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Dynamics of Bond and Stock Returns

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Abstract

I present a production-based general equilibrium model that jointly prices bond and stock returns. The model produces time-varying correlation between stock and long-term default-free real bond returns that changes in both magnitude and sign. The real term premium is also time-varying and changes sign. To generate these results, the model incorporates time-varying risk aversion within Epstein-Zin preferences and two physical technologies with different exposure to cash-flow risk. Bonds hedge risk-aversion (discount-rate) shocks and command negative term premium through this channel. Capital (cash-flow) shocks produce comovement of bond and stock returns and positive term premium. The relative strength of these two mechanisms varies over time.

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1 Introduction

During the 2008 financial crisis US treasury bonds proved to be excellent hedges against stock market risk. Frequent "flight-to-safety" episodes pushed stock prices and bond yields down simultaneously. Bonds were thus particularly valuable and, according to standard asset pricing theory, were likely to command low or even negative term premium. In 1990s, however, bond and stock prices tended to comove, making bonds risky and therefore term premia were likely positive. Figure 1 shows that correlation of stock returns with *real* long-term bond (TIPS) returns is variable and changes sign[1]. No existing general equilibrium models, to my knowledge, are able to obtain this result through a purely "real" channel[2]. In fact, in most of consumption-based, habit[3], and long-run-risk model calibrations, the correlation between real bond and stock returns and real bond term premium are always negative and the real yield curve is always downward-sloping[4]. In this paper I present a production-based general equilibrium model that produces time-varying correlation between stock and real bond returns and real term premium that change in both magnitude and sign.

The model features two physical technologies with different amounts of capital risk and adjustment costs to investment. Both technologies produce the same good, but differ in their risk. One technology is more productive, but also has higher exposure to capital risk. The two technologies are intended to model cross-sectional heterogeneity in cash-flow risk across firms (e.g., low-risk "utility" companies vs. high-risk innovative "hi-tech" companies).

[1]Because reliable data on TIPS starts in early 2000s, it might not be evident that the correlation of stock returns with TIPS can be positive in the data. To shed light on this issue I show similar evidence from the UK in Figure 14. Correlation between UK inflation-protected government bonds and UK MSCI stock market index is positive before late 90s and negative since then.

[2]Bansal and Shaliastovich (2012); David and Veronesi (2009); Rudebusch and Swanson (2008) specify models in which correlation between stock and *nominal* bond returns can change sign.

[3]Bekaert et al. (2010) and Wachter (2006) are exceptions. In these models consumption-smoothing effect of habits dominate the precautionary-savings effect and results in an upward-sloping yield curve and positive real bond risk premium. On the downside, however, these models counterfactually imply that interest rates are high when surplus-consumption ratio is high (in recessions) and low otherwise.

[4]The intuition behind this result can be easily illustrated with a simple consumption-based model with log utility. In such a model a one-period bond price is $\log P_t^{(1)} = -\mathbb{E}_t[\Delta c_{t+1}]$ and term premium on a two-year bond is thus $brx_t^{(2)} = -\text{cov}_t\left(\text{SDF}_{t+1}, \log P_{t+1}^{(1)}\right) = -\text{cov}_t\left(\Delta c_{t+1}, \mathbb{E}_{t+1}[\Delta c_{t+2}]\right)$. When consumption growth is positively auto-correlated (as in the data), bond risk premium is always negative.

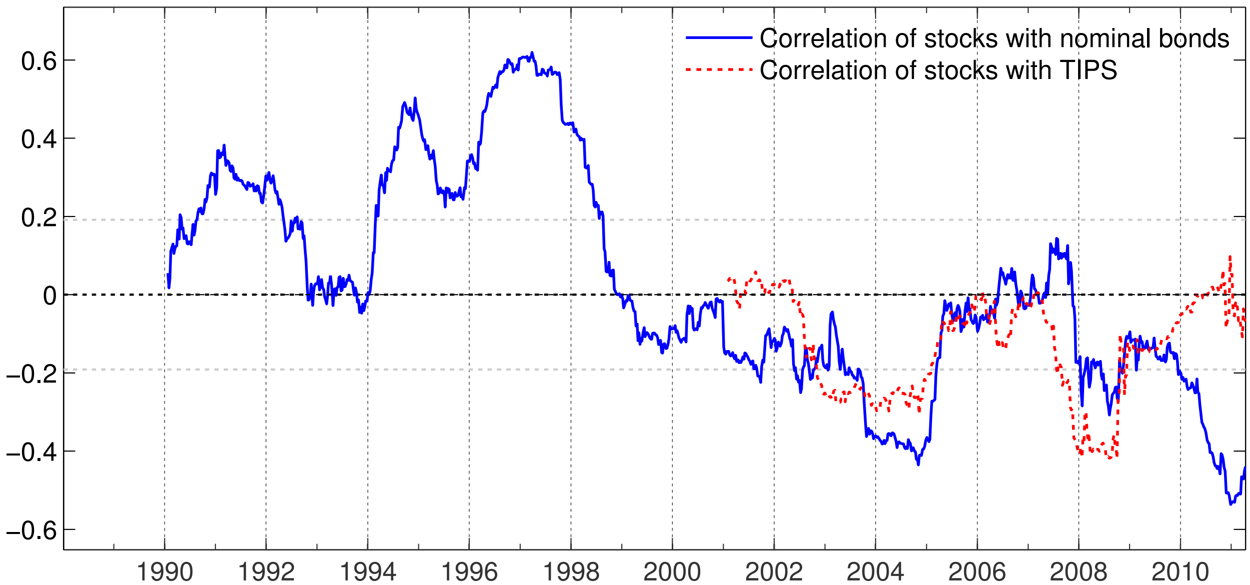
When one technology has a good shock (more capital; constant productivity), investors want to rebalance towards the other technology to diversify their capital investment. However, they face adjustment costs, which drives up the price of the technology with no shock. This mechanism, *technology diversification*, produces positive correlation between returns on the two technologies, even when their cash flows are independent. Technology diversification also produces positive correlation between returns on real bonds and stocks (total wealth) in the model. The mechanism is similar to “two trees” in Cochrane et al. (2008) or two technologies in Wang and Eberly (2012) and operates through discount-rate effects induced by general equilibrium market clearing.

