ESTIMATING MEAN SURVIVAL TIME: WHEN IS IT POSSIBLE? (Supplementary Material)

YING DING

Department of Biostatistics, University of Pittsburgh

BIN NAN

Department of Biostatistics, University of Michigan

1 Proofs of technical lemmas

We prove Theorems 2.1-2.2 in this supplementary material. Firstly, we provide several lemmas that will be used in the proofs.

Lemma 1.1. *For every $\varepsilon > 0$, with probability 1 we have*

$$\sup_{\beta\in\mathcal{B},-\infty<s<\infty} n^{1/2}|H_n^{(k)}(\beta,s)-h^{(k)}(\beta,s)| = o(n^{\varepsilon}),$$

where $H_n^{(k)}(\beta,s)$ and $h^{(k)}(\beta,s)$, $k=0,1$, are defined in equations (2.2) and (2.3) in the main text respectively.

Proof. We apply the empirical process theory to prove this result. Since the class of indicator functions of half spaces is a VC-class, see e.g. Exercise 9 on page 151 and Exercise 14 on page 152 in van der Vaart & Wellner (1996), and thus a Donsker class, the sets of functions $\mathcal{F}_0 = \{1(\epsilon_\beta \le s, \Delta = 1)\}$=$\{\Delta 1(\epsilon_\beta \le s)\}$ and $\mathcal{F}_1 = \{1(\epsilon_\beta \ge s)\}$ are both Donsker classes. Let $\bar{\mathcal{F}}_k$ be the closure of $\mathcal{F}_k$, $k = 0,1$, respectively. Then $H_n^{(k)}(\beta,s)$ and $h^{(k)}(\beta,s)$ are in the convex hull of $\bar{\mathcal{F}}_k$, $k=0,1$, and thus belong to Donsker classes (see e.g. Theorems 2.10.2 and 2.10.3 in van der Vaart & Wellner (1996)). Hence by their Theorem 2.6.7 and Theorem 2.14.9, it follows that for every $t>0$,

$$P\left(\sup_{\beta\in\mathcal{B},-\infty<s<\infty} n^{1/2}|H_n^{(k)}(\beta,s)-h^{(k)}(\beta,s)| > t\right) \le Mt^V e^{-2t^2},$$

where $M>0$ is a constant and $V = 2V(\mathcal{F})-2$ with $V(\mathcal{F})$ being the index of the VC-class $\mathcal{F}$, which is 4 in this case for one-dimensional β_0, hence $V=6$. When

$\beta_0 \in \mathbb{R}^d$ for a fixed d, the index of the VC-class is $V(\mathcal{F}) = d+3$ and the following argument still holds. Then for any $\varepsilon > 0$, let

$$A_{n,\varepsilon} = \sup_{\beta \in \mathcal{B}, -\infty < s < \infty} n^{1/2-\varepsilon} |H_n^{(k)}(\beta, s) - h^{(k)}(\beta, s)|.$$

Since $t^V \le e^{1.5t^2}$ for large enough $t > 0$ and a fixed $V > 0$, then

$$\sum_{n=1}^{\infty} P(|A_{n,\varepsilon} - 0| > t) \le M \sum_{n=1}^{\infty} \exp\{-0.5(n^{\varepsilon} t)^2\} < \infty.$$

By the Borel-Cantelli lemma we have $P\big(\lim_{n\to\infty} A_{n,\varepsilon} = 0\big) = 1$. We then have obtained the desired result. $\square$

Lemma 1.2. *Assume Conditions 1-3 hold, then for every* $\varepsilon \ge 0$ *we have*

$$\sup_{|\beta-\beta'|+|s-s'| \le n^{-\varepsilon}} |h^{(k)}(\beta, s) - h^{(k)}(\beta', s')| = O(n^{-\varepsilon}),$$

where $h^{(k)}(\beta, s)$, $k = 0, 1$ *and* 2, *are defined in equations (2.2), (2.3) and (3.1) in the main text respectively.*

Proof. Since $e_0 = T - \beta_0 X$ is independent of (X, C), the joint density function of (T, C, X) can then be decomposed as

$$f_{T,C,X}(t, c, x) = f_{e_0,C,X}(t - \beta_0 x, c, x) = f(t - \beta_0 x) f_{C,X}(c, x)$$

where f is the density of e_0. So

$$f(t - \beta_0 x) = f_{T|C,X}(t|C = c, X = x) = f_{T|X}(t|X = x).$$

Then the joint density function of (Y, Δ, X) follows

$$\begin{aligned} & f_{Y,\Delta,X}(y, \delta, x) \\ & \quad = f(y - \beta_0 x)^{\delta} \bar{F}(y - \beta_0 x)^{1-\delta} g_{C|X}(y|X = x)^{1-\delta} \bar{G}_{C|X}(y|X = x)^{\delta} f_X(x), \end{aligned}$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$ and $\bar{G}_{C|X}(\cdot|X = x) = 1 - G_{C|X}(\cdot|X = x)$.

