of using s_t^A to serve d_t^B is similar. Additionally, for $q_t = \underline{\underline{q}}_t$, although remote storage is not used, we have $\mathbf{s}_{t+1}^* = \underline{\underline{\mathbf{s}}}_t$ and the proof is essentially the same.

There are a few degenerative cases to consider: If $s_t^G = 0$, then $\mathbf{s}_{t+1}^*$ is on face 5 instead of $5'$, but it is still locally optimal as $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along $\mathbf{e}_{45}$ (Lemma II.27(iii)). If $d_t^A = 0$, then $\mathbf{s}_{t+1}^*$ is on face $4'$, but still locally optimal as $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along $\mathbf{e}_{44'}$ (Lemma II.27(iii)). If $q_t = \underline{\underline{\underline{q}}}_t$, $\mathcal{A}(q_t)$ shrinks to a point $\underline{\underline{\underline{\mathbf{s}}}}_t$, which is the only choice for $\mathbf{s}_{t+1}^*$.

In proving the remaining portion of the theorem, all degenerative cases (i.e., when some faces don't exist) can be similarly proven, but due to the length of the proof we omit the details.

For $q_t \in (\underline{\underline{q}}_t, \underline{q}_t)$, per theorem, $s_{t+1}^{G*} < s_t^G$ and either $(s_{t+1}^{A*}, s_{t+1}^{B*}) = (0, s_t^B - d_t^B/\alpha)$ or $(s_{t+1}^{A*}, s_{t+1}^{B*}) = (s_t^A - d_t^A/\alpha, 0)$. For case of $(s_{t+1}^{A*}, s_{t+1}^{B*}) = (0, s_t^B - d_t^B/\alpha)$, $\mathbf{s}_{t+1}^*$ is on the intersection of faces $0'$ and $5'$, shown in Figure 2.14(b). It is locally optimal because $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases as $\mathbf{s}_{t+1}$ moves along $\mathbf{e}_{0'5'}$, $\mathbf{e}_{4'5'}$, and $\mathbf{e}_{0'7'}$. The proof is parallel for the case of

$(s_{t+1}^{A*}, s_{t+1}^{B*}) = (s_t^A - d_t^A/\alpha, 0)$.

For $q_t \in (\underline{q}_t, q_t^o)$, per theorem, $\mathbf{s}_{t+1}^*$ is on $\mathfrak{B}(\underline{\mathbf{s}}_t, \mathbf{s}_t, z)$, implying $s_{t+1}^{G*} = s_t^G$ and $s_{t+1}^{i*} \in [(s_t^i - d_t^i/\alpha)^+, s_t^i]$. Hence, $\mathbf{s}_{t+1}^*$ lies on the edge formed by faces 0 and 0′, shown in Figure 2.14(c)-(d) as the dashed line segment. By the definition of $\mathfrak{B}(\underline{\mathbf{s}}_t, \mathbf{s}_t, z)$, $\mathbf{s}_{t+1}^*$ minimizes $\mathsf{E}_t[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})]$ within this line segment. To show that $\mathbf{s}_{t+1}^*$ is a local minimizer for $\mathsf{E}_t[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})]$ within $\mathcal{A}(q_t)$ we only need to show that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases as $\mathbf{s}_{t+1}$ moves along the directions shown in Figure 2.14(c)-(d), but this follows immediately from Lemma II.27. Additionally, for $q = \underline{q}_t$, the line segment in Figure 2.14(c) shrinks to the point intersecting face 6, and along the additional directions $\mathbf{e}_{56}$ and $\mathbf{e}_{67}$, $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ still increases. Similarly, for $q = q_t^o$, the line segment in Figure 2.14(d) shrinks to the point intersecting face 2′, and along the additional directions $\mathbf{e}_{1'2'}$ and $\mathbf{e}_{2'3'}$, $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ still increases.

For $q_t \in (q_t^o, \overline{q}_t)$, per theorem, $\mathbf{s}_{t+1}^* = \big(s_t^A, s_t^B, s_t^G + \alpha(q - q_t^o)\big)$, which is exactly the intersection of faces 0, 1 , 2, and 3. Lemma II.27 asserts that $V_{t+1}(\mathbf{s}_t, \mathbf{d}_t)$ increases along directions $\mathbf{e}_{01}$, $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, and $\mathbf{e}_{03}$, which ensures the local optimality of $\mathbf{s}_{t+1}^*$.

For $q_t \in (\overline{q}_t, \overline{\overline{\overline{q}}}_t)$, per theorem, $\mathbf{s}_{t+1}^*$ is on $\mathfrak{B}(\overline{\mathbf{s}}_t, \mathbf{S}, z)$, implying $s_{t+1}^{G*} = S^G$ and $s_{t+1}^{i*} \geq s_t^i$, $i = A, B$. Hence, $\mathbf{s}_{t+1}^*$ is on the top edge (the edge with $s_{t+1}^G = S^G$) of face 2. The proof for this case as well the case of $q_t = \overline{q}_t$ is similar to the proof for $q_t \in (\underline{q}_t, q_t^o)$. □

Proof of Theorem II.11: Regardless of the relation of α and β, the surface $\mathcal{A}(q_t)$ has the structures shown in Figure 2.13. Thus, we use the same graphs to illustrate the optimal decisions. Further, the proof for the case of $q_t \in [\underline{\underline{\underline{q}}}_t, \underline{q}_t]$ remains same as Theorem II.9 because $V_{t+1}(\mathbf{s}_t, \mathbf{d}_t)$ increases along all directions when $q_t \in [\underline{\underline{\underline{q}}}_t, \underline{q}_t]$ irrespective of the relative value of α and β.

In Theorem II.9, the condition $\alpha \leq \beta$ (thus $\alpha^2/\beta \leq \beta$) is crucial for $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ to increase along $\mathbf{e}_{01} = \mathbf{e}_{05} = (-1,\, 0,\, \alpha^2/\beta)$ or $\mathbf{e}_{03} = \mathbf{e}_{07} = (0,\, -1,\, \alpha^2/\beta)$; see Figure 2.13. When $\alpha > \beta$, however, this result may not hold. In fact, if $\alpha = 1$, these directions become (scaled by β):

$$\mathbf{e}_{01} = \mathbf{e}_{05} = (-\beta,\, 0,\, 1) \quad \text{and} \quad \mathbf{e}_{03} = \mathbf{e}_{07} = (0,\, -\beta,\, 1), \qquad \text{if } \alpha = 1. \tag{2.40}$$

Lemma II.5 confirms that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ *decreases* when $\mathbf{s}_{t+1}$ moves in the directions in (2.40).

When $\beta < \alpha < 1$, the objective value may increase or decrease as $\mathbf{s}_{t+1}$ moves along the directions in (2.40). Thus, we need to search more extensively than the cases of $\alpha \leq \beta$ and $\beta < \alpha = 1$.

Equations (2.17b) and (2.18b): Case of $\underline{q}_t < q_t \leq \underline{\overline{q}}_t$, illustrated in Figure 2.15(a). In Figure 2.15(a), the dashed line contains $\mathbf{s}^*_{t+1}$ when $\alpha \leq \beta$ (see Figure 2.6(a)). However, when $\beta < \alpha = 1$, moving along $\mathbf{e}_{05}$ and $\mathbf{e}_{07}$ further reduces $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$. Thus, $\mathbf{s}^*_{t+1}$ is the intersection of faces 0, 5, 6, and 7. That is, $\mathbf{s}^*_{t+1} = \big((s^A_t - d^A_t)^+,\, (s^B_t - d^B_t)^+,\, s^G_t + q_t - \underline{q}_t\big)$. When $\beta < \alpha < 1$, the solution $\mathbf{s}^*_{t+1}$ is in face 0, defined as F_t in the theorem. The exact location depends on the demand distribution.

Equations (2.17c) and (2.18c): Case of $\underline{\overline{q}}_t < q_t < \overline{q}_t$, illustrated in Figure 2.15(b) and (c).
As q_t increases, the entire surface $\mathcal{A}(q_t)$ rises. As $q_t > \underline{\overline{q}}_t$, face 6 no longer exists (cf. Figure 2.15(a)) and we have the situation in Figure 2.15(b). The dashed line contains $\mathbf{s}^*_{t+1}$ if $\alpha \leq \beta$. When $\beta < \alpha = 1$, moving along $\mathbf{e}_{01}$ and $\mathbf{e}_{03}$ reduces the objective value, and thus $\mathbf{s}^*_{t+1}$ is on the top edge of face 0 (i.e., the edge with $s^G_{t+1} = S^G$). If $\beta < \alpha < 1$, then $\mathbf{s}^*_{t+1}$ is in face 0.

As q_t increases even further, faces 5 and 7 no longer exist and the situation is illustrated in Figure 2.15(c). We need to consider not only the top edge of face 0, but also the top edges of faces 1 and 3, because moving along $\mathbf{e}_{11'}$ or $\mathbf{e}_{33'}$ may reduce the objective value. These three edges form the set E_t defined in the theorem. We do not need to consider the entire faces 1 and 3 because the objective value increases as $\mathbf{s}_{t+1}$ moves into faces 1 or 3 along directions $\mathbf{e}_{12}$ or $\mathbf{e}_{23}$.

Therefore, when $\beta < \alpha < 1$, the solution $\mathbf{s}^*_{t+1}$ belongs to $F_t \cup E_t$. When $\beta < \alpha = 1$, $\mathbf{s}^*_{t+1}$ belongs to E_t, which becomes a straight line segment because $\mathbf{e}_{11'} = (-1, 1, 0)$ and $\mathbf{e}_{33'} = (1, -1, 0)$. Minimizing $\mathsf{E}_t[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})]$ within this line segment gives $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$ per (2.15), which is the result in (2.17c) in Theorem II.11.

Equations (2.17c) and (2.18d): Case of $\overline{q}_t \leq q_t < \overline{\overline{\overline{q}}}_t$, illustrated in Figure 2.15(d). When $q_t \geq \overline{q}_t$, the surface $\mathcal{A}(q_t)$ rises such that face 0 does not exist, illustrated in Figure 2.15(d). Thus, $\mathbf{s}_{t+1}^*$ is on the top edges of faces 1, 2, and 3, which is the set E_t in the theorem. Further, when $\alpha = 1$, E_t is a straight line segment and thus $\mathbf{s}_{t+1}^*$ can be expressed using $\mathfrak{B}(\mathbf{x}, \mathbf{y}, z)$. $\square$

Proof of Theorem II.26:

The proof of Theorem II.26 follows exactly along the lines of Theorems II.9 and II.11. We first identify all faces of $\mathcal{A}(q_t)$ and edges, as in Figure 2.16 (cf. Figure 2.13). Here, a_k^A and a_k^B have three possible values ($\alpha\beta$, $\alpha^{-1}\beta$, and $\alpha^{-1}\beta^{-1}$), and a_k^G is either α or α^{-1}. For the part of $\mathcal{A}(q_t)$ with $s_{t+1}^G > s_t^G$ (storing in G), we have $a_k^G = \alpha^{-1}$, while a_k^A and a_k^B have 9 combinations, as shown in Figure 2.16(a). The other part of $\mathcal{A}(q_t)$ with $s_{t+1}^G < s_t^G$ consists of faces with $a_k^G = \alpha$. Note that the lower-left area cannot be part of $\mathcal{A}(q_t)$ because $s_{t+1}^G < s_t^G$ and $s_{t+1}^A < (s_t^A - d_t^A/\alpha)$ implies that $s_{t+1}^B \geq s_t^B + |d_t^B|\alpha$ due to (2.38).

Next, we identify that directions in which $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing and show local optimality for each case of the Theorem.

Applying similar logic as Lemma II.27, we make the following observations to the directions referenced in Figure 2.16: For given state $(\mathbf{s}_t, \mathbf{d}_t)$, consider any $\mathbf{s}_{t+1} \in \mathcal{A}(q_t)$, we have:

(i) $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edges $\mathbf{e}_{55'},\ \mathbf{e}_{44'},\ \mathbf{e}_{88'},\ \mathbf{e}_{33'}$ from Lemma II.5(i) and Lemma II.4.

(ii) $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edges, $\mathbf{e}_{12},\ \mathbf{e}_{18},\ \mathbf{e}_{23},\ \mathbf{e}_{34},\ \mathbf{e}_{0'1'},\ \mathbf{e}_{0'5'},\ \mathbf{e}_{4'5'},\ \mathbf{e}_{7'8'}$ from Lemma II.5(ii).

(iii) $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edges $\mathbf{e}_{45},\ \mathbf{e}_{56},\ \mathbf{e}_{67},\ \mathbf{e}_{78},\ \mathbf{e}_{1'2'},\ \mathbf{e}_{1'8'},\ \mathbf{e}_{2'3'},\ \mathbf{e}_{3'4'}$ from Lemma II.5(ii) and Lemma II.4.

(iv) If $\alpha \leq \beta$, then $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edges $\mathbf{e}_{01},\ \mathbf{e}_{05},\ \mathbf{e}_{11'},\ \mathbf{e}_{0'3'}$ from Lemma II.5 and Lemma II.4. Further, if $\alpha \leq \beta^2$, then $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edge $\mathbf{e}_{00'}$.

(v) If $\mathbf{s}_{t+1} = \mathbf{b}(s_{t+1}^A + s_{t+1}^B; s_{t+1}^G, \mathbf{d}_{t+1})$, then $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ increases along the edges, $\mathbf{e}_{22'},\ -\mathbf{e}_{22'}$.

Part (i): Case when $\alpha \leq \beta$.

The proof for $q_t \in [\underline{\underline{\underline{q}}}_t, \underline{\underline{q}}_t]$ remains same as Theorem II.9 as all the equivalent directions in Figure 2.16 (cf. Figure 2.13) discussed are increasing for the case with $d_t^B < 0$.

For $q_t \in (\underline{\underline{q}}_t, \underline{q}_t)$, per theorem, $s_{t+1}^{G*} < s_t^G$ and $s_{t+1}^{B*} = s_{t+1}^B,\ s_{t+1}^{A*} = 0$ implying $\mathbf{s}_{t+1}^*$ is on the intersection of faces $0', 5'$. $\mathbf{s}_{t+1}^*$ is locally optimal because, $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along,

$\mathbf{e}_{55'}$, $\mathbf{e}_{4'5'}$, $\mathbf{e}_{0'3'}$. For $q = \underline{q}_t$, we note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is additionally increasing along $\mathbf{e}_{56}$. Notice that although the equivalent edges between Figures 2.13 and 2.16 are not always equal, the same arguments apply. It can be confirmed that similar arguments as Theorem II.9 apply for $q_t \in (q_t^o, \overline{\overline{\overline{q}}}_t]$.

It remains to show the case when $q_t \in (\underline{q}_t, q_t^o]$. Here, $s_{t+1}^{G*} = s_t^G$ and the set $L_{AB}(q_t)$ follows the dashed line in Figure 2.14(c) and (d). Note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along the directions: $\mathbf{e}_{03}$, $\mathbf{e}_{07}$, $\mathbf{e}_{01}$, $\mathbf{e}_{05}$, $\mathbf{e}_{0'7'}$, $\mathbf{e}_{0'3'}$, $\mathbf{e}_{0'1'}$, $\mathbf{e}_{0'5'}$, $\mathbf{e}_{56}$, $\mathbf{e}_{78}$, $\mathbf{e}_{11'}$, $\mathbf{e}_{33'}$, $\mathbf{e}_{44'}$, $\mathbf{e}_{55'}$, $\mathbf{e}_{77'}$, $\mathbf{e}_{88'}$. Further, when $\alpha < \beta^2$, we observe that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along, $\mathbf{e}_{00'}$. Hence, when $\alpha \leq \beta^2$, for $q_t \in (\underline{q}_t, \widetilde{q}_t]$, $\mathbf{s}_{t+1}^*$ is at the intersection of faces $0, 0', 5, 5'$ and for $q_t \in (\widetilde{q}_t, q_t^o]$, $\mathbf{s}_{t+1}^*$ is at the intersection of faces $0, 0', 3, 3'$. However, for $\beta^2 < \alpha \leq \beta$, the solution can be on any point on the set $L_{AB}(q_t)$.

Part (ii): Case when $\beta < \alpha \leq 1$.

The proof for $q_t \in [\underline{\underline{\underline{q}}}_t, \underline{\underline{q}}_t]$ remains same as Theorem II.11 as the equivalent directions discussed are still increasing.

For $q_t \in (\underline{\underline{q}}_t, \underline{q}_t)$, per Theorem, $s_{t+1}^{A*} = 0$, $s_{t+1}^{G*} \leq s_{t+1}^G$, $s_{t+1}^{B*} \in [s_t^B, \ddot{s}_t^B]$. Hence, $\mathbf{s}_{t+1}^*$ is on the edge of face $0'$ and $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along edges $\mathbf{e}_{0'5'}$, $\mathbf{e}_{4'5'}$, $\mathbf{e}_{0'1'}$, and $\mathbf{e}_{1'2'}$ (or $\mathbf{e}_{1'8'}$). Hence, deviating to any point on face $0'$ or $1'$ increases $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$. Further, for $q_t = \underline{q}_t$, we note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is additionally increasing along $\mathbf{e}_{55'}$, as the prescribed $\mathbf{s}_{t+1}^*$ is at the intersection of faces $0, 5, 5'$.

For $q_t \in (\underline{q}_t, \underline{\ddot{q}}_t]$, we additionally consider the directions $\mathbf{e}_{05}$, $\mathbf{e}_{03}$ as the prescribed $\mathbf{s}_{t+1}^*$ may be on the edge intersecting faces $0, 0'$ ($L_{AB}(q_t)$) (See Figure 2.17(a)).

For $q_t \in (\underline{\ddot{q}}_t, \overline{q}_t)$, $\mathbf{s}_{t+1}^*$ may be on a set of up to three contiguous line segments directions on $\mathcal{A}(q_t)$. Figure 2.17(b),(c) confirm that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along the feasible directions to all other points on faces $0, 0', 1, 1', 3, 3', 8$. Further, if $q_t = q_t^o$, we note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing additionally along, $\mathbf{e}_{1'2'}$, $\mathbf{e}_{2'3'}$, as prescribed $\mathbf{s}_{t+1}^*$ intersects with face 2.

For $q_t \in (\overline{q}_t, \overline{\overline{\overline{q}}}_t)$, $\mathbf{s}_{t+1}^*$ may be on the set $\ddot{E}_0$, as shown in Figure 2.17(d). We note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$ is increasing along directions $\mathbf{e}_{12}$, $\mathbf{e}_{23}$, $\mathbf{e}_{33'}$. □

Proof of Lemma II.12: For given $\mathbf{S}$, we use $\mathbf{s}_{t+1}^*$ as short for the optimal decision rule

$\mathbf{s}^*_{t+1}(\mathbf{s}_t, \mathbf{d}_t; \mathbf{S})$, and let $\{\mathbf{s}^*_t : t \in \mathcal{T}\}$ be the optimal policy. For any $\widetilde{\mathbf{S}} \geq \mathbf{S}$, we can construct a feasible policy: $\{\widetilde{\mathbf{s}}^*_t = \mathbf{s}^*_t + \widetilde{\mathbf{S}} - \mathbf{S} : t \in \mathcal{T}\}$. The two policies yield the same inventory changes, $\Delta\widetilde{\mathbf{s}}^*_t = \Delta\mathbf{s}^*_t$, and thus the same expected operating cost. Therefore, $V(\widetilde{\mathbf{S}}) \leq V(\mathbf{S})$.

Using similar inductive arguments as Lemma II.4, we can show that $V_t(\mathbf{s}_t, \mathbf{d}_t; \mathbf{S})$ is convex in $(\mathbf{s}_t, \mathbf{S})$ for any $t \in \mathcal{T}$. In particular, $V_1(\mathbf{s}_1, \mathbf{d}_1; \mathbf{S})$ is convex in $(\mathbf{s}_1, \mathbf{S})$. Hence, $V(\mathbf{S}) = \mathsf{E}_0 V_1(\mathbf{S}, \mathbf{d}_1; \mathbf{S})$ is convex in $\mathbf{S}$. The concavity of $B(\mathbf{S})$ follows immediately. $\square$

Proof of Lemma II.13: (i) Under investment $\mathbf{S}$, let $\{\mathbf{s}^*_t : t \in \mathcal{T}\}$ be the optimal inventory policy, and $u^{i*}_t = d^i_t + \psi_\alpha(\Delta s^{i*}_t)$, $i = A, B$, be the corresponding transmission flows according to (2.3).

Under investment $\mathbf{S}_g = \big(0,\, 0,\, S^G + \beta^{-1}(S^A + S^B)\big)$, we construct an operating policy $\{\widetilde{\mathbf{s}}_t : t \in \mathcal{T}\}$:

$$\widetilde{s}^A_t = \widetilde{s}^B_t = 0, \qquad \widetilde{s}^G_t = s^{G*}_t + g_t, \qquad \forall\, t \in \mathcal{T}. \tag{2.41}$$

$$g_1 = \beta^{-1}(S^A + S^B), \tag{2.42}$$

$$g_{t+1} = \min\left\{\beta^{-1}(S^A + S^B),\;\; g_t + \beta^{-1}(\Delta s^{A*}_t + \Delta s^{B*}_t) + \frac{\beta - \beta^{-1}}{\alpha}\min\{u^{A*}_t, u^{B*}_t, 0\}\right\}. \tag{2.43}$$

Since $\mathbf{s}^*_1 = \mathbf{S}$, we have $\widetilde{\mathbf{s}}_1 = \mathbf{S}_g$. The definition in (2.43) implies $g_t \in \big[\beta^{-1}(s^{A*}_t + s^{B*}_t),\ \beta^{-1}(S^A + S^B)\big]$ for all $t \in \mathcal{T}$.[6] Hence, $\mathbf{0} \leq \widetilde{\mathbf{s}}_t \leq \mathbf{S}_g$, thus the constructed policy $\{\widetilde{\mathbf{s}}_t : t \in \mathcal{T}\}$ is feasible under $\mathbf{S}_g$. We do not require the non-negative production constraint as in (2.6), because for any inventory decision that results in $q_t < 0$, there exists another inventory decision that results in $q_t \geq 0$ and the same objective value, which is shown in the proof of Lemma II.4.

We now prove that under $\mathbf{S}_g$ and $\{\widetilde{\mathbf{s}}_t : t \in \mathcal{T}\}$, the production $\widetilde{q}_t = \psi_\alpha(\Delta\widetilde{s}^G_t) + \beta^{-1}(d^A_t + d^B_t)$ does not exceed the optimal production under $\mathbf{S}$: $q^*_t = \psi_\alpha(\Delta s^{G*}_t) + \sum_{i=A,B} \psi_\beta(u^{i*}_t)$. Consider three cases:

1) $u^{A*}_t \geq 0$ and $u^{B*}_t \geq 0$. In this case, (2.43) implies $\Delta g_t = g_{t+1} - g_t \leq \beta^{-1}(\Delta s^{A*}_t + \Delta s^{B*}_t)$.

[6] We can inductively show $g_t \geq \beta^{-1}(s^{A*}_t + s^{B*}_t)$. This is true for $t = 1$. Suppose $g_t \geq \beta^{-1}(s^{A*}_t + s^{B*}_t)$. Then, $g_t + \beta^{-1}(\Delta s^{A*}_t + \Delta s^{B*}_t) + \frac{\beta - \beta^{-1}}{\alpha}\min\{u^{A*}_t, u^{B*}_t, 0\} \geq \beta^{-1}(s^{A*}_t + s^{B*}_t) + \beta^{-1}(\Delta s^{A*}_t + \Delta s^{B*}_t) = \beta^{-1}(s^{A*}_{t+1} + s^{B*}_{t+1})$. This, together with $\beta^{-1}(S^A + S^B) \geq \beta^{-1}(s^{A*}_{t+1} + s^{B*}_{t+1})$, implies that $g_{t+1} \geq \beta^{-1}(s^{A*}_{t+1} + s^{B*}_{t+1})$.

Then,

$$
\begin{aligned}
q_t^* &= \psi_\alpha(\Delta s_t^{G*}) + \sum_{i=A,B} \beta^{-1}(d_t^i + \psi_\alpha(\Delta s_t^{i*})) = \psi_\alpha(\Delta s_t^{G*}) + \sum_{i=A,B} \beta^{-1} d_t^i + \psi_\alpha(\beta^{-1}\Delta s_t^{i*}) \\
&\geq \psi_\alpha(\Delta s_t^{G*} + \beta^{-1}\Delta s_t^{A*} + \beta^{-1}\Delta s_t^{B*}) + \beta^{-1}(d_t^A + d_t^B) \\
&\geq \psi_\alpha(\Delta s_t^{G*} + \Delta g_t) + \beta^{-1}(d_t^A + d_t^B) \;=\; \widetilde{q}_t,
\end{aligned}
$$

where the first inequality utilizes the subadditivity of $\psi_\alpha(\cdot)$, i.e., $\psi_\alpha(x)+\psi_\alpha(y) \geq \psi_\alpha(x+y)$.

2) $u_t^{A*} < 0$ and $u_t^{B*} \geq 0$, i.e., some energy is transmitted from A to B. The condition $u_t^{A*} < 0$ implies $\Delta s_t^{A*} < 0$, which in turn implies $\Delta s_t^{G*} \leq 0$ and $\Delta s_t^{B*} \leq 0$, because Lemma II.5 suggests that energy should not be withdrawn from one node only to store it in another node. These conditions, together with $d_t^A \geq 0$, imply that $\Delta g_t = \beta\Delta s_t^{A*} + \beta^{-1}\Delta s_t^{B*} + \dfrac{\beta - \beta^{-1}}{\alpha} d_t^A < 0$.[7] Then,

$$
\widetilde{q}_t = \alpha(\Delta s_t^{G*} + \Delta g_t) + \beta^{-1}(d_t^A + d_t^B) \;=\; \alpha\Delta s_t^{G*} + \beta(\alpha\Delta s_t^{A*} + d_t^A) + \beta^{-1}(\alpha\Delta s_t^{B*} + d_t^B) \;=\; q_t^*.
$$

3) $u_t^{A*} \geq 0$ and $u_t^{B*} < 0$. This case is parallel to case 2.

The case of $u_t^{A*} < 0$ and $u_t^{B*} < 0$ does not exist, because the corresponding inventory changes are suboptimal by Lemma II.5. Therefore, in all cases, we have $\widetilde{q}_t \leq q_t^*$, implying that the policy $\{\widetilde{\mathbf{s}}_t : t \in \mathcal{T}\}$ achieves an operating cost no higher than $V(\mathbf{S})$. Therefore, $V(\mathbf{S}_g) \leq V(\mathbf{S})$.

(ii) Let $\{\mathbf{s}_t^* : t \in \mathcal{T}\}$ be the optimal policy under $\mathbf{S}$. Under $\mathbf{S}_l = (S^A + \beta S^G, S^B + \beta S^G, 0)$,

[7] To see this, note that the last two terms in (2.43) are $\beta^{-1}(\Delta s_t^{A*} + \Delta s_t^{B*}) + \frac{\beta-\beta^{-1}}{\alpha}\min\{u_t^{A*}, u_t^{B*}, 0\} = \beta^{-1}(\Delta s_t^{A*} + \Delta s_t^{B*}) + \frac{\beta-\beta^{-1}}{\alpha}(d_t^A + \alpha\Delta s_t^{A*}) = \beta\Delta s_t^{A*} + \beta^{-1}\Delta s_t^{B*} + \frac{\beta-\beta^{-1}}{\alpha} d_t^A \;<\; 0.$

we construct an operating policy $\{\widehat{\mathbf{s}}_t : t \in \mathcal{T}\}$ as follows:

$$\widehat{s}_t^A = s_t^{A*} + g_t^A, \qquad \widehat{s}_t^B = s_t^{B*} + g_t^B, \qquad \widehat{s}_t^G = 0, \qquad \forall\, t \in \mathcal{T}, \tag{2.44}$$

$$g_1^A = \beta S^G, \tag{2.45}$$

$$\Delta g_t^A = g_{t+1}^A - g_t^A = \begin{cases} \max\left\{\beta \Delta s_t^{G*},\ -(u_t^{A*})^+/\alpha\right\}, & \text{if } \Delta s_t^{G*} < 0, \\ \min\left\{\beta \Delta s_t^{G*},\ \beta S^G - g_t^A\right\}, & \text{if } \Delta s_t^{G*} \geq 0, \end{cases} \qquad t \in \mathcal{T}, \tag{2.46}$$

$$g_t^B = \beta(s_t^{G*} + S^G) - g_t^A, \qquad t \in \mathcal{T}. \tag{2.47}$$

We first show the following properties for g_t^A and g_t^B: (P1) $g_t^i \in \left[\beta s_t^{G*}, \beta S^G\right]$, $i = A, B$; (P2) $\Delta g_t^A + \Delta g_t^B = \beta \Delta s_t^{G*}$; (P3) $\Delta g_t^A \Delta g_t^B \geq 0$.

It can be shown by induction that $g_t^A \in \left[\beta s_t^{G*}, \beta S^G\right]$ and the details are omitted. This range for g_t^A, together with (2.47), implies that $g_t^B \in [\beta s_t^{G*}, \beta S^G]$ for all $t \in \mathcal{T}$. This proves (P1) and thus $\{\widehat{\mathbf{s}}_t : t \in \mathcal{T}\}$ is a feasible policy. Property (P2) follows directly from (2.47). From (2.46), if $\Delta s_t^{G*} \geq 0$, then $\Delta g_t^A \in [0, \beta \Delta s_t^{G*}]$; if $\Delta s_t^{G*} < 0$, then $\Delta g_t^A \in [-\beta \Delta s_t^{G*}, 0]$. These ranges for Δg_t^A, together with property (P2), imply that Δg_t^A and Δg_t^B have the same sign, hence property (P3).

We now prove that under $\mathbf{S}_l$ and $\{\widehat{\mathbf{s}}_t : t \in \mathcal{T}\}$, the production $\widehat{q}_t = \sum_{i=A,B} \psi_\beta(\widehat{u}_t^i)$ does not exceed the optimal production under $\mathbf{S}$: $q_t^* = \psi_\alpha(\Delta s_t^{G*}) + \sum_{i=A,B} \psi_\beta(u_t^{i*})$. The subadditivity of $\psi_\alpha(\cdot)$ leads to

$$\widehat{u}_t^i \equiv d_t^i + \psi_\alpha(\Delta s_t^{i*} + \Delta g_t^i) \leq d_t^i + \psi_\alpha(\Delta s_t^{i*}) + \psi_\alpha(\Delta g_t^i) = u_t^{i*} + \psi_\alpha(\Delta g_t^i). \tag{2.48}$$

If $\Delta s_t^{G*} \geq 0$, then $\Delta g_t^i \geq 0$ due to properties (P2) and (P3). Using (2.48), the subadditivity of $\psi_\beta(\cdot)$, and property (P2), we have

$$\widehat{q}_t \leq \sum_{i=A,B} \psi_\beta(u_t^{i*} + \psi_\alpha(\Delta g_t^i)) \leq \sum_{i=A,B} \left[\psi_\beta(u_t^{i*}) + \Delta g_t^i/(\alpha\beta)\right] = \sum_{i=A,B} \psi_\beta(u_t^{i*}) + \Delta s_t^{G*}/\alpha = q_t^*.$$

If $\Delta s_t^{G*} < 0$, then $\Delta g_t^i \leq 0$, $i = A, B$, and we consider three cases:

1) $u_t^{A*} \geq 0$ and $u_t^{B*} \geq 0$. Because Lemma II.5 suggests that energy should not be withdrawn from G only to store it in another node, we have $\Delta s_t^{i*} \leq 0$, $i = A, B$. Thus, $\widehat{u}_t^A =$

$d_t^A + \alpha(\Delta s_t^{A*} + \Delta g_t^A) = u_t^{A*} + \alpha\Delta g_t^A \geq 0$, where the inequality is due to (2.46). It can be shown that $\widehat{u}_t^B \geq 0$.[8] Hence,

$$\widehat{q}_t = \beta^{-1}(\widehat{u}_t^A + \widehat{u}_t^B) = \beta^{-1}(u_t^{A*} + u_t^{B*} + \alpha\Delta g_t^A + \alpha\Delta g_t^B) = \beta^{-1}(u_t^{A*} + u_t^{B*}) + \alpha\Delta s_t^{G*} = q_t^*.$$

2) $u_t^{A*} < 0$ and $u_t^{B*} \geq 0$. Definition (2.46) suggests $\Delta g_t^A = 0$. Hence, $\widehat{u}_t^A = u_t^{A*} < 0$ and $\Delta g_t^B = \beta\Delta s_t^{G*}$. We must have $\widehat{u}_t^B > 0$ to balance the flows at node G. Thus, using (2.48), we have

$$\widehat{q}_t = \beta u_t^{A*} + \beta^{-1}\widehat{u}_t^B \leq \beta u_t^{A*} + \beta^{-1}u_t^{B*} + \beta^{-1}\alpha\Delta g_t^B = \beta u_t^{A*} + \beta^{-1}u_t^{B*} + \alpha\Delta s_t^{G*} = q_t^*.$$

3) $u_t^{B*} < 0$ and $u_t^{A*} \geq 0$. Since $q_t^* = \alpha\Delta s_t^{G*} + \beta^{-1}u^{A*} + \beta u_t^{B*} \geq 0$, we have $\beta\Delta s_t^{G*} \geq -u^{A*}/\alpha$. Thus, using (2.46), we have $\Delta g_t^A = \beta\Delta s_t^{G*}$, which in turn implies that $\Delta g_t^B = 0$ due to property (P2). Then, a similar logic as in case 2 gives $\widehat{q}_t \leq q_t^*$.

The case of $u_t^{A*} < 0$ and $u_t^{B*} < 0$ does not exist. Hence, in all cases, we have $\widehat{q}_t \leq q_t^*$, implying that the policy $\{\widehat{\mathbf{s}}_t : t \in \mathcal{T}\}$ achieves an operating cost no higher than $V(\mathbf{S})$. Therefore, $V(\mathbf{S}_l) \leq V(\mathbf{S})$. □

Proof of Theorem II.14: For any given $\mathbf{S} \geq 0$ and the associated optimal operating policy $\{\mathbf{s}_t^* : t \in \mathcal{T}\}$, we construct a two-node system with node G and a single demand node with demand $d_t^L = d_t^A + d_t^B$. The demand node has storage capacity $S^L = S^A + S^B$ with operating policy $s_t^L = s_t^{A*} + s_t^{B*}$. The storage capacity and operations at node G remain the same. The subadditivity of ψ_α and ψ_β implies

$$\psi_\beta\big(d_t^L + \psi_\alpha(\Delta s_t^L)\big) \leq \psi_\beta\big(d_t^L + \psi_\alpha(\Delta s_t^{A*}) + \psi_\alpha(\Delta s_t^{B*})\big) \leq \sum_{i=A,B} \psi_\beta\big(d_t^i + \psi_\alpha(\Delta s_t^{i*})\big), \quad t \in \mathcal{T},$$

which in turn implies that the two-node system produces no more than the three-node system. Thus,

$$\widetilde{B}(S^A + S^B, S^G) \geq B(S^A, S^B, S^G), \tag{2.49}$$

[8] If $\widehat{u}_t^A = 0$, then the flow balance at node G requires $\widehat{u}_t^B \geq 0$. If $\widehat{u}_t^A = u_t^{A*} + \alpha\Delta g_t^A > 0$, then $\Delta g_t^A > -u_t^{A*}/\alpha$. Consequently, (2.46) gives $\Delta g_t^A = \beta\Delta s_t^{G*}$. Thus, $\Delta g_t^B = 0$. Hence, $\widehat{u}_t^B = u_t^{B*} \geq 0$.

where $\widetilde{B}(S^L, S^G)$ is the net benefit of investment (S^L, S^G) for the two-node system.

Furthermore, (2.49) holds with equality if $d_t^A = k\, d_t^B$ and $S^A = k\, S^B$. This can be shown by using the optimal policy for the two-node system to construct a feasible operating policy for the three-node system that yield the same operating cost. The construction maintains the leaf storage at the constant ratio k; the details are omitted. Therefore, under $d_t^A = k\, d_t^B$, we have

$$B\Big(\frac{k\, S^{L*}}{k+1}, \frac{S^{L*}}{k+1}, 0\Big) = \widetilde{B}(S^{L*}, 0) \geq \widetilde{B}(S^A + S^B, S^G) \geq B(S^A, S^B, S^G), \tag{2.50}$$

where the first inequality claims that for the two-node system, localized investment is optimal (S^{L*} is the optimal localized investment), which follows from the discussion after Theorem II.14. Noting that $\mathbf{S}$ is arbitrary, we conclude from (2.50) that the localized investment $\big(\frac{k\, S^{L*}}{k+1}, \frac{S^{L*}}{k+1}, 0\big)$ is optimal. $\square$

The proof of Lemma II.15 requires some properties of the optimal operating policy and the value function when $d_{\min}^A > 0$, as stated in the following lemma.

Lemma II.28. *Suppose $d_{\min}^A > 0$. For given storage capacity $\mathbf{S} = (S^A, S^B, S^G)$ with $S^A < \alpha^{-1} d_{\min}^A$, (i) There exists an optimal policy satisfying $\Delta s_t^{A*} \cdot \Delta s_t^{G*} \geq 0$ for all $t \in \mathcal{T}$;*
(ii) If $\mathbf{s}_t, \widetilde{\mathbf{s}}_t \in \mathcal{A}$ and $\widetilde{\mathbf{s}}_t - \mathbf{s}_t = (\beta\delta, 0, -\delta)$ for some $\delta > 0$, then $V_t(\mathbf{s}_t, \mathbf{d}_t) = V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t)$ for any $\mathbf{d}_t$.

