Volumes and integer points of multi-index transportation polytopes

by

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To Tessa, for always supporting me

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ABSTRACT

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Counting the integer points of transportation polytopes has important applications in statistics for tests of statistical significance, as well as in several applications in combinatorics. In this dissertation, we derive asymptotic formulas for the number of integer and binary integer points in a wide class of multi-index $k_1 \times k_2 \times \ldots \times k_\nu$ transportation polytopes. A simple closed form approximation is given as the k_js go to infinity. A formula for the volume of 4-index transportation polytopes is also given.

We follow the approach of Barvinok and Hartigan to estimate the quantities through a type of local Central Limit Theorem. By carefully estimating eigenvalues and eigenvectors of certain quadratic forms, we are able to prove strong concentration results for the corresponding multivariate Gaussian random variables. We also estimate correlations between linear functions of Gaussian random variables to produce concentration results for the distribution of certain exponential functions. Combined, these techniques allow us to give a complete accounting of the integrals of several functions that are key to estimating the number of integer or binary integer points in multi-index transportation polytopes. As an additional result, we transform a stan-

dard concentration of measure on the sphere argument to a concentration result for Gaussian measures whose quadratic forms contain several small eigenvalues, allowing a similar calculation for the volume of multi-index transportation polytopes.

CHAPTER I

Introduction

A *ν-index transportation polytope* is a set of $k_1 \times \ldots \times k_\nu$ arrays of non-negative numbers with fixed hyper hypercube arrays of non-negative numbers of the form

$$(\xi_{m_1,\ldots,m_\nu})_{m_1,\ldots,m_\nu=1}^{k_1,\ldots,k_\nu}$$

satisfying the following relations: Fix some arbitrary j with $1 \leq j \leq \nu$, and some arbitrary m_j with $1 \leq m_j \leq k_j$. Then there are constants $S^j_{m_j}$ for each such j and m_j such that

$$\sum_{m_1,\ldots,m_{j-1},m_{j+1},\ldots,m_\nu} \xi_{m_1,m_2,\ldots,m_\nu} = S^j_{m_j}.$$

For such a j and m_j we call $S^j_{m_j}$ the m_jth *margin* in the jth direction. In the literature this is often referred to as a multi-index transportation polytope with *fixed* 1-*margins*, for example in [LO04], or a *planar* transportation polytope in the 3-index case, for example in [L+09].

The main results in this paper are asymptotic formulas to approximately count the number of integer and binary integer points in a wide class of ν-index transportation polytopes. for $\nu \geq 3$, and an asymptotic formula to approximate the volume of certain 4-index transportation polytopes.

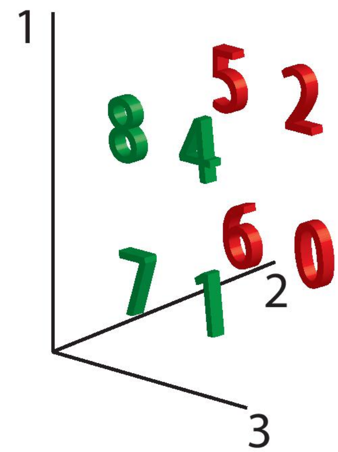

Figure 1.1: An integer point in a $2 \times 2 \times 2$ three-index transportation polytope. The entries corresponding to the first margin in the second direction are colored green, and the entries corresponding to the second margin in the second direction are colored red. The polytope this point comes from has margins $S^1_1 = 14$, $S^1_2 = 19$, $S^2_1 = 20$, $S^2_2 = 13$, $S^3_1 = 26$, and $S^3_2 = 7$.

Counting the number of binary integer points in a ν-index transportation polytope is a special case of counting ν uniform, ν-partite hypergraphs. The vertices of the jth partition are labeled 1 through k_j, and an entry in the $m_1, m_2, \ldots, m_\nu$ position says there exists an edge connecting vertices $m_1, m_2, \ldots, m_\nu$. Work has gone into counting asymptotically the number of hypergraphs of certain forms, see [DFRS13].

Counting the number of integer and binary integer points in 3-way contingency tables has applications in algebraic combinatorics. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ for partitions λ, μ and ν of some integer n are defined by the identity

$$\chi^\lambda \otimes \chi^\mu = \sum_\nu g(\lambda, \mu, \nu)\chi^\nu,$$

where χ^α is the irreducible representation of S_n indexed by partition α. It is known that the values of $g(\lambda, \mu, \nu)$ are non-negative, but a combinatorial interpretation or simple counting formula is not known. In [AV12] it is shown that the the value of

$g(\lambda, \mu, \nu)$ can be bounded from above by the number of integer points of a 3-way contingency table whose margins are given by λ, μ and ν. It is also shown that $g(\lambda, \mu, \nu)$ is bounded from above by the number of binary integer points of a 3-way contingency table whose margins are given by λ', μ and ν, where λ' is the conjugate partition of λ. In [PP3] it is shown that $g(\lambda, \mu, \nu)$ can be calculated exactly in terms of the number of integer points in 3-way contingency tables of various margins.

In statistics, points in a ν-index transportation polytopes tables are constructed from a given dataset in the following way: N objects have ν qualities divided into k_1 categories for the first quality, k_2 categories for the second, through k_ν categories for the last. The entry $x_{m_1,\ldots,m_\nu}$ is the number of objects that have quality 1 fall into category m_1, quality 2 into category m_2, through quality ν in category m_ν. Estimating the number of integer points contained within the corresponding transportation polytope is critical for tests of significance in the distribution of contingency tables and interpretation of those results. See [DE85] for an exposition in the $\nu = 2$ case.

Much work has been done in calculating asymptotic formulas for the number of integer points of multi-index transportation polytopes in special cases. Examples include two-directional transportation polytopes, in the sparse case in [GM08], and in the case of all equal margins in [CM10]. An asymptotic formula for the number of integer points in certain "smooth" - or close in a certain technical sense to the case of all equal margins - two-directional multi-index transportation polytopes has been calculated in [BH12]. Formulas for the volume and number of integer points and binary points for smooth multi-index transportation polytope of five or more directions was found in in [BH10]. It was not previously known that smooth three and four directional transportation polytopes allowed the same asymptotic formula. In addition, the asymptotic error for the case of having five or more directions is improved over that calculated in [BH10]. We will combine the approaches of these last two papers, along with improved estimates on the variance of certain Gaussian random variables to achieve the result.

We also extend the results of Barvinok and Hartigan in [BH10] to estimate the volume of a large class of 4-index transportation polytopes defined by a similar condition to being smooth. In [BH10] they estimate the volume of ν-index transportation polytopes for $\nu \geq 5$. We apply well known concentration of measure results on the sphere to improve estimates of integrals and allow for the general proof technique to be used on 4-index transportation polytopes as well.

The layout of this paper is: the remainder of Chapter I states the three main theorems, and discusses future potential work related to them. In the proof of all three theorems we rely heavily on the eigenspace of quadratic forms of a certain type. Chapter II calculates the eigenspace of these quadratic forms, and proves several lemmas and theorems common to the proofs of all three main theorems. The theorems are then proven in Chapters III, IV, and V.

1.1 The Polytope Constraints

In what follows, a set $P \subset \mathbb{R}^n$ is called a polyhedron - and a polytope, if it is bounded - if it can be defined as

$$P = \{x = (\xi_1, \ldots, \xi_n) \ : \ Ax = b \quad \text{and} \quad \xi_j \geq 0 \text{ for all } j\}$$

for some A a $d \times n$ matrix of real numbers, and $b \in \mathbb{R}^d$. In this case the columns of A will be denoted $a_1, \ldots, a_n$. For the ν-index transportation polytope defined earlier, we write a point in our hypercube array as

$$(\xi_{11\ldots1}, \xi_{11\ldots2}, \ldots, \xi_{11\ldots k_1}, \xi_{11\ldots121}, \ldots, \xi_{k_1k_2\ldots k_\nu})$$

with the coordinate $\xi_{m_1\ldots m_\nu}$ being the coordinate lying in the m_jth margin of the jth direction. The transportation polytope then fits the above definition with $n = k_1k_2\ldots k_\nu$, and each $a_{m_1\ldots m_\nu}$ being a vector of length $k_1 + k_2 + \ldots + k_\nu$ that has all 0s, except for a 1 in positions m_1, $k_1 + m_2$, $k_1 + k_2 + m_3$,..., and $k_1 + \ldots + k_{\nu-1} + m_\nu$. In this case the entry of b in position $k_1 + \ldots + k_{j-1} + m_j$ is $S^j_{m_j}$ for each j and m_j.

It is important to note that the constraint matrix A for a multi-index transportation polytope does not have full rank. This is easily seen by observing that for each j, the sum

$$\sum_{m_j=1}^{k_j} S^j_{m_j}$$

must be the same value, as it gives the sum of all entries in the hypercube array. This is the only linear dependency amongst the constraints, and a basis of the constraints consists of removing the constraint on the k_jth margin in the jth direction for $j = 2, \ldots, \nu$. If $\mathcal{L} \subset \mathbb{R}^{k_1+\ldots+k_\nu}$ is the subspace

$$\mathcal{L} = \left\{(\xi_1, \ldots, \xi_{k_1+\ldots+k_\nu}) \ : \ \xi_{k_1+\ldots+k_j} = 0 \quad \text{for all} \quad 2 \leq j \leq \nu\right\}, \tag{1.1}$$

and $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ is the orthogonal projection onto $\mathcal{L}$, then QA is a full rank linear transformation from $\mathbb{R}^{k_1+\ldots+k_\nu} \to \mathcal{L}$ and the system of constraints $QAx = Qb$ is equivalent to selecting a linearly independent set of constraints for P.

1.2 Quadratic Forms and Inner Products

Recall if $q(t)$ is a positive semidefinite quadratic form on $\mathbb{R}^d$, then there exists a positive semidefinite symmetric matrix B such that $q(t) = \frac{1}{2}\langle t, Bt\rangle$. We define the eigenvalues and eigenvectors of q to simply be the eigenvalues and eigenvectors of B. If $\mathcal{V} \subset \mathbb{R}^d$ is a linear subspace and Q the orthogonal projection onto $\mathcal{V}$, then $q|_{\mathcal{V}}(t)$ will denote the quadratic form $\frac{1}{2}\langle t, QBQt\rangle$. For $t \in \mathcal{V}$ this conforms with the original definition, but we will occasionally decompose t into vectors not contained in $\mathcal{V}$ which will make this definition convenient.

If B is positive semidefinite symmetric $d \times d$ matrix, then QBQ is a positive semidefinite symmetric $d\times d$ matrix whose kernel includes $\mathcal{V}^{\perp}$. Therefore there exists a basis of orthogonal eigenvectors that all lie in $\mathcal{V}$ or $\mathcal{V}^{\perp}$. By $\det(q)$ we mean the product of the eigenvalues of B, and by $\det(q|_{\mathcal{V}})$ we mean the product of the eigenvalues of the eigenvectors of QBQ that lie in $\mathcal{V}$.

Lastly, we recall that if q is a positive definite quadratic form on some subspace $\mathcal{V} \subset \mathbb{R}^n$, then

$$\int_{\mathcal{V}} e^{-q(t)} dt = \frac{(2\pi)^{\dim(\mathcal{V})/2}}{\sqrt{\det(q|_{\mathcal{V}})}}. \tag{1.2}$$

1.3 Maximum Entropy in the Counting Problem

In many counting and volume measurement problems, the problem is reduced to calculating an integral. Examples include [BH12] and [BH10] in counting integer points of general polytopes. In [BH13], the number of graphs satisfying certain conditions on the degrees of its vertices is counted in a similar manner.

The principle that allows the construction of the integral is inspired by the standard 'Monte Carlo' or random sampling method. To count the number of integer points in a polytope $P \subset \mathbb{R}^n_+$ defined by the system of equations $Ax = b$, we first construct a random variable X that takes values in $\mathbb{Z}^n_+$, for which $\mathbf{E}X \in P$. We then express $|P \cap \mathbb{Z}^n|$ as a function of $\mathbf{Pr}(X \in P)$. Rather than a numerical sampling to estimate $\mathbf{Pr}(X \in P)$, we use $X \in P$ if and only if $AX = b$. If A is $d \times n$ with $n \gg d$, and each row of A has sufficiently many nonzero entries, and each entry of X is picked independently, then the entries of AX are approximately Gaussian by the Central Limit Theorem. The integrand of the integral we use to estimate the number of integer points is simply $e^{-q(t)}$, where $q(t)$ is a certain quadratic form that we construct later.

It turns out that a useful choice of X is the random variable of maximum entropy whose expected value lies in P that takes the relevant values. In the integer point case the entries of X are independent geometric random variables, in the binary integer point case the entries of X are independent Bernoulli random variables, and in the volume case the entries of X are independent exponential random variables. In Sections 3.1, 4.1, and 5.1, we cite several lemmas and theorems of Barvinok and Hartigan that describe the choice of a random variable X that is appropriate for counting integer points, counting binary integer points, and measuring volumes, and how to construct the probability mass function or density of the distribution explicitly, but the focus of this paper will be on the application of these theorems to the specific example of multi-index transportation polytopes. See [BH10] for more details on the general case.

1.4 Counting Integer Points of Transportation Polytopes

In this section we state the main theorem estimating the number of integer points in a multi-index transportation polytope. The theorem is

Theorem I.1. *Let P be a $k_1 \times \ldots \times k_\nu$ multi-index transportation polytope with $\nu \geq 3$ defined as in Section 1.1 by the overdetermined linear equations $Ax = b$ with $\mathcal{L}$ the subspace defining a linearly independent subset of equations, and let $n = k_1 \times \ldots \times k_\nu$. Let $z = (\zeta_1, \ldots, \zeta_n)$ be the unique point in P on which the strictly concave function*

$$g(x) = \sum_{j=1}^{n} (\xi_j + 1) \ln(\xi_j + 1) - \xi_j \ln(\xi_j) \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)$$

attains its maximum value. Let D be the matrix whose columns are $(\zeta_j + \zeta_j^2)^{1/2} a_j$, where a_j are the columns of A, and let $q(t) = \frac{1}{2} \langle t, DD^t t \rangle$. Suppose there exist numbers $0 < \omega < 1$, along with $k > 0$, and $R > r > 0$ such that the following inequalities hold:

$$\omega k \leq k_j \leq k \qquad \text{for} \qquad j = 1, \ldots, \nu, \qquad \text{and}$$

$$r \leq \zeta_j^2 + \zeta_j \leq R \qquad \text{for} \qquad j = 1, \ldots, n,$$

along with the inequalities $\omega k \geq 2$, and $R > 1$. If k is large enough to satisfy the following inequalities:

$$\frac{\pi^2 2^\nu \nu^3}{\omega^\nu \ln(1 + \frac{2}{5}\pi^2 r)} \left(\frac{1}{2} \nu^2 k \ln(k) + \frac{1}{2} \nu k \ln(R) \right) k^{-\nu+1} \leq \frac{1}{4\nu^2 R}, \quad \text{and}$$

$$\frac{8\pi^2 2^\nu \nu^7 R^2}{\omega^\nu r} \ln(k) k^{-\nu+2} \leq 3/4,$$

then $|P \cap \mathbb{Z}^n|$ is approximated by

$$\frac{e^{g(z)}}{(2\pi)^{(k_1 + \ldots + k_\nu - \nu + 1)/2}} \det(q|_{\mathcal{L}})^{-1/2}$$

to within relative error

$$\Gamma k^{-\nu+2.5}$$

for some constant $\Gamma = \Gamma(R, r, \omega, \nu)$. In particular, if r, R, ω and ν are fixed, there *exists* $N = N(r, R, \omega, \nu)$ *such that for all* $k \geq N$*, we have*

$$\Gamma = \frac{4R^2\pi^4 4^\nu \nu^{10}}{\omega^{2\nu}}.$$

The conditions of Theorem I.1 essentially say that P looks similar to the most symmetric case possible. P is called a *polystochastic tensor* if $k_1 = k_2 = \ldots = k_\nu$ and every entry of b is equal to $k^{\nu-1}$. In this case we can take $k = k_j$ for all $j = 1, \ldots, \nu$, and $\omega = 1$. Furthermore, by the symmetry of the problem, we get $\zeta_j = 1$ for $j = 1, \ldots, n$. The value of ω measures how far from a hypercube the shape of the polytope's arrays are. The values R/r essentially measure how far from equal the entries of b are, and the magnitude of r (or R) is a measure of how large the entries of b are.

The assumption that $R > 1$ is trivial, as R is simply an upper bound on the values of $\zeta_{m_1,\ldots,m_\nu}$ and can be chosen to be larger if needed. If any of the k_js are equal to 1, then P is a $\nu-1$ index transportation polytope, so $\omega k \geq 2$ is also a trivial assumption. The two non-trivial assumptions on how large k is are generated by the specific proof we use. Informally, they say if R is too large compared to k, the theorem is not valid. Fixing R/r and letting r, R go to infinity is equivalent to letting the margin sums go to infinity. In this case the number of integer points well-approximates the volume of P [KV97]. As we will see in Theorem I.3, estimating the volume of multi-index transportation polytopes is more difficult. As we will discuss in Section 1.8 this restriction is likely artificial. Under the heuristic described in Section 1.3, we would expect the problem of estimating the number of integer points to become easier as the margins go to infinity.

To prove Theorem I.1, we show that the number of integer points in P can be expressed using

$$\int\limits_{\Pi} F(t)dt,$$

where $\Pi \subset \mathcal{L}$ for $\mathcal{L}$ as in (1.1) is a cube centered at the origin whose sides have length 2π, and $F(t)$ is a function that will be defined later. We then split $\mathcal{L}$ into three regions X_1, X_2, and X_3. We show that

$$\int\limits_{(X_2 \cup X_3) \cap \Pi} |F(t)|dt \quad \text{and} \quad \int\limits_{X_2 \cup X_3} e^{-q(t)}dt \ll \int\limits_{\mathcal{L}} e^{-q(t)}dt,$$

where $q(t)$ is the quadratic form constructed in Theorem I.1, and that

$$\int\limits_{X_1} F(t)dt \approx \int\limits_{X_1} e^{-q(t)}dt \approx \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

To facilitate these calculations we will require several results about the probability distribution whose density is proportional to $e^{-q(t)}$ on $\mathcal{L}$. Chapter II will contain these, and the proof of Theorem I.1 will take place in Chapter III.

1.5 Counting Binary Integer Points in Transportation Polytopes

In this section we state the main theorem estimating the number of binary integer points in a multi-index transportation polytope. We use several pieces of notation that overlap with Theorem I.1. The usage as in Theorem I.1 will be restricted to its proof in Chapter III, and the usage in Theorem I.2 will be restricted to its proof in Chapter IV.

Theorem I.2. *Let P be a $k_1 \times \ldots \times k_\nu$ transportation polytope with $\nu \geq 3$ defined by the overdetermined linear equations $Ax = b$ as described in Section 1.1, with $\mathcal{L}$ the subspace defining a linearly independent set of equations, and let $n = k_1 \times \ldots \times k_\nu$.*

Let $z = (\zeta_1, \ldots, \zeta_n)$ be the unique point in $P \cap [0,1]^n$ on which the strictly concave function

$$g(x) = \sum_{j=1}^{n} \xi_j \ln \frac{1}{\xi_j} + (1-\xi_j) \ln \frac{1}{1-\xi_j} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n)$$

attains its maximum value. Let D be the matrix whose columns are $(\zeta_j - \zeta_j^2)^{1/2} a_j$, where a_j are the columns of A, and let $q(t) = \frac{1}{2}\langle t, DD^t t\rangle$. Suppose there exist numbers $0 < \omega < 1$, along with $k > 0$ and $r > 0$ such that

$$\omega k \le k_j \le k \quad \text{for} \quad j = 1, \ldots, \nu, \quad \text{and}$$

$$r \le \zeta_j - \zeta_j^2 \quad \text{for} \quad j = 1, \ldots, n,$$

along with $\omega k \ge 2$. If k is large enough so that

$$\frac{5\nu^5 2^{\nu-1}}{2r\omega^\nu} \ln(k) k^{-\nu+2} \le \frac{1}{4\nu^2} \quad \text{and}$$

$$\frac{10\nu^7 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+2} \le 3/4,$$

then $|P \cap \{0,1\}^n|$ is approximated by

$$\frac{e^{g(z)}}{(2\pi)^{(k_1+\ldots+k_\nu-\nu+1)/2}} \det(q|_{\mathcal{L}})^{-1/2}$$

to within relative error

$$\Gamma k^{-\nu+2.5}$$

for some constant $\Gamma = \Gamma(r,\omega,\nu)$. There exists some constant $N = N(r,\omega,\nu)$ such that if $k \ge N$, then Γ may be chosen to be

$$\Gamma = \frac{25\nu^{14}4^{\nu-1}}{4r^2\omega^{2\nu}}.$$

The conditions of the theorem essentially say that P looks similar to the most symmetric case possible. Suppose $k_1 = k_2 = \ldots = k_\nu$ and every entry of b is equal to $k^{\nu-1}/2$. In this case we can take $k = k_j$ for all $j = 1, \ldots, \nu$, and $\omega = 1$. Furthermore, by the symmetry of the problem, we get $\zeta_j = 1/2$ for $j = 1, \ldots, n$, and $\zeta_j - \zeta_j^2 = \frac{1}{4}$ for all values of j. The value of ω measures how far from a hypercube the shape of the polytope's arrays are. The value of r measures how far from $k^{\nu-1}/2$ the entries of b are - as the entries of b approach the extremal permissible values of $k^{\nu-1}$ and 0 where the counting problem is trivial, the value of r goes to zero. It is easily seen r can never be larger than 1/4 as by hypothesis, if $z \in P \cap [0,1]^n$ then $0 \leq \zeta_j \leq 1$ for all $j = 1, \ldots, n$. The inequality $\omega k \geq 2$ is trivial, as if any $k_j = 1$, then P is a $\nu - 1$-index transportation polytope. The non-trivial relationship between k, r, ω and ν is a consequence of how the proof of the theorem is constructed, and is likely not optimal. However, under the heuristic described in Section 1.3, we also would not expect Theorem I.2 to hold if r is small enough compared to k.