The model also features changes in risk attitudes, generated by a time-varying risk-aversion coefficient within Epstein-Zin preferences or, equivalently, by time-varying uncertainty about model misspecification as in Drechsler (2013). When risk aversion rises, investors want to move from riskier to less risky assets, a *flight-to-safety* effect, which produces a negative correlation between returns on real bonds and stocks.

Technology diversification and flight-to-safety together result in time-varying correlation between returns on financial assets with different amounts of cash-flow risk even when their cash flows are independent. When applied to bonds and stocks, the model produces time-varying correlation between stock and default-free real bond returns, that changes sign. Further, the model produces a time-varying real term premium, which also changes sign. When risk aversion is high, the flight-to-safety mechanism dominates and the return correlation is negative. When risk aversion is low, the technology diversification mechanism results in a positive correlation between bond and stock returns and a positive term premium. Technology diversification relies on slow physical capital reallocation and thus drives low frequency dynamics. Flight-to-safety operates at a higher frequency and is mostly responsible for variation in financial variables in the model.

I calibrate the model and find the ICAPM representation of expected returns. This allows me to learn what fraction of the risk premium on any asset comes from technology

Figure 1: Correlation between market excess return and 10-year bond return (weekly)



Notes: Rolling 1 year correlation between weekly stock excess returns and 10-year bond excess returns. The blue line depicts correlation of excess stock returns with nominal excess returns on a 10-year bond. The red line depicts correlation of excess stock returns with real returns on a 10-year TIPS.

diversification and flight-to-safety mechanisms. The contribution of technology diversification is positive for all assets, resulting in a "common factor" in returns. The contribution of the flight-to-safety mechanism is negative for bonds and positive for stocks. Bonds therefore hedge some stock market risk because they pay well when market discount rates (risk aversion) increase. The relative magnitudes of the two components change over time across all assets. When discount rates are high, flight-to-safety dominates and we see high stock risk premium, negative bond risk premium, and negative return correlations. The opposite holds when discount rates are low.

2 Literature Review

The paper builds on several strands of existing literature. First, many papers analyze bond and stock risk premia separately. Others find evidence linking both markets. Fama and French (1993) note that the term spread predicts the stock market returns. Similarly, Cochrane and Piazzesi (2005) find that a linear combination of forward rates, the "CP factor", is a good forecaster of government bond risk premia in the cross-section and time-series, and also forecasts stock excess returns. Van Binsbergen et al. (2010) build a DSGE model to jointly price bonds and stocks. Kozak and Santosh (2015) price a cross-section of bond and stock returns within a three-factor empirical ICAPM specification. The last two papers also show decompositions of bond risk premia by maturity.

The paper also relates to the literature that studies the correlation of bond and stock returns. Campbell et al. (2012) explicitly embed time-varying bond-stock covariance in a reduced-form model to explore the changes in risk of nominal government bonds over time. Baele et al. (2010) design a dynamic factor model to analyze economic sources of bond-stock comovement. Connolly et al. (2005) find a negative relation between implied volatility and bond-stock correlation, which they attribute to flight-to-quality. Finally, Campbell et al. (2012) describe a puzzle: "some papers have also modeled stock and bond prices jointly, but no existing models allow bond-stock covariances to change signs". In my model, bond-stock covariances change signs endogenously, due to changing relative strength of two mechanisms, rebalancing and flight-to-safety.

Many general equilibrium macroeconomic models produce a negative real bond risk premium and a negatively sloped real yield curve (and a negative correlation of bond and stock returns). Some exceptions, such as Bekaert et al. (2010) and Wachter (2006), produce a positively sloped yield curve within an external habit model by making the consumption smoothing motive dominate the precautionary savings motive. These papers, however, imply that the real short rate is high when the consumption-surplus ratio is low. Others (David and Veronesi, 2009; Rudebusch and Swanson, 2008, 2012) have tried to use the dynamics of inflation to reconcile the failure of standard models to produce a positive term premium. Rich dynamics of inflation within a classical macroeconomic model that implies a negative real term premium can potentially resolve some of the puzzles for *nominal* bonds. Although inflation is an important determinant of prices of nominal bonds, I focus exclusively on *real* bond prices in this paper. Empirical evidence suggests the salient features of the dynamics of real and nominal bond prices are similar.[5]

On a more technical note, this paper's two main mechanisms heavily depend on several ingredients in the literature. First, the rebalancing mechanism, borrows a lot from Cochrane et al. (2008), who show that having two technologies with infinite adjustment costs produces a comovement, or a "common factor" in returns on assets with independent cash flows in general equilibrium. Martin (2012) generalized this setup to multiple "trees". My paper differs from these by endogenizing the size of "trees" and modeling technologies with different amounts of cash-flow risk. The latter feature leads to an endogenous time-varying quantity of risk in the economy, which is essential for producing positive bond risk premia and a positive correlation of bond and stock returns. Wang and Eberly (2012) while also endogenize the size of "trees", do not allow for this endogenous risk-taking. Nevertheless, I borrow a lot from Wang and Eberly (2012) in terms of modeling two technologies and using similar analytically tractable functional forms. None of the papers mentioned also allowed for an exogenous variation in risk aversion.