For $h^{(0)}(\beta, s)$, the joint sub-density function of $(Y, \Delta = 1, X)$ can be written as $f_{Y,\Delta,X}(y, 1, x) = f(y - \beta_0 x) \bar{G}_{C|X}(y|X = x) f_X(x)$. So

$$\begin{aligned} h^{(0)}(\beta, s) &= P\{1(\epsilon_\beta \le s, \Delta = 1)\} \\ &= \int_{\mathcal{X}} \left\{ \int_{-\infty}^{s} f(u + (\beta - \beta_0)x) \bar{G}_{C|X}(u + \beta x|X = x)\, du \right\} f_X(x)\, dx. \end{aligned}$$

Then for any $\beta, \beta' \in \mathcal{B}$ and $-\infty < s < \infty$, by the mean value theorem, there exists a value $\tilde{\beta}$ between β and β' such that

$$
\begin{aligned}
&|h^{(0)}(\beta, s) - h^{(0)}(\beta', s)| \\
&= \Big| \int_{\mathcal{X}} \Big\{ \int_{-\infty}^{s} \big[\dot{f}(u + (\tilde{\beta} - \beta_0)x) \bar{G}_{C|X}(u + \tilde{\beta}x | X = x) \\
&\qquad - f(u + (\tilde{\beta} - \beta_0)x) g_{C|X}(u + \tilde{\beta}x | X = x) \big] (\beta - \beta')x \, du \Big\} f_X(x) \, dx \Big| \\
&\leq |\beta - \beta'| \int_{\mathcal{X}} \Big\{ \int_{-\infty}^{s} \big| \dot{f}(u + (\tilde{\beta} - \beta_0)x) \bar{G}_{C|X}(u + \tilde{\beta}x | X = x) \\
&\qquad - f(u + (\tilde{\beta} - \beta_0)x) g_{C|X}(u + \tilde{\beta}x | X = x) \big| \, du \Big\} |x| f_X(x) \, dx \\
&\leq C_1 |\beta - \beta'| \int_{\mathcal{X}} \Big\{ \int_{-\infty}^{\infty} \{ |\dot{f}(u)| + f(u) \} \, du \Big\} |x| f_X(x) \, dx \\
&\leq C_1 C_2 |\beta - \beta'| \int_{\mathcal{X}} |x| f_X(x) \, dx,
\end{aligned}
$$

where the second inequality holds for some finite constant $C_1 \geq 1$ such that $g_{C|X}(\cdot | X = x) \leq C_1$ uniformly, which is guaranteed by Condition 3; and the third inequality holds by Condition 2 and the following Cauchy-Schwartz inequality

$$
\begin{aligned}
\Big\{ \int_{-\infty}^{\infty} |\dot{f}(u)| \, du \Big\}^2 &\leq \int_{-\infty}^{\infty} \Big(\frac{|\dot{f}(u)|}{\sqrt{f(u)}} \Big)^2 du \cdot \int_{-\infty}^{\infty} \big(\sqrt{f(u)} \big)^2 du \\
&= \Big\{ \int_{-\infty}^{\infty} \Big(\frac{\dot{f}(u)}{f(u)} \Big)^2 f(u) \, du \Big\} \cdot 1 < \infty
\end{aligned}
$$

such that

$$
\int_{-\infty}^{\infty} \{ |\dot{f}(u)| + f(u) \} \, du = \int_{-\infty}^{\infty} |\dot{f}(u)| \, du + 1 \leq C_2
$$

for a constant $C_2 < \infty$. Therefore, by Condition 1 that X has a finite second moment and thus a finite first moment, it follows that

$$
|h^{(0)}(\beta, s) - h^{(0)}(\beta', s)| \leq K_1 |\beta - \beta'|
$$

for a constant $K_1 < \infty$.