To prove Theorem I.2, we show that the number of binary integer points in P can be expressed using

$$\int_{\Pi} F(t)dt,$$

where $\Pi \subset \mathcal{L}$ for $\mathcal{L}$ as in (1.1) is a cube centered at the origin whose sides have length 2π, and $F(t)$ is a function that will be defined later. Note that $F(t)$ is a different but similar function to the one described after Theorem I.1, and each function's definition will be restricted to the chapter containing the respective theorems' proofs. We then split $\mathcal{L}$ into three regions X_1, X_2, and X_3. We show that

$$\int_{(X_2 \cup X_3) \cap \Pi} |F(t)|dt \quad \text{and} \quad \int_{X_2 \cup X_3} e^{-q(t)}dt \ll \int_{\mathcal{L}} e^{-q(t)}dt,$$

where $q(t)$ is the quadratic form constructed in Theorem I.2, and that

$$\int\limits_{X_1} F(t)dt \approx \int\limits_{X_1} e^{-q(t)}dt \approx \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

To facilitate these calculations we will require several results about the probability distribution whose density is proportional to $e^{-q(t)}$ on $\mathcal{L}$. Chapter II will contain these, and the proof of Theorem I.2 will take place in Chapter IV.

1.6 Measuring Volumes of Transportation Polytopes

In this section we state the main theorem estimating the volume of a 4-index transportation polytope. As in the case of Theorem I.2, we will allow conflicting notation whose usage will be restricted to the proof of Theorem I.3 in Chapter V.

Theorem I.3. *Let P be a $k_1 \times k_2 \times k_3 \times k_4$ transportation polytope defined by the overdetermined linear equations $Ax = b$ as described in Section 1.1, with $\mathcal{L}$ the subspace defining a linearly independent subset of equations, and let $n = k_1k_2k_3k_4$. Let $z = (\zeta_1, \ldots, \zeta_n)$ be the unique point in P on which the strictly concave function*

$$g(x) = n + \sum_{j=1}^{n} \ln \xi_j \quad \textit{for} \quad x = (\xi_1, \ldots, \xi_n)$$

attains its maximum value. Let D be the matrix whose columns are $\zeta_j a_j$, where a_j are the columns of A, and let $q(t) = \frac{1}{2}\langle t, DD^t t\rangle$ and $s(t) = \frac{1}{2}\langle t, AA^t t\rangle$. Suppose there exist numbers $0 < \omega < 1$ along with $k > 0$ and $R > r > 0$, such that

$$\omega k \leq k_j \leq k \quad \textit{for} \quad j = 1, \ldots, \nu, \textit{and}$$

$$r \leq \zeta_j^2 \leq R \quad \textit{for} \quad j = 1, \ldots, n,$$

along with the inequalities $\omega k \geq 2$, $R > 1$ and $k > 4$. Suppose

$$\frac{3072R^2}{\omega^3 r}\left(8k^{-1}\ln(k) + 2k^{-1}\ln(R)\right) \leq 1,$$

$$1 - \frac{3072R}{r\omega^3 k} > 0,$$

$$\frac{245760R^2}{\omega^3 r}k^{-1}\ln(k) \leq 1, \quad \text{and}$$

$$k \geq \frac{2}{\omega}(2\pi)^{4/\omega}.$$

Then $\mathrm{vol}(P)$ *is approximated by*

$$\frac{e^{g(z)}}{(2\pi)^{(k_1+k_2+k_3+k_4-3)/2}}\frac{\det(s|_{\mathcal{L}})^{1/2}}{\det(q|_{\mathcal{L}})^{1/2}}$$

to within relative error

$$\Gamma k^{-.2}$$

for some constant $\Gamma = \Gamma(R, r, \omega) > 0$. Furthermore, there exists some constant $N = N(R, r, \omega)$ such that for $k \geq N$, we can let $\Gamma = \frac{2R^{3/2}}{3}$.

The conditions of the theorem essentially say that P looks similar to the most symmetric case possible. P is called a *polystochastic tensor* if $k_1 = k_2 = \ldots = k_\nu$ and every entry of b is equal to $k^{\nu-1}$. In this case we can take $k = k_j$ for all $j = 1, \ldots, \nu$, and $\omega = 1$. Furthermore, by the symmetry of the problem, we get $\zeta_j = 1$ for $j = 1, \ldots, n$. The value of ω measures how far from a hypercube the shape of the polytope's arrays are, and the values r and R essentially measure how far from equal the entries of b are.

The inequality $\omega k \geq 2$ is trivial because if any $k_j = 1$, then P is a 3-index transportation polytope. The inequalities $R > 1$ and $k > 4$ are also trivial as R and k are upper bounds on values, and can always be taken to be larger if necessary. The other inequalities are non-trivial and consequences of how the proof is constructed.

One consequence of them is that if R is too large compared to k, the theorem is not valid. As R goes to infinity the volume is well-approximated by the number of integer points in P, so for similar reasons as in the description following Theorem I.1, we expect this restriction to be artificial, and replaceable with a relationship between k and R/r.

Similar to the proofs of Theorems I.1 and I.2, we show that the volume in P can be expressed using

$$\int_{\mathcal{L}} F(t)dt,$$

for $\mathcal{L}$ as in (1.1) and $F(t)$ a function that will be defined later. The $F(t)$ in the proof of Theorem I.3 will be different than the function $F(t)$ discussed after Theorems I.1 and I.2, and the notation will be restricted to the proof in Chapter V. We then split $\mathcal{L}$ into three regions X_1, X_2, and X_3. We show that

$$\int_{(X_2 \cup X_3)} |F(t)|dt \quad \text{and} \quad \int_{X_2 \cup X_3} e^{-q(t)}dt \ll \int_{\mathcal{L}} e^{-q(t)}dt,$$

where $q(t)$ is the quadratic form constructed in Theorem I.3, and that

$$\int_{X_1} F(t)dt \approx \int_{X_1} e^{-q(t)}dt \approx \int_{\mathcal{L}} e^{-q(t)}dt.$$

Unlike the proofs of the other main theorems, this last approximate equality will require splitting X_1 into several pieces as well - this complication is why the volume calculation cannot be extended to $\nu = 3$ readily. To facilitate these calculations we will require several results about the probability distribution whose density is proportional to $e^{-q(t)}$ on $\mathcal{L}$. Chapter II will contain these, and the proof of Theorem I.3 will take place in Chapter V.

1.7 Polynomial Time Calculations

In all of Theorems I.1, I.2, and I.3, to calculate the given estimate one must calculate the determinant of a known quadratic form, which can be done in time polynomial in n, and find the extremal value of a strictly concave function. This can be calculated to within error ϵ in time polynomial in n and $\ln(1/\epsilon)$, see [NN94]. Combined, this says that all three theorems give polynomial time algorithms for estimating the number of integer points, binary integer points, or volume of transportation polytopes.

1.8 Future Work

In Theorems I.1 and I.3, the value R cannot be too large or the hypothesis of the theorem is not satisfied. This is likely an artifact of the proof technique and not a hard requirement. In [BH12], Barvinok and Hartigan count the number of integer points in 2-way transportation polytopes as long as R/r is held constant, and R is bounded by any arbitrary polynomial in k. For very large values of r and R, the polytope itself is large enough that the volume and number of integer points approximate each other quite well. The authors use scaling of the polytope to show that any value of R is admissible, and to estimate the volume of 2-way transportation polytopes as well with no upper bound on the value of R.

The extra flexibility comes from being able to show that

$$\int_{X_2 \cup X_3} F(t)dt \ll \int_{\mathcal{L}} e^{-q(t)}dt$$

for a much larger set $X_2 \cup X_3$ than we are able to construct for $\nu \geq 3$. It is an open question if for $\nu \geq 3$ there is no upper bound on how large R can be for the number of integer points and volume calculations. It is also an open question if there exists a formula for the volume of 3-way transportation polytopes even in the case when R is held constant as k grows.

CHAPTER II

Correlations and Variances of the Quadratic Form

For the entirety of this chapter, we will let $q(t) : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}$ be the quadratic form

$$q(t) = \frac{1}{2} \sum_{m_1,\ldots,m_\nu=1}^{k_1,\ldots,k_\nu} \alpha_{m_1,\ldots,m_\nu} (t_{1m_1} + \ldots t_{\nu m_\nu})^2, \tag{2.1}$$

where $\alpha_{m_1,\ldots,m_\nu}$ are arbitrary positive constants, such that each k_i is at least 2. Note that the quadratic forms in Theorems I.1, I.2 and I.3 are of this form with $\alpha_{m_1,\ldots,m_\nu} = \zeta_{m_1,\ldots,m_\nu} + \zeta^2_{m_1,\ldots,m_\nu}$ in the first case, $\alpha_{m_1,\ldots,m_\nu} = \zeta_{m_1,\ldots,m_\nu} - \zeta^2_{m_1,\ldots,m_\nu}$ in the second, and $\alpha_{m_1,\ldots,m_\nu} = \zeta^2_{m_1,\ldots,m_\nu}$ in the last. We will let B be the unique positive semidefinite matrix such that

$$q(t) = \frac{1}{2} \langle t, Bt \rangle .$$

Note that for D as in Theorems I.1, I.2, or I.3, we have $B = DD^t$. Suppose X is a random variable whose density is proportional to $e^{-q(t)}$ restricted to $\mathcal{L}$, where $\mathcal{L}$ is as defined in (1.1). The objective of this chapter is to calculate correlations of random variables of the form $\langle X, e_i \rangle$ for any standard basis vector $e_i \in \mathcal{L}$. To do so, we will carefully bound the eigenvalues of $q(t)$ and estimate the eigenvectors of $q(t)$.

The application of these results will be applied in the proofs of the main theorems in two ways. It will allow us to show that the integral of $e^{-q(t)}$ outside of a neighborhood of the origin is negligible. It will also allow us to place bounds on $\mathbf{E}\left(e^{if(t)}\right)$ for a certain cubic polynomial $f(t)$ when t is drawn according to the distribution given by X. In the proofs of each of the main theorems we will show the quantity to be measured is equal to the integral of a function $F(t)$ (different for each theorem), which we will approximate near the origin via Taylor polynomial approximations as $F(t) \approx e^{-q(t)+if(t)}$. The results of this chapter will allow us to then estimate the integral of F in a neighborhood of the origin.

We introduce some notation and concepts for the chapter. If C is a symmetric matrix, we write $\lambda_i(C)$ to be the ith largest eigenvalue of C. Throughout the entire chapter we assume that there are values $R > r > 0$ such that

$$r \leq \alpha_{m_1,\ldots,m_\nu} \leq R \quad \text{for all} \quad m_1, \ldots, m_\nu.$$

We also assume there exists ω and k such that

$$1 \leq \omega k \leq k_1, \ldots, k_\nu \leq k.$$

For notational convenience we will define

$$k'_j = \prod_{\substack{i=1,\ldots,\nu \\ i \neq j}} k_i.$$

We will also denote $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ to be the orthogonal projection onto $\mathcal{L}$.

2.1 Eigenvalues and Eigenvectors of $q(t)$

In this section we calculate the eigenvalues and eigenvectors of $q(t)$ (that is, the eigenvalues and eigenvectors of B), and the eigenvalues and eigenvectors of $q(t)$ when restricted to the subspace $\mathcal{L}$ defined in Equation (1.1). Throughout we will let $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ be the orthogonal projection onto $\mathcal{L}$. The main result of this section is the following:

Theorem II.1. *For $\nu \geq 2$, there exists a set of eigenvectors and eigenvalues of QBQ as follows: there are $\nu - 1$ eigenvectors with eigenvalue 0 lying in the kernel of Q, $\nu - 1$ unit eigenvectors with eigenvalues that lie between*

$$r\frac{\omega^{\nu-1}}{\nu(\nu-1)}k^{\nu-2} \quad \textit{and} \quad R\omega^{-1}k^{\nu-2}$$

such that the square of the distance of each eigenvector to $\ker(B)$ is smaller than

$$\frac{R}{r}\omega^{-\nu}k^{-1},$$

one eigenvalue which lies between

$$\frac{r}{2}\omega^{\nu-1}\nu k^{\nu-1} \quad \textit{and} \quad R\nu k^{\nu-1},$$

and the remaining eigenvalues all lie between

$$r\omega^{\nu-1}k^{\nu-1} \quad \textit{and} \quad Rk^{\nu-1}.$$

This theorem describes the eigenvalues and eigenvectors of the quadratic form $q|_{\mathcal{L}}$. The outline of the proof is as follows: we first calculate all the eigenvectors and eigenvalues of B, and see the eigenvalues are all $\Theta(k^{\nu-1})$. Most of these eigenvectors will lie in $\mathcal{L}$ and hence be eigenvectors of $q|_{\mathcal{L}}$ as well. The remaining few eigenvectors will be nearly orthogonal to $\mathcal{L}$, which we will use to show that the remaining eigenvalues are $\Theta(k^{\nu-2})$.

We require the use of two well known lemmas on comparing eigenvalues of symmetric matrices.

Lemma II.2. *Let C and D be symmetric positive semidefinite $m \times m$ matrices such that $C - D$ is positive semidefinite. Then*

$$\lambda_i(C) \geq \lambda_i(D) \text{ for all } i = 1, \ldots, m.$$

Proof. This is Corollary 7.7.4 of [HJ85]. □

Lemma II.3. (The Weyl Inequalities)*: Let C and D be $m \times m$ real symmetric matrices. Then*

$$\lambda_{i+j-1}(D + C) \leq \lambda_i(C) + \lambda_j(D)$$

as long as $1 \leq i, j \leq m$ such that $i + j - 1 \leq m$.

Proof. This inequality is shown in Section 1.3.3 of [Ta12]. □

Lemma II.4. *The matrix B has a basis of orthogonal eigenvectors such that $\nu - 1$ lie in the kernel of B, one has eigenvalue between $r\omega^{\nu-1}\nu k^{\nu-1}$ and $R\nu k^{\nu-1}$, and the remaining eigenvalues lie between $r\omega^{\nu-1}k^{\nu-1}$ and $Rk^{\nu-1}$.*

Proof. If $q(t) = \frac{1}{2}\langle t, Bt\rangle$ then

$$\nabla q(t) = Bt.$$

Calculating the gradient in (2.1) gives us

$$\frac{\partial q}{\partial \tau_{jm_j}} = \sum_{m_1,\ldots,\hat{m}_j,\ldots,m_\nu} \alpha_{m_1,\ldots,m_\nu}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu}). \tag{2.2}$$

First we consider the case when $\alpha_{m_1,\ldots,m_\nu} = 1$ for all $m_1, \ldots, m_\nu$. Then for all $1 \leq$

$j \leq \nu$ and $1 \leq m_j \leq k_j$,

$$\frac{\partial q}{\partial \tau_{jm_j}} = k'_j \tau_{jm_j} + \sum_{m_1,\ldots,\hat{m}_j,\ldots,m_\nu=1}^{k_1,\ldots,\hat{k}_j,\ldots,k_\nu} \left(\tau_{1m_1} + \ldots + \hat{\tau}_{jm_j} + \ldots + \tau_{\nu m_\nu} \right). \tag{2.3}$$

From this it is immediately clear by substituting in the relevant vectors that for any $1 \leq j \leq \nu$, any non-zero vector contained in the subspace

$$\mathcal{W}_j = \left\{ (\underbrace{0,\ldots,0}_{k_1+\ldots+k_{j-1}}, \tau_{j1},\ldots,\tau_{jk_j}, \underbrace{0,\ldots,0}_{k_{j+1}+\ldots+k_\nu}) \;:\; \sum_{i=1}^{k_j} \tau_{ji} = 0 \right\} \tag{2.4}$$

is an eigenvector of B with eigenvalue k'_j. As B has a basis of orthogonal eigenvectors, we can complete the description of its eigenspace by considering vectors orthogonal to each $\mathcal{W}_j$, which are of the form

$$(\underbrace{\sigma_1,\ldots,\sigma_1}_{k_1}, \underbrace{\sigma_2,\ldots,\sigma_2}_{k_2}, \ldots, \underbrace{\sigma_\nu,\ldots,\sigma_\nu}_{k_\nu}).$$

If $\sigma_1 + \ldots + \sigma_\nu = 0$ then this vector lies in the kernel of B, so is an eigenvector with eigenvalue 0. By dimension counting there is one remaining eigenvector of B, which has $\sigma_j = k'_j$ for all j and has an eigenvalue of

$$\sum_{j=1}^{\nu} k'_j.$$

If instead we have $r < \alpha_{m_1\ldots m_\nu} < R$, the set of vectors of the form

$$(\underbrace{\sigma_1,\ldots,\sigma_1}_{k_1}, \underbrace{\sigma_2,\ldots,\sigma_2}_{k_2}, \ldots, \underbrace{\sigma_\nu,\ldots,\sigma_\nu}_{k_\nu}) \quad \text{such that} \quad \sigma_1 + \ldots + \sigma_\nu = 0$$

still form the kernel of B, and the remaining eigenvectors will be orthogonal to this space. Applying Lemma II.2 and comparing the eigenvalues of $q(t)$ with the quadratic forms

$$\tilde{q}_Z(t) = \frac{Z}{2} \sum_{m_1,\ldots,m_\nu=1}^{k_1,\ldots,k_\nu} (\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})^2$$

for $Z = R$ and $Z = r$ completes the proof. $\square$

At this point we are ready to prove Theorem II.1.

Proof. We will first prove the result when $\alpha_{m_1,\ldots,m_\nu} = 1$ for all $m_1, \ldots, m_\nu$. Then for $\mathcal{W}_j$ as defined in Equation (2.4) we immediately get that $\mathcal{W}_j \cap \mathcal{L}$ are eigenspaces of QBQ with eigenvalues k_j'. Furthermore, $\mathcal{W}_1 \cap \mathcal{L} = \mathcal{W}_1$ has dimension k_1, and for $j \geq 2$, the subspace $\mathcal{W}_j \cap \mathcal{L}$ has codimension 1 in $\mathcal{W}_j$ so is a subspace of $\mathcal{L}$ of dimension $k_j - 1$. By dimension counting we are left with ν linearly independent eigenvectors of QBQ in $\mathcal{L}$ that are unaccounted for. There must exist a set of eigenvectors of the form

$$s = (\underbrace{\sigma_1, \ldots, \sigma_1}_{k_1}, \underbrace{\sigma_2, \ldots, \sigma_2}_{k_2-1}, 0, \underbrace{\sigma_3, \ldots, \sigma_3}_{k_3-1}, 0, \ldots, \underbrace{\sigma_\nu, \ldots, \sigma_\nu}_{k_\nu-1}, 0) \tag{2.5}$$

as they are orthogonal to the eigenspaces of QBQ that we have calculated so far. Let $\mathcal{V}$ be the subspace of all vectors of the form defined in Equation (2.5). We decompose s into a linear combination of the $2\nu - 1$ remaining eigenvectors of B orthogonal to $\mathcal{W}_j \cap \mathcal{L}$ for all j. These are the kernel vectors

$$(\underbrace{\sigma_1, \ldots, \sigma_1}_{k_1}, \underbrace{\sigma_2, \ldots, \sigma_2}_{k_2}, \ldots, \underbrace{\sigma_\nu, \ldots, \sigma_\nu}_{k_\nu}) \quad \text{with} \quad \sum_{j=1}^{\nu} \sigma_j = 0, \tag{2.6}$$

the vector

$$v_0 = (\underbrace{k_1', \ldots, k_1'}_{k_1}, \underbrace{k_2', \ldots, k_2'}_{k_2}, \ldots, \underbrace{k_\nu', \ldots, k_\nu'}_{k_\nu}) \tag{2.7}$$

and one vector from each $\mathcal{W}_j$ for $j \neq 1$ of the form

$$v_j = (\underbrace{0, 0 \ldots, 0}_{k_1+\ldots+k_{j-1}}, \underbrace{1, 1, \ldots, 1}_{k_j-1}, 1 - k_j, \underbrace{0, 0, \ldots, 0}_{k_{j+1}+\ldots+k_\nu}). \tag{2.8}$$

The projection of s onto the span of v_j by definition is

$$\frac{\langle v_j, s\rangle}{\langle v_j, v_j\rangle} v_j.$$

For $s \in \mathrm{im}(Q)$, we have $\langle v_j, s\rangle = \langle Qv_j, s\rangle$. If P_j is the projection onto v_j, then

$$QBQs = QBQ\left(P_0 s + \sum_{j=2}^{\nu} P_j s\right)$$

can be rewritten as

$$QBQs = \frac{\langle Qv_0, s\rangle}{\langle v_0, v_0\rangle} QBQv_0 + \sum_{j\geq 2} \frac{\langle Qv_j, s\rangle}{\langle v_j, v_j\rangle} QBQv_j. \tag{2.9}$$