Second, the flight-to-safety mechanism relies on the time variation in risk aversion. The general idea goes back to Constantinides (1990) and Campbell and Cochrane (1999), who studied models with internal and external habits, capable of generating endogenous fluctuations in the curvature of the value function. In these papers, fluctuations in the curvature come purely from the past consumption dynamics. Alternatively, Gârleanu and Panageas (2008) show time-varying risk aversion naturally arises in a general equilibrium with heterogeneous agents. Menzly et al. (2004) specify a process for the consumption-surplus ratio

[5] **??** plots the correlation between stock excess returns and real and nominal long-term bond returns. The correlations look similar.

directly, which simplifies the modeling. Bekaert et al. (2010) goes further by allowing the curvature of the value function depend also on exogenous shocks, producing essentially exogenous variation in risk aversion. Finally, Dew-Becker (2011) specify an exogenous process for risk aversion directly, and is the closest setup to the way I model the flight-to-safety mechanism in my paper.

Third, several papers (Rudebusch, 2010; Tallarini, 2000) pointed out that keeping the elasticity of intertemporal substitution detached from risk aversion is important for fitting both macro and financial moments in general equilibrium models. I therefore employ Epstein and Zin (1989) preferences in continuous time, by borrowing from Duffie and Epstein (1992b) and Duffie and Epstein (1992a). Although many papers employed the Duffie and Epstein (1992b) stochastic differential utility specification, I use an *unnormalized* aggregator of Duffie and Epstein (1992b), which is relatively non-standard in the literature. The aggregator allows me to preserve the homogeneity of the value function when risk aversion is stochastic. This property delivers a more analytically tractable version of Dew-Becker (2011) setup.

Finally, I employ small-noise expansions to get approximate analytic solutions of the stylized model in the paper. These expansions have been studied in the control-theory literature, namely, Fleming (1971), Fleming and Yang (1994), and James and Campi (1996). Anderson et al. (2012) and Kogan and Uppal (2001) have analyzed a similar type of expansions. These expansions, however, differ from the ones typically used in economic literature where an expansion takes place in the shock standard deviation and around some deterministic steady state (imposing steady-state values of variables).

3 The Model

3.1 Setup

3.1.1 Production

There are two technologies indexed by $n = \{0, 1\}$. Production function takes a simple AK form: $Y_{n,t} = A_n K_{n,t}$, where $Y_{n,t}$ is the total output, A_n is a constant productivity multiplier, and $K_{n,t}$ is the equilibrium capital of technology n.

A process for evolution of capital features adjustment costs and is given by

$$dK_{n,t} = \phi_n \left(i_{n,t}\right) K_{n,t} dt + K_{n,t} \sigma_{K,n} dZ_{K,t}, \tag{1}$$

where $i_{n,t} = \frac{I_{n,t}}{K_{n,t}}$ is the investment-capital ratio of each of two technologies, $\phi_n \left(i_{n,t}\right) \times K_{n,t}$ is a concave in $i_{n,t}$ installation function (so that an adjustment-cost function is convex), which is homogeneous of degree one in capital, $Z_{K,t}$ is a capital shock, and $\sigma_{K,n}$ is a constant loading of technology n on the shock. I thus assume the capital shock is the same for two technologies, which leads to a single source of cash-flow news in the model. The presence of the shock in the capital-accumulation process of an AK-technology is identical to shocks to production possibilities in Cox et al. (1985).

Each unit of investment increases the capital stock by $\phi_n^{'} \left(i_n\right)$ and is valued at q_n, the Tobin's (marginal) q. Competitive firms therefore optimally choose to equate $\phi_n^{'} \left(i_n\right) \times q_n$ to unity, the cost of investment. The Tobin's q is hence given by $q_n = Q_n = \frac{1}{\phi_n^{'}(i_n)}$. See Appendix for a more formal argument.

For expositional purposes I assume that technology indexed by $n = 0$ has no cash-flow risk exposure in the rest of the paper. This assumption does not change the main mechanisms of the model but simplifies algebra and intuition significantly and amplifies the technology-diversification effect.

Definition 3.1. The *riskless technology* has no risk in its capital accumulation process,

$\sigma_{K,0} = 0$.

Although the capital-accumulation process contains no risk, returns on the "riskless" technology are not instantaneously risk free and are exposed to the discount-rate risk due to the built-in adjustment costs. With adjustment costs, the price of installed capital (Tobin's q) changes over time, affecting the overall value of the technology.

Definition 3.2. The *risky technology* has a non-zero loading on the capital shock, $\sigma_{K,1} \equiv \varsigma_K \neq 0$.

In principle, both technologies can be risky. None of the mechanisms of the model requires one technology to be riskless. I make this assumption for analytical convenience only. As long as two technologies differ in their exposure to the capital shock ($\sigma_{K,1} > \sigma_{K,0}$), the main results and implications of the model continue to hold.

Finally, suppose there are two types of competitive firms and that each type can invest in a single type of capital. Firms choose investment to maximize their value,

$$P_{n,t} \equiv p_{n,t} \times K_{n,t} = \sup_{\{i_{n,t}\}} \mathbb{E}_t \int_t^{\infty} \frac{\Lambda_{t+\tau}}{\Lambda_t} \left[A_n K_{n,t+\tau} - i_{n,t+\tau} K_{n,t+\tau}\right] d\tau, \tag{2}$$

where Λ_t is a stochastic discount factor (SDF) that is determined in equilibrium.