Proof of Lemma II.28: The condition $\alpha S^A < d_{\min}^A$ means that the demand at node A cannot be met solely by storage A in a period. Thus, energy is transmitted from G to A in every period.

Suppose part (ii) holds for period $t+1$ (it clearly holds for period $T+1$). In period t, we consider any given state $(\mathbf{s}, \mathbf{d})$ and any decision $\mathbf{s}_{t+1}$ with with inventory change $\Delta s^A < 0$ and $\Delta s^G > 0$. We now show that a strictly better decision is $\widehat{\mathbf{s}}_{t+1} = \mathbf{s}_{t+1} + (\beta\delta, 0, -\delta)$, where $\delta = \min\{-\beta^{-1}\Delta s^A, \Delta s^G\}$. This new decision satisfies $\Delta\widehat{s}^A \cdot \Delta\widehat{s}^G = 0$, $\Delta\widehat{s}^A = \Delta s^A + \beta\delta \leq 0$, and $\Delta\widehat{s}^G = \Delta s^G - \delta \geq 0$. To show the superiority of $\widehat{\mathbf{s}}_{t+1}$, note that $V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1}) = V_{t+1}(\widehat{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})$ by the induction hypothesis and

$$q(\Delta\widehat{\mathbf{s}}, \mathbf{d}) - q(\Delta\mathbf{s}, \mathbf{d}) = \beta^{-1}(d^A + \alpha\Delta\widehat{s}^A) + \alpha^{-1}\Delta\widehat{s}^G - \beta^{-1}(d^A + \alpha\Delta s^A) - \alpha^{-1}\Delta s^G = \alpha\delta - \alpha^{-1}\delta < 0.$$

Similarly, any decision $\mathbf{s}_{t+1}$ with $\Delta s^A > 0$ and $\Delta s^G < 0$ can also be improved. Thus, part (i) holds for period t. We next prove part (ii) for period t.

Consider states $(\mathbf{s}, \mathbf{d})$ and $(\widetilde{\mathbf{s}}, \mathbf{d})$ in period t, with $\widetilde{\mathbf{s}} = \mathbf{s} + (\beta\delta,\, 0,\, -\delta)$ for some $\delta > 0$. Lemma II.5 implies that $V_t(\mathbf{s}, \mathbf{d}) \le V_t(\widetilde{\mathbf{s}}, \mathbf{d})$. Thus, we only need to show $V_t(\widetilde{\mathbf{s}}, \mathbf{d}) \le V_t(\mathbf{s}, \mathbf{d})$. Let $\mathbf{s}^*_{t+1}$ be the optimal decision for $(\mathbf{s}, \mathbf{d})$ and denote $\Delta\mathbf{s}^* = \mathbf{s}^*_{t+1} - \mathbf{s}$. For state $(\widetilde{\mathbf{s}}, \mathbf{d})$, we construct a decision $\widetilde{\mathbf{s}}_{t+1} = \mathbf{s}^*_{t+1} + (\beta\widetilde{\delta},\, 0,\, -\widetilde{\delta})$, where $\widetilde{\delta} = \min\left\{\delta,\, s^{G*}_{t+1},\, \beta^{-1}(S^A - s^{A*}_{t+1})\right\}$. We next show that $\widetilde{\mathbf{s}}_{t+1}$ for $(\widetilde{\mathbf{s}}, \mathbf{d})$ gives the same operating cost as $\mathbf{s}^*_{t+1}$ for $(\mathbf{s}, \mathbf{d})$. First, by the induction hypothesis, $V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1}) = V_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})$. Second, we show the production quantities are the same. Let $\Delta\widetilde{\mathbf{s}} = \widetilde{\mathbf{s}}_{t+1} - \widetilde{\mathbf{s}} = \Delta\mathbf{s}^* - (\beta\varepsilon,\, 0,\, -\varepsilon)$, where $\varepsilon = \delta - \widetilde{\delta}$. Consider two cases:

- Case 1: $\Delta s^{A*} \ge 0$ and $\Delta s^{G*} \ge 0$. We have $s^{G*}_{t+1} \ge s^G = \widetilde{s}^G + \delta \ge \delta$. Thus, either $\widetilde{\delta} = \delta$ or $\widetilde{\delta} = \beta^{-1}(S^A - s^{A*}_{t+1})$. In either case, we can verify that $\Delta\widetilde{s}^A \ge 0$. Also, $\Delta\widetilde{s}^G \ge 0$. Hence,

$$q(\Delta\widetilde{\mathbf{s}}, \mathbf{d}) - q(\Delta\mathbf{s}^*, \mathbf{d}) = \beta^{-1}(d^A + \alpha^{-1}\Delta\widetilde{s}^A) + \alpha^{-1}\Delta\widetilde{s}^G - \beta^{-1}(d^A + \alpha^{-1}\Delta s^{A*}) - \alpha^{-1}\Delta s^{G*} \tag{2.51}$$

$$= -\beta^{-1}\alpha^{-1}\beta\varepsilon + \alpha^{-1}\varepsilon \;=\; 0.$$

- Case 2: $\Delta s^{A*} \le 0$ and $\Delta s^{G*} \le 0$. Using similar logic, we can show $\Delta\widetilde{s}^A \le 0$ and $\Delta\widetilde{s}^G \le 0$, and $q(\Delta\widetilde{\mathbf{s}}, \mathbf{d}) = q(\Delta\mathbf{s}^*, \mathbf{d})$.

These are the only cases we need to consider, as indicated by part (i). Equal production and equal future expected cost together imply that $V_t(\widetilde{\mathbf{s}}, \mathbf{d}) \le V_t(\mathbf{s}, \mathbf{d})$, completing the proof.

Proof of Lemma II.15: Under investment $\mathbf{S}$, let $\{\mathbf{s}^*_t : t \in \mathcal{T}\}$ be an optimal policy satisfying $\Delta s^{A*}_t \cdot \Delta s^{G*}_t \ge 0$, which follows from Lemma II.28(i). Under investment $\widetilde{\mathbf{S}}$, we construct a policy $\widetilde{\mathbf{s}}_t = \mathbf{s}^*_t + (\beta\delta_t,\, 0,\, -\delta_t)$, where $\delta_t = \min\{\delta,\, s^{G*}_t\}$, for all $t \in \mathcal{T}$. The policy $\{\widetilde{\mathbf{s}}_t : t \in \mathcal{T}\}$ is feasible under $\widetilde{\mathbf{S}}$ because $\widetilde{s}^A_t \ge 0$, $\widetilde{s}^A_t \le s^{A*}_t + \beta\delta \le \widetilde{S}^A$, and $\widetilde{s}^G_t = s^{G*}_t - \delta_t = \max\{s^{G*}_t - \delta,\, 0\} \in [0, \widetilde{S}^G]$.

We next show that the two policies yields the same production quantities. If $\Delta s^{A*}_t \ge 0$ and $\Delta s^{G*}_t \ge 0$, we have $\delta_{t+1} - \delta_t \in [0, \Delta s^{G*}_t]$, which implies $\Delta\widetilde{s}^A_t = \Delta s^{A*}_t + \beta(\delta_{t+1} - \delta_t) \ge 0$ and $\Delta\widetilde{s}^G_t = \Delta s^{G*}_t - (\delta_{t+1} - \delta_t) \ge 0$. Then, following exactly the same logic in (2.51), $q(\Delta\widetilde{\mathbf{s}}_t, \mathbf{d}_t) = q(\Delta\mathbf{s}^*_t, \mathbf{d}_t)$. If $\Delta s^{A*}_t \le 0$ and $\Delta s^{G*}_t \le 0$, similar logic applies. Therefore,

$q(\Delta\widetilde{\mathbf{s}}_t, \mathbf{d}_t) = q(\Delta\mathbf{s}_t^*, \mathbf{d}_t)$ for all $t \in \mathcal{T}$, and consequently the total operating costs are the same for both policies, which implies $V(\widetilde{\mathbf{S}}) \le V(\mathbf{S})$. The opposite inequality $V(\widetilde{\mathbf{S}}) \ge V(\mathbf{S})$ can be proved similarly.

The proof of part (ii) follows similar logic, but note that $V(\widetilde{\mathbf{S}}) \ge V(\mathbf{S})$ may not be true because the condition for $d^B_{\min}$ is not given. □

Proof of Theorem II.16: Because $B(\mathbf{S})$ is concave in $\mathbf{S}$ (Lemma II.12), it suffices to show that $\mathbf{S}_l^*$ achieves a local maximum. Consider deviating from $\mathbf{S}_l^*$ by $(\delta^A, \delta^B, \delta^G)$, where $\delta^i \in \big(-S_l^{i*}, (\alpha^{-1}d^i_{\min} - S_l^{i*})/2\big)$, for $i = A, B$, and $\delta^G \in \big[0, (\alpha^{-1}d^A_{\min} - S_l^{A*})/2\big)$. We have

$$\begin{aligned} B(\mathbf{S}_l^*) - B(S_l^{A*} + \delta^A,\, S_l^{B*} + \delta^B,\, \delta^G) &= V(S_l^{A*} + \delta^A,\, S_l^{B*} + \delta^B,\, \delta^G) - V(\mathbf{S}_l^*) + p(\delta^A + \delta^B + \delta^G) \\ &\ge V(S_l^{A*} + \delta^A + \beta\delta^G,\, S_l^{B*} + \delta^B,\, 0) - V(\mathbf{S}_l^*) + p(\delta^A + \delta^B + \beta\delta^G) \\ &= B(\mathbf{S}_l^*) - B(S_l^{A*} + \delta^A + \beta\delta^G,\, S_l^{B*} + \delta^B,\, 0) \;\ge\; 0, \end{aligned}$$

where the first inequality follows from Lemma II.15(i) and $\delta^G \ge \beta\delta^G$, and the last inequality follows from optimality of $\mathbf{S}_l^*$ for the constrained investment problem (2.20). This proves the optimality of $\mathbf{S}_l^*$. Furthermore, if $\delta^G > 0$, then the first inequality holds strictly, which implies that investing in $S^G > 0$ is strictly dominated by investing $\mathbf{S}_l^*$. □

Proof of Lemma II.20: The proof for part (i) is straightforward and omitted. To prove part (ii), consider any p_1 and p_2 with $p_1 < p_2$. The optimality of $\mathbf{S}^*(p_1)$ suggests $p_1\,|\mathbf{S}^*(p_1)| + V(\mathbf{S}^*(p_1)) \le p_1\,|\mathbf{S}^*(p_2)| + V(\mathbf{S}^*(p_2))$. Similarly, $p_2\,|\mathbf{S}^*(p_2)| + V(\mathbf{S}^*(p_2)) \le p_2\,|\mathbf{S}^*(p_1)| + V(\mathbf{S}^*(p_1))$. Combining these two inequalities, we have

$$p_1(|\mathbf{S}^*(p_1)| - |\mathbf{S}^*(p_2)|) \le V(\mathbf{S}^*(p_2)) - V(\mathbf{S}^*(p_1)) \le p_2(|\mathbf{S}^*(p_1)| - |\mathbf{S}^*(p_2)|),$$

which implies $(p_1 - p_2)(|\mathbf{S}^*(p_1)| - |\mathbf{S}^*(p_2)|) \le 0$. Because $p_1 < p_2$, we have $|\mathbf{S}^*(p_1)| \ge |\mathbf{S}^*(p_2)|$. □

Lemma II.29. *For $n = 1, 2, \ldots,$ suppose $a_n \ge 0$, $b_n > 0$, $b_n \ge b_{n+1}$, $\lim\limits_{n\to\infty} b_n = 0$, and $\sum\limits_{n=1}^{\infty} a_n b_n < \infty$. Then, $\lim\limits_{n\to\infty} \left(b_n \sum\limits_{i=1}^{n} a_i\right) = 0$.*

Proof of Lemma II.29: First, $a_n b_n \ge 0$ and $\sum\limits_{n=1}^{\infty} a_n b_n < \infty$ imply $\sum\limits_{n=1}^{\infty} a_n b_n$ exists. Let

$\sum_{n=1}^{\infty} a_n b_n = M$. For any $\varepsilon > 0$, there exists N_1 such that $\sum_{n=N_1}^{\infty} a_n b_n < \frac{\varepsilon}{2}$. Because $b_n > 0$ decreases in n and converges to zero, there exists $N_2 > N_1$ such that $\frac{b_{N_2}}{b_{N_1}} < \frac{\varepsilon}{2M}$. Then, for any $N > N_2$, we have

$$b_N \sum_{n=1}^{N} a_n = b_N \left[\sum_{n=1}^{N_1} a_n + \sum_{n=N_1+1}^{N} a_n \right] < \frac{b_N}{b_{N_1}} \sum_{n=1}^{N_1} a_n b_n + \sum_{n=N_1+1}^{N} a_n b_n < \frac{\varepsilon}{2M} M + \frac{\varepsilon}{2} = \varepsilon. \quad (2.52)$$

Hence the limiting result holds. □

Proof of Theorem II.21: To prove this theorem, we first show

$$\lim_{p \to 0} p \, |\mathbf{S}^*(p)| = 0. \quad (2.53)$$

Let $\{p_n\}$ be a sequence of positive prices such that p_n decreases in n and converges to zero. For simplicity, let $\mathbf{S}_n \equiv \mathbf{S}^*(p_n)$. Lemma II.20(ii) implies that $|\mathbf{S}_n| - |\mathbf{S}_{n-1}| \geq 0$.

By optimality of $\mathbf{S}_n$, we have $p_n|\mathbf{S}_n| + V(\mathbf{S}_n) \leq p_n|\mathbf{S}_{n-1}| + V(\mathbf{S}_{n-1})$ or

$$p_n(|\mathbf{S}_n| - |\mathbf{S}_{n-1}|) \leq V(\mathbf{S}_{n-1}) - V(\mathbf{S}_n).$$

Summing over n, we have

$$\sum_{n=1}^{\infty} p_n(|\mathbf{S}_n| - |\mathbf{S}_{n-1}|) \leq V(\mathbf{S}_0) - \lim_{n \to \infty} V(\mathbf{S}_n) < \infty.$$

Applying Lemma II.29, we have $\lim_{n \to \infty} p_n(|\mathbf{S}_n| - |\mathbf{S}_0|) = 0$. Since $\lim_{n \to \infty} p_n|\mathbf{S}_0| = 0$, we have $\lim_{n \to \infty} p_n|\mathbf{S}_n| = 0$. Because $\{p_n\}$ is chosen arbitrarily, we have $\lim_{p \to 0} p \, |\mathbf{S}^*(p)| = 0$.

(i) Given an optimal investment $\mathbf{S}^* = (S^{A*}, S^{B*}, S^{G*})$, consider a localized investment $\widetilde{\mathbf{S}} = (S^{A*} + \beta S^{G*}, S^{B*} + \beta S^{G*}, 0)$. Lemma II.13(ii) suggests that $V(\widetilde{\mathbf{S}}) \leq V(\mathbf{S}^*)$. In addition, as the optimal localized investment is $\mathbf{S}_l^*$, we have $B(\widetilde{\mathbf{S}}) \leq B(\mathbf{S}_l^*)$. Utilizing these inequalities, we have

$$
\begin{aligned}
0 \le B(\mathbf{S}^*) - B(\mathbf{S}_l^*) \le B(\mathbf{S}^*) - B(\widetilde{\mathbf{S}}) \;&=\; V(\widetilde{\mathbf{S}}) + p|\widetilde{\mathbf{S}}| - V(\mathbf{S}^*) - p|\mathbf{S}^*| \\
&\le p|\widetilde{\mathbf{S}}| - p|\mathbf{S}^*| \;=\; p(S^{A*} + \beta S^{G*} + S^{B*} + \beta S^{G*}) - p(S^{A*} + S^{B*} + S^{G*}) \\
&= (2\beta - 1)p\, S^{G*}.
\end{aligned}
$$

Note that S^{G*} is a function of p, and $\lim_{p\to 0} p\, S^{G*}(p) = 0$ due to (2.53). Hence,

$$
\lim_{p\to 0} B(\mathbf{S}^*(p)) - B(\mathbf{S}_l^*(p)) \;=\; 0.
$$

(ii) Consider a pooled investment $\widehat{\mathbf{S}} = \big(0,\, 0,\, S^{G*} + \beta^{-1}(S^{A*} + S^{B*})\big)$. Using similar logic and the result in Lemma II.13(i) (which requires non-negative demand), we have

$$
\begin{aligned}
0 \le B(\mathbf{S}^*) - B(\mathbf{S}_g^*) \le B(\mathbf{S}^*) - B(\widehat{\mathbf{S}}) \;&=\; V(\widehat{\mathbf{S}}) + p|\widehat{\mathbf{S}}| - V(\mathbf{S}^*) - p|\mathbf{S}^*| \\
&\le p|\widehat{\mathbf{S}}| - p|\mathbf{S}^*| \;=\; p(S^{G*} + \beta^{-1}(S^{A*} + S^{B*})) - p(S^{A*} + S^{B*} + S^{G*}) \\
&= (\beta^{-1} - 1)p\,(S^{A*} + S^{B*}).
\end{aligned}
$$

Because $\lim_{p\to 0} p\,(S^{A*}(p) + S^{B*}(p)) = 0$ due to (2.53), we have

$$
\lim_{p\to 0} B(\mathbf{S}^*(p)) - B(\mathbf{S}_g^*(p)) \;=\; 0.
$$

□

Proof of Theorem II.23: For part (i), it suffices to prove, $V(\mathbf{S})$ is decreasing with α. Clearly, any policy feasible for α is feasible when the storage efficiency is $\alpha_1 > \alpha$ and has lower production costs, hence, $V(\mathbf{S})$ cannot increase for α_1.

For part (ii), similar to part (i), suppose $\mathbf{s}_t^*, t \in \mathcal{T}$ is the optimal policy for system with transmission efficiency β. Note that $\mathbf{s}_t^*$ is feasible for system with transmission efficiency $\beta_1 > \beta$ and the equivalent production quantity is lesser in each period. This and Theorem II.25 together imply part (ii).

For part (iii), suppose $\mathbf{S}^*$ is an optimal solution and $S^{G*} > 0$. We consider an alternative feasible policy with investment $\mathbf{S}' = (S^{A*} + \beta S^{G*}, S^{B*} + \beta S^{G*}, 0)$. Lemma II.13(ii) implies $V(\mathbf{S}') \le V(\mathbf{S}^*)$. Further, as $\beta < 0.5$, $p|\mathbf{S}'| < p|\mathbf{S}^*|$. These two inequalities together imply,

$B(\mathbf{S}^*) < B(\mathbf{S}')$ which contradicts the optimality of $\mathbf{S}^*$ for (2.7). $\square$

Proof of Theorem II.25: We prove the theorem jointly with the property that $V_t(\mathbf{s}_t, \mathbf{d}_t)$ decreases in $\mathbf{s}_t$. The property holds trivially for $t = T$, as $V_T(\cdot, \cdot) = 0$. Consider the function minimized in (2.24):

$$U(\mathbf{s}_{t+1}) \stackrel{\text{def}}{=} c(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big]. \tag{2.54}$$

(i) When $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) \geq 0$, if $\mathbf{s}'_{t+1}$ minimizes $U(\mathbf{s}_{t+1})$ but $q(\mathbf{s}'_{t+1} - \mathbf{s}_t, \mathbf{d}_t) < 0$, then using the intermediate value theorem, there exists $\mathbf{s}''_{t+1}$ such that $\mathbf{s}'_{t+1} \leq \mathbf{s}''_{t+1} \leq \mathbf{S}$, and $q(\mathbf{s}''_{t+1} - \mathbf{s}_t, \mathbf{d}_t) = 0$. Because $U(\mathbf{s}''_{t+1}) \leq U(\mathbf{s}'_{t+1})$ by the induction hypothesis, $\mathbf{s}''_{t+1}$ must also minimize $U(\mathbf{s}_{t+1})$. Hence, there exists $\mathbf{s}^*_{t+1}$ satisfying $q(\mathbf{s}^*_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \geq 0$ and minimizing (2.54), (i) holds and equation (2.24) minimizes total cost.
(ii) If $q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t) < 0$, then

$$U(\mathbf{S}) = \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{S}, \mathbf{d}_{t+1})\big] \leq \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})\big] \leq U(\mathbf{s}_{t+1}), \qquad \text{for all } \mathbf{s}_{t+1} \in \mathcal{A}.$$

Hence, $\mathbf{s}^*_{t+1} = \mathbf{S}$ and produce nothing at G is optimal. These actions involve zero production cost in period t and minimum expected cost from period $t+1$ onward. Therefore, the minimum of (2.54) is indeed $V_t(\mathbf{s}_t, \mathbf{d}_t)$, leading to equation (2.24).

Finally, we find the amount of curtailment. The maximum curtailment at leaf node i is the excess energy that cannot be stored locally, i.e., $(-d^i_t - (S^i - s^i_t)/\alpha)^+$. Let $w^i_t \in \big[0, (-d^i_t - (S^i - s^i_t)/\alpha)^+\big]$ be the distributed generation curtailed at node i. If $w^i_t > 0$, then clearly, $d^i_t + w^i_t + (S^i - s^i_t)/\alpha \leq 0$. The curtailment w^i_t must be such that the total flows at G sum up to zero: $(S^G - s^G_t)/\alpha + \sum_{i=A,B} \psi_\beta\big(d^i_t + w^i_t + (S^i - s^i_t)/\alpha\big) = 0$. Thus,

$$(S^G - s^G_t)/\alpha + \sum_{i=A,B} \big[\psi_\beta\big(d^i_t + (S^i - s^i_t)/\alpha\big) + \beta w^i_t\big] = 0,$$

which is equivalent to $w^A_t + w^B_t = -q(\mathbf{S} - \mathbf{s}_t, \mathbf{d}_t)/\beta$, as stated in part (ii) of the theorem.

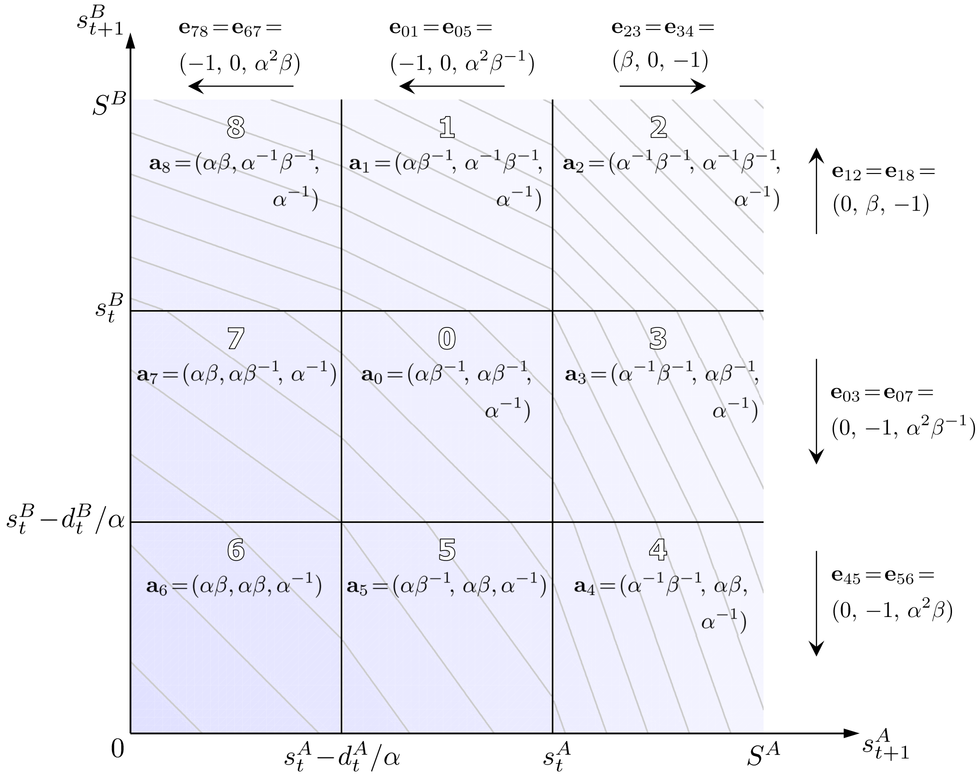
To complete the induction, we show $V_t(\mathbf{s}_t, \mathbf{d}_t)$ decreases in $\mathbf{s}_t$ for any $\mathbf{d}_t$. Let $\mathbf{s}^*_{t+1}$ be an optimal decision for state $(\mathbf{s}_t, \mathbf{d}_t)$. For any $\widetilde{\mathbf{s}}_t \geq \mathbf{s}_t$, we have

$$
\begin{aligned}
V_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) &\leq c(q(\mathbf{s}^*_{t+1} - \widetilde{\mathbf{s}}_t, \mathbf{d}_t)) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})\big] \\
&\leq c(q(\mathbf{s}^*_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + \gamma \mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})\big] = V_t(\mathbf{s}_t, \mathbf{d}_t).
\end{aligned}
$$

□

Figure 2.13: Contours of $\mathcal{A}(q_t)$, faces, and edges

(a) $s^G_{t+1} > s^G_t$: store energy in G



(b) $s^G_{t+1} < s^G_t$: release energy from G

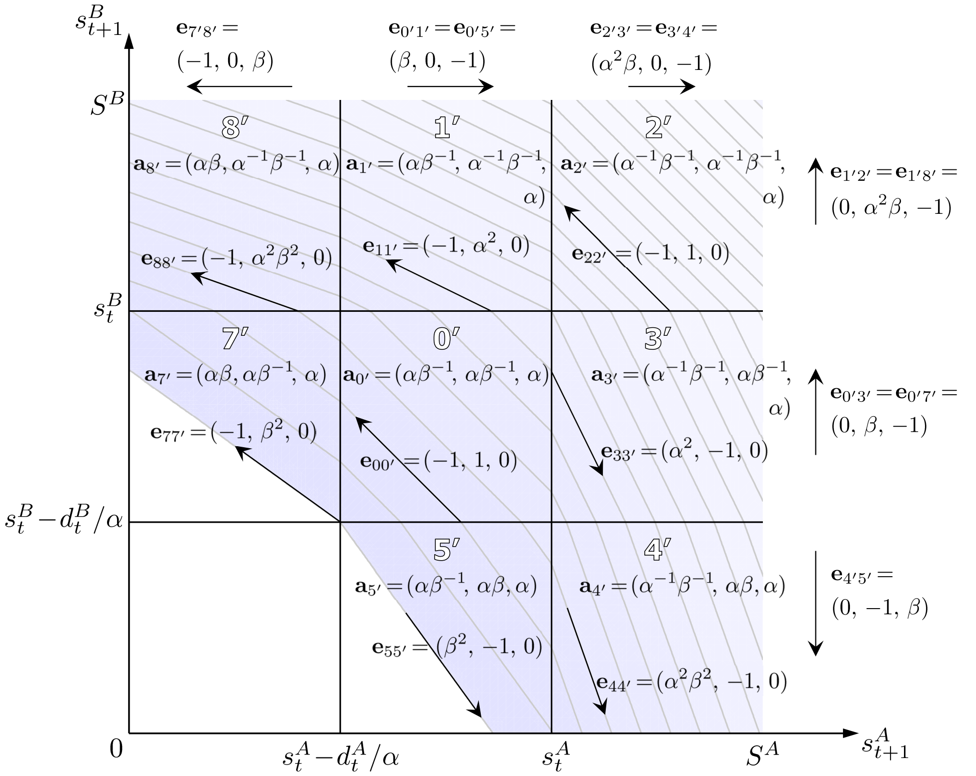


Figure 2.14: Iso-production surface $\mathcal{A}(q_t)$ for Theorem II.9

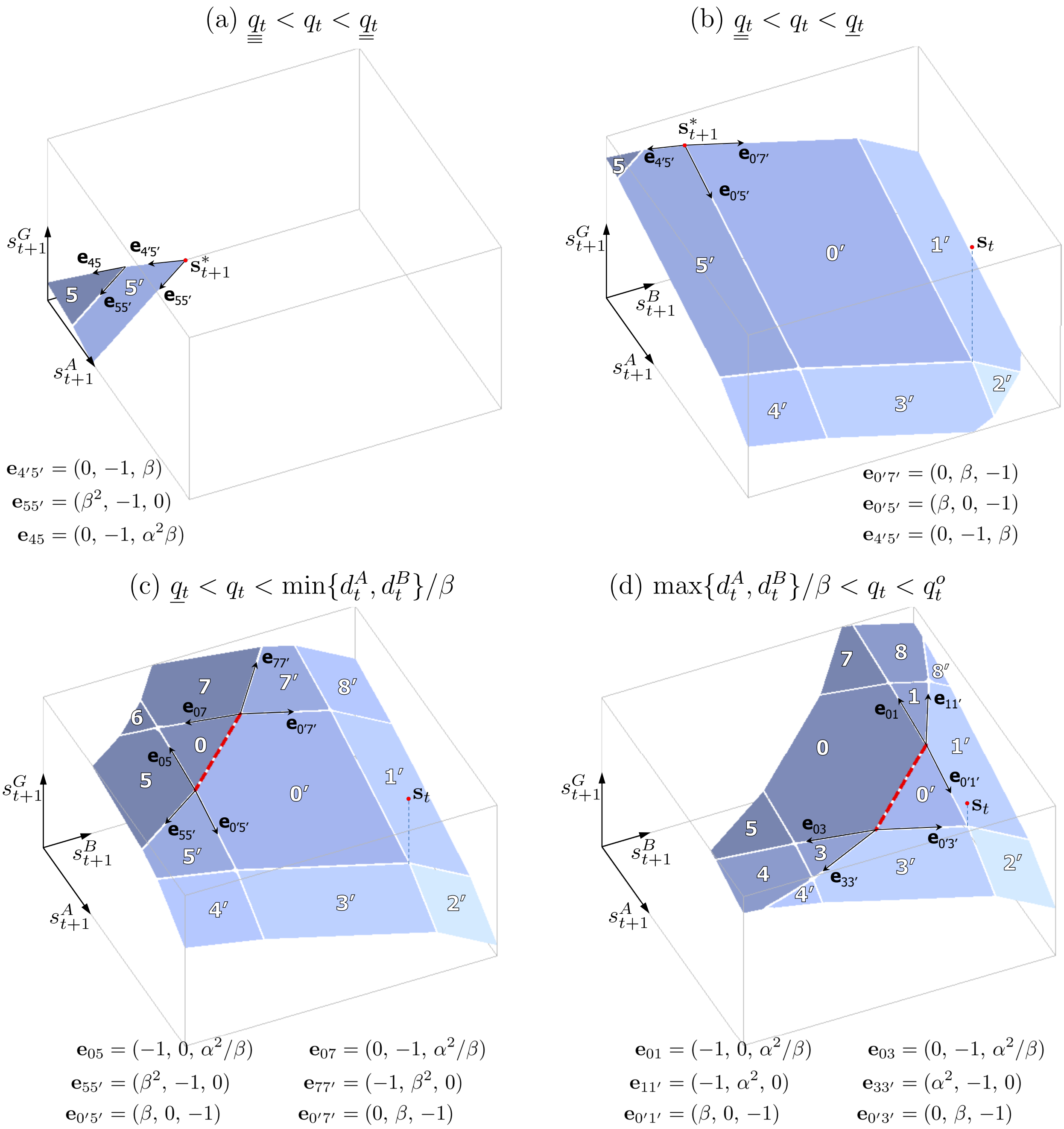


Figure 2.15: Iso-production surface $\mathcal{A}(q_t)$ for Theorem II.11

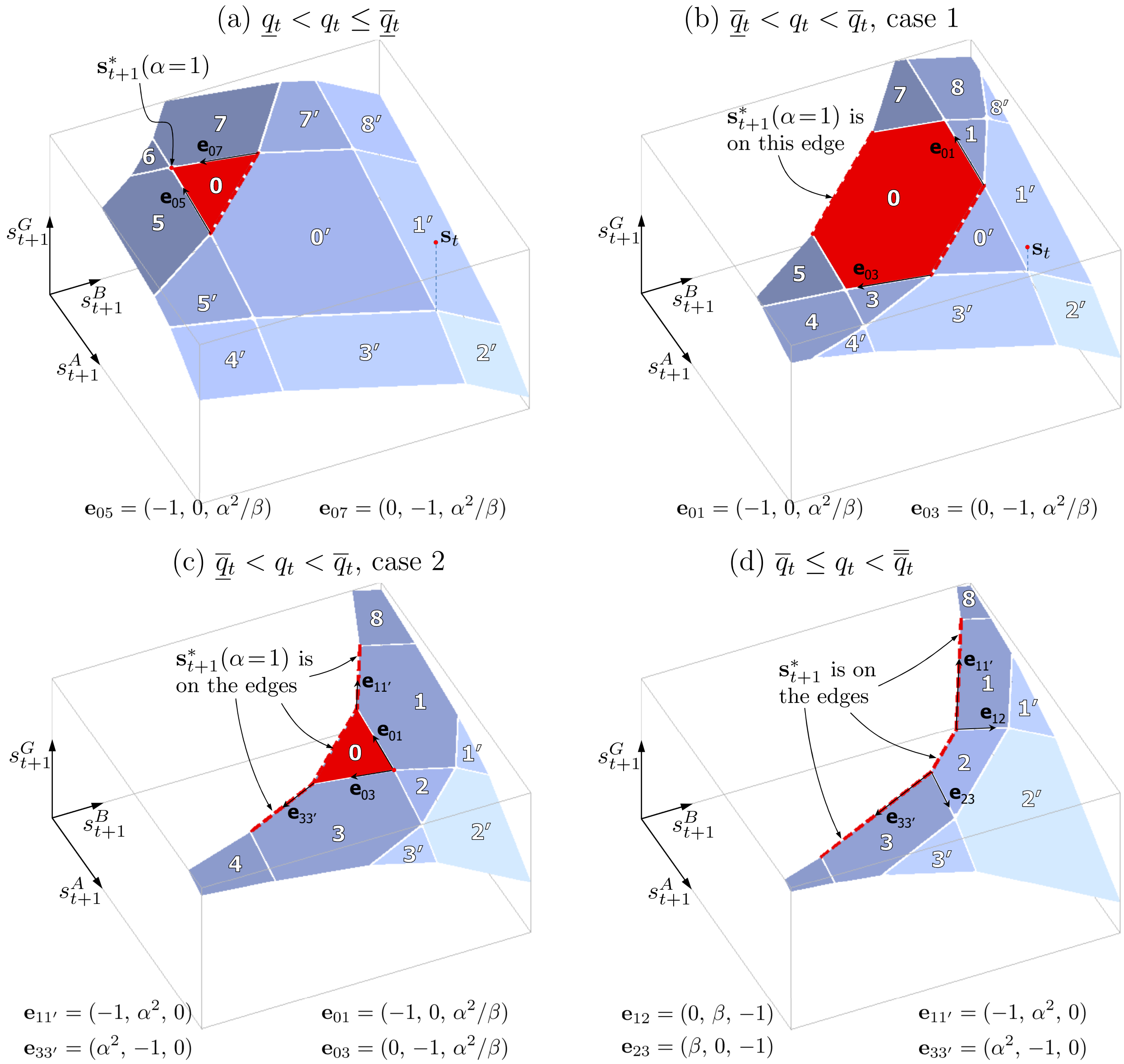
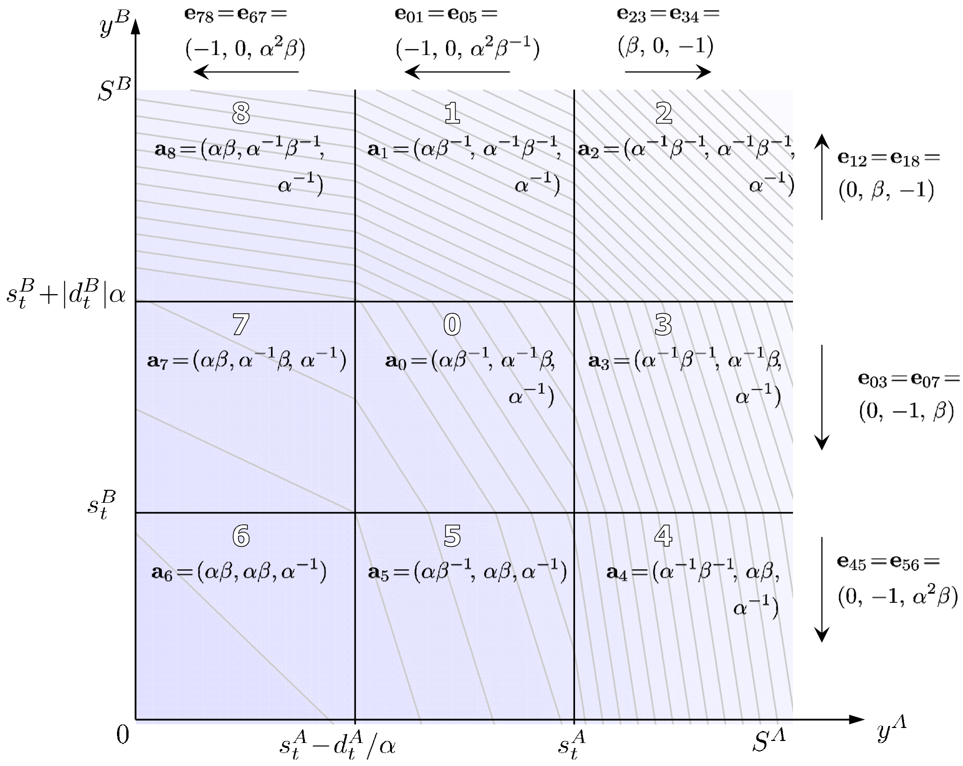


Figure 2.16: Contours of $\mathcal{A}(q_t)$, faces, and edges (under Theorem II.26 only)

(a) $s^G_{t+1} > s^G_t$: store energy in G



(b) $s^G_{t+1} < s^G_t$: release energy from G

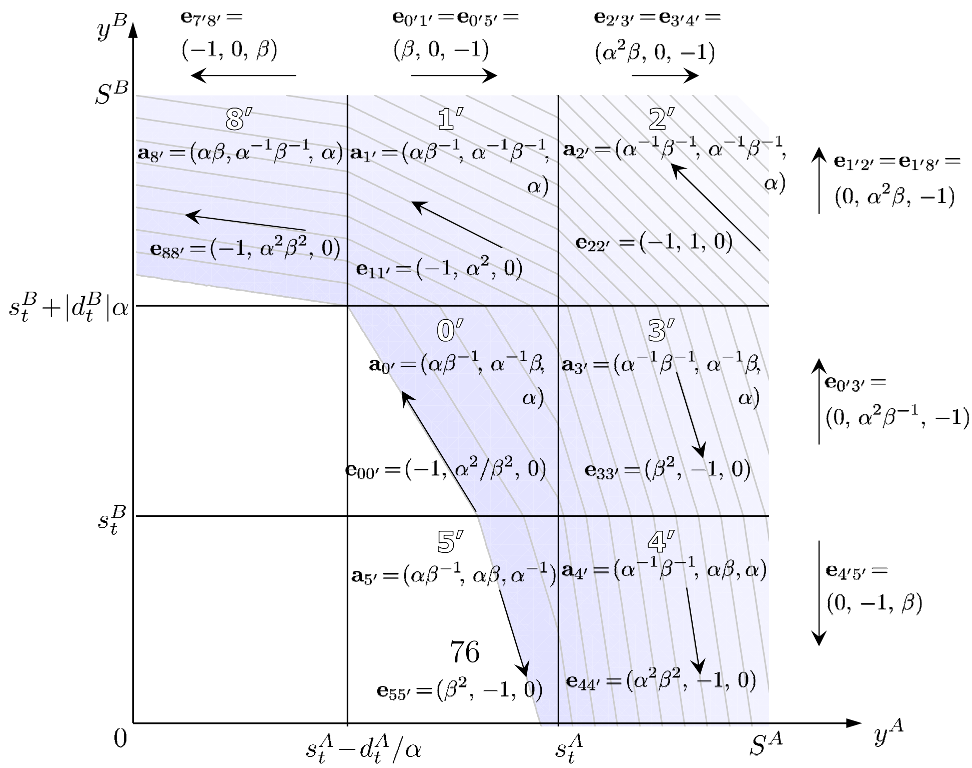
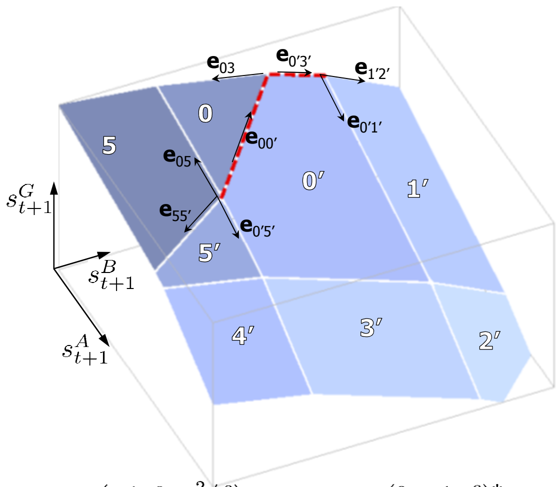


Figure 2.17: Iso-production surface $\mathcal{A}(q_t)$ for Theorem II.26 (ii) and (iii).