Let

$$u_0 = Qv_0 = (\underbrace{k_1', .., k_1'}_{k_1}, \underbrace{k_2', \ldots, k_2'}_{k_2-1}, 0, \underbrace{k_3', \ldots, k_3'}_{k_3-1}, 0, \ldots, \underbrace{k_\nu', \ldots, k_\nu'}_{k_\nu-1}, 0), \tag{2.10}$$

and for $j > 1$,

$$u_j = Qv_j = (\underbrace{0, \ldots, 0}_{k_1+\ldots+k_{j-1}}, \underbrace{1, \ldots, 1}_{k_j-1}, 0, \underbrace{0, \ldots, 0}_{k_{j+1}+\ldots+k_\nu}). \tag{2.11}$$

By the eigenvalues of the v_js calculated in Lemma II.4, plugging Equations (2.10) and (2.11) into Equation (2.9), we get

$$QBQ|_{\mathcal{V}} = \frac{\sum_{j=1}^{\nu} k_j'}{\langle v_0, v_0\rangle} u_0 u_0^t + \sum_{j\geq 2} \frac{k_j'}{\langle v_j, v_j\rangle} u_j u_j^t.$$

We can then write $QBQ|_{\mathcal{V}} = C + D$, where

$$C = \frac{\sum_{j=1}^{\nu} k_j'}{\langle v_0, v_0\rangle} u_0 u_0^t,$$

is a rank one symmetric matrix with nonzero eigenvector u_0 and eigenvalue

$$\lambda_C = \frac{\langle u_0, u_0 \rangle}{\langle v_0, v_0 \rangle} \sum_{j=1}^{\nu} k'_j,$$

and

$$D = \sum_{j \geq 2} \frac{k'_j}{\langle v_j, v_j \rangle} u_j u_j^t$$

is a rank $\nu - 1$ symmetric matrix that has nonzero eigenvectors u_j for $j \geq 2$ of eigenvalue

$$\lambda_j = \frac{\langle u_j, u_j \rangle k'_j}{\langle v_j, v_j \rangle}.$$

By Equations (2.7) and (2.10),

$$\left(1 - \frac{1}{k}\right) \nu \omega^{\nu-1} k^{\nu-1} \leq \lambda_C \leq \left(1 - \frac{\omega}{k}\right) \nu k^{\nu-1}, \tag{2.12}$$

and by Equations (2.8) and (2.11), for all $j \geq 2$ we have

$$\omega^{\nu-1} k^{\nu-2} \leq \lambda_j \leq \omega^{-1} k^{\nu-2}. \tag{2.13}$$

By Lemma II.3, we get by plugging in $i = 2$ and $j = 1$ that

$$\lambda_2 \left(QBQ|_{\mathcal{V}}\right) \leq \lambda_2 \left(C|_{\mathcal{V}}\right) + \lambda_1 \left(D|_{\mathcal{V}}\right) \leq \omega^{-1} k^{\nu-2}. \tag{2.14}$$

Furthermore,

$$\lambda_1 \left(QBQ|_{\mathcal{V}}\right) \geq \lambda_1 \left(C|_{\mathcal{V}}\right) \geq \left(1 - \frac{1}{k}\right) \nu \omega^{\nu-1} k^{\nu-1}.$$

Also, the largest eigenvalue of $QBQ|_{\mathcal{V}}$ must be smaller than the largest eigenvalue of B, which is $\nu k^{\nu-1}$. Taking the largest eigenvalue of $QBQ|_{\mathcal{V}}$ and the eigenvalues we have calculated previously, we find that all but $\nu - 1$ eigenvalues of QBQ on $\mathcal{L}$ lie between

$$\omega^{\nu-1}k^{\nu-1} \quad \text{and} \quad \nu k^{\nu-1}.$$

Furthermore, by (2.14), to complete the proof on the magnitude of the eigenvalues it suffices to show that

$$\lambda_\nu\left(QBQ|_{\mathcal{V}}\right) \geq \frac{\omega^{\nu-1}}{\nu(\nu-1)}k^{\nu-2}.$$

If $t \in \mathcal{L}$ is an eigenvector of $QBQ|_{\mathcal{V}}$ with eigenvalue λ, then $q(t) = \frac{1}{2}\lambda||t||^2$. Decomposing $t = w_1 + w_2$ with $w_1 \in \ker(B)$, and $w_2 \perp \ker(B)$, we also have $q(t) = q(w_2) \geq \frac{1}{2}\omega^{\nu-1}k^{\nu-1}||w_2||^2$ by Lemma II.4. Combining these gives

$$\lambda \geq \omega^{\nu-1}k^{\nu-1}\frac{||w_2||^2}{||t||^2}. \tag{2.15}$$

Assume t is of the form given in Equation (2.5). Let T be the orthogonal projection onto the kernel of B, so $w_1 = Tt$. Then

$$\frac{||w_2||^2}{||t||^2} = 1 - \frac{\langle t, u\rangle^2}{||t||^2}. \tag{2.16}$$

Furthermore,

$$Tt = \langle t, u\rangle\, u \quad \text{for} \quad u = \frac{1}{||w_1||}w_1,$$

and

$$\frac{||Tt||^2}{||t||^2} = \frac{\langle t, u\rangle^2}{||t||^2}.$$

We also use the fact that

$$||Qu||^2 \geq \frac{\langle t, u\rangle^2}{||t||^2}$$

as the projection of u onto $\mathcal{L}$ is at least as large as the projection of u onto the span of t. Combining these along with (2.15), we get that

$$\lambda_\nu\left(QBQ|_{\mathcal{V}}\right) \geq \omega^{\nu-1}k^{\nu-1}\left(1 - ||Qu||^2\right). \tag{2.17}$$

We consider the problem

$$\text{minimize} \quad ||(\mathrm{Id} - Q)u||^2 \quad \text{given} \quad u \in \ker(B),\ ||u|| = 1.$$

Recall that every $u \in \ker(B)$ is in the form given in Equation (2.6), so the problem reduces to finding a lower bound for

$$||(\mathrm{Id} - Q)u||^2 = \sum_{j=2}^{\nu} \sigma_j^2$$

under the conditions that

$$\sum_{j=1}^{\nu} k_j \sigma_j^2 = 1 \quad \text{and} \quad \sum_{j=1}^{\nu} \sigma_j = 0.$$

Substituting $-\sigma_1 = \sigma_2 + \ldots + \sigma_\nu$, we reduce the problem to

$$\text{minimize} \quad \sum_{j=2}^{\nu} \sigma_j^2 \quad \text{given} \quad k_1 \left(\sum_{j=2}^{\nu} \sigma_j \right)^2 + \sum_{j=2}^{\nu} k_j \sigma_j^2 = 1.$$

If the minimum is achieved at $(\sigma_2, \ldots, \sigma_\nu) = (\beta_2, \ldots, \beta_\nu)$, then every β_j must have the same sign. If not, then

$$k_1 \left(\sum_{j=2}^{\nu} |\beta_j| \right)^2 + \sum_{j=2}^{\nu} k_j \beta_j^2 \geq k_1 \left(\sum_{j=2}^{\nu} \beta_j \right)^2 + \sum_{j=2}^{\nu} k_j \beta_j^2 = 1,$$

and therefore we can take the vector $(|\beta_2|, \ldots, |\beta_\nu|)$ and scale it down to find a smaller minimum satisfying the constraints. By taking negatives if necessary, assume $\beta_j > 0$ for all $j \geq 2$. If for some i we have $\beta_i \geq \beta_j$ for all $j \geq 2$, then

$$\left((\nu - 1)^2 k_1 + \sum_{j=2}^{\nu} k_j \right) \beta_i^2 \geq k_1 \left(\sum_{j=2}^{\nu} \beta_j \right)^2 + \sum_{j=2}^{\nu} k_j \beta_j^2 = 1,$$

so

$$\beta_i^2 \geq \frac{1}{(\nu-1)^2 k_1 + \sum_{j=2}^{\nu} k_j}$$

and hence

$$\sum_{j=2}^{\nu} \beta_j^2 \geq \frac{1}{(\nu-1)^2 k_1 + \sum_{j=2}^{\nu} k_j}.$$

Therefore

$$||(\mathrm{Id} - Q)u||^2 \geq \frac{1}{\nu(\nu-1)k} ||u||^2,$$

which combined with Equation (2.17) completes the proof that on $\mathcal{V}$,

$$\lambda_\nu(QBQ|_{\mathcal{V}}) \geq \frac{\omega^{\nu-1}}{\nu(\nu-1)k^{\nu-2}}.$$

Combining this with (2.14) completes the proof of the theorem in the case that $\alpha_{m_1,\ldots,m_\nu} = 1$ for all $m_1, \ldots, m_\nu$.

By Lemma II.2, if $r \leq \alpha_{m_1,\ldots,m_\nu} \leq R$, the eigenvalues of $q|_{\mathcal{L}}$ are bounded from above by the previously calculated eigenvalues multiplied by R, and bounded from below by the previously calculated eigenvalues multiplied by r. Furthermore if $t \in \mathcal{L}$ is a unit eigenvector whose eigenvalue is smaller than $R\omega^{-1}k^{\nu-2}$, then writing

$$t = \alpha_t t_1 + \beta_t t_2$$

with $t_1 \in \ker(B)$, $t_2 \in \mathrm{im}(B)$, we get that

$$\frac{1}{2} R\omega^{-1} k^{\nu-2} \geq q(t) \geq \frac{1}{2} \beta_t^2 r \omega^{\nu-1} k^{\nu-1},$$

and hence

$$\beta_t^2 \leq \frac{R}{r} \omega^{-\nu} k^{-1},$$

which completes the proof.

$\square$

We immediately get the following corollary:

Corollary II.5.

$$\int_{\mathcal{L}} e^{-q(t)} dt \geq \exp\left(-\frac{1}{4}\nu^2 k \ln(k) - \frac{1}{2}\nu k \ln(R)\right).$$

Proof. First, we observe by (1.2) and Theorem II.1 that

$$\int_{\mathcal{L}} e^{-q(t)} dt \geq (2\pi)^{(k_1+\ldots+k_\nu-\nu+1)/2} \sqrt{\left(\frac{\omega}{Rk^{\nu-2}}\right)^{(\nu-1)} \frac{1}{R\nu k^{\nu-1}} \left(\frac{1}{Rk^{\nu-1}}\right)^{k_1+\ldots+k_\nu-2\nu+1}}.$$

Observing that $\omega k \geq 1$, we can simplify this to

$$\int_{\mathcal{L}} e^{-q(t)} dt \geq (2\pi)^{(k_1+\ldots+k_\nu-\nu+1)/2} \frac{1}{\sqrt{\nu}} \left(Rk^{\nu-1}\right)^{-(k_1+\ldots+k_\nu-\nu+1)/2},$$

which we can further simplify by using the fact that $k \geq 2$ to the claim of the corollary.

$\square$

2.2 Variances

In this section we continue to use the notation introduced at the beginning of the chapter before Section 2.1. In particular the quadratic form $q(t)$, the subspace $\mathcal{L}$, and the constants r, R, ω and k introduced there are used extensively, as well as the notation for k'_j. We consider the probability density on $\mathcal{L}$ proportional to $e^{-q(t)}$, and show the total measure outside of a small box around the origin in $\mathcal{L}$ is negligible. The main result is the following:

Lemma II.6. *Let*

$$X_\delta = \{t \in \mathcal{L} \ : \ ||t||_\infty \geq \delta\}.$$

Then

$$\int\limits_{X_\delta} e^{-q(t)} dt \leq \nu k \exp\left(-\delta^2 k^{\nu-1}/\Gamma\right) \int\limits_{\mathcal{L}} e^{-q(t)} dt,$$

where dt is the Lebesgue measure on $\mathcal{L}$, and

$$\Gamma = \frac{2\nu^4 R}{\omega^{6\nu-3} r^2}.$$

To prove Lemma II.6, we consider random variables of the form $\langle t, v\rangle$ for a fixed vector v when t is drawn from the distribution with density proportional to $e^{-q(t)}$ restricted to $\mathcal{L}$. In general, if $\psi(t)$ is a positive definite quadratic form on a vector space $\mathcal{V}$ of dimension d with unit eigenvectors $v_1, \ldots, v_d$ and eigenvalues $\lambda_1, \ldots, \lambda_d$, and t is drawn randomly from the distribution whose density is proportional to $e^{-\psi(t)}$ on $\mathcal{V}$, and $u \in \mathcal{V}$ is fixed, then $\langle u, t\rangle$ is a normal random variable, and

$$\mathbf{Var}\left(\langle u, t\rangle\right) = \sum_{j=1}^{d} \frac{1}{\lambda_i} \left\langle u, v_i\right\rangle^2. \tag{2.18}$$

Before we begin the proof of the main result we require a technical lemma.

Lemma II.7. *Let e_j be a standard basis vector of $\mathbb{R}^{k_1+\ldots+k_\nu}$, and let T be the orthogonal projection onto the kernel of $q(t)$. Then there exist constants $\gamma(\omega,\nu) > 0$, $\Gamma(\omega,\nu) > 0$ such that*

$$\frac{\gamma}{k} \ \leq \ ||Te_j||_\infty \ \leq \ \frac{\Gamma}{k}.$$

The constants may be chosen to be

$$\gamma = \frac{\omega^{2\nu-1}}{2}, \quad \textit{and} \quad \Gamma = \frac{1}{\omega^{2\nu-1}}.$$

Proof. For notational simplicity we assume that e_j corresponds to one of the first k_1 entries. An orthogonal basis of the kernel of $q(t)$ can be written as follows:

$$u_{\nu-1} = (\underbrace{0,\dots,0}_{k_1+\dots+k_{\nu-2}}, \underbrace{-k'_\nu,\dots,-k'_\nu}_{k_{\nu-1}}, \underbrace{k'_\nu,\dots,k'_\nu}_{k_\nu}),$$

$$u_{\nu-2} = (\underbrace{0,\dots,0}_{k_1+\dots+k_{\nu-3}}, \underbrace{-(k'_{\nu-1}+k'_\nu),\dots,-(k'_{\nu-1}+k'_\nu)}_{k_{\nu-2}}, \underbrace{k'_{\nu-1},\dots,k'_{\nu-1}}_{k_{\nu-1}}, \underbrace{k'_\nu,\dots,k'_\nu}_{k_\nu}),$$

and for any $2 \le i \le \nu$, we have u_{i-1} given by

$$(\underbrace{0,\dots,0}_{k_1+\dots+k_{i-2}}, \underbrace{-(k'_i+\dots+k'_\nu),\dots,-(k'_i+\dots+k'_\nu)}_{k_{i-1}}, \underbrace{k'_i,\dots,k'_i}_{k_i},\dots,\underbrace{k'_\nu,\dots,k'_\nu}_{k_\nu}),$$

culminating with

$$u_1 = (\underbrace{-(k'_2+\dots+k'_\nu),\dots,-(k'_2+\dots+k'_\nu)}_{k_1}, \underbrace{k'_2,\dots,k'_2}_{k_2}, \underbrace{k'_3,\dots,k'_3}_{k_3},\dots,\underbrace{k'_\nu,\dots,k'_\nu}_{k_\nu}).$$

It is easy to see by construction that each u_i lies in the kernel of $q(t)$, and by dimension counting they therefore form a basis. To see that they form an orthogonal set, for any $i < l$ we have

$$\langle u_i, u_l \rangle = -k_l k'_l \left(\sum_{p=l+1}^{\nu} k'_p \right) + \sum_{p=l+1}^{\nu} k_p (k'_p)^2.$$

As $k_p k'_p = k_1 k_2 \dots k_\nu$ for any p, we can re-write this as

$$-(k_1 k_2 \dots k_\nu) \left(\sum_{p=l+1}^{\nu} k'_p \right) + (k_1 k_2 \dots k_\nu) \sum_{p=l+1}^{\nu} k'_p = 0.$$

Then e_j is orthogonal to u_i for all $i > 1$, and therefore the projection of e_j onto the kernel of $q(t)$ is simply

$$\frac{\langle e_j, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = \frac{-(k_2' + \ldots + k_\nu')}{k_1(k_2' + \ldots + k_\nu')^2 + k_2 k_2'^2 + \ldots + k_\nu k_\nu'^2} u_1. \tag{2.19}$$

Using the inequality $\omega k \leq k_1, \ldots, k_\nu \leq k$, we get

$$\frac{(\nu-1)\omega^\nu}{(\nu-1)^2 + \nu - 2} k^{-\nu} \leq \left| \frac{-(k_2' + \ldots + k_\nu')}{k_1(k_2' + \ldots + k_\nu')^2 + k_2 k_2'^2 + \ldots + k_\nu k_\nu'^2} \right|, \quad \text{and}$$

$$\left| \frac{-(k_2' + \ldots + k_\nu')}{k_1(k_2' + \ldots + k_\nu')^2 + k_2 k_2'^2 + \ldots + k_\nu k_\nu'^2} \right| \leq \frac{(\nu-1)}{\omega^{2\nu-1}\left((\nu-1)^2 + \nu - 2\right)} k^{-\nu}.$$

Combining these with

$$(\nu-1)\omega^{\nu-1} k^{\nu-1} \leq ||u_1||_\infty \leq (\nu-1) k^{\nu-1}$$

in (2.19) and simplifying bounds completes the proof. □

Lemma II.8. *Suppose t is drawn from the distribution with density proportional to $e^{-q(t)}$ restricted to $\mathcal{L}$. Then if e_j is a standard basis vector contained in $\mathcal{L}$, there exists a constant $\Gamma = \Gamma(\omega, \nu, r, R) > 0$ such that*

$$\mathbf{Var}\left(\langle t, e_j\rangle\right) \leq \frac{\Gamma}{k^{\nu-1}}.$$

The constant Γ may be chosen to be

$$\Gamma = \frac{14\nu^4 R}{\omega^{6\nu-3} r^2}.$$

Proof. We apply Equation (2.18) with $\mathcal{V} = \mathcal{L}$, letting $u = e_j$ be any standard basis vector contained in $\mathcal{L}$ and $\psi(t) = q(t)$ restricted to $\mathcal{L}$. Let $v_1, \ldots v_{\nu-1}$ be the unit eigenvectors whose distance to $\ker(B)$ was calculated in Theorem II.1. Substituting in the lower bound for the eigenvalues of the remaining eigenvectors from Theorem II.1

into Equation (2.18), we get the variance is bounded from above by

$$\frac{1}{r\omega^{\nu-1}k^{\nu-1}} + \frac{\nu(\nu-1)}{r\omega^{\nu-1}k^{\nu-2}} \sum_{i=1}^{\nu-1} \langle e_j, v_i \rangle^2 . \tag{2.20}$$

For any such v_i, we can decompose it as

$$v_i = a_i + b_i, \text{ where}$$

$$a_i \in \ker(B),\ ||a_i|| \leq 1, \text{ and}$$

$$b_i \perp \ker(B),\ ||b_i|| \leq \sqrt{\frac{R}{r}\omega^{-\nu}k^{-1/2}}.$$

Then

$$\langle e_j, v_i \rangle^2 = \langle e_j, a_i \rangle^2 + \langle e_j, b_i \rangle^2 + 2 \langle e_j, a_i \rangle \langle e_j, b_i \rangle . \tag{2.21}$$

If T is the orthogonal projection onto the kernel of B, then

$$|\langle e_j, a_i \rangle| \leq ||Te_j|| \leq \sqrt{\nu k}||Te_j||_\infty.$$

Applying Lemma II.7,

$$|\langle e_j, a_i \rangle| \leq \frac{\sqrt{\nu}}{\omega^{2\nu-1}}k^{-1/2}.$$

Also,

$$|\langle e_j, b_i \rangle| \leq ||b_i|| \leq \sqrt{\frac{R}{\omega^\nu r}}k^{-1/2}.$$

Combining these into Equation (2.21) yields

$$\langle e_j, v_i \rangle^2 \leq \left(\frac{\nu}{\omega^{2\nu-1}} + \frac{R}{\omega^\nu r} + 2\sqrt{\frac{\nu R}{\omega^{5\nu-2}r}} \right) k^{-1}.$$

Simplifying the bound to be

$$\langle e_j, v_i \rangle^2 \leq \frac{4\nu R}{\omega^{5\nu-2} r} k^{-1},$$

plugging this into Equation (2.20) and simplifying again completes the proof. $\square$

To calculate bounds on probabilities we use the following lemma.

Lemma II.9. *Let X be a normal variable with variance σ^2 and $\mathbf{E}(X) = 0$. Then*

$$\mathbf{Pr}\left(|X| \geq \tau\right) \leq e^{-\tau^2/(2\sigma^2)}.$$

Proof. We use the well known result that if X is a standard normal variable then

$$\mathbf{Pr}(|X| \geq \tau) \leq e^{-\tau^2/2}.$$

If X has variance σ^2, then X/σ is the standard normal variable, so

$$\mathbf{Pr}(|X| \geq \tau) = \mathbf{Pr}(|X|/\sigma \geq \tau/\sigma) \leq e^{-\tau^2/(2\sigma^2)}.$$

$\square$

Combining Lemmas II.8 and II.9, along with a union bound, gives Lemma II.6.