3.1.2 Preferences

I specify a stochastic differential utility of Duffie and Epstein (1992b). This utility is a continuous-time version of Epstein-Zin discrete-time specification. Following Duffie and Epstein (1992a) I define a stochastic differential utility by two primitive functions, $f(C_t, J_t) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $A(J_t) : \mathbb{R} \to \mathbb{R}$. For a given consumption process C, the utility process J is a unique Ito process that satisfies a stochastic differential equation,

$$dJ_t = \left[-f(C_t, J_t) - \frac{1}{2} A(J_t) \left\|\boldsymbol{\sigma_{J,t}}\right\|^2\right] dt + \boldsymbol{\sigma_{J,t}} d\boldsymbol{Z_t}, \tag{3}$$

where $\boldsymbol{\sigma_{J,t}}$ is an $\mathbb{R}^2$-valued square-integrable utility-"volatility" process, J_t is a continuation utility for C at time t, conditional on current information, $f(C_t, J_t)$ is the flow utility, $A(J_t)$ is a variance multiplier that penalizes the variance of the utility "volatility" $\|\boldsymbol{\sigma_{J,t}}\|$, and $\boldsymbol{Z_t} \equiv (Z_K, Z_\alpha)^\top$ is a vector of shocks. A pair (f, A) is called an aggregator. I use a Kreps-Porteus (Epstein-Zin-Weil) aggregator, defined as

$$\begin{aligned} f(C, J) &= \frac{\delta}{\rho}\frac{C^\rho - J^\rho}{J^{\rho-1}} = \frac{\delta}{\rho}J\left[\left(\frac{C}{J}\right)^\rho - 1\right] && (4) \\ A(J) &= -\frac{\alpha}{J}, && (5) \end{aligned}$$

where $\rho = 1 - \frac{1}{\psi}$ and ψ is the elasticity of intertemporal substitution; δ is a subjective discount factor, and α is the risk-aversion parameter. Duffie and Epstein (1992a,b) show finding an ordinally equivalent aggregator $\left(\bar{f}, \bar{A}\right)$ is possible such that $\bar{A} = 0$, a *normalized* aggregator. Most papers that employ Epstein-Zin-Weil preferences use this type of aggregator.

I take a different approach and use an *unnormalized* aggregator as defined explicitly in Eq. 4 and Eq. 5. Although such a representation requires computing an additional variance term in Eq. 3, it allows me to separate the effect of elasticity of intertemporal substitution (EIS) and risk aversion in the stochastic differential utility. In particular, the first term, $f(C, J)$, depends only on EIS, whereas the second term, $\frac{1}{2}\frac{\alpha}{J}\|\boldsymbol{\sigma_{J,t}}\|^2$ depends only on risk aversion and is linear in it (it might depend on the EIS indirectly through the $\boldsymbol{\sigma_{J,t}}$ term, however).

I further extend the utility specification when the risk-aversion parameter α is stochastic. The specification results in stochastic differential utility being linear in risk aversion and therefore tractable. In particular, it preserves the homogeneity property of the value function and thus allows me to scale everything with the level of total capital.

Finally, agents face a total wealth constraint, which is given by

$$dW_t = \left[W_t \boldsymbol{\theta_t'} \boldsymbol{\lambda_t} + W_t r_t - C_t\right] dt + W_t \boldsymbol{\theta_t'} \boldsymbol{\sigma_R} d\boldsymbol{Z}, \tag{6}$$

where W denotes total wealth, $\boldsymbol{\theta} = (\theta_0,\ \theta_1)^\top$ denotes the vector of shares of wealth in assets that span the market, $\boldsymbol{\lambda}$ denotes a 2×1 vector of risk premia on the two assets, r_t is the equilibrium risk-free rate, and $\boldsymbol{\sigma_R}$ is a 2×2 covariance matrix of returns on the two assets. The formulation in Eq. 3 and Eq. 6 describes a standard portfolio allocation problem of an infinitely lived investor who can freely participate in complete financial markets.

3.1.3 State variables

Because of homogeneity of utility specification in Eq. 3, the use of the unnormalized aggregator, and homogeneity of the capital accumulation processes in Eq. 1, I am able rescale all variables by the total capital or total wealth. I establish this result in section 6.3.1. I need therefore only two state variables: the share of capital in the risky technology, which I will denote with x, and risk aversion, α.

The risk-aversion process The process for risk aversion α_t is taken as exogenous,

$$d\alpha_t \quad = \quad \phi\left(\bar{\alpha} - \alpha_t\right) dt + \alpha_t \boldsymbol{\sigma_\alpha} d\boldsymbol{Z_t}, \tag{7}$$

where $\boldsymbol{\sigma_\alpha} = (\lambda\varsigma_K,\ \varsigma_\alpha)^\top$ is a vector of loadings on the capital and risk-aversion shocks $\boldsymbol{Z_t} \equiv (Z_K,\ Z_\alpha)^\top$ in the economy, λ, ς_K, and ς_α are constants, and $\lambda \leqslant 0$ controls the strength of the risk-aversion response to capital shocks. The volatility is scaled by α_t for two reasons. First, it guarantees, α_t is always positive and hence the agents are always risk averse. Second, it makes the size of shocks scale with current level of risk aversion. When α_t is high, shocks to risk aversion matter more than when α_t is low. This assumption is standard in the literature. Campbell and Cochrane (1999) and Bekaert et al. (2010), for example, specify the process for the consumption surplus ratio and risk aversion in logs, which produces heteroskedasticity of the same kind in levels.

This parameter can be interpreted as either time-varying risk aversion (Campbell and Cochrane, 1999; Dew-Becker, 2011) or time-varying ambiguity aversion with respect to model specification (Drechsler, 2013; Hansen and Sargent, 2008)[6].

The share of capital in the risky technology Let the share of risky capital to total capital be $x = \frac{K_1}{K_0+K_1} = \frac{K_1}{K}$, where $K \equiv K_0 + K_1$ is total capital. With this specification $x \in [0,1]$ with $x = 0$ corresponding to the case with only riskless technology and $x = 1$ corresponding to the case with only risky technology. Note $x = 0$ and $x = 1$ states are absorbing: once one technology vanishes, it cannot be rebuilt, because loadings on shocks as well as the drift (due to homogeneity of the adjustment-cost function) in Eq. 1 are proportional to the stock of capital, which is zero.

