TAUTOLOGICAL BUNDLES ON THE HILBERT SCHEME OF POINTS AND THE NORMALITY OF SECANT VARIETIES

by

Brooke Susanna Ullery

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Doctoral Committee:

Professor Karen E. Smith, Co-Chair
Professor Robert K. Lazarsfeld, Stony Brook University, Co-Chair
Professor Mircea I. Mustaţă
Professor David E. Speyer
Professor Martin J. Strauss

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CHAPTER I

Introduction

In this dissertation, we study the geometry of secant varieties and their connections to certain tautological bundles on Hilbert schemes of points. The main theorem, detailed in chapter IV, shows that the first secant variety to a projective variety embedded by a sufficiently positive line bundle is a normal variety. In particular, this confirms the vision and completes the results of Vermeire in [30] and renders unconditional the results in [27], [26], [32], and [31].[1]

Let

$$X \subset \mathbb{P}(H^0(X, \mathcal{L})) = \mathbb{P}^r$$

be a smooth variety over an algebraically closed field of characteristic zero, embedded by the complete linear system corresponding to a very ample line bundle $\mathcal{L}$. We define the kth secant variety

$$\Sigma_k(X, \mathcal{L}) \subset \mathbb{P}^r$$

to be the Zariski closure of the union of k-planes intersecting X in $k+1$ points (counting multiplicity) in $\mathbb{P}^r$. We will typically omit the subscript when discussing the first secant variety. As secant varieties are classical constructions in algebraic geometry, there has been a great deal of work done in an attempt to understand

[1] This question of the normality of the secant variety came up in 2001 when a proof was proposed by Vermeire [30]. However, in 2011, Adam Ginensky and Mohan Kumar pointed out that the proof was erroneous, as explained in Remark 4 of [27].

their geometry. Recently, there has been interest in determining defining equations and syzygies of secant varieties [5] [6] [7] [26] [27] [33], motivated in part by questions in algebraic statistics [12] [28] and algebraic complexity theory [19] [20]. In this dissertation, we focus on the singularities of the first secant variety and higher secant varieties to curves, using the comprehensive geometric description developed by Bertram [3] and Vermeire [30].

If the embedding line bundle $\mathcal{L}$ is not sufficiently positive, the behavior of the singularities of $\Sigma_k(X, \mathcal{L})$ can be quite complicated. For example, the first secant variety is generally singular along X, but if four points of X lie on a plane, then three pairs of secant lines will intersect away from X. In some cases this will create additional singularities at those intersection points on $\Sigma(X, \mathcal{L})$. In more degenerate cases, the secant variety may simply fill the whole projective space, e.g. the first secant variety to any non-linear plane curve. However, if $\mathcal{L}$ is sufficiently positive, we will see that $\Sigma(X, \mathcal{L})$ will be singular precisely along X. More generally, $\Sigma_k(X, \mathcal{L})$ will be singular precisely along $\Sigma_{k-1}(X, \mathcal{L})$. As $\mathcal{L}$ becomes increasingly positive, it is natural to predict that the singularities of the secant variety will become easier to control.

We start by stating some concrete special cases of the main theorem. In the case of curves, normality of the first secant variety only depends on a degree condition:

Corollary A. *Let X be a smooth projective curve of genus g and $\mathcal{L}$ a line bundle on X of degree d. If $d \geq 2g+3$, then $\Sigma(X, \mathcal{L})$ is a normal variety.*

Moreover, in the example of canonical curves, we have a stronger result not covered by the above proposition:

Corollary B. *Let X be a curve of genus g which is neither a plane sextic nor a* *four-fold cover of* $\mathbb{P}^1$*. Then* $\Sigma(X, \omega_X)$*, the secant variety of the canonical embedding of* X*, is a normal variety.*

In particular, the above implies that the first secant variety to a general canonical curve of genus at least 7 is normal.

More generally, we can also give a positivity condition on embeddings of higher dimensional varieties to ensure that the first secant variety is normal:

Corollary C. *Let* X *be a smooth projective variety of dimension* n*. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be very ample and nef, respectively, and*

$$\mathcal{L} = \omega_X \otimes \mathcal{A}^{\otimes 2(n+1)} \otimes \mathcal{B}.$$

Then $\Sigma(X, \mathcal{L})$ *is a normal variety.*

Before we state the main theorem, we must define k-very ampleness, a rough measure of the positivity of a line bundle:

A line bundle $\mathcal{L}$ on X is ***k-very ample*** if every length $k+1$ 0-dimensional subscheme $\xi \subseteq X$ imposes independent conditions on $\mathcal{L}$, i.e.

$$H^0(\mathcal{L}) \to H^0(\mathcal{L} \otimes \mathcal{O}_\xi)$$

is surjective.[2] In other words, $\mathcal{L}$ is 1-very ample if and only if it is very ample, and for any positive k, $\mathcal{L}$ is k-very ample if and only if no length $k+1$ 0-dimensional subscheme of X lies on a $(k-1)$-plane in $\mathbb{P}(H^0(\mathcal{L}))$.

Our main result is the following:

Theorem D. *Let* X *be a smooth projective variety, and* $\mathcal{L}$ *a 3-very ample line bundle on* X*. Let* m_x *be the ideal sheaf of* $x \in X$*. Suppose that for all* $x \in X$ *and* $i > 0$*, the natural map*

$$\mathrm{Sym}^i H^0(\mathcal{L} \otimes m_x^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes i} \otimes m_x^{\otimes 2i})$$

[2]Some sources, e.g. [27], [26], [30], and [33], call this property $(k+1)$-very ampleness.

is surjective.[3] *Then* $\Sigma(X, \mathcal{L})$ *is a normal variety.*

In chapter IV we prove the above theorem and corollaries.

Higher secant varieties tend to be more complicated. In Chapter IV, we see that even when restricting our attention to curves, it is significantly more difficult to control the singularities of the higher secant varieties. This is in part due to the fact that the singular locus is no longer just the original variety X, but rather the next lower secant variety, as mentioned above. Though we are unable to prove normality, we conjecture that it holds given a high enough degree embedding of the curve.

Conjecture E. *If* X *is a smooth projective curve of genus* g *and* $\mathcal{L}$ *a very ample line bundle on* X *such that* $\deg \mathcal{L} \geq 2g + 2n + 1$*, then* $\Sigma_n(X, \mathcal{L})$ *is a normal variety.*

To date, the best evidence toward the conjecture is our theorem below.

Theorem F. *Let* X *be a smooth projective curve, and* $\mathcal{L}$ *a* $(2n+1)$*-very ample line bundle on* X*, where* $n \geq 2$*. Suppose* $\Sigma_{n-1}(X, \mathcal{L}(-2x))$ *is projectively normal for all* $x \in X$*. Then* $\Sigma_n(X, \mathcal{L})$ *is normal along* X*.*

The above theorem shows that the normality along the curve is controlled by the projective normality of the next lower secant variety. According to a theorem of Sidman and Vermeire [26], under some hypotheses, the first secant variety is projectively normal. This leads to the following corollary.

Corollary G. *If* X *is a smooth projective curve of genus* g *and* $\mathcal{L}$ *a very ample line bundle on* X *such that* $\deg \mathcal{L} \geq 2g + 5$*, then* $\Sigma_2(X, \mathcal{L})$ *is normal along* X*.*

As described above, chapters IV and V are devoted to our results on the normality of secant varieties. In chapter II, we introduce our main piece of machinery:

[3] Note that this map is surjective for every i if and only if $b_x^*\mathcal{L} \otimes \mathcal{O}(-2E_x)$ (or simply $\mathcal{L}(-2x)$ when X is a curve) is normally generated, where b_x is the blow-up map of X at x, and E_x is the corresponding exceptional divisor.

tautological bundles on Hilbert schemes of points. In chapter III, we give some exposition and examples of secant varieties. We also describe the geometric setup relating Hilbert schemes to secant varieties that we will use in chapters IV and V.

CHAPTER II

Tautological Bundles on Hilbert schemes

In this entirely expository chapter, we introduce tautological bundles on Hilbert schemes and state some well-known results and examples. These bundles are the primary tools that we will use to understand the geometry of secant varieties in chapter III. Our notation and conventions will be the same as in the introduction.

2.1 The Hilbert scheme of points

2.1.1 Definitions

Let X be a smooth projective variety of dimension m. The ***Hilbert scheme of*** n ***points on*** X, denoted $X^{[n]}$, represents the functor of 0-dimensional length n subschemes of X. As such, there exists a universal family of subschemes, $\Phi_{X,n}$ called the ***universal subscheme*** of $X^{[n]}$. Set theoretically, it is the incidence variety

$$\Phi_{X,n} := \{(x, \xi) \in X \times X^{[n]} : x \in \xi\},$$

or just Φ when the context is clear. Let q and σ be the two projections as shown below:

$$\begin{tikzcd} & \Phi \arrow[dl, "q"'] \arrow[dr, "\sigma"] & \\ X & & X^{[n]} \end{tikzcd}.$$

Note that the fiber of σ over a subscheme $\xi \in X^{[n]}$ is isomorphic to the subscheme ξ itself.

Let $X^{(n)}$ denote the ***nth symmetric power of*** X, which parametrizes unordered n-tuples of points on X. The Hilbert scheme $X^{[n]}$ is equipped with a natural map called the ***Hilbert-Chow morphism***

$$\rho : X^{[n]} \to X^{(n)}.$$

Set-theoretically the map is obvious; it sends a subscheme to the corresponding 0-cycle, forgetting the scheme structure. In fact, it is also a morphism of schemes (see, for example, section 7.1 of [10] for the construction of the morphism). It fits into a diagram

$$\begin{tikzcd} & X^n \arrow[d] \\ X^{[n]} \arrow[r, "\rho"'] & X^{(n)} \end{tikzcd}$$

where the vertical map is the quotient by the S_n-action. Let $X_0^{[n]} \subset X^{[n]}$ and $X_0^{(n)} \subset X^{(n)}$ be the open loci parametrizing reduced subschemes and distinct n-tuples of points, respectively. Note that restricting ρ yields an isomorphism between $X_0^{[n]}$ and $X_0^{(n)}$. Thus,

$$\dim X_0^{[n]} = \dim X_0^{(n)} = \dim X^n = mn.$$

Furthermore, consider the open subset $X_*^{(n)} \subset X^{(n)}$ consisting of the 0-cycle supported on at least $n-1$ points. Define $X_*^{[n]} \subset X^{[n]}$ and $X_*^n \subset X^n$ to be the preimages of $X_*^{(n)}$ in the above diagram. Define

$$B_*^n := X_*^{[n]} \times_{X_*^{(n)}} X_*^n$$

so that we have the fiber square

$$\begin{tikzcd} B_*^n \arrow[r] \arrow[d] & X_*^n \arrow[d] \\ X_*^{[n]} \arrow[r, "\rho"'] & X_*^{(n)} \end{tikzcd} . \tag{2.1}$$

The following lemma provides a nice geometric description of B_*^n:

Lemma II.1 ([1] p. 60 and [11] Lemma 4.4)**.** *The map*

$$B^n_* \to X^n_*$$

is the blowup along $\Delta = \{(x_i) : x_i = x_j$ *for some* $i \neq j\}$ *and the map*

$$B^n_* \to X^{[n]}_*$$

is the quotient by the action of the symmetric group S_n*.*

In general, $X^{[n]}$ is very singular. In fact, it is generally reducible. Even understanding the geometry and singularities of the punctual Hilbert scheme, or $\rho^{-1}(n \cdot x)$ for any $x \in X$, is an enormous task that is far from complete. However, for small m and n, the geometry of $X^{[n]}$ is more understandable. In particular, when $n \leq 3$ or $\dim X = m \leq 2$, $X^{[n]}$ is smooth. We will be primarily concerned with the two simplest cases for the remainder of the dissertation: the case where $n = 2$ and the case where $m = 1$, or X is a curve.

2.1.2 The Hilbert Scheme of two points

Again let X be a variety of dimension m. The length two zero-dimensional subschemes of X come in two types: the reduced subschemes and the subschemes supported at a single point. Intuitively, we can think of the latter case as the choice of a point and a direction in the tangent space at that point. In fact, the universal subscheme of $X^{[2]}$ is

$$\Phi = \{(x, \xi) \in X \times X^{[2]} : x \in \xi\} \cong bl_\Delta(X^2), \tag{2.2}$$

the blowup of X^2 along the diagonal.

Moreover, applying lemma II.1, we have the Cartesian square

$$\begin{tikzcd} bl_\Delta(X^2) \arrow[r] \arrow[d] & X^2 \arrow[d] \\ X^{[2]} \arrow[r] & X^{(2)} \end{tikzcd} \tag{2.3}$$

where the vertical arrows are quotients by the involution, and the horizontal maps are the natural ones.