2.3 Correlations

In this section we continue to use the notation introduced at the beginning of this chapter before Section 2.1, in particular the quadratic form $q(t)$, the subspace $\mathcal{L}$, the constants r, R, ω and k, and the definition of k'_j. If we draw

$$t = (\tau_{11}, \ldots, \tau_{1k_1}, \tau_{21}, \ldots, \tau_{2k_2}, \ldots, \tau_{\nu 1}, \ldots, \tau_{\nu k_\nu})$$

from $\mathcal{L}$ with density proportional to $e^{-q(t)}$, then we can treat the individual coordinates τ_{ij} as random variables. In this section we will calculate the correlation between pairs of coordinates. The main result is the following:

Theorem II.10. *Let $\mathcal{M}$ be any subspace of $\mathbb{R}^{k_1+\ldots+k_\nu}$ of codimension $\nu-1$ such that $\mathcal{M}\cap\ker(q)=\{0\}$. Suppose that $t\in\mathcal{M}$ is drawn from the distribution with density proportional to $e^{-q(t)}$ restricted to $\mathcal{M}$. Then there exists some constant $\Gamma(r,R,\omega,\nu)>0$ such that*

$$|\mathbf{E}(\tau_{1m_1}+\tau_{2m_2}+\ldots+\tau_{\nu m_\nu})(\tau_{1p_1}+\tau_{2p_2}+\ldots+\tau_{\nu p_\nu})|\leq\frac{\Gamma}{k^{\nu-1}}$$

for all $m_1,\ldots,m_\nu,p_1,\ldots,p_\nu$, and

$$|\mathbf{E}(\tau_{1m_1}+\tau_{2m_2}+\ldots+\tau_{\nu m_\nu})(\tau_{1p_1}+\tau_{2p_2}+\ldots+\tau_{\nu p_\nu})|\leq\frac{\Gamma}{k^{\nu}}$$

as long as $m_j\neq p_j$ for all $j=1,\ldots,\nu$. The constant Γ may be chosen to be

$$\Gamma=\frac{4\nu^4R^2}{r^3\omega^{7\nu-5}}.$$

We will make use of two basic lemmas.

Lemma II.11. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be any subspaces of codimension $\nu-1$ such that $\mathcal{M}_1\cap\ker(q)=\mathcal{M}_2\cap\ker(q)=\{0\}$. Then*

$$\mathbf{E_1}(\tau_{1m_1}+\ldots+\tau_{\nu m_\nu})(\tau_{1p_1}+\ldots+\tau_{\nu p_\nu})=\mathbf{E_2}(\tau_{1m_1}+\ldots+\tau_{\nu m_\nu})(\tau_{1p_1}+\ldots+\tau_{\nu p_\nu}),$$

where $\mathbf{E_1}$ is taking the expected value over the distribution with density proportional to $e^{-q(t)}$ restricted to $\mathcal{M}_1$, and $\mathbf{E_2}$ the expected value over the distribution with density proportional to $e^{-q(t)}$ restricted to $\mathcal{M}_2$.

Proof. Let $S:\mathcal{M}_1\to\mathcal{M}_2$ be the restriction of the orthogonal projection onto $\mathcal{M}_2$

whose kernel is $\ker(q)$:

$$St = t + u \quad \text{with} \quad u \in \ker(q) \quad \text{for all} \quad t \in \mathcal{M}_1.$$

As $\det(q|_{\mathcal{M}_1}) \det(S) = \det(q|_{\mathcal{M}_2})$, and $e^{-q(t)} = e^{-q(St)}$, we get that the push forward of the probability measure with density proportional to $e^{-q(t)}$ restricted to $\mathcal{M}_1$ by S is equal to the probability measure with density proportional to $e^{-q(t)}$ restricted to $\mathcal{M}_2$. Furthermore, $(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})$ for any $m_1, \ldots, m_\nu$ is unchanged when replacing t by St. Therefore

$$\mathbf{E_1}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})(\tau_{1p_1} + \ldots + \tau_{\nu p_\nu}) = \mathbf{E_2}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})(\tau_{1p_1} + \ldots + \tau_{\nu p_\nu})$$

as required. $\square$

Lemma II.12. *Let $v_1, v_2 \in \mathbb{R}^n$, and $C : \mathbb{R}^n \to \mathbb{R}^n$ be a positive definite self-adjoint linear transformation, and let there exist absolute constants γ_1, γ_2, Γ_1, Γ_2, Γ_3, and Γ_4 so that*

1.
$$||Cv_i - \gamma_i v_i|| \leq \Gamma_1 \quad \textit{for} \quad i = 1, 2,$$

2.
$$|\langle Cv_i - \gamma_i v_i, v_j \rangle| \leq \Gamma_2 \quad \textit{for} \quad i \neq j,$$

3.
$$|\langle v_1, v_2 \rangle| \leq \Gamma_3, \quad \textit{and}$$

4.
$$\lambda_n(C) \geq \Gamma_4$$

where $\lambda_n(C)$ is the smallest eigenvalue of C. Then

$$\left|\left\langle C^{-1}v_1, v_2\right\rangle\right| \leq \frac{\Gamma_3}{\gamma_1} + \frac{\Gamma_2}{\gamma_1\gamma_2} + \frac{\Gamma_1^2}{\gamma_1\gamma_2\Gamma_4}.$$

Informally, the lemma states that if v_1 and v_2 are very close to being orthogonal eigenvectors of C, then $C^{-1}v_1$ is close to being orthogonal to v_2.

Proof. We can write this expression as

$$\left|\left\langle C^{-1}v_1, v_2\right\rangle\right| = \frac{1}{\gamma_1}\left|\left\langle C^{-1}\left(\gamma_1 v_1 - Cv_1 + Cv_1\right), v_2\right\rangle\right|.$$

By linearity and the triangle inequality,

$$\left|\left\langle C^{-1}v_1, v_2\right\rangle\right| \leq \frac{1}{\gamma_1}\left|\left\langle v_1, v_2\right\rangle\right| + \frac{1}{\gamma_1}\left|\left\langle C^{-1}\left(\gamma_1 v_1 - Cv_1\right), v_2\right\rangle\right|. \tag{2.22}$$

Using that C^{-1} is self adjoint, and by linearity and the triangle inequality again we get

$$\left|\left\langle C^{-1}\left(\gamma_1 v_1 - Cv_1\right), v_2\right\rangle\right| \leq \frac{1}{\gamma_2}\left(\left|\left\langle \gamma_1 v_1 - Cv_1, v_2\right\rangle\right| + \left|\left\langle \gamma_1 v_1 - Cv_1, C^{-1}\left(\gamma_2 v_2 - Cv_2\right)\right\rangle\right|\right).$$

By conditions (1) and (4), we have

$$\left|\left\langle \gamma_1 v_1 - Cv_1, C^{-1}\left(\gamma_2 v_2 - Cv_2\right)\right\rangle\right| \leq \frac{\Gamma_1^2}{\Gamma_4}.$$

Combining this with condition (2) yields

$$\left|\left\langle C^{-1}\left(\gamma_1 v_1 - Cv_1\right), v_2\right\rangle\right| \leq \frac{\Gamma_2}{\gamma_2} + \frac{\Gamma_1^2}{\Gamma_4\gamma_2}.$$

This along with condition (3) and Equation (2.22) completes the proof.

□

We use this lemma to prove the following:

Lemma II.13. *Let $B_{\mathcal{M}}$ be the linear transformation B restricted to $\mathcal{M} = \text{im}(B)$, and let $S : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ be the orthogonal projection onto $\mathcal{M}$. Then there exist constants $\kappa_1 = \kappa_1(r, \omega, \nu) > 0$ and $\kappa_2 = \kappa_2(r, R, \nu, \omega) > 0$ such that for any choice of i and j, we have*

$$\left|\left\langle B_{\mathcal{M}}^{-1} Se_j, Se_i \right\rangle\right| \leq \kappa_1 \delta_{ij} k^{-\nu+1} + \kappa_2 k^{-\nu}.$$

If $\nu \geq 3$, the constants may be chosen such that

$$\kappa_1 = \frac{2R}{r^2 \omega^{2\nu-2}}, \quad \textit{and} \quad \kappa_2 = \frac{7\nu^2 R^2}{r^3 \omega^{7\nu-5}}.$$

If $\nu = 2$ the same result holds, but the algebraic simplifications to arrive at κ_2 requires an extra multiplicative factor greater than 7.

Proof. We apply Lemma II.12, with $v_1 = Se_j$, $v_2 = Se_i$, and $C = B_{\mathcal{M}}$. By Lemma II.4, we get condition (4) is satisfied with

$$\lambda_{k_1+\ldots+k_\nu-\nu+1}\left(B_{\mathcal{M}}\right) \geq \Gamma_4 = \omega^{\nu-1} r k^{\nu-1}. \tag{2.23}$$

By Lemma II.7, we can write

$$Se_j = e_j + w_j, \quad \text{where} \tag{2.24}$$

$$||w_j||_\infty \leq \frac{1}{\omega^{2\nu-1}} k^{-1}.$$

This gives condition (3) of Lemma II.7,

$$|\langle Se_j, Se_i \rangle| \leq \Gamma_3, \quad \text{with}$$

$$\Gamma_3 = \delta_{ij} + \frac{2}{\omega^{2\nu-1}}k^{-1} + \frac{1}{\omega^{4\nu-2}}k^{-2}.$$

We can simplify this to be

$$\Gamma_3 = \delta_{ij} + \frac{3}{\omega^{4\nu-2}}k^{-1}. \tag{2.25}$$

By (2.2), we can see that the entries of B are bounded by the following: the diagonal entries are bounded from above by $Rk^{\nu-1}$ and the off-diagonal entries are bounded by $Rk^{\nu-2}$. Hence,

$$Be_j = \gamma_1 e_j + w'_j, \quad \text{and} \quad Be_i = \gamma_2 e_i + w'_i, \quad \text{where} \tag{2.26}$$

$$||w'_j||_\infty,\ ||w'_i||_\infty \leq Rk^{\nu-2}, \quad \text{and} \tag{2.27}$$

$$r\omega^{\nu-1}k^{\nu-1} \leq \gamma_1,\ \gamma_2 \leq Rk^{\nu-1}.$$

Therefore, using that $B_{\mathcal{M}}Se_j = BSe_j$ and applying B to (2.24),

$$B_{\mathcal{M}}Se_j = \gamma_1 e_j + w'_j + Bw_j.$$

Applying (2.24) again, we get

$$||B_{\mathcal{M}}Se_j - \gamma_1 Se_j|| \leq ||w'_j|| + ||Bw_j|| + ||\gamma_1 w_j||,$$

and similarly for e_i. By (2.27),

$$||w'_j||,\ ||w'_i|| \leq \sqrt{\nu k}Rk^{\nu-2},$$

and by (2.24) and Lemma II.4,

$$||Bw_j||,\ ||Bw_i|| \leq \sqrt{\nu k}\frac{1}{\omega^{2\nu-1}}R\nu k^{\nu-2}.$$

Also, by (2.24) and (2.27),

$$||\gamma_1 w_j||,\ ||\gamma_2 w_i|| \leq \frac{\sqrt{\nu k}R}{\omega^{2\nu-1}} k^{\nu-2}.$$

Therefore we get condition (1) is satisfied with

$$\Gamma_1 = \sqrt{\nu}R\left(1 + \frac{2}{\omega^{2\nu-1}}\right) k^{\nu-1.5},$$

and γ_1, γ_2 as in (2.26). We simplify the bound to be

$$\Gamma_1 = \frac{3\sqrt{\nu}R}{\omega^{2\nu-1}} k^{\nu-1.5}. \tag{2.28}$$

Also,

$$|\langle BSe_j - \gamma_1 Se_j, Se_i\rangle| \leq ||w_j'||_\infty + ||Bw_j||_\infty + \left|\left\langle w_j', w_i\right\rangle\right| + |\langle Bw_j, w_i\rangle| + \gamma_1 \langle e_j, e_i\rangle + \gamma_1 ||w_i||_\infty.$$

Using (2.24) and the bounds on the entries of B described above we get

$$||Bw||_\infty \leq \frac{(\nu+1)R}{\omega^{2\nu-1}} k^{\nu-2}.$$

This along with (2.27), and the fact that for $u, v \in \mathbb{R}^n$ we have $|\langle u, v\rangle| \leq n||u||_\infty ||v||_\infty$, gives that condition (2) holds with

$$\Gamma_2 \leq \left(1 + \frac{\nu}{\omega^{2\nu-1}} + \frac{\nu+1}{\omega^{2\nu-1}} + \frac{\nu(\nu+1)}{\omega^{2\nu-1}} + \frac{1}{\omega^{2\nu-1}}\right) Rk^{\nu-2} + \delta_{ij} Rk^{\nu-1}.$$

We can simplify this bound to be

$$\Gamma_2 = \frac{3\nu^2 R}{\omega^{2\nu-1}} k^{\nu-2} + \delta_{ij} Rk^{\nu-1}. \tag{2.29}$$

Taking (2.23), (2.25), (2.28), (2.26), and (2.29), and applying Lemma II.12 gives us

$$\left|\left\langle B_{\mathcal{M}}^{-1}Se_j, Se_i\right\rangle\right| \le \delta_{ij}\left(\frac{1}{r\omega^{\nu-1}} + \frac{R}{r^2\omega^{2\nu-2}}\right)k^{-\nu+1} + \left(\frac{3}{r\omega^{5\nu-3}} + \frac{3\nu^2 R}{r^2\omega^{4\nu-3}} + \frac{9\nu R^2}{r^3\omega^{7\nu-5}}\right)k^{-\nu}.$$

Simplifying the coefficients completes the proof.

□

The last observation we need is:

Lemma II.14. *Let $\psi(z) : \mathbb{R}^d \to \mathbb{R}$ be a positive definite quadratic form, and let D be the positive definite matrix such that $\psi(z) = \frac{1}{2}\langle z, Dz\rangle$. Let $l_1(z) = \langle v_1, z\rangle$ and $l_2(z) = \langle v_2, z\rangle$ for some fixed $v_1, v_2 \in \mathbb{R}^d$. If z is drawn from the distribution with density proportional to $e^{-\psi(z)}$, then*

$$\mathbf{E}(l_1(z)l_2(z)) = \left\langle v_1, D^{-1}v_2\right\rangle.$$

Proof. Let

$$v_i = (v_i^1, \ldots, v_i^d) \quad \text{for} \quad i = 1, 2.$$

By linearity of expectation we can write

$$\mathbf{E}(l_1(z)l_2(z)) = \sum_{i,j=1}^{d} v_1^i v_2^j \mathbf{E}(z^i z^j) \quad \text{for} \quad z = (z^1, \ldots, z^d).$$

D^{-1} is exactly the matrix whose entries are $\mathbf{E}(z^i z^j)$, so the sum composes into $\langle v_1, D^{-1}v_2\rangle$ which completes the proof. □

We are now ready to prove Theorem II.10. For notational purposes it will be convenient to write t as

$$t = (\chi_1, \ldots, \chi_{k_1+k_2+\ldots+k_\nu}),$$

so for example $\tau_{11} = \chi_1$ and $\tau_{22} = \chi_{k_1+2}$.

Proof. By Lemma II.11, it suffices to prove the result only for $\mathcal{M} = \ker(q)^{\perp}$. By Lemma II.14, if t is drawn from $\mathcal{M}$ with distribution with density proportional to $e^{-q(t)}$ then for $t = (\chi_1, \ldots, \chi_{k_1+k_2+\ldots+k_\nu})$,

$$\mathbf{E}(\chi_i \chi_j) = \left\langle Se_i, B_{\mathcal{M}}^{-1} Se_j \right\rangle .$$

We apply Lemma II.13, noting that we can simplify the bound to be

$$\left| \left\langle B_{\mathcal{M}}^{-1} Se_j, Se_i \right\rangle \right| \leq \frac{14\nu^2 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu+\delta_{ij}}.$$

We distribute and use the linearity of expectation and the triangle inequality to get

$$|\mathbf{E}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})(\tau_{1p_1} + \ldots + \tau_{\nu p_\nu})| \leq \sum_{i,j=1}^{\nu} \mathbf{E} \left| \tau_{im_i} \tau_{jp_j} \right| .$$

In the event that at least one $m_j = p_j$, we can bound all ν^2 terms of the form $|\mathbf{E}(\chi_i \chi_l)|$ by

$$\frac{14\nu^2 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu+1},$$

giving

$$|\mathbf{E}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})(\tau_{1p_1} + \ldots + \tau_{\nu p_\nu})| \leq \frac{14\nu^4 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu+1}.$$

If there is no j such that $m_j = p_j$, we can bound each expression of the form $|\mathbf{E}(\chi_i \chi_l)|$ by

$$\frac{14\nu^2 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu},$$

giving

$$|\mathbf{E}(\tau_{1m_1} + \ldots + \tau_{\nu m_\nu})(\tau_{1p_1} + \ldots + \tau_{\nu p_\nu})| \leq \frac{14\nu^4 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu}.$$

This completes the proof. $\square$

2.4 The Third Degree Term

In this section we use the notation and concepts introduced at the beginning of the chapter before Section 2.1, in particular the quadratic form $q(t)$, the subspace $\mathcal{L}$, and the constants r, R, ω and k. The main result of this section is:

Lemma II.15. *Assume $\nu \geq 3$. Let*

$$u_{m_1 \dots m_\nu} = \beta_{m_1,\dots,m_\nu}(\tau_{1m_1} + \dots + \tau_{\nu m_\nu})$$

be random variables for $1 \leq m_j \leq k_j$ for each $j = 1, \dots, \nu$, where $t = (\tau_{11}, \tau_{12}, \dots \tau_{\nu k_\nu})$ is drawn from the distribution with probability density proportional to $e^{-q(t)}$ restricted to $\mathcal{L}$. Let $\theta > 0$ be chosen such that

$$|\beta_{m_1,\dots,m_\nu}| \leq \theta \quad \textit{for all } m_1, \dots, m_\nu,$$

and let

$$U = \sum_{m_1,\dots,m_\nu=1}^{k_1,\dots,k_\nu} u^3_{m_1 \dots m_\nu}.$$

Then there exists a constant $\Gamma = \Gamma(\theta, \nu, \omega, R, r) > 0$ such that

$$\left|\mathbf{E} e^{iU} - 1\right| \leq \Gamma k^{2-\nu}.$$

The constant Γ may be chosen to be

$$\Gamma = \frac{3360 \theta^6 \nu^{13} R^6}{r^9 \omega^{21\nu - 15}}.$$

We will apply this lemma in the proof of Theorems I.1 and I.2. In the proof of these theorems, we will show the points to be counted can be expressed as the integral of a function $F(t)$ (different for each theorem). We will construct a neighborhood, which in the proof will be called X_1, of the origin in which we can use Taylor polynomial approximations to express F as $F(t) = e^{-q(t)+if(t)+h(t)}$, where $f(t)$ is a pure cubic function in the form of U in Lemma II.15, and $h(t)$ is small. We will also show that the asymptotically all of the integral of $e^{-q(t)}$ is contained in X_1. Combining these will allow us to approximate $\int_{X_1} F(t)dt$, and show it is asymptotically equal to $\int_{\mathcal{L}} e^{-q(t)}$. The proof of Theorem I.3 will proceed similarly but not use Lemma II.15.

A version of Lemma II.15 exists for $\nu = 2$ as well. In this case the upper bound does not go to zero as k goes to infinity. This qualitative difference introduces an extra factor called the Edgeworth correction into the formula for counting integer points in 2-way transportation polytopes, see [BH12].

The proof of Lemma II.15 relies on a more general result based on Wick's formula, see for example [Zv97], for the expected value of a product of Gaussian random variables. Let $w_1, \ldots, w_l$ be Gaussian random variables with expected value of 0. Then

$$\mathbf{E}(w_1 \ldots w_l) = 0 \quad \text{if } l \text{ is odd, and}$$

$$\mathbf{E}(w_1 \ldots w_l) = \sum \left(\mathbf{E} w_{i_1} w_{i_2}\right) \ldots \left(\mathbf{E} w_{i_{l-1}} w_{i_l}\right) \quad \text{if } l \text{ is even,}$$

where the sum is taken over all unordered pairings of the set of indices $1, 2, \ldots, l$. In particular,

$$\mathbf{E} w_1^3 w_2^3 = 9\left(\mathbf{E} w_1^2\right)\left(\mathbf{E} w_2^2\right)\left(\mathbf{E} w_1 w_2\right) + 6\left(\mathbf{E} w_1 w_2\right)^3. \tag{2.30}$$

Note that the random variables $u_{m_1,\ldots,m_\nu}$ are Gaussian random variables by construction. We are now ready to prove Lemma II.15.