Moreover, for any $\beta \in \mathcal{B}$ and $-\infty < s, s' < \infty$, we have

$$
\begin{aligned}
&|h^{(0)}(\beta, s) - h^{(0)}(\beta, s')| \\
&= \Big| \int_{\mathcal{X}} \Big\{ \int_{s}^{s'} f(u + (\beta - \beta_0)x) \bar{G}_{C|X}(u + \beta x | X = x) \, du \Big\} f_X(x) \, dx \Big| \\
&\leq C_3 |s - s'|,
\end{aligned}
$$

where C_3 is a constant such that $f(\cdot) \le C_3$, which is guaranteed by Condition 2. Hence, for any $\beta, \beta' \in \mathcal{B}$ and $-\infty < s, s' < \infty$, it follows that

$$\sup_{|\beta-\beta'|+|s-s'|\le n^{-\varepsilon}} |h^{(0)}(\beta, s) - h^{(0)}(\beta', s')| = O(n^{-\varepsilon}).$$

For $h^{(1)}(\beta, s)$, it is easy to obtain that

$$P\{1(\epsilon_\beta \ge s)|X = x\} = \bar{F}(s + (\beta - \beta_0)x)\bar{G}_{C|X}(s + \beta x|X = x).$$

Then for any $\beta, \beta' \in \mathcal{B}$ and $-\infty < s < \infty$, by the mean value theorem, there exists a value $\tilde{\beta}$ between β and β' such that

$$\begin{aligned}
&|h^{(1)}(\beta, s) - h^{(1)}(\beta', s)| \\
&= \Big|\int_{\mathcal{X}} \big\{\bar{F}(s + (\beta - \beta_0)x)\bar{G}_{C|X}(s + \beta x|X = x) \\
&\qquad - \bar{F}(s + (\beta' - \beta_0)x)\bar{G}_{C|X}(s + \beta' x|X = x)\big\} f_X(x)\, dx\Big| \\
&= \Big|\int_{\mathcal{X}} \big\{-f(s + (\tilde{\beta} - \beta_0)x)\bar{G}_{C|X}(s + \tilde{\beta} x|X = x) \\
&\qquad - \bar{F}(s + (\tilde{\beta} - \beta_0)x)g_{C|X}(s + \tilde{\beta} x|X = x)\big\}(\beta - \beta')x f_X(x)\, dx\Big| \\
&\le |\beta - \beta'| \int_{\mathcal{X}} \big\{f(s + (\tilde{\beta} - \beta_0)x) + g_{C|X}(s + \tilde{\beta} x|X = x)\big\}|x| f_X(x)\, dx \\
&\le (C_1 + C_3)|\beta - \beta'| \int_{\mathcal{X}} |x| f_X(x)\, dx \\
&= K_2|\beta - \beta'|
\end{aligned}$$

for some constant $K_2 = (C_1+C_3)E|X| < \infty$, where C_1 and C_3 are defined before. Moreover, for any $\beta \in \mathcal{B}$ and $-\infty < s, s' < \infty$, by the mean value theorem, there exists a value $\tilde{s}$ between s and s' such that

$$\begin{aligned}
&|h^{(1)}(\beta, s) - h^{(1)}(\beta, s')| \\
&= \Big|\int_{\mathcal{X}} \big\{-f(\tilde{s} + (\beta - \beta_0)x)\bar{G}_{C|X}(\tilde{s} + \beta x|X = x) \\
&\qquad - \bar{F}(\tilde{s} + (\beta - \beta_0)x)g_{C|X}(\tilde{s} + \beta x|X = x)\big\}(s - s') f_X(x)\, dx\Big| \\
&\le |s - s'| \int_{\mathcal{X}} \big\{f(\tilde{s} + (\beta - \beta_0)x) + g_{C|X}(\tilde{s} + \beta x|X = x)\big\} f_X(x)\, dx \\
&\le (C_1 + C_3)|s - s'|.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, we have

$$\sup_{|\beta-\beta'|+|s-s'|\le n^{-\varepsilon}} |h^{(1)}(\beta,s) - h^{(1)}(\beta',s')| = O(n^{-\varepsilon}).$$