The fixed locus of $bl_\Delta(X^2)$ under the S_2 action is the exceptional locus, which is a divisor. Thus, the following lemma follows from the classical Chevalley-Shephard-Todd Theorem (see [4], §5 Theorem 4).

Lemma II.2. *If X is a smooth projective variety, then $X^{[2]}$ is smooth as well.*

2.1.3 Symmetric powers of curves

When X is a smooth curve, it can be shown that $X^{[n]} = X^{(n)}$ (see, for example, [10], Proposition 7.3.3). Furthermore, as mentioned above, we have the following lemma:

Lemma II.3. *Let X be a smooth projective curve. Then $X^{(n)}$ is a smooth projective variety of dimension n.*

This is again a classical lemma and has been proved many times over. One method of proof involves calculating the dimension of the tangent space using deformation theory. Another reduces to an analytic coordinate open subset of the curve and looks at S_n-invariant holomorphic functions. (See, for example, [10] Theorem 7.2.3 and [2] page 18, respectively.)

Just as in the case of the Hilbert scheme of two points, the universal subscheme of the Hilbert scheme of points on a curve has a nice geometric description. Since we can think of the points of $X^{(n)}$ as effective divisors of degree n on X, a point of $\Phi_{X,n}$ is of the form $(Q, D+Q)$, where $Q \in X$ and D is an effective divisor of degree $n-1$. Thus, we get a canonical isomorphism

$$\Phi_{X,n} \overset{\cong}{\to} X \times X^{(n-1)}$$

given by the map

$$(Q, Q+D) \mapsto (Q, D).$$

The natural map $\sigma : X \times X^{(n-1)} \to X^{(n)}$ is then given by addition of the coordinates. That is,

$$\sigma(Q, D) = Q + D.$$

Example II.4 ($X = \mathbb{P}^1$)**.** In the case where $X = \mathbb{P}^1$, all divisors of degree n are linearly equivalent. So, if D is a divisor of degree n and $|D|$ the corresponding linear system, then

$$(\mathbb{P}^1)^{(n)} \cong |D| \cong \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(n)) \cong \mathbb{P}^n.$$

2.2 Tautological bundles

In this section, assume that $\dim X \leq 2$ or $n \leq 3$. That is, we want to make sure that $X^{[n]}$ is smooth and irreducible.

2.2.1 Definition and basic properties

Just as in the previous section, $\Phi \subset X \times X^{[n]}$ is the universal subscheme of $X^{[n]}$, and we have the two projection maps below.

$$\begin{tikzcd} & \Phi \arrow[dl, "q"'] \arrow[dr, "\sigma"] & \\ X & & X^{[n]} \end{tikzcd}$$

Let $\mathcal{L}$ be a line bundle on X. Define the sheaf

$$\mathcal{E}_{n+1,\mathcal{L}} = \sigma_* q^* \mathcal{L},$$

or just $\mathcal{E}_{\mathcal{L}}$ when the context is clear. Since σ is flat (all of the fibers are finite and of the same length), $\mathcal{E}_{n+1,\mathcal{L}}$ is a locally free sheaf of rank n. The bundle $\mathcal{E}_{n+1,\mathcal{L}}$ is tautological in the sense that the fiber of $\mathcal{E}_{n+1,\mathcal{L}}$ over $\xi \in X^{[n]}$ is the global sections of $\mathcal{L}$ restricted the the corresponding subscheme of X. That is,

$$\text{fiber of } \mathcal{E}_{\mathcal{L}} \text{ over } \xi = H^0(X, \mathcal{L} \otimes \mathcal{O}_\xi).$$

Using the projection formula, we can compute the space of global sections of $\mathcal{E}_{\mathcal{L}}$:

$$H^0(\mathcal{E}_{\mathcal{L}}) = H^0(q^*\mathcal{L}) = H^0(\mathcal{L} \otimes q_*\mathcal{O}_\Phi).$$

Since q is proper with connected fibers (at least in the cases with which we are concerned), Stein factorization [14] implies that $q_*\mathcal{O}_\Phi = \mathcal{O}_X$. Thus

$$H^0(\mathcal{E}_{\mathcal{L}}) = H^0(\mathcal{L}). \tag{2.4}$$

Notice that by pushing forward the map

$$H^0(L) \otimes \mathcal{O}_\Phi \to q^*\mathcal{L}$$

and composing it with the natural map $\mathcal{O}_{X^{[n]}} \to \sigma_*\mathcal{O}_\Phi$, we get an evaluation map

$$ev : H^0(\mathcal{L}) \otimes \mathcal{O}_{X^{[n]}} \to \mathcal{E}_{\mathcal{L}}.$$

Over $\xi \in X^{[n]}$, the evaluation map on fibers is the restriction

$$H^0(\mathcal{L}) \to H^0(\mathcal{L}|_\xi).$$

Thus, the evaluation map is surjective on every fiber (and therefore surjective) if and only if every n-tuple of points imposes independent conditions on $\mathcal{L}$. Or, more precisely:

Lemma II.1. *The evaluation map $ev : H^0(\mathcal{L}) \otimes \mathcal{O}_{X^{[n]}} \to \mathcal{E}_{\mathcal{L}}$ is surjective if and only if $\mathcal{L}$ is $(n-1)$-very ample.*

Just as in the last section, we compute the example in which $X = \mathbb{P}^1$:

Example II.2. From example II.4, we recall that $(\mathbb{P}^1)^{[n]} = (\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$, where we think of $\mathbb{P}^n$ as the space of homogeneous n-forms on $\mathbb{P}^1$ up to scaling. Let

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(b),$$

where $b \geq n-1$. Then $\mathcal{L}$ is $(n-1)$-very ample.

We can think of $\Phi \subset \mathbb{P}^1 \times \mathbb{P}^n$ as pairs consisting of a point and a homogeneous n-form on $\mathbb{P}^1$ vanishing at that point. Since $\dim \Phi = n$, Φ is a divisor on $\mathbb{P}^1 \times \mathbb{P}^n$. Let F be a homogeneous n-form on $\mathbb{P}^1$. Then Φ intersects $\mathbb{P}^1 \times \{F\}$ at the n points (counting multiplicity) along which F vanishes. If P is a point in X, then Φ intersects $\{P\} \times \mathbb{P}^n$ at the homogeneous forms which vanishing along P, i.e. a hyperplane. Thus,

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(\Phi) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(n, 1).$$

We thus have a short exact sequence on $\mathbb{P}^1 \times \mathbb{P}^n$:

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(-n, -1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n} \to \mathcal{O}_\Phi \to 0.$$

Pulling back $\mathcal{L}$ along the projection map $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^1$, we get $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(b, 0)$. Thus, tensoring the above short exact sequence by this pullback yields the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(b-n, -1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(b, 0) \to q^*\mathcal{L} \to 0.$$

By definition, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(b-n, -1) = \mathcal{O}_{\mathbb{P}^1}(b-n) \boxtimes \mathcal{O}_{\mathbb{P}^n}(-1)$. Thus, by base change and the projection formula, the push-forward of this line bundle along the projection to the second factor is $H^0(\mathcal{O}_{\mathbb{P}^1}(b-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$. Using an analogous argument for the middle term yields the short exact sequence of sheaves on $\mathbb{P}^n$

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^1}(b-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to H^0(\mathcal{O}_{\mathbb{P}^1}(b)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}_{\mathcal{L}} \to 0.$$

The first map is multiplication by homogeneous n-forms on $\mathbb{P}^1$. The second map is the evaluation map, and thus surjective. (Equivalently, the left term of the previous short exact sequence has no nonzero higher direct images.) Therefore, the sequence above is indeed exact, and we have a description of $\mathcal{E}_{\mathcal{L}}$ as a quotient of decomposable vector bundles.

Lastly, it follows that the determinant bundle is

$$\det \mathcal{E}_{\mathcal{L}} \cong \mathcal{O}_{\mathbb{P}^n}(b-n+1).$$

2.2.2 Maps to Grassmannians

Let $\mathcal{L}$ be an $(n-1)$-very ample line bundle on X. Denote

$$\mathbb{G} = \mathrm{Gr}\,(n, H^0(L)),$$

the Grassmannian of n-dimensional quotients of $H^0(L)$.

The evaluation map

$$ev : H^0(L) \otimes \mathcal{O}_{X^{[n]}} \to \mathcal{E}_{n+1,\mathcal{L}}$$

is surjective by $(n-1)$-very ampleness. Thus, the maps on fibers are also surjective, and the evaluation map induces a morphism

$$\phi : X^{[n]} \to \mathbb{G},$$

where each point is sent to the corresponding map on fibers. That is,

$$\phi(\xi) = \left(H^0(L) \twoheadrightarrow H^0(L \otimes \mathcal{O}_\xi)\right).$$

So by the universal property of Grassmannians, the evaluation map is the pullback of the map to the universal quotient on the Grassmannian:

$$H^0(L) \otimes \mathcal{O}_{\mathbb{G}} \to Q \to 0.$$

Taking the nth exterior power of both maps, we see that $\bigwedge^n H^0(\mathcal{L}) \to \det \mathcal{E}_{n+1,\mathcal{L}}$ is the pullback of the map defining the Plücker embedding, and we get the commutative diagram

$$\begin{tikzcd} X^{[n]} \arrow[r, "\phi"] \arrow[dr, "\det \mathcal{E}_{n+1,\mathcal{L}}"'] & \mathbb{G} \arrow[d, "\det Q"] \\ & \mathbb{P}\left(\bigwedge^n H^0(\mathcal{L})\right) \end{tikzcd} .$$

Note that the map ϕ will be injective when no two length n subschemes determine the same $(n-1)$ plane. Thus, as long as $\mathcal{L}$ is n-very ample, ϕ will be injective.

CHAPTER III

Secant varieties

In this chapter, we introduce secant varieties, the primary objects of our focus. In section 3.1 we give some definitions and discuss a few examples of mathematical areas in which secant varieties have shown up. In sections 3.2 and 3.3, we discuss the geometry of secant varieties in terms of tautological bundles, originally described by Bertram and Vermeire in [3] and [30], respectively.

3.1 Secant Varieties

3.1.1 Introduction to secant varieties

Let $\mathcal{L}$ be a very ample line bundle on X so that

$$X \subset \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^r,$$

thinking of the points of $\mathbb{P}(H^0(\mathcal{L}))$ as the one-dimensional quotients of $H^0(\mathcal{L})$. Recall from the introduction that the kth secant variety to X,

$$\Sigma_k(X, \mathcal{L}) \subset \mathbb{P}^r,$$

is the Zariski closure of the union of secant k-planes passing through $k+1$ distinct points of X.

In some cases, for instance when $k = 1$ or X is curve, $\Sigma_k(X, \mathcal{L})$ is the union of the secant and tangent k-planes. However, this will not hold more generally, in part due to the complicated geometry of the punctual Hilbert scheme. This will become more clear when we illustrate the connection to Hilbert schemes in the next section.

The secant k-planes are spanned (sometimes in multiple ways) by $k+1$ distinct points of X. Thus

$$\dim \Sigma_k(X, \mathcal{L}) \leq mk + k, \tag{3.1}$$

where $m = \dim X$. When equality holds, we say that $\Sigma_k(X, \mathcal{L})$ has the expected dimension. When equality does not hold, $\Sigma_k(X, \mathcal{L})$ is ***defective***. There has been a great deal of work done in trying to understand defective secant varieties, and much is still unknown. In fact, it is unknown whether there is a bound on the deficiency (the difference between the actual and expected dimensions) of secant varieties. Throughout the rest of the dissertation, we will primarily be concerned with non-defective secant varieties. In fact, we will impose a stronger condition, detailed in sections 3.2 and 3.3.

Although secant varieties are classical constructions, and it is always beneficial to understand the geometry of known examples of algebraic varieties, it may still seem somewhat arbitrary to study this single family of varieties. However, secant varieties do in fact appear in many areas of algebraic geometry, some quite unexpected. In the next few sections, we discuss a few of these applications.