Proof. By Theorem II.10,

$$|\mathbf{E} u_{m_1,\ldots,m_\nu} u_{p_1,\ldots,p_\nu}| \leq \frac{14\theta^2 \nu^4 R^2}{r^3 \omega^{7\nu-5}} k^{-\nu+1} \quad \text{for all} \quad m_1, \ldots, m_\nu, p_1, \ldots, p_\nu, \quad \text{and}$$

$$|\mathbf{E}u_{m_1,\ldots,m_\nu}u_{p_1,\ldots,p_\nu}| \le \frac{14\theta^2\nu^4R^2}{r^3\omega^{7\nu-5}}k^{-\nu} \quad \text{if} \quad m_j \ne p_j \quad \text{for all} \quad 1 \le j \le \nu.$$

By (2.30), with $w_1 = u_{m_1,\ldots,m_\nu}$ and $w_2 = u_{p_1,\ldots,p_\nu}$,

$$\left|\mathbf{E}(u^3_{m_1\ldots m_\nu}u^3_{p_1\ldots p_\nu})\right| \le \frac{3360\theta^6\nu^{12}R^6}{r^9\omega^{21\nu-15}}k^{-3\nu+2} \quad \text{for all} \quad m_1,\ldots,m_\nu,p_1,...,p_\nu, \quad \text{and}$$

$$\left|\mathbf{E}(u^3_{m_1\ldots m_\nu}u^3_{p_1\ldots p_\nu})\right| \le \frac{3360\theta^6\nu^{12}R^6}{r^9\omega^{21\nu-15}}k^{-3\nu+1} \quad \text{if} \quad m_j \ne p_j \quad \text{for} \quad j = 1,\ldots,\nu.$$

There are no more than $k^{2\nu}$ total choices of $m_1,\ldots,m_\nu,p_1,\ldots,p_\nu$, and no more than $\nu k^{2\nu-1}$ of them in which there exists j such that a pair m_j and p_j are equal, so

$$\mathbf{E}U^2 \le \frac{3360\theta^6\nu^{12}(\nu+1)R^6}{r^9\omega^{21\nu-15}}k^{-\nu+2}. \tag{2.31}$$

By the Taylor series estimate

$$\left|e^{i\xi} - (1+i\xi)\right| \le \frac{1}{2}\xi^2 \quad \text{for} \quad \xi \in \mathbb{R},$$

along with the triangle inequality for expected values, we get that

$$\left|\left(\mathbf{E}e^{iU}\right) - 1\right| \le \frac{1}{2}\mathbf{E}U^2 \ \text{ yields}$$

$$\left|\mathbf{E}\left(e^{iU}\right) - 1\right| \le \frac{1680\theta^6\nu^{12}(\nu+1)R^6}{r^9\omega^{21\nu-15}}k^{-\nu+2}.$$

Applying the simplification $\nu + 1 \le 2\nu$ completes the proof. $\square$

CHAPTER III

Proof of Theorem I.1

In this chapter, we complete the proof of Theorem I.1. For the entirety of this chapter we use the notation introduced in the statement of the theorem, most importantly the quadratic form $q(t)$ and the constants r, R, ω and k. We also recall the overdetermined system of equations for a multi-index transportation polytope of the form $Ax = b$, where A has columns $a_1, \ldots, a_n$ as described in Section 1.1, along with the subspace $\mathcal{L}$ that describes a linearly independent set of equations. The matrix $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ will be the orthogonal projection onto $\mathcal{L}$. The outline of the proof is as follows: we construct a function $F(t)$, and show that for a multi-index transportation polytope P as in Theorem I.1,

$$|P \cap \mathbb{Z}^n| = \frac{e^{g(z)}}{(2\pi)^{(k_1+\ldots+k_\nu-\nu+1)/2}} \int\limits_{\Pi} F(t)dt,$$

where $\Pi = \mathcal{L} \cap [-\pi, \pi]^{k_1+\ldots+k_\nu}$. We then split Π up into three regions: an outside region X_3, a middle region X_2, and an inner region X_1. We show that

$$\int\limits_{X_2 \cup X_3} F(t)dt \quad \text{and} \quad \int\limits_{\mathcal{L} \setminus X_1} e^{-q(t)}dt$$

are negligible compared $\int_{\mathcal{L}} e^{-q(t)}dt$. We show through use of Taylor polynomial approximations that in X_1, $F(t) \approx e^{-q(t)+if(t)+h(t)}$, where $h(t)$ is small in X_1, and $f(t)$ is a cubic polynomial in t of the form given in Lemma II.15. We finish the proof by applying Lemma II.15 to show that

$$\int\limits_{X_1} \left|F(t) - e^{-q(t)}\right| dt \ll \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

3.1 Integral Expression of the Counting Problem

We use two results of [BH10] to express the number of integer points of P as an integral of a function $F(t)$. Let $\Pi \subset \mathcal{L}$ be the cube centered at the origin:

$$\Pi = \{t \in \mathcal{L} \ : \ ||t||_\infty \leq \pi\}.$$

We will show that for multi-index transportation polytopes P satisfying the conditions of Theorem I.1, the number of integer points satisfies

$$|P \cap \mathbb{Z}^n| = \frac{e^{g(z)}}{(2\pi)^{k_1+\ldots+k_\nu-\nu+1}} \int\limits_{\Pi} e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j,t\rangle}} dt. \tag{3.1}$$

Before we do, we recall the concept of a geometric random variable. We say x is a geometric random variable if for some $0 < p < 1$,

$$\mathbf{Pr}(x = j) = (1-p)p^j \quad \text{for all} \quad j \in \mathbb{Z}_{\geq 0}.$$

In this case, $\mathbf{E}x = \frac{p}{1-p}$. Conversely, if $\mathbf{E}x = \zeta$, then $p = \frac{\zeta}{1+\zeta}$. The first theorem we need is the following:

Theorem III.1. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace in $\mathbb{R}^n$ and the non-negative orthant $\mathbb{R}^n_+$. Suppose that P is bounded and has a non-empty interior,* *that is a point $y = (\eta_1, \ldots, \eta_n)$ where $\eta_i > 0$ for $i = 1, \ldots, n$. Then the strictly concave function*

$$g(x) = \sum_{j=1}^{n} \left((\xi_j + 1) \ln(\xi_j + 1) - \xi_j \ln(\xi_j) \right)$$

attains its maximum value in P at a unique point $z = (\zeta_1, \ldots, \zeta_n)$ such that $\zeta_j > 0$ for $j = 1, \ldots, n$. Furthermore, suppose $x_1, \ldots, x_n$ are independent geometric random variables with $\mathbf{E}x_j = \zeta_j$, and let $X = (x_1, \ldots, x_n)$. Then the probability mass function of X is constant on $P \cap \mathbb{Z}^n$ and equal to $e^{-g(z)}$ at every $x \in P \cap \mathbb{Z}^n$. In particular,

$$|P \cap \mathbb{Z}^n| = e^{g(z)} \mathbf{Pr}\left(X \in P \right).$$

This is Theorem 4 of [BH10]. This theorem lets us reduce counting the number of integer points in P to calculating $\mathbf{Pr}\left(X \in P \right)$. We combine this result with the following:

Lemma III.2. *Let p_j, q_j be positive numbers such that $p_j + q_j = 1$ for $j = 1, \ldots, n$ and let μ be the geometric measure on the set $\mathbb{Z}_+^n$ of non-negative integer vectors with*

$$\mu\{x\} = \prod_{j=1}^{n} p_j q_j^{\xi_j} \quad \text{for} \quad x = (\xi_1, \ldots, \xi_n).$$

Let P be defined by the linear equalities $Ax = b$, where A has columns $a_1, \ldots, a_n$, and $a_1, \ldots, a_n, b \in \mathbb{R}^d$. Let $\Pi = [-\pi, \pi]^d$ be a cube centered at the origin in $\mathbb{R}^d$. Then

$$\mu\left(P\right) = \frac{1}{(2\pi)^d} \int\limits_{\Pi} e^{-i\langle t, b \rangle} \prod_{j=1}^{n} \frac{p_j}{1 - q_j e^{i\langle a_j, t \rangle}} dt.$$

Here, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$ and dt is the Lebesgue measure.

This is Lemma 13 of [BH10]. We combine this with Theorem III.1 to derive (3.1) in the following way: we identify $\mathcal{L}$ with $\mathbb{R}^{k_1 + \ldots + k_\nu - \nu + 1}$ in the natural way by identifying the non-zero coordinates of L with the coordinates of $\mathbb{R}^{k_1 + \ldots + k_\nu - \nu + 1}$. Then P is defined by the linear equations $QAx = Qb$ where Q is the orthogonal projection onto $\mathcal{L}$. We note that for $t \in \mathcal{L}$, $\langle Qa_j, t\rangle = \langle a_j, t\rangle$ and $\langle Qb, t\rangle = \langle b, t\rangle$ so we can write the integrand using the vectors $a_1, \ldots, a_n, b$ instead of $Qa_1, \ldots, Qa_n, b$. The random variable X in Theorem III.1 has probability mass function equal to the geometric measure μ in Lemma III.2 when $\zeta_j = q_j/p_j = (1-p_j)/p_j$. This turns the integrand of Lemma III.2 into

$$e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j,t\rangle}},$$

which proves (3.1). Let

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j,t\rangle}}.$$

The bulk of the proof is dedicated to showing that

$$\int_{\Pi} F(t)dt \approx \int_{\mathcal{L}} e^{-q(t)}dt.$$

3.2 A Bound on $F(t)$ Away from the Origin

The main result of this section is the following:

Lemma III.3. *Let*

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j,t\rangle}}.$$

Then there exists a constant $\gamma = \gamma(\omega, \nu, R) > 0$ such that

$$|F(t)| \leq \exp\left(-\gamma ||t||_\infty^2 k^{\nu-1}\right) \quad \text{for all} \quad t \in \mathcal{L}.$$

If we restrict t such that

$$\frac{\omega^\nu ||t||_\infty^2}{\pi^2 2^{\nu-1}\nu^3} k^{\nu-1} \geq 2,$$

then γ may be chosen to be

$$\gamma = \frac{\omega^\nu}{\pi^2 2^\nu \nu^3} \ln\left(1 + \frac{2}{5}\pi^2 r\right).$$

We apply this lemma in the following way: we construct a region, which in the proof will be called X_3, which is the complement of a neighborhood of the origin in Π. Then we use Lemma III.3 and Lemma II.6 to show that

$$\left|\int_{X_3\cap\Pi} F(t)dt\right|, \int_{X_3} e^{-q(t)} \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

In $\Pi \setminus X_3$ we will then be able to express $F(t)$ as $F(t) = e^{-q(t)+if(t)+h(t)}$ and show that $f(t)$ and $h(t)$ have a negligible effect on the integral.

To prove Lemma III.3 we use the following:

Lemma III.4. *Let D be a $d \times n$ integer matrix with columns $d_1, \ldots d_n \in \mathbb{Z}^d$. For each $1 \le l \le d$, let $Y_l \subset \mathbb{Z}^n$ be a non-empty finite set such that for all $y \in Y_l$, we have $Dy = e_l$, where e_l is the lth standard basis vector. Let $\psi_l : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form*

$$\psi_l(x) = \frac{1}{|Y_l|}\sum_{y\in Y_l} \langle y, x\rangle^2 \quad \text{for} \quad x \in \mathbb{R}^n,$$

and let ρ_l be a constant such that

$$\psi_l(x) \le \rho_l ||x||^2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Suppose further that for $\zeta_1, \ldots, \zeta_n > 0$ we have

$$\zeta_j + \zeta_j^2 \ge \alpha \quad \text{for some} \quad \alpha > 0 \quad \text{and} \quad j = 1, \ldots, n.$$

Then for any $t = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d$, and for each l, we have

$$\left| \prod_{j=1}^{n} \frac{1}{1 + \zeta_j - \zeta_j e^{i\langle d_j, t \rangle}} \right| \leq \left(1 + \frac{2}{5}\alpha\pi^2\right)^{-\gamma_l} \quad \text{where} \quad \gamma_l = \left\lfloor \frac{\tau_l^2}{\pi^2 \rho_l} \right\rfloor.$$

This is Lemma 14 of [BH10]. We are now ready to prove Lemma III.3. We do so by constructing arrays Y_l for each $e_l \in \mathcal{L}$, following a similar construction in a different coordinate system presented in [BH10]. We then find a uniform bound on ρ_l and apply Lemma III.4 to every coordinate uniformly.

Proof. We identify L with $\mathbb{R}^{k_1 + \ldots + k_\nu - \nu + 1}$ in the natural way, and construct a set Y_l for each $e_l \in L$.

For fixed $1 \leq p < k_2$, let Y_{k_1+p} (corresponding to a margin in the second direction) be the set of hypercube arrays labeled with $m_1, m_3, m_4, \ldots, m_\nu$ with $1 \leq m_j \leq k_j$ for each $j \neq 2$, and let $y_{m_1 m_3 m_4 \ldots m_\nu}$ that have a 1 in the $m_1 p m_3 \ldots m_\nu$ position and a -1 in the $m_1 k_2 m_3 \ldots m_\nu$ position, and a 0 in every other position. There are k_2' such arrays, and the corresponding quadratic form is

$$\psi_{k_1+p}(x) = \frac{1}{k_2'} \sum_{m,p} \left(\xi_{m_1 p m_2 \ldots m_\nu} - \xi_{m_1 k_2 m_3 \ldots m_\nu}\right)^2.$$

No two terms $\left(\xi_{m_1 p m_2 \ldots m_\nu} - \xi_{m_1 k_2 m_3 \ldots m_\nu}\right)^2$ of the above sum share any variables, so the eigenvalues of ψ_{k_1+p} are simply the non-zero eigenvalues of the simpler quadratic forms

$$\frac{1}{k_2'} \left(\xi_{m_1 p m_2 \ldots m_\nu} - \xi_{m_1 k_2 m_3 \ldots m_\nu}\right)^2,$$

along with 0. The eigenvalues of this quadratic form are $\frac{2}{k_2'}$. Furthermore, for every $y \in Y_{k_1+p}$, we have $Ay = e_{k_1+p} - e_{k_1+k_2}$, so $QAy = e_{k_1+p}$.

Similarly for fixed $1 \leq p < k_3$, let $Y_{k_1+k_2+p}$ (corresponding to a margin in the third direction) be the set of all hypercubes labeled by $m_1 m_2 m_4 \ldots m_\nu$ with $1 \leq m_j \leq k_j$

for all $j \neq 3$, and let $y_{m_1 m_2 p m_4 \ldots m_\nu}$ have a 1 in an $m_1 m_2 p m_4 \ldots m_\nu$ position and a -1 in the $m_1 m_2 k_3 m_4 \ldots m_\nu$ position, and a 0 in every other position. There are k_3' such arrays, the corresponding quadratic form has largest eigenvalue $\frac{2}{k_3'}$, and for each $y \in Y_{k_1+k_2+p}$, we have $Ay = e_{k_1+k_2+p} - e_{k_1+k_2+k_3}$, so $QAy = e_{k_1+k_2+p}$.

This process can be repeated for any $2 \leq j \leq \nu$ and $1 \leq p < k_j$ to get an array $Y_{k_1+\ldots+k_{j-1}+p}$ of hypercubes such that the corresponding quadratic form has largest eigenvalue $\frac{2}{k_j'}$ and for all $y \in Y_{k_1+\ldots+k_{j-1}+p}$, we have $QAy = e_{k_1+\ldots+k_{j-1}+p}$.

We now construct Y_p corresponding to any margin in the first direction. For any choice of $m_1, m_2, m_3, \ldots, m_\nu$ with $1 \leq m_j < k_j$ for each $2 \leq j \leq \nu$ and $1 \leq m_1 \leq k_1$ with $m_1 \neq p$, let $y_{m_1 m_2 \ldots m_\nu}$ be the array which contains a $-(\nu - 1)$ in the $m_1 m_2 m_3 \ldots m_\nu$ position, a 1 in the $p m_2 m_3 \ldots m_\nu$ position, and a 1 in every $m_1 m_2 \ldots m_{j-1} k_j m_{j+1} \ldots m_\nu$ position. Then the sum over every margin except for the pth margin in the first direction and the last margin in every other direction are zero, and the sum over the pth margin in the first direction is 1. Therefore for all $y \in Y_p$, we have $QAy = e_p$ as required. Furthermore there are $\prod_j (k_j - 1)$ such points, and the corresponding quadratic form is

$$\psi_p(x) = \prod_{j=1}^{\nu} \frac{1}{k_j - 1} \sum_{m_1, \ldots m_\nu} \left((1-\nu)\xi_{m_1 \ldots m_\nu} + \xi_{p m_2 \ldots m_\nu} + \xi_{p k_2 m_3 \ldots m_\nu} + \ldots + \xi_{p m_2 \ldots m_{\nu-1} k_\nu} \right)^2 .$$

In general for real numbers $\gamma_1, \ldots, \gamma_{\nu+1}$

$$\left(\sum_{i=1}^{\nu+1} \gamma_i \right)^2 \leq (\nu + 1) \sum_{i=1}^{\nu+1} \gamma_i^2 ,$$

so

$$\psi_p(x) \leq \frac{\nu+1}{\prod_{j=1}^{\nu}(k_j-1)^2} \sum_{m_1\neq p,\ldots,m_\nu} (\nu-1)^2\xi_{m_1\ldots m_\nu}^2 + \xi_{pm_2\ldots m_\nu}^2 + \xi_{pk_2m_3\ldots m_\nu}^2 + \ldots + \xi_{pm_2\ldots m_{\nu-1}k_\nu}^2.$$

This latter quadratic form has as its eigenvectors the standard unit basis vectors, and the largest eigenvalue it has is bounded by

$$\frac{(\nu-1)^2(\nu+1)}{\prod_{j=1}^{\nu}(k_j-1)} \max_{j=1,\ldots,\nu} \{k_j - 1\}.$$

The subspace $\mathcal{L}$ is spanned by all the standard basis vectors with the exception of $e_{k_1+\ldots+k_j}$ for each $j = 1, \ldots, \nu$. For every other e_l we have constructed a set Y_l and a corresponding quadratic form ψ_l with maximum eigenvalues all no larger than

$$\frac{(\nu-1)^2(\nu+1)(k-1)}{(\omega k-1)^\nu}$$

satisfying the hypothesis of Lemma III.4. Furthermore, if λ_l is the largest eigenvalue of ψ_l as defined in Section 1.2, then $\rho_l = \frac{1}{2}\lambda_l$ satisfies the hypothesis of Lemma III.4. Assuming $\omega k - 1 \geq \omega k/2$ and noting $(\nu-1)(\nu+1) \leq \nu^2$, we can simplify this bound to

$$\rho_l \leq \frac{\nu^3 2^{\nu-1}}{\omega^\nu} k^{-\nu+1}.$$

Applying Lemma III.4 uniformly over all values of l with $D = QA$, and observing we can let $\alpha = r$, we arrive at

$$|F(t)| \leq \left(1 + \frac{2}{5} r\pi^2\right)^{-\gamma}, \quad \text{where}$$

$$\gamma = \left\lfloor \frac{||t||_\infty^2}{\pi^2} \frac{\omega^\nu}{\nu^3 2^{\nu-1}} k^{\nu-1} \right\rfloor.$$

As long as

$$\frac{||t||_\infty^2}{\pi^2} \frac{\omega^\nu}{\nu^3 2^{\nu-1}} k^{\nu-1} \geq 2,$$

we can apply the inequality

$$\left\lfloor \frac{||t||_\infty^2}{\pi^2} \frac{\omega^\nu}{\nu^3 2^{\nu-1}} k^{\nu-1} \right\rfloor \geq \frac{1}{2} \left(\frac{||t||_\infty^2}{\pi^2} \frac{\omega^\nu}{\nu^3 2^{\nu-1}} \right) k^{\nu-1}.$$

This completes the proof. $\square$

3.3 The Proof of Theorem I.1

At this point we are ready to prove Theorem I.1. The outline of the proof is as follows: we first construct a region $X_3 \subset \mathcal{L}$ which is of the form

$$X_3 = \{t \in \mathcal{L} \ : \ ||t||_\infty \geq \beta\}$$

for some $\beta \in \mathbb{R}$. We apply Lemma III.3 to show that

$$\int\limits_{X_3 \cap \Pi} |F(t)| dt \ll \int\limits_{\mathcal{L}} e^{-q(t)} dt.$$

For $||t||_\infty < \beta$, we express $F(t)$ as

$$F(t) = e^{-q(t) - if(t) + h(t)},$$

where $q(t)$ is the quadratic form as in Theorem I.1, $f(t)$ is a cubic polynomial, and $h(t)$ is bounded by a quartic polynomial. We use Lemma II.6 along with an inequality comparing $q(t)$ to $h(t)$ to show that for some set $X_2 \subset \mathcal{L}$ of the form

$$X_2 = \{t \in \mathcal{L} \ : \ \delta \leq ||t||_\infty \leq \beta\},$$

we have

$$\int_{X_2} |F(t)|dt \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

We also use Lemma II.6 to show that

$$\int_{X_2 \cup X_3} e^{-q(t)}dt \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

We then let

$$X_1 = \{t \in \mathcal{L} \ : \ ||t||_\infty \leq \delta\}.$$

We show that $|h(t)|$ is small for all $t \in X_1$, and then use Lemma II.15 to show that

$$\left| \int_{X_1} F(t) - e^{-q(t)}dt \right| \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

Combining the calculations over the three regions X_1, X_2, X_3 will allow us to show that

$$\int_{\Pi} F(t)dt \approx \int_{\mathcal{L}} e^{-q(t)}dt.$$

Proof. By (3.1) and (1.2), it suffices to show that

$$\left| \int_{\mathcal{L}} e^{-q(t)}dt - \int_{\Pi} F(t) \right| \leq \Gamma k^{-\nu+2.5}$$

for some constant Γ. Let

$$X_3 = \left\{ t \in \mathcal{L} \ : \ ||t||_\infty^2 \geq \frac{\pi^2 2^\nu \nu^3}{\omega^\nu \ln\left(1 + \frac{2}{5}\pi^2 r\right)} \left(\frac{1}{2}\nu^2 k \ln(k) + \frac{1}{2}\nu k \ln(R) \right) k^{-\nu+1} \right\}. \quad (3.2)$$

By Lemma III.3, observing that

$$2\left(\frac{1}{2}\nu^2 k \ln(k) + \frac{1}{2}\nu k \ln(R) \right) \geq 2$$

always holds as $k \geq 2$, $\nu \geq 3$ and $R \geq 1$, we have

$$\int_{X_3 \cap \Pi} |F(t)| dt \leq (2\pi)^{\nu k} \exp\left(-\frac{1}{2}\nu^2 k \ln(k) - \frac{1}{2}\nu k \ln(R)\right).$$

By Corollary II.5 , we have

$$\int_{X_3 \cap \Pi} |F(t)| dt \leq \exp\left(-\frac{1}{4}\nu^2 k \ln(k) + \nu k \ln(2\pi)\right) \int_{\mathcal{L}} e^{-q(t)} dt, \qquad (3.3)$$

which is negligible compared to $\int_{\mathcal{L}} e^{-q(t)} dt$.