I apply Ito's lemma in Appendix to show that the share of risky technology x in equilibrium follows an endogenous stochastic process:

$$dx_t = x_t\left(1-x_t\right)\left[\phi_1\left(i_{1,t}\right) - \phi_0\left(i_{0,t}\right) - x_t\varsigma_K^2\right]dt + x_t\left(1-x_t\right)\varsigma_K dZ_{K,t}.$$

3.1.4 Competitive equilibrium

Definition 3.3. Given the vector of state variables $\boldsymbol{X} = (x,\ \alpha)^\top$, the SDF process $\frac{d\Lambda}{\Lambda}$ and price functions $\{p_n \equiv p_n\left(\boldsymbol{X}\right)\}_{n=0}^{1}$, the total wealth of the representative agent W, the vector of risk premia $\lambda\left(\boldsymbol{X}\right)$, the risk-free rate $r\left(\boldsymbol{X}\right)$, and the covariance matrix $\boldsymbol{\sigma_R}\left(\boldsymbol{X}\right)$ of returns on two assets that span the markets, a *stochastic competitive equilibrium* is a set of allocations $\{C,\ \theta,\ I_0,\ I_1,\ K_0,\ K_1\}_{t=0}^{\infty}$ such that

(i) agents maximize utility given by the stochastic differential utility process in Eq. 3, subject to their budget wealth evolution in Eq. 6, by choosing how much to consume, C, and how much to invest in either of the two assets, $\boldsymbol{\theta}$;

(ii) firms solve an optimal investment problem in Eq. 2 subject to Eq. 1;

(iii) risk aversion follows an exogenous stochastic process given by Eq. 7;

[6] Variability in α generates time-varying prices of risk in the model. The main results survives if we instead model time-varying quantity of risk (through time-varying volatility of consumption growth).

(iv) capital accumulation grows according to Eq. 1;

(v) aggregate wealth evolves according to Eq. 6;

(vi) financial markets clear, $\boldsymbol{\theta} = \left(\frac{p_n(\boldsymbol{X})K_n}{W}\right)^{\top}$, $n = \{0,\ 1\}$.

3.1.5 Solution

The first and the second Welfare Theorems hold in the model. I therefore start by solving the planner's problem and then decentralizing the economy to find prices.

The planner chooses investment and consumption to maximize the agent's lifetime utility. Due to homogeneity, I guess that the solution is linear in total capital: $J(K_0, K_1, \alpha) = K \times F(x, \alpha)$. I establish the following result in Appendix, section 6.3.

Theorem 3.1. *The solution to the planner's problem is given by the system of a PDE,*

$$\frac{\delta}{\rho}\left[\left(\frac{c(x,\alpha)}{F(x,\alpha)}\right)^{\rho} - 1\right] - \frac{1}{2}\alpha\left\|\boldsymbol{\sigma_F}\right\|^2 + (1-x)\left(1 - \frac{F_x}{F}x\right)\phi_0(i_0) + x\left(1 + \frac{F_x}{F}(1-x)\right)\phi_1(i_1)$$
$$+\frac{F_\alpha}{F}\phi(\bar{\alpha}-\alpha) + \frac{1}{2}\frac{F_{xx}}{F}(1-x)^2 x^2 \varsigma_K^2 + \frac{1}{2}\frac{F_{\alpha\alpha}}{F}\alpha^2\left\|\boldsymbol{\sigma_\alpha}\right\|^2 + \left[\frac{F_\alpha}{F} + \frac{F_{x\alpha}}{F}(1-x)\right]x\alpha\lambda\varsigma_K^2 = 0,$$

with boundary conditions listed in section 6.3.3, first-order conditions for optimal investment,

$$\delta\left(\frac{c}{F}\right)^{\rho-1} = (F - F_x x)\,\phi_0'(i_0)$$
$$\delta\left(\frac{c}{F}\right)^{\rho-1} = (F + F_x(1-x))\,\phi_1'(i_1),$$

and the aggregate resource constraint

$$c = (A_0 - i_0)(1-x) + (A_1 - i_1)\,x,$$

where F_x denotes a derivative of the rescaled value function w.r.t. x, $\boldsymbol{\sigma_F} = \left(1 + \frac{F_x}{F}(1-x)\right)x\boldsymbol{\sigma_K} + \frac{F_\alpha}{F}\alpha\boldsymbol{\sigma_\alpha}$, and $\boldsymbol{\sigma_K} = (\varsigma_K,\ 0)^{\top}$.

Proof. Appendix, section 6.3 provides the details and derivations. □

Next, I decentralize the economy and solve a portfolio allocation problem, which is detailed in section 6.2.2. The following theorem summarized the result for the SDF of the economy.

Theorem 3.2. *The stochastic discount factor (SDF) of the economy is given by*

$$\frac{d\Lambda}{\Lambda} = -r\left(\boldsymbol{X}\right)dt + \mathcal{L}\left(dlnf_C - \alpha_t dlnJ\right)d\boldsymbol{Z}, \tag{8}$$

where $r\left(\boldsymbol{X}\right)$ is the equilibrium interest rate, $\mathcal{L}\left(d\tilde{s}\right)$ denotes the vector of loadings on shocks of a stochastic process $d\tilde{s}$, and f_C is the derivative of the flow utility in Eq. 3 with respect to consumption C.