3.1.2 A Whitney-type Embedding Theorem for varieties

Possibly one of the first appearances of secant varieties was in connection with projection maps. By simply applying the upper bound of the dimension of secant varieties, we get an almost immediate proof of a Whitney-type embedding theorem for varieties. The Whitney embedding theorem is a classical result in differential topology stating that any smooth m-dimensional real manifold can be embedded in

$\mathbb{R}^{2m}$. An analogous result holds for algebraic varieties:

Theorem III.1 ([25], chapter II, §5.4)**.** *Let X be a smooth projective variety of dimension m. Then there exists an embedding $X \hookrightarrow \mathbb{P}^{2m+1}$.*

Proof. Since X is projective, we have an embedding $X \subset \mathbb{P}^N$, for some N. If $N \leq 2m+1$, we are done, so assume $N > 2m+1$. Let $y \in \mathbb{P}^N \backslash X$. The projection map from y, $\mathbb{P}^N \to \mathbb{P}^{N-1}$ will map X isomorphically onto its image if and only if y is not on a secant or tangent line; this is because it will be a bijection, but also an immersion since it won't collapse any tangent vectors. Since the first secant variety is the union of the secant and tangent lines of X, this is equivalent to y not lying on the secant variety. By the inequality (3.1),

$$\dim \Sigma \leq 2m+1 < N.$$

Thus, such an y exists, and in fact, a general point of $\mathbb{P}^N$ will work. The statement of the theorem follows by induction. □

3.1.3 Secant varieties to Segre varieties

As mentioned in the introduction, a motivation for some recent work on secant varieties is to answer questions in algebraic statistics [12] [28] and algebraic complexity theory [19] [20]. In particular, most of these questions deal with secant varieties to Segre embeddings.

Recall that a Segre variety is the image of the Segre embedding

$$s : \mathbb{P}^{m+1} \times \mathbb{P}^{n+1} \to \mathbb{P}^{mn+1}.$$

More generally (and more intrinsically), given a collection of vector spaces $V_1, \dots, V_m$, the Segre is given by the natural map

$$s : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) \to \mathbb{P}(V_1 \otimes \cdots \otimes V_m).$$

The image of the map, the Segre variety, corresponds to the rank 1, or simple, tensors, up to scaling. Furthermore, the rank two tensors are those which can be written as a linear combination of two simple tensors. Of course, in the projectivization, these correspond to points along the secant line connecting the corresponding points on the Segre variety. More generally, the rank k tensors correspond to points on secant $(k-1)$-planes.

However, we have to be a bit careful. Just like the union of the secant $(k-1)$-planes, the variety of rank k tensors is generally not Zariski closed when the Segre variety is the product of at least three projective spaces. A tensor has ***border rank*** k if it is the limit of rank k tensors but is not the limit of rank $k-1$ tensors. This definition thus leads to the following well-known fact:

Lemma III.2. *The kth secant variety of the image of $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m)$ under the Segre embedding is equal to the locus of tensors in $\mathbb{P}(V_1 \otimes \cdots \otimes V_m)$ of border rank at most $k+1$.*

3.1.4 Secant varieties to Veronese varieties

Another example that is related to the previous one is the Veronese embedding and secant varieties to Veronese varieties. Recall that a Veronese embedding is the embedding of $\mathbb{P}^m$ into a larger projective space by a the complete linear system of $\mathcal{O}_{\mathbb{P}^m}(n)$. In other words, it is the natural map

$$v : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d V).$$

The image of the map is the Veronese variety. We can think of the elements of $\mathrm{Sym}^d V$ as homogenous degree d polynomials. The Veronese variety then corresponds to rank 1 polynomials, i.e. those that can be written as the dth power of a linear form. Just as in the case of Segre varieties, we can thus describe the kth secant variety of the

Veronese variety as the closure of rank n degree d polynomials.

Lemma III.3. *The kth secant variety of the image of $\mathbb{P}(V)$ under the Veronese embedding is equal to the locus of homogeneous degree d polynomials of border rank at most $k+1$. That is, limits of polynomials of the form*

$$L_1^d + L_2^d + \cdots + L_{k+1}^d,$$

where each L_i is linear.

The first nontrivial example is, naturally, the twisted cubic.

Example III.4. Let $X = \mathbb{P}^1$, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(3)$. The complete linear system corresponding to $\mathcal{L}$ embeds $\mathbb{P}^1$ into $\mathbb{P}^3$ as the twisted cubic. As pointed out previously, $\mathcal{O}(3)$ is 3-very ample. That is, no length 3 subscheme of X is collinear in $\mathbb{P}^3$. This is the same as saying that two secant or tangent lines will never meet off X. (We will go into more depth about this idea in the next section.) Thus, $\Sigma(X, \mathcal{L})$ is not defective, so, since $\dim X = 3$, $\Sigma(X, \mathcal{L})X = \mathbb{P}^3$. More over, every point in $\Sigma(X, \mathcal{L})\backslash X = \mathbb{P}^3\backslash X$ lies on a unique tangent or secant line.

What does this mean in terms of homogeneous polynomials? It means that every degree 3 homogeneous polynomial in two variables that is not already the power of a linear form can be written *uniquely* in the form $L_1^3 + L_2^3$, where each L_i is linear.

In general, questions about whether a point is contained in a secant variety can be very difficult. In fact, there is a famous open question called Waring's problems for polynomials [17]: if F is a homogeneous polynomial of degree d, what is the minimum k so that

$$F = L_1^d + L_2^d + \cdots + L_k^d?$$

Of course, from the above exposition, this is equivalent to asking which secant variety of a Veronese variety a given point is on.

3.1.5 Application to vector bundle stability

One of the most surprising applications of secant varieties is to the stability of vector bundles on curves. This application is due to Bertram [3].

First we recall some definitions. Let X be a smooth projective curve. Let $\mathcal{E}$ be a vector bundle on X of rank r. The ***degree*** of $\mathcal{E}$ is the degree of the determinant, or top exterior power, of $\mathcal{E}$, which is a line bundle. That is,

$$\deg \mathcal{E} = \deg(\det \mathcal{E}).$$

The ***slope*** of $\mathcal{E}$ is

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{r}.$$

$\mathcal{E}$ is ***stable*** (respectively ***semistable***) if for every quotient vector bundle $\mathcal{E} \twoheadrightarrow \mathcal{F}$, $\mu(\mathcal{E}) < \mu(\mathcal{F})$ (respectively $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$).

The moduli space $\mathscr{M}_{r,\mathcal{L}}$ of semistable bundles of rank r and determinant $\mathcal{L}$ is a very widely studied object. In fact, according to a well-known result of Mumford [23], $\mathscr{M}_{r,\mathcal{L}}$ is a projective variety.

In this example, we will be concerned with rank 2 vector bundles. Let $\mathcal{L}$ be a very ample line bundle and ω the canonical line bundle of X. The extension group $\operatorname{Ext}^1(\mathcal{L}, \omega_X)$ parametrizes short exact sequences of the form

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{B} \to 0,$$

where $\mathcal{B} = \mathcal{L} \otimes \omega_X^*$ and $\mathcal{E}$ is a rank two vector bundle. Let $b = \deg \mathcal{B}$.

By Serre duality,

$$\operatorname{Ext}^1(\mathcal{L}, \omega_X) \cong H^0(\mathcal{L})^*.$$

Thus, since $\mathcal{L}$ is very ample, we have an embedding

$$X \hookrightarrow \mathbb{P}(\operatorname{Ext}^1(\mathcal{L}, \omega_X)).$$

Note that scaling an element of $\operatorname{Ext}^1(\mathcal{L}, \omega_X)$ by a constant does not affect the rank two vector bundle in the corresponding short exact sequence; it merely alters the maps. Thus, each point of the projective space $\mathbb{P}(\operatorname{Ext}^1(\mathcal{L}, \omega_X))$ corresponds to some rank 2 vector bundle. In order to get a rational map $\phi_{\mathcal{L}} : \mathbb{P}(\operatorname{Ext}^1(\mathcal{L}, \omega_X)) \to \mathscr{M}_{2,\mathcal{L}}$, we need to show that a general vector bundle in $\operatorname{Ext}^1(\mathcal{L}, \omega_X)$ is semistable.

Let

$$(*) : 0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{B} \to 0$$

be a class in $\operatorname{Ext}^1(\mathcal{L}, \omega_X)$. Let $\mathcal{M}$ be a line bundle on X of minimum degree such that there is a quotient $\mathcal{E} \twoheadrightarrow \mathcal{M}$. Then we have the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, two heads] & \mathcal{B} \arrow[r] & 0 \\ & & \mathcal{M} & & \end{tikzcd}$$

Of course, $\deg \mathcal{M} \leq b = \deg \mathcal{B}$, so the composite map $\mathcal{O}_X \to \mathcal{M}$ must be nonzero. Otherwise, $\mathcal{E} \to \mathcal{M}$ would factor through $\mathcal{B}$. Let s be the section of $\mathcal{M}$ obtained by this construction. The section s corresponds to a specific effective divisor in $|\mathcal{M}|$. Call this divisor D. By inspection of the isomorphism $\operatorname{Ext}^1(\mathcal{L}, \omega_X) \cong H^0(\mathcal{L})^*$, one can show that the point corresponding to (*) in the projective space lies on the span of the points of D. This means that the "maximally unstable" bundles, i.e. those destabilized by a line bundle of degree one, correspond to points along the image of X in the projective space. The "second most unstable" bundles correspond to points that lie on the first secant variety $\Sigma(X, \mathcal{L})$, and so on.

Furthermore, the unstable bundles in $\mathbb{P}(\operatorname{Ext}^1(\mathcal{L}, \omega_X))$ are those which correspond to points on the span of fewer than $b/2$ points. Let $b' = \lfloor b/2 \rfloor$. Then the unstable bundles are those those which correspond to points on $\Sigma_{b'-1}(X,\mathcal{L})$. Notice that

$$\dim \Sigma_{b'-1}(X,\mathcal{L}) = 2(b'-1)+1 = 2b'-1 = \begin{cases} b-1 & b \text{ even} \\ \\ b-2 & b \text{ odd} \end{cases}.$$

By the Riemann-Roch theorem,

$$\dim \mathbb{P}(H^0(\mathcal{L})^*) \geq \deg \mathcal{L} - g = b + (2g-2) - g = b + g - 2,$$

which is greater than the dimension of $\mathrm{Ext}^1(\mathcal{L},\omega_X))$ as long as the genus g is at least 2 (or at least 1 when b is odd). Thus, in these cases, $\phi_{\mathcal{L}}$ is in fact a rational map to $\mathscr{M}_{2,\mathcal{L}}$, as desired.

3.2 Geometry of the first secant variety

In this section, we describe the geometric connection between Hilbert schemes of two points and the first secant variety, detailed in the case of curves in [3], and extended to higher dimensions in [30]. This will be the geometric setup for chapter IV.

3.2.1 Geometric setup

Let $\mathcal{L}$ be a very ample line bundle on a smooth variety X and

$$X \hookrightarrow \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^r$$

the corresponding embedding, again treating the points of $\mathbb{P}(H^0(\mathcal{L}))$ as the one dimensional quotients of $H^0(\mathcal{L})$.

Recall that $X^{[2]}$ is smooth, and its universal subscheme is the incidence variety

$$\Phi = \{(x,\xi) \in X \times X^{[2]} : x \in \xi\} \cong bl_{\Delta}(X \times X), \tag{3.2}$$

the blowup of $X \times X$ along the diagonal. Moreover, we have the Cartesian square

$$\begin{tikzcd} bl_\Delta(X \times X) \arrow[r] \arrow[d] & X \times X \arrow[d] \\ X^{[2]} \arrow[r] & \mathrm{Sym}^2 X \end{tikzcd} \tag{3.3}$$

where the vertical arrows are quotients by the involution, and the horizontal maps are the natural ones. Note that when X is a curve, the horizontal maps are isomorphisms.

Let q and σ be the two projections as shown below:

$$\begin{tikzcd} & \Phi \arrow[dl, "q"'] \arrow[dr, "\sigma"] & \\ X & & X^{[2]} \end{tikzcd} .$$

Recall the tautological vector bundle

$$\mathcal{E}_{\mathcal{L}} = \sigma_* q^* \mathcal{L}.$$

Since $\mathcal{L}$ is very ample, the map

$$H^0(\mathcal{L}) \otimes \mathcal{O}_{X^{[2]}} \to \mathcal{E}_{\mathcal{L}}$$

is surjective and induces a morphism

$$f : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \to \mathbb{P}^r.$$

We can think of the points of $\mathbb{P}(\mathcal{E}_{\mathcal{L}})$ as pairs $(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow Q)$, where Q is a one-dimensional quotient, and ξ is a point of $X^{[2]}$. Thus,

$$f(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow Q) = (H^0(\mathcal{L}) \twoheadrightarrow Q) \in \mathbb{P}^r. \tag{3.4}$$

Notice that the image of f is $\Sigma(X, \mathcal{L})$, since the surjections in the image are precisely those which factor through $H^0(\mathcal{L} \otimes \mathcal{O}_\xi)$ for some $\xi \in X^{[2]}$.