(a) $\underline{q}_t < q_t \le \ddot{q}_t$



$\mathbf{e}_{05} = (-1,\ 0,\ \alpha^2/\beta)$

$\mathbf{e}_{55'} = (\beta^2,\ -1,\ 0)$

$\mathbf{e}_{0'5'} = (\beta,\ 0,\ -1)$

$\mathbf{e}_{00'} = (-1,\ \alpha^2/\beta^2,\ 0)*$

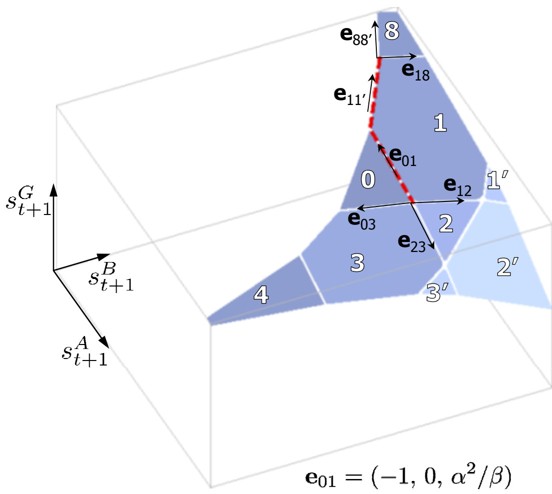
$\mathbf{e}_{03} = (0,\ -1,\ \beta)*$

$\mathbf{e}_{0'3'} = (0,\ \alpha^2/\beta,\ -1)*$

$\mathbf{e}_{1'2'} = (0,\ \alpha^2\beta,\ -1)$

$\mathbf{e}_{0'1'} = (\beta,\ 0,\ -1)$

(b) $\ddot{q}_t < q_t \le \overline{q}_t$, case 1



$\mathbf{e}_{01} = (-1,\ 0,\ \alpha^2/\beta)$

$\mathbf{e}_{03} = (0,\ -1,\ \beta)*$

$\mathbf{e}_{11'} = (-1,\ \alpha^2,\ 0)$

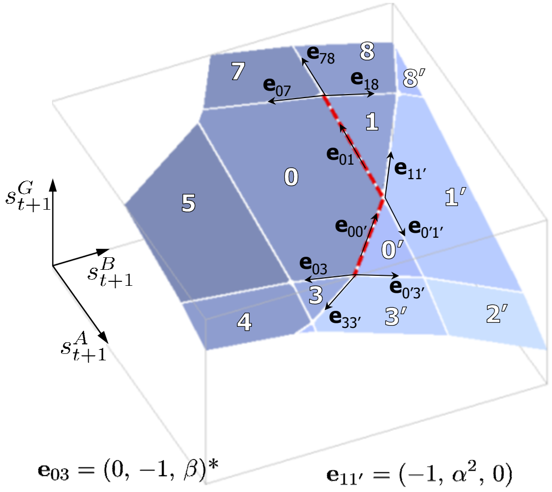
$\mathbf{e}_{23} = (\beta,\ 0,\ -1)$

$\mathbf{e}_{18} = (0,\ \beta,\ -1)$

$\mathbf{e}_{12} = (0,\ \beta,\ -1)$

$\mathbf{e}_{88'} = (-1,\ \alpha^2\beta^2,\ 0)$

(c) $\ddot{q}_t < q_t \le \overline{q}_t$, case 2



$\mathbf{e}_{03} = (0,\ -1,\ \beta)*$

$\mathbf{e}_{33'} = (\beta^2,\ -1,\ 0)*$

$\mathbf{e}_{0'3'} = (0,\ \alpha^2/\beta,\ -1)*$

$\mathbf{e}_{00'} = (-1,\ \alpha^2/\beta^2,\ 0)*$

$\mathbf{e}_{01} = (-1,\ 0,\ \alpha^2/\beta)$

$\mathbf{e}_{11'} = (-1,\ \alpha^2,\ 0)$

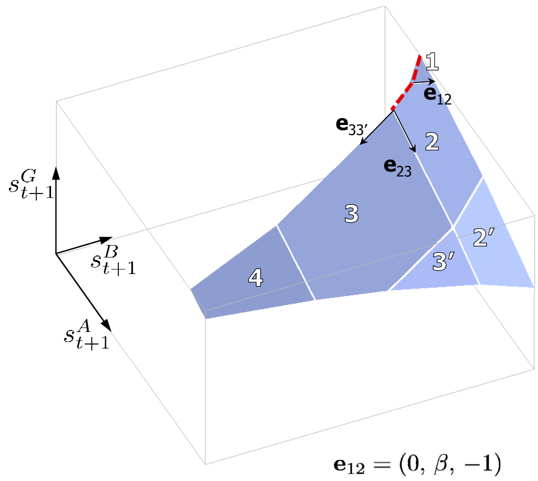
$\mathbf{e}_{0'1'} = (\beta,\ 0,\ -1)$

$\mathbf{e}_{07} = (0,\ -1,\ \beta)*$

$\mathbf{e}_{78} = (-1,\ 0,\ \alpha^2\beta)$

$\mathbf{e}_{18} = (0,\ \beta,\ -1)$

(d) $\overline{q}_t < q_t \le \overline{\overline{\overline{q}}}_t$



$\mathbf{e}_{12} = (0,\ \beta,\ -1)$

$\mathbf{e}_{33'} = (\beta^2,\ -1,\ 0)*$

$\mathbf{e}_{23} = (\beta,\ 0,\ -1)$

* Directions different from Theorems 1 and 2

CHAPTER III

Choice of Storage Technology for an Electrical System

3.1. Introduction

The previous chapter considers the operation and investment of storage units across the grid. It assumes, however, the same storage technology at all locations on the grid. In reality, there is a multitude of storage technologies available and energy companies are often deciding on the technologies they choose to invest in. Each of these technologies have different cost and operational parameters. Some of the parameters to compare storage technologies include power output, energy storage capacity, cost per unit capacity, conversion efficiency, system power ratings, lifetime charge-discharge cycles, weight energy density, volume energy density, maintenance and operational costs. Depending on these parameters, different storage technologies are better suited for different grid services (San Martin et al. 2011). Interestingly, in order to increase revenues, storage owners have begun to simultaneously provide multiple services to the grid, such as frequency regulation and arbitrage (Hobby 2012). We observe, interestingly, that there may be some synergies in using multiple technologies simultaneously to provide the same service.

Through out this dissertation, we consider the storage service of production smoothing, or 'arbitrage'. Arbitrage (or load shifting) is one of the key roles of storage in the grid today, (Sreedharan et al. 2012). Storage technology can prevent the usage of expensive natural gas power plants, by shifting peak load during high demand periods, to low demand periods when traditional coal power plants are under utilized. Providing this service, results in arbitrage revenues to storage owners. In this chapter, we aim to understand how technology parameters of cost per unity capacity and conversion efficiency affect the technology choice for this service. In particular, we question whether it is beneficial to use multiple technologies to provide the same service in tandem. We consider this question from both a centralized decision maker and a decentralized storage investor perspective.

The centralized decision maker perspective serves as a benchmark to provide investment insights to policy makers on the types of technologies that are better suited for a grid with given system parameters, while the decentralized perspective allows us to provide insights to storage investors on the types of storage technologies to choose.

We classify technologies based on two significant operational metrics, the cost per unit capacity and efficiency.[1] While storage technology differ on several parameters, their conversion efficiency and cost per unit capacities are major differentiating factors, deciding the economic viability of these technologies (Divya and Østergaard 2009). Hence, we consider the trade-off of choosing between expensive and efficient technology, like flywheels, as opposed to cheaper but inefficient technology like Pumped Storage. We address the questions related to optimization of a portfolio of storage technologies, providing arbitrage service to the grid. This is analogous to the investment in multiple technologies in manufacturing and fleet optimization (Wang et al. 2013).

Our work provides several insights for both the centralized and decentralized perspectives:

Under the centralized perspective, we investigate situations where it may be beneficial to invest in and operate multiple technologies (or 'mix' investment), as illustrated by the example later in this section. However, while it is beneficial to invest in multiple technologies, we show evidence that the marginal benefit of the flexibility of being able to invest in and operate multiple technologies (or 'mixing' technologies) is decreasing in the number of technologies, both numerically and analytically. As decision makers are often making trade-offs on the purchase of storage technologies, and there are operational overheads of purchasing, building expertise on and operating multiple technologies; our result offers some insights for decision makers while choosing the portfolio of technologies to invest in. We show various properties of the optimal storage portfolio of technologies, allowing us to improve the

[1] We assume the technology life cycles are long enough that they do not affect the costs in our horizon. We do not consider other technology parameters in our model. This is a limitation of this model.

algorithmic efficiency of choosing technologies. In particular, we identify the 'convex hull' property of the optimal set of technologies, which allows us to neglect any technology that is in the convex hull formed by plotting the remaining technologies on a cost per unit capacity vs. loss rate plot. Here, loss rate refers to the inverse of round trip efficiency.

Next, we also study the impact of various system parameters on the optimal technology choice selection problem using a series of examples and numerical study. Specifically, we observe that the cheaper technology is invested more when renewable penetration increases. Interestingly, some counter intuitive messages also arise. We observe that, for instance, under fixed investment budget scenarios, in some technology portfolios, if one technology gets better (i.e., greater efficiency or cheaper cost/capacity due to research and development), we may invest less in that technology.

Further, in order to understand the investment problem, we also attempt the operation problem. We identify the structure of the optimal policy of operating multiple storage units in tandem using a stochastic dynamic program framework, reduce the dimensionality of the operational problem.

Under the decentralized scenario, we consider the reality of the industry today, and observe that most companies choose to invest in not more than one technology for a given service, and for load shifting applications in particular. We observe that while there may be synergies of 'mixing' storage technologies from the central planner perspective, an individual storage owner may not be persuaded because of their beliefs on the price path of energy that is traded in the system. This is also the predominant modeling in most academic literature today. We consider two of the most common assumptions in the literature: small storage (inelastic prices) and large storage (myopic markets). We show that it is always optimal to invest in a single technology under the 'inelastic prices' assumption (i.e., storage owners actions don't affect the price of energy) and the benefit of mixing is substantially reduced under the 'myopic markets assumption' (i.e., the price of energy is the current period marginal cost of energy production). This leads to one explanation why companies may choose not to invest in a portfolio of storage technologies, even though it may be beneficial to 'mix' from a central planner perspective.

The rest of this chapter is organized as follows: We end this section with a simple motivating example demonstrating the synergies of simultaneously investing and operating in multiple technologies. Section 4.2 comprises a survey of the relevant literature, Section 3.3 describes the model, Section 3.4 considers the analysis from the perspective of the central decision maker, providing several insights on the benefits of operating a portfolio of storage facilities. Section 3.5 describes the analysis under the small storage and large storage assumptions. We conclude this chapter in Section 4.6 with a brief summary of our learnings and some discussion on future research.

3.1.1 Motivating Example

Consider a simple 2-period system representing low-demand night and the high-demand day scenario. We consider total cost function of producing Q units of energy, in a format of a simple quadratic function $C(Q, K) = 10K + 19Q + 6Q^2/K$, defined for $Q \leq K$, where $K =$ 1000 MWh represents the total production capacity of the aggregate coal-fired intermediate gas plants. The natural gas peaking power plant is used to supplement demand in excess of K during any period at a rate of 50$/MWh. Given a limited budget B of $ 180 million, the objective is to invest in a subset of the available storage technologies: flywheels or lead-acid batteries so as to minimize the net operating cost. The basic attributes of the storage technologies are as follows (ES-Select 2012):

- Flywheels: Investment Cost= 1.6 Million $/MWh, efficiency= 86%
- Lead Acid Batteries: Investment Cost= 0.56 Million $/MWh , efficiency= 65%

Consider three potential investment choices: either invest in only flywheels or only in lead acid batteries or invest half the capital in each; in order to run the system for 2 periods.
Period 1 (Night) demand: 500 MWh
Period 2 (Day) demand: 1500 MWh

Table 3.1 summarizes the costs in each of three scenarios as well as the default case with no storage investment.

In this simplest deterministic two period scenario, with linear investment and convex generating costs. We observe an 8.57% cost decrease due to use of both types of storage. This example demonstrates the benefits of both the technologies: for the same budget,

	No Storage Case	Only Flywheels	Only Lead Acid Battery	Using both
Period 1 Production (MWh)	500.0	630.8	994.5	812.7
Period 2 Production (MWh)	1500.0	1387.5	1178.6	1283.0
Total Cost (M $)	81.0	78.7	78.8	78.6

Table 3.1: This table illustrates the benefit of investing in multiple storage technologies.

using solely the more efficient technology results in minimal conversion losses, while using solely the cheaper technology provides greater smoothing due to higher storage capacity. However, while mixing, initially, it appears that the inefficient technology has higher value per dollar (= value per capacity * capacity per dollar) and as the capacity invested increases, the efficient technology will have higher value for dollar. This is because, even though the efficient technology always has higher benefit per unit capacity, the relative difference in the benefits per unit capacity between the two technologies increases as more storage capacity is invested (due to the convexity of the value function), making the more efficient technology have higher value per dollar as storage capacity increases.

3.2. Literature Survey

This paper deals with capacity investment and operation of multiple storage technologies. Two streams of literature are relevant: the literature on energy storage and the traditional OM literature on capacity investment in multiple technologies.

3.2.1 Energy storage

In this subsection, we first discuss the various literature on energy storage services, and specifically, the arbitrage service that we study in our paper. Then, we discuss the problem of 'mixing' storage technologies and related literature.

Energy storage is gaining increasing attention due to its applications to several grid services. The grid services include frequency regulation (Oudalov et al. 2007), system stability (Mercier et al. 2009), load shifting (Even et al. 1993) and spinning reserve (Kottick et al. 1993). Interestingly, each of these services is historically served by significantly different storage technologies (Denholm et al. 2010), with different parameters for cost per unit capacity, conversion efficiency, charging and discharging rates etc. A review of the different technologies best suited for each of these different services is in Hadjipaschalis et al. (2009). Further, Schoenung and Hassenzahl (2003) consider an extensive comparison of the different storage technologies in terms of capital cost, operation and maintenance, efficiency, parasitic cost, replacement costs to perform a life-cycle cost of this analysis. Recently, Xi et al. (2011) consider using the same technology to provide multiple services in tandem, since each of these services is relevant to the grid at different times of the day. Of the several services of storage, the use of load shifting is increasing because of growing use of intermittent renewables to provide energy to the grid (Zeng et al. 2006, Arulampalam et al. 2006, Teleke et al. 2010). Such renewable generation systems are typically built in conjunction with a storage unit in order to smooth the output flow. This service is typically provided by Compressed Air Energy Systems (CAES), Battery Energy Storage Systems (BESS) and Pumped Storage Systems (PSS) technologies. While the current storage market is dominated by Pumped Storage Systems, Dunn et al. (2011) suggest that battery technologies are beginning to offer high value opportunities.

The problem of choosing technologies to provide arbitrage service is practically important, especially because both the capital costs of storage and the potential benefits can range to the millions of dollars (Alt et al. 1997). Several papers consider the optimal technology selection, sizing and operating of storage under different operating conditions (Lee and Chen 1993, Yoshimoto et al. 2006, Banos et al. 2006, Oudalov et al. 2006), but the focus is on choosing a single best technology, while we investigate the possibility of investing and simultaneously operating storage facilities with multiple technologies or 'mixing' technologies. We are not aware of any other paper other than Kraning et al. (2011) that model the simultaneous operation of multiple technologies. They consider a similar model and demonstrate numerically, that using multiple technologies is beneficial using Receding Horizon Control (RHC) methodology. We extend their work, by considering the features of the set of technologies and describe the 'efficient frontier' of the technology set. We also provide analytical structure to the optimal operating and investment policy to the grid.

Optimal operation and investment of storage technologies in the grid is a well studied problem in the Energy Storage literature. See Mokrian and Stephen (2006) for a comprehensive review of methodology related to evaluation of the economic benefit of energy storage from arbitrage, operating in a market with exogenous prices. In contrast, our work focuses on both the system perspective and the decentralized perspective. In order to understand the system perspective, we consider one of the main reasons for the variation in energy prices in markets: the variation in marginal cost of energy due to the use of expensive sources of energy, such as natural gas power plants, during high demand periods. We model this as convexity in the production cost function similar to Wu and Kapuscinski (2013). Several other recent papers consider the perspective of revenue maximizing storage owner. These papers also differ from our paper in the methodology. Korpaas et al. (2003), Castronuovo and Lopes (2004), Brunetto and Tina (2007), attempt this problem as a deterministic optimization problem given a particular sample path over a finite horizon and then averaging the results over the sample paths. Bitar et al. (2010), Bitar et al. (2011), Kim and Powell (2011) consider a stochastic generalization of this problem. They derive closed form expressions for the value of storage under certain special cases of the energy price and wind distributions, to help evaluate storage investments. We focus on deriving operational insights in more general demand situations.

3.2.2 Inventory Control models

The electric energy storage optimization problem is similar to a classical multi-period inventory problem with stochastic demand. While most of the inventory literature assumes linear production costs, energy markets have convex production costs. A treatment of the traditional inventory model with convex production costs can be found in Karlin (1960). Another significant difference between electric energy storage and traditional inventory models is the upfront efficiency loss incurred when inventory is added to a buffer. Thus, storage conversion losses act as a one time non-linear holding cost, which depends on the production cost function as well as the production quantity during that period. In energy markets, production is generally load-following and we cannot allow for unsatisfied demand, unlike traditional inventory models where there's a lead time for production and stock outs are permitted. We extend the inventory control literature by providing another model with practical applications in Energy markets.

3.2.3 Multiple Technologies in Operations Management

We are not aware of any paper in OM literature that considers inventory in multiple storage technologies, but there are papers that consider multiple technologies in a variety of other settings including manufacturing and fleet control. All the papers we discuss in this subsection deal with capacity adjustment in multiple technologies, similar to our work. However, we capacity of consider storage technologies where as these papers consider production technology capacities.

In their seminal paper, Crew and Kleindorfer (1976) consider the problem of investing in different plant types (technologies) when demand is stochastic and price dependent. They describe the efficient technological frontier and the optimal production policy. They also discuss the optimal pricing scheme from the perspective of a public utility maximizing the welfare of society. Chao (1983) and Kleindorfer and Fernando (1993) extend this work to include supply uncertainty. The production cost is linear in these papers. Drake et al. (2010) consider capacity investment into two technologies with different emission intensities and uncertainty in the emissions allowances of the future. In contrast, our model considers a non-linear production cost with multiple technologies in an energy setting. We also model multiple storage technologies with different cost structures from standard OR literature.

In a linear production cost setting, several papers consider capacity size adjustment. Dixit (1997) and Eberly and Van Mieghem (1997) include capacity adjustment cost associated with changing capacity during the horizon. They, however do not consider technology selection. More recently, Kleindorfer et al. (2012) and Wang et al. (2013) consider the problem of co-investing in more than one technology, with uncertainty in operating costs and dynamic capacity adjustment and identify a control-limit policy structure for capacity adjustment. In contrast, our work considers a static setting with multiple technologies but focus on technology selection. Our application for the electric grid implies a different cost structure with a non-linear production cost model. The trade-off between cost of investment in storage and operating efficiency, is different from the traditional fixed cost-variable cost trade-off considered in the papers above.

Our paper also considers the uncertainty in future demand and it's impact on the storage investment decision. Uncertain demand leading to uncertain operating cost in production planning has been considered, in different settings, in Ding et al. (2007), Kazaz et al. (2005), Plambeck and Taylor (2011). Several papers consider dynamic capacity adjustment of a single technology under uncertainty settings (Chao et al. 2009). However, we consider capacity adjustment of multiple technologies under a static setting. Our paper also identifies the intuition of decreasing marginal benefit of the flexibility of being able to invest in, and operate multiple technologies in tandem. Similar decreasing marginal benefits of flexibility in queuing systems have been shown in Bassamboo et al. (2012). In contrast to all these papers, our model considers a non-linear cost function. Hence, our trade-off of investing in a portfolio of technologies applies under deterministic as well as under stochastic demand.

3.3. The Model

We consider a problem of investing in energy storage facilities built from $M(\geq 2)$ available technologies. The objective is to minimize the combined cost of investment and operations of the system serving demands, that can use storage facility. We use $t \in \{1, 2, \ldots, T\}$ to index time periods. Storage facilities are built prior to $t = 1$, and once built, the storage size is fixed throughout the horizon. First, we describe the setting of storing and releasing energy. In the subsections that follow, we consider the objective function. Both the system operator and storage investor perspectives will be considered.

Assumption III.1 (Storage)**.** *(i) The storage size is a continuous decision variable and the cost of storage facility is linear in its size. (ii) Storage can be filled up or emptied within one period. (iii) Energy loss takes place when injecting energy to storage. The loss is linear in the amount of energy injected.*

In this paper, "storage level" or "inventory level" refer to the amount of energy that a storage facility can release until empty. We denote a storage investment decision by $\mathbf{S} = (S^j)_{j=1}^M$, where $S^j \geq 0$ is the storage size (i.e. maximum storage level) of the facility with technology j. Note that $S^j = 0$ implies we choose not to invest in technology j.

The round-trip efficiency of technology j is the product of the charging and discharging efficiencies of that technology. We denote the loss factor σ^j as the inverse of the efficiency. Thus, storing $\sigma^j(> 1)$ units of energy into storage facility of technology j accounts for an inventory level increase of 1 unit. $\boldsymbol{\sigma} = (\sigma^j)_{j=1}^M$ represents the vector of loss factors. The energy flow into storage corresponding to change of storage by δ (positive for storing, negative for releasing), is:[2]

$$\psi^j(\delta) \stackrel{\text{def}}{=} \begin{cases} \delta\sigma^j, & \text{if } \delta \geq 0, \\ \delta, & \text{if } \delta < 0. \end{cases} \tag{3.1}$$

We denote cost of storage technology by $\mathbf{c} = (c^j)_{j=1}^M$, where investing in technology j with facility size S^j costs $c^j S^j$. Each technology j is represented by the parameters (σ^j, c^j). We define $\Omega = \{(\sigma^j, c^j) : 1 \leq j \leq M\}$ as the set of available technologies. Without loss of generality, we assume $\sigma^1 < \sigma^2 < \cdots < \sigma^M$ and $c^1 > c^2 > \ldots > c^M$ (i.e., more expensive and less efficient technology can be eliminated from consideration).

The storage level at the beginning of period t is denoted as $\mathbf{s}_t = (s_t^j)_{j=1}^M$. The feasible storage levels are in the set $\mathcal{A} \stackrel{\text{def}}{=} \{\mathbf{s} : 0 \leq \mathbf{s} \leq \mathbf{S}\}$.

We consider an electrical system with stochastic demand. Let d_t denote the total demand across the grid in period t. We assume d_t is a deterministic function of a vector of Markovian states, $\mathbf{d}_t$, which include the factors driving the demand. We assume there are no transmission losses or transmission constraints in the grid.

3.3.1 Central Decision Maker Perspective

We first formulate storage investment problem from the perspective of a central decision maker who considers given set of technologies Ω and invests in storage capacity S^j for $j = 1, \ldots, M$ and then operates the system across multiple periods. We consider this as a benchmark model, to help understand the perspective of the first best for the system. The storage investment decision is made prior to period 1 and no additional investment or divestment can be made over the planning horizon. When operating the system, the sequence of events in each period is as follows; at the beginning of period t, the system operator observes the period-starting storage level $\mathbf{s}_t$ and the state $\mathbf{d}_t$. The corresponding demand $d_t(> 0)$ must be satisfied in period t. The system operator decides the period-ending storage level $\mathbf{s}_{t+1} = (s_{t+1}^j)_{j=1}^M \in \mathcal{A}$. Energy balance during period t implies the total

[2]Note that the storage efficiency is accounted for differently in this chapter, as compared to the previous chapters.

power plant generation, denoted as q_t, as a function of demand d_t and inventory change $\Delta \mathbf{s}_t = \mathbf{s}_{t+1} - \mathbf{s}_t$:

$$q_t = q(\Delta \mathbf{s}_t, \mathbf{d}_t) \stackrel{\text{def}}{=} d_t + \sum_{j=1}^{M} \psi^j(\Delta s_t^j) \tag{3.2}$$

Let $C(q_t)$ denote the production cost during period t, for $q_t \geq 0$.

Assumption III.2 (Production). *(i) $C(q)$ is strictly convex and increasing in q for $q \geq 0$. (ii) The total power plant generation, q, can be adjusted to any desired level at negligible cost;*

The convexity holds in practice and is typically assumed in the literature, e.g., Bessembinder and Lemmon (2006), Madrigal and Quintana (2000), Dominguez-Garcia et al. (2012), Alvarez Lopez et al. (2010). The seminal reference for power generation and operation in the industry, Wood and Wollenberg (1996), use a convex approximation for costs of power generation in thermal plants.

We aim to decide an energy storage investment strategy and corresponding operating policy that satisfies the demand at minimum cost. To evaluate a storage investment decision $\mathbf{S}$, we define $V_t^o(\mathbf{s}_t, \mathbf{d}_t)$ as the minimum expected discounted operating costs of the system from period t onward when the initial state is $(\mathbf{s}_t, \mathbf{d}_t)$. $\gamma \in (0, 1]$ is the discount factor. The optimal operating policy is determined by the following stochastic dynamic program:

$$V_t^o(\mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1} \in \mathcal{A}} \Big\{ C(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + \gamma \mathsf{E}_t[V_{t+1}^o(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})] \Big\}, \qquad t = 1, \ldots, T, \tag{3.3}$$

$$s.t. \quad q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \geq 0, \tag{3.4}$$

where E_t denotes the expectation conditioning with respect to $\mathbf{d}_t$. The terminal condition is $V_{T+1}^o(\cdot, \cdot) = 0$. We define $U^o(\mathbf{S}, \boldsymbol{\sigma}) = \mathsf{E}_0[V_1^o(\mathbf{0}, \mathbf{d}_1 | \mathbf{S}, \boldsymbol{\sigma})]$, as the minimum expected operating cost at the time storage investment is made, for given storage size $\mathbf{S}$, whose loss factor is given by $\boldsymbol{\sigma}$. The storage investment decision trades off between the upfront investment cost $\mathbf{c} \cdot \mathbf{S} = \sum_{j=1}^{M} c^j S^j$ and the ongoing operating cost $U^o(\mathbf{S}, \boldsymbol{\sigma})$. To capture the liquidity issues faced by decision makers, we set a budget constraint B for the storage investment. Thus, the investment is decided by solving

$$\overline{V}^o(B) = \min_{\mathbf{S}\geq 0} \{\mathbf{c}\cdot\mathbf{S} + U^o(\mathbf{S},\boldsymbol{\sigma})\}, \tag{3.5}$$

$$s.t. \;\; \mathbf{c}\cdot\mathbf{S} \leq B, \tag{3.6}$$

The objective is to make storage portfolio decision taking into account storage investment costs, operating costs, and storage efficiencies, within a limited budget constraint.

3.3.2 Storage Investor Perspective

The perspective of the independent storage investor is different from the central decision maker. Since storage investor invests and operates the storage, we refer to them as storage operator. In a decentralized system, the storage operator invests in storage capacity with the objective of maximizing his profit from buying and selling energy (arbitrage). However, a different entity, the system operator, generates the energy and satisfies the demand, while also buying and selling energy from the storage investor.

The investment decisions $\mathbf{S}$ for the M technologies are made up front. Then, at the beginning of each period, after observing the current storage level $\mathbf{s}_t$ and demand factor $\mathbf{d}_t$, the storage operator decides the final storage level $\mathbf{s}_{t+1} \in \mathcal{A}$. Note that storage decisions imply the power plant generation q_t per (3.2) for the system and the resulting market price for energy. The net energy traded by the storage operator is given by $\sum_{j=1}^{M} \psi^j(\Delta s^j)$ (positive for energy purchased, negative for energy sold).

We convert the maximization problem to the equivalent minimization for ease of comparison of the central decision maker and storage investor perspectives. For given storage investment decision $\mathbf{S}$, we define $V_t^I(\mathbf{s}_t,\mathbf{d}_t)$ to be the storage operator's minimum expected discounted operating costs less revenues from period t onward when the state is $(\mathbf{s}_t,\mathbf{d}_t)$. The optimal operating policy is determined by the following stochastic dynamic program:

$$V_t^I(\mathbf{s}_t,\mathbf{d}_t) = \min_{\mathbf{s}_{t+1}\in\mathcal{A}} \{p_t(\mathbf{s}_{t+1}-\mathbf{s}_t,\mathbf{d}_t)\sum_{j=1}^{M}\psi^j(\Delta s^j) + \gamma\mathsf{E}_t[V_{t+1}^I(\mathbf{s}_{t+1},\mathbf{d}_{t+1})]\}, \qquad t=1,\ldots,T, \tag{3.7}$$

$$s.t. \;\; q(\mathbf{s}_{t+1}-\mathbf{s}_t,\mathbf{d}_t) \geq 0. \tag{3.8}$$

where $p_t(\Delta\mathbf{s}_t, \mathbf{d}_t)$ is the unit price of energy to be stored or released in period t. The terminal condition is $V^I_{T+1}(\cdot,\cdot) = 0$. Note that the storage operator's current period cost is given by $\big(p_t(\Delta\mathbf{s}_t, \mathbf{d}_t) \sum_{j=1}^{M} \psi^j(\Delta s^j)\big)$, and also $V^I_t(.,.)$ may be negative. Similar to the previous section, the net out-flow of cash, for given storage capacity investment decision $\mathbf{S}$, is given by, $U^I(\mathbf{S},\boldsymbol{\sigma}) = \mathsf{E}_0[V^I_1(\mathbf{0}, \mathbf{d}_1|\mathbf{S},\boldsymbol{\sigma})]$. The investment cost $\mathbf{c}\cdot\mathbf{S}$, is bounded by budget B and storage operator's objective is:

$$\overline{V}^I(B,\Omega) = \min_{\mathbf{S}\geq 0} \big\{\mathbf{c}\cdot\mathbf{S} + U^I(\mathbf{S},\boldsymbol{\sigma})\big\}, \tag{3.9}$$

$$s.t. \;\; \mathbf{c}\cdot\mathbf{S} \leq B. \tag{3.10}$$

3.4. Optimal Investment under Central Decision Maker Perspective

We consider here the perspective of the central decision maker trying to minimize total costs of the entire system. We first discuss the structure of the optimal operating policy. The result will help in selecting technologies optimally. We also provide a canonical example to demonstrate the insights and conclude the section with numerical evaluations.

3.4.1 Optimal Operating Policy for given Storage Investment

The constraint in (3.4) defines a non-convex feasible region for $\mathbf{s}_{t+1}$, which complicates the analysis. Below we show that the problem in (3.3)-(3.4) is equivalent to a problem without constraint (3.4).

Lemma III.3. *(i) For fixed* $\mathbf{S}$*, if* $\{\mathbf{s}^*_{t+1}\}$ *is an optimal policy for* (3.3)-(3.4)*, then it is also optimal for the following problem:*

$$V^r_t(\mathbf{s}_t, \mathbf{d}_t) = \min_{\mathbf{s}_{t+1}\in\mathcal{A}} \big\{C(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + \gamma\mathsf{E}_t[V^r_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})]\big\}, \qquad t = 1,\ldots,T, \tag{3.11}$$

where $C(q) \equiv 0$ *for* $q < 0$*, and the terminal condition is* $V^r_{T+1}(\cdot,\cdot) = 0$.
(ii) $V^o_t(\mathbf{s}_t, \mathbf{d}_t) = V^r_t(\mathbf{s}_t, \mathbf{d}_t)$ *for any* $(\mathbf{s}_t, \mathbf{d}_t)$ *and* $t = 1,\ldots,T$.
(iii) $V^o_t(\mathbf{s}_t, \mathbf{d}_t)$ *is jointly convex and decreasing in* $\mathbf{s}_t$*, for any fixed* $\mathbf{d}_t$*, and any* $t = 1,\ldots,T$.

The proof of this Lemma mimics proof of Lemma 1 from Chapter II. The above lemma shows that relaxing the constraint (3.4) has no effect on the value function and optimality of policies. Thus, from this point onward, we focus on problem (3.11). We next prove additional properties of the value function, which are useful in deriving the optimal policy.

Lemma III.4. *Let $\mathbf{s}_t, \widetilde{\mathbf{s}}_t \in \mathcal{A}$ satisfy one of the two conditions below,*
(i) $\widetilde{s}^i_t = s^i_t + \varepsilon/\sigma^i$, $\widetilde{s}^j_t = s^j_t - \varepsilon/\sigma^j$ for some $1 \leq i < j \leq M$, $\varepsilon \geq 0$ and $\widetilde{s}^k_t = s^k_t$ for all $k \neq i, j$,
(ii) $\widetilde{s}^i_t = s^i_t - \varepsilon$, $\widetilde{s}^j_t = s^j_t + \varepsilon$ for some $1 \leq i < j \leq M$, $\varepsilon \geq 0$ and $\widetilde{s}^k_t = s^k_t$ for all $k \neq i, j$,
Then, we have, $V^o_t(\widetilde{\mathbf{s}}_t) \leq V^o_t(\mathbf{s}_t)$, for any $\mathbf{d}_t$ and $t = 1, \ldots, T$.

Lemma III.4 shows the relative change in the value function for changes in the storage level in each of the facilities 1 through M. Part (i) says that, for $i < j$, storing energy at technology i is more economical than storing at technology j. Part (ii) shows that withdrawing from storage i is preferred compared to storage j. Using these properties, the structure of the optimal policy follows.