Proof. See section 6.5 for the derivation. □

3.1.6 Returns

I choose two assets that span the markets as follows. The first asset's instantaneous total return process is given by

$$dR_{0,t} = \frac{A_0 - i_{0,t}}{q_{0,t}}dt + \phi_0\left(i_{0,t}\right)dt + \frac{dq_{0,t}}{q_{0,t}}, \tag{9}$$

where $q_{0,t}$ is the Tobin's q of riskless technology. The expression gives the total return on the riskless technology.

The second asset corresponds to the total return on the risky technology. The instantaneous return is given by

$$dR_{1,t} = \frac{A_1 - i_{1,t}}{q_{1,t}}dt + \phi_1\left(i_{1,t}\right)dt + \frac{dq_{1,t}}{q_{1,t}} + \left(\frac{dK_{1,t}}{K_{1,t}} - \mathbb{E}\frac{dK_{1,t}}{K_{1,t}}\right) + \left\langle\frac{dq_{1,t}}{q_{1,t}},\ \frac{dK_{1,t}}{K_{1,t}}\right\rangle.$$

The first component is the dividend-price ratio. The second is a capital increase due to new investment. The third gives the change in the value per unit of installed capital. The fourth is the change in the return due to the shock to the physical capital. The last component reflects the Ito term due to changing total value.

The final step to complete the mapping to the competitive equilibrium in Definition 3.3 is to compute the risk-free rate. Appendix, section 6.3.1 provides details on these calculations.

3.1.7 Stylized Model

Unfortunately, the planner's problem does not have an analytic solution in general. In the Appendix, section 6.4 I show pseudo-analytic solutions to the stilized model in which EIS is set to 1 and installation function $\phi_n(\cdot)$ is the same for two technologies and takes a log form. The solution relies on small-noise expansions (perturbations around the non-stochastic steady state). It allows us to establish analytic propositions characterizing the economy which I briefly summarize here:

1. Shocks to risk aversion that are orthogonal to shocks to the capital-accumulation process move returns on risky and riskless technologies in *opposite* directions;

2. Shocks to the risky capital-accumulation process that are orthogonal to shocks to risk aversion move returns on risky and riskless technologies in the *same* direction;

3. One can find a calibration of the model that produces a positive correlation between returns on riskless and risky technologies for low levels of risk aversion and negative correlation when risk aversion is high.

For more formal argument and proofs, refer to Appendix, section 6.4. In section 4 I will also show evidence that these propositions work numerically in a calibration of the model.

3.2 Pricing Bonds and Stocks

The price of a real long-term bond of maturity T is given by

$$P_{B,t}^{(T)} = \mathbb{E}_t \left[\frac{\Lambda_T}{\Lambda_t} \times 1 \right]. \tag{10}$$

In models of such complexity, analytical solutions for the term structure are typically unavailable. I therefore take a different approach that allows me to use analytical solutions for returns on the two technologies as good approximations of bond and stock prices.

The method relies on a specific choice of technologies I have made. The returns on the riskless technology, given by Eq. 9, are similar to the returns on a perpetuity that pays $A_0 dt$ at each instant, which can be defined as $dR_c = \frac{A_0}{q_c} dt + \frac{dq_c}{q_c}$. The difference lies in the investment adjustment, $\left[-\frac{i_0}{q_0} + \phi(i_0)\right] dt$, which reflects the fact that not all of the new investment is installed at the marginal cost. In the formula for the riskless technology, the "profits" due to this fact are split among all existing shareholders equally. In practice, the adjustment turns out to be small and thus has little impact on the dynamics of returns on the riskless technology. This allows me to use an analytic solution for the returns on the riskless technology as a good approximation of returns on a perpetuity that pays $A_0 dt$ every instant. In the empirical section of the paper, I further verify the dynamics of returns of the riskless technology are indeed similar to those of a perpetuity, which in turn are similar to those of a long-term real default-free bond, which I price in the model by explicitly computing the expectation of the SDF in Eq. 10. I am therefore able to use an analytic solution for returns on the riskless technology as a good approximation of the returns on a bond with some high duration.

I define a stock as a levered claim to a portfolio of two technologies. I assume the amount of leverage is time-varying and determined by the relative sizes of the two technologies (the Modigliani-Miller theorem holds in my model; therefore, any amount of leverage is consistent with firms' capital-structure decisions). As a consequence, the aggregate leverage of the economy is pinned down by the aggregate risk-taking and general equilibrium, rather than an individual firm's decision. The asset being shorted in the leveraging process in this case is a perpetuity (a claim on the riskless technology) rather than an instantaneously risk-free bond. With this definition of the leveraging process, the returns on a stock are equal to returns on the risky technology itself. To make this interpretation viable quantitatively, I calibrate the process for capital accumulation in Eq. 1 and other parameters of the model in such a way that the volatility and dynamics of returns and risk premium produced by the risky technology closely resemble the respective dynamics generated by the stock market index. With this interpretation, I can therefore use an analytical solution for returns on the risky technology as a good approximation to returns on the stock market index.

Note, however, although the specific assumption on technologies made facilitates the analysis and allows me to get approximate analytic formulas for bond and stock prices, the mechanisms of the model do not require one technology to be riskless. As long as two technologies differ in their exposure to the capital shock, the main results of the paper continue to hold. In this case, I can numerically solve a model with two risky technologies that differ in their amounts of cash-flow risk, and price long-term bonds by explicitly computing the expectation of the SDF in Eq. 10. Because the riskless and risky technologies differ in the amounts of their cash-flow risk and returns on the technologies co-move more strongly with returns on bonds and stocks respectively, changes in the relative sizes of the technologies affect the prices of bonds and stocks (through an SDF).

3.3 Mechanisms

With this intuition in mind, I now use the analytical results of section 6.4, to revisit the main mechanisms of the model.