3.2.2 Resolution of singularities of $\Sigma(X, \mathcal{L})$

Let

$$t : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \twoheadrightarrow \Sigma(X, \mathcal{L})$$

be f with its target restricted.

The following lemma is adapted from [3] in the case of curves and [30] for higher dimensions.

Lemma III.1. *Suppose $\mathcal{L}$ is 3-very ample. Then $t : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \to \Sigma(X, \mathcal{L})$ is an isomorphism away from $t^{-1}(X)$. In particular, t is a resolution of singularities.*

Proof. For clarity, we first show that t is a bijection away from $t^{-1}(X)$, which follows nearly immediately from the 3-very ampleness of $\mathcal{L}$:

Given a length two 0-dimensional subscheme ξ, points of the form $(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow Q) \in \mathbb{P}(\mathcal{E}_{\mathcal{L}})$ map bijectively to the secant line spanned by ξ. Since $\mathcal{L}$ is 3-very ample, no two distinct length two subschemes will correspond to the same secant line. Thus, the only way for t not to be a bijection away from $t^{-1}(X)$ would be for two secant lines of X to intersect away from X. This would cause four points of X to lie on a plane in $\mathbb{P}^r$, which contradicts the 3-very ampleness of $\mathcal{L}$.

In order to show that t is actually an isomorphism away from $t^{-1}(X)$, we need to check that it is an immersion. This follows in the curve case from Lemma 1.4 of [3], and in the higher dimensional case from Theorem 3.9 of [30]. In the former, Bertram proves that it is an immersion directly. In the latter, Vermeire shows that $\mathbb{P}(\mathcal{E}_{\mathcal{L}})$ is isomorphic to the blowup of $\Sigma(X, \mathcal{L})$ along X, which clearly implies what we need. $\square$

For our purposes, it will be useful to also understand $t^{-1}(X)$. Looking at (3.4), we see that

$$t^{-1}(X) = f^{-1}(X) = \{(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_x) : x \in \xi\},$$

which is set-theoretically equal to Φ (defined in (3.2)). In fact, a lemma of Vermeire implies that it is actually an isomorphism:

Lemma III.2 ([30], Lemma 3.8)**.** *The scheme-theoretic inverse image $t^{-1}(X)$ is isomorphic to $bl_\Delta(X \times X)$.*

From now on, we will refer to $t^{-1}(X)$ as simply Φ. Notice that $t\big|_\Phi = q$, and for $x \in X$, the fiber is

$$F_x := t^{-1}(x) = \{\xi : x \in \xi\} \cong bl_x(X),$$

which is simply X when X is a curve.[1]

Let

$$\pi : \mathbb{P}(\mathcal{E}_\mathcal{L}) \to X^{[2]}$$

be the projection map. Notice that $\pi\big|_\Phi = \sigma$. Furthermore, $\pi\big|_{F_x}$ is an isomorphism, as F_x is a section over $\pi(F_x)$. When the context is clear, we will refer to $\pi(F_x)$, the points of $X^{[2]}$ whose corresponding subschemes contain x, as simply F_x.

3.2.3 Useful diagrams

To summarize, we have the following two commutative diagrams, to which we will refer back in chapter IV:

(3.5)
$$\begin{tikzcd}
F_x \cong bl_x(X) \arrow[r, hook] \arrow[d] & \Phi \cong bl_\Delta(X^2) \arrow[r, hook] \arrow[d, "q"] & \mathbb{P}(\mathcal{E}_\mathcal{L}) \arrow[d, "t"] \arrow[dr, "f"] & \\
\{x\} \arrow[r, hook] & X \arrow[r, hook] & \Sigma(X, \mathcal{L}) \arrow[r, hook] & \mathbb{P}^r
\end{tikzcd}$$

[1] All of the arguments for the remainder of this section and chapter IV go through in the case of curves by replacing E_x with x. From now on, this will be assumed.

and

$$\begin{tikzcd} F_x \arrow[r, hook] \arrow[d, "\cong"] & \Phi \arrow[r, hook] \arrow[d, "\sigma"] & \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \arrow[dl] \\ F_x \arrow[r, hook] & X^{[2]} & \end{tikzcd} \tag{3.6}$$

Since t is a resolution of singularities, and hence a birational map from a normal variety, our strategy for showing $\Sigma(X, \mathcal{L})$ is normal is to show $t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})} = \mathcal{O}_{\Sigma(X,\mathcal{L})}$ by exploiting the geometry of Φ and F_x.

3.3 Geometry of higher secant varieties to curves

In this section, we set up the parallel story of the geometry of the higher secant varieties of curves. Though at some level, the framework is very similar to the previous section, the geometry of higher secant varieties of a curve is substantially more complicated than the geometry of the first secant of a higher dimensional variety. In fact, our main theorem, detailed in chapter IV, only holds in the latter case. As such, we treat the two cases separately. We will use the material from this section in chapter V when we give some lemmas and conjectures toward the normality of higher secant varieties to curves. The material in this section is also based on Bertram's paper [3].

3.3.1 Geometric setup

Let X be a smooth projective curve of genus g, and $\mathcal{L}$ an n-very ample line bundle embedding X into $\mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^r$. Recall that in this case, $X^{[n+1]}$ is smooth, and $X^{[n+1]} = X^{(n+1)}$. It's universal subscheme is

$$\Phi = \Phi_{X,n+1} \overset{\cong}{\to} X \times X^{(n)},$$

and as before we have the two maps

$$\begin{tikzcd} & X \times X^{(n)} \arrow[dl, "q"'] \arrow[dr, "\sigma"] & \\ X & & X^{(n+1)} \end{tikzcd}.$$

The map q is the projection, and σ takes the sum of the two factors.

Recall that the vector bundle $\mathcal{E}_{n+1,\mathcal{L}} = \sigma_* q^* \mathcal{L}$ has rank $n+1$. By n-very ampleness, the evaluation map

$$H^0(\mathcal{L}) \otimes \mathcal{O}_{X^{(n+1)}} \to \mathcal{E}_{n+1,\mathcal{L}}$$

is surjective and again induces a morphism

$$f : \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \to \mathbb{P}^r.$$

Notice that the fiber over a subscheme $\xi \in X^{(n+1)}$ is sent by f to the n-plane spanned by ξ. Thus, the image of f is $\Sigma_n(X, \mathcal{L})$.

3.3.2 Resolution of singularities of $\Sigma_n(X, \mathcal{L})$

As in the previous section, let $t : \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \twoheadrightarrow \Sigma_n(X, \mathcal{L})$ be equal to f with its target restricted.[2] Again, t is a resolution of singularities, with slightly stronger hypotheses than in the case of the first secant variety.

This lemma is adapted from [3].

Lemma III.1. *Suppose $\mathcal{L}$ is $(2n+1)$-very ample. Then $t : \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \to \Sigma_n(X, \mathcal{L})$ is an isomorphism away from $t^{-1}(\Sigma_{n-1}(X, \mathcal{L}))$. In particular, t is a resolution of singularities.*

Proof. First we show that t is a bijection away from $t^{-1}(\Sigma_{n-1}(X, \mathcal{L}))$. Given a degree $n+1$ divisor ξ of X, points of the form $(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow Q) \in \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})$ are sent

[2] We recognize the slight abuse of notation, since we also named the analogous maps in the previous section f and t. This is to avoid excess notation. However, there should be no confusion since we are treating the two cases entirely separately.

bijectively, via t, to the n-plane spanned by ξ. This follows from n-very ampleness. Thus, we just need to show that if the n-planes spanned by two different divisors meet, they meet along the smaller secant varieties.

Let $\xi \neq \xi'$ be two degree $n+1$ divisors, spanning the n-planes H and H', respectively. Their intersection

$$Z = \xi \cap \xi'$$

has degree m, where $0 \leq m \leq n$. Since $\mathcal{L}$ is certainly $(m-1)$-very ample, Z spans an $(m-1)$-plane, ℓ. We will show that H and H' don't meet away from ℓ. The union of ξ and ξ' has degree $2n+2-m \leq 2n+2$. Thus, since $\mathcal{L}$ is $(2n+1-m)$-very ample, H and H' span a $2n+1-m$ dimensional space, which means that there intersection has dimension exactly $m-1$. Thus,

$$H \cap H' = \ell \subset \Sigma_{m-1}(X, \mathcal{L}) \subseteq \Sigma_{n-1}(X, \mathcal{L}).$$

The fact that t is an immersion away from $t^{-1}(\Sigma_{n-1}(X, \mathcal{L}))$ follows from Lemma 1.4 of [3], and we are done. $\square$

It follows from this lemma that $\Sigma_n(X, \mathcal{L})$ is smooth away from $\Sigma_{n-1}(X, \mathcal{L})$. However, it is important to note that one can show it is singular at every point of $\Sigma_{n-1}(X, \mathcal{L})$.

Now that we have this resolution of singularities, it will be useful to get a better understanding of the exceptional locus. The exceptional locus itself, $t^{-1}(\Sigma_{n-1}(X, \mathcal{L}))$, is singular and quite complicated. In fact, the intersection with each fiber of $\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})$ is the union of $n+1$ $(n-1)$-planes, counting multiplicity. However, the preimage of X is more readily understandable. Thinking as points of $\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})$ as pairs of a subscheme and a one dimensional quotient, we have

$$t^{-1}(X) = f^{-1}(X) = \{(\xi, H^0(\mathcal{L} \otimes \mathcal{O}_\xi) \twoheadrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_x)) : x \in \xi\}.$$

Just as in the previous section, this is set-theoretically equal to the incidence variety $\Phi \cong X \times X^{(n)}$. By the discussion in Section 1 of [3], this is actually an isomorphism. That is,

$$t^{-1}(X) \cong X \times X^{(n)}.$$

As such, we will from now on refer to $t^{-1}(X)$ as Φ.

Notice that $t\big|_{\Phi} = q$, the projection onto the first factor. Thus, for $x \in X$, the fiber is

$$F_x := t^{-1}(x) = q^{-1}(x) = \{x\} \times X^{(n)} \cong X^{(n)}.$$

Let

$$\pi : \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \to X^{(n+1)}$$

be the projection map. Then, analogous to the previous section, $\pi\big|_{\Phi} = \sigma$, the addition map, and $\pi\big|_{F_x}$ is an isomorphism, as F_x is a section over $\pi(F_x)$, the locus of divisors that contain x. Again, we may abuse notation and refer to $\pi(F_x)$ as F_x when the context is clear.

Of course, F_x isn't the only kind of fiber over a singular point. Let $y \in \Sigma_{n-1}(X, \mathcal{L})$ be a point in the singular locus, but not in X. Let m be the minimum number so that $y \in \Sigma_{m-1}(X, \mathcal{L})$. Then there is a unique degree m divisor D such that y lies in the $(m-1)$-plane spanned by D. Let $H^0(\mathcal{L} \otimes \mathcal{O}_D) \twoheadrightarrow Q_y$ be the one dimensional quotient corresponding to y. Then we have

$$F_{y,D} := t^{-1}(y) = \{(\xi, H^0(\mathcal{L}\otimes\mathcal{O}_\xi) \twoheadrightarrow Q_y) : \xi \supset D\} \cong \{\xi \in X^{(n+1)} : D \subset \xi\} \cong X^{(n+1-m)}.$$

3.3.3 More useful diagrams

For completeness and clarity, we will reproduce the diagrams from the previous section, identical in notation, but very different in geometry as we saw above:

(3.7)
$$
\begin{tikzcd}
F_x \cong \{x\} \times X^{(n)} \arrow[r, hook] \arrow[d, two heads] & \Phi \cong X \times X^{(n)} \arrow[r, hook] \arrow[d, two heads, "q"] & \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \arrow[d, two heads, "t"] \arrow[dr, two heads, "f"] & \\
\{x\} \arrow[r, hook] & X \arrow[r, hook] & \Sigma_n(X, \mathcal{L}) \arrow[r, hook] & \mathbb{P}^r
\end{tikzcd}
$$

and

(3.8)
$$
\begin{tikzcd}
F_x \cong \{x\} \times X^{(n)} \arrow[r, hook] \arrow[d, "\cong"] & \Phi \arrow[r, hook] \arrow[d, two heads, "\sigma"] & \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \arrow[dl, two heads] \; . \\
F_x \cong X^{(n)} \arrow[r, hook] & X^{(n+1)} &
\end{tikzcd}
$$

We will refer back to these diagrams in chapter V.

CHAPTER IV

Normality of the first secant variety

In this chapter, we present our results about the first secant variety, following the geometric setup in section 3.2. In the first section, we prove the main theorem. In section 4.2, we prove some corollaries that help illustrate the power of the theorem. In the last section of this chapter, we discuss a few theorems and conjectures of Sidman and Vermeire that use the normality of secant varieties as a hypothesis. This chapter is taken from our paper [29].