Finally for $h^{(2)}(\beta,s)$, by using the similar argument to that for $h^{(1)}(\beta,s)$, we can easily obtain that

$$|h^{(2)}(\beta,s) - h^{(2)}(\beta',s)| \le (C_1+C_3)|\beta-\beta'| \int_{\mathcal{X}} x^2 f_X(x)\ dx = K_3|\beta-\beta'|$$

and

$$|h^{(2)}(\beta,s) - h^{(2)}(\beta,s')| \le (C_1+C_3)|s-s'| \int_{\mathcal{X}} |x| f_X(x)\ dx = K_2|s-s'|,$$

where $K_3 = (C_1+C_3)EX^2 < \infty$. Therefore, for any $\varepsilon > 0$, we have

$$\sup_{|\beta-\beta'|+|s-s'|\le n^{-\varepsilon}} |h^{(2)}(\beta,s) - h^{(2)}(\beta',s')| = O(n^{-\varepsilon}).$$

Thus, we have proved Lemma 1.2. □

Lemma 1.3. *Let $U_n(\beta,s)$ be random variables for which there exist non-random Borel functions $u_n(\beta,s)$ such that for every $\varepsilon > 0$,*

(A1) $\sup_{\beta\in\mathcal{B},-\infty<s<\infty} |U_n(\beta,s) - u_n(\beta,s)| = o(n^{-1/2+\varepsilon})$ *almost surely.*

(A2) $U_n(\beta,s)$ has a bounded variation in s uniformly on $\mathcal{B}$, that is,

$$\sup_{\beta\in\mathcal{B}} \int_{s=-\infty}^{\infty} |dU_n(\beta,s)| = O(1) \text{ almost surely.}$$

(A3) u_n satisfies

$$\sup_{\beta\in\mathcal{B},-\infty<s<\infty} |u_n(\beta,s)| = O(1).$$

Then under Conditions 1-3, for every $0 < \varepsilon \le 1/2$, with probability 1 we have

$$\sup_{\beta\in\mathcal{B},-\infty<y<\infty} \left| \int_{s=-\infty}^{y} U_n(\beta,s)\ dH_n^{(0)}(\beta,s) - \int_{s=-\infty}^{y} u_n(\beta,s)\ dh^{(0)}(\beta,s) \right|$$
$$= o(n^{-1/2+\varepsilon}).$$

Proof. By the triangle inequality and integration by parts, we have

$$\begin{aligned}
&\left|\int_{s=-\infty}^{y} U_n(\beta,s)\ dH_n^{(0)}(\beta,s) - \int_{s=-\infty}^{y} u_n(\beta,s)\ dh^{(0)}(\beta,s)\right| \\
&\quad \le \int_{s=-\infty}^{y} |U_n(\beta,s)-u_n(\beta,s)|\ dh^{(0)}(\beta,s) \\
&\qquad + |U_n(\beta,y)\big(H_n^{(0)}(\beta,y)-h^{(0)}(\beta,y)\big)| \\
&\qquad + \int_{s=-\infty}^{y} |H_n^{(0)}(\beta,s)-h^{(0)}(\beta,s)|\ |dU_n(\beta,s)|.
\end{aligned}$$

Then it is easy to see that each term on the right hand side of the above inequality is $o(n^{-1/2+\varepsilon})$ almost surely under (A1)-(A3) and Lemma 1.1. □

2 Proof of Theorem 2.1

Proof. By the first order Taylor expansion of function $\log(1-x)$, for large n we have

$$\begin{aligned}
\hat{F}_{n,\beta}(t) &= 1-\exp\Big\{\sum_{i:\epsilon_{\beta,i}\le t}\log\Big(1-\frac{\Delta_i/n}{H_n^{(1)}(\beta,\epsilon_{\beta,i})}\Big)\Big\} \\
&= 1-\exp\Big\{-\int_{u\le t}\frac{dH_n^{(0)}(\beta,u)}{H_n^{(1)}(\beta,u)} - \sum_{i:\epsilon_{\beta,i}\le t} O\big(\{nH_n^{(1)}(\beta,\epsilon_{\beta,i})\}^{-2}\big)\Big\}.
\end{aligned}$$