Theorem III.5. *For each period $t = 1, 2, \ldots, T$, for given $\mathbf{d}_t$, $\mathbf{s}_t \in \mathcal{A}$, optimal policy $\mathbf{s}_{t+1}$ for state $(\mathbf{s}_t, \mathbf{d}_t)$ satisfies,*

(i) If $s^j_{t+1} > s^j_t$ for any given $1 < j \leq M$, then $s^i_{t+1} = S^i$ for all $i < j$,
(ii) If $s^j_{t+1} < s^j_t$ for any given $1 < j \leq M$, then $s^i_{t+1} = 0$ for all $i < j$,
(iii) For $M = 2$, for any $\widehat{\mathbf{s}}_t \in \mathcal{A}$, if $\widehat{\mathbf{s}}_t \leq \mathbf{s}_t$, then corresponding optimal policy $\widehat{\mathbf{s}}_{t+1}$ satisfies, $q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t) \leq q(\widehat{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t, \mathbf{d}_t)$, i.e., optimal production quantity is monotonically decreasing and $V(\mathbf{s}_t, \mathbf{d}_t)$ is supermodular in $\mathbf{s}_t$ for given $\mathbf{d}_t$.

Part (i)-(ii) describes a structure in storage operations for given production quantity. If production exceeds demand, part (i) implies excess energy is stored in the most efficient facility first until it is full or we run out of excess energy. If more excess energy remains to be stored, the next most efficient storage facility is used. Similarly, if demand exceeds production, excess demand is satisfied using stored energy in the most efficient facility. Energy from a less efficient technology is used only when all storage facilities with greater efficiency are empty. This also implies that either $\mathbf{s}_{t+1} \geq \mathbf{s}_t$ or $\mathbf{s}_{t+1} \leq \mathbf{s}_t$. Clearly, ending states may be non monotonic and thus, the optimal policy is not monotonic. For example, $\widehat{\mathbf{s}}_t = (0, 0)$, $\mathbf{s}_t = (0, 10)$ may have an optimal policy $\widehat{\mathbf{s}}_{t+1} = (1, 0)$, $\mathbf{s}_{t+1} = (0, 10)$ resulting in $\mathbf{s}_{t+1} \ngeq \widehat{\mathbf{s}}_{t+1}$ even though $\mathbf{s}_t \geq \widehat{\mathbf{s}}_t$.

This simple form of optimal policy for storage operations reduces the computational complexity of the problem to one-dimensional decision of production quantity. Furthermore, the monotonicity in optimal production quantity and supermodularity of the value function with respect to $\mathbf{s}_t$, means that the benefit of increasing storage level in one facility decreases as the storage level in another facility increases.

3.4.2 Selection of Technologies and Capacity Investment

Given a set of available technologies, it may be sub-optimal to invest non-zero amounts in all the technologies. In this section, we discuss the properties of the technologies that may be excluded for investment, given a set of technologies. $\Omega = \{(\sigma^j, c^j) : 1 \leq j \leq M\}$; $\overline{V}^o(B, \Omega)$ is the total operating and investment cost, defined in (3.6), for given Ω and B. From this section on, we do not impose the assumption that $\sigma^1 < \sigma^2 < \cdots < \sigma^M$ for a given Ω.

For given budget B and set of technologies Ω, let $\Omega_E^o(B, \Omega)$ be the set of technologies that should not be considered for co-investment with the set of technologies Ω. These are the technologies, (σ^{M+1}, c^{M+1}), that do not decrease the cost. Formally,

$$\Omega_E^o(B, \Omega) \stackrel{\text{def}}{=} \Big\{(\sigma^{M+1}, c^{M+1}) : \overline{V}^o\big(B, \, \Omega \cup \{(\sigma^{M+1}, c^{M+1})\}\big) \geq \overline{V}^o(B, \Omega)\Big\}. \tag{3.12}$$

Obviously, $\overline{V}^o\big(B, \, \Omega \cup \{(\sigma^{M+1}, c^{M+1})\}\big) \geq \overline{V}^o(B, \Omega)$ implies, $\overline{V}^o\big(B, \, \Omega \cup \{(\sigma^{M+1}, c^{M+1})\}\big) = \overline{V}^o(B, \Omega)$ as adding another technology can not increase the costs. Clearly $(\sigma^j, c^j) \in \Omega$ implies $(\sigma^j, c^j) \in \Omega_E^o(B, \Omega)$ for all B. Note that in the optimal investment that gives $\overline{V}^o(B, \Omega)$, not all technologies in Ω, may be invested. We discuss some useful properties of the set $\Omega_E^o(B, \Omega)$ and $\overline{V}^o(B, \Omega)$ in the following theorem.

Theorem III.6. *Let Ω_1, Ω_2 be two finite sets of technologies. The following statements are equivalent:*

a. $\overline{V}^o\big(B, \, \Omega \cup \Omega_1\big) \geq \overline{V}^o(B, \Omega)$ *and* $\overline{V}^o\big(B, \, \Omega \cup \Omega_2\big) \geq \overline{V}^o(B, \Omega)$

b. $\overline{V}^o\big(B, \, \Omega \cup \Omega_1 \cup \Omega_2\big) \geq \overline{V}^o(B, \Omega)$.

c. $\Omega_1 \cup \Omega_2 \subset \Omega_E^o(B, \Omega)$

The basic idea of the benefit of mixing is that, when using multiple technologies, one can achieve costs lower than when using the technologies individually. The above theorem describes that if one set of technologies Ω_1 is dominated by a set Ω of technologies, and another set Ω_2 is dominated by the same set Ω, the combined set $\Omega_1 \cup \Omega_2$ cannot be used to achieve better costs i.e., if two sets individually do not improve costs, then their combined benefit, still does not improve costs. This theorem allows us to consider one technology at a time, with Ω and exclude it entirely from consideration, if it does not help individually.

Now, using the above properties, we describe a useful property of the set, $\Omega_E^o(B, \Omega)$.

Lemma III.7. *Let* $(\sigma, c) = \kappa(\sigma^1, c^1) + (1-\kappa)(\sigma^2, c^2)$ *for some* $0 \le \kappa \le 1$*, then,* $\overline{V}^o(B, \Omega \cup \{(\sigma, c)\}) \ge \overline{V}^o(B, \Omega \cup \{(\sigma^1, c^1), (\sigma^2, c^2)\})$.

Note that the above lemma holds for σ but would not hold for $1/\sigma$. Thus, our definition of parameters was imposed for choosing from the given set of technologies.

From Lemma III.7, we have,

Theorem III.8. $\Omega_E^o(B, \Omega)$ *is convex for budget* $B \ge 0$ *and any given set of technologies* Ω*.*

Theorem III.8 helps to identify dominated technologies, independent of demand distributions. Consider a graph with σ on horizontal axis and c on vertical axis. Each coordinate in the first quadrant with $\sigma \ge 1$ represents a feasible storage technology. The Theorem implies a non-trivial way to eliminate technologies from consideration prior to taking into account the demand and budget information. Lemma III.7 suggests that any technology that is dominated by a technology on the straight line connecting any two technologies of the set Ω may be neglected from consideration. Consequently, any technology in the "convex hull" of the given technologies Ω under consideration and the points $(\min_{1 \le j \le M} \sigma^j, \infty), (\infty, \min_{1 \le j \le M} c^j)$ may be neglected, as shown in Figure 3.1.

We observe that the 'convex hull' defined in Theorem III.8 provides a tight bound as illustrated in the following example.

Example III.9. *Consider,* $\Omega = \{(\sigma^1, c^1), (\sigma^2, c^2)\}$ *and* $T = 2$*. Let* $d_1 = 0$ *and* $d_2 > 0$ *and* $C(q) = q^2/2$*.*

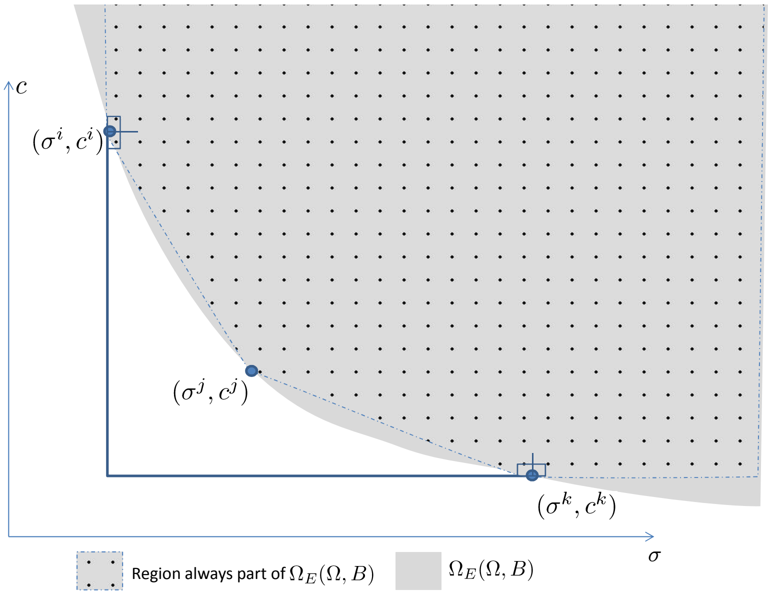


Figure 3.1: The dotted region ("convex hull" of Ω) is always part of $\Omega^o_E(B, \Omega)$ for all $B \geq 0$.

For this example, note that, $\Omega^E(\infty, \{c^1, f^1\}) = \{(c, f)|c > 0, f > 1, c^1 + \sigma^1(d_2 - c^1)(\sigma^1 - \sigma)/(1 + (\sigma^1)^2) \geq c\}$*. Hence, it can be shown that there exist* $d_2 > 0$ *such that,* $\Omega^E(\infty, \Omega) = \{(c, f)|c > 0, f > 1, c^1 - (\sigma - \sigma^1)(c^1 - c^2)/(\sigma^2 - \sigma^1) \leq c\}$*. In other words, for any given point below the line connecting* (σ^1, c^1) *and* (σ^2, c^2)*, we can always find a demand scenario, which does not belong to* $\Omega^E(B, \Omega)$*.*

We now show decreasing marginal benefit from increasing number of technologies. That is, for given a set of two technologies, the benefit of investing in at most one technology, exceeds the incremental benefit of using both technologies instead of one.

Lemma III.10. *For given set of technologies* Ω*, with cost vector* $\mathbf{c}$ *and loss factors* $\boldsymbol{\sigma}$*, we have that,*

(i) $V^o_t(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ *is convex in* $\mathbf{S}$ *for* $t = 1, \ldots, T$ *and therefore,* $U^o(\mathbf{S}, \boldsymbol{\sigma})$ *is convex in* $\mathbf{S}$*.*

Consider $\Omega = \{(\sigma^1, c^1), (\sigma^2, c^2)\}$*, we have the following:*

(ii) For all $t = 0, 1, \ldots, T+1$*, given* $\mathbf{S} = (S^1, S^2)$*, we have,* $V^o_t((s^1_t, s^2_t), \mathbf{d}_t|\mathbf{S}) + V^o_t((0, 0), \mathbf{d}_t|\mathbf{0}) \geq V^o_t((s^1_t, 0), \mathbf{d}_t|(S^1, 0)) + V^o_t((0, s^2_t), \mathbf{d}_t|(0, S^2))$*.*

(iii) Given $\mathbf{S} = (S^1, S^2)$*, we have that,* $U^o(\mathbf{0}, \boldsymbol{\sigma}) + U^o(\mathbf{S}, \boldsymbol{\sigma}) \geq U^o((S^1, 0), \boldsymbol{\sigma}) + U^o((0, S^2), \boldsymbol{\sigma})$

Lemma III.10 shows convexity with respect to $\mathbf{S}$ and supermodularity when $M = 2$. In other words, the benefit of adding one unit of S^1 is decreasing with S^2 and vice versa. This intuitive property is used to show the marginal decreasing benefit of mixing, which we discuss below.

Theorem III.11. *Given a set of technologies, $\Omega = \{(\sigma^1, c^1), (\sigma^2, c^2)\}$ and any $B > 0$, then, $\overline{V}^o(B, \phi) + \overline{V}^o(B, \Omega) \geq 2 \times \min\{\overline{V}^o(B, \{(\sigma^1, c^1)\}), \overline{V}^o(B, \{(\sigma^2, c^2)\})\}$.*

Note that Theorem III.11 means that the benefit of having one technology over no technology is greater than the benefit of having two technologies over one technology, when choosing the best possible of the two available technologies. This shows decreasing marginal benefit of including more flexibility in choosing technologies for two technologies. Operating multiple technologies requires certain costs of operation, maintenance, as well as building the required technological expertise. These additional costs would further reinforce decreasing benefit of multiple technologies.

Further, we show a useful property of the optimal investment decision $\mathbf{S}^*$ and it's sensitivity to the price per unit capacity of storage technologies.

Lemma III.12. *Let $\mathbf{S}^*$ be the optimal investment decision of (3.5) for a set of technologies $\Omega = (\mathbf{c}, \boldsymbol{\sigma})$, when budget B is not binding. Then the optimal investment decision of technology j, S^{j*} is decreasing in c^j for all $1 \leq j \leq M$.*

Proof of Lemma III.12. We prove the lemma by contradiction. Assume that $c_2^j > c_1^j$ and $c_1^k = c_2^k$ for all $k \neq j$. Let $\mathbf{S}_1$, $\mathbf{S}_2$ be the optimal decision for capacity per unit costs $\mathbf{c}_1$ and $\mathbf{c}_2$ respectively. Because of their respective optimalities, we have,

$$U^o(\mathbf{S}_1, \boldsymbol{\sigma}) + \mathbf{c}_1.\mathbf{S}_1 \leq U^o(\mathbf{S}_2, \boldsymbol{\sigma}) + \mathbf{c}_1.\mathbf{S}_2,$$
$$U^o(\mathbf{S}_2, \boldsymbol{\sigma}) + \mathbf{c}_2.\mathbf{S}_2 \leq U^o(\mathbf{S}_1, \boldsymbol{\sigma}) + \mathbf{c}_2.\mathbf{S}_1.$$

Adding the two inequalities gives, that $(c_1^j - c_2^j)(S_1^j - S_2^j) \leq 0$. This implies, $S_1^j \leq S_2^j$. $\square$

Interestingly, for given budget B, while it is tempting to believe that the optimal investment decision $\mathbf{S}^*(B)$ is monotonically increasing, the example in the next section will demonstrate otherwise.

3.4.3 Canonical example

In order to demonstrate the nature of the trade-offs affecting the decision to invest in multiple storage units, we consider a two-period deterministic demand setting with two technologies, i.e., $M = 2$, $i = 1$, $j = 2$.[3] This example, shows how changing parameters affects the storage investment decision and leads to some counter intuitive insights. This example also allows us to revisit (from the motivating example in Section 3.1.1) the trade-off of 'mixing' under deterministic demand, because of non-linear costs.

Let $T = 2$ and $d_1 = d_l$ and $d_2 = d_h$ be the low and high demands of this horizon ($d_l << d_j$). Hence, (3.5) reduces to $\min_{\mathbf{S}\geq 0} \left\{c^i S^i + c^j S^j + C(q_1) + \gamma C(q_2)\right\}$ where $q_1 = d_l + S^i\sigma^i + S^j\sigma^j$ and $q_2 = d_h - S^i - S^j$. The optimal allocation must also satisfy the constraint $c^i S^i + c^j S^j \leq B$. As is standard in the literature (Lu and Shahidehpour 2004), we assume that the cost function $C(q) = Xq^2 + Yq + Z$ is quadratic. Without loss of generality, we assume, $c^i \geq c^j$ and $\sigma^i \leq \sigma^j$.

Note that in a 2 period deterministic setting, it can be shown that it is optimal to invest in at most 2 technologies, from any portfolio of available technologies. We consider two cases in the solution to the problem: when budget B is binding and when budget B is not binding.

3.4.3.1 Case 1: Budget is Binding

We assume that the budget B is binding, i.e., $c^i S^i + c^j S^j = B$. In this case, the optimal fraction of capital invested in technology i, $\xi^{i*} = c^i S^{i*}/B$, given by (assuming $\gamma = 1$),

$$\xi^{i*} = \frac{\left(c^i(c^i - c^j) + c^i\sigma^j(c^i\sigma^j - c^j\sigma^i) - \frac{c^i c^j}{B}((d_h + \frac{Y}{X})(c^i - c^j) - (d_l + \frac{Y}{X})(c^i\sigma^j - c^j\sigma^i))\right)^+}{(c^i - c^j)^2 + (c^i\sigma^j - c^j\sigma^i)^2} \tag{3.13}$$

and $\xi^{j*} = 1 - \xi^{i*}$.

$c^i - c^j$ represents the marginal benefit of choosing the cheaper technology j over the more efficient technology i during the withdrawal period. $c^i\sigma^j - \sigma^i c^j$ represents the benefit of choosing the more efficient technology over the cheaper one during the storage period. The

[3] We use i, j instead of $1, 2$ in order to avoid confusion between the power square x^2 and index j representing technology j.

capital allocation ratio ξ^{i*}, depends on the ratio of these benefits. The higher the d_h, the more the benefit during the withdrawal period and hence more capital is invested in the cheaper storage. Similarly, from the above expressions, we observe the following:

- As invested capital B increases, ξ^{i*} increases.
- As high period demand d_h increases, ξ^{i*} decreases.
- As low period demand d_l increases, ξ^{i*} increases.
- As convexity of generation cost, X increases, ξ^{i*} decreases.
- As base cost of generation, Y increases, ξ^{i*} increases.

It seems that the two attributes of storage technologies, price per unit capacity and efficiency provide cost benefit to grid operation under different circumstances. We notice that the cheaper less efficient storage is preferred when there is a greater need for storage (i.e., high d_h, low d_l, high X, low B), by focusing on volume of energy stored rather than the efficacy of storage. Similarly, as the more efficient storage is preferred when the need for storage is lesser, focusing on capturing the benefit more efficiently.

This example succinctly captures the primary trade-off in mixing. We observe, through robust numerical analysis, that these lessons continue to hold for more general demand scenarios.

Counter-intuitive situations

For this example, a number of counter intuitive situations arise:

- We observe for cases with high ξ^{i*} values, increasing σ^i (making technology i less efficient), counter intuitively increases the amount of money invested in technology i. This is because, increasing σ^i in this range, substantially increases q_1 and decreases q_2. Under such a situation, the inefficient technology becomes less useful. As the gap between q_1 and q_2 is quite small, efficiency becomes more important and we invest more in the efficient technology.

  A numerical example where it can be seen that $\frac{d\xi^{i*}}{d\sigma^i}$ is positive is: $d_h = 1$, $\sigma^i = 1.047$, $\sigma^j = 1.25$, $c^i = 0.625$ and $c^j = 0.5$, $B = 0.174$.

- We observe in cases with moderate $\xi^{i*}(> 0.5)$ values and low σ^i (very efficient technology), increasing σ^j (making technology 2 more inefficient), counter intuitively increases the amount of money invested in technology 2. This is because, increasing σ^j in this range, substantially reduces the smoothing achieved by the portfolio of technologies, while increasing q_1 and reducing q_2, but not to a large extent. Under such a situation, the cheaper technology becomes relatively more beneficial, as it is more valuable to achieve greater smoothing.

  A numerical example where it can be seen that $\frac{d\xi^{i*}}{d\sigma^j}$ is negative is: $d_h = 1$, $\sigma^i = 1.004$, $\sigma^j = 1.625$, $c^i = 0.5$, $c^j = 0.25$ and $B = 0.125$.

- As another example, we observe, counter intuitively, that as c^i increases, it is possible that ξ^{i*} may increase. In other words, we invest more in the expensive technology when it becomes more expensive. This idea is quite similar to fixed budget optimization decisions in economics. As c^i becomes more expensive, when ξ^1 is close to 1, the greater need for efficacy causes greater investment in c^1.

  A numerical example where it can be seen that $\frac{d\xi^{i*}}{dc^i}$ is negative is: $d_h = 1$, $\sigma^i = 1.004$, $\sigma^j = 1.17$, $c^i = 0.3755$, $c^j = 0.375$ and $B = 0.25$.

It is interesting to note that such counter intuitive insights may arise under limited budget investment decisions, analogous to the concept of a Giffen good in economics (Spiegel 1994).

3.4.3.2 Case 2: Budget is not Binding

Here, we assume the budget is sufficient, i.e., $c^iS^i + c^jS^j < B$. We have that, the solution may be one of four cases (for simplicity, we assume $Y = d_l = 0$),

- Invest in no storage at all, i.e., $\mathbf{S}^* = \mathbf{0}$
- Invest only in technology 1, i.e., $\mathbf{S}^* = \left(\frac{2d_h - c^i}{2X(1+(\sigma^i)^2)}, 0\right)$
- Invest only in technology 2, i.e., $\mathbf{S}^* = \left(0, \frac{2d_h - c^j}{2X(1+(\sigma^j)^2)}\right)$
- Invest non zero capital in both technologies, i.e.,

$$\mathbf{S}^* = \left(\frac{2d_h\sigma^jX(\sigma^j - \sigma^i) - (c^i - c^j) - \sigma^j(c^i\sigma^j - c^j\sigma^i)}{2X(\sigma^j - \sigma^i)^2}, \frac{(c^i - c^j) + \sigma^i(c^i\sigma^j - c^j\sigma^i) - 2d_h\sigma^iX(\sigma^i - \sigma^j)}{2X(\sigma^j - \sigma^i)^2}\right)$$

The intuition of the sensitivities with respect to d_h, X etc. from the above results remains similar to Case 1. However, we do not find any counter intuitive relationships when the budget is not binding. In fact, Lemma III.12 shows that increasing the cost of a technology decreases the capacity invested in that technology. The investment in a technology may decrease as σ increases.

3.4.4 Numerical Example of the Benefits of Mixing Multiple Technologies

In this subsection, we discuss a series of numerical examples, where it is optimal to invest in several technologies (> 2). Through these examples, we also observe a pattern of decreasing marginal benefit in the flexibility of using a number of technologies, consistent with Theorem III.11.

Consider the following four period example, which shows the benefit of mixing four technologies.

Example III.13. *Consider,* $\Omega = \{(\sigma^1, c^1), (\sigma^2, c^2), (\sigma^3, c^3), (\sigma^4, c^4)\}$ *and* $T = 4$, $\gamma = 1$. *Let* $d_1 = 0$, $d_2 = 100$, $d_3 = 0$, $d_4 = 10$ *and* $C(q) = q^2$ *and the technology parameters per the table below.*

Technology j	1	2	3	4
Cost/Unit Capacity c^j	99	89	39.2	27
Loss Rate σ^j	1	1.134	1.818	2

We consider optimally investing in a storage portfolio from this set of technologies Ω restricting the number of technologies with non-zero investment. Let $W(n)$ be the minimum operating and investment cost of investing in at most n technologies from the set Ω. Further, let $MB(n) = \frac{W(n-1)-W(n)}{W(0)-W(1)}$ represent the relative marginal benefit of the additional flexibility of using the n^{th} technology, compared to the benefit of adding the first technology.

For Example III.13, applying the structure of the optimal policy discussed in Section 3.4.1, we consider the optimal policy and investments for $n = 0$ to 4.

By Theorem III.11, we have that $MB(2) \leq MB(1) = 1$. For this particular example, it is interesting to note that the initial benefits of mixing are decreasing very quickly after $M = 2$ technologies.

Table 3.2: Marginal Benefit of multiple technologies

n	$W(n)$	$W(n-1)-W(n)$	$MB(n)\%$	**Technologies Used**
0	10100	-	-	None
1	8575.502	1524.49781	100	3
2	8569.709	5.79355578	0.38003	2, 3
3	8569.536	0.17238215	0.011307	1, 3, 4
4	8569.532	0.00458542	0.000301	1, 2, 3, 4

This example provides a methodology to construct similar examples that show non-zero (but tiny) benefit of mixing multiple technologies ($n > 4$). For example, in order to show an example that allows optimally mixing 5 or 6 technologies, we may consider a six period deterministic example, with three cycles of low and high demands. Each consecutive cycle has decreasing average demand. Hence, the most efficient technologies may be used in all three cycles, while the least efficient technology is only used in the first cycle. This presents a set of simultaneous equations for the investment quantities and there always exist a set of technologies and demand distributions satisfying that it is optimal to mix multiple technologies. Interestingly, the canonical example in Section 3.4.3 suggests that in a two period deterministic case, only two technologies can be mixed. This subsection and the following section together provide insights that the marginal benefit of mixing is decreasing.

3.4.5 Numerical analysis

In this subsection, we consider a numerical example of realistic demand to evaluate the benefit of using multiple technologies. The goal of this section is to investigate the impact of demand distribution parameters on the storage technology choice problem. Theorem III.5 provides an explicit solution for $\mathbf{s}^*_{t+1}$ given q_t, allowing for efficient computational study.

3.4.6 Model Parameters and Simulation Details

We consider, specifically, the choice of mixing under two technologies. The two technologies we consider have the parameters: $c^1 = 2.96, \sigma^1 = 1.004016$ and $c^2 = 0.931, \sigma^2 = 1.041666667$. Aggregate production cost of the grid is often approximated as quadratic in the total energy produced in a given period (Bessembinder and Lemmon 2006). We assume the cost function as follows, satisfying Assumption III.2:

$$c(q) = a\,q^2 + 10q \tag{3.14}$$

where a is the co-efficient of the quadratic term, indicating the convexity.

The evolutions of load exhibit predictable patterns and random fluctuations. Let L_t be the predictable load at time t and l_t be the unpredictable load variations. The net demand is given by:

$$d_t = L_t + l_t$$

The predictable component $\{L_t\}$ is a deterministic processes whose values are known prior to time zero. The stochastic processes $\{l_t\}$ represents the deviations from the deterministic levels and evolve according to preset probability distributions. We model l_t to be influenced by the wind variations, caused by the penetration of wind energy; hence l_t can be negative.

We consider a cycle of T periods. For given investment in storage capacities, denoted by $\mathbf{S}$, we employ an infinite-horizon average-cost model assuming every cycle faces the same distribution of demands. In the second stage of the optimization, we search for optimal investment capacity $\mathbf{S}$ that minimizes the total investment and operating costs.

The following demand model, while stylized, illustrates the trade-offs in a more realistic system. Consider 8 periods per day (of 3 hrs each), with each period representing 3 hours. Predictable components of load and wind are cyclic over these 8 periods.

Table 3.3: Predictable Components of Load

Time (hour of the day)	0-3	3-6	6-9	9-12	12-15	15-18	18-21	21-24	Average
Predictable load L_t (MWh)	12	14	25	36	38	36	25	14	25

The unpredictable variations in each period is given by, $l_t = \xi l_t^0$, where ξ is a measure of the variability in the demand, here ξ changes from 1 to 5.

$$l_t = \begin{cases} -2 & p = 0.015618765, \\ -1.33 & p = 0.093747594, \\ -0.66 & p = 0.234379485, \\ 0 & p = 0.312508313, \\ 0.66 & p = 0.234379485, \\ 1.33 & p = 0.093747594, \\ 2 & p = 0.015618764. \end{cases} \tag{3.15}$$

Our objective is to study the impact of changes in the demand parameters on the optimal investment decisions in this scenario. We plot the following metrics against change in ξ in Figure 3.2. Observe that l_t^0 suggests huge swings in the variability of demand, where as L_t presents smaller periodic variations in the variability of demand. As expected, the investment in the cheaper technology 2 increases as ξ increases, and for these parameters, the benefit of mixing is also increasing. This confirms an important intuition of the benefit of mixing. Investing in the cheaper technology allows to smooth large variations in demand, that happen occasionally, while the more efficient expensive technology can smooth tiny variations, especially when frequent.

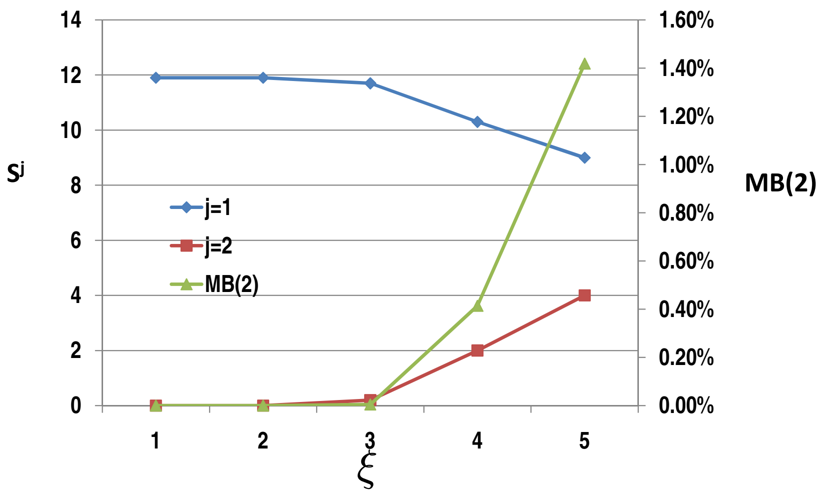


Figure 3.2: Impact of change in ξ for $a = 0.1$.

We also choose to study the impact of variation in quadratic co-efficient a. We observe that the impact of changes in a are non-linear.

3.5. Decentralized Storage Owner Perspective under Inelastic Prices Assumption

For the decentralized storage owner perspective discussed in Section 3.3.2, we consider two possibilities: (a) Small Storage Investor, (b) Large Storage Investor. For each case, we estimate the value function and optimal decisions of the storage investor (See Chapter 6 of Carlton and Perloff (2010)).

3.5.1 Small Storage Investor

Consider the case where the storage facility is considered to be small compared to the size of the market, as typically assumed in the literature (Zhou et al. 2011, Kim and Powell 2011, Denholm and Sioshansi 2009), and $p_t(\Delta\mathbf{s}_t, \mathbf{d}_t) = p_t(\mathbf{d}_t)$ for all $\Delta\mathbf{s}_t$ for all t in $\{1, 2, 3, \ldots, T\}$ through out this subsection. Here, the price of energy transfer is independent of the actions of the storage owner. Under this case, the constraint of $q(\Delta\mathbf{s}_t, \mathbf{d}_t) \geq 0$ may be neglected.

Storage investment decisions are made by independent profit maximizing players under their individual beliefs about the price path. The objective of this section is to yield analytical evidence to justify the dynamics observed in the market today, and understand how storage investment decisions are affected by the nature of the price paths.

We first observe some properties of the function $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$, per (3.7).

Lemma III.14. *For all $t = 1, \ldots, T$, we have the following properties:*

(i) $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is jointly convex and decreasing in $\mathbf{s}_t$.

(ii) $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is jointly concave in $\boldsymbol{\sigma}$.

Proof of Lemma III.14. The proof of part (i) is similar to the proof of Lemma III.3(iii). We now show part (ii) by induction on t. It can be seen that $V_{T+1}^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is concave in $\boldsymbol{\sigma}$. Now we prove the hypothesis for period t, assuming it is true for period $t+1$. Note further that the term $p_t \sum_{j=1}^{M} \psi^j(\Delta s_t)$ is linear in $\boldsymbol{\sigma}$ for any Δs_t. Hence, the term inside the minimization of (3.7) is concave in $\boldsymbol{\sigma}$ in small storage investor case. Since the minimization of a concave function is always concave, we have that the $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is concave in $\boldsymbol{\sigma}$. $\square$

Because the price is independent of the actions of the storage operator, we have that the optimal operating revenue of a portfolio of facilities is the sum of the optimal operating revenues of each of the individual facilities. Further, for given technology, each unit of capacity can garner the same revenue. This gives us some simplifying properties of the value function.

Lemma III.15. *For all* $t = 1, \dots, n, n+1$*,* $\mathbf{s}_t \in \mathcal{A}$ *we have the following:*

(i) $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma}) = \sum_{j=1}^{M} V_t^I(s_t^j, \mathbf{d}_t|S^j, \sigma^j)$.

(ii) $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma}) = kV_t^I(\mathbf{s}_t/k, \mathbf{d}_t|\mathbf{S}/k, \boldsymbol{\sigma})$ *for all* $k > 0$.

Proof of Lemma III.15. Part (i) follows by construction. We prove part (ii) by induction. It can be seen that both statements are true for $t = T+1$ as $V_{T+1}^I(.,.) = 0$. We now assume the lemma is true for period $t+1$. Applying (3.7), for period t, we have:

$$\begin{aligned}
V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma}) =& \min\Big\{\mathsf{E}_t[\gamma V_{t+1}^I(\mathbf{s}_{t+1}, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})] - p_t\big(\sum_{j=1}^{M}\psi^j(\Delta s_t^j)\big)\Big\} \\
=& \min\Big\{\mathsf{E}_t[\gamma k V_{t+1}^I(\mathbf{s}_{t+1}/k, \mathbf{d}_t, \mathbf{S}/k, \boldsymbol{\sigma})] - kp_t\big(\sum_{j=1}^{M}\psi^j(\Delta s_t^j/k)\big)\Big\} \\
=& k\min\Big\{\mathsf{E}_t[\gamma V_{t+1}^I(\mathbf{s}_{t+1}/k, \mathbf{d}_t, \mathbf{S}/k, \boldsymbol{\sigma})] - p_t\big(\sum_{j=1}^{M}\psi^j(\Delta s_t^j/k)\big)\Big\} \\
=& kV_t^I(\mathbf{s}_t/k, \mathbf{d}_t, \mathbf{S}/k, \boldsymbol{\sigma})
\end{aligned}$$

where the second equality follows from the induction and the linearity of the $\psi^j()$ function, and the third equality follows from the fact that the action $\mathbf{s}_{t+1}/k$ is feasible per (3.7). This proves part (ii). $\square$

Because the size of the storage does not directly affect the prices in this model, each dollar that is invested in storage garners the highest revenue per dollar invested independent of the other investments. Hence, one does not observe any synergy of mixing, the total capital is invested in the technology that gives the highest expected net present value of the profits per dollar invested.

Lemma III.16. *For given investment* B *and set of technologies* $\Omega = \{(\sigma^j, c^j) : j =$

$1, \ldots, M\}$, *there exists an optimal solution to* (3.9) *such that* $S^j > 0$ *for at most one of the* M *technologies.*

3.5.1.1 Technology Choice

While at most one technology will be chosen, it is still a non-trivial decision to make the choice. Given capacity cost per unit investment c and conversion loss factor σ, if a technology is worse in both parameters, it may directly be eliminated from the choice set. Consider a set of non-dominated technologies Ω; similar to (3.12), we define,

$$\Omega_E^I(B, \Omega) \stackrel{\text{def}}{=} \Big\{(\sigma^{M+1}, c^{M+1}) : \overline{V}^I\big(B, \, \Omega \cup (\sigma^{M+1}, c^{M+1})\big) \geq \overline{V}^I(B, \Omega)\Big\}. \tag{3.16}$$

Theorem III.17. *For any set of technologies,* Ω*, and budget* $B \geq 0$*, we have that the set,* $\Omega_E^I(B, \Omega)$ *is convex.*

Proof. From Lemma III.14(ii) and separability of investment and operations, we have that, $\overline{V}^I(B, \Omega)$ is jointly concave in $(\boldsymbol{\sigma}, \mathbf{c})$. Hence, $\overline{V}^I\big(B, \, \Omega \cup (\sigma^{M+1}, c^{M+1})\big)$ is concave in (σ^{M+1}, c^{M+1}). If we consider this to be a function on (σ^{M+1}, c^{M+1}), then Ω_E^I is a set of the form $\{(x, y) | f(x, y) \geq 0\}$. Hence, Lemma III.20 implies the set is convex. $\square$

Similar to the discussion in Section 3.4, this allows to eliminate any technologies in Ω that are in the convex hull of a subset of technologies (See Figure 3.1).

3.5.1.2 Canonical Example

In order to gain further insights into the type of technology that may be chosen based on the system metrics, we consider a cyclic price scenario with periodicity two, where the storage operator acts as a price taker in the market. We also assume that the distribution of prices for each cycle is stationary, hence, our analysis on a 2 period model will extend to the entire horizon. Let $p_1 = p_l$ and $p_2 = p_h$, with $p_l << p_h$. The objective of the storage operator is to invest a given amount of capital B (often, decided by other constraints, such as company budgets, Public Relations efforts and government grants and bills etc.) in order to maximize the net present value of the investment. We address the question of which technology to choose based on the prices. Note that $\mathbf{s}_1^* = \mathbf{S},\ \mathbf{s}_2^* = 0$, hence,

$$\overline{V}^I(1, \{(\sigma, c)\}) = \min\{1 - \mathsf{E}[(p^h\gamma - p^l\sigma)]\frac{1-\gamma^T}{c(1-\gamma^2)}, 0\} \tag{3.17}$$

We assume the distribution of prices is such that the random variable $p^h\gamma - p^l\sigma$ is always positive, hence ensuring that is optimal to buy and sell during each cycle of the horizon. Hence choosing the optimal technology from a set Ω, depends on the particular expression for each technology, $(\theta - \sigma)/c$, where $\theta = \mathsf{E}[p_h]\gamma/[p_l]$. Specifically, when θ is high, suggesting huge disparity between the high and low price periods, we prefer the cheaper storage to capitalize on the arbitrage as much as possible. If θ is low, we choose the more efficient technology, preferring to make more efficient use of the price differences.

In summary, this section considers a simplistic but common assumption about price paths, showing that 'mixing' is in fact not beneficial under this assumption. Further, using a simplistic canonical example, we learn that higher the variation of prices, we choose the cheaper technology and vice versa.

3.5.2 Large Storage Owner Perspective

If the amount of storage invested influences market prices, price should influence quantity produced. In this subsection, we assume that there is only one storage investor in the market, and that his actions determine q_t. Thus $p_t = C'(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t))$, price of energy traded is equal to the marginal cost of energy in the system during that period. This assumption is also common in the literature (Lynch and Law 2004). We begin this section by considering the properties of the value function, $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ under this price path assumption.

Lemma III.18. *Suppose $C(q)$ is quadratic in q, For all $t = 1, \ldots, T$, we have the following properties,*

(i) $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is jointly convex and decreasing in $(\mathbf{s}_t, \mathbf{S})$.

(ii) $U^I(\mathbf{S}, \boldsymbol{\sigma})$ is jointly convex in $\mathbf{S}$.