4.1 Proof of the main theorem

In this section, we give the proof of Theorem D, which we have restated below:

Theorem D. *Let X be a smooth projective variety, and $\mathcal{L}$ a 3-very ample line bundle on X. Let m_x be the ideal sheaf of $x \in X$. Suppose that for all $x \in X$ and $i > 0$, the natural map*

$$\mathrm{Sym}^i H^0(\mathcal{L} \otimes m_x^{\otimes 2}) \to H^0(\mathcal{L}^{\otimes i} \otimes m_x^{\otimes 2i})$$

is surjective.[1] *Then $\Sigma(X, \mathcal{L})$ is a normal variety.*

[1] Note that this map is surjective for every i if and only if $b_x^*\mathcal{L} \otimes \mathcal{O}(-2E_x)$ (or simply $\mathcal{L}(-2x)$ when X is a curve) is normally generated, where b_x is the blow-up map of X at x, and E_x is the corresponding exceptional divisor.

4.1.1 Preliminary lemmas

We begin by observing that the normality of the secant variety $\Sigma(X, \mathcal{L})$ is controlled by the geometry of the conormal bundle to F_x. Recall that

$$F_x = t^{-1}(x) \cong bl_x X,$$

where $x \in X$ and

$$t : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \to \Sigma(X, \mathcal{L})$$

is the resolution of singularities.

Lemma IV.1. *Let $\mathcal{L}$ be a 3-very ample line bundle on X. Let $x \in X$, and let $\alpha_{x,k}$ be the natural map*

$$\alpha_{x,k} : \mathrm{Sym}^k(T_x^* \mathbb{P}^r) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}).$$

If $\alpha_{x,k}$ is surjective for all $k > 0$ and all $x \in X$, then $\Sigma(X, \mathcal{L})$ is a normal variety.

Proof. We have the following natural maps of sheaves:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^r} \arrow[r, two heads] \arrow[d] & \mathcal{O}_{\Sigma(X,\mathcal{L})} \arrow[dl, hook] \\
t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})} &
\end{tikzcd}.$$

As pointed out at the end of section 3.2, if $t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})} = \mathcal{O}_{\Sigma(X,\mathcal{L})}$, then $\Sigma(X, \mathcal{L})$ is normal. So we need to show $\mathcal{O}_{\Sigma(X,\mathcal{L})} \to t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}$ is surjective. Thus, by the above diagram, it suffices to show $\mathcal{O}_{\mathbb{P}^r} \to t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}$ is surjective.

The map $\mathcal{O}_{\mathbb{P}^r} \to t_*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}$ is surjective if and only if the completion of the map is surjective at every point $x \in \Sigma(X, \mathcal{L})$. However, we only need to check this for $x \in X$, since $\mathbb{P}(\mathcal{E}_{\mathcal{L}})$ is smooth, and t is an isomorphism away from $t^{-1}(X)$ by Lemma III.1.

Let

$$\mathcal{I}_x = \text{the ideal sheaf of } F_x \subseteq \mathbb{P}(\mathcal{E}_{\mathcal{L}})$$

and

$$m_x = \text{the ideal sheaf of } x \in \mathbb{P}^r.$$

Then by the theorem on formal functions (see [15] Theorem 11.1), we need to show that the map

$$\Psi_x : \varprojlim \left(\mathcal{O}_{\mathbb{P}^r}/m_x^k\right) \to \varprojlim \left(H^0\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}/\mathcal{I}_x^k\right)\right)$$

is surjective for each $x \in X$.

Consider the following diagram:

(4.1)

$$\begin{tikzcd}
0 \arrow[r] & m_x^k/m_x^{k+1} \arrow[r] \arrow[d, "\alpha_{x,k}"] & \mathcal{O}_{\mathbb{P}^r}/m_x^{k+1} \arrow[r, "a"] \arrow[d, "\Psi_{x,k+1}"] & \mathcal{O}_{\mathbb{P}^r}/m_x^k \arrow[r] \arrow[d, "\Psi_{x,k}"] & 0 \\
0 \arrow[r] & H^0\left(\mathcal{I}_x^k/\mathcal{I}_x^{k+1}\right) \arrow[r] & H^0\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}/\mathcal{I}_x^{k+1}\right) \arrow[r, "b"] & H^0\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}/\mathcal{I}_x^k\right) \arrow[r, "c"] & H^1\left(\mathcal{I}_x^k/\mathcal{I}_x^{k+1}\right) \arrow[r] & \cdots
\end{tikzcd}.$$

Note that we have canonical isomorphisms

$$m_x^k/m_x^{k+1} \cong \mathrm{Sym}^k(T_x^*\mathbb{P}^r)$$

and

$$\mathcal{I}_x^k/\mathcal{I}_x^{k+1} \cong \mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}.$$

We claim that it suffices to show all the vertical maps are surjective for all k: Assume the vertical maps are surjective. Then the snake lemma says that

$$\ker \Psi_{x,k+1} \to \ker \Psi_{x,k}$$

is surjective for all k. In particular, the inverse system $(\ker \Psi_{x,k})$ satisfies the Mittag-Leffler condition (see II.9 of [15]). Thus, by Prop II.9.1(b) of [15], Ψ_x is surjective. Thus, we are reduced to showing that the vertical arrows are surjections.

We claim that if the left vertical arrow $\alpha_{x,k}$ is surjective for all k, then $\Psi_{x,k}$ is surjective for all k. We show this by induction.

The base case is $k = 1$: Consider the map

$$\Psi_{x,1} : \mathcal{O}_{\mathbb{P}^r}/m_x \to H^0\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}/\mathcal{I}_x\right) = H^0(\mathcal{O}_{F_x}).$$

Since F_x is reduced and irreducible, $h^0(\mathcal{O}_{F_x}) = 1$, and since $\Psi_{x,1}$ is certainly nonzero, it must be surjective.

Now assume $\Psi_{x,k}$ is surjective. Then, looking back at (4.1), the composition $\Psi_{x,k} \circ a$ is surjective. Thus, by commutativity, $b \circ \Psi_{x,k+1}$ is surjective. Therefore, c must be the zero map, so that the bottom sequence of maps between global sections is actually short exact. Thus, by the five lemma, the center vertical map $\Psi_{x,k+1}$ is surjective. Thus, only the left vertical map $\alpha_{x,k}$ needs to be surjective in order to guarantee the normality of $\Sigma(X, \mathcal{L})$, as desired. $\square$

For the remainder of the section, we will focus on finding the conditions under which $\alpha_{x,k}$ is surjective. The next lemma will help us better understand the target space. Recall that n is the dimension of X.

Lemma IV.2. *Suppose $\mathcal{L}$ is 3-very ample. Then for all $x \in X$,*

$$N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \cong \mathcal{O}_{F_x}^{\oplus n} \oplus (b_x^*\mathcal{L}(-2E_x)),$$

where b_x is the blow-up map of X at x, and E_x is the corresponding exceptional divisor.

Proof. Since F_x is a section over its image $\pi(F_x)$, we have the following short exact sequence:

$$0 \to T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}\big|_{F_x} \to N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \to N_{F_x/X^{[2]}} \to 0. \tag{4.2}$$

First we will try to understand the left term, $T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}\big|_{F_x}$ by looking at the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \to \pi^* {\mathcal{E}_{\mathcal{L}}}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(1) \to T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}} \to 0. \tag{4.3}$$

Since $T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}$ is a line bundle, taking determinants yields

$$T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}} \cong \det(\pi^* {\mathcal{E}_{\mathcal{L}}}^*) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(2) \cong (\pi^* \det \mathcal{E}_{\mathcal{L}})^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(2).$$

So

$$T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}\big|_{F_x} \cong (\pi^* \det \mathcal{E}_{\mathcal{L}})^*\big|_{F_x} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(2)\big|_{F_x} \cong \det {\mathcal{E}_{\mathcal{L}}}^*\big|_{F_x} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(2)\big|_{F_x}.$$

To calculate $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(2)\big|_{F_x}$, consider the diagram (3.5). First note that by construction of the map $f : \mathbb{P}(\mathcal{E}_{\mathcal{L}}) \to \mathbb{P}^r$ via maps of vector bundles, it follows that the pullback of the tautological bundle is also the tautological bundle. That is,

$$f^* \mathcal{O}_{\mathbb{P}^r}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(1).$$

Thus, $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(1)\big|_{F_x}$ is isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}^r}(1)\big|_x \cong \mathcal{O}_x$ to F_x. So

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})}(1)\big|_{F_x} \cong \mathcal{O}_{F_x}.$$

Thus,

$$T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}\big|_{F_x} \cong \det {\mathcal{E}_{\mathcal{L}}}^*\big|_{F_x},$$

so the next step is to understand the restriction of $\mathcal{E}_{\mathcal{L}}$ to F_x.

Consider the diagram

$$\begin{tikzcd} \sigma^{-1}(F_x) \arrow[r, hook, "i"] \arrow[d, "\sigma"'] & \Phi \arrow[d, "\sigma"] \\ F_x \arrow[r, hook, "j"'] & X^{[2]} \end{tikzcd} . \tag{4.4}$$

We have temporarily named the inclusion maps so that we can easily refer to them. Note that $\sigma^{-1}(F_x)$ is two copies of F_x intersecting along E_x. Since the above is a Cartesian square and σ is flat and finite, base change yields

$$\mathcal{E}_{\mathcal{L}}\big|_{F_x} = j^*\sigma_* q^*\mathcal{L} \cong \sigma_* i^* q^*\mathcal{L}.$$

If we think of Φ as $bl_\Delta(X \times X)$, then q is the blowup morphism followed by projection to the first factor. Thus, $i^*q^*\mathcal{L}$ is isomorphic to $\mathcal{O}_{F_x}$ when restricted to one reducible component, and $b_x^*\mathcal{L}$ when restricted to the other. Thus, pushing forward, we have a natural map

$$\mathcal{E}_{\mathcal{L}}\big|_{F_x} \cong \sigma_* i^* q^*\mathcal{L} \to \mathcal{O}_{F_x} \oplus b_x^*\mathcal{L}, \tag{4.5}$$

which is an injection that drops rank along E_x.

As an aside, it is useful to recall that the fiber of $\mathcal{E}_{\mathcal{L}}\big|_{F_x}$ over a point $\xi \in F_x$ is $H^0(X, L \otimes \mathcal{O}_\xi)$, where ξ is some length two subscheme of X which contains x. So over generic ξ, the map (4.5) on fibers is the sum of restrictions

$$H^0(X, L \otimes \mathcal{O}_\xi) \to H^0(X, L \otimes \mathcal{O}_x) \oplus H^0(X, L \otimes \mathcal{O}_y),$$

where $\{x, y\} = \mathrm{Supp}(\xi)$.

Since the vector bundles in (4.5) have the same rank, taking determinants yields

$$\det \mathcal{E}_{\mathcal{L}}\big|_{F_x} \cong b_x^*\mathcal{L}(-E_x),$$

which means

$$T_{\mathbb{P}(\mathcal{E}_{\mathcal{L}})/X^{[2]}}\big|_{F_x} \cong b_x^*\mathcal{L}^*(E_x).$$

We can now rewrite (4.2) as

$$0 \to b_x^*\mathcal{L}^*(E_x) \to N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \to N_{F_x/X^{[2]}} \to 0$$

and turn our attention to $N_{F_x/X^{[2]}}$. The map induced by σ on normal bundles $N_{F_x/\Phi} \to N_{F_x/X^{[2]}}$ is an isomorphism away from the ramification locus, which intersects F_x in E_x. Thus,

$$\det N_{F_x/X^{[2]}} \cong (\det N_{F_x/\Phi})(E_x).$$

Now $q : \Phi \to X$ is a smooth map of which F_x is a fiber, so $N_{F_x/\Phi}$ is isomorphic to the pullback of $N_{x/X}$. Thus,

$$N_{F_x/\Phi} \cong \mathcal{O}^n_{F_x},$$

which means that

$$\det N_{F_x/X^{[2]}} \cong \mathcal{O}_{F_x}(E_x).$$

Looking back at our short exact sequence, this means that

$$\det N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \cong b_x^*\mathcal{L}^*(2E_x).$$

Now consider the following short exact sequence on normal bundles, again involving $N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}$:

$$0 \to N_{F_x/\Phi} \to N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \to N_{\Phi/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\big|_{F_x} \to 0. \tag{4.6}$$

We have already established that the left term is the trivial bundle of rank n. Since $\Phi \subset \mathbb{P}(\mathcal{E}_{\mathcal{L}})$ has codimension one, $N_{\Phi/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\big|_{F_x}$ must be a line bundle. Thus, taking determinants, we obtain

$$N_{\Phi/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\big|_{F_x} \cong \det N_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \cong b_x^*\mathcal{L}^*(2E_x).$$

We take the dual and rewrite (4.6) as

$$0 \to b_x^*\mathcal{L}(-2E_x) \to N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})} \to \mathcal{O}^n_{F_x} \to 0.$$

Our final goal is to show that the above sequence splits. Since the right term is trivial, this is the same as showing that the map on global sections

$$H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right) \to H^0\left(N^*_{F_x/\Phi}\right)$$

is a surjection. Consider the commutative diagram

$$\begin{tikzcd}
T_x^*\mathbb{P}^r \arrow[r, two heads] \arrow[d, "\alpha_{x,1}"'] & T_x^*X \arrow[d] \\
H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right) \arrow[r] & H^0\left(N^*_{F_x/\Phi}\right)
\end{tikzcd} .$$

As mentioned earlier,

$$N_{F_x/\Phi} \cong T_x^*X \otimes \mathcal{O}_{F_x}.$$

Thus, the right vertical map is an isomorphism, so the bottom horizontal map must be a surjection, as desired. Therefore, the desired sequence splits, which completes the proof.