For the middle and inside regions, we can use the Taylor polynomial estimate

$$\left| e^{i\xi} - 1 - i\xi + \frac{\xi^2}{2} + i\frac{\xi^3}{6} \right| \leq \frac{\xi^4}{24} \quad \text{for all} \quad \xi \in \mathbb{R}$$

to write

$$e^{i\langle a_j, t\rangle} = 1 + i\langle a_j, t\rangle - \frac{\langle a_j, t\rangle^2}{2} - i\frac{\langle a_j, t\rangle^3}{6} + g_j(t)\langle a_j, t\rangle^4,$$

where $|g_j(t)| \leq \frac{1}{24}$ for all $j = 1, \ldots, n$ for $n = k_1 \times k_2 \times \ldots \times k_\nu$. Therefore

$$F(t) = e^{-i\langle b, t\rangle} \prod_{j=1}^{n} \left(1 - \zeta_j + i\zeta_j \langle a_j, t\rangle - \zeta_j \frac{\langle a_j, t\rangle^2}{2} - i\zeta_j \frac{\langle a_j, t\rangle^3}{6} + \zeta_j g_j(t) \langle a_j, t\rangle^4\right)^{-1}.$$

Furthermore, using

$$\left| \ln(1+\xi) - \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} \right| \leq \frac{|\xi|^4}{2} \quad \text{for all complex} \quad |\xi| \leq 1/2,$$

plus

$$\sum_{j=1}^{n} \zeta_j a_j = b,$$

we can write

$$F(t) = e^{-q(t)-if(t)+h(t)}, \quad \text{where}$$

$$q(t) = \frac{1}{2} \sum_{m_1,\dots,m_\nu} \left(\zeta^2_{m_1\dots m_\nu} + \zeta_{m_1\dots m_\nu}\right) \left(\tau_{m_1 1} + \tau_{m_2 2} + \dots + \tau_{m_\nu \nu}\right)^2,$$

$$f(t) = \frac{1}{6} \sum_{m_1,\dots,m_\nu} \left(\zeta_{m_1,\dots,m_\nu} + \zeta^2_{m_1,\dots,m_\nu}\right) \left(2\zeta_{m_1,\dots,m_\nu} + 1\right) \left(\tau_{m_1 1} + \tau_{m_2 2} + \dots + \tau_{m_\nu \nu}\right)^3,$$

is a cubic polynomial of the form in Section 2.4, and

$$|h(t)| \leq 2 \sum_{m_1\dots m_\nu} \left(1 + \zeta^4_{m_1\dots m_\nu}\right) \left(\tau_{m_1 1} + \tau_{m_2 2} + \dots + \tau_{m_\nu \nu}\right)^4. \tag{3.4}$$

This expansion is valid as long as $||t||_\infty \leq 1/(2\nu\sqrt{R})$. For $t \in \Pi \setminus X_3$, this inequality is true as long as

$$\frac{\pi^2 2^\nu \nu^3}{\omega^\nu \ln\left(1 + \frac{2}{5}\pi^2 r\right)} \left(\frac{1}{2}\nu^2 k \ln(k) + \frac{1}{2}\nu k \ln(R)\right) k^{-\nu+1} \leq \frac{1}{4\nu^2 R},$$

which is assumed by hypothesis. Let

$$X_2 = \left\{ t \in \mathcal{L} \ : \ \frac{\pi^2 2^\nu \nu^5}{\omega^\nu} k^{-\nu+1.25} \leq ||t||^2_\infty \leq \frac{\pi^2 2^\nu \nu^5 R}{\omega^\nu r} \ln(k) k^{-\nu+2} \right\}.$$

We have simplified the upper bound on $||t||_\infty$ from the X_3 lower bound by making it strictly larger, using $\ln(R)/\ln(1 + 2\pi^2 r/5) \leq R/r$, and $\nu \ln(k) \geq 2$ as long as $k \geq 2$ and $\nu \geq 3$. Then we get

$$\left| \int_{X_2} F(t) dt \right| \leq \int_{X_2} |F(t)| dt = \int_{X_2} e^{-q(t)+h(t)} dt.$$

As

$$(\tau_{m_1 1} + \dots + \tau_{m_\nu \nu})^2 \leq \frac{\pi^2 2^\nu \nu^7 R}{\omega^\nu r} \ln(k) k^{-\nu+2} \quad \text{for} \quad t \in X_2, \quad \text{and}$$

$$\frac{1+\zeta^4_{m_1\dots m_\nu}}{\zeta^2_{m_1\dots m_\nu}+\zeta_{m_1\dots m_\nu}} \le \frac{R^2+1}{R} \le 2R \quad \text{as} \quad R \ge 1,$$

we get for $t \in X_2$ that

$$|h(t)| \le \frac{8\pi^2 2^\nu \nu^7 R^2}{\omega^\nu r} \ln(k) k^{-\nu+2} q(t).$$

Assuming as in the hypothesis of Theorem I.1 that

$$\delta = \frac{8\pi^2 2^\nu \nu^7 R^2}{\omega^\nu r} \ln(k) k^{-\nu+2} \le 3/4,$$

for $t \in X_2$ we get $|F(t)| = e^{-q(t)+h(t)} \le e^{-(1-\delta)q(t)}$. Therefore,

$$\left| \int_{X_2} F(t)dt \right| \le \int_{X_2} e^{-(1-\delta)q(t)} dt.$$

Doing the change of variables $t \mapsto (\sqrt{1-\delta})t$ we get

$$\left| \int_{X_2} F(t)dt \right| \le (1-\delta)^{-\nu k/2} \int_{\sqrt{1-\delta}X_2} e^{-q(t)} dt.$$

We use the bound

$$\left(1 - \frac{8\pi^2 2^\nu \nu^7 R^2}{\omega^\nu r} \ln(k) k^{-\nu+2}\right)^{-\nu k/2} \le \exp\left(\frac{16\pi^2 2^\nu \nu^6 R^2}{\omega^\nu r} \ln(k) k^{-\nu+3}\right),$$

and by Lemma II.6 and the choice of the lower bound in the definition of X_2, we get

$$\left| \int_{X_2} F(t)dt \right| \le \nu k \exp\left(\frac{16\pi^2 2^\nu \nu^6 R^2}{\omega^\nu r} \ln(k) k^{-\nu+3} - \frac{\pi^2 \omega^{5\nu-3} 2^\nu r^2 \nu}{2R} k^{.25}\right) \int_{\mathcal{L}} e^{-q(t)} dt. \quad (3.5)$$

Similarly, by Lemma II.6, we get

$$\int\limits_{X_2 \cup X_3} e^{-q(t)} dt \leq \nu k \exp\left(-\frac{\pi^2 \omega^{5\nu-3} 2^\nu r^2 \nu}{R} k^{.25}\right) \int\limits_{\mathcal{L}} e^{-q(t)} dt. \tag{3.6}$$

For the inner region, we define

$$X_1 = \left\{ t \in \mathcal{L} \ : \ ||t||_\infty^2 \leq \frac{\pi^2 2^\nu \nu^5}{\omega^\nu} k^{-\nu+1.25} \right\}.$$

For $t \in X_1$, the inequality $|\langle a_j, t \rangle|^4 \leq \nu^4 ||t||_\infty^4$ gives us

$$|h(t)| \leq 2R^2 \frac{\pi^4 4^\nu \nu^{10}}{\omega^{2\nu}} k^{-\nu+2.5}. \tag{3.7}$$

Hence, writing

$$\left| \int\limits_{X_1} F(t) - e^{-q(t)} dt \right| = \left| \int\limits_{X_1} e^{-q(t)+if(t)+h(t)} - e^{-q(t)} dt \right|,$$

we use the triangle inequality to get

$$\left| \int\limits_{X_1} F(t) - e^{-q(t)} dt \right| \leq \left| \int\limits_{X_1} e^{-q(t)-if(t)+h(t)} - e^{-q(t)-if(t)} dt \right| + \left| \int\limits_{X_1} e^{-q(t)-if(t)} - e^{-q(t)} dt \right|.$$

By Holder's inequality,

$$\left| \int\limits_{X_1} e^{-q(t)-if(t)+h(t)} - e^{-q(t)-if(t)} dt \right| \leq \sup_{t \in X_1} \left| e^{h(t)} - 1 \right| \int\limits_{X_1} \left| e^{-q(t)-if(t)} \right| dt.$$

Applying (3.7) yields

$$\left| \int\limits_{X_1} e^{-q(t)-if(t)+h(t)} - e^{-q(t)-if(t)} dt \right| \leq \left(\exp\left(2R^2 \frac{\pi^4 4^\nu \nu^{10}}{\omega^{2\nu}} k^{-\nu+2.5} \right) - 1 \right) \int\limits_{\mathcal{L}} e^{-q(t)} dt. \tag{3.8}$$

Furthermore, applying Lemma II.15 with $\beta^3_{m_1,\ldots,m_\nu} = (\zeta^2_{m_1,\ldots,m_\nu} + \zeta_{m_1,\ldots,m_\nu})(2\zeta_{m_1,\ldots,m_\nu} + 1) \leq 2R^{3/2}$, and noting that almost all of the measure of $e^{-q(t)}$ is contained in X_1 by (3.6) gives us

$$\left| \int\limits_{X_1} e^{-q(t)-if(t)} - e^{-q(t)} dt \right| \leq \\ \left(2\nu k \exp\left(-\frac{\pi^2 \omega^{5\nu-3} 2^\nu r^2 \nu}{R} k^{.25} \right) + \frac{13440 \nu^{13} R^9}{r^9 \omega^{21\nu-15}} k^{2-\nu} \right) \int\limits_{\mathcal{L}} e^{-q(t)} dt. \tag{3.9}$$

Combining Equations (3.3), (3.5), (3.6), (3.8) and (3.9) completes the proof. If k is large enough, the $k^{-2.5+\nu}$ term from (3.8) dominates, and doubling it gives us the example value for Γ. $\square$

CHAPTER IV

Proof of Theorem I.2

In this chapter, we complete the proof of Theorem I.2. For the entirety of this chapter we use the notation introduced in the statement of the theorem, most importantly the quadratic form $q(t)$ and the constants r, R, ω and k. We also recall the overdetermined system of equations for a multi-index transportation polytope of the form $Ax = b$, where A has columns $a_1, \ldots, a_n$ as described in Section 1.1, along with the subspace $\mathcal{L}$ that describes a linearly independent set of equations. The matrix $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ will be the orthogonal projection onto $\mathcal{L}$. The outline of the proof is as follows: we construct a function $F(t)$, and show that for a multi-index transportation polytope P as in Theorem I.2,

$$|P \cap \{0,1\}^n| = \frac{e^{g(z)}}{(2\pi)^{(k_1+\ldots+k_\nu-\nu+1)/2}} \int\limits_{\Pi} F(t)dt,$$

where $\Pi \subset \mathcal{L}$ is the set $\{t \in \mathcal{L} \ : \ ||t||_\infty \leq \pi\}$. We then split Π up into three regions: an outside region X_3, a middle region X_2, and an inner region X_1. We show that

$$\int\limits_{X_2 \cup X_3} F(t)dt \quad \text{and} \quad \int\limits_{\mathcal{L}\setminus X_1} e^{-q(t)}dt$$

are negligible compared $\int_{\mathcal{L}} e^{-q(t)}dt$. We show through use of Taylor polynomial approximations that in X_1, $F(t) \approx e^{-q(t)+if(t)+h(t)}$, where $h(t)$ is small in X_1, and $f(t)$

is a cubic polynomial in t of the form given in Lemma II.15. We finish the proof by applying Lemma II.15 to show that

$$\int_{X_1} \left|F(t) - e^{-q(t)}\right| dt \ll \int_{\mathcal{L}} e^{-q(t)} dt.$$

4.1 Integral Expression of the Counting Problem

We use two results of [BH10] to express the number of binary integer points of P as an integral of a function $F(t)$. Let $\Pi \subset \mathcal{L}$ be the cube centered at the origin:

$$\Pi = \{t \in \mathcal{L} \;:\; ||t||_\infty \leq \pi\}.$$

We will show that for multi-index transportation polytopes P satisfying the conditions of Theorem I.2, the number of binary integer points satisfies

$$|P \cap \{0,1\}^n| = \frac{e^{g(z)}}{(2\pi)^{k_1+\ldots+k_\nu-\nu+1}} \int_{\Pi} e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right) dt. \tag{4.1}$$

Before we do, we recall the concept of a Bernoulli random variable. We say x is a Bernoulli random variable if for some $0 < p < 1$,

$$\mathbf{Pr}(x = 0) = p \quad \text{and} \quad \mathbf{Pr}(x = 1) = (1 - p).$$

In this case, $\mathbf{E}x = 1 - p$. Conversely, if $\mathbf{E}x = \zeta$, then $p = 1 - \zeta$. The first theorem we need is the following:

Theorem IV.1. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace in $\mathbb{R}^n$ and the unit cube $[0,1]^n$. Suppose that P is bounded and has a non-empty interior, that is a point $y = (\eta_1, \ldots, \eta_n)$ where $\eta_i > 0$ for $i = 1, \ldots, n$. Then the strictly concave* *function*

$$g(x) = \sum_{j=1}^{n} \left(\xi_j \ln \frac{1}{\xi_j} + (1 - \xi_j) \ln \frac{1}{1 - \xi_j} \right)$$

attains its maximum value in P at a unique point $z = (\zeta_1, \ldots, \zeta_n)$ such that $0 < \zeta_j < 1$ for $j = 1, \ldots, n$. Furthermore, suppose $x_1, \ldots, x_n$ are independent Bernoulli random variables with $\mathbf{E}x_j = \zeta_j$, and let $X = (x_1, \ldots, x_n)$. Then the probability mass function of X is constant on $P \cap \{0,1\}^n$ and equal to $e^{-g(z)}$ at every $x \in P \cap \{0,1\}^n$. In particular,

$$|P \cap \{0,1\}^n| = e^{g(z)} \mathbf{Pr}\left(X \in P\right).$$

This is Theorem 5 of [BH10]. This lets us reduce counting the number of binary integer points in P to calculating $\mathbf{Pr}\left(X \in P\right)$. We combine this result with the following:

Lemma IV.2. *Let p_j, q_j be positive numbers such that $p_j + q_j = 1$ for $j = 1, \ldots, n$, and let μ be the Bernoulli measure on the set $\{0,1\}^n$ of non-negative integer vectors with*

$$\mu\{x\} = \prod_{j=1}^{n} p_j^{1-\xi_j} q_j^{\xi_j} \quad \textit{for} \quad x = (\xi_1, \ldots, \xi_n).$$

Let P be defined by the linear equalities $Ax = b$, where A has columns $a_1, \ldots, a_n$, and $a_1, \ldots, a_n, b \in \mathbb{R}^d$. Let $\Pi = [-\pi, \pi]^d$ be a cube centered at the origin in $\mathbb{R}^d$. Then

$$\mu\left(P\right) = \frac{1}{(2\pi)^d} \int\limits_{\Pi} e^{-i\langle t, b\rangle} \prod_{j=1}^{n} \left(p_j + q_j e^{i\langle a_j, t\rangle}\right) dt.$$

Here, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$ and dt is the Lebesgue measure.

This is Lemma 11 of [BH10]. We combine this with Theorem IV.1 to derive (4.1) as follows: we identify $\mathcal{L}$ with $\mathbb{R}^{k_1+\ldots+k_\nu-\nu+1}$ in the natural way by identifying the non-zero coordinates of L with the coordinates of $\mathbb{R}^{k_1+\ldots+k_\nu-\nu+1}$. Then P is defined by the linear equations $QAx = Qb$ where Q is the orthogonal projection onto L. As

$\langle Qa_j, t\rangle = \langle a_j, t\rangle$ and $\langle Qb, t\rangle = \langle b, t\rangle$ for $t \in \mathcal{L}$, we use the columns of A and the vector b in the integrand instead of QA and Qb. The random variable X in Theorem IV.1 induces the Bernoulli measure μ in Lemma IV.2 when $\zeta_j = 1 - p_j$. This turns the integrand of Lemma IV.2 into

$$e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right).$$

Let

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right).$$

The bulk of the proof is dedicated to showing that

$$\int_{\Pi} F(t)dt \approx \int_{\mathcal{L}} e^{-q(t)}dt.$$

4.2 A Bound on $F(t)$ Away from the Origin

The main result of this section is the following:

Lemma IV.3. *Let*

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right).$$

Then there exists a constant $\gamma = \gamma(\omega, \nu, r) > 0$ such that

$$|F(t)| \leq \exp\left(-\gamma ||t||_\infty^2 k^{\nu-1}\right).$$

The constant γ may be chosen to be

$$\gamma = \frac{r\omega^\nu}{5\nu^3 2^{\nu-1}}.$$

We apply this lemma in the following way: we construct a region, which in the proof will be called X_3, which is the complement of a neighborhood of the origin in Π. Then we use Lemma III.3 and Lemma II.6 to show that

$$\left|\int\limits_{X_3\cap\Pi} F(t)dt\right|, \int\limits_{X_3} e^{-q(t)}dt \ll \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

To prove Lemma III.3 we use the following:

Lemma IV.4. *Let D be a $d \times n$ integer matrix with columns $d_1, \ldots d_n \in \mathbb{Z}^d$. For each $1 \leq l \leq d$, let $Y_l \subset \mathbb{Z}^d$ be a non-empty finite set such that for all $y \in Y_l$, we have $Dy = e_l$, where e_l is the lth standard basis vector. Let $\psi_l : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form*

$$\psi_l(x) = \frac{1}{|Y_l|}\sum_{y\in Y_l} \langle y, x\rangle^2 \quad \textit{for} \quad x \in \mathbb{R}^n,$$

and let ρ_l be constants such that

$$\psi_l(x) \leq \rho_l||x||^2.$$

Suppose further that for $\zeta_1, \ldots, \zeta_n > 0$ we have

$$\zeta_j - \zeta_j^2 \geq \alpha \quad \textit{for some} \quad \alpha > 0 \quad \textit{and} \quad j = 1, \ldots, n.$$

Then for any $t = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d$, and for each l, we have

$$\left|\prod_{j=1}^{n}\left(1 - \zeta_j + \zeta_j e^{i\langle d_j, t\rangle}\right)\right| \leq \exp\left(-\frac{\alpha\tau_l^2}{5\rho_l}\right).$$

This is Lemma 12 of [BH10]. We are now ready to prove Lemma IV.3.

Proof. We identify $\mathcal{L}$ with $\mathbb{R}^{k_1+\ldots+k_\nu-\nu+1}$ in the natural way. We use the sets Y_l

constructed in the proof of III.3, to get sets Y_l satisfying the hypothesis with

$$\rho_l \leq \frac{\nu^3 2^{\nu-1}}{\omega^\nu} k^{-\nu+1}.$$

Applying Lemma IV.4 uniformly over all values of l with $D = QA$ and $\alpha = r$, we arrive at

$$|F(t)| \leq \exp\left(-\frac{r\omega^\nu ||t||_\infty^2}{5\nu^3 2^{\nu-1}} k^{\nu-1} \right).$$

$\square$

4.3 The Proof of Theorem I.2

At this point we are ready to prove Theorem I.2. The outline of the proof is as follows: we first construct a region $X_3 \subset \mathcal{L}$ which is of the form

$$X_3 = \{ t \in \mathcal{L} \ : \ ||t||_\infty \geq \beta \}$$

for some $\beta \in \mathbb{R}$. We apply Lemma IV.3 to show that

$$\int_{X_3 \cap \Pi} |F(t)| dt \ll \int_{\mathcal{L}} e^{-q(t)} dt.$$

For $||t||_\infty < \beta$, we express $F(t)$ as

$$F(t) = e^{-q(t) + if(t) + h(t)},$$

where $q(t)$ is the quadratic form as in Theorem I.2, $f(t)$ is a cubic polynomial, and $h(t)$ is bounded by a quartic polynomial. We use Lemma II.6 along with an inequality comparing $q(t)$ to $h(t)$ to show that for some set $X_2 \subset \mathcal{L}$ of the form

$$X_2 = \{ t \in \mathcal{L} \ : \ \delta \leq ||t||_\infty \leq \beta \},$$

we have

$$\int_{X_2} |F(t)|dt \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

We also use Lemma II.6 to show that

$$\int_{X_2 \cup X_3} e^{-q(t)}dt \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

We then let

$$X_1 = \{t \in \mathcal{L} \;:\; ||t||_\infty \leq \delta\}.$$

We show that $|h(t)|$ is small for all $t \in X_1$, and then use Lemma II.15 to show that

$$\left| \int_{X_1} F(t) - e^{-q(t)}dt \right| \ll \int_{\mathcal{L}} e^{-q(t)}dt.$$

Combining the calculations over the three regions X_1, X_2, X_3 will allow us to show that

$$\int_{\Pi} F(t)dt \approx \int_{\mathcal{L}} e^{-q(t)}dt.$$

We observe that for all the calculations in Chapter II, we can replace R with 1 as $\zeta_{m_1,\ldots,m_\nu} - \zeta^2_{m_1,\ldots,m_\nu} \leq 1/4$ always.