Proof of Lemma III.18. We prove part (i) by induction. $V_{T+1}^I(\mathbf{s}, \mathbf{d}|\mathbf{S}, \boldsymbol{\sigma})$ is convex and decreasing in $(\mathbf{s}, \mathbf{S})$ by assumption. We now prove convexity for period t assuming that part

(i) is true for period $t+1$. Suppose $C(q) = Xq^2 + Yq + Z$, then $C'(q_t) = 2Xq_t + Y$. Hence, the current period cost in (3.7), is given by $(2X(\sum_{j=1}^{M} \psi^j(\Delta s^j) + d_t) + Y)(\sum_{j=1}^{M} \psi^j(\Delta s^j))$. It can be seen that this expression is jointly convex in $\mathbf{s}_t,\ \mathbf{s}_{t+1}, \mathbf{S}$. Hence, the expression inside the minimization in (3.7) is convex in both the state and the decision variables. By the theorem on convexity preservation under minimization Heyman and Sobel (1984, p. 525), we conclude that $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is convex in $\mathbf{s}_t, \mathbf{S}$.

Suppose that the optimal solution for $\mathbf{s}_t, \mathbf{d}_t$ is $\mathbf{s}_{t+1}^*$. To prove that $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is decreasing in $\mathbf{s}_t$, the intermediate value theorem gives the existence of action $\mathbf{s}'_{t+1} \in [\mathbf{s}_{t+1}^*, \mathbf{S}]$ for given storage level $\mathbf{s}'_t \geq \mathbf{s}_t$ such that $0 \leq q(\Delta\mathbf{s}'_t, \mathbf{d}_t) \leq q(\Delta\mathbf{s}_t, \mathbf{d}_t)$. This and the inductive hypothesis prove that $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is decreasing in $\mathbf{s}_t$. Further, it can be seen that $V_t^I(\mathbf{s}_t, \mathbf{d}_t|\mathbf{S}, \boldsymbol{\sigma})$ is decreasing in $\mathbf{S}$ as any feasible action under $\mathbf{S}$ would be feasible for given $\mathbf{S}' \geq \mathbf{S}$.

Part (ii) follows from applying part (i) to the definition of $U^I(\mathbf{S}, \boldsymbol{\sigma})$. □

Note that $V_t^I(.,.)$ under a general convex cost function $C(q)$ is not necessarily convex in $\mathbf{s}$. Further still, $U^I(\mathbf{S}, \boldsymbol{\sigma})$ may not also be convex in $\mathbf{S}$. For example, in the two period deterministic problem with $d_1 = 0$ and $d_2 = d_h$, and $C(q) = e^q$, it can be verified that the $U^I(S, \sigma)$ is not convex in S.

While it is useful to note that convexity is guaranteed only under quadratic generation cost function, we observe that, in two period deterministic setting with multiple technologies, the optimal investment under the decentralized is always less than the optimal investment under centralized perspective.

Example III.19. *For a given set of technologies Ω, and $T = 2$, with deterministic demands $d_1 << d_2$, let $\mathbf{S}^o$ be the optimal solution to* (3.5) *and $\mathbf{S}^I$ be an optimal solution* (3.9). *We have that $\mathbf{S}^I \leq \mathbf{S}^o$, when B is not binding.*

It can be seen that the optimal operating policy, for given investment $\mathbf{S}$ (assuming it is optimal) is to produce to fill up storage in period 1, i.e., $q_1 = d_1 + \sum_{j=1}^{M} \psi^j(S^j)$ and empty *storage in period 2, i.e.,* $q_2 = d_2 - \sum_{j=1}^{M} S^j$. *Further, we have,*

$$\frac{d\overline{V}^I(\mathbf{S},\Omega)}{dS^k} = (C''(q_1)\sigma^k(\sum_{j=1}^{M}\psi^j(S^j)) + C''(q_2)(\sum_{j=1}^{M}S^j))(\sum_{j=1}^{M}\psi^j(S^j)) + C'(q_1)\sigma^k - C'(q_2) + c^k \tag{3.18}$$

$$\frac{d\overline{V}^o(\mathbf{S},\Omega)}{dS^k} = C'(q_1)\sigma^k - C'(q_2) + c^k \tag{3.19}$$

Due to the convexity of $C()$ *per Assumption III.2, we have that* $\frac{d\overline{V}^I(\mathbf{S},\Omega)}{dS^k} - \frac{d\overline{V}^o(\mathbf{S},\Omega)}{dS^k} \geq 0$. *Further, since,* $V^o(\mathbf{S},\Omega)$ *is convex in* $\mathbf{S}$ *per Lemma III.10(i), we have that if* $\mathbf{S}^{o*}$ *is a solution to the problem in* (3.5), $\frac{d\overline{V}^I(\mathbf{S},\Omega)}{dS^k} \geq 0$ *for all* $\mathbf{S} > \mathbf{S}^{o*}$, *hence* $\mathbf{S}^{I*} \leq \mathbf{S}^{o*}$, *where* $\mathbf{S}^{I*}$ *is optimal solution to* (3.9). *Note that this discussion does not assume convexity of* $\overline{V}^I(\mathbf{S},\Omega)$ *with respect to* $\mathbf{S}$.

Note that, when B is binding, the total investment is the same in both centralized and decentralized scenarios, so this result cannot be extended to the case when B is binding. For each dollar of capacity invested, the benefit gained under the centralized case is the difference between the marginal cost in the high demand period and the low demand period (adjusted for efficiency). However, under the decentralized case, this difference is further decreased, proportionate to the rate of change of marginal cost, as the price of all the energy purchased by the storage unit increases at the rate of $C''(q_1)$ and the price of all energy sold by the storage unit decreases at the rate of $C''(q_2)$. Figure 3.3 demonstrates this result.

Notice that Example III.19 describes an elegant intuition. If the optimal investment under $\mathbf{S}^I$ is vested in more than one technology, then so is $\mathbf{S}^o$. In other words, if we 'mix' technologies under the decentralized case, then we mix technologies under the centralized case. This suggests that we mix more under the centralized decision maker perspective than the decentralized perspectives.

3.5.2.1 Canonical Example

We now consider the system with demands and costs described in Section 3.4.3. We further assume $X = 1$, $Y = 0$, $d_l = 0$ and $\gamma = 1$. Again, here, since $M = 2$, we have that $i = 1$ and $j = 2$. For this system, we again consider the problem under two cases, with the

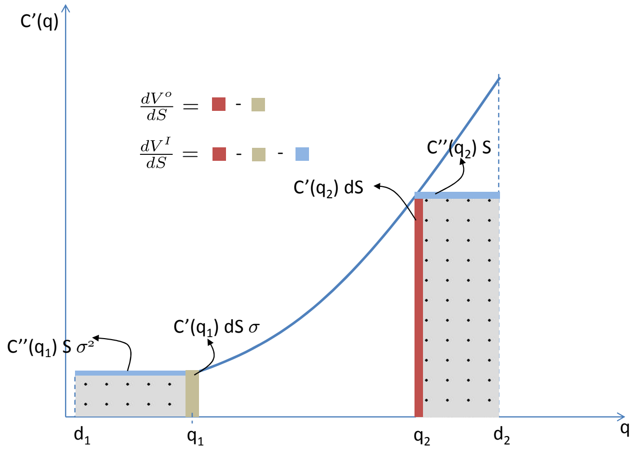


Figure 3.3: The marginal benefit of each infinitesimal unit of storage capacity in a two period deterministic demand system under the decentralized case is less than the marginal benefit under the centralized system by the area of the light blue rectangles.

'myopic markets' assumption about the price paths:

Case 1: Budget B is binding, here, the total investment is the entire available budget B and we similarly measure the optimal investment in the more efficient technology as follows,

$$\xi^{i*} = \frac{(c^i(c^i - c^j) + c^i\sigma^j(c^i\sigma^j - c^j\sigma^i) - \frac{c^ic^j}{2B}((d_h)(c^i - c^j)))^+}{(c^i - c^j)^2 + (c^i\sigma^j - c^j\sigma^i)^2} \tag{3.20}$$

Note that the ξ^{i*} described above is higher than the ξ^{i*} that is described in (3.13), explaining a preference for the more efficient technology. We observe that the decentralized system inherently prefers the more efficient storage under binding budget because there is an additional penalty due to the inefficiency under the 'myopic markets' assumption. For each additional unit of storage capacity, we pay not only the additional increase in marginal cost for that unit but also the rate of increase of $c'(q)$ is felt on the entire purchased energy, so, the effect of inefficiency σ is felt twice, effecting a double taxation.

Case 2: In this case, we assume that the budget B is not binding. Here, the optimal investment when investing in both technologies, given by,

$$\mathbf{S}^* = \Big(\frac{2d_h\sigma^j X(\sigma^j - \sigma^i) - (c^i - c^j) - \sigma^j(c^i\sigma^j - c^j\sigma^i)}{4X(\sigma^j - \sigma^i)^2}, \frac{(c^i - c^j) + \sigma^i(c^i\sigma^j - c^j\sigma^i) - 2d_h\sigma^i X(\sigma^i - \sigma^j)}{4X(\sigma^j - \sigma^i)^2}\Big) \tag{3.21}$$

Compared to the centralized example in Section 3.4.3, notice that while the ratio of investments in the two technologies remains the same, the absolute capacity invested is halved. This result is consistent with Example III.19.

In this section, we demonstrate that having more generic price path assumptions can create non-convex optimizations, making them harder to solve. However, for a deterministic two period setting, we provide intuition that the storage investment under decentralized case is substantially smaller than under the centralized case and that we invest smaller amounts. However, (3.21) suggests that the ratio of mixing remains the same.

3.6. Conclusion and Future Research

In summary, this chapter demonstrates the benefit of using multiple storage technologies in tandem. While the traditional fixed cost/variable cost trade-off presents synergies in 'mixing' of technologies in a variety of settings, we further observe that the convexity of the cost function allows for synergies of 'mixing' even in a two period deterministic setting, for our problem. Apart from Kraning et al. (2011), to the best of our knowledge, we are the only other work to consider using multiple technologies simultaneously to provide a single service to the grid. While their focus is on computational methodologies for evaluating the optimal operating policies and investment decision, we focus on providing structural insights by identifying the structure of the optimal operating policy and establish identifiable properties of the optimal investment portfolio.[4] We further demonstrate that the marginal benefit of this flexibility of 'mixing' is decreasing in the number of technologies available. To the best of our knowledge, we are the first to extend such insights from other settings to the non-linear setting of energy markets.

This chapter also identifies some situations where investment decisions under limited budget can be counter-intuitive, when technologies in the portfolio under consideration change.

[4]They additionally consider the discharge rate of storage units.

In particular, when a technology becomes worse (either in the cost or efficiency metric), it may be optimal to invest more in that technology because of the increased relative scarcity of the budget. Finally, our work offers a nascent explanation to the lack of ‘mixing’ in the grid today. We posit that the incentives of individual storage operators maximizing their own profits does not present synergies to invest in multiple technologies in tandem. Specifically, we observe no benefits of mixing under the ‘small storage’ assumption, which is common in the literature.

For future research, it may be apt to consider a storage model where we compare technologies differing in their power ratings and number of life cycles as well. It would be a stronger analytical result if we can conclude that the marginal benefit of ‘mixing’ is decreasing for $M > 2$ as well, extending Theorem III.11.

3.7. Appendix: Definitions and Proofs

In this section, we define some functions and discuss their properties in order to prove the theorems in the paper. We first discuss some preliminary results.

Lemma III.20. *Let $f(x, y)$ be a function that is jointly concave in x, y, then the set described by $\Xi = \{(x, y)|f(x, y) \geq 0\}$ is convex.*

Proof of Lemma III.20. Suppose $(x_1, y_1), (x_2, y_2) \in \Xi$, Consider $(\widetilde{x}, \widetilde{y}) = \tau(x_1, y_1) + (1 - \tau)(x_2, y_2)$ for some $\tau \in [0, 1]$. Observe that $f(\widetilde{x}, \widetilde{y}) \geq \tau f(x_1, y_1) + (1 - \tau)f(x_2, y_2) \geq 0$, where the first inequality follows from concavity and the second inequality follows from the assumption. Hence, $(\widetilde{x}, \widetilde{y}) \in \Xi$. □

Lemma III.21. *Let $f(x, y)$ be a non-negative function that is jointly convex and decreasing in x, y, then the* $\lim_{m\to\infty} \lim_{n\to\infty} \dfrac{f(x + 1/n, y + 1/m)}{1/n} = \lim_{n\to\infty} \lim_{m\to\infty} \dfrac{f(x + 1/n, y + 1/m)}{1/n}$.

Proof of Lemma III.21. Let $a_{mn} = \frac{f(x+1/n,y+1/m)}{1/n}$ for $m, n \in \mathcal{N}$ and $c_{mn} = a_{m(n+1)} - a_{mn}$. Suppose further that $c_{mn} \leq 0$ for all m, n and that c_{mn} is increasing in n. Then, by the monotone convergence theorem, we have, $\lim_{m\to\infty} \sum_{j=1}^{\infty} c_{mj} = \sum_{j=1}^{\infty} \lim_{m\to\infty} c_{mj}$, which simplifies to, $\lim_{m\to\infty} \lim_{n\to\infty} a_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} a_{mn}$.

Hence, it suffices to show that $c_{mn} \leq 0$, and c_{mn} is increasing with n. Because $f(., .)$ is convex, it can be seen that $\frac{f(x+\Delta,y)-f(x,y)}{\Delta}$ is increasing with respect to Δ. Hence, if

$\Delta_n = 1/n$ is decreasing with increasing n, we have that a_{mn} is decreasing with respect to n, giving $c_{mn} \leq 0$.

Further, we have that $f(x + 1/\Delta, y)/\Delta$[5] is convex in Δ when $\Delta \geq 0$, hence, we observe that $c_{m(n+1)} - c_{mn} = (n+2)f(x+1/(n+2), y) + nf(x+1/n, y) - 2(n+1)f(x+1/(n+1), y) \geq 0$.

$\square$

We define the function Γ as follows to represent the marginal increase in total costs by adding infinitesimal capacities across the M technologies along the vector $\mathbf{v}$. For given set of technologies $\Omega = \{(\sigma^j, c^j) : 1 \leq j \leq M\}$, let $|\Omega| = M$ be the cardinality of the set of technologies, we have,

$$\Gamma(\mathbf{v}, \mathbf{S}, \Omega) = \lim_{r \to 0^+} \mathbf{c} \cdot \mathbf{v} + \frac{U^o(\mathbf{S} + r\mathbf{v}, \boldsymbol{\sigma}) - U^o(\mathbf{S}, \boldsymbol{\sigma})}{r} \tag{3.22}$$

where $\mathbf{S} \in (\Re^+)^M$, and $\mathbf{v} \in \Re^M$, such that, there exists $\varepsilon > 0$ satisfying, $\mathbf{S} + r\mathbf{v} \in (\Re^+)^M$ for all $0 < r < \varepsilon$. Similarly, for all $1 \leq j \leq M$, we define,

$$\Gamma_j^+(\mathbf{S}, \Omega) = \Gamma(\mathbf{1}_j, \mathbf{S}, \Omega), \;\; \Gamma_j^-(\mathbf{S}, \Omega) = \Gamma(-\mathbf{1}_j, \mathbf{S}, \Omega)$$

where, $\mathbf{1}_j$ is the unit vector along the axis of S^j. Note that it can be shown that $\Gamma_j^+(\mathbf{S}, \Omega) = \Gamma_j^-(\mathbf{S}, \Omega)$ when $C(.)$ is differentiable.

We now show the properties of these functions, in order to help us understand the properties of the set Ω^E defined in (3.12).

Lemma III.22. *Given Ω and that $C()$ is continuous and twice differentiable in q and $C'(0) = 0$, we have that $V_t^o(\mathbf{s}, \mathbf{d}|\mathbf{S})$ as in (3.3) is convex and semi-differentiable in each variable of $(\mathbf{s}, \mathbf{S})$ for all $t \in \{1, 2, \ldots, n, n+1\}$, $\mathbf{S} > 0$ and $\mathbf{s} \in (0, \mathbf{S})$.*

Proof of Lemma III.22. Since $V_t^o(.,.)$ is convex (Lemma III.3) and on the open interval I of each variable of $(\mathbf{s}, \mathbf{S})$ as per Lemma statement. Hence, the left and right derivatives with respect to $\mathbf{s}$ and $\mathbf{S}$ exist and are continuous in all but countably many points (set D, say) on the open interval, per Theorem 25.3 in Rockafellar (1997). In each point of set D, the left

[5] Check double derivative with respect to Δ

and right derivatives exist because of continuity and existence of V' on the set $I - D$. This proves that it is semi-differentiable in the entire open set described by the Lemma. $\square$

Lemma III.23. *The function* $\Gamma(.,.,.)$ *defined per* (3.22) *satisfies the following properties:*

(i) Γ *exists for given* $\Omega, \mathbf{v}$ *for all feasible* $\mathbf{S}$*, and for all* $1 \leq i, j \leq M$*,* $\lim_{h \to 0^+} \Gamma_i^+(\mathbf{S}+h\mathbf{1}_j, \Omega) = \Gamma_i^+(\mathbf{S}, \Omega)$*,* $\lim_{h \to 0^+} \Gamma_i^-(\mathbf{S}+h\mathbf{1}_j, \Omega) = \Gamma_i^-(\mathbf{S}, \Omega)$.

(ii) $\Gamma(\mathbf{v}, \mathbf{S}, \Omega) = \sum_{j=1}^{M} |v^j| \big(\mathbf{1}_{\{v^j > 0\}} \Gamma_j^+(\mathbf{S}, \Omega) + \mathbf{1}_{\{v^j < 0\}} \Gamma_j^-(\mathbf{S}, \Omega) \big)$.

(iii) Suppose $\widetilde{\Omega} \subset \Omega$ *,* $\widetilde{\mathbf{v}} \in \Re^{|\widetilde{\Omega}|}, \widetilde{\mathbf{S}} \in (\Re^+)^{|\Omega|}$ *and* $\mathbf{v} = (\widetilde{\mathbf{v}}, \mathbf{0}^{|\Omega| - |\widetilde{\Omega}|}), \mathbf{S} = (\widetilde{\mathbf{S}}, \mathbf{0}^{|\Omega| - |\widetilde{\Omega}|})$ *are the extensions of* $\widetilde{\mathbf{v}}, \widetilde{\mathbf{S}}$ *respectively in the extended space of* Ω *technologies and that* $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{S}}, \widetilde{\Omega})$ *is in the domain of the function* Γ*, then,* $\Gamma(\widetilde{\mathbf{v}}, \widetilde{\mathbf{S}}, \widetilde{\Omega}) = \Gamma(\mathbf{v}, \mathbf{S}, \Omega)$.

Proof of Lemma III.23. First, we prove part (i). Existence is implied from Lemma III.22 and (3.5). Consider,

$$\begin{aligned} \lim_{h \to 0^+} \Gamma_i^+(\mathbf{S}+h\mathbf{1}_j, \Omega) &= c^i + \lim_{h \to 0^+} \lim_{r \to 0^+} \frac{U^o(\mathbf{S}+r\mathbf{1}_i+h\mathbf{1}_j, \boldsymbol{\sigma}) - U^o(\mathbf{S}+h\mathbf{1}_j, \boldsymbol{\sigma})}{r} \\ &= c^i + \lim_{r \to 0^+} \lim_{h \to 0^+} \frac{U^o(\mathbf{S}+r\mathbf{1}_i+h\mathbf{1}_j, \boldsymbol{\sigma}) - U^o(\mathbf{S}+h\mathbf{1}_j, \boldsymbol{\sigma})}{r} \\ &= \Gamma_i^+(\mathbf{S}, \Omega). \end{aligned} \tag{3.23}$$

where the first equality follows from the definition and the second equality follows from Lemma III.21. The proof is similar for $\Gamma_i^-(\mathbf{S}, \Omega)$.

Now, to prove part (ii), consider, $\mathbf{v} = a\mathbf{1}_i + b\mathbf{1}_j$.

$$\begin{aligned} \Gamma(\mathbf{v}, \mathbf{S}, \Omega) &= (ac^i + bc^j) + \lim_{r \to 0^+} \frac{U^o(\mathbf{S}+r(a\mathbf{1}_i+b\mathbf{1}_j), \boldsymbol{\sigma}) - U^o(\mathbf{S}, \boldsymbol{\sigma})}{r} \\ &= ac^i + a \lim_{r \to 0^+} \frac{U^o(\mathbf{S}+r(a\mathbf{1}_i+b\mathbf{1}_j), \boldsymbol{\sigma}) - U^o(\mathbf{S}+r(b\mathbf{1}_j), \boldsymbol{\sigma})}{ar} \\ &+ bc^j + b \lim_{r \to 0^+} \frac{U^o(\mathbf{S}+r(b\mathbf{1}_j), \boldsymbol{\sigma}) - U^o(\mathbf{S}, \boldsymbol{\sigma})}{br} \\ &= \lim_{r \to 0^+} a\Gamma_i^+(\mathbf{S}+br, \Omega) + b\Gamma_j^+(\mathbf{S}, \Omega) \\ &= a\Gamma_i^+(\mathbf{S}, \Omega) + b\Gamma_j^+(\mathbf{S}, \Omega), \end{aligned} \tag{3.24}$$

where the first two equalities follow from the definition and the last equality follows from part (i). The proof is similar for all other $\mathbf{v}$.

Now, to prove part (iii), note that $\mathbf{c}\cdot\mathbf{v} = \widetilde{\mathbf{c}}\cdot\widetilde{\mathbf{v}}$, $U^o(\mathbf{S},\boldsymbol{\sigma}) = U^o(\widetilde{\mathbf{S}},\widetilde{\boldsymbol{\sigma}})$ and $U^o(\mathbf{S}+r\mathbf{v},\boldsymbol{\sigma}) = U^o(\widetilde{\mathbf{S}}+r\widetilde{\mathbf{v}},\widetilde{\boldsymbol{\sigma}})$. This implies part (iii). □

Proof of Theorem III.6.

We first show (a) $\Longrightarrow$ (b). Let $\mathbf{S}^*$ be the optimal solution to (3.5) using the set of technologies Ω. Let $\mathbf{S}_1$ be a projection of $\mathbf{S}^*$ on the set of technologies $\Omega\cup\Omega_1$, i.e., $S_1^j = S^{j*}$ for $1\le j\le M$ and $S_1^j = 0$ for $M+1\le j\le|\Omega\cup\Omega_1|$. Similarly, let $\mathbf{S}_2$, $\widetilde{\mathbf{S}}$ be the projections of $\mathbf{S}^*$ on the sets $\Omega\cup\Omega_2$ and $\Omega\cup\Omega_1\cup\Omega_2$ respectively. Given the optimality of $\mathbf{S}^*$ in the space of Ω, $\overline{V}^o\big(B,\,\Omega\cup\Omega_1\big)\ge\overline{V}^o(B,\Omega)$ and $\overline{V}^o\big(B,\,\Omega\cup\Omega_2\big)\ge\overline{V}^o(B,\Omega)$ together imply (from the equivalence of local minimality with global minimality in a convex optimization),

$$\Gamma(\mathbf{v}_1,\mathbf{S}_1,\Omega\cup\Omega_1)\ge 0,\qquad \Gamma(\mathbf{v}_2,\mathbf{S}_2,\Omega\cup\Omega_2)\ge 0,\qquad \Gamma(\mathbf{v}_0,\mathbf{S}^*,\Omega)\ge 0 \tag{3.25}$$

for all $\mathbf{v}_1$ in the space of $\Omega\cup\Omega_1$ whose projection on the space of Ω is $\mathbf{0}$, $\mathbf{v}_2$ in the space of $\Omega\cup\Omega_2$ whose projection of Ω is $\mathbf{0}$, and all $\mathbf{v}$ in the space of Ω, while still part of the domain of Γ (per (3.22)). Hence, we have,

$$\begin{aligned}
0 &\le \Gamma(\mathbf{v}_1,\mathbf{S}_1,\Omega\cup\Omega_1)+\Gamma(\mathbf{v}_2,\mathbf{S}_2,\Omega\cup\Omega_2)+\Gamma(\mathbf{v}_0,\mathbf{S}^*,\Omega)\\
&= \Gamma(\widetilde{\mathbf{v}}_0,\widetilde{\mathbf{S}},\Omega\cup\Omega_1\cup\Omega_2)+\Gamma(\widetilde{\mathbf{v}}_1,\widetilde{\mathbf{S}},\Omega\cup\Omega_1\cup\Omega_2)+\Gamma(\widetilde{\mathbf{v}}_2,\widetilde{\mathbf{S}},\Omega\cup\Omega_1\cup\Omega_2)\\
&= \Gamma(\widetilde{\mathbf{v}}_0+\widetilde{\mathbf{v}}_1+\widetilde{\mathbf{v}}_2,\widetilde{\mathbf{S}},\Omega\cup\Omega_1\cup\Omega_2),
\end{aligned} \tag{3.26}$$

where the first equality follows from Lemma III.23 (iii) and the second equality follows from Lemma III.23(ii). Note that any vector in the space of $\Omega\cup\Omega_1\cup\Omega_2$ which has a projection $\mathbf{0}$ on the space of Ω can be expressed as a sum of the vectors $\widetilde{\mathbf{v}}_1,\widetilde{\mathbf{v}}_2$ which are the projections of $\mathbf{v}_1,\mathbf{v}_2$ on the space of Ω_1 and Ω_2 respectively extended to the entire set $\Omega\cup\Omega_1\cup\Omega_2$. Hence (3.35) implies part (b). Hence, (a) $\Longrightarrow$ (b). Part (b) trivially implies (c).

Now we show (c) $\Longrightarrow$ (a), completing the equivalence. (c) implies that for every technology $(\sigma^j,c^j)\in\Omega_1\cup\Omega_2$, $\overline{V}^o(B,\Omega\cup(\sigma^j,c^j))\ge\overline{V}^o(B,\Omega)$. Since (a) $\Longrightarrow$ (b) is true for any two subsets of Ω, this implies (b). □

Proof of Lemma II.4: The statement of the lemma holds for period T because $V_T(\cdot,\cdot)=0$. Suppose $V_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_{t+1})$ is decreasing and convex in $\mathbf{s}_{t+1}$ for any $\mathbf{d}_{t+1}$.

For any $\mathbf{d}_t$, the objective function in (2.5) is defined on a non-convex set $\{(\mathbf{s}_t,\mathbf{s}_{t+1})\in\mathcal{A}\times\mathcal{A} : q(\mathbf{s}_{t+1}-\mathbf{s}_t,\mathbf{d}_t)\geq 0\}$. We introduce the following auxiliary function, which is an extension of the objective function in (2.5) to a larger convex set:

$$f_t(\mathbf{s}_{t+1},\mathbf{s}_t,\mathbf{d}_t) \stackrel{\text{def}}{=} c\big([q(\mathbf{s}_{t+1}-\mathbf{s}_t,\mathbf{d}_t)]^+\big)+\gamma\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_{t+1})\big], \qquad \text{for } (\mathbf{s}_{t+1},\mathbf{s}_t)\in\mathcal{A}\times\mathcal{A}.$$

For state $(\mathbf{s}_t,\mathbf{d}_t)$, let $\mathbf{s}^*_{t+1}$ be an optimal decision found by (2.5)-(2.6). Consider any $\widetilde{\mathbf{s}}_t\geq\mathbf{s}_t$. If $\mathbf{s}^*_{t+1}$ is feasible for state $(\widetilde{\mathbf{s}}_t,\mathbf{d}_t)$, i.e., $q(\mathbf{s}^*_{t+1}-\widetilde{\mathbf{s}}_t,\mathbf{d}_t)\geq 0$, then

$$V_t(\widetilde{\mathbf{s}}_t,\mathbf{d}_t)\leq f_t(\mathbf{s}^*_{t+1},\widetilde{\mathbf{s}}_t,\mathbf{d}_t)\leq f_t(\mathbf{s}^*_{t+1},\mathbf{s}_t,\mathbf{d}_t)=V_t(\mathbf{s}_t,\mathbf{d}_t).$$

If $q(\mathbf{s}^*_{t+1}-\widetilde{\mathbf{s}}_t,\mathbf{d}_t)<0$ (infeasible), then using $q(\mathbf{S}-\widetilde{\mathbf{s}}_t,\mathbf{d}_t)\geq 0$ and applying the intermediate value theorem, we can find a feasible decision $\widetilde{\mathbf{s}}_{t+1}$ with $\mathbf{s}^*_{t+1}\leq\widetilde{\mathbf{s}}_{t+1}\leq\mathbf{S}$ and $q(\widetilde{\mathbf{s}}_{t+1}-\widetilde{\mathbf{s}}_t,\mathbf{d}_t)=0$. Thus,

$$V_t(\widetilde{\mathbf{s}}_t,\mathbf{d}_t)\leq\gamma\mathsf{E}_t\big[V_{t+1}(\widetilde{\mathbf{s}}_{t+1},\mathbf{d}_{t+1})\big]\leq\gamma\mathsf{E}_t\big[V_{t+1}(\mathbf{s}^*_{t+1},\mathbf{d}_{t+1})\big]\leq f_t(\mathbf{s}^*_{t+1},\mathbf{s}_t,\mathbf{d}_t)=V_t(\mathbf{s}_t,\mathbf{d}_t),$$

where the second inequality follows from the induction hypothesis and $\widetilde{\mathbf{s}}_{t+1}\geq\mathbf{s}^*_{t+1}$. Using the intermediate value theorem, we can also show that $\min_{\mathbf{s}_{t+1}\in\mathcal{A}} f_t(\mathbf{s}_{t+1},\mathbf{s}_t,\mathbf{d}_t)=V_t(\mathbf{s}_t,\mathbf{d}_t)$.

Note that $c\big([q(\Delta\mathbf{s},\mathbf{d}_t)]^+\big)$ is a composition of convex increasing functions, and thus it is convex in $\Delta\mathbf{s}$. From the induction hypothesis, $\mathsf{E}_t\big[V_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_{t+1})\big]$ is convex in $\mathbf{s}_{t+1}$. Therefore, $f_t(\mathbf{s}_{t+1},\mathbf{s}_t,\mathbf{d}_t)$ is jointly convex in $(\mathbf{s}_{t+1},\mathbf{s}_t)$ on closed convex set $\mathcal{A}\times\mathcal{A}$. Then, using the theorem on convexity preservation under minimization from Heyman and Sobel (1984, p. 525), we conclude that $V_t(\mathbf{s}_t,\mathbf{d}_t)=\min_{\mathbf{s}_{t+1}\in\mathcal{A}} f_t(\mathbf{s}_{t+1},\mathbf{s}_t,\mathbf{d}_t)$ is convex in $\mathbf{s}_t$. □

Proof of Lemma III.4. Part (i) holds trivially for $T+1$ since $V^o_{T+1}(.,.)=0$. Suppose the statement of the Lemma holds for $t+1$. For period t, we consider states $(\mathbf{s}_t,\mathbf{d}_t)$ and $(\widetilde{\mathbf{s}}_t,\mathbf{d}_t)$, with $\widetilde{\mathbf{s}}_t,\mathbf{s}_t$ satisfying part (i) of the Lemma for some $\varepsilon\geq 0$. Let $\mathbf{s}^*_{t+1}$ be the optimal action for state $(\mathbf{s}_t,\mathbf{d}_t)$, and denote $\Delta\mathbf{s}_t=\mathbf{s}^*_{t+1}-\mathbf{s}_t$ and $q^*=q_t(\Delta\mathbf{s}_t,\mathbf{d}_t)$. For state $(\widetilde{\mathbf{s}}_t,\mathbf{d}_t)$, consider action $\widetilde{\mathbf{s}}_{t+1}\in\mathcal{A}$ and denote $\Delta\widetilde{\mathbf{s}}_t=\widetilde{\mathbf{s}}_{t+1}-\widetilde{\mathbf{s}}_t$ and $\widetilde{q}_t=q_t(\Delta\widetilde{\mathbf{s}}_t,\mathbf{d}_t)$. Consider for some $\widetilde{\varepsilon}\geq 0$

such that $\widetilde{s}^i_{t+1} = s^{i*}_{t+1} + \widetilde{\varepsilon}/\sigma^i, \widetilde{s}^j_{t+1} = s^{j*}_{t+1} - \widetilde{\varepsilon}/\sigma^j, \widetilde{s}^k_{t+1} = s^{k*}_{t+1}$ for all $k \neq i, j$, and $\widetilde{q}_t \leq q^*_t$, then the feasibility of the action $\widetilde{\mathbf{s}}_{t+1}$ and the induction hypothesis leads to the intended result:

$$V^o_t(\widetilde{\mathbf{s}}_t, \mathbf{d}_t) \leq C(\widetilde{q}_t) + \mathsf{E}_t[V^o_{t+1}(\widetilde{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1})] \leq C(q^*_t) + \mathsf{E}_t[V^o_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1})] = V^o_t(\mathbf{s}_t, \mathbf{d}_t). \quad (3.27)$$

Below, we identify such $\widetilde{\varepsilon}$. Consider two cases:

Case 1: If $s^{i*}_{t+1}+\varepsilon/\sigma^i \leq S^i$ and $s^{j*}_{t+1}-\varepsilon/\sigma^j \geq 0$, we set $\widetilde{\varepsilon} = \varepsilon$. We have $\mathbf{s}^*_{t+1}-\mathbf{s}_t = \widetilde{\mathbf{s}}_{t+1}-\widetilde{\mathbf{s}}_t$; hence, $\widetilde{q}_t = q^*_t$.

Case 2: If $s^{i*}_{t+1} + \varepsilon/\sigma^i > S^i$ or $s^{j*}_{t+1} - \varepsilon/\sigma^j < 0$ or both inequalities hold, we set $\widetilde{\varepsilon} = \min\left\{(S^i - s^{i*}_{t+1})\sigma^i,\, s^{j*}_{t+1}\sigma^j\right\}$. Note that $\widetilde{\mathbf{s}}_{t+1} \in \partial\mathcal{A}$ (the boundary of $\mathcal{A}$), i.e., $\widetilde{s}^i_{t+1} = S^i$ or $\widetilde{s}^j_{t+1} = 0$. This implies

$$\Delta\widetilde{s}^i_t = \widetilde{s}^i_{t+1} - \widetilde{s}^i_t \geq 0 \quad \text{or} \quad \Delta\widetilde{s}^j_t = \widetilde{s}^j_{t+1} - \widetilde{s}^j_t \leq 0. \quad (3.28)$$

Let $\delta = \varepsilon - \widetilde{\varepsilon}$. Note that $\widetilde{\varepsilon} < \varepsilon$, hence $\delta > 0$. Then, by definitions, we have

$$\Delta\mathbf{s}_t - \Delta\widetilde{\mathbf{s}}_t \;=\; \mathbf{s}^*_{t+1} - \widetilde{\mathbf{s}}_{t+1} - \mathbf{s}_t + \widetilde{\mathbf{s}}_t \quad (3.29)$$

That is, $\Delta s^i_t = \Delta\widetilde{s}^i_t + \delta/\sigma^i$, $\Delta s^j_t = \Delta\widetilde{s}^j_t - \delta/\sigma^j$ and $\Delta s^k_t = \Delta\widetilde{s}^k_t$ for $k \neq i, j$. Applying (3.2), we have

$$q_t - \widetilde{q}_t = \psi^i(\Delta s^i_t) - \psi^i(\Delta\widetilde{s}^i_t) - \left[\psi^j(\Delta\widetilde{s}^j_t) - \psi^j(\Delta s^j_t)\right] \;\equiv\; \Gamma,$$

Now consider the two conditions derived in (3.28):

- If $\Delta\widetilde{s}^i_t \geq 0$, then $\Gamma = \delta - \left[\psi^j(\Delta\widetilde{s}^j_t) - \psi^j(\Delta s^j_t)\right] \;\geq\; \delta - \sigma^j\delta/\sigma^j \;=\; 0.$
- If $\Delta\widetilde{s}^j_t \leq 0$, then $\Gamma = \left[\psi^i(\Delta s^i_t) - \psi^i(\Delta\widetilde{s}^i_t)\right] - \delta/\sigma^j \;\geq\; \delta/\sigma^i - \delta/\sigma^j \;\geq\; 0.$

Hence, $q_t \geq \widetilde{q}_t$ and the result in (3.27) holds.

For part (ii) of the lemma, the proof follows similar lines as above. However, we choose $\widetilde{\varepsilon} = \min\left\{s^{i*}_{t+1}, (S^j - s^{j*}_{t+1})\right\}$ for case 2, and $\widetilde{s}^i_{t+1} = 0$ or $\widetilde{s}^j_{t+1} = S^i$ implies $\Delta\widetilde{s}^i_t \leq 0$ or $\Delta\widetilde{s}^j_t \geq 0$, and $(\Delta s_t^i, \Delta s_t^j) - (\Delta \widetilde{s}_t^i, \Delta \widetilde{s}_t^j) = (-\delta, \delta)$. The corresponding inequalities become

$$
\begin{aligned}
q_t - \widetilde{q}_t &= \left[\psi^j(\Delta s_t^j) - \psi^j(\Delta \widetilde{s}_t^j)\right] - \left[\psi^i(\Delta \widetilde{s}_t^i) - \psi^i(\Delta s_t^i)\right], \\
&\geq \begin{cases} \delta - \delta \ = \ 0, & \text{if } \Delta \widetilde{s}_t^i \leq 0, \\ \delta\sigma^j - \delta\sigma^i \ \geq \ 0, & \text{if } \Delta \widetilde{s}_t^j \geq 0. \end{cases}
\end{aligned}
$$

□

Proof of Theorem III.5.