□

Now we return to showing that $\alpha_{x,k}$ is surjective. In the case $k = 1$, it is actually an isomorphism, which follows from a straight-forward geometric argument.

Lemma IV.3. *Suppose $\mathcal{L}$ is 3-very ample. Then*

$$\alpha_{x,1} : T_x^*\mathbb{P}^r \to H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right)$$

is an isomorphism for all $x \in X$.

Proof. First we show $\alpha_{x,1}$ is injective. Let $w \in T_x^*\mathbb{P}^r$ be a nonzero covector. Call the kernel hyperplane in the tangent space $H \subset \mathbb{P}^r$. Since $X \in \mathbb{P}^r$ is non-degenerate, we can pick some $y \in X$ such that $y \notin H$. Define ℓ to be the secant line through x and y. Now define

$$\tilde{\ell} := f^{-1}(\ell) \subset \mathbb{P}(\mathcal{E}_\mathcal{L}).$$

Note that $\tilde{\ell}$ is all points in $\mathbb{P}(\mathcal{E}_{\mathcal{L}})$ in the fiber over the subscheme $x+y \in X^{[2]}$. That is,

$$\tilde{\ell} = \pi^{-1}(x+y).$$

Thus, $\tilde{\ell}$ intersects $F_x \cong bl_x(X)$ at the point corresponding to y, i.e. at the point $(x+y, H^0(\mathcal{L} \otimes \mathcal{O}_{x+y}) \to H^0(\mathcal{L} \otimes \mathcal{O}_x)) \in \mathbb{P}(\mathcal{E}_{\mathcal{L}})$. Call this point P_y.

Consider the commutative diagram of tangent spaces

$$\begin{tikzcd} T_{P_y}\tilde{\ell} \arrow[r, "\cong"] \arrow[d, hook] & T_x\ell \arrow[d, hook] \\ T_{P_y}\mathbb{P}(\mathcal{E}_{\mathcal{L}}) \arrow[r, "df"] & T_x\mathbb{P}^r \end{tikzcd},$$

where the top horizontal map is an isomorphism since f is an isomorphism on $\tilde{\ell}$. Let $v \in T_{P_y}\tilde{\ell}$ be a nonzero vector. Looking the above diagram, $df(v)$ is nonzero and sits inside $T_x\ell$. Thus, since ℓ is not contained in H, we know that

$$\langle f^*w, v\rangle_{P_y} = \langle w, df(v)\rangle_x \neq 0,$$

which means that $f^*w \neq 0$.

Notice that the pullback map $T_x^*\mathbb{P}^r \to T_{P_y}^*\mathbb{P}(\mathcal{E}_{\mathcal{L}})$ factors through $H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})})$ as follows:

$$\begin{tikzcd} T_x^*\mathbb{P}^r \arrow[r, "f^*"] \arrow[d, "\alpha_{x,1}"'] & T_{P_y}^*\mathbb{P}(\mathcal{E}_{\mathcal{L}}) \\ H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\right) \arrow[r, "\text{restr.}"] & H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\big|_{P_y}\right) \arrow[u, hook] \end{tikzcd}$$

Thus, since $f^*w \neq 0$, we know $\alpha_{x,1}(w) \neq 0$. Thus, $\alpha_{x,1}$ is injective.

Now to show that $\alpha_{x,1}$ is an isomorphism, we show that $T_x^*\mathbb{P}^r$ and $H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{\mathcal{L}})}\right)$ have the same dimension.

First of all,

$$\dim T_x^*\mathbb{P}^r = r = h^0(\mathcal{L}) - 1.$$

Next, by Lemma IV.2,

$$h^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right) = h^0(\mathcal{O}^n_{F_x}) + h^0(b_x^*\mathcal{L}(-2E_x)).$$

Of course, $h^0(\mathcal{O}^n_{F_x}) = n$. To calculate $h^0(b_x^*\mathcal{L}(-2E_x))$, consider the natural short exact sequence

$$0 \to \mathcal{O}_{F_x}(-2E_x) \to \mathcal{O}_{F_x} \to \mathcal{O}_{2E_x} \to 0.$$

Tensoring by $b_x^*\mathcal{L}$ and taking cohomology yields

$$0 \to H^0(b_x^*\mathcal{L}(-2E_x)) \to H^0(b_x^*\mathcal{L}) \to H^0(b_x^*\mathcal{L} \otimes \mathcal{O}_{2E_x}) \to \cdots .$$

Pushing forward, the second map on global sections is equal to the map

$$H^0(\mathcal{L}) \to H^0(\mathcal{L} \otimes \mathcal{O}/m_x^2),$$

which is surjective by very ampleness of $\mathcal{L}$. Thus,

$$h^0(b_x^*\mathcal{L}(-2E_x)) = h^0(\mathcal{L}) - h^0(\mathcal{L} \otimes \mathcal{O}/m_x^2) = h^0(\mathcal{L}) - (n+1).$$

So

$$h^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right) = n + h^0(\mathcal{L}) - n - 1 = h^0(\mathcal{L}) - 1 = \dim T_x^*\mathbb{P}^r,$$

as desired, which completes the proof.

$\square$

4.1.2 The proof

Now we prove the main theorem by showing that the higher $\alpha_{x,k}$ are surjective.

Proof of Theorem D. By Lemma IV.1, showing that

$$\alpha_{x,k} : \mathrm{Sym}^k(T_x^*\mathbb{P}^r) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})$$

is surjective will prove the theorem.

Notice that we can build $\alpha_{x,k}$ from $\alpha_{x,1}$ as follows:

$$\begin{tikzcd}
\mathrm{Sym}^k(T_x^*\mathbb{P}^r) \arrow[r, "\mathrm{Sym}^k\alpha_{x,1}"] \arrow[dr, "\alpha_{x,k}"'] & \mathrm{Sym}^k H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}) \arrow[d] \\
& H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})
\end{tikzcd},$$

where the vertical map is the natural one. By Lemma IV.3, $\alpha_{x,1}$ is an isomorphism, so the induced map $\mathrm{Sym}^k\alpha_{x,1}$ must be as well. Thus, $\alpha_{x,k}$ is surjective if and only if

$$\mathrm{Sym}^k(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})$$

is surjective.

By Lemma IV.2,

$$\mathrm{Sym}^k\left(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})\right) \cong \mathrm{Sym}^k\left(H^0(\mathcal{O}_{F_x})^{\oplus n} \oplus H^0(b_x^*\mathcal{L}(-2E_x))\right)$$

and

$$H^0\left(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})}\right) \cong H^0\left(\mathrm{Sym}^k\left(\mathcal{O}_{F_x}^{\oplus n} \oplus (b_x^*\mathcal{L}(-2E_x))\right)\right).$$

By construction of the map,

$$\mathrm{Sym}^k(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})$$

decomposes as the sum of maps of the form

$$\mathrm{Sym}^i H^0(b_x^*\mathcal{L}(-2E_x)) \to H^0\left((b_x^*\mathcal{L}(-2E_x))^{\otimes i}\right).$$

These maps are surjective for all i if and only if $b_x^*\mathcal{L}(-2E_x)$ is normally generated, which is equivalent to hypothesis (1) of the theorem. Thus,

$$\mathrm{Sym}^k(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_\mathcal{L})})$$

is surjective, and we are done. $\square$

4.2 Examples and Corollaries

This section is devoted to proving the corollaries from the introduction. These corollaries help show the strength of the theorem, as they are more immediately applicable than the rather abstract hypotheses of the main theorem.

4.2.1 Degree condition on line bundles of curves

Corollary A. *Let X be a smooth projective curve of genus g and $\mathcal{L}$ a line bundle on X of degree d. If $d \geq 2g+3$, then $\Sigma(X,\mathcal{L})$ is a normal variety.*

Proof. Let D be an effective divisor on X of degree 4. Then both $\mathcal{L}$ and $\mathcal{L}(-D)$ have degree greater than $2g-2$, so they are both non-special. Thus, Riemann-Roch implies that

$$h^0(\mathcal{L}(-D)) = h^0(\mathcal{L}) - 4.$$

Thus, $\mathcal{L}$ is 3-very ample.

Let $x \in X$. Then

$$\deg \mathcal{L}(-2x) \geq 2g+1.$$

A classical result of Castelnuovo, Mattuck [22], and Mumford [24] states that line bundles on curves with degree at least $2g+1$ are normally generated, which means the maps in the hypothesis of the theorem are surjective, as desired. □

4.2.2 Canonical curves

Next, we prove the corollary involving canonical curves. Note that this example is not covered by Corollary A.

Corollary B. *Let X be a curve of genus g with Clifford index* $\mathrm{Cliff}(X) \geq 3$. *Then $\Sigma(X,\omega_X)$ is a normal variety.*

Proof. Let $c = \mathrm{Cliff}(X)$. The following classification is given in [9]:

$c = 0 \iff X$ is hyperelliptic.

$c = 1 \iff X$ has a g^1_3 or X is a plane quintic.

$c = 2 \iff X$ has a g^1_4 or X is a plane sextic.

Thus, $c \geq 3$ if and only if X has no g^1_4 and is not a plane sextic.

First we will show that ω_X is 3-very ample. Let D be an effective divisor of degree 4. Then Riemann-Roch gives

$$h^0(\omega_X(-D)) = h^0(D) + (2g-2-4) - g + 1 = h^0(\omega_X) + h^0(D) - 5.$$

Thus, ω_X is 3-very ample if and only if $h^0(D) = 1$, i.e. X has no g^1_4, which follows from the hypothesis.

Next we show that $\omega_X(-2x)$ is normally generated. A theorem of Green and Lazarfeld (Theorem 1 in [13]) states that if $\mathcal{L}$ is very ample, and

$$\deg \mathcal{L} \geq 2g + 1 - 2h^1(\mathcal{L}) - c,$$

then $\mathcal{L}$ is normally generated. In the situation of interest, $\deg \omega_X(-2x) = 2g - 4$, and by Serre duality $h^1(\omega_X(-2x)) = h^0(2x)$, which is 1 since X is not hyperelliptic. Thus, the Green-Lazarsfeld theorem implies $\omega_X(-2x)$ is normally generated as long as $c \geq 3$. ☐

In the above proof, the lack of a g^1_4 was equivalent to 3-very ampleness. However, $c \geq 3$ merely implies the normal generation condition. This raises the question: do we need the hypothesis that X is not a plane sextic, or does the lack of a g^1_4 suffice? In fact, if X is a plane sextic, $\omega(-2x)$ is not normally generated. This follows from a proof analogous to one due to Konno (Lemma 2.2 of [18]), setting $D = \omega_X(-2x)$ and $k = 2$. We won't restate the proof, as it is nearly identical to Konno's proof except we replace ℓ with a line tangent to X at x and blow up twice at the intersection of X and ℓ rather than once. Thus, to satisfy the hypotheses of our theorem, it is necessary that X is not a plane sextic. However, our theorem only gives sufficient conditions for normality, so we ask the following question:

Question IV.1. *If X is a smooth plane sextic, is $\Sigma(X, \omega_X)$ a normal variety?*

4.2.3 Main corollary

Now we turn to our final corollary, which deals with higher dimensional X.