Proof. By (4.1) and (1.2), it suffices to show

$$\left| \int_{\Pi} F(t)dt - \int_{\mathcal{L}} e^{-q(t)}dt \right| \leq \Gamma k^{-\nu+2.5}$$

for some constant $\Gamma > 0$. Let

$$X_3 = \left\{ t \in \mathcal{L} \;:\; ||t||^2_\infty \geq \frac{5\nu^3 2^{\nu-1}}{2r\omega^\nu}\nu^2 \ln(k)k^{-\nu+2} \right\}. \tag{4.2}$$

By Lemma IV.3, we have

$$\int_{X_3 \cap \Pi} |F(t)| dt \leq (2\pi)^{\nu k} \exp\left(-\frac{1}{2}\nu^2 k \ln(k)\right).$$

By Corollary II.5 , we have

$$\int_{X_3 \cap \Pi} |F(t)| dt \leq \exp\left(-\frac{1}{4}\nu^2 k \ln(k) + \nu k \ln(2\pi)\right) \int_{\mathcal{L}} e^{-q(t)} dt, \tag{4.3}$$

which is negligible compared to $\int_{\mathcal{L}} e^{-q(t)} dt$.

For the middle and inside regions, we can use the Taylor polynomial estimate

$$\left| e^{i\xi} - 1 - i\xi + \frac{\xi^2}{2} + i\frac{\xi^3}{6} \right| \leq \frac{\xi^4}{24} \quad \text{for all} \quad \xi \in \mathbb{R}$$

to write

$$e^{i\langle a_j, t\rangle} = 1 + i\langle a_j, t\rangle - \frac{\langle a_j, t\rangle^2}{2} - i\frac{\langle a_j, t\rangle^3}{6} + g_j(t)\langle a_j, t\rangle^4,$$

where $|g_j(t)| \leq \frac{1}{24}$ for all $j = 1, \ldots, k_1 k_2 \ldots k_\nu$. Therefore

$$F(t) = e^{-i\langle b, t\rangle} \prod_{j=1}^{n} \left(1 + i\zeta_j \langle a_j, t\rangle - \zeta_j \frac{\langle a_j, t\rangle^2}{2} - i\zeta_j \frac{\langle a_j, t\rangle^3}{6} + \zeta_j g_j(t) \langle a_j, t\rangle^4\right).$$

Furthermore, using

$$\left| \ln(1+\xi) - \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} \right| \leq \frac{|\xi|^4}{2} \quad \text{for all complex} \quad |\xi| \leq 1/2,$$

plus

$$\sum_{j=1}^{n} \zeta_j a_j = b,$$

we can write

$$F(t) = e^{-q(t)+if(t)+h(t)}, \quad \text{where}$$

$$q(t) = \frac{1}{2} \sum_{m_1,\ldots,m_\nu} \left(\zeta_{m_1\ldots m_\nu} - \zeta^2_{m_1\ldots m_\nu}\right) \left(\tau_{m_1 1} + \tau_{m_2 2} + \ldots + \tau_{m_\nu \nu}\right)^2,$$

$$f(t) = \frac{1}{6} \sum_{m_1,\ldots,m_\nu} \left(\zeta_{m_1,\ldots,m_\nu} - \zeta^2_{m_1,\ldots,m_\nu}\right) \left(2\zeta_{m_1,\ldots,m_\nu} - 1\right) \left(\tau_{m_1 1} + \tau_{m_2 2} + \ldots + \tau_{m_\nu \nu}\right)^3,$$

is a cubic polynomial of the form in Section 2.4, and

$$|h(t)| \leq 2 \sum_{m_1\ldots m_\nu} \left(\tau_{m_1 1} + \tau_{m_2 2} + \ldots + \tau_{m_\nu \nu}\right)^4. \tag{4.4}$$

This representation is valid as long as $||t||_\infty \leq 1/(2\nu)$. For $t \in \Pi \setminus X_3$, this inequality is true as long as

$$\frac{5\nu^4 2^{\nu-1}}{2r\omega^\nu} \ln(k) k^{-\nu+2} \leq \frac{1}{4\nu^2},$$

which is assumed by hypothesis. Let

$$X_2 = \left\{ t \in \mathcal{L} \ : \ \frac{5\nu^5 2^{\nu-1}}{2r\omega^\nu} k^{-\nu+1.25} \leq ||t||^2_\infty \leq \frac{5\nu^5 2^{\nu-1}}{2r\omega^\nu} \ln(k) k^{-\nu+2} \right\}.$$

Then we get

$$\left| \int_{X_2} F(t) dt \right| \leq \int_{X_2} |F(t)| dt = \int_{X_2} e^{-q(t)+h(t)} dt.$$

As

$$(\tau_{m_1 1} + \ldots + \tau_{m_\nu \nu})^2 \leq \nu^2 ||t||^2_\infty,$$

we get for $t \in X_2$ that

$$|h(t)| \leq \frac{10\nu^7 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+2} q(t).$$

Assuming as in the hypothesis of Theorem I.2 that

$$\delta = \frac{10\nu^7 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+2} \leq 3/4,$$

for $t \in X_2$ we get $|F(t)| = e^{-q(t)+h(t)} \leq e^{-(1-\delta)q(t)}$. Therefore,

$$\left| \int_{X_2} F(t) dt \right| \leq \int_{X_2} e^{-(1-\delta)q(t)} dt.$$

Doing the change of variables $t \mapsto (\sqrt{1-\delta})t$ we get

$$\left| \int_{X_2} F(t) dt \right| \leq (1-\delta)^{-\nu k/2} \int_{\sqrt{1-\delta} X_2} e^{-q(t)} dt.$$

We use the bound

$$\left(1 - \frac{10\nu^7 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+2}\right)^{-\nu k/2} \leq \exp\left(\frac{20\nu^6 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+3}\right),$$

and by Lemma II.6 and the choice of the lower bound in the definition of X_2, we get

$$\left| \int_{X_2} F(t) dt \right| \leq \nu k \exp\left(\frac{20\nu^6 2^{\nu-1}}{r^2\omega^\nu} \ln(k) k^{-\nu+3} - \frac{5\omega^{5\nu-3} r \nu 2^{\nu-1}}{16} k^{.25}\right) \int_{\mathcal{L}} e^{-q(t)} dt. \quad (4.5)$$

Similarly, by Lemma II.6, we get

$$\int_{X_2 \cup X_3} e^{-q(t)} dt \leq \nu k \exp\left(-\frac{5\omega^{5\nu-3} r \nu 2^{\nu-1}}{4} k^{.25}\right) \int_{\mathcal{L}} e^{-q(t)} dt. \quad (4.6)$$

We define

$$X_1 = \left\{ t \in \mathcal{L} \ : \ ||t||_\infty^2 \leq \frac{5\nu^5 2^{\nu-1}}{2r\omega^\nu} k^{-\nu+1.25} \right\}.$$

For $t \in X_1$, the inequality $|\langle a_j, t\rangle|^4 \leq \nu^4 ||t||_\infty^4$ gives us

$$|h(t)| \leq \frac{25\nu^{14}4^{\nu-1}}{4r^2\omega^{2\nu}}k^{-\nu+2.5}. \tag{4.7}$$

Hence, writing

$$\left|\int\limits_{X_1} F(t) - e^{-q(t)}dt\right| = \left|\int\limits_{X_1} e^{-q(t)+if(t)+h(t)} - e^{-q(t)}dt\right|,$$

we can use the triangle inequality to get

$$\left|\int\limits_{X_1} F(t) - e^{-q(t)}dt\right| \leq \left|\int\limits_{X_1} e^{-q(t)+if(t)+h(t)} - e^{-q(t)+if(t)}dt\right| + \left|\int\limits_{X_1} e^{-q(t)+if(t)} - e^{-q(t)}dt\right|.$$

We apply Holder's inequality to get

$$\left|\int\limits_{X_1} e^{-q(t)+if(t)+h(t)} - e^{-q(t)+if(t)}dt\right| \leq \sup_{t\in X_1} |e^{h(t)} - 1| \int\limits_{X_1} \left|e^{-q(t)+if(t)}\right| dt.$$

By (4.7), we can bound this by

$$\left|\int\limits_{X_1} e^{-q(t)+if(t)+h(t)} - e^{-q(t)+if(t)}dt\right| \leq \left(\exp\left(\frac{25\nu^{14}4^{\nu-1}}{4r^2\omega^{2\nu}}k^{-\nu+2.5}\right) - 1\right) \int\limits_{\mathcal{L}} e^{-q(t)}dt. \tag{4.8}$$

Applying Lemma II.15 with $\left|\beta^3_{m_1,\ldots,m_\nu}\right| = \left|(\zeta_{m_1,\ldots,m_\nu} - \zeta^2_{m_1,\ldots,m_\nu})(2\zeta_{m_1,\ldots,m_\nu} - 1)\right| \leq 1/4$, and noting that almost all the measure of $e^{-q(t)}$ is contained in X_1 by (4.6), we get

$$\begin{gathered}\left|\int\limits_{X_1} e^{-q(t)+if(t)} - e^{-q(t)}dt\right| \leq \\ \left(2\nu k \exp\left(-\frac{5\omega^{5\nu-3}r\nu 2^{\nu-1}}{4}k^{.25}\right) + \frac{224\nu^{13}}{r^9\omega^{21\nu-15}}k^{2-\nu}\right) \int\limits_{\mathcal{L}} e^{-q(t)}dt.\end{gathered} \tag{4.9}$$

Combining Equations (4.3), (4.5), (4.6), (4.8), and (4.9) completes the proof. If k is large enough, the $k^{-2.5+\nu}$ term from (4.8) dominates, and doubling it gives us the example value for Γ. $\square$

CHAPTER V

Proof of Theorem I.3

In this chapter we prove Theorem I.3, counting the volume of certain 4-way transportation polytopes. For the entirety of this chapter we use the notation introduced in the statement of the theorem, most importantly the quadratic form $q(t)$ and the constants r, R, ω and k. We also recall the overdetermined system of equations for a multi-index transportation polytope of the form $Ax = b$, where A has columns $a_1, \dots, a_n$ as described in Section 1.1, along with the subspace $\mathcal{L}$ that describes a linearly independent set of equations. The matrix $Q : \mathbb{R}^{k_1+\ldots+k_\nu} \to \mathbb{R}^{k_1+\ldots+k_\nu}$ will be the orthogonal projection onto $\mathcal{L}$. To prove Theorem I.3, we write the volume as

$$\mathrm{vol}(P) = \frac{1}{(2\pi)^{k_1+k_2+k_3+k_4-3}} \int\limits_{\mathcal{L}} F(t)dt \quad \text{for some} \quad F : \mathbb{R}^{k_1+\ldots+k_\nu-\nu+1} \to \mathbb{R}.$$

We then partition $\mathcal{L}$ into $\mathcal{L} = X_1 \cup X_2 \cup X_3$, and show that

$$\left| \int\limits_{X_2 \cup X_3} F(t)dt \right|, \quad \int\limits_{X_2 \cup X_3} e^{-q(t)}dt \ll \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

The proof then diverges from the pattern followed by the proofs of I.1 and I.2, as X_1 is further partitioned and additional analytic results are required. To prove Theorem I.3 we use a concept called concentration of measure to show that the integral of $e^{-q(t)}$ near the subspace of L spanned by the eigenvectors of $q(t)$ whose eigenvalues are

$\Theta(k^2)$ is negligible as well.

5.1 Concentration of the Gaussian Integral

The main crux of the proof that improves on [BH10]'s proof for $\nu \geq 5$ is certain concentration of measure arguments. The main result of this section is:

Theorem V.1. *Let $q(t) : \mathbb{R}^{k_1+k_2+k_3+k_4} \to \mathbb{R}$ satisfy the conditions of Theorem II.1 for $\nu = 4$. Then for all $0 < \epsilon < 1/2$ and all $\sigma > 0$, there exist constants $\gamma = \gamma(R, \omega) > 0$ and $N = N(\omega, R/r, \epsilon)$ such that if $k \geq N$,*

$$\int\limits_{\substack{t \in \mathcal{L} \\ q(t) < \sigma \\ q(t) \geq r k^{2+\epsilon} ||t||^2}} e^{-q(t)} dt \geq \left(1 - 3 \exp\left(-\frac{\omega^7 r}{384 R} k^{1-\epsilon}\right)\right) \int\limits_{\substack{t \in \mathcal{L} \\ q(t) < \sigma}} e^{-q(t)} dt.$$

The theorem essentially says that the measure of $e^{-q(t)}$ is concentrated around the subspace of $\mathcal{L}$ spanned by the eigenvector whose eigenvalues are $\Theta(k^3)$. To show this, we require several results on concentration of measure on the sphere. For notation, we will let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$.

Lemma V.2. *Let μ be the rotationally invariant Borel probability measure on $\mathbb{S}^{d-1}$. Let $\mathcal{H} \subset \mathbb{R}^d$ be a hyperplane passing through the origin. For $\gamma < 1$, let $X \subset \mathbb{S}^{d-1}$ be defined by*

$$X = \left\{ x \in \mathbb{S}^{d-1} \ : \ \mathrm{dist}(x, \mathcal{H}) < \gamma \right\},$$

where $\mathrm{dist}(x, \mathcal{H})$ is the shortest Euclidean distance between x and any point in $\mathcal{H}$. Then

$$\mu(X) \geq 1 - \exp\left(-\frac{\gamma^2}{2} d\right).$$

This is Corollary 2.2 of [MS86]. The lemma says if you take any hyperplane slice of the sphere, almost all of the sphere's measure is close to the hyperplane. We use this to prove the following:

Lemma V.3. *For some* $0 < \kappa < 1$*, let* $Y \subset \mathbb{S}^{d-1}$ *be defined as*

$$Y = \left\{x = (\xi_1, \ldots, \xi_d) \in \mathbb{S}^{d-1} \;:\; \xi_1^2 + \xi_2^2 + \xi_3^2 \leq \kappa^2\right\}.$$

Then

$$\mu(Y) \geq 1 - 3\exp\left(-\frac{1}{6}d\kappa^2\right).$$

Proof. Define the subsets of the sphere

$$H_r = \left\{x = (\xi_1, \ldots, \xi_d) \in \mathbb{S}^{d-1} \;:\; |\xi_r| \leq \frac{1}{\sqrt{3}}\kappa\right\}.$$

By Lemma V.2, for μ the rotationally invariant Borel measure probability measure, we get

$$\mu\left(H_r\right) \geq 1 - \exp\left(-\frac{1}{6}d\kappa^2\right).$$

Therefore, by a simple union bound we can bound the measure of the intersection by

$$\mu\left(H_1 \cap H_2 \cap H_3\right) \geq 1 - 3\exp\left(-\frac{1}{6}d\kappa^2\right). \tag{5.1}$$

Moreover, $H_1 \cap H_2 \cap H_3 \subset Y$, since if $x = (\xi_j)$ is contained in the intersection, then

$$\xi_1^2 + \xi_2^2 + \xi_3^2 \leq 3\frac{1}{3}\kappa^2 = \kappa^2,$$

which is exactly what is required for $x \in Y$. Hence by (5.1),

$$\mu\left(Y\right) \geq 1 - 3\exp\left(-\frac{1}{6}d\kappa^2\right).$$

□

We are now ready to prove Theorem V.1. To do so, we perform a change of coordinates to turn the integral of $e^{-q(t)}$ over the region $q(t) < \sigma$ into an integral over a sphere of $e^{-||u||^2}$. We calculate explicitly where the set

$$\left\{t \;:\; q(t) < \sigma,\; q(t) \geq rk^{2+\epsilon}||t||^2\right\}$$

is mapped to under this change of coordinates, and show it contains a set of the form Y as in Lemma V.3.

Proof. Let D be the unique symmetric positive semidefinite square root of B, so that $q|_{\mathcal{L}}(t) = \frac{1}{2}\langle t, QBQt\rangle = \frac{1}{2}\langle QDQt, QDQt\rangle$. Picking an orthonormal basis of eigenvectors of QDQ, we can assume without loss of generality that QDQ is diagonal. The diagonal entries are the square roots of the eigenvalues calculated in Theorem II.1. We will denote the non-zero eigenvalues of QBQ corresponding to eigenvectors in $\mathcal{L}$ as $\lambda_1 < \lambda_2 < \ldots < \lambda_{k_1+k_2+k_3+k_4-3}$.

For notational simplicity we omit the $t \in \mathcal{L}$ subscript on all integrals - it is assumed for the remainder of the proof. We also identify $\mathcal{L}$ with $\mathbb{R}^{k_1+k_2+k_3+k_4-3}$ in the natural way, by matching non-zero coordinates in order. We perform the change of coordinates $u = Dt$, making $q(t) = ||u||^2$ and

$$\int\limits_{q(t)<\sigma} e^{-q(t)} dt = \frac{1}{\det D} \int\limits_{||u||^2<\sigma} e^{-||u||^2} du.$$

Define the functions $a(t)$ and $b(t)$ by

$$a(t) = \sqrt{\sum_{j=4}^{k_1+k_2+k_3+k_4-3} \tau_j^2}, \quad \text{and} \quad b(t) = \sqrt{\sum_{j=1}^{3} \tau_j^2}.$$

These functions define the square of the distance from t to the subspaces spanned by the eigenvectors of QBQ whose eigenvalues are $\Theta(k^3)$ and the three eigenvectors whose eigenvalues are $\Theta(k^2)$. Then

$$q(t) \geq r\omega^3 k^3 a^2(t) + \frac{r\omega^3}{12} k^2 b^2(t), \quad \text{and} \quad ||t||^2 = a^2(t) + b^2(t). \tag{5.2}$$

If

$$\frac{r\omega^3 k^3 a^2(t) + \frac{r\omega^3}{12} k^2 b^2(t)}{a^2(t) + b^2(t)} \geq rk^{2+\epsilon},$$

we get $q(t) \geq rk^{2+\epsilon}||t||^2$. For $u = (\eta_1, \eta_2, \ldots, \eta_{k_1+k_2+k_3+k_4-3})$, we define the auxiliary functions

$$\tilde{a}(u) = \sqrt{\sum_{j=4}^{k_1+k_2+k_3+k_4-3} \frac{\eta_j^2}{\lambda_j}}, \quad \text{and} \quad \tilde{b}(u) = \sqrt{\sum_{j=1}^{3} \frac{\eta_j^2}{\lambda_j}}.$$

Then for $u = Dt$, we have $\tilde{a}(u) = a(t)$ and $\tilde{b}(u) = b(t)$. Let

$$W = \left\{ u \in \mathcal{L} \ : \ ||u||^2 < \sigma \quad \text{and} \quad \left(r\omega^3 k^3 - rk^{2+\epsilon}\right) \tilde{a}^2(u) \geq \left(rk^{2+\epsilon} - \frac{r\omega^3}{12} k^2\right) \tilde{b}^2(u) \right\}.$$

By (5.2) we get that if $u \in W$, then $t = D^{-1}u$ satisfies $q(t) \geq rk^{2+\epsilon}||t||^2$, and hence

$$\int\limits_{\substack{||u||^2<\sigma \\ u \in W}} e^{-||u||^2} du \Bigg/ \int\limits_{||u||^2<\sigma} e^{-||u||^2} du) \leq \int\limits_{\substack{q(t)<\sigma \\ q(t) \geq rk^{2+\epsilon}||t||^2}} e^{-q(t)} dt \Bigg/ \int\limits_{q(t)<\sigma} e^{-q(t)} dt. \tag{5.3}$$

As W is conical -that is if $u \in W$, $\alpha u \in W$ for all $\alpha > 0$ - and the integrals on the left hand side of (5.3) are symmetric under rotations, we let $W_1 = W \cap \mathbb{S}^{k_1+k_2+k_3+k_4-4}$ and get that

$$\mu(W_1) \leq \left(\int\limits_{\substack{q(t)<\sigma \\ q(t) \geq rk^{2+\epsilon}||t||^2}} e^{-q(t)} dt \right) \Bigg/ \left(\int\limits_{q(t)<\sigma} e^{-q(t)} dt \right), \tag{5.4}$$

where μ is the rotationally invariant Borel probability measure on $\mathbb{S}^{k_1+k_2+k_3+k_4-4}$. By the bounds on the eigenvalues λ_j given in Theorem II.1, we get that

$$\tilde{a}^2(u) \geq \frac{a^2(u)}{4Rk^3} \quad \text{and} \quad \tilde{b}^2(u) \leq \frac{12b^2(u)}{r\omega^3 k^2}.$$

Therefore, letting

$$W_2 = \left\{ u \in \mathbb{S}^{k_1+k_2+k_3+k_4-4} \ : \ \frac{r\omega^3 k^3 - rk^{2+\epsilon}}{4Rk^3} a^2(u) \geq \frac{12rk^{2+\epsilon} - r\omega^3 k^2}{r\omega^3 k^2} b^2(u) \right\},$$

we have $W_2 \subset W_1$ and by (5.4),

$$\mu(W_2) \leq \left(\int\limits_{\substack{q(t)<\sigma \\ q(t)\geq rk^{2+\epsilon}||t||^2}} e^{-q(t)} dt \right) \Bigg/ \left(\int\limits_{q(t)<\sigma} e^{-q(t)} dt \right). \tag{5.5}$$

For $u \in W_2$, $a^2(u) + b^2(u) = 1$ by (5.2), so we can rewrite W_2 as

$$W_2 = \left\{ u \in \mathbb{S}^{k_1+k_2+k_3+k_4-4} \ : \ b^2(u) \leq \frac{\omega^6 r^2 - \omega^3 r^2 k^{\epsilon-1}}{\omega^6 r^2 - \omega^3 r^2 k^{\epsilon-1} + 48Rrk^{\epsilon} - 4Rr\omega^3} \right\}. \tag{5.6}$$

There exists a constant $N = N(\omega, R/r, \epsilon)$ such that for all $k \geq N$, the numerator and denominator are dominated by the $\omega^6 r^2/2$ and $48Rrk^{\epsilon}$. In particular, the numerator is at least $\omega^6 r^2/2$ and the denominator is no more than $96Rrk^{\epsilon}$. For $k \geq N$, we let

$$W_3 = \left\{ u \in \mathbb{S}^{k_1+k_2+k_3+k_4-4} \ : \ b^2(u) \leq \frac{\omega^6 r}{192R} k^{-\epsilon} \right\}.$$

Then $W_3 \subset W_2$, and hence

$$\mu(W_3) \leq \left(\int\limits_{\substack{q(t)<\sigma \\ q(t)\geq rk^{2+\epsilon}||t||^2}} e^{-q(t)} dt \right) \Bigg/ \left(\int\limits_{q(t)<\sigma} e^{-q(t)} dt \right). \tag{5.7}$$

Applying Lemma V.3 to W_3 and noting the dimension is at least as large as ωk completes the proof. $\square$

5.2 Integral Representation of Measuring Volume

We use two results of [BH10] to express the volume of P as an integral of a function $F(t)$. We will show that for multi-index transportation polytopes P satisfying the conditions of Theorem I.3, the volume satisfies (recall $s(t)$ from the statement of Theorem I.3):

$$\operatorname{vol}(P) = \det(s|_{\mathcal{L}})^{1/2} \frac{e^{g(z)}}{(2\pi)^{k_1+k_2+k_3+k_4-3}} \int_{\mathcal{L}} e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1 - i\zeta_j \langle a_j, t\rangle} dt. \tag{5.8}$$

Before we prove this, we recall the concept of an exponential random variable. We say x is an exponential random variable with $\mathbf{E}x = \zeta$ if the density function ψ of x is

$$\psi(\tau) = \frac{1}{\zeta} e^{-\zeta\tau} \quad \text{for} \quad \tau \geq 0.$$

The first theorem we need to prove (5.8) is the following:

Theorem V.4. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace with the non-negative orthant $\mathbb{R}^n_+$. Suppose that P is bounded and has a non-empty interior. Then the strictly concave function*

$$g(x) = n + \sum_{j=1}^{n} \ln \xi_j \quad \textit{for} \quad x = (\xi_1, \ldots, \xi_n)$$

attains its unique maximum on P at a point $z = (\zeta_1, \ldots, \zeta_n)$ where $\zeta_j > 0$ for $j = 1, \ldots, n$. Let $x_1, \ldots, x_n$ be independent exponential random variables with $\mathbf{E}x_j = \zeta_j$ for $j = 1, \ldots, n$. Let $X = (x_1, \ldots, x_n)$. Then the density of X is constant on P and for every $x \in P$, is equal to $e^{-g(z)}$.