Parts (i) and (ii) follow directly from Lemma III.4. For part (iii), since $M = 2$, we assume $i = 1$, $j = 2$. We prove part (iii) by induction. We assume that $V_{t+1}^o(.,.)$ is supermodular in $\mathbf{s}_t$ (true for $T+1$ since $V_{T+1}(.,.) = 0$). Let $q_t = q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)$, $\widehat{q}_t = q(\widehat{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t, \mathbf{d}_t)$.

We prove, $q_t \leq \widehat{q}_t$ by contradiction. Suppose that $q_t > \widehat{q}_t$; for optimal actions satisfying part (i)-(ii), we have three cases:

- If $s_{t+1}^i < \widehat{s}_{t+1}^i$, then $\Delta s_t^i < \Delta \widehat{s}_t^i$ and (3.2) imply $\Delta s_t^j > \Delta \widehat{s}_t^j$, hence $\Delta s_t^j > 0$ or $\Delta \widehat{s}_t^j < 0$; part (i)-(ii) implies either $s_{t+1}^i = S^i$ or $\widehat{s}_{t+1}^i = 0$, hence, $s_{t+1}^i \geq \widehat{s}_{t+1}^i$, which gives a contradiction.

- If $s_{t+1}^j < \widehat{s}_{t+1}^j$, then $\Delta s_t^j < \Delta \widehat{s}_t^j$, hence, $\Delta s_t^j < 0$ or $\Delta \widehat{s}_t^j > 0$; part (i)-(ii) implies, $s_{t+1}^i = 0$ or $\widehat{s}_{t+1}^i = S^i$; hence, $s_{t+1}^i \leq \widehat{s}_{t+1}^i$. This gives $q_t < \widehat{q}_t$ from (3.2), which is a contradiction.

- Else if, $\mathbf{s}_{t+1} \geq \widehat{\mathbf{s}}_{t+1}$, note that $\mathbf{s}_{t+1} \neq \widehat{\mathbf{s}}_{t+1}$ as $q_t > \widehat{q}_t$ per our assumption and $\mathbf{s}_t \geq \widehat{\mathbf{s}}_t$. We find a $\overline{\mathbf{s}}_{t+1}, \underline{\mathbf{s}}_{t+1}$ s.t. $\mathbf{s}_{t+1} \geq \overline{\mathbf{s}}_{t+1} \geq \underline{\mathbf{s}}_{t+1} \geq \widehat{\mathbf{s}}_{t+1}$, $\mathbf{s}_{t+1} - \overline{\mathbf{s}}_{t+1} = \underline{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_{t+1}$, $q_t - q(\overline{\mathbf{s}}_{t+1} - \mathbf{s}_t) \geq q(\underline{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t) - \widehat{q}_t$ and $q_t \geq q(\overline{\mathbf{s}}_{t+1} - \mathbf{s}_t), q(\underline{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t) \geq \widehat{q}_t$, which when combined with the strict convexity of $C(.)$ implies,

$$
C(q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) - C(q(\overline{\mathbf{s}}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) > C(q(\underline{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t, \mathbf{d}_t)) - C(q(\widehat{\mathbf{s}}_{t+1} - \widehat{\mathbf{s}}_t, \mathbf{d}_t)). \tag{3.30}
$$

Consider,

$$C(q(\mathbf{s}_{t+1}-\mathbf{s}_t,\mathbf{d}_t)) - C(q(\bar{\mathbf{s}}_{t+1}-\mathbf{s}_t,\mathbf{d}_t)) \le \mathsf{E}_t[V^o_{t+1}(\bar{\mathbf{s}}_{t+1},\mathbf{d}_{t+1})] - \mathsf{E}_t[V^o_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_{t+1})]$$
$$\le \mathsf{E}_t[V^o_{t+1}(\widehat{\mathbf{s}}_{t+1},\mathbf{d}_{t+1})] - \mathsf{E}_t[V^o_{t+1}(\underline{\mathbf{s}}_{t+1},\mathbf{d}_{t+1})] \le C(q(\underline{\mathbf{s}}_{t+1}-\widehat{\mathbf{s}}_t,\mathbf{d}_t)) - C(q(\widehat{\mathbf{s}}_{t+1}-\widehat{\mathbf{s}}_t,\mathbf{d}_t)), \tag{3.31}$$

where the first inequality is from the optimality of $\mathbf{s}_{t+1}$ for state $(\mathbf{s}_t,\mathbf{d}_t)$, the second inequality follows from the supermodularity and convexity of $V_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_t)$ w.r.t $\mathbf{s}_{t+1}$ (inductive hypothesis), and the third inequality follows from the optimality of $\widehat{\mathbf{s}}_{t+1}$ for state $(\widehat{\mathbf{s}}_t,\mathbf{d}_t)$. (3.30) and (3.31) together imply a contradiction. Now we propose such $\bar{\mathbf{s}}_{t+1},\underline{\mathbf{s}}_{t+1}$ in two cases:

- If $s^i_{t+1} > \widehat{s}^i_{t+1}$, we choose a $\delta \in (0,(q_t-\widehat{q}_t)/(2\sigma^i))$ which is small enough s.t, $\psi^i(s^i_{t+1}-s^i_t) - \psi^i(s^i_{t+1}-\delta-s^i_t) \ge \psi^i(\widehat{s}^i_{t+1}-\widehat{s}^i_t) - \psi^i(\widehat{s}^i_{t+1}-\delta-\widehat{s}^i_t)$. Such δ exists as $\widehat{q}_t < q_t$.[6] We choose $\bar{\mathbf{s}}_{t+1} = (s^i_{t+1}-\delta, s^j_{t+1})$ and $\underline{\mathbf{s}}_{t+1} = (\widehat{s}^i_{t+1}+\delta, \widehat{s}^j_{t+1})$. This implies $q_t - q(\bar{\mathbf{s}}_{t+1}-\mathbf{s}_t) \ge q(\underline{\mathbf{s}}_{t+1}-\widehat{\mathbf{s}}_t) - \widehat{q}_t$ and $q_t \ge q(\bar{\mathbf{s}}_{t+1}-\mathbf{s}_t), q(\underline{\mathbf{s}}_{t+1}-\widehat{\mathbf{s}}_t) \ge \widehat{q}_t$.
- Else, it means $s^i_{t+1} = \widehat{s}^i_{t+1} = 0$ or $s^i_{t+1} = \widehat{s}^i_{t+1} = S^i$ (due to part (i)-(ii)). In either case, we choose a $\delta \in (0,(q_t-\widehat{q}_t)/(2\sigma^j))$ which is small enough such that, $\psi^j(s^j_{t+1}-s^j_t) - \psi^i(s^j_{t+1}-\delta-s^j_t) \ge \psi^j(\widehat{s}^j_{t+1}-\widehat{s}^j_t) - \psi^j(\widehat{s}^j_{t+1}-\delta-\widehat{s}^j_t)$ and set $\bar{\mathbf{s}}_{t+1} = (s^i_{t+1}, s^j_{t+1}-\delta)$ and $\underline{\mathbf{s}}_{t+1} = (\widehat{s}^i_{t+1}, \widehat{s}^j_{t+1}+\delta)$. Again, such δ exists and this satisfies the above conditions establishing the contradiction.

Now, we prove supermodularity of $V^o(\mathbf{s}_t,\mathbf{d}_t)$ in $\mathbf{s}_t$. Consider any, $\mathbf{s}_t > \widehat{\mathbf{s}}_t$. Let $\bar{\mathbf{s}}_t = (s^i_t, \widehat{s}^j_t)$, $\underline{\mathbf{s}}_t = (\widehat{s}^i_t, s^j_t)$. It suffices to prove, $V^o_t(\mathbf{s}_t,\mathbf{d}_t) + V^o_t(\widehat{\mathbf{s}}_t,\mathbf{d}_t) \ge V^o_t(\bar{\mathbf{s}}_t,\mathbf{d}_t) + V^o_t(\underline{\mathbf{s}}_t,\mathbf{d}_t)$ for any $\mathbf{d}_t$. It is sufficient to find feasible policies $\underline{\mathbf{s}}_{t+1}$ for state $(\underline{\mathbf{s}}_t,\mathbf{d}_t)$ and $\bar{\mathbf{s}}_{t+1}$ for state $(\bar{\mathbf{s}}_t,\mathbf{d}_t)$ satisfying:

$$C(q_t) + C(\widehat{q}_t) \ge C(q(\bar{\mathbf{s}}_{t+1}-\bar{\mathbf{s}}_t,\mathbf{d}_t)) + C(q(\underline{\mathbf{s}}_{t+1}-\underline{\mathbf{s}}_t,\mathbf{d}_t)), \tag{3.32}$$

$$V^o_{t+1}(\mathbf{s}_{t+1},\mathbf{d}_{t+1}) + V^o_{t+1}(\widehat{\mathbf{s}}_{t+1},\mathbf{d}_{t+1}) \ge V^o_{t+1}(\bar{\mathbf{s}}_{t+1},\mathbf{d}_{t+1}) + V^o_{t+1}(\bar{\mathbf{s}}_{t+1},\mathbf{d}_{t+1}), \tag{3.33}$$

[6] If $s^i_{t+1} > s^i_t$, then we choose $\delta < \Delta s^i_t$, implying, $\psi^i(s^i_{t+1}-s^i_t) - \psi^i(s^i_{t+1}-\delta-s^i_t) = \delta\sigma^i \ge \psi^i(\widehat{s}^i_{t+1} - \widehat{s}^i_t) - \psi^i(\widehat{s}^i_{t+1}-\delta-\widehat{s}^i_t)$. Else, $\widehat{q}_t < q_t$ implies $\widehat{s}^i_{t+1} < \widehat{s}^i_t$, giving, $\psi^i(s^i_{t+1}-s^i_t) - \psi^i(s^i_{t+1}-\delta-s^i_t) \ge \delta = \psi^i(s^i_{t+1}-s^i_t) - \psi^i(s^i_{t+1}-\delta-s^i_t) \ge \delta = \psi^i(\widehat{s}^i_{t+1}-\widehat{s}^i_t) - \psi^i(\widehat{s}^i_{t+1}-\delta-\widehat{s}^i_t)$.

for any $\mathbf{d}_{t+1}$. We find such $\underline{\mathbf{s}}_{t+1}, \overline{\mathbf{s}}_{t+1}$ in three cases:

Case 1: If $s^j_{t+1} - \widehat{s}^j_{t+1} > s^j_t - \widehat{s}^j_t$, then part (i) implies, $s^i_{t+1} \geq \widehat{s}^i_{t+1}$; hence, we choose $\overline{\mathbf{s}}_{t+1} = (s^i_{t+1}, s^j_{t+1} - (s^j_t - \widehat{s}^j_t))$ and $\underline{\mathbf{s}}_{t+1} = (\widehat{s}^i_{t+1}, \widehat{s}^j_{t+1} + (s^j_t - \widehat{s}^j_t))$. The assumptions ensure $\overline{\mathbf{s}}_{t+1}, \underline{\mathbf{s}}_{t+1} \in \mathcal{A}$, $q(\overline{\mathbf{s}}_{t+1} - \overline{\mathbf{s}}_t, \mathbf{d}_t) = q(\mathbf{s}_{t+1} - \mathbf{s}_t, \mathbf{d}_t)$ and $q(\underline{\mathbf{s}}_{t+1} - \underline{\mathbf{s}}_t, \mathbf{d}_t) = q(\mathbf{s}_{t+1} - \mathbf{s}, \mathbf{d}_t)$. Hence, this policy satisfies (3.32). Further, $V^o_{t+1}(\widehat{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1}) - V^o_{t+1}(\underline{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1}) \geq V^o_{t+1}((s^i_{t+1}, \widehat{s}^j_{t+1}), \mathbf{d}_{t+1}) - V^o_{t+1}((s^i_{t+1}, \widehat{s}^j_{t+1} + (s^j_t - \widehat{s}^j_t)), \mathbf{d}_{t+1}) \geq V^o_{t+1}(\overline{\mathbf{s}}_{t+1}, \mathbf{d}_{t+1}) - V^o_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_{t+1})$, where the first inequality follows from supermodularity and the second inequality follows from convexity of $V^o_{t+1}(\mathbf{s}_{t+1}, \mathbf{d}_t)$ w.r.t $\mathbf{s}_{t+1}$ which verifies (3.33).

Case 2: Else if $s^j_{t+1} - \widehat{s}^j_{t+1} < s^j_t - \widehat{s}^j_t$, part (i) implies $s^i_{t+1} = 0$ or $\widehat{s}^i_{t+1} = S^i$, giving $s^i_{t+1} \leq \widehat{s}^i_{t+1}$; We choose $\overline{\mathbf{s}}_{t+1} = \widehat{\mathbf{s}}_{t+1}$ and $\underline{\mathbf{s}}_{t+1} = \mathbf{s}_{t+1}$. By applying (3.2), we have: have: $q_t = \psi^i(s^i_{t+1} - s^i_t) + \psi^j(s^j_{t+1} - s^j_t)$, $\widehat{q}_t = \psi^i(\widehat{s}^i_{t+1} - \widehat{s}^i_t) + \psi^j(\widehat{s}^j_{t+1} - \widehat{s}^j_t)$, $q(\overline{\mathbf{s}}_{t+1} - \overline{\mathbf{s}}_t, \mathbf{d}_t) = \psi^i(\widehat{s}^i_{t+1} - s^i_t) + \psi^j(\widehat{s}^j_{t+1} - \widehat{s}^j_t)$, $q(\underline{\mathbf{s}}_{t+1} - \underline{\mathbf{s}}_t, \mathbf{d}_t) = \psi^i(s^i_{t+1} - \widehat{s}^i_t) + \psi^j(s^j_{t+1} - s^j_t)$. From our assumptions, it can be verified that, $q_t + \widehat{q}_t \geq q(\overline{\mathbf{s}}_{t+1} - \overline{\mathbf{s}}_t, \mathbf{d}_t) + q(\underline{\mathbf{s}}_{t+1} - \underline{\mathbf{s}}_t, \mathbf{d}_t)$ and $\widehat{q}_t \geq q(\overline{\mathbf{s}}_{t+1} - \overline{\mathbf{s}}_t, \mathbf{d}_t), q(\underline{\mathbf{s}}_{t+1} - \underline{\mathbf{s}}_t, \mathbf{d}_t) \geq q_t$. Hence, convexity of $C(.)$ w.r.t q_t implies (3.32). Further, (3.33) reduces to equality, proving supermodularity.

Case 3: Else if $s^j_{t+1} - \widehat{s}^j_{t+1} = s^j_t - \widehat{s}^j_t$, we consider two subcases:

- If $s^i_{t+1} \geq \widehat{s}^i_{t+1}$, we choose $\overline{\mathbf{s}}_{t+1}, \underline{\mathbf{s}}_{t+1}$ similar to Case 1.
- Else if, $s^i_{t+1} < \widehat{s}^i_{t+1}$, we choose $\overline{\mathbf{s}}_{t+1}, \underline{\mathbf{s}}_{t+1}$ similar to Case 2. □

Proof of Theorem III.6.

We first show (a) $\implies$ (b). Let $\mathbf{S}^*$ be the optimal solution to (3.5) using the set of technologies Ω. Let $\mathbf{S}_1$ be a projection of $\mathbf{S}^*$ on the set of technologies $\Omega \cup \Omega_1$, i.e., $S^j_1 = S^{j*}$ for $1 \leq j \leq M$ and $S^j_1 =$ for $M + 1 \leq j \leq M + |\Omega_1|$. Similarly, let $\mathbf{S}_2$, $\widetilde{\mathbf{S}}$ be the projections of $\mathbf{S}^*$ on the sets $\Omega \cup \Omega_2$ and $\Omega \cup \Omega_1 \cup \Omega_2$ respectively. Given the optimality of $\mathbf{S}^*$ in the space of Ω, $\overline{V}^o\big(B, \Omega \cup \Omega_1\big) \geq \overline{V}^o(B, \Omega)$ and $\overline{V}^o\big(B, \Omega \cup \Omega_2\big) \geq \overline{V}^o(B, \Omega)$ together imply (from the equivalence of local minimality with global minimality in a convex optimization),

$$\Gamma(\mathbf{v}_1, \mathbf{S}_1, \Omega \cup \Omega_1) \geq 0, \qquad \Gamma(\mathbf{v}_2, \mathbf{S}_2, \Omega \cup \Omega_2) \geq 0 \tag{3.34}$$

for all $\mathbf{v}_1$ in the space of $\Omega \cup \Omega_1$ whose projection on the space of Ω is $\mathbf{0}$, while still part of the domain of Γ (per (3.22)) and for all $\mathbf{v}_2$ in the space of $\Omega \cup \Omega_2$, feasible in the domain of Γ (per (3.22)) whose projection in the space of Ω is $\mathbf{0}$. Hence, we have,

$$\begin{aligned} 0 \leq \Gamma(\mathbf{v}_1, \mathbf{S}_1, \Omega \cup \Omega_1) + \Gamma(\mathbf{v}_2, \mathbf{S}_2, \Omega \cup \Omega_2) &= \Gamma(\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{S}}, \Omega \cup \Omega_1 \cup \Omega_2) + \Gamma(\widetilde{\mathbf{v}}_2, \widetilde{\mathbf{S}}, \Omega \cup \Omega_1 \cup \Omega_2) \\ &= \Gamma(\widetilde{\mathbf{v}}_1 + \widetilde{\mathbf{v}}_2, \widetilde{\mathbf{S}}, \Omega \cup \Omega_1 \cup \Omega_2), \end{aligned} \tag{3.35}$$

where the first equality follows from Lemma III.23 (iii) and the second equality follows from Lemma III.23(ii). Note that any vector in the space of $\Omega \cup \Omega_1 \cup \Omega_2$ which has a projection $\mathbf{0}$ on the space of Ω can be expressed as a sum of the vectors $\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2$ which are the projections of $\mathbf{v}_1, \mathbf{v}_2$ on the space of Ω_1 and Ω_2 respectively extended to the entire set $\Omega \cup \Omega_1 \cup \Omega_2$. Hence (3.35) implies part (b). Hence, (a) $\Longrightarrow$ (b).

Now, if (c) is not true, there exists at least one technology $(\sigma^j, c^j) \in \Omega_1 \cup \Omega_2$ which satisfies $\overline{V}^o(B, \Omega \cup (\sigma^j, c^j)) < \overline{V}^o(B, \Omega)$. By using the same storage size that achieves the value in $\overline{V}^o(B, \Omega \cup (\sigma^j, c^j))$, and investing 0 in the remaining storage technologies of $\Omega_1 \cup \Omega_2$, we have a solution to (3.5) using the set of technologies $\Omega \cup \Omega_1 \cup \Omega_2$. This contradicts (b). Hence (b) $\Longrightarrow$ (c).

Now we show (c) $\Longrightarrow$ (a), completing the equivalence. (c) implies that for every technology $(\sigma^j, c^j) \in \Omega_1 \cup \Omega_2$, $\overline{V}^o(B, \Omega \cup (\sigma^j, c^j)) \geq \overline{V}^o(B, \Omega)$. Since (a) $\Longrightarrow$ (b) is true for any two subsets of Ω, this implies (b). $\square$

Proof of Lemma III.7.

For given optimal solution $\mathbf{S}$ to (3.5) with the set of technologies $\Omega \cup \{(\sigma, c)\}$, consider solution to (3.5) with the set of technologies $\Omega \cup \{(\sigma^1, c^1), (\sigma^2, c^2)\}$, with $\widetilde{\mathbf{S}}$, where $\widetilde{S}^j = S^j$ for $1 \leq j \leq |\Omega|$, and $\widetilde{S}^{|\Omega|+1} = \kappa S^{|\Omega|+1}$ and $\widetilde{S}^{|\Omega|+2} = (1-\kappa) S^{|\Omega|+1}$. The investment costs under both investments are the same, i.e., $\widetilde{\mathbf{c}} \cdot \widetilde{\mathbf{S}} = \mathbf{c} \cdot \mathbf{S}$.

Further, we also claim that: $V_t^o(\mathbf{s}_t, \mathbf{d}_t | \mathbf{S}, \Omega \cup \{(\sigma, c)\}) \geq V_t^o(\widetilde{\mathbf{s}}_t, \mathbf{d}_t | \widetilde{\mathbf{S}}, \Omega \cup \{(\sigma^1, c^1), (\sigma^2, c^2)\})$. The proof follows by induction. It can be verified that the statement is true for period $T+1$. If in period t, the optimal policy under for state $(\mathbf{s}_t, \mathbf{d}_t)$ is $\mathbf{s}_{t+1}$ for investment $\mathbf{S}$ with, set of technologies, $\Omega \cup \{(\sigma, c)\}$, consider the equivalent policy, $\widetilde{\mathbf{s}}_{t+1}$ such that, $\widetilde{s}^j_{t+1} = s^j_{t+1}$ for $1 \leq j \leq |\Omega|$ and $\widetilde{s}^{|\Omega|+1}_{t+1} = \kappa s^{|\Omega|+1}_{t+1}$ and $\widetilde{s}^{|\Omega|+2}_{t+1} = (1-\kappa) s^{|\Omega|+1}_{t+1}$. Note that, this and (3.2) implies, $q(\Delta \mathbf{s}_t, \mathbf{d}_t) = q(\Delta \widetilde{\mathbf{s}}_t, \mathbf{d}_t)$, which is sufficient for the induction hypothesis for period t to hold.

$\square$

Proof of Lemma III.10.

The proof of part (i) is similar to the proof of Lemma III.3(iii) applied on the relaxed problem. We further have that $U^o(\mathbf{S}, \boldsymbol{\sigma})$ is convex by applying the definition.

We prove part (ii) by induction. Clearly, the statement is true for period $T+1$. Assume that the statement is true for period $t+1$. Let $\mathbf{s}^*_{t+1} = (s^{1*}_{t+1}, s^{2*}_{t+1})$ be the optimal solution for state $(\mathbf{s}_t, \mathbf{d}_t)$ for the system with storage capacity $\mathbf{S}$. Clearly, the optimal policy for the system with capacity $\mathbf{0}$ is $(0,0)$. Consider the following inequalities,

$$\gamma \mathsf{E}_t[V^o_{t+1}(\mathbf{s}^*_{t+1}, \mathbf{d}_{t+1}|\mathbf{S}) + V^o_{t+1}(\mathbf{0}, \mathbf{d}_{t+1}|\mathbf{0})] \geq \gamma \mathsf{E}_t[V^o_{t+1}((s^{1*}_{t+1}, 0), \mathbf{d}_t|(S^1, 0)) + V^o_{t+1}((0, s^{2*}_{t+1}), \mathbf{d}_t|(0, S^2))]$$
$$C(q(\mathbf{s}^*_{t+1} - \mathbf{s}_t, \mathbf{d}_t)) + C(d_t) \geq C(d_t + \psi^1(s^{1*}_{t+1} - s^1_t)) + C(d_t + \psi^2(s^{2*}_{t+1} - s^2_t))$$

where the first inequality follows from the induction hypothesis and the second inequality follows from the property that $\Delta s^{i*}_t \Delta s^{j*}_t \geq 0$ for all $1 \leq i, j \leq M$ for an optimal policy (consequence of Theorem III.5 parts (i),(ii)) and convexity of $C(.)$.

Adding these two inequalities and (3.3) together implies the inductive hypothesis on part (i), because $(s^{1*}_{t+1}, 0)$ is a feasible policy for state $\big((s^1_t, 0),\ \mathbf{d}_t\big)$ with investment $(S^1, 0)$ (similarly for state $(0, s^2_t), \mathbf{d}_t)$.

Part (iii) follows directly from applying part (ii) to the definition of $U^o(\mathbf{S}, \boldsymbol{\sigma})$. $\square$

Proof of Theorem III.11.

Let $(\overline{S}_1, \overline{S}_2)$ be optimal investments from the set Ω. From part (iii) of Lemma III.10,

$$U^o(\overline{S}^1, \overline{S}^2) + U^o(0,0) \geq U^o(\overline{S}^1, 0) + U^o(0, \overline{S}^2)$$

Adding the above 3 inequalities, we get,

$$U^o(S^1, 0) + c^1 S^1 + U^o(0, S^2) + c^2 S^2 \leq U^o(\overline{S}^1, \overline{S}^2) + c^1 \overline{S}^1 + c^2 \overline{S}^2 + U^o(0,0)$$

This implies, $\overline{V}^o(B, \phi) + \overline{V}^o(B, \Omega) \geq \overline{V}^o(B, \{\sigma^1, c^1\}) + \overline{V}^o(B, \{\sigma^2, c^2\})$, which proves the result. $\square$

3.8. Appendix: Alternative Investment Problem formulation

In this section, we consider the investment decision in order to minimize operating costs given a fixed budget for initial investment B.

$$V^*(B) = \min_{0 \le \mathbf{c}^I \cdot \mathbf{S} \le B} \big\{ \mathsf{E}_0[V_1^o(0, \mathbf{d}_1)|\mathbf{S}] \big\}, \tag{3.36}$$

where $\mathbf{S}$ is the storage size that defines the action space $\mathcal{A}$ in (3.3) and $V_1(.,.)$ is the value function per (3.3). Note that investment portfolio that solves $\min_{B \ge 0} V^*(B)$ will also solve (3.5). In that sense, the two models are equivalent when B is *large enough.* Let $\mathbf{S}^*(B)$ be the optimal storage investment that solves (3.36).

We define $\widetilde{c}^j = |\frac{\partial \mathsf{E}_0[V_1(0,\mathbf{d}_1)]}{\partial S_j}|_{\mathbf{S}=\mathbf{S}^*(B)}$ (Lagrangian of (3.36)) for all $j = 1, \ldots, M$. It can be verified that the solution to (3.5) with the investment cost vector $\mathbf{c} = \widetilde{\mathbf{c}}$ is $\mathbf{S}^*(B)$. Hence, the two formulations are structurally equivalent.

CHAPTER IV

Enhancing the effect of Social Comparisons in Energy markets

4.1. Introduction

The previous two chapters focused on reducing the cost of Electric Energy Supply. In this chapter, we shift our focus to the demand side of the supply chain and consider the potential methods to reduce the cost of satisfying consumer preferences. Reducing 'demand' or reducing the cost of serving consumers, known as 'behavioral energy efficiency,' may be accomplished in primarily two ways: motivating consumers to upgrade to energy-efficient equipment and/or motivating consumers to modify their consumption behaviors.

Upgrading to energy-efficient equipment may include simple changes such as upgrading to programmable thermostats or CFL light bulbs as well as more extensive projects such as winterizing homes. These are recognized as one time actions with a cost associated with the upgrade. Motivating consumers to manage their consumption behaviors, include sending variety of messages, e.g. patriotic messages to reduce energy dependence of the nation, messages describing economic benefits to consumers of consuming less, 'Go Green' messages to incite saving the environment or peer-consumption information intended to trigger a healthy competition on being energy efficient. Cialdini and Schlutz (2004) describe a study in San Marcos, California where it was found that peer comparisons was the most effective motivator. In fact, Opower, an Arlington based company, involved in data mining of energy usage of households, combines both these methods in their monthly Home Energy Reports (HER) to consumers, providing upgrade information and peer-comparison information. Several studies have shown these reports have decreased average energy consumption by as much as 3% (Todd et al. 2014).

The objective of our work is to model and study each of the two above methods (modifying behaviors and upgrading equipment) and the interaction between them. In particular, we characterize the situation where these two outcomes are synergetic. We provide a parsimonious model to evaluate the effects of both these methods and also extend the literature on modelling the impact of social comparisons from the perspective of the population with continuous consumer types. Finally, we provide one additional reason explaining the effectiveness of Opower's HER program: its combination of economic (i.e., upgrades) and social information (i.e., peer comparisons) can reinforce each other justifying the joint use of both of them.

Through the rest of this chapter, Section 4.2 reviews the current literature, Section 4.3 discusses the model combining both social and economic incentives, Section 4.4 derives the outcomes under the various incentives, Section 4.5 discusses the numerical results and the chapter concludes in Section 4.6.

4.2. Literature Review

Our work considers the interaction of economic and social incentives provided to consumers under the Opower HER program. Hence, it is related to three streams of literature: the empirical research on the impact of Opower's HER program, the modelling of social comparisons (as an incentive), and the literature on the impact of economic incentives for Energy consumers. We discuss each of these streams in detail below.

4.2.1 Opower's HER program

There are a few recent empirical research papers analyzing the effects of Opower's HER program. Cialdini and Schlutz (2004) show that normative messages coupled with appeals to reduce consumption of energy can have a significant impact on energy consumption. In particular, peer consumption information was found to be the most effective. Applying this concept, Opower began sending monthly energy reports to consumers comparing their consumption with that of similar households. Several papers report that Opower HER program has reduced energy consumption by around 2% (Allcott 2011, Ayres et al. 2013).

The longevity of these effects have been questioned in Ferraro and Price (2013), with the effects depending on the political inclinations of the consumer (Costa and Kahn 2013). HER program, however, has been widely accepted as effective in helping to achieve the state and federal energy efficiency requirements, to the extent that many of the nations largest utilities (like PG&E, ComEd, AEP) have begun to ramp up their behavioural energy efficiency portfolios (Opower 2012). While the above papers study the effects of Opower's HER program empirically, we model these effects analytically.

4.2.2 Utility from Social Comparisons

In general, there are several papers that consider the outcome of social comparisons as a part of utility of consumers. Similar to our work, a majority of the literature assumes that such utility is linear and proportional to the difference between the individual's outcome and the reference point. The main types of social comparison models are as follows,

- Status seeking (Frank 1985): The original hypothesis was that individuals are generally spiteful or envious of others and derive utility from being ahead of others performance, in order to seek status relative to other individuals. Frank (1985) proposed that individuals seek small 'ponds' in which they are relatively big fish rather than big ponds, in which they are relatively small fish. The paper models the utility as linear function of the difference between performance of an individual and the reference point x_0 in that 'pond'. If x_i is the performance of an individual, then individuals are either *ahead-seeking*, where their utility from being ahead of a reference point is given by, $\alpha(x_i - x_0)^+$ or *behind-averse* where they experience disutility from being below the reference point, contributing to utility as: $-\beta(x_0 - x_i)^+$.

- Inequity averse (Fehr and Schmidt 1999): In this model, individuals are generally averse to all kinds of inequity, whether they are ahead or behind compared to other individuals in the pool.[1] The standard form of utility of an individual i is given by,

$$U_i(\{x_i, x_j\}) = x_i - \frac{\alpha}{n-1}\sum(x_j - x_i)^+ - \frac{\beta}{n-1}\sum(x_i - x_j)^+, \tag{4.1}$$

[1] There is often the assumption that individuals are more averse to being behind than being ahead.

where $\alpha \geq \beta$. This model is often considered the gold standard for social comparisons and its variations are used frequently in the literature (Charness and Rabin 2002, Bolton and Ockenfels 2006).

- Risk aversion models (Linde and Sonnemans 2012, Bolton and Ockenfels 2010): These models extend the Fehr and Schmidt (1999) model by including uncertainty in the outcome of an individual's action due to a state of nature or the actions of other individuals. Bolton and Ockenfels (2010) argue that consumers would rather take a higher risk due to a state of the nature than risk of being 'betrayed' by another individual (in a teamwork setting). Linde and Sonnemans (2012) consider the implications of Prospect theory and risk aversion when utility derived from social comparisons is uncertain because of uncertain outcomes.

- General Utility function (Levitt and List 2007): Some authors consider a general utility function where each individual's utility depends on the performance of all individuals in the economy and may be increasing or decreasing non linearly in the parameters associated with other individuals.

Roels and Su (2013) belongs to the first group and is methodologically closest to our paper, while asking different research questions. Motivated by Opower, Roels and Su (2013) study how social comparisons may affect performance of consumers when consumer population consists of two types. Their research assumes that consumers are either *ahead-seeking* or *behind-averse*. The focus of their work is to consider the most effective reference system to provide the consumer population, in order to achieve the social planner's objectives. We extend their model to consider continuous population types and analyze the combined effect of the social comparisons (that they consider) and economic incentives (that they do not consider). Like Roels and Su (2013), we restrict our attention to the "ahead-seeking" and "behind-averse" social utility models. The current empirical research on social comparisons is often based on decisions of players in ultimatum games, responder games and such similar set ups where players make a decision to up-hold/allocate pay-offs (Fehr and Schmidt 2006). In these contexts, we observe that the "behind-averse" utility may be applicable (Bolton and Ockenfels 2006). Our context is to evaluate the effort levels of the players in relation to the social motivation they receive from comparison to a reference, akin to students efforts when provided reference information of other student's performances (Stevenson et al. 1990). Here, in some cultural contexts, students strive to be at the top of their class, i.e., we observe 'ahead-seeking' behavior. A similar behavior is observed in income distribution preferences of German citizens (Schwarze and Härpfer 2003). We hypothesize that a population may consist of both 'ahead-seeking' and 'behind-averse' consumers.

There are two other social utility models that are related: inequity averse and concave utility functions. However, inequity aversion is not applicable in our context as it is unlikely that customers feel a penalty by consuming far less than the average. While concave utility is a likely option, we do not consider it in this chapter, for the sake of analytical tractability.[2]

4.2.3 Energy Efficiency Incentives to Energy Consumers

One of key decisions consumers in power markets can make is to invest in more energy-efficient equipment. The energy savings constitute an economic incentive for investment. The Opower HER program provides consumers with necessary information about investments in energy savings and upgrades. This subsection discusses the stream of literature that models energy efficiency upgrades.

To entice consumers to be energy efficient, utilities and government agencies have been providing Energy Efficiency (EE) or Demand Side Management (DSM) programs for decades. These programs include Appliance recycling, HVAC upgrade initiations, Energy Star campaigns, Green lights programs etc, which result in significant reductions in energy consumption (Geller et al. 2006, Eto et al. 1994). These programs typically cost between 3.9-5 cents per KWh of Energy reduction (Arimura et al. 2011), and are estimated as being overall profitable (Friedrich et al. 2009). Additionally, energy efficiency programs have other tangential benefits including improved health and comfort for the consumer and generation of jobs (Clinch and Healy 2000, Mills and Rosenfeld 1996).

Interestingly, while these programs have been effective, the effectiveness observed is generally much lower than their predicted effectiveness. This is known as the 'energy efficiency' gap (Jaffe and Stavins 1994). The empirical evidence for this gap is well established, showing

[2] Concave social utility makes the consumption decision maximizing a non-concave function

that consumers make decisions based on an implied discount rate of 25-100%, which is far above the market discount rate (Sanstad et al. 2006, Train 1985). There are a wide variety of explanations for this gap, such as hidden costs of switching (Jaffe et al. 2004), other product attributes (e.g., lower lighting quality), heterogeneity of consumers (Hausman and Joskow 1982), and future uncertainties (Sutherland 1991). Reddy (1991) suggest that these barriers may be because of liquidity constraints with the consumers, indifference, bounded rationality, legacy issues, and imperfect information (Howarth and Andersson 1993). See Gillingham et al. (2009) for a complete survey of potential behavioral and market failures to explain this energy efficiency gap.

Our paper attempts to model the barriers that cause this gap analytically. Due to the higher discount rate, adoption rates of EE programs are higher for programs with faster payback and greater annual savings (Anderson and Newell 2004). The initial investment costs may be significant and include, for example, transactions costs associated with making the switch to low-carbon technologies (Mundaca et al. 2013), search costs, deciding from the array of available options and making life style changes etc. B Howarth et al. (2000) argue that EPA programs such as 'Green Lights' and 'Energy Star Office Products' have been successful because their campaigns address the imperfect information and bounded rationality issues (i.e., huge costs of decision making) well. Note that Opower's HER program provides this benefit with the information sent in the reports. We explicitly model these transaction costs in the context of making the decision to switch and the additional costs of the new technology. We capture these costs as a one time fixed cost of making the switch and a linear cost with increase in energy efficiency ability. Related to our work, Gillingham et al. (2009) model consumers' Energy Efficiency decisions as a trade-off between capital costs of making the switch and cost of energy, while providing the same level of energy service. In contrast, we assume a concave utility function for the service provided by energy consumption and assign the energy efficiency ability as a parameter that affects the utility from energy consumption. Additionally, we also consider the effects of social comparisons while they focus on the design of energy efficiency programs.

4.3. The Model

In this section we describe the model, which we use to compare the effects of different incentives.

We consider a population of consumers with a continuous distribution of their types, indexed by θ. Intuitively, θ stands for the energy efficiency. Each player chooses a demand quantity to consume $x \in [0, X]$, where minimum demand is normalized to 0 and X is the maximum possible demand. From the system perspective, lower net demand is environmentally friendly and more desirable. However, consumer of type θ receives a utility from their consumption, given by $V_\theta(x)$, with decreasing marginal benefit. We assume $V_\theta(x)$ is strictly increasing and strictly concave. Also, each player incurs a financial cost of consuming demand x, given by γx. Thus each player has a unique demand consumption that maximizes the net utility $V_\theta(x) - \gamma x$.

The customer types arise from their inherent differences in consumption utility from given demand. We consider a continuum of player types θ, per an atomless distribution, given by the Probability Distribution Function (PDF), $f(\theta) : [0, \theta^{\max}] \to \Re^+$, and the differentiable Cumulative Distribution Function (CDF) given by $F(\theta)$. We assume that a player of higher type-θ requires to put a lesser amount of effort to achieve the same utility as a player of lower type. Specifically, we assume that the consumption utility for type-θ player is given by, $V_\theta(x) = V(x + \theta)$ where $V(.) = V_0(.)$ is the utility function of player of type 0. Note that players with higher θ value are predisposed to have lower demand consumption, as their marginal utility of consuming x units is the same as the marginal utility of consuming $x + \theta$ for player of type 0. This allows us to consider the well-defined function $v_1(y) = (V')^{-1}(y)$ on $[-\infty, V'(0)]$, which represents the inverse of $V'(x)$.[3] We further assume X is large enough to be not binding.