Corollary C. *Let X be a smooth projective variety of dimension n. Let $\mathcal{A}$ and $\mathcal{B}$ be very ample and nef, respectively, and*

$$\mathcal{L} = \omega_X \otimes \mathcal{A}^{\otimes 2(n+1)} \otimes \mathcal{B}.$$

Then $\Sigma(X, \mathcal{L})$ is a normal variety.

Proof. When $n = 1$, $\mathcal{L}$ already has sufficiently high degree so that it satisfies the hypothesis of Corollary A. We will assume from now on that n is at least 2.

The line bundle $\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{B}$ is very ample when $k \geq n+2$ (see [21], Example 1.8.23). On the other hand, the product of an i-very ample line bundle with a j-very ample line bundle will be $(i+j)$-very ample [16]. Thus $\omega_X \otimes \mathcal{A}^{\otimes k} \otimes \mathcal{B}$ will be 3-very ample for $k \geq n+4$. For $n \geq 2$, we have $2(n+1) \geq n+4$, so $\mathcal{L} = \omega_X \otimes \mathcal{A}^{\otimes 2(n+1)} \otimes \mathcal{B}$ must be 3-very ample.

Now we check the remaining hypotheses on $\widetilde{X} = bl_x X$. First we calculate $b_x^* \mathcal{L}(-2E_x)$.

$$\begin{aligned} b_x^* \mathcal{L} &= b_x^* \omega_X \otimes b_x^* \mathcal{A}^{\otimes 2(n+1)} \otimes b_x^* \mathcal{B} \\ &= \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(-(n-1)E) \otimes b_x^* \mathcal{A}^{\otimes 2(n+1)} \otimes b_x^* \mathcal{B}. \end{aligned}$$

Thus, we get

$$\begin{aligned} b_x^*\mathcal{L}(-2E_x) &= \omega_{\widetilde{X}} \otimes b_x^*\mathcal{A}^{\otimes 2(n+1)} \otimes \mathcal{O}_{\widetilde{X}}(-(n+1)E_x) \otimes b_x^*\mathcal{B} \\ &= \omega_{\widetilde{X}} \otimes (b_x^*\mathcal{A}^{\otimes 2}(-E_x))^{\otimes(n+1)} \otimes b_x^*\mathcal{B}. \end{aligned}$$

$\mathcal{A}$ is very ample, so it is the restriction of $\mathcal{O}(1)$ of the corresponding projective space $\mathbb{P}^m$. Consider the blowup $\widetilde{\mathbb{P}^m}$ of $\mathbb{P}^m$ at $x \in X$. It is well-known that $2\widetilde{H} - E$ is very ample, where $\widetilde{H}$ is the pullback of a hyperplane. Thus,

$$\mathcal{O}_{\widetilde{X}}(2\widetilde{H} - E) = b_x^*\mathcal{A}^{\otimes 2}(-E_x)$$

is also very ample. Furthermore, the pullback of a nef line bundle is again nef. A theorem of Ein and Lazarsfeld in [8] states that line bundles of the form $\omega \otimes \mathcal{M}^{\otimes(n+1)} \otimes \mathcal{N}$, where $\mathcal{M}$ is very ample and $\mathcal{N}$ is nef, are normally generated. Thus, $b_x^*\mathcal{L}(-2E_x)$ is normally generated, so $\Sigma(X, \mathcal{L})$ must be normal. $\square$

4.3 Further applications of the main theorem

After Vermeire proposed a proof of the normality of secant varieties in [30], he and Sidman used the purported normality to prove theorems and pose conjectures about the first secant variety of curves in [27], [26], [32], [31]. Our main theorem thus confirms these results, eliminating the hypotheses requiring the secant varieties to be normal. In this section, we state some of the most powerful of these theorems.

In [26], Sidman and Vermeire show that for high degree line bundles on curves, $\Sigma(X, \mathcal{L})$ is arithmetically Cohen-Macaulay:

Theorem IV.1 ([26], Theorem 1.1)**.** *If $C \subset \mathbb{P}^n$ is a smooth curve of genus g and degree $d \geq 2g + 3$, then its secant variety $\Sigma(X, \mathcal{L})$ is ACM.*

Also in [26], Sidman and Vermeire give a result on the vanishing of higher cohomology on $\Sigma(X, \mathcal{L})$:

Theorem IV.2 ([26], Theorem 3.2)**.** *If $C \subset \mathbb{P}^n$ is a smooth curve of genus g and degree $d \geq 2g+3$, then $H^i(\Sigma(X,\mathcal{L}), \mathcal{O}_{\Sigma(X,\mathcal{L})}(k)) = 0$ for $k < 0$ and $i = 1, 2$.*

Finally, in [32], Vermeire gives a regularity bound for the secant variety:

Theorem IV.3 ([32], Corollary 11)**.** *If $C \subset \mathbb{P}^n$ is a smooth curve of genus g and degree $d \geq 2g+3$, then $\Sigma(X,\mathcal{L})$ is 5-regular.*

Note that in the above theorems, the hypothesis $d \geq 2g+3$ implies normality of the secant variety of a curve (cf. Corollary A), thus we do not need to impose any additional hypotheses.

CHAPTER V

Higher secant varieties to curves

In this chapter, we present our results about the normality of higher secant varieties to curves, following the geometric setup in section 3.3. In this situation, we do not yet have a proof of normality, but we present some preliminary lemmas and conjectures. This story parallels that of the previous chapter, which will help make it more obvious when we reach an obstacle.

5.1 Toward a proof of normality

5.1.1 Preliminary results

Let X be a smooth projective curve, and $\mathcal{L}$ a $(2n+1)$-very ample line bundle on X. Just as in the previous chapter, the normality of the secant variety $\Sigma_n(X, \mathcal{L})$ of X is controlled by the geometry of the conormal bundle to the fiber F_x. Recall that

$$F_x = t^{-1}(x) \cong X^{(n)},$$

where

$$t : \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \to \Sigma_n(X, \mathcal{L})$$

is the resolution of singularities, and $\mathbb{P}^r = \mathbb{P}(H^0(\mathcal{L}))$.

Lemma V.1. *Let $\mathcal{L}$ be a (2n+1)-very ample line bundle on X. Let $x \in X$, and let*

$\alpha_{x,k,n}$ be the natural map

$$\alpha_{x,k,n} : \mathrm{Sym}^k(T_x^*\mathbb{P}^r) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}).$$

If $\alpha_{x,k,n}$ is surjective for all $k > 0$ and all $x \in X$, then $\Sigma_n(X,\mathcal{L})$ is normal along X.

This lemma is analogous to Lemma IV.1. In fact, the notation that we have chosen makes the proof identical when we replace $\mathcal{E}_{\mathcal{L}}$ with $\mathcal{E}_{n+1,\mathcal{L}}$.

As we pointed out in Section 3.3, $\Sigma_n(X,\mathcal{L})$ is not singular only along X. Rather, it is singular along $\Sigma_{n-1}(X,\mathcal{L})$. However, the above lemma actually does hold for points $y \in \Sigma_{n-1}(X,\mathcal{L})\backslash X$ as well, replacing F_x with $F_{y,D}$, the fiber over y. We will not state the lemma in full detail since, as we will soon see, this is the direction in which we face our main obstacles.

Now we will focus on understanding the conormal bundle $N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}$.

Lemma V.2. *Suppose $\mathcal{L}$ is $(2n+1)$-very ample. Then for all $x \in X$,*

$$N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \cong \mathcal{O}_{F_x} \oplus \mathcal{E}_{n,\mathcal{L}(-2x)}.$$

Proof. The fiber F_x is a section over its image $\pi(F_x)$, so we have the following short exact sequence:

$$0 \to T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}}\big|_{F_x} \to N_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \to N_{F_x/X^{(n+1)}} \to 0. \tag{5.1}$$

First we calculate the left term in the above sequence, $T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n)}}\big|_{F_x}$. Consider the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \to \pi^*\mathcal{E}^*_{n+1,\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(1) \to T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}} \to 0.$$

The vector bundle $T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}}$ has rank n. Taking determinants, we get

$$\begin{aligned} \det(T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}}) &\cong \det\left(\pi^*\mathcal{E}^*_{n+1,\mathcal{L}}\right) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(n+1) \\ &\cong (\pi^* \det \mathcal{E}_{n+1,\mathcal{L}})^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(n+1). \end{aligned}$$

So restricting to F_x yields

$$\begin{aligned}\det(T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}})\big|_{F_x} &\cong (\pi^* \det \mathcal{E}_{n+1,\mathcal{L}})^*\big|_{F_x} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(n+1)\big|_{F_x} \\ &\cong \det \mathcal{E}^*_{n+1,\mathcal{L}}\big|_{F_x} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(n+1)\big|_{F_x}.\end{aligned}$$

For the same reason as described in the proof of Lemma IV.2,

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}(1)\big|_{F_x} \cong \mathcal{O}_{F_x}.$$

Thus,

$$\det(T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}})\big|_{F_x} \cong \det \mathcal{E}^*_{n+1,\mathcal{L}}\big|_{F_x}.$$

Now we need to understand the restriction of $\mathcal{E}_{n+1,\mathcal{L}}$ to F_x. Consider the fiber square

$$\begin{tikzcd} \Phi \times_{X^{(n+1)}} F_x \arrow[r, hook, "i"] \arrow[d, "\sigma"'] & \Phi \cong X \times X^{(n)} \arrow[d, "\sigma"] \\ F_x = x + X^{(n)} \arrow[r, hook, "j"'] & X^{(n+1)} \end{tikzcd}.$$

The key observation here is that

$$\begin{aligned}\Phi \times_{X^{(n+1)}} F_x &= \{(x, D) : D \in X^{(n)}\} \bigcup \{(y, x + C) : y \in X, C \in X^{(n-1)}\} \\ &\cong \left(\{x\} \times X^{(n)}\right) \bigcup \left(X \times (x + X^{(n-1)})\right) \\ &\cong X^{(n)} \bigcup (X \times X^{(n-1)}).\end{aligned}$$

From this fiber square, we get a natural map

$$\mathcal{E}_{n+1,\mathcal{L}}\big|_{F_x} \to \mathcal{O}_{F_x} \oplus \mathcal{E}_{n,\mathcal{L}}$$

which is an injection that drops rank along the divisor

$$F'_x := 2x + X^{(n-1)} \subset x + X^{(n)} = F_x \subset X^{(n+1)}.$$

(For more details regarding how we get this map via base change, see the proof of Lemma IV.2.) Both the source and target vector bundles have rank $n+1$. Thus, taking determinants gives

$$\det(\mathcal{E}_{n+1,\mathcal{L}}\big|_{F_x}) \cong \det(\mathcal{E}_{n,\mathcal{L}}) \otimes \mathcal{O}(-F'_x),$$

which means

$$\det(T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}})\big|_{F_x} \cong \det(\mathcal{E}^*_{n,\mathcal{L}}) \otimes \mathcal{O}(F'_x).$$

Now we turn our attention to the line bundle $N_{F_x/X^{(n+1)}}$. The map induced by σ on normal bundles $N_{F_x/\Phi} \to N_{F_x/X^{(n+1)}}$ is an isomorphism away from the ramification locus, which intersects F_x in F'_x. Thus,

$$N_{F_x/X^{(n+1)}} \cong N_{F_x/\Phi}(F'_x).$$

Recall that F_x sits inside Φ as follows:

$$F_x = \{x\} \times X^{(n)} \subset X \times X^{(n)} = \Phi.$$

That is, it is just a fiber over the projection onto the first factor. Thus,

$$N_{F_x/\Phi} \cong \mathcal{O}_{F_x}$$

and

$$N_{F_x/X^{(n+1)}} \cong \mathcal{O}_{F_x}(F'_x).$$

Looking back at the short exact sequence (5.1), taking determinants gives us

$$\det N_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \cong \det(T_{\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})/X^{(n+1)}})\big|_{F_x} \otimes N_{F_x/X^{(n+1)}} \cong \det(\mathcal{E}^*_{n,\mathcal{L}}) \otimes \mathcal{O}_{F_x}(2F'_x)$$

Now consider the following short exact sequence on normal bundles:

$$0 \to N_{F_x/\Phi} \to N_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \to N_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x} \to 0 \tag{5.2}$$

We have already established that the left term is a trivial line bundle. Thus,

$$\det N_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x} \cong \det N_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \cong \det(\mathcal{E}^*_{n,\mathcal{L}}) \otimes \mathcal{O}_{F_x}(2F'_x).$$

By lemma 1.3(b) of [3],

$$\mathbb{P}(N^*_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x}) \cong \mathbb{P}(\mathcal{E}_{n,\mathcal{L}(-2x)}).$$

This means that

$$N^*_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x} \cong \mathcal{E}_{n,\mathcal{L}(-2x)} \otimes \mathcal{M},$$

where $\mathcal{M}$ is some line bundle. However, we know the determinant of $N^*_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x}$, so we can figure out what $\mathcal{M}$ is.