Then by the mean value theorem and the fact that $e^x \le 1$ for any $x \le 0$, it follows that

$$\begin{aligned}
&|\hat{F}_{n,\beta}(t) - F(\beta,t)| \\
&= \Big|\exp\Big\{-\int_{-\infty}^{t}\frac{dh^{(0)}(\beta,u)}{h^{(1)}(\beta,u)}\Big\} \\
&\qquad - \exp\Big\{-\int_{-\infty}^{t}\frac{dH_n^{(0)}(\beta,u)}{H_n^{(1)}(\beta,u)} - n^{-2}\sum_{i:\epsilon_{\beta,i}\le t} O\big(H_n^{(1)}(\beta,\epsilon_{\beta,i})^{-2}\big)\Big\}\Big| \\
&\le \Big|\int_{-\infty}^{t}\frac{dH_n^{(0)}(\beta,u)}{H_n^{(1)}(\beta,u)} - \int_{-\infty}^{t}\frac{dh^{(0)}(\beta,u)}{h^{(1)}(\beta,u)} + n^{-2}\sum_{i:\epsilon_{\beta,i}\le t} O\big(H_n^{(1)}(\beta,\epsilon_{\beta,i})^{-2}\big)\Big|.
\end{aligned}$$

Under the condition $H_n^{(1)}(\beta,t)\ge n^{-\varepsilon}$, we have

$$n^{-2}\sum_{i:\epsilon_{\beta,i}\le t} O\big(H_n^{(1)}(\beta,\epsilon_{\beta,i})^{-2}\big) \le n^{-2}\cdot O(n^{2\varepsilon})\cdot n = O(n^{-1+2\varepsilon}) = o(n^{-\frac{1}{2}+3\varepsilon}).$$

So in order to show equation (2.9) in the main text, we only need to show

$$\sup\left\{\left|\int_{-\infty}^{t}\frac{dH_n^{(0)}(\beta,u)}{H_n^{(1)}(\beta,u)}-\int_{-\infty}^{t}\frac{dh^{(0)}(\beta,u)}{h^{(1)}(\beta,u)}\right|:\right. \qquad (2.1)$$
$$\left.\beta\in\mathcal{B},H_n^{(1)}(\beta,t)\geq n^{-\varepsilon}\right\}=o(n^{-\frac{1}{2}+3\varepsilon})$$

almost surely. Now we define $\tilde{T}_n=\sup\{t:\beta\in\mathcal{B},H_n^{(1)}(\beta,t)\geq n^{-\varepsilon}\}$, and let

$$\tilde{H}_n^{(1)}(\beta,t)=\begin{cases}H_n^{(1)}(\beta,t), & \text{if } t\leq\tilde{T}_n,\\ H_n^{(1)}(\beta,\tilde{T}_n), & \text{if } t>\tilde{T}_n.\end{cases}$$

Then $\tilde{H}_n^{(1)}(\beta,t)\geq n^{-\varepsilon}$ for all $\beta\in\mathcal{B}$ and $-\infty<t<\infty$. Define $\tilde{h}^{(1)}(\beta,t)$ similarly as $\tilde{H}_n^{(1)}(\beta,t)$ and apply Lemma 1.3 to $U_n(\beta,u)=n^{-2\varepsilon}\{\tilde{H}_n^{(1)}(\beta,u)\}^{-1}$ and $u_n(\beta,u)=n^{-2\varepsilon}\{\tilde{h}^{(1)}(\beta,u)\}^{-1}$, we obtain (2.1) and thus equation (2.9) in the main text holds.

We now show equation (2.10) in the main text. Notice that $F(t)=F(\beta_0,t)$, then under the restriction $\{|\beta-\beta_0|\leq n^{-3\varepsilon},h^{(1)}(\beta,t)\geq n^{-\varepsilon}\}$, by the mean value theorem we obtain

$$\begin{aligned}
&|F(\beta,t)-F(t)|\\
&=\left|\exp\left\{-\int_{u\leq t}\frac{dh^{(0)}(\beta,u)}{h^{(1)}(\beta,u)}\right\}-\exp\left\{-\int_{u\leq t}\frac{dh^{(0)}(\beta_0,u)}{h^{(1)}(\beta_0,u)}\right\}\right|\\
&\leq\left|\int_{u\leq t}\frac{dh^{(0)}(\beta,u)}{h^{(1)}(\beta,u)}-\int_{u\leq t}\frac{dh^{(0)}(\beta_0,u)}{h^{(1)}(\beta_0,u)}\right|\\
&\leq\left|\int_{u\leq t}\frac{d\{h^{(0)}(\beta,u)-h^{(0)}(\beta_0,u)\}}{h^{(1)}(\beta,u)}\right|\\
&\qquad+\left|\int_{u\leq t}\left(\frac{h^{(1)}(\beta_0,u)-h^{(1)}(\beta,u)}{h^{(1)}(\beta,u)h^{(1)}(\beta_0,u)}\right)dh^{(0)}(\beta_0,u)\right|\\
&\leq n^{\varepsilon}\sup\{|h^{(0)}(\beta_0,t)-h^{(0)}(\beta,t)|\}\\
&\qquad+n^{2\varepsilon}h^{(0)}(\beta_0,t)\sup\{|h^{(1)}(\beta_0,t)-h^{(1)}(\beta,t)|\}\\
&=O(n^{-\varepsilon}),
\end{aligned}$$