The social planner has two potential methods of incentivizing consumers to consume less demand. We label them as economic and social incentives.

- Under economic incentives, the social planner provides information to consumers about investments that improve energy efficiency, may be in the form of making investments to

[3]We assume $\gamma < V'(0)$, else the optimal consumption is 0 for all cases.

caulk windows, improve insulation, upgrade to more energy efficient equipment etc. The HER reports radically simplify the process by providing relevant information. Given this information, these decisions would typically involve significant transactions costs making the choice to upgrade. Once the consumers have acquired the information, they may upgrade to any level, i.e., up to $\theta^{\max}$, with a fixed cost c, and a cost of δ per unit improvement in θ. Let the binary decision variable, i, reflect consumers' investment decision, with the net utility given by,

$$U_\theta(x,i) = \begin{cases} \max\limits_{\theta \le \theta^* \le \theta^{\max}} \{V(x+\theta^*) - \gamma x - c - \delta(\theta^* - \theta)\}, & \text{if } i = 1, \\ V(x+\theta) - \gamma x, & \text{else,} \end{cases} \tag{4.2}$$

• Under social incentives, the social planner may choose to provide a reference consumption $\widehat{x}$, which typically is the average consumption of the consumers. A consumer may derive utility from social comparisons with the outputs of other consumers in the economy and thus, the reference consumption provided to the consumers, may influence consumers' decision. In our model, the utility gains/losses from social comparisons are brought in addition to the economic value achieved from consumption, $V_\theta(x) - \gamma x$. The reference consumption provided to the consumers, may influence consumers' decision. We consider two types of social comparison utilities. First, the player may face disutility (or guilty feeling) from higher consumption compared to other players. Hence, the utility of a player of type θ, similar to Fehr and Schmidt (1999), is given by,

$$U_\theta(x,\widehat{x}) = V_\theta(x) - \gamma x - \beta\big(x - \widehat{x}\big)^+, \tag{4.3}$$

where, $\beta\big(x - \widehat{x}\big)^+$ is called "behind loss" of this particular player and players in such model are "behind-averse." Alternatively, a player may receive additional utility from consuming less than other players. We label these players as "ahead-seeking" and their utility function is given by,

$$U_\theta(x,\widehat{x}) = V_\theta(x) - \gamma x + \alpha\big(\widehat{x} - x\big)^+. \tag{4.4}$$

The optimal consumption decision function for each player of type θ, depends on other players' decisions and the reference point, i.e., $x^*(\theta) \in \arg\max_{i\in\{0,1\}} U_\theta(x, \widehat{x}, i)$, where $\widehat{x} = \int_{\theta=0}^{\theta^{\max}} f(\theta)x^*(\theta)d\theta$. The equilibrium characterizes the consumption of all players in the population. We assume that all players have the same parameters α and β. When the players receive both social and economic incentives, their utility depends on the investment θ, consumption x and the benchmark consumption $\widehat{x}$,

$$U_\theta(x, i, \widehat{x}) = \begin{cases} \max\limits_{\theta \le \theta^* \le \theta^{\max}} \{V(x+\theta^*) - \gamma x - c + g(x, \widehat{x}) - \delta(\theta^* - \theta)\}, & \text{if } i = 1, \\ V(x+\theta) - \gamma x + g(x, \widehat{x}), & \text{else,} \end{cases} \tag{4.5}$$

where $g(x, \widehat{x}) = \alpha(\widehat{x} - x)^+$ or $-\beta(x - \widehat{x})^+$, in the ahead-seeking and behind-averse cases respectively.

We compare the impact of providing social (in both ahead-seeking and behind-averse environments) or economic incentives, separately or together. That is, we compare the equilibrium consumption profiles under 4 cases: case without economic or social incentives, case with only economic incentives, case with only social incentives and case with both economic and social incentives. Among social incentives, we consider both ahead-seeking and behind-averse cases. Additionally, let $\theta' = \min\{v_1(\alpha+\gamma), \theta^{\max}\}$ and $\theta'' = \min\{v_1(\gamma), \theta^{\max}\}$ represent two thresholds of customer types. If $\theta' < \theta^{\max}$ or $\theta'' < \theta^{\max}$, then we will describe in the next section that it may be optimal to consume 0 units (the lower limit). We also assume that $\alpha, \beta, \gamma, c > 0$.

4.4. Equilibria characterization

In this section, we characterize the equilibria under the various scenarios. First, we consider the benchmark case of no incentives and compare it to the case with economic incentives.

Proposition IV.1. *(i) Under no social or economic incentives, the optimal decision function is, $x^*(\theta) = (v_1(\gamma) - \theta)^+$, with average consumption $\widehat{x} = \int_0^{\theta'} (v_1(\gamma) - \theta) f(\theta) d\theta$.*

(ii) Under economic incentives, the optimal decision function is,

$$(x^*(\theta), \theta^*(\theta)) = \begin{cases} (v_1(\gamma) - \theta', \theta'), & \text{if } \theta \leq \widetilde{\theta}, \\ ((v_1(\gamma) - \theta)^+, \theta), & \text{else}, \end{cases} \tag{4.6}$$

where $\widetilde{\theta} = (\theta' - c/(\gamma - \delta))^+$ *if* $\delta < \gamma$*, else* $\widetilde{\theta} = 0$.

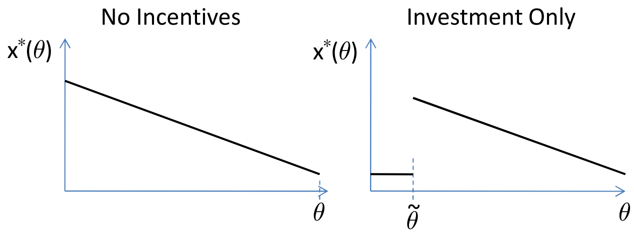


Figure 4.1: Pattern of optimal consumption function under the vanilla case and the case with only economic incentives.

Clearly, when we have economic incentives, the average consumption decreases. There exists a threshold $\widetilde{\theta}$, below which all consumers choose to invest. However, consumers who have reasonably high ability choose to not invest. Those that upgrade, choose their technology level $\theta^* = \theta'$ if $\delta < \gamma$.

Now, we consider the optimal policies under only social incentives, in both ahead-seeking and behind-averse scenarios:

Proposition IV.2. *The equilibrium under social incentives is given by,*

(i) Under ahead-seeking scenario, the optimal decision function is given by,

$$x^*(\theta) = \begin{cases} v_1(\gamma) - \theta, & \text{if } 0 \leq \theta \leq \widehat{\theta}, \\ (v_1(\gamma + \alpha) - \theta)^+, & \text{if } \widehat{\theta} < \theta < \theta^{\max}, \end{cases}$$

where $\widehat{\theta} = v_1(\alpha + \gamma) - \widehat{x} + k^\alpha$*, where* $k^\alpha = (V(v_1(\gamma)) - V(v_1(\alpha + \gamma)) + \gamma(v_1(\alpha + \gamma) - v_1(\gamma)))/\alpha$ *for some unique* $\widehat{x}$.

(ii) Under behind-averse scenario, the optimal decision function is as follows,

If $v_1(\gamma) - \theta' \leq v_1(\gamma + \beta)$,

$$x^*(\theta) = \begin{cases} v_1(\gamma+\beta) - \theta, & \text{if } 0 \leq \theta < \underline{\theta}, \\ \widehat{x} & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta}, \\ (v_1(\gamma) - \theta)^+, & \text{if } \overline{\theta} < \theta \leq \theta^{\max}, \end{cases}$$

where $\underline{\theta} = v_1(\gamma+\beta) - \widehat{x}$, $\overline{\theta} = v_1(\gamma) - \widehat{x}$, *for some unique* $\widehat{x} \in (v_1(\gamma) - \theta', v_1(\gamma+\beta))$. *Else,* $x^*(\theta) = \widehat{x}$ *for all* $\theta \in [0, \theta^{\max}]$ *is optimal for all* $\widehat{x} \in [v_1(\gamma+\beta)^+, v_1(\gamma) - \theta']$.

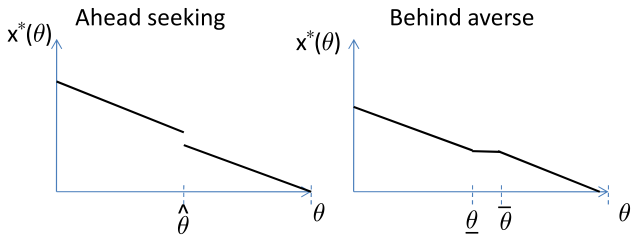


Figure 4.2: Pattern of optimal consumption function under the case with only social incentives.

In both ahead-seeking and behind-averse scenarios, unique equilibria exist and are as described above. The average consumption, in each case is smaller than the case with no incentives. However, the relative ordering of the average consumption under economic only incentive and social only incentive depends on the value of the parameters c and α.

Now, we consider the case when both incentives are applied.

Proposition IV.3. *Given* $\delta < \gamma$, *the optimal decision function under social and economic incentives is as follows:*

(i) Under ahead-seeking scenario,

$$(x^*(\theta), \theta^*(\theta)) = \begin{cases} (v_1(\gamma+\alpha) - \theta'', \theta''), & \text{if } 0 \leq \theta \leq \widetilde{\theta}, \\ (v_1(\gamma) - \theta, \theta), & \text{if } \widetilde{\theta} < \theta \leq \widehat{\theta}, \\ (v_1(\gamma+\alpha) - \theta, \theta), & \text{if } \widehat{\theta} < \theta \leq \theta^{\max}, \end{cases}$$

where $\widehat{\theta} = \min\{v_1(\alpha+\gamma) - \widehat{x} + k^\alpha, \theta^{\max}\}$, *and* $\widetilde{\theta} = (\theta'' - c/(\gamma-\delta) + \alpha(\theta'' - (v_1(\alpha+\gamma) - \widehat{x} + k^\alpha))/(\gamma-\delta))^+$, *and* $k^\alpha = (V(v_1(\gamma)) - V(v_1(\alpha+\gamma))) + \gamma(v_1(\alpha+\gamma) - v_1(\gamma)))/\alpha$ *for some unique* $\widehat{x}$.

(ii) Under behind-averse scenario, if $v_1(\gamma) - \theta' \leq v_1(\gamma+\beta)$,

$$(x^*(\theta), \theta^*(\theta)) = \begin{cases} (v_1(\gamma) - \theta', \theta'), & \text{if } 0 \leq \theta < \widetilde{\theta}, \\ (v_1(\gamma+\beta) - \theta, 0), & \text{if } \widetilde{\theta} \leq \theta < \max\{\underline{\theta}, \widetilde{\theta}\}, \\ (\widehat{x}, 0), & \text{if } \max\{\underline{\theta}, \widetilde{\theta}\} \leq \theta \leq \overline{\theta}, \\ (v_1(\gamma) - \theta, 0), & \text{if } \overline{\theta} < \theta \leq \theta^{\max}, \end{cases}$$

where $\underline{\theta} = (v_1(\gamma+\beta) - \widehat{x})^+$, $\overline{\theta} = v_1(\gamma) - \widehat{x}$, $\widetilde{\theta} = \big((\gamma-\delta)\theta' + \beta(v_1(\gamma+\beta) - \widehat{x}) + \beta k^\beta - c\big)^+/(\gamma+\beta-\delta)$ *, and* $k^\beta = (V(v_1(\gamma)) - V(v_1(\beta+\gamma)) + \gamma(v_1(\beta+\gamma) - v_1(\gamma)))/\beta$ *for some unique* $\widehat{x} \in [v_1(\gamma) - \theta', v_1(\gamma+\beta)]$*. Else,* $x^*(\theta) = \widehat{x} = v_1(\gamma) - \theta'$ *for all* $\theta \in [0, \theta^{\max}]$ *is optimal.*

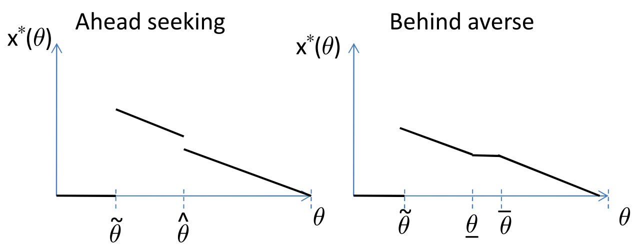


Figure 4.3: Pattern of optimal consumption function under the case with both economic and social incentives.

The above case is more involved and contains more intricacies than the previous two propositions. However, we observe that there ends up being a social pressure for "investing" in more energy efficient equipment, hence the threshold of investment $\widetilde{\theta}$, goes up dramatically, especially in the ahead-seeking scenario. Applying Propositions IV.1-IV.3, we have the following result about the relative values of the average consumptions under the various incentive schemes.

Proposition IV.4. *Assume population with uniform probability distribution function* $f(\theta) = 1/\theta^{\max}$ *for all* $\theta \in [0, \theta^{\max}]$, $c > \sqrt{2\gamma^2\theta^{\max}k^\alpha + (\gamma(v_1(\gamma) - v_1(\alpha+\gamma)))^2} - \gamma(v_1(\gamma) - v_1(\alpha+\gamma))$ *and* $v_1(\alpha+\gamma) \geq \theta^{\max}$ *and the value function* $V(.)$ *quadratic. In ahead-seeking scenarios, the reduction in average consumption due to the combined social and economic incentives is higher than the sum of the reductions due to each of the incentives individually.*

Proposition IV.4 states that the two types of incentives, when combined provide larger reduction in the average consumption of the population, as compared to the sum of reductions under the individual incentives, under ahead-seeking scenarios. This behavior is illustrated in Figure 4.4. Observe the relatively large decrease in the average consumption under both incentives. The grayed region and the tiled region under both incentives is larger then the respective regions under social and economic incentives respectively.

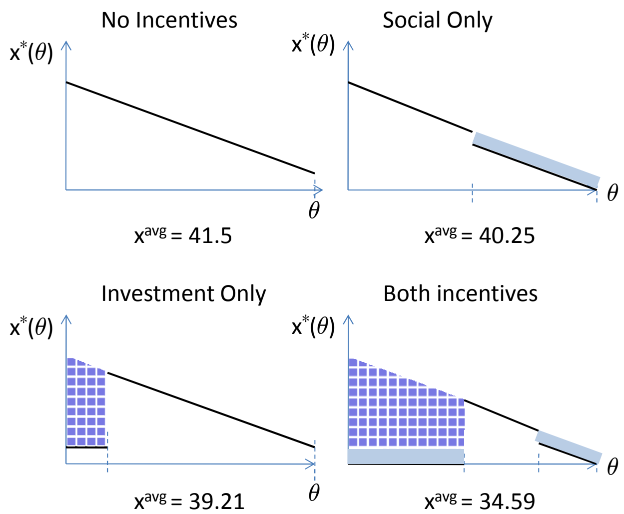


Figure 4.4: Output consumptions under all four incentive schemes for an example with $\theta \sim U[0, 15]$, $V(x) = 100x - x^2$, $\alpha = 5$, $\gamma = 2$, $c = 25$, $\theta^{\max} = 15$. The tiled region represents the magnitude of the decrease in consumption because of consumers who upgrade. The gray region denotes the decrease in consumption due to social utility achieved by consumers who are below the reference point.

Interestingly, under behind-averse scenarios, we discuss in the following section that depending on the relative values of β and c, the benefit of both methods can be sub-additive or super-additive.

We note that Proposition IV.4 requires the additional constraint that $v_1(\alpha+\gamma)-\theta^{\max} > 0$. This constraint implies that conditional on making the decision to invest, consumers upgrade to the highest ability level $\theta^{\max}$. Indeed, the varying level of investment for different customer types is a more realistic reflection of the markets, i.e., only some consumers may choose to upgrade and their upgrade levels would depend on their starting level θ. We observe that under these relaxed constraints, the proposition no longer holds. This is because, for high enough α and low enough c, it may be that each of the incentives individually may be sufficient to motivate all consumers to consume the lower limit 0. Hence, introducing both incentives provides no additional benefit compared to introducing a single incentive, as in the following example.

Example IV.5. *Consider the case when $V(x) = 20x - x^2$, $c = 5$, $\delta = 1$, $\gamma = 10$, $\alpha = 10$, $\theta^{\max} = 15$ and $f(\theta) = 1/15$ for $\theta \in [0, \theta^{\max}]$.*

For this example, observe that the optimal consumption function under the vanilla case is given by, $x^*(\theta) = (5 - \theta)^+$, i.e., $\theta' = 5$. Interestingly, given that $\alpha = 10$, it can be shown that $x^*(\theta) = 0$ for all $\theta \in [0, \theta^{\max}]$ for the case with social incentives only. However, for the case with economic incentives, the optimal consumption function is given by $x^*(\theta) = (5 - \theta)$ if $\theta \in [40/9, 5]$ and $x^*(\theta) = 0$ otherwise. Observe that in the case with both incentives the optimal consumption function is still $x^*(\theta) = 0$ for all $\theta \in [0, \theta^{\max}]$. Hence, the net reduction in average consumption under both incentives is -0.823 less than the sum of reductions under the individual incentives, contradicting the superadditivity hypothesis.

4.5. Impact of Parameters on superadditivity

In this section, we consider the effect of varying the parameters, $c, \delta, \gamma, \alpha (= \beta)$ and the PDF $f(.)$ on the superadditivity of the two incentive schemes. We measure the superadditivity in terms of the difference between the sums of the outcomes: $\Delta = (\widehat{x}_i + \widehat{x}_\alpha) - (\widehat{x}_0 + \widehat{x}_{\alpha,i})$, in other words, this describes the additional decrease in the average consumption under providing both incentives when compared to the sum of the benefits under both incentives. We first describe the intuition behind the superadditivity under the ahead-seeking social utilities. Similarly, we describe why superadditivity may not be true under the behind-averse scenario. Then, we compare the changes in the output Δ and the benefit of the incentives to changes in the parameters.

4.5.1 Mechanism of superadditivity

We first discuss the intuition behind the superadditivity in the ahead-seeking case. Observe that under economic incentives, several consumers of lower type θ switch to the higher type, where-as under the social incentives, the broad upper range of types θ under ahead-seeking are willing to consume less, for the benefit of increasing how much they are ahead.

This can be seen in Figure 4.4. Under both incentives combined, the social incentives motivates more consumers to make the upgrade, hence, the threshold $\widetilde{\theta}$ up to which consumers upgrade increases (i.e., the size of the tiled region in the Figure increases). Interestingly not only do the lower types decide to upgrade, but they also consume less because they have the additional incentive of being ahead of the average (i.e., the size of the shaded region in Figure 4.4 increases). We identify these as the two sources of superadditivity. Through the rest of this section, we explain the changes in the superadditivity to changes in parameters by attributing to the change in size of these areas.

Under the behind-averse scenario, applying social incentives benefits the low types of consumers, as these customers decrease their consumption to get closer to the average. Similarly, the economic incentives also provides the low type consumers possibility to upgrade to a higher technology. However, under both incentives, consumers either upgrade to the highest level or work extra hard to consume less (to reduce the behind loss), in other words, they choose one of the two incentives to motivate them as once they upgrade to the highest level, they do not have any motivation to consume even less. Hence, we observe in our extensive numerical study that often, $\Delta < 0$, i.e., the two benefits are sub-additive. However, when c is quite high or β is quite low, providing both benefits presents the additional nudge for more consumers to upgrade further or pushing down the average causing the 'guilty feeling' on more consumers. In these cases, the benefits may be super-additive.

4.5.2 Explanation of numerical study

To illustrate the interaction of incentives, we consider a simple quadratic model with $V(x) = 100x - x^2$ and a range for the values of parameters c, δ, α, β. We assume that the distribution of consumer types is triangular and follows the PDF $f(.)$ given by,

$$f(\theta) = \frac{g}{\theta^{\max}} + \begin{cases} \frac{4\theta(1-g)}{(\theta^{\max})^2} & \text{if } \theta < \frac{\theta^{\max}}{2} \\ \frac{4(\theta^{\max}-\theta)(1-g)}{(\theta^{\max})^2} & \text{else} \end{cases} \tag{4.7}$$

where $g \in [0, 2]$ is an index of variance of the distribution, while the mean remains the same.

4.5.3 Impact of c and δ

First consider the impact of the fixed cost of upgrading, c. Recall that superadditivity in ahead-seeking scenarios stems from the increase in the areas of the tiled and shaded regions, as seen in Figure 4.4. Superadditivity is high for very low c and very high c. This is because, at very low c, the increase in the area of the shaded region is high, because the threshold $\widetilde{\theta}$ is high. At very high c, the relative increase in the area of the tiled region is high because of the increase in the threshold $\widetilde{\theta}$ from the case with economic incentives to the case with combined incentives.

In behind-averse scenario, as c increases, fewer consumers react to economic incentives. Recall that in behind-averse scenarios sub-additivity arises because the ones who upgrade lose the motivation due to social incentives as they are ahead of the average (due to economic incentives). However, as c increases, more consumers have insufficient incentives to upgrade and combining it with social incentives helps. Thus, we observe superadditivity increases.

Interestingly, as δ represents the unit cost of upgrading while c represents the fixed cost of upgrading, the effect on the output parameters to changes in δ follows similar patterns as c, providing the same insights.

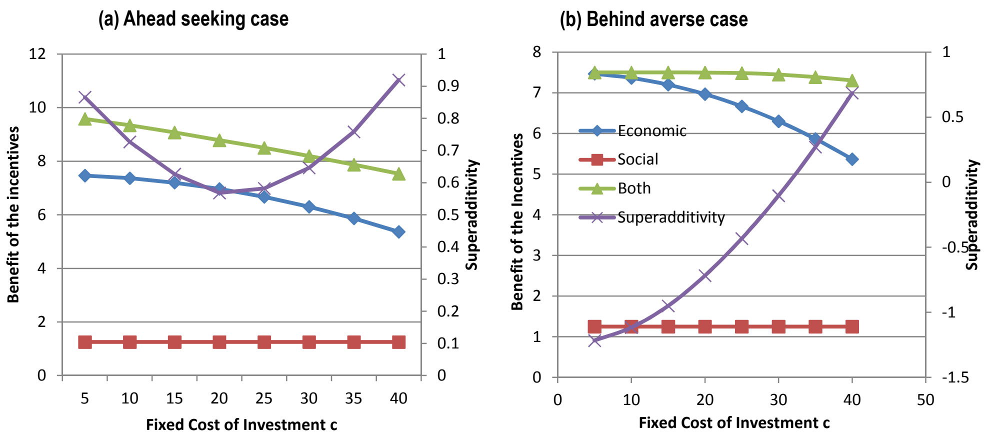


Figure 4.5: Impact of fixed cost of investment when $\delta = 1,\ \gamma = 6,\ g = 1, \alpha(= \beta) = 5$.

4.5.4 Impact of γ

As γ increases, the threshold $\widetilde{\theta}$ up to which consumers upgrade increases.[4] Hence, the area of the shaded region is increasing, i.e., superadditivity, which captures the additional decrease in consumption of those who upgrade due to social incentives, is increasing. In contrast, under behind-averse scenario, increasing the threshold up to which consumers upgrade, implies more consumers (who upgrade) lose out on the social sensitivity benefits. Hence, the superadditivity factor Δ decreases with increase in γ.

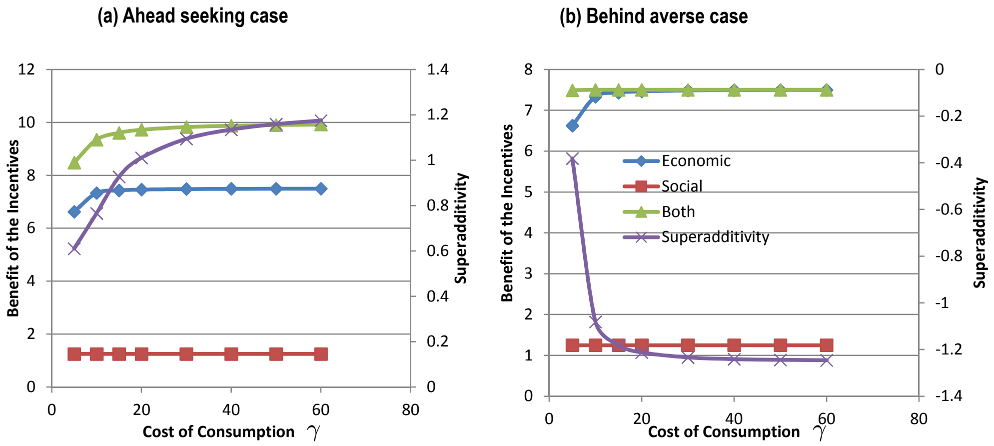


Figure 4.6: Impact of cost of consumption γ when $\delta = 1.1,\ c = 20,\ g = 1, \alpha(=\beta) = 5$.

4.5.5 Impact of PDF $f(.)$

In this subsection, we measure the impact of the variance in the distribution $f(.)$ of the consumer types.

Observe that the value of the threshold $\widetilde{\theta}$ up to which consumers upgrade does not change with g. We first consider the case when $\widetilde{\theta}$ is larger than the median of the distribution. In this case, observe that larger g implies that a smaller proportion of the population is less than the threshold. Hence, the weighted area of the shaded region (i.e., the amount by which consumers who upgrade reduce their consumption) decreases as g increases for the ahead-seeking scenario. Hence, we observe that superadditivity is decreasing when g increases. In contrast, for the behind-averse scenario, as g increases, the number of consumers upgrading decreases. Hence, in line with the arguments in section 4.5.1, the super additivity benefit is

[4]All consumers who upgrade choose the same final ability level.

increasing as fewer consumers are missing out on the double benefits.

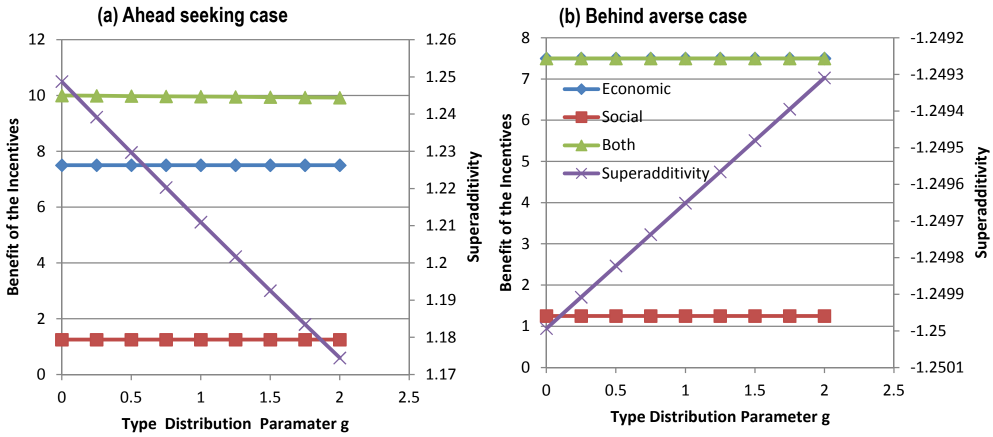


Figure 4.7: Impact of Distribution parameter g when the threshold of upgrading $\widetilde{\theta}$ is higher than the median, and $\delta = 1.1,\ \gamma = 50,\ c = 30, \alpha(= \beta) = 5$.

For the behind-averse scenario, when the threshold $\widetilde{\theta}$ is below the median, increasing g increases the proportion of the population who upgrade when subject to economic incentives. Hence, using the same arguments as above, the superadditivity benefit is decreasing. However, for the ahead-seeking scenario, when g increases, while the proportion of population who upgrade is increasing the proportion of the population in the higher distribution of types who are above the average is also increasing, neutralizing the effects of the change in area of the shaded region. More interestingly, increasing g reduces the impact of the increase in the area of the tiled region. Hence, superadditivity decreases when g increases in this case.

4.5.6 Impact of $\alpha(= \beta)$

For ahead-seeking case, as α increases, the relative increase in the upgrade threshold due to applying both incentives increases, because there's more motivation to upgrade, as upgrading allows the consumer to be ahead of the average. Hence superadditivity is increasing, until α becomes high enough that the constraint $x \geq 0$ becomes binding. At this point, the two benefits become redundant as one benefit is already beginning to achieve the maximum possible reduction in the consumption. Hence, Δ starts to decrease at higher α.

In contrast, under the behind-averse scenario, the sub additivity comes from low ability consumers who either upgrade or reduce behind loss by consuming less. As β increases, consumers who upgrade lose relatively more to the social case when they would consume

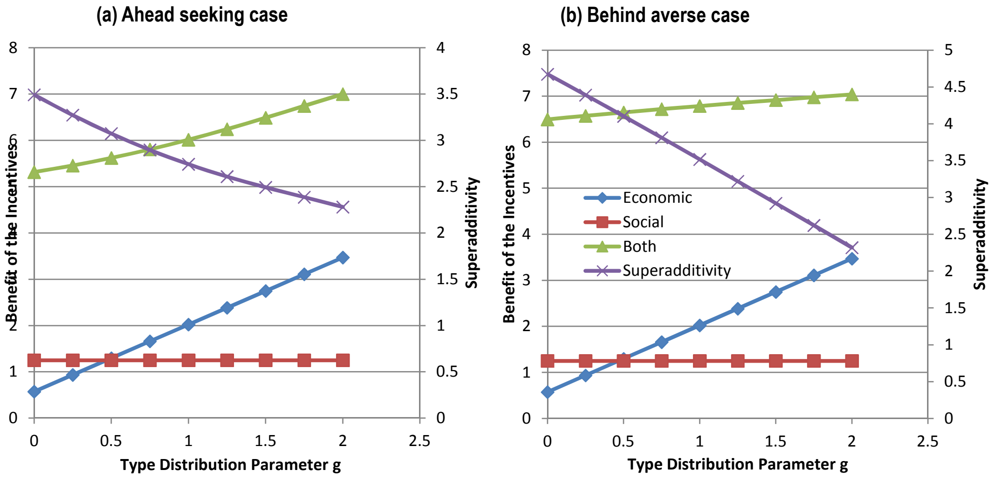


Figure 4.8: Impact of Distribution parameter g when the threshold of upgrading $\widetilde{\theta}$ is less than the median, and $\delta = 1.1,\ \gamma = 5,\ c = 50, \alpha(= \beta) = 5$.

lesser for being closer to the average. Hence, the sub-additivity increases, as β increases.

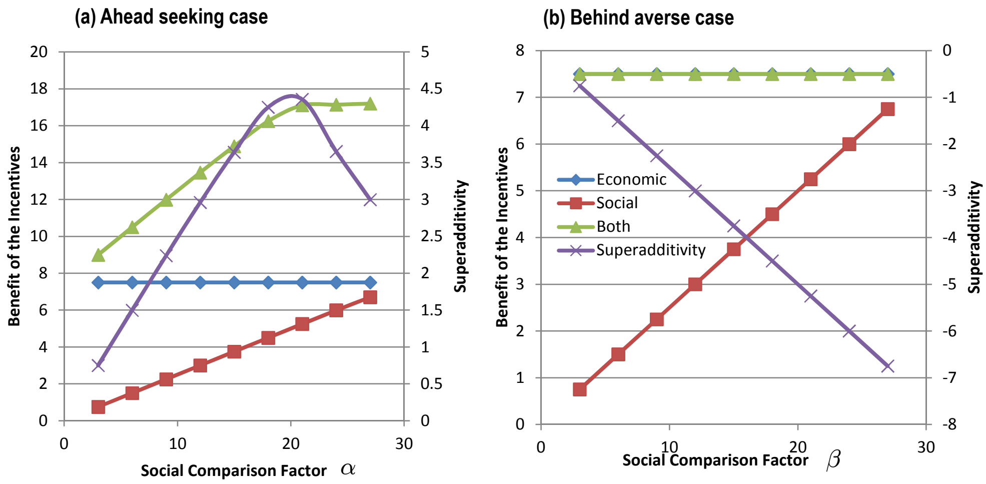


Figure 4.9: Impact of social comparison factor $\alpha(= \beta)$ when $\delta = 1.1,\ \gamma = 50,\ g = 0, c = 5$.

4.6. Conclusion and Extensions

The Home Energy Reports (HER) program of Opower have proved very effective in reducing the average demand consumption of the energy customer population. This work attempts to provide one explanation as to why the combination of actionable information (ideas for upgrading to energy efficient equipment etc) and appropriate social comparisons can be motivating to consumers rather than the individual effects of these two 'incentive' schemes, within the framework of the literature on social comparisons.

We consider a stylized model which compares the impact of economic and social incentives on the consumption levels of energy consumers. We prove the structure of the optimal consumption functions for consumers as a function of their type θ. Proposition IV.4 analytically verifies the observation that the net reduction under both incentives is higher than the sum of the separate reductions under the individual incentive schemes, for ahead-seeking social comparisons, under some constraints.

Further, we conduct an extensive numerical study to measure the effect of changes in parameters on the 'superadditivity' of the two effects, in both ahead-seeking scenarios. In general, we note that often when a change of parameters increases superadditivity for ahead-seeking, it is likely to reduce superadditivity for the behind-averse scenario. Specifically, we find that as γ, the cost of consumption increases, the threshold level up to which consumers upgrade increases, hence the superadditivity increases under ahead-seeking and decreases under behind-averse.

Finally, when the consumption level begins to hit the lower bound, the super additivity may not be true in ahead-seeking scenarios. In behind-averse scenarios, superadditivity is not generally true.

There are several interesting extensions to strengthen the results. One may consider generalizations of Proposition IV.4 to more general distributions of consumer types. One may consider a concave utility from social comparisons rather than linear in the difference. One may also consider the case when there are customer types associated with the social sensitivity parameter α or β. This work will benefit from these extensions to provide further insights on the effectiveness of social comparisons in energy markets.

There is some evidence to believe that certain energy efficiency improvements are easier and often more fruitful than other energy efficiency improvements, i.e., the cost of increasing ability θ may be convex instead of concave. This kind of extensions may also be considered.

4.7. Appendix: Proofs

Proof of Proposition IV.1. Part (i) follows directly from the fact that $U_\theta(x) = V_\theta(x) - \gamma x$ is strictly concave in x, for given θ, giving a unique maximizer, $x^*(\theta) = (v_1(\gamma) - \theta)^+$ and

$\widehat{x} = \int_{\theta=0}^{\theta^{\max}} f(\theta)x^*(\theta)d\theta$, per the definition.

The proof of part (ii) is similar. Observe first that $U_\theta(x,1) = V(x+\theta') - c - \gamma x - \delta(\theta' - \theta)$ as long as $\delta < \gamma$, else $U_\theta(x,1) = V(x+\theta) - c - \gamma x$. Further, note that $U_\theta(x,1)$ and $U_\theta(x,0)$ are both strictly concave in x respectively per (4.2). Further, $x_1^*(\theta) = v_1(\gamma) - \theta^{\max}$ and $x_0^*(\theta) = v_1(\gamma) - \theta$ maximizes $U_\theta(x,1)$ and $U_\theta(x,0)$ respectively. Finally, we note that, $U_\theta(x_1^*(\theta),1) - U_\theta(x_2^*(\theta),0) = \gamma(\theta' - \theta) - c$ is greater than 0 when $\theta < \widetilde{\theta}$ and less than 0, when $\theta > \widetilde{\theta}$. Hence, we have the optimal policy per (ii). Further, the expression of $\widehat{x}$ follows directly from the expression for the average. □

Proof of Proposition IV.2.

First, we prove part (i). Observe that concavity of $V(.)$, implies, $(V(v_1(\gamma)) - V(v_1(\alpha+\gamma))/(v_1(\gamma) - v_1(\alpha+\gamma)) \geq V'(v_1(\gamma)) = \gamma$, hence, we have, $k^\alpha \geq 0$. For given mean $\widehat{x}$, in order to satisfy rationality, the utility function, given by $U_\theta(x,\widehat{x}) = V(x+\theta) + \alpha(\widehat{x} - x)^+$ must be maximized. This function is clearly concave. Hence, if $x < \widehat{x}$, then the optimal decision satisfies, $x_1^*(\theta) = (v_1(\gamma+\alpha) - \theta)^+$ and if $x > \widehat{x}$, then $x_2^*(\theta) = v_1(\gamma) - \theta$. Note further that, $x^*(\theta) \neq \widehat{x}$ for any θ, as such a value cannot be optimal (except when $\widehat{x} = 0$). For each $\widehat{x}$, there exists $\widehat{\theta} = (v_1(\alpha+\gamma) - \widehat{x}) + k$ at which the optimal solutions from both $x_1^*(\theta)$ and $x_2^*(\theta)$ yields the same value. This shows optimality. Now, we show the existence of a unique $\widehat{x}$, which achieves optimality. Consider,

$$-\widehat{x} + \int_0^{\widehat{\theta}} f(\theta)x_2^*(\theta)d\theta + \int_{\widehat{\theta}}^{\theta^{\max}} f(\theta)x_1^*(\theta)d\theta = 0 \tag{4.8}$$

For $\theta'' = \theta^{\max}$, observe that the first derivative of the LHS of (4.8) is $-1 - f(\widehat{\theta})(v_1(\gamma) - v_1(\gamma+\alpha))$, which is strictly negative. Further the LHS of (4.8) is positive at $\widehat{x} = v_1(\alpha+\gamma) - \theta^{\max} + k^\alpha$, and negative at $\widehat{x} = v_1(\gamma+\alpha) + k^\alpha$. If $\widehat{x} > 0$, then stability requires that $x_1^*(\widehat{\theta}) < \widehat{x} < x_2^*(\widehat{\theta})$, i.e., there does not exist θ for which $x^*(\theta) = \widehat{x}$. We observe that this is true due to the strict concavity of $V(.)$.