The technology diversification mechanism relies on having two technologies that differ in their cash-flow risk, in positive supply, and adjustment costs to investment. With an endogenous investment choice, a time-varying endogenous supply of risk emerges. When one technology has a good shock, investors want to rebalance (diversify) to the other technology, but face adjustment costs, which drives up the price of the other technology. Prices adjust because quantities are fixed in the short run. In fact, I show in section 4 that the model behaves as "two trees" of Cochrane et al. (2008) in the short run, when quantities are essentially fixed, and as CIR-type model in the longer run, when quantities are free to adjust and prices stay constant. Unlike “two trees,” however, the mechanism endogenizes the size of technologies and the aggregate risk-taking decision. An endogenous size is important for preserving the stationarity of the model and sustaining an equilibrium in which two technologies persist. An endogenous risk-taking leads to an SDF that produces a co-movement of bond and stock returns in the model. It also produces realistic investment dynamics in the model.

The “two trees” mechanism induces a co-movement of returns on any “trees”, whether they are i.i.d. or different in some important dimension. I assume technologies differ in their amounts of cash-flow risk, which leads to a co-movement of returns on low- and high-risk technologies. Because a low-risk technology is more “bond-like” (has lower cash-flow risk), its returns co-move much stronger with returns on a perpetuity (and hence with returns on a long-term bond) than returns of a high-risk technology, which is more “stock-like”. This built-in feature of the model design, delivers an SDF that generates a co-movement of bond and stock returns that are priced in the model using the SDF. With only the technology diversification mechanism present, the model would always produce positive correlation of bond and stock returns, and always positive bond risk premia at all maturities.

The flight-to-safety mechanism relies on exogenous variation in risk aversion. When risk aversion rises, investors want to move from the riskier to the less risky technology, which produces a negative correlation between returns on the technologies. The mechanism produces time-varying preference for risk. With only flight-to-safety mechanism present, the model would always produce negative correlation of bond and stock returns, and always negative bond risk premia at all maturities.

Time-variation and relative strength of the two mechanisms determines the overall correlation between returns on risky and riskless securities as well as the sign of risk premium on the riskless technology. When risk aversion is low, the technology diversification mechanism dominates, producing a positive correlation between bond and stock returns on average. During those times, bonds are exposed to the same discount-rate risk as stocks, and this exposure is higher than opposing hedging motives. Riskless securities, therefore, tend to command mildly positive term premium during tranquil times. On the other hand, when risk aversion rises, the flight-to-safety mechanism becomes stronger and often overturns the technology diversification mechanism, to produce a negative correlation between bond and stock returns. During such times, bonds are typically perceived as good hedges against the stock market risk and the hedging motive dominates the common exposure to a discount-rate risk. Bonds thus command a negative risk premium.

The change in the relative importance of two mechanisms occurs because in response to a capital shock, discount rates on the risky asset move in a way that dampens the cash-flow effect on the asset. This dampening becomes stronger as risk aversion rises, leading to a relatively weaker comovement of bond and stock returns. At the same time, an increase in risk aversion leads to a stronger "decoupling" of bond and stock returns due to flight-to-safety. As a result, the flight-to-quality mechanism starts dominating the technology diversification mechanism at higher levels of risk aversion.

4 Dynamics

I now explore the dynamics the general model generates, and verify the results obtained are in line with the intuition that we acquired by analyzing small-noise expansions of the stylized model.

4.1 Calibration

I calibrate the model to qualitatively match a set of stylized facts about the dynamics of real and financial variables, such as the mean and standard deviations of consumption and output growth, risk-free rate, equity mean return, standard deviation of equity return, Sharpe ratio, and price-dividend ratio. Additionally, I compare moments for the consumption-to-wealth ratio and total wealth excess returns implied by the model with the estimates by Lustig et al.

Table 1: Calibration

	Variable	Value
	Preferences	
Time discouting	δ	0.03
EIS	ψ	2
Mean reversion of α	ϕ	0.25
Mean of risk aversion	$\bar{\alpha}$	25
Volatility of risk aversion	ς_α	0.17
Propagation of capital shocks	λ	0
	Technology	
Volatility of capital	ς_K	0.062
MPK of bond technology	A_0	0.03
MPK of stock technology	A_1	0.085
Adjustment cost of riskless technology	ξ_0	0.03
Adjustment cost of risky technology	ξ_1	0.02

Notes: Parameters used in the calibration of the general model in section 3.1. All parameter values are annualized.

(2008). I also consider some conditional moments and implications the model produces (see section 4.3). Because of the simplistic design of the model aimed at producing time-varying correlation with as few ingredients as possible, some moments of the data, unsurprisingly, cannot be exactly matched. In particular, I have difficulties matching high average volatility of stock and long-term bonds, although the volatility in the model is highly time-varying and occasionally does get high. One reason for this problem is that I do not have any permanent productivity shocks in the model.

I assume two shocks are uncorrelated. If anything, I found a non-zero correlation of shocks makes returns more volatile and stocks more risky relative to bonds, which somewhat improves the general fit. When the shocks are independent, on the other hand, we do not lose a lot in terms of qualitative features of the fit, but gain much in terms of understanding the impact of either of the two main mechanisms independently. This benefit will be most vivid in section 4.3.3, where I study impulse responses to each of the shocks, and in section 4.3.2, where I study policy functions.

Table 2: Fitted moments

	Data	Model
Mean consumption growth	2%	1.8%
Standard deviation of consumption growth	1.9%	2.5%
Mean output growth	1.7%	1.7%
Standard deviation of output growth	3.7%	3.6%
Mean price-dividend ratio	26	20
Standard deviation of the log price-dividend ratio	0.29	0.26
Mean equity risk premium	6.4%	5.3%
Standard deviation of equity returns	17%	8.2%
Mean equity Sharpe ratio	0.32	0.61
Standard deviation of the equity Sharpe ratio	0.22	0.3
Mean risk-free rate	3%	3.4%
Standard deviation of the risk-free rate	2%	0.5%
Mean total wealth excess return	2.4%	1.7%
Mean wealth-consumption ratio	83	53

Notes: The Data column shows the empirical estimates of moments of interests in the data. The Model column shows the values implied by the calibration of the model in Table 1. All moments are annualized.