This is Theorem 7 of [BH10]. We combine this with:

Lemma V.5. *Let $x_1, \ldots, x_n$ be independent exponential random variables such that $\mathbf{E}x_j = \zeta_j$ for $j = 1, \ldots, n$. Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be vectors which span $\mathbb{R}^d$, and let $Y = x_1 a_1 + \ldots + x_n a_n$. Then the density of Y at $b \in \mathbb{R}^d_+$ is equal to*

$$\frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1 - i\zeta_j \langle a_j, t\rangle} dt.$$

This is Lemma 8 of [BH10]. Identify $\mathcal{L}$ with $\mathbb{R}^{k_1+k_2+k_3+k_4-3}$ by matching nonzero coordinates in order. Let $Y = QAX$ for $X = (x_1, \ldots, x_n)$ as in Theorem V.4. Then Theorem V.4 says that

$$\mathrm{vol}(P) = e^{f(z)} \det \left(QA(QA)^t\right)^{1/2} \mathbf{Pr}(Y = Qb).$$

We combine this with Lemma V.5, noting that $\det(s|_{\mathcal{L}}) = \det((QA)(QA)^t)$. We also use the fact that the columns of QA are $Qa_1, \ldots, Qa_n$, and that for $t \in \mathcal{L}$, $\langle Qa_j, t\rangle = \langle a_j, t\rangle$ and $\langle Qb, t\rangle = \langle b, t\rangle$ to replace $Qa_1, \ldots, Qa_n, Qb$ in the integrand with $a_1, \ldots, a_n, b$. This proves Equation (5.8).

5.3 A Bound on $F(t)$ Away from the Origin

The main result of this section is the following:

Lemma V.6. *Let*

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1 - i\zeta_j \langle a_j, t\rangle}.$$

Let $0 < \rho < 1$. Then if

$$k \geq \frac{192R}{\omega^3 r},$$

we have

$$\int\limits_{\substack{t\in\mathcal{L}\\ ||t||\geq\rho}} |F(t)|\,dt \leq \exp\left(-\frac{\omega^3 r}{384R}k^2\rho^2\right).$$

We will use this lemma to show that for some $\rho > 0$,

$$\int\limits_{||t||\geq\rho} |F(t)|dt \ll \int\limits_{\mathcal{L}} e^{-q(t)}dt.$$

For $||t|| < \rho$ we will express $F(t) = e^{-q(t)+if(t)+h(t)}$ for $f(t)$ a cubic polynomial and $h(t)$ a small error function, and show that the effects of $f(t)$ and $h(t)$ on the integral of $F(t)$ are negligible. We reconstruct the proof of Lemma V.6 which is broken into pieces in [BH10].

Proof. We have

$$|F(t)| \leq \left(\prod_{j=1}^{n}\frac{1}{1+\zeta_j^2\left\langle a_j,t\right\rangle^2}\right)^{1/2}.$$

Letting $\xi_j = \zeta_j^2\left\langle a_j,t\right\rangle^2$, we get

$$\sum_{j=1}^{n}\xi_j = 2q(t),$$

so in particular by Theorem II.1,

$$\sum_{j=1}^{n}\xi_j \geq \frac{r\omega^3}{6}k^2||t||^2.$$

We also have that $0 \leq \xi_j \leq 4R||t||^2$ for $j = 1,\ldots,n$. Fix $||t||^2$. The minimum of the log-concave function

$$\prod_{j=1}^{n}(1+\xi_j)$$

on the polytope with constraints

$$0 \leq \xi_j \leq 4R||t||^2 \quad \text{and} \quad \sum_{j=1}^{n}\xi_j \geq \frac{r\omega^3}{6}k^2||t||^2$$

must occur on a vertex of the polytope, where every ξ_j is either 0 or $4R||t||^2$ such that the sum of the coordinates is as small as possible satisfying the last constraint. Therefore, $|F(t)|$ is bounded from above when we replace every $\zeta_j^2 \langle a_j, t\rangle^2$ with either a 0 or a $4R||t||^2$ as well. To satisfy the last constraint at least

$$\frac{\omega^3 r}{24R} k^2$$

of the ξ_js must be non-zero. Hence,

$$|F(t)| \leq (1 + ||t||^2)^{-\omega^3 r k^2/(48R)}.$$

We then find a bound on

$$\int\limits_{||t|| \geq \rho} (1 + ||t||^2)^{-\omega^3 r k^2/(48R)}.$$

Let $\mathbb{S}^{k_1+k_2+k_3+k_4-4} \subset \mathcal{L}$ be the unit sphere. As the integrand is rotationally invariant, we can rewrite the integral in polar coordinates as

$$\int\limits_{||t|| \geq \rho} (1 + ||t||^2)^{-\omega^3 r k^2/(48R)} dt = \left|\mathbb{S}^{k_1+k_2+k_3+k_4-4}\right| \int\limits_{\rho}^{\infty} (1 + s^2)^{-\omega^3 r k^2/(48R)} s^{k_1+k_2+k_3+k_4-4} ds.$$

Using that for any m a positive number, $s^m = s s^{m-1} \leq s(1 + s^2)^{(m-1)/2}$ we rewrite this as

$$\int\limits_{||t|| \geq \rho} (1+||t||^2)^{-\omega^3 r k^2/(48R)} dt \leq \left|\mathbb{S}^{k_1+k_2+k_3+k_4-4}\right| \int\limits_{\rho}^{\infty} (1+s^2)^{-\omega^3 r k^2/(48R)+(k_1+k_2+k_3+k_4-4)/2} s ds.$$

We use the bound $(k_1 + k_2 + k_3 + k_4 - 4)/2 \leq 2k$ along with the hypothesis bound on k and perform integration by substitution to get

$$\int\limits_{||t||\geq\rho} (1+||t||^2)^{-\omega^3 rk^2/(48R)}dt \leq \left|\mathbb{S}^{k_1+k_2+k_3+k_4-4}\right| \frac{24R(1+\rho^2)^{-\omega^3 rk^2/(96R)+1}}{\omega^3 r} k^{-2}.$$

Noting that the Lebesgue measure of the sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is $(2\pi)^{n/2}/\Gamma(n/2)$, and that in this case we have

$$\omega k \leq n \leq 4k,$$

we get

$$\int\limits_{||t||\geq\rho} |F(t)|dt \leq \frac{(2\pi)^{2k}(1+\rho^2)^{-\omega^3 rk^2/(96R)+1}}{\omega^3 r\Gamma(\omega k/2)} k^{-2}.$$

The remainder of the proof is simplifying the terms. We use the bound

$$\Gamma(\omega k/2) \leq \left(\frac{\omega k}{2}\right)^{\omega k/2}$$

as long as $\omega k \geq 2$ to write everything in exponentials as

$$\int\limits_{||t||\geq\rho} |F(t)|dt \leq$$

$$\exp\left(2k\ln(2\pi) - \left(\frac{\omega^3 rk^2}{96R}+1\right)\ln(1+\rho^2) - \frac{\omega k}{2}\ln\left(\frac{\omega k}{2}\right) - \ln(\omega^3 r) - 2\ln(k)\right).$$

As long as $k \geq \frac{2}{\omega}(2\pi)^{4/\omega}$, we get $2k\ln(2\pi) \leq \frac{\omega k}{2}\ln\left(\frac{\omega k}{2}\right)$, and we can simplify this to be

$$\int\limits_{||t||\geq\rho} |F(t)|dt \leq \exp\left(-\left(\frac{\omega^3 rk^2}{96R}+1\right)\ln(1+\rho^2)\right).$$

As long as $\rho^2 \leq 1$, we get $\ln(1+\rho^2) \geq \frac{\rho^2}{2}$. Furthermore, by

$$k \geq \frac{192R}{\omega^3 r},$$

we get that

$$1 \leq \frac{\omega^3 r k^2}{192R}$$

which completes the proof.

$\square$

5.4 The Proof Of Theorem I.3

Before we begin the proof of Theorem I.3 we require one more technical lemma:

Lemma V.7. *Let $q : \mathbb{R}^d \to \mathbb{R}$ be a positive definite quadratic form, and let $\kappa \geq 3$ be a number. Then*

$$\int\limits_{q(t)\geq \kappa d} e^{-q(t)} dt \leq e^{-\kappa d/2} \int\limits_{\mathbb{R}^d} e^{-q(t)} dt.$$

This is Lemma 9 of [BH10]. We are now ready to prove Theorem I.3. The outline of the proof is as follows: we construct sets $X_1, X_2, X_3 \subset \mathcal{L}$ such that

$$\int\limits_{X_2 \cup X_3} |F(t)| dt, \int\limits_{X_2 \cup X_3} e^{-q(t)} dt \ll \int\limits_{\mathcal{L}} e^{-q(t)} dt.$$

We then show that

$$\left| \int\limits_{X_1} F(t) - e^{-q(t)} dt \right| \ll \int\limits_{\mathcal{L}} e^{-q(t)} dt$$

through a sequence of dividing X_1 into smaller parts to complete the proof.

Proof. First, we observe by Corollary II.5 and Equation (5.8) that it suffices to show

$$\left| \int\limits_{\mathcal{L}} F(t) dt - \int\limits_{\mathcal{L}} e^{-q(t)} dt \right| \leq \Gamma k^{-.2}.$$

Let

$$X_3 = \left\{ t \in \mathcal{L} \ : \ ||t||^2 \geq \frac{384R}{\omega^3 r} \left(8k^{-1} \ln(k) + 2k^{-1} \ln(R) \right) \right\}.$$

As long as

$$\frac{384R}{\omega^3 r}\left(8k^{-1}\ln(k)+2k^{-1}\ln(R)\right)\leq 1,$$

which is a weaker condition than the hypothesis, we can apply Lemma V.6 and Corollary II.5 to get

$$\int\limits_{X_3}|F(t)|dt\leq \exp\left(-4k\ln(k)\right)\int\limits_{\mathcal{L}} e^{-q(t)}dt. \tag{5.9}$$

This gives us that $\int_{X_3} F(t)dt$ is negligible compared to $\int_{\mathcal{L}} e^{-q(t)}dt$.

For the middle and inside regions, we can use the Taylor polynomial estimate

$$\left|\ln(1+\xi)-\xi+\frac{\xi^2}{2}-\frac{\xi^3}{3}\right|\leq\frac{|\xi|^4}{2}\quad\text{for all complex}\quad |\xi|\leq 1/2$$

to write

$$\ln(1-i\zeta_j\left\langle a_j,t\right\rangle)=-i\zeta_j\left\langle a_j,t\right\rangle+\frac{1}{2}\zeta_j^2\left\langle a_j,t\right\rangle^2+\frac{i}{3}\zeta_j^3\left\langle a_j,t\right\rangle^3+h_j(t)\zeta_j^4\left\langle a_j,t\right\rangle^4$$

where $|h_j(t)|\leq 1/2$ for $j=1,\ldots,n$. Since

$$\sum_j \zeta_j a_j = b,$$

we have

$$F(t)=e^{-q(t)-if(t)+h(t)},\qquad\text{where}$$

$$q(t)=\frac{1}{2}\sum_{m_1,\ldots,m_\nu}\zeta_{m_1\ldots m_\nu}^2+\left(\tau_{m_1 1}+\tau_{m_2 2}+\ldots+\tau_{m_\nu \nu}\right)^2,$$

$$f(t)=\frac{1}{3}\sum_{m_1,\ldots,m_\nu}\zeta_{m_1,\ldots,m_\nu}^3\left(\tau_{m_1 1}+\tau_{m_2 2}+\ldots+\tau_{m_\nu \nu}\right)^3,$$

is a cubic polynomial of the form in Section 2.4, and

$$|h(t)| \leq \frac{1}{2} \sum_{m_1 \ldots m_\nu} \zeta^4_{m_1 \ldots m_\nu} \left(\tau_{m_1 1} + \tau_{m_2 2} + \ldots + \tau_{m_\nu \nu}\right)^4 . \tag{5.10}$$

This holds as long as $|\zeta_j \langle a_j, t\rangle| \leq 1/2$ for $j = 1, \ldots, n$. As each a_j is a binary vector with at most four 1's, this holds as long as

$$||t||^2 \leq \frac{1}{64R}.$$

By hypothesis for all $t \notin X_3$ this holds.

As long as $k \geq R$, which always holds by hypothesis, for $t \notin X_3$ we can weaken the condition on t to be

$$||t||^2 \leq \frac{3840R}{\omega^3 r} \frac{\ln(k)}{k}.$$

We let

$$X_2 = \left\{ t \in \mathcal{L} \ : \ ||t||^2 \leq \frac{3840R}{\omega^3 r} \frac{\ln(k)}{k} \quad \text{and} \quad q(t) \geq 16k \right\}.$$

We observe that for $t \in \mathcal{L}$ with $q(t) < 16k$, by Theorem II.1 we have

$$||t||^2 \leq \frac{192}{r\omega^3 k}. \tag{5.11}$$

This is strictly smaller than $\frac{3840R}{\omega^3 r} \frac{\ln(k)}{k}$ as long as $R \geq 1$, $\omega < 1$ and $k \geq 2$, so the three sets X_3, X_2, and

$$X_1 = \{t \in \mathcal{L} \ : \ q(t) \leq 16k\} \tag{5.12}$$

cover $\mathcal{L}$ completely, and X_1 and X_3 have an empty intersection.

For $t \in X_2$, we have $(\tau_{m_1 1} + \tau_{m_2 2} + \tau_{m_3 3} + \tau_{m_4 4})^2 \leq 16||t||^2$. Bounding one $(\tau_{m_1 1} + \tau_{m_2 2} + \tau_{m_3 3} + \tau_{m_4 4})^2$ in each summand of $h(t)$ and leaving the other, this gives us

$$|h(t)| \le \frac{61440R^2}{\omega^3 r}\frac{\ln(k)}{k}q(t).$$

By hypothesis of the theorem, we have for $t \in X_2$ that

$$|F(t)| = e^{-q(t)+h(t)} \le e^{-\frac{3}{4}q(t)}.$$

Hence,

$$\left|\int_{X_2} F(t)dt\right| \le \int_{q(t)>16k} e^{-3q(t)/4}dt.$$

Performing the change of variables $u = \sqrt{3/4}t$, we get

$$\left|\int_{X_2} F(t)dt\right| \le \left(\frac{4}{3}\right)^{2k} \int_{q(t)>12k} e^{-q(t)}dt.$$

By Lemma V.7, and noting that $\dim(\mathcal{L}) \le 4k$, we get

$$\left|\int_{X_2} F(t)dt\right| \le e^{-6k+2\ln(4/3)k} \int_{\mathcal{L}} e^{-q(t)}dt, \tag{5.13}$$

which is negligible as $6 > 2\ln(4/3)$. Similarly,

$$\int_{X_2 \cup X_3} e^{-q(t)}dt \le \int_{q(t)>16k} e^{-q(t)}dt \le e^{-8k}\int_{\mathcal{L}} e^{-q(t)}dt. \tag{5.14}$$

Letting X_1 be as defined in (5.12), we split it further into two regions:

$$V_1 = \left\{t \in X_1 \ : \ q(t) \ge rk^{2.5}||t||^2\right\}, \quad \text{and}$$

$$V_2 = \left\{t \in X_1 \ : \ q(t) \le rk^{2.5}||t||^2\right\}.$$

For $t \in V_2$, we use (5.11) to derive a similar bound on h to the one in X_2:

$$|h(t)| \leq \frac{3072R}{r\omega^3 k} q(t).$$

Hence,

$$\left| \int\limits_{V_2} F(t)dt \right| \leq \int\limits_{V_2} e^{-(1-3072R/(r\omega^3 k))q(t)} dt.$$

Performing the change of variables $t \mapsto \sqrt{1 - 3072R/(r\omega^3 k)}$, and assuming this is positive to ensure this is well-defined, we get

$$\left| \int\limits_{V_2} F(t)dt \right| \leq \left(1 - \frac{3072R}{r\omega^3 k}\right)^{-2k} \int\limits_{q(t) \leq rk^{2.5}||t||^2} e^{-q(t)} dt.$$

The Jacobian $(1 - \frac{3072R}{r\omega^3 k})^{-2k}$ is bounded from above by $e^{6148R/(r\omega^3)}$. Then applying Theorem V.1 gives us

$$\left| \int\limits_{V_2} F(t)dt \right| \leq 3 \exp\left(\frac{6148R}{r\omega^3} - \frac{\omega^7 r}{384R} k^{.5}\right) \int\limits_{\mathcal{L}} e^{-q(t)} dt. \tag{5.15}$$

Similarly,

$$\left| \int\limits_{V_2} e^{-q(t)} dt \right| \leq 3 \exp\left(-\frac{\omega^7 r}{384R} k^{.5}\right) \int\limits_{\mathcal{L}} e^{-q(t)} dt. \tag{5.16}$$

For $t \in V_1$, we use $16k \geq q(t) \geq rk^{2.5}||t||^2$ to get

$$||t||^2 \leq \frac{16}{rk^{1.5}}.$$

Combining this with $|h(t)| \leq 16R||t||^2 q(t)$ and $q(t) < 16k$ yields

$$|h(t)| \leq \frac{256R}{rk^{.5}}. \tag{5.17}$$

We split V_1 into two further regions: Let $W_1, W_2 \subset V_1$ be the sets

$$W_1 = \left\{ t \in V_1 : \ ||t||_\infty \geq \frac{1}{k^{1.4}} \right\},$$

and

$$W_2 = V_1 \setminus W_1.$$

By (5.17),

$$\left| \int_{W_1} F(t) dt \right| \leq \exp\left(\frac{256R}{r} k^{-.5} \right) \int_{W_1} e^{-q(t)} dt$$

and applying Lemma II.6 we get

$$\left| \int_{W_1} F(t) dt \right| \leq 4k \exp\left(\frac{256R}{r} k^{-.5} - \frac{\omega^{15} r^2}{512R} k^{.2} \right) \int_{\mathcal{L}} e^{-q(t)} dt. \tag{5.18}$$

Similarly, by Lemma II.6,

$$\left| \int_{W_1} e^{-q(t)} dt \right| \leq 4k \exp\left(-\frac{\omega^{15} r^2}{512R} k^{.2} \right) \int_{\mathcal{L}} e^{-q(t)} dt. \tag{5.19}$$

Lastly, for $t \in W_2$, we have by bounding the summands in $f(t)$ that

$$|f(t)| \leq \frac{R^{3/2}}{3} k^{-.2} \tag{5.20}$$

Therefore, by (5.20) and (5.17), along with Holder's inequality,

$$\begin{aligned} &\left| \int_{W_2} F(t) - e^{-q(t)} dt \right| \leq \\ &\left| \exp\left(\frac{256R}{r} k^{-.5} + i \frac{R^{3/2}}{3} k^{-.2} \right) - 1 \right| \int_{\mathcal{L}} e^{-q(t)} dt. \end{aligned} \tag{5.21}$$

Combining (5.9), (5.13), (5.14), (5.15), (5.16), (5.18), (5.19), and (5.21) completes the proof. The $k^{-.2}$ term in (5.21) is the dominant term as k goes to infinity and the other terms are held constant, and doubling it gives the example value of Γ given in Theorem I.3.

$\square$

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