Proof of part (ii) is similar to the proof of part (i). Note that the utility function is concave in x even in the behind-averse scenario. □

Proof of Proposition IV.3. Observe, similar to the proof in Proposition IV.2, that

$k^\alpha, k^\beta > 0$. First, we prove part (i) for the case when $c > (v_1(\gamma) - v_1(\gamma+\alpha))\gamma$. Note that the function remains concave in the decision variable x, for both $i = 0, 1$. Hence, the expressions for $x^*(\theta)$ are similar to the proofs of Proposition IV.1 and IV.2. Further, substituting the expressions gives $\widehat{\theta} - \widetilde{\theta} \geq 0$ for all $\widehat{x} \in \left[v_1(\alpha+\gamma) - \theta^{\max} + k^\alpha, v_1(\gamma+\alpha) - \theta^{\max} + c/(\alpha+\gamma) + k^\alpha\right]$. Clearly, $\widehat{x}$ solves,

$$-x + \int_0^{\widetilde{\theta}} f(\theta) x_1^*(\theta'') d\theta + \int_{\widetilde{\theta}}^{\widehat{\theta}} f(\theta) x_2^*(\theta) d\theta + \int_{\widehat{\theta}}^{\theta^{\max}} f(\theta) x_1^*(\theta) d\theta = 0 \tag{4.9}$$

where $x_1^*(\theta), x_2^*(\theta)$ are as defined in the proof of Proposition IV.2. Uniqueness is established because the first derivative of the LHS of the above expression is negative, similar to (4.8). The proofs for the other cases is similar.

The proof of part (ii) is similar. □

Proof of Proposition IV.4. Firstly, for the given range of c, it can be seen that $\widehat{\theta} = v_1(\alpha+\gamma) - \widehat{x} + k^\alpha$. We consider the proof in three cases:

- If $\delta \leq \gamma$, in this case, the variable cost per unit of increasing ability is lower than the benefit received from higher ability. Hence, if a consumer decides to invest, they will invest to increase their ability to the maximum level. So, the structure of equilibria will remain the same, however, the expression for $\widetilde{\theta}$ for cases under Proposition IV.1(ii) and IV.3(i) become $\theta^{\max} - c/(\gamma-\delta)$ and $\theta^{\max} - c/(\gamma-\delta) + \alpha(\theta^{\max} - \widehat{\theta})/(\gamma-\delta)$, while the rest of the expressions remain exactly the same. Substituting these changes into the expression for, $\widehat{x}_0 + \widehat{x}_{\alpha,i} - (\widehat{x}_i + \widehat{x}_\alpha)$ is a positive factor of, $\alpha^2\left(-\left(c^2 r - \theta^{\max}(\gamma-\delta)^3\right)\right) + \theta^{\max} r(\gamma - \delta)^2 \left(2\gamma^2\theta^{\max} - 4\gamma\delta\theta^{\max} - \sqrt{\frac{(\gamma-\delta)^2(\alpha+2\theta^{\max} r)(\alpha^3+2\alpha^2(\gamma-\delta)+\alpha((\gamma-\delta)^2+4cr)+2\theta^{\max} r(\gamma-\delta)^2)}{r^2}} + 2\delta^2\theta^{\max}\right) + \alpha\theta^{\max}(\gamma-\delta)^2\left((\gamma-\delta)^2 + 2cr\right)$, where r is the negative of the co-efficient for the quadratic term in $V(.)$. It can be shown that this term is always negative, for the given conditions, proving the result.

- Else if $\gamma < \delta \leq \gamma + \alpha$, then $\widehat{x}_i = \widehat{x}_0$, proving the result trivially.

- Else if $\delta > \gamma + \alpha$, in this case, the variable cost per unit of θ is higher than the benefit received from higher ability, giving, $\widehat{x}_i = \widehat{x}_0$ and $\widehat{x}_\alpha = \widehat{x}_{\alpha,i}$. Hence, in this case, the reduction due to providing both incentives is exactly the same as the reduction due to providing only social incentives. □

4.8. Appendix: Model Extension from Roels and Su (2013)

In this section, we extend the model in Roels and Su (2013) to show that their insights continue to hold for more general distributions of consumer types. While most of the notation remains similar to the original model in section 4.3, we apply the notation in Roels and Su (2013) in order to be consistent.

We attempt to use the same notation as Roels and Su (2013) as much as possible in order to facilitate comparison. Each player chooses an output $x \in [0, X]$, where X is the maximum possible out. Higher outputs are more valuable but come at a cost, for example, students prefer to receive higher grades, but will require to put more effort. We summarize all costs and benefits using a strictly concave (net) value function $V_\theta(x)$, so each player has a unique utility maximizing output.

The customer types arise from their inherent costs of achieving a given output. We assume that a player of type θ requires to put an additional amount of effort to achieve the same output as a player of type 0. Hence, the utility to a type θ player for an output x is given by, $V_\theta(x) = V(x+\theta)$ where $V(.)$ is the utility function of player of type 0. Note that players with higher θ value are predisposed to have lower outputs, as their marginal utility of achieving output x is the same as the marginal utility of achieving output $x+\theta$ for player of type 0. We also assume that $V''(x) < -1$ for all x and $V'(0) > 0$. Note that strict concavity anyway implies, $V''(x) < 0$ for all x. Further, this allows us to consider the well defined function $v_1(y) = (V')^{-1}(y)$ on $[-\infty, V'(0)]$, which represents the inverse of $V'(x)$.

Each player also derives utility from social comparisons with the outputs of other players in the economy, which is evaluated based on the Cumulative Distribution Function (CDF) $G(x) : [0, X] \to [0, 1]$ of outputs of the other players in the system, where $G(X) = 1$ and $G(.)$ is non-negative, increasing and upper semi-continuous. For the sake of analytical convenience, we further assume that $0, X$ are not binding solutions.

We consider two models of social comparisons. First, the player may face disutility from achieving a lower output compared to other players. Hence, the utility of a player of type θ, similar to Fehr and Schmidt (1999), is given by,

$$U_\theta(x, G(.)) = V_\theta(x) - \beta \lim_{h \to 0^+} \sum_{i=1}^{n} (x - x_i)^+ (G(x_i) - G((x_{i-1})) \tag{4.10}$$

where $n = \lfloor \frac{X}{h} \rfloor$, $x_i = ih$, and $x^+ \stackrel{\text{def}}{=} \max\{x, 0\}$ and β represents the parameter associated with the social sensitivity of the population. Note that we use Reimann integrals to describe the social comparison utility in order to succinctly capture the possibility of distributions with atoms. If $G()$ were differentiable, we have, that $U_\theta(x, G(.)) = V_\theta(x) - \beta \int_x^X (\tilde{x} - x) g(\tilde{x}) d\tilde{x}$, where $g(x) = dG(x)/dx$. We refer to the term as $\beta \int_x^X (\tilde{x} - x) g(\tilde{x}) d\tilde{x}$ as the behind loss of this particular player and say that the player is behind-averse.

Alternatively, a player may receive additional utility from being ahead of other players in the population. We label these players as ahead-seeking and their utility function (for continuous $G(.)$, the definition would be equivalent for a distribution with atoms) is given by,

$$U_\theta(x, G(.)) = V_\theta(x) + \alpha \int_0^X (x - \tilde{x})^+ g(\tilde{x}) d\tilde{x} = V_\theta(x) + \alpha \int_0^x (x - \tilde{x}) g(\tilde{x}) d\tilde{x} \tag{4.11}$$

We refer to these players as ahead-seeking and the term, $\alpha \int_0^x (x - \tilde{x}) g(\tilde{x}) d\tilde{x}$ as the ahead gain of this particular player for output x.

In our model, the utility gains/losses are brought about from social comparisons on top of the value achieved from the outputs, $V_\theta(x)$. Further, the social planner can actively influence the outputs of players by providing the appropriate reference structure. We consider two possible reference structures that may be provided to the consumers: the average output of population $\widehat{x} = \int_0^X (1 - G(x)) dx$[5] or the entire distribution of the population (i.e. the distribution $G(x)$). We label these as the aggregate reference point or the full reference distribution cases.

The utilities under the full reference distributions are described above. Similarly, the utilities under the aggregate reference point for the behind-averse and ahead-seeking type θ

[5]Fubini's theorem shows that this is an expression for the mean of a random variable, as long as the CDF $G(x)$ is integrable.

players respectively can be written as,

$$U_\theta(x, \widehat{x}) = V_\theta(x) - \beta\big(\widehat{x} - x\big)^+, \tag{4.12}$$

$$U_\theta(x, \widehat{x}) = V_\theta(x) + \alpha\big(x - \widehat{x}\big)^+. \tag{4.13}$$

Our analysis hopes to compare the impact of the reference structures through the two extreme cases (average point vs. entire distribution), in both ahead-seeking and behind-averse environments. Hence, we consider a total of four possible cases. We consider a continuum of player types θ, per an atomless distribution, given by the Probability Distribution Function (PDF), $f(\theta) : [0, \theta^{\max}] \rightarrow \Re^+$, and the differentiable Cumulative Distribution Function (CDF) given by $F(\theta)$. We assume that all players have the same parameters α and β.

4.8.1 Definition of Nash Equilibrium

Using simplified notation, we describe the notion of an equilibrium for a game with infinite players. See Housman (1988) for a rigorous analytical treatment of the same. In the game derived from our model, we describe the equilibrium by means of the family of CDFs $H_\theta(.) : [0, X] \rightarrow [0, 1]$ for each $\theta \in [0, \theta^{\max}]$, which represents the distribution of the outputs of the players of type θ. Since the players are nameless (Housman 1988), we are only concerned with the distribution of their outputs. We label $H_\theta(.)$ as the best response distribution of players of type θ.

Equivalent to the constraints of an equilibrium strategy satisfying the principle of no deviation in a Nash Equilibrium, we formulate the following three conditions that must be true for any equilibrium that makes 'sense'.

1. *Rationality:* Each player enters the game with a belief about the distribution of the outputs of the other players, represented by the Cumulative Distribution Function $G(x) : [0, X] \rightarrow [0, 1]$. This is equivalent to the strategy set of all players in a finite player game. Note here, that since there are an infinite number of players, the individual player's actions are assumed to not affect the distribution, hence this is a non-atomic game (Khan and Papageorgiou 1987). However, the player will choose from the set of actions that maximizes his utility. Hence, the if the CDF H_θ has positive

support at some $\tilde{x}$, then it maximizes $U_\theta(x, G())$, formally,

$$\text{if } \lim_{r \to 0^+} \frac{(H_\theta(\tilde{x}) - H_\theta(\tilde{x} - r))}{r} > 0, \text{ then}, \tilde{x} \in \arg\max_{x \in [0,X]} U_\theta(x, G(.)), \tag{4.14}$$

In Behind Averse environments, it can be seen that the term $U_\theta(x, G())$ is strictly concave in x. In such a case, there is a unique maximum for $U_\theta(x, G(.))$. For these cases, we express the optimal action simply as a function θ and $G(.)$, i.e., $x^*(\theta, G(.))$, the best response function, which corresponds to $H_\theta()$ as follows,

$$H_\theta(x) = \begin{cases} 0, & \text{if } x < x^*(\theta, G(.)), \\ 1, & \text{else}, \end{cases} \tag{4.15}$$

We label an equilibrium whose 'best response distribution' H_θ satisfies the above expression as a 'smooth equilibrium'.

2. *Fairness:* We posit that equilibriums that make sense also conform that the actions taken by players must be concomitant with their ability. Hence players with higher θ (less ability) must have weakly lower output. More precisely, if $\theta_1 > \theta_2$, the highest x with positive support in H_{θ_2} must be weakly less than the lowest x with positive support in H_{θ_1}. Formally, let,

$$\begin{aligned} x^{\max}(\theta) &= \max\left\{\tilde{x} | \tilde{x} \in [0, X] \text{ and } \lim_{r \to 0^+} (H_\theta(\tilde{x}) - H_\theta(\tilde{x} - r))/r > 0\right\}, \\ x^{\min}(\theta) &= \min\left\{\tilde{x} | \tilde{x} \in [0, X] \text{ and } \lim_{r \to 0^+} (H_\theta(\tilde{x}) - H_\theta(\tilde{x} - r))/r > 0\right\}. \end{aligned} \tag{4.16}$$

We describe, by a sense of fairness that if $\theta_1 > \theta_2$, then $x^{\max}(\theta_1) \leq x^{\min}(\theta_2)$. Hence, for a smooth equilibrium, we require that $x^*(\theta, G(.))$ is weakly decreasing in θ, for given $G(.)$.

3. *Stability:* The equilibrium is also required to satisfy a stability condition. In other words, if all the players choose an output from their optimal set per $H_\theta()$, the output distribution must be consistent with the original assumption $G(x)$. Hence, the

equilibrium output must also satisfy,

$$G(x) - \int_{\theta=0}^{\theta^{\max}} f(\theta)H_\theta(x)d\theta = 0, \text{ for all } x \in [0, X]. \tag{4.17}$$

Definition IV.6. *Rational Expectations Equilibrium (REE): A Rational Expectations Equilibrium is a correspondence between from $\theta \in [0, \theta^{\max}]$ to a family of functions $H_\theta() : [0, X] \to [0, 1]$ that satisfies the above the conditions of Rationality, $Fairness$ and Stability.*

Smoothness is also a useful property (per (4.15)), which implies a sense of equality among all participants of equal ability. For the sake of ease of analysis, we restrict our initial results to smooth REEs. Note that a smooth equilibrium can be represented by the function $x^*(\theta)$, which we label as the 'strategy profile'. We assume that X is sufficiently large, so that it is never a binding solution. In this rest of this section, we characterize smooth REEs under 4 scenarios, arising from a combination of Full reference distribution or aggregate reference point, either ahead-seeking or behind-averse scenarios. We observe differences in the REEs in each of the four scenarios. Note that we identify smooth equilibria in all four cases.

4.8.2 Equilibrium Characterization under full reference distribution

In order to better understand the implications of these results, we lead the discussion with an example, with $V(x) = 100x - x^2$, with $\alpha = \beta = 5$ and $\theta^{\max} = 15$ and $f(\theta) = 1/15$ for all $\theta \in [0, 15]$ (i.e., uniformly distributed). Notice this in contrast to the example in Roels and Su (2013), where they assume the two types with $\theta = 0$ and $\theta = 15$, with rest of the numbers being the same.

Proposition IV.7. *Suppose that players are ahead-seeking in an infinite population with type distribution given by the probability distribution function $f(\theta) : [0, \theta^{\max}] \to [0, \infty)$, that $V(.)$ is quadratic and have the full reference distribution, then there exists a smooth REE satisfying,*

(i) The unique optimal action for every player of type θ is given by the strictly decreasing function, $x^*(\theta) = v_1(\alpha F(\theta) - \alpha) - \theta$, defined on $[0, \theta^{\max}] \to [0, X]$.

(ii) The stable output distribution CDF is given by,

$$G(x) = \begin{cases} 0, & \textit{if } x < x^*(\theta^{\max}), \\ 1 - F(\theta^{-1}(x)) & \textit{if } x^*(\theta^{\max}) \le x \le x^*(0), \\ 1, & \textit{else}, \end{cases} \tag{4.18}$$

where $\theta^{-1}(x) : [x^*(\theta^{\max}), x^*(0)] \to [0, \theta^{\max}]$ *is the inverse of* $x^*(\theta)$*.*

Proof of Proposition IV.7. As part of the proof, we show that the $x^*(\theta)$ and the corresponding output distribution $G(x)$ satisfy, the three properties of Rationality, Fairness and Stability.

First, part (i) implies,

$$V'(x^*(\theta) + \theta) = (\alpha F(\theta) - \alpha) \tag{4.19}$$

Differentiating, both sides, we have,

$$\frac{dx^*(\theta)}{d\theta} = \frac{\alpha f(\theta)}{V''(x^*(\theta) + \theta)} - 1. \tag{4.20}$$

Hence, we observe that $x^*(\theta)$ per part (i) is strictly decreasing in θ, satisfying Fairness. Because of strict monotonicity, the distribution resulting from this strategy profile, can be expressed using part (ii). Hence, it satisfies the Stability condition.

We observe that the CDF in part (ii) is differentiable, giving

$$g(x) = \begin{cases} 0, & \text{if } x < x^*(\theta^{\max}), \\ -f(\theta^{-1}(x))\frac{d\theta^{-1}(x)}{dx}, & \text{if } x^*(\theta^{\max}) \le x \le x^*(0), \\ 0, & \text{else}, \end{cases} \tag{4.21}$$

where $\theta^{-1}(x)$ is the inverse of the function $x^*(\theta)$ and $\frac{d\theta^{-1}(x)}{dx} = 1/(\frac{dx^*(\theta)}{d\theta})$. Note that $\theta^{-1}(x)$ is a well defined function on the set $[x^*(\theta^{\max}), x^*(0)]$, hence $g(x)$ is well defined on this set as well. For the sake of convenience, we allow, for $g(x) = 0$ for $x < x^*(\theta^{\max})$ and $x > x^*(0)$.

Hence, $U_\theta(.)$ can be expressed as $U_\theta(x, G(.)) = V(x + \theta) + \alpha \int_0^x (x - \tilde{x}) g(\tilde{x}) d\tilde{x}$. Further, we have, the derivatives of this function as follows,

$$\begin{aligned}\frac{dU_\theta(x,G(.))}{dx} &= V'(x+\theta)+\alpha G(x)\\ \frac{d^2U_\theta(x,G(.))}{dx^2} &= V''(x+\theta)+\alpha g(x)\end{aligned}\tag{4.22}$$

Since $V()$ is quadratic, it can be shown that the second derivative is always negative by substituting for $g(.)$ in (4.22). Note that $x^*(\theta)$ per part (i) satisfies the first order condition per (4.22). Note further that this is the only point at which the first derivative is zero including the ranges $[0, x^*(\theta^{\max})]$ and $[x^*(0), X]$ as the expression for the first derivative is the same across the entire domain $[0, X]$. This proves Rationality property of the equilibrium.

□

Observe that, for the introductory numerical example, the stable output distribution will be uniformly distributed over the range of $x \in [35, 52.5]$, spread over a range of 17.5, with a mean of 43.75.

We note that $v_1(.)$ is a well defined function on $[-\infty, V'(0)]$ as $V(x)$ is strictly concave, implying that $V'(.)$ is monotonically decreasing. Because of strict monotonicity, the inverse functions are generally well defined across the respective ranges for all the functions discussed in this paper. Proposition IV.7 provides a smooth REE for the game in ahead-seeking environments. Note that, in an ahead-seeking environment, the stable output distribution cannot have any atoms. This is because, for a point x to be an optimal solution for a player of type θ, we need,

$$\begin{aligned}\lim_{h\to 0^+}\frac{U_\theta(x+h,G())-U_\theta(x,G())}{h} &= V(x+\theta)+\alpha\lim_{h\to 0^+}G(x+h)\le 0\\ \lim_{h\to 0^-}\frac{U_\theta(x+h,G())-U_\theta(x,G())}{h} &= V(x+\theta)+\alpha\lim_{h\to 0^-}G(x+h)\ge 0\end{aligned}\tag{4.23}$$

However, if x' is an atom, then $\lim_{h\to 0^+} G(x'+h) - \lim_{h\to 0^-} G(x'+h) = P(x') > 0$, hence, x' can not be the optimal solution for any θ, contradicting the stability condition.

However, the setting described in Proposition IV.7 is a supermodular game. Hence, the existence of a pure strategy 'Nash Equilibrium', is guaranteed. Now, we discuss the behind-averse environment, where in a submodular game, such an existence is not guaranteed.

Proposition IV.8. *Suppose that players are behind-averse in an infinite population with type distribution given by the probability distribution function $f(\theta) : [0, \theta^{\max}] \to [0, \infty)$ and have the full reference distribution and the distribution also satisfies $|V''(v_1(-\beta F(\theta)))| > \beta f(\theta)$ for all $(\theta) \in [\theta^{\min}, \theta^{\max}]$, then there exists a unique REE satisfying,*

(i) The unique optimal action for every player of type θ is given by the strictly decreasing function, $x^*(\theta) = v_1(-\beta F(\theta)) - \theta$

(ii) The stable output distribution CDF is given by,

$$G(x) = \begin{cases} 0, & \text{if } x < x^*(\theta^{\max}), \\ 1 - F(\theta^{-1}(x)) & \text{if } x^*(\theta^{\max}) \le x \le x^*(0), \\ 1, & \text{else,} \end{cases} \tag{4.24}$$

where $\theta^{-1}(x) : [x^*(\theta^{\max}), x^*(0)] \to [0, \theta^{\max}]$ is the inverse of $x^*(\theta)$.

Proof of Proposition IV.8. As part of the proof, we show that the $x^*(\theta)$ and the corresponding output distribution $G(x)$ satisfy, the three properties of Rationality, Fairness and Stability.

First, part (i) implies,

$$V'(x^*(\theta) + \theta) = (-\beta F(\theta)) \tag{4.25}$$

Differentiating, both sides, we have,

$$\frac{dx^*(\theta)}{d\theta} = \frac{-\beta f(\theta)}{V''(x^*(\theta) + \theta)} - 1. \tag{4.26}$$

Hence, we observe that $x^*(\theta)$ per part (i) is strictly decreasing in θ, per assumption, satisfying Fairness. Because of strict monotonicity, the distribution resulting from this strategy profile, can be expressed using part (ii). Hence, it satisfies the Stability condition.

We observe that the CDF in part (ii) is differentiable, giving $g(x) = -f(\theta^{-1}(x))\frac{d\theta^{-1}(x)}{dx}$, where $\theta^{-1}(x)$ is the inverse of the function $x^*(\theta)$ and $\frac{d\theta^{-1}(x)}{dx} = 1/(\frac{dx^*(\theta)}{d\theta})$. Note that $\theta^{-1}(x)$ is a well defined function on the set $[x^*(\theta^{\max}), x^*(0)]$, hence $g(x)$ is well defined on this set as well. For the sake of convenience, we allow, for $g(x) = 0$ for $x < x^*(\theta^{\max})$ and $x > x^*(0)$.

Hence, $U_\theta(.)$ can be expressed as $U_\theta(x, G(.)) = V(x+\theta) - \beta \int_x^X (\tilde{x} - x) g(\tilde{x}) d\tilde{x}$. Further, we have, the derivatives of this function as follows,

$$\begin{aligned} \frac{dU_\theta(x, G(.))}{dx} &= V'(x+\theta) + \beta(1 - G(x)) \\ \frac{d^2 U_\theta(x, G(.))}{dx^2} &= V''(x+\theta) - \beta g(x) \end{aligned} \tag{4.27}$$

Clearly, the utility function $U_\theta(x, G(.))$ is strictly concave. Hence, there is only one $x^*(\theta)$ which maximizes the utility, satisfying the first order of the condition. Note that $x^*(\theta)$ per part (i) satisfies the first order condition for part (i). This proves Rationality property of the equilibrium.

The uniqueness can be verified from the fact that the first derivative of the utility, is strictly decreasing in x for given θ, per (4.27), ensuring unique maximum. Hence, all REEs must be smooth. Further, observe that $x^*(\theta)$ which satisfies the first order condition must be strictly decreasing in θ. Hence, all REEs must satisfy part (ii) of the proposition. □

Observe that, for the example being discussed, the stable output distribution will be uniformly distributed over the range of $x \in [37.5, 50]$. Compared to the ahead-seeking case, the range has decreased to 12.5 units, while the average remains the same at 43.75. However, we observe that, the effect of polarization vs. clustering that is discussed in Roels and Su (2013) is not as profound in this example with Uniform distribution of customer types.

Note that Proposition IV.8 has a restriction of distributions of θ, for which these equilibira may exist. Note that behind-averse scenarios represent submodular games in the finite player setting, hence, in some sense, the existence of a Pure Strategy Nash Equilibrium is not guaranteed.

4.8.3 Case Under Aggregate Reference Point

In this subsection, we consider a more generic reference system, where each player is provided only with the aggregate reference point, $\widehat{x}$. Note that the players operate under the available information. In a sense, it is sufficient to give information about the mean to all the players, as the players are indifferent to the distribution of $G(x)$, if they have the same mean $\widehat{x}$. Under these simplified settings, we are able to describe the structure of all the REE that exist. We start with the ahead-seeking scenario.

Proposition IV.9. *Suppose that players are ahead-seeking in an infinite population with type distribution given by the probability distribution function $f(\theta) : [0, \theta^{\max}] \to [0, \infty)$ and an aggregate reference point, then there exist two smooth REE represented by the strategy profile,*

$$x^*(\theta) = \begin{cases} v_1(-\alpha) - \theta, & \textit{if } 0 \leq \theta < \widehat{\theta}, \\ v_1(0) - \theta, & \textit{if } \widehat{\theta} < \theta < \theta^{\max}, \end{cases}$$

and $x^*(\widehat{\theta}) = v_1(-\alpha) - \widehat{\theta}$ or $v_1(0) - \widehat{\theta}$, where[6] *$\widehat{\theta} = (v_1(-\alpha) - \widehat{x}) - k$, $k = (V(v_1(0)) - V(v_1(-\alpha)))/\alpha$, for some unique $\widehat{x} \in \left[v_1(-\alpha) - \theta^{\max} - k, v_1(-\alpha) - k\right]$.*[7]

Proof of Proposition IV.9. Note that the consumers are indifferent between two output distributions $G_1(.), G_2(.)$ if $\int_0^X (1 - G_1(x))dx = \int_0^X (1 - G_2(x))dx = \widehat{x}$.

Hence, for any distribution with fixed mean $\widehat{x}$, in order to satisfy rationality, the utility function, given by $U_\theta(x, \widehat{x}) = V(x + \theta) + \alpha(x - \widehat{x})^+$ must be maximized. This function is not necessarily concave. In fact, it has up to two maxima. Precisely, (from the first order conditions) if $x < \widehat{x}$, then it satisfies, $x_1^*(\theta) = v_1(0) - \theta$ and if $x > \widehat{x}$, then $x_2^*(\theta) = v_1(-\alpha) - \theta$. Note further that, $x^*(\theta) \neq \widehat{x}$ for any θ, as such a value cannot be optimal. For each $\widehat{x}$, there exists $\widehat{\theta} = (v_1(-\alpha) - \widehat{x}) - k$ at which the optimal solutions from both $x_1^*(\theta)$ and $x_2^*(\theta)$ yields the same value. Hence, the expression in the proposition satisfies the Rationality condition.

Fairness follows directly as the strategy profile is decreasing in θ. Now, we show the existence of a unique $\widehat{x}$, which achieves this stability. The stability condition is,

$$-\widehat{x} + \int_0^{\widehat{\theta}} f(\theta) x_2^*(\theta) d\theta + \int_{\widehat{\theta}}^{\theta^{\max}} f(\theta) x_1^*(\theta) d\theta = 0 \tag{4.28}$$

Observe that the first derivative of the LHS of (4.28) is $-1 - f(\widehat{\theta})(v_1(-\alpha) - v_1(0))$, which

[6] Note that any valid CDF $H_\theta(.)$ with finite support at only these two values, would be a non-smooth REE for this game. In fact, we are able to show that these are the only non-smooth REEs possible.

[7] We assume that $[0, X]$ are not binding solutions, i.e., $\theta^{\max} < v_1(-\alpha) - k < X$

is strictly negative. Further the LHS of (4.28) is positive at $\widehat{x} = v_1(-\alpha) - \theta^{\max} - k$, and negative at $\widehat{x} = v_1(-\alpha) - k$. Stability requires that $x_1^*(\widehat{\theta}) < \widehat{x} < x_2^*(\widehat{\theta})$. We observe that this is true due to the strict concavity of $V(.)$. This satisfies the stability condition. □

Here, we observe that the canonical example gives a distribution that has equal support through the range of $[35, 52.5]$ with the exception of $[42.5, 45]$, where it has zero support. Notice that the range remains the same 17.5 units as in the continuous case, with a break in the middle, while the average remains the same as 43.75. This represents remnants of the polarization effect that was discussed by Roels and Su (2013).

Proposition IV.10. *Suppose that players are behind-averse in an infinite population with type distribution given by the probability distribution function $f(\theta) : [0, \theta^{\max}] \to [0, \infty)$ and an aggregate reference point,*

- *If $v_1(-\beta) - \theta^{\max} \leq v_1(0)$, then there exists a unique REE and this REE is smooth and can be represented by the following strategy profile,*

$$x^*(\theta) = \begin{cases} v_1(0) - \theta, & \text{if } 0 \leq \theta < \underline{\theta}, \\ \widehat{x} & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta}, \\ v_1(-\beta) - \theta, & \text{if } \overline{\theta} < \theta \leq \theta^{\max}, \end{cases}$$

  where $\underline{\theta} = v_1(0) - \widehat{x}$, $\overline{\theta} = v_1(-\beta) - \widehat{x}$ for some unique $\widehat{x} \in [v_1(-\beta) - \theta^{\max}, v_1(0)]$.

- *Else, $x^*(\theta) = \widehat{x}$ for all $\theta \in [0, \theta^{\max}]$ is optimal for all $\widehat{x} \in [v_1(0), v_1(-\beta) - \theta^{\max}]$*

Proof of Proposition IV.10. Note that the consumers are indifferent between two output distributions $G_1(.), G_2(.)$ if $\int_0^X (1 - G_1(x))dx = \int_0^X (1 - G_2(x))dx = \widehat{x}$.

Hence, for any distribution with fixed mean $\widehat{x}$, in order to satisfy rationality, the utility function, given by $U_\theta(x, \widehat{x}) = V(x + \theta) - \beta(\widehat{x} - x)^+$ must be maximized. Clearly, it is strictly concave, hence unique maximum exists. It can be observed that the expression in the Proposition satisfies the first order condition of this utility function. Hence, this is the unique response for any strategy profile with mean $\widehat{x}$, guaranteeing smoothness. Observe further, that the expression also confirms fairness.

Finally, we now show that there exists a unique $\widehat{x}$ satisfying stability, if $v_1(-\beta) - \theta^{\max} \leq v_1(0)$. Else, we show that each $\widehat{x}$ is an REE as it is sub-optimal to deviate.

For the strategy profile described, the average response can be evaluated, based on the distribution $f(\theta)$, and this must be equal to $\widehat{x}$. Hence, any stable REE must have $\widehat{x}$ satisfying,

$$-\widehat{x} + \int_0^{\underline{\theta}} (v_1(0) - \theta) f(\theta) d\theta + \widehat{x} \int_{\underline{\theta}}^{\overline{\theta}} f(\theta) d\theta + \int_{\overline{\theta}}^{\theta^{\max}} (v_1(-\beta) - \theta) f(\theta) d\theta = 0 \tag{4.29}$$

Note that the expression on the right hand side has a derivative of, $-1 + F(\overline{\theta}) - F(\underline{\theta})$, which is strictly negative except if, $\overline{\theta} = \theta^{\max}$ and $\underline{\theta} = 0$, which happens only when $\widehat{x} = v_1(-\beta) - \theta^{\max} = v_1(0)$. Hence, either way, there can be at most one solution to (4.29). Further, it can be seen that the expression on the LHS of (4.29) is positive for $\widehat{x} = v_1(-\beta) - \theta^{\max}$ and negative for $\widehat{x} = v_1(0)$. Hence, such $\widehat{x}$ exists for $v_1(-\beta) - \theta^{\max} \leq v_1(0)$. This confirms stability. □

Note that Propositions IV.9 and IV.10 both describe sufficient information to find the stable output distribution. Interestingly, the introductory example gives an interesting output distribution here, with an atom at $x = 43.75$, $P(x = 43.75) = 1/6$, and the rest of the probabilities uniformly spread across the range $[37.5, 50] - \{43.75\}$. Clearly, the effect of clustering is more pronounced in the aggregate reference point case, relative to the full distribution reference (See Proposition IV.8).

The application of the above propositions on the introductory example are summarized in Figure 4.10. The example, demonstrates the major learning from these propositions, i.e., the clustering and polarization effects in behind-averse and ahead-seeking environments respectively, remain in the case with continuous distributions. However, their effect is less pronounced. Further, their effect is substantially mitigated from providing the entire reference distribution (ensuring customers' anonymity), instead of the aggregate reference point.

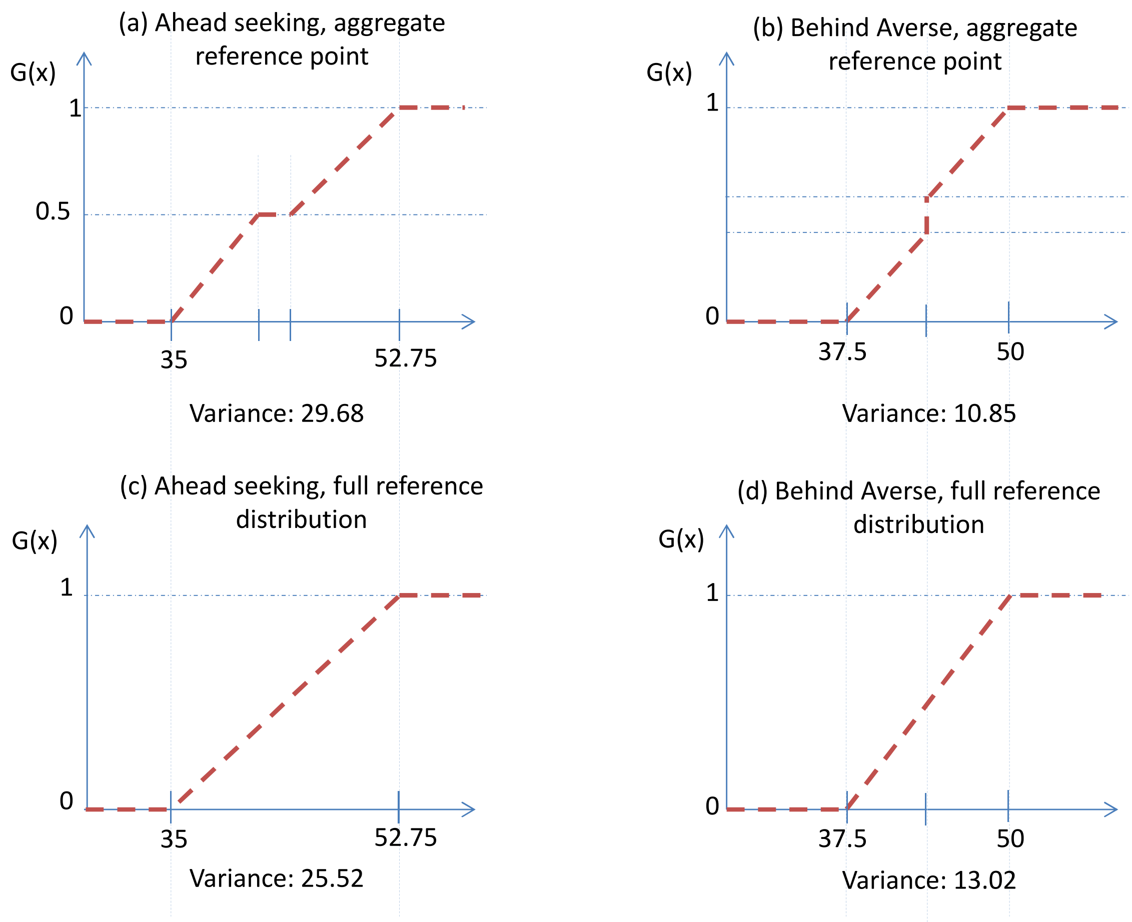


Figure 4.10: Equilibrium Output distributions under the four scenarios, for the introductory example.

CHAPTER V

Conclusion

This dissertation contains three essays. The first two deal with the investment and operations of energy storage technologies, while the third essay deals with the mechanisms to reduce consumption, by providing social and economic incentives to consumers.

In the storage investment problem, we consider a stylized model of the grid, to model the localizing vs. centralizing trade-off in storage investments. We evaluate that the high cost of storage capacity and high minimum demand, largely favors localizing of energy storage capacity due to the increased efficacy of localizing. Our analytical conclusions are supported by managerial decisions undertaken by American Electric Power (AEP) (Nourai 2007). We manipulate this model to consider the question of energy storage technologies as well. We consider a variation of the traditional fixed cost/variable cost trade-off and present synergies in 'mixing' of technologies in a variety of settings. We provide structural insights by identifying the structure of the optimal operating policy and establish identifiable properties of the optimal investment portfolio. We further demonstrate that the marginal benefit of this flexibility of 'mixing' is decreasing in the number of technologies available. To the best of our knowledge, we are the first to extend such insights from other settings to the non-linear setting of energy markets.

In the third essay, we investigate the effectiveness of Opower's HER program by applying a model to compare the potential benefits of social incentives and economic incentives. Opower's HER program provides both actionable investment information and social comparison information. Consumers react to both information and modify their consumption behaviours to be more environmentally friendly. We show that the combined effect of both these sets of information can be dramatically higher than the sum of the individual effects of providing separate pieces of information, in some circumstances.

Several extensions to the above essays are possible. For the second essay, the incentives of individual storage operators maximizing their own profits to invest in multiple technologies needs to be investigated. For future research, it may be apt to consider a storage model where we compare technologies differing in their power ratings and number of life cycles as well. For the third essay, we may consider varying utility functions which are non-linear in the social comparisons. It is likely, that the benefit of being ahead has decreasing marginal benefits to the consumer. Further, the social sensitivities may vary among different members of the population. Our model is also able to consider the effects of varying the distribution of the parameters, and how it affects the appropriate incentive schemes and comparison benchmarks. Several analytical generalizations of the results in the dissertation may be feasible and appropriate.

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