where the third inequality holds because for any $u\leq t$, $\{h^{(1)}(\beta,u)\}^{-1}\leq\{h^{(1)}(\beta,t)\}^{-1}\leq n^{\varepsilon}$, and the last equality holds because $h^{(0)}(\beta_0,t)\leq 1$ and $\sup\{|h^{(k)}(\beta,t)-h^{(k)}(\beta_0,t)|\}=O(|\beta-\beta_0|)$, $k=0,1$, by Lemma 1.2. Thus equation (2.10) in the main text holds. Finally, equation (2.11) in the main text can be easily obtained by applying the triangle inequality to equations (2.9) and (2.10) together with Lemma 1.1 provided that $-\frac{1}{2}+3\varepsilon \le -\varepsilon$, i.e., $0<\varepsilon\le \frac{1}{8}$. □

3 Proof of Theorem 2.2

Proof. Notice that

$$\alpha_0 = \int_{-\infty}^{\infty} t\,dF(t) = \int_0^{\infty} \{1-F(t)\}\;dt - \int_{-\infty}^{0} F(t)\,dt.$$

We thus have

$$\begin{aligned}\int_{-\infty}^{\infty} t\,d\hat{F}_{n,\beta}(t) - \alpha_0 &= \int_{-\infty}^{\infty} t\,d\hat{F}_{n,\beta}(t) - \int_{-\infty}^{\infty} t\,dF(t)\\ &= \left\{\int_0^{\infty}\{1-\hat{F}_{n,\beta}(t)\}\,dt - \int_0^{\infty}\{1-F(t)\}\,dt\right\} \qquad (3.1)\\ &\quad -\left\{\int_{-\infty}^{0}\hat{F}_{n,\beta}(t)\,dt - \int_{-\infty}^{0}F(t)\,dt\right\}.\end{aligned}$$

With $\beta_0\neq 0$, when β satisfies $|\beta-\beta_0|\le n^{-3\varepsilon}$, we have $\beta\neq 0$ for sufficiently large n. For any $\beta\neq 0$ and $t\in(-\infty,\infty)$, one can always find a range of x such that $\bar{G}_{C|X}(t+\beta x|X=x)>0$ and $\bar{F}(t+(\beta-\beta_0)x)>0$ since $\bar{F}(t)>0$ for all $t<\infty$ under the assumption $f_X(x)>0$ for all $-\infty<x<\infty$ and $\beta_0\neq 0$. Therefore, we have $h^{(1)}(\beta,t)>0$ for all $t\in(-\infty,\infty)$ from the following equation that is obtained in the proof of Lemma 1.2:

$$h^{(1)}(\beta,t) = \int_{-\infty}^{\infty} \bar{F}(t+(\beta-\beta_0)x)\bar{G}_{C|X}(t+\beta x|X=x)f_X(x)\,dx.$$

Moreover, since $H_n^{(1)}(\beta,t)\to h^{(1)}(\beta,t)$ almost surely as $n\to\infty$, then with n sufficiently large, we have $H_n^{(1)}(\beta,t)>0$ almost surely for any $\beta\neq 0$ and $t\in(-\infty,\infty)$. Hence $T_n\to\infty$ almost surely as $n\to\infty$, where $T_n=\sup\big\{t: H_n^{(1)}(\beta,t)\ge n^{-\varepsilon}, |\beta-\beta_0|\le n^{-3\varepsilon}\big\}$, as defined in equation (2.12) in the main text.