Consider the following short exact sequence on X:

$$0 \to L(-2x) \to L \to \mathcal{O}_{2x} \to 0.$$

The maps q and σ are flat and finite, respectively, so pulling back the sequence along q and pushing it forward along σ preserves exactness:

$$0 \to \mathcal{E}_{n,\mathcal{L}(-2x)} \to \mathcal{E}_{n,\mathcal{L}} \to \mathcal{O}_{2F'_x} \to 0.$$

So we get

$$\det \mathcal{E}_{n,\mathcal{L}(-2x)} = \det(\mathcal{E}_{n,\mathcal{L}}) \otimes \mathcal{O}_{F_x}(-2F'_x) \cong N^*_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x}.$$

Thus, $\mathcal{M}$ is trivial, so

$$N^*_{\Phi/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_{F_x} \cong \mathcal{E}_{n,\mathcal{L}(-2x)}.$$

We can now rewrite the dual of the short exact sequence (5.2) as

$$0 \to \mathcal{E}_{n,\mathcal{L}(-2x)} \to N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})} \to \mathcal{O}_{F_x} \to 0.$$

All that is left is to show this sequence splits. This follows by the same argument as in the last paragraph of the proof of Lemma IV.2, and we are done.

□

Now we return to our main goal, which is to show $\alpha_{x,k,n}$ is surjective for all k. In the case $k = 1$, it is an isomorphism. To show this, we follow the same argument as in the proof of Lemma IV.3.

Lemma V.3. *Suppose $\mathcal{L}$ is (2n+1)-very ample. Then*

$$\alpha_{x,1,n} : T_x^*\mathbb{P}^r \to H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right)$$

is an isomorphism for all $x \in X$.

Proof. First we show $\alpha_{x,1,n}$ is injective. Let $w \in T_x^*\mathbb{P}^r$ be a nonzero covector. Call the kernel hyperplane in the tangent space $H \subset \mathbb{P}^r$. Since $X \in \mathbb{P}^r$ is non-degenerate, we can pick some $y \in X$ such that $y \notin H$. Define ℓ to be the secant line through x and y, and define J to be the unique secant n-plane determined by the divisor $x + ny$.

Now define

$$\tilde{J} := f^{-1}(J) \subset \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}).$$

Note that $\tilde{J}$ is all points in $\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})$ in the fiber over the subscheme $x+ny \in X^{(n+1)}$. That is,

$$\tilde{J} = \pi^{-1}(x + ny).$$

Define

$$\tilde{\ell} := f^{-1}(\ell) \cap \tilde{J}.$$

Note that $\tilde{\ell}$ is the line connecting the preimages of x and y in $\tilde{J}$. More explicitly, $\tilde{J}$ connects the points

$$\left(x + ny, H^0(\mathcal{L} \otimes \mathcal{O}_{x+ny}\right) \to H^0(\mathcal{L} \otimes \mathcal{O}_x))$$

and

$$\left(x + ny, H^0(\mathcal{L} \otimes \mathcal{O}_{x+ny}) \to H^0(\mathcal{L} \otimes \mathcal{O}_y)\right).$$

By construction, f maps $\tilde{J}$ and $\tilde{\ell}$ isomorphically onto their images. Let P be the preimage of x in $\tilde{J}$ and $\tilde{\ell}$. That is,

$$P = \big(x + ny, H^0(\mathcal{L} \otimes \mathcal{O}_{x+ny}\big) \to H^0(\mathcal{L} \otimes \mathcal{O}_x)) \in \mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}).$$

Consider the commutative diagram of tangent spaces

$$\begin{tikzcd} T_P\tilde{\ell} \arrow[r, "\cong"] \arrow[d, hook] & T_x\ell \arrow[d, hook] \\ T_P\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \arrow[r, "df"] & T_x\mathbb{P}^r \end{tikzcd},$$

where the top horizontal map is an isomorphism since f is an isomorphism on $\tilde{\ell}$. Let $v \in T_P\tilde{\ell}$ be a nonzero vector. Looking the above diagram, $df(v)$ is nonzero and sits inside $T_x\ell$. Thus, since ℓ is not contained in H (because $y \notin H$), we know that

$$\langle f^*w, v\rangle_P = \langle w, df(v)\rangle_x \neq 0,$$

which means that $f^*w \neq 0$.

Notice that the pullback map $T_x^*\mathbb{P}^r \to T_P^*\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})$ factors through $H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})$ as follows:

$$\begin{tikzcd} T_x^*\mathbb{P}^r \arrow[r, "f^*"] \arrow[d, "\alpha_{x,1,n}"'] & T_P^*\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}}) \\ H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right) \arrow[r, "\text{restr.}"] & H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\big|_P\right) \arrow[u, hook] \end{tikzcd}$$

Thus, since $f^*w \neq 0$, we know $\alpha_{x,1,n}(w) \neq 0$. Thus, $\alpha_{x,1,n}$ is injective.

Now to show that $\alpha_{x,1,n}$ is an isomorphism, we show that $T_x^*\mathbb{P}^r$ and $H^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right)$ have the same dimension.

First of all,

$$\dim T_x^*\mathbb{P}^r = r = h^0(\mathcal{L}) - 1.$$

Next, by Lemma V.2,

$$h^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right) = h^0(\mathcal{O}_{F_x}) + h^0(\mathcal{E}_{n,\mathcal{L}(-2x)}).$$

Of course, $h^0(\mathcal{O}_{F_x}) = 1$. By (2.4) and very ampleness of $\mathcal{L}$,

$$h^0(\mathcal{E}_{n,\mathcal{L}(-2x)}) = h^0(\mathcal{L}(-2x)) = h^0(L) - 2.$$

Thus,

$$h^0\left(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right) = h^0(L) - 1 = \dim T^*_x\mathbb{P}^r,$$

as desired, and we are done. □

The only remaining thing we need in order for $\Sigma_n(X,\mathcal{L})$ to be normal along X is for the higher $\alpha_{x,k,n}$ to be surjective. It turns out that the hypothesis we need is that a lower secant variety be projectively normal, as described in this next theorem.

Theorem F. *Let X be a smooth projective curve, and $\mathcal{L}$ a $(2n+1)$-very ample line bundle on X, where $n \geq 2$. Suppose $\Sigma_{n-1}(X,\mathcal{L}(-2x))$ is projectively normal for all $x \in X$. Then $\Sigma_n(X,\mathcal{L})$ is normal along X.*

Proof. By Lemma V.1, showing that

$$\alpha_{x,k,n} : Sym^k(T^*_x\mathbb{P}^r) \to H^0(Sym^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})$$

is surjective will prove the lemma.

Notice that we can build $\alpha_{x,k,n}$ from $\alpha_{x,1,n}$ as follows:

$$\begin{tikzcd}
\mathrm{Sym}^k(T^*_x\mathbb{P}^r) \arrow[r, "\mathrm{Sym}^k\alpha_{x,1,n}"] \arrow[dr, "\alpha_{x,k}"'] & \mathrm{Sym}^k H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}) \arrow[d] \\
& H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})
\end{tikzcd},$$

where the vertical map is the natural one. By Lemma V.3, $\alpha_{x,1,n}$ is an isomorphism, so the induced map $\mathrm{Sym}^k\alpha_{x,1,n}$ must be as well. Thus, $\alpha_{x,k,n}$ is surjective if and only if

$$\mathrm{Sym}^k(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})$$

is surjective.

By Lemma V.2,

$$\mathrm{Sym}^k\left(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})\right) \cong \mathrm{Sym}^k\left(H^0(\mathcal{O}_{F_x}) \oplus H^0(\mathcal{E}_{n,\mathcal{L}(-2x)})\right)$$

and

$$H^0\left(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}\right) \cong H^0\left(\mathrm{Sym}^k\left(\mathcal{O}_{F_x} \oplus \mathcal{E}_{n,\mathcal{L}(-2x)}\right)\right).$$

By construction of the map,

$$\mathrm{Sym}^k(H^0(N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})) \to H^0(\mathrm{Sym}^k N^*_{F_x/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})})$$

decomposes as the sum of maps of the form

$$\mathrm{Sym}^i H^0(\mathcal{E}_{n,\mathcal{L}(-2x)}) \to H^0\left(\mathrm{Sym}^i \mathcal{E}_{n,\mathcal{L}(-2x)}\right).$$

We want to show these maps are surjective for all i.

Now consider the secant variety $\Sigma_{n-1}(X, L(-2x))$. Let $\mathcal{M}$ be the embedding line bundle. Then the hypothesis of this lemma means the map

$$\mathrm{Sym}^i H^0(\mathcal{M}) \twoheadrightarrow H^0(\mathcal{M}^{\otimes i})$$

is surjective for all i. Since $\mathcal{M}$ is the restriction of $\mathcal{O}_{\mathbb{P}^r}$, pulling back this map along f yields the surjective map

$$\mathrm{Sym}^i H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{n,\mathcal{L}(-2x)})}(1)) \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{n,\mathcal{L}(-2x)})}(i)).$$

Recall that if we pushforward $\mathcal{O}(i)$ along the projection $\pi : \mathbb{P}(\mathcal{E}_{n,\mathcal{L}(-2x)}) \to X^{(n)}$, we get $\mathrm{Sym}^i \mathcal{E}_{n,\mathcal{L}(-2x)}$. Thus, we have a natural isomorphism

$$H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{n,\mathcal{L}(-2x)})}(i)) \cong H^0(\mathrm{Sym}^i \mathcal{E}_{n,\mathcal{L}(-2x)}).$$

Therefore, the map

$$\mathrm{Sym}^i H^0(\mathcal{E}_{n,\mathcal{L}(-2x)}) \to H^0\left(\mathrm{Sym}^i \mathcal{E}_{n,\mathcal{L}(-2x)}\right)$$

is surjective, as desired, and we are done. $\square$

5.1.2 Corollaries and conjectures

Now the question becomes: when is $\Sigma_{n-1}(X, L(-2x))$ projectively normal? According to a result of Sidman and Vermeire (Corollary 3.4 of [26]), $\Sigma_1(X, \mathcal{B})$ is projectively normal as long as $\deg(\mathcal{B}) \geq 2g+3$. This immediately leads to the following corollary.

Corollary G. *If X is a smooth projective curve of genus g and $\mathcal{L}$ a very ample line bundle on X such that $\deg \mathcal{L} \geq 2g+5$, then $\Sigma_2(X, \mathcal{L})$ is normal along X.*

Note that we do not need to add the condition that $\mathcal{L}$ be 5-very ample in the above, as the degree condition will imply that.

It is unknown whether higher secant varieties are projectively normal. However, we quote a conjecture of Vermeire below.

Conjecture V.4 ([32], Conjecture 5)**.** *Let $C \subset \mathbb{P}^n$ be an embedding of a smooth curve of genus g by a line bundle $\mathcal{B}$. If $\deg \mathcal{B} \geq 2g+1+2k$, $k \geq 0$, then $\Sigma_k(C, \mathcal{B})$ is projectively normal.*

As we have already mentioned, $\Sigma_n(X, \mathcal{L})$ is singular along $\Sigma_{n-1}(X, \mathcal{L})$, not just along X. However, the place that we run into a dead end is calculating the conormal bundle $N^*_{F_{y,D}/\mathbb{P}(\mathcal{E}_{n+1,\mathcal{L}})}$, where $y \in \Sigma_{n-1}(X, \mathcal{L}) \backslash X$. Intuition tells us that the singularities should be the "worst" along X and get better as we move to higher secant varieties. Thus, since we have strong evidence that $\Sigma_n(X, \mathcal{L})$ is normal along X for sufficiently high degree $\mathcal{L}$, we combine our intuition with our theorem and Vermeire's conjecture to get the following conjecture.

Conjecture E. *If X is a smooth projective curve of genus g and $\mathcal{L}$ a very ample line bundle on X such that $\deg \mathcal{L} \geq 2g+2n+1$, then $\Sigma_n(X, \mathcal{L})$ is a normal variety.*

5.2 Further considerations

As mentioned in previous chapters, the Hilbert scheme $X^{[n]}$ is smooth when $n \leq 3$ or $\dim X \leq 2$. Thus, in these cases, we would also get a resolution of singularities of the corresponding secant variety. However, as we've seen above, higher secant varieties can get very complicated, even in the simplest case of curves. We conclude with the following question.

Question V.1. *Is* $\Sigma_n(X, \mathcal{L})$ *normal when* $\dim X = 2$ *or when* $n = 2$*?*

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