Then at $\beta=\beta_0$, by the independence of e_0 and $C-\beta_0 X$ and the Markov's inequality, it follows that

$$\begin{aligned}h^{(1)}(\beta_0,T_n) &= P\{1(e_0\ge T_n)\}\cdot P\{1(C-\beta_0X\ge T_n)\}\\ &\le P\{1(e_0\ge T_n)\} \le \frac{Ee_0^2}{T_n^2}.\end{aligned}$$

Since $H_n^{(1)}(\beta_0, T_n) \geq n^{-\varepsilon}$ implies $h^{(1)}(\beta_0, T_n) \geq n^{-\varepsilon}$, together with Condition 4 that $Ee_0^2 < \infty$, we have $T_n^2 \leq Ee_0^2\{h^{(1)}(\beta_0, T_n)\}^{-1} \leq O(n^{\varepsilon})$, i.e., $T_n \leq O(n^{\varepsilon/2})$. This implies that $T_n \to \infty$ in a rate no faster than $n^{\varepsilon/2}$.

Since the Kaplan-Meier estimator $\hat{F}_{n,\beta}(t)$ is set to 1 for $t > T_n$, (3.1) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} t\, d\hat{F}_{n,\beta}(t) - \alpha_0 \quad &= \quad \int_0^{T_n} \{F(t) - \hat{F}_{n,\beta}(t)\}\, dt - \int_{T_n}^{\infty} \{1 - F(t)\}\, dt \\ &\quad - \int_{-\infty}^{0} \{\hat{F}_{n,\beta}(t) - F(t)\}\, dt.\end{aligned}$$

Then by Theorem 2.1, we have

$$\sup\left\{\int_0^{T_n} |F(t) - \hat{F}_{n,\beta}(t)|\, dt : |\beta - \beta_0| \leq n^{-3\varepsilon}\right\} \leq T_n \cdot O(n^{-\varepsilon}) \leq O(n^{-\frac{\varepsilon}{2}})$$

almost surely. For the second term on the right hand side of above equation, applying the Markov's inequality we obtain

$$\int_{T_n}^{\infty} \{1 - F(t)\}\, dt \leq \int_{T_n}^{\infty} P\{1(|e_0| \geq t)\}\, dt \leq \int_{T_n}^{\infty} \frac{Ee_0^2}{t^2}\, dt \leq \frac{Ee_0^2}{T_n} = o(1)$$

almost surely. For the third term, we have

$$\begin{aligned}\int_{-\infty}^{0} \{\hat{F}_{n,\beta}(t) - F(t)\}\, dt \quad &= \quad \int_{-T_n}^{0} \{\hat{F}_{n,\beta}(t) - F(t)\}\, dt \\ &\quad + \int_{-\infty}^{-T_n} \{\hat{F}_{n,\beta}(t) - F(t)\}\, dt,\end{aligned}$$

where

$$\sup\left\{\int_{-T_n}^{0} |F(t) - \hat{F}_{n,\beta}(t)|\, dt : |\beta - \beta_0| \leq n^{-3\varepsilon}\right\} \leq T_n \cdot O(n^{-\varepsilon}) \leq O(n^{-\frac{\varepsilon}{2}})$$

almost surely, and

$$\begin{aligned}\int_{-\infty}^{-T_n} |F(t) - \hat{F}_{n,\beta}(t)|\, dt \quad &\leq \quad \int_{-\infty}^{-T_n} F(t)\, dt + \int_{-\infty}^{-T_n} \hat{F}_{n,\beta}(t)\, dt \\ &= \quad \int_{T_n}^{\infty} F(-t)\, dt + \int_{-\infty}^{-T_n} \hat{F}_{n,\beta}(t)\, dt \\ &\leq \quad \frac{Ee_0^2}{T_n} + o(1) = o(1)\end{aligned}$$

almost surely, where the last inequality holds because of the Markov's inequality

$$F(-t) = P\{1(e_0 \leq -t)\} \leq P\{1(|e_0| \geq t)\} \leq \frac{Ee_0^2}{t^2}$$

and the fact $\int_{-\infty}^{-T_n} \hat{F}_{n,\beta}(t)\, dt \to 0$ almost surely as $n \to \infty$. Therefore,

$$\sup\left\{ \left| \int_{-\infty}^{\infty} t\, d\hat{F}_{n,\beta}(t) - \alpha_0 \right| : |\beta - \beta_0| \leq n^{-3\varepsilon} \right\} = o(1)$$

almost surely. We now have proved Theorem 2.2. □

References

van der Vaart, A. W. & Wellner, J. A. (1996). *Weak convergence and empirical processes.* Springer, New York.

Ying Ding, 130 DeSoto Street, 318C Parran Hall, Pittsburgh, 15261, USA.
E-mail: yingding@pitt.edu