Further, I specify a functional form of the installation function similar to the one in Assumption 2. In particular, I assume an installation function for a technology n is given by $\phi_n(i_n) = \xi_n \times ln\left(1 + \frac{i_n}{\xi_n}\right)$. Note that unlike in Assumption 2, constants ξ_n are different for two technologies. I also considered different functional forms of installation functions, quadratic and power. The latter gives more flexibility in jointly matching investment and price dynamics, because it includes an additional parameter that controls the elasticity of investment with respect to Tobin's q. I chose to use the log functional form to emphasize that results hold true even with the simplest specifications of adjustment costs.

Calibrated values of parameters of the model are shown in Table 1. I set the time discounting, δ to match the level of real bond yields, $\delta = 0.03$ (annualized). The elasticity of intertemporal substitution (EIS), ψ, is chosen to equal 2. This value is consistent with the estimates in Van Binsbergen et al. (2010). Wide debate argues whether the EIS is less than or higher than 1, but most of the recent asset-pricing literature lean towards the value higher than 1, with some estimates being above 2 (Van Binsbergen et al. (2010) get an estimate of $\psi = 1.731$ when using inflation data and $\psi = 2.087$ without it). I set $\phi = 0.25$ to generate high-frequency variation of prices in the model. Campbell and Cochrane (1999) use a much lower value to match high persistence of the price-dividend ratio. In my model, high persistence is generated by an endogenous process for the share of two technologies, x, whereas the risk-aversion process is mostly responsible for high-frequency movements. I set $\bar{\alpha} = 25$ and $\varsigma_\alpha = 0.17$ to target the Sharpe ratio and volatility of stock and bond returns. Finally, I choose $\lambda = 0$ to make shocks uncorrelated for expository purposes. I also considered calibrations with $\lambda < 0$, which do not change the qualitative pattern of the results much. Calibrating $\lambda < 0$ makes risk aversion react negatively to direct shocks to capital. This direct impact is often modeled in the literature under the guise of external habit as in Campbell and Cochrane (1999). Direct shocks to risk aversion have been analyzed less often, but are used by Bekaert et al. (2010) and Dew-Becker (2011).

I calibrate the production side of the model to roughly fit real moments. The variance of consumption growth is used to find the volatility of the stock technology, $\varsigma_K = 0.062$. I choose $A_1 = 0.085$ to match the variance of output growth and the level of the price-dividend ratio. The productivity of bond technology, A_0, is set to approximate the average yield on a constant-maturity long-term real bond (30 years) traded at par with semi-annual coupons, $A_0 = 0.03$. Finally, the values for adjustment costs are set to match the mean of consumption growth, standard deviation of investment, and distributions of means and volatilities of bond and stock returns.

The calibration implies the mean value of risky capital $x \approx 0.32$. This value is consistent with the empirical findings of Lustig et al. (2008). In particular, they find the risk premium on total wealth portfolio is about a third of the risk premium on stocks and the wealth-to-consumption ratio of 83 (compared to 26 for the price-dividend ratio). The value of x in my model implies the sizes of estimates that are roughly similar to these moments. The mean value of x is also roughly consistent with the amount of leverage that Bansal and Yaron (2005) use to price stocks ($\phi = 3.0$). Finally, in the empirical section of the paper, I will show a rough estimate of the dynamics of x from the distribution of capital across portfolios of less and more risky stocks.

Table 2 reports empirical and fitted values of a subset of moments I am considering. Empirical estimates of mean and standard deviation of consumption growth and mean and standard deviation of output growth are from Van Binsbergen et al. (2010). Sharpe ratio estimates are taken from Dew-Becker (2011). All other empirical moments are from Lustig et al. (2008). Overall, the model is able to fit the moments reported in Table 2 quite well, with the exception of high volatilities of returns on stocks and long-term real bonds. Alvarez et al. (2002) show bond risk premium relative to total risk premium tells us about the fraction of the variance of the pricing kernel that arises from the martingale component. Koijen et al. (2010) further point out the dynamics of the wealth-consumption ratio constitutes an important empirical test of any model and is related to the fraction of transitory and permanent shocks in the SDF. My model includes the moments of the wealth-consumption ratio in the calibration and does a decent job fitting them.

4.2 Solution Method

I find a numerical solution to the system of equations in Theorem 3.1 using high-order projection methods. I parametrize the value function and two investment functions as a complete product of 20^{th} order Chebyshev polynomials in two state variables, x and α. Next, I evaluate the system of equations in Theorem 3.1 at 30×30 points on the state space. Points are chosen as Chebyshev's zeroes. I then search for coefficients of three policy functions to minimize the L_1-norm of PDE errors. In practice, the algorithm is iterative. I start by fitting low order polynomials on the grid of 30×30 Chebyshev zeroes and iteratively increase the order of the fit until the desired precision is reached. I impose boundary conditions as given in section 6.3.3. The resulting fit is the global solution on the entire state space.

The problem is therefore formulated as a sequence of standard constrained optimization problems with thousands of constraints (one at each grid node) and thousands of unknowns (Chebyshev coefficients). I use the GAMS modeling language together with CONOPT and SNOPT non-linear constrained optimizers to find a solution. Once the solution to the optimization problem is found, I import the results in Matlab to perform simulations and report the results.

Once the approximation to the value and investment functions is found, I can use them to calculate the aggregates and prices in the model. Refer to section 6.3 for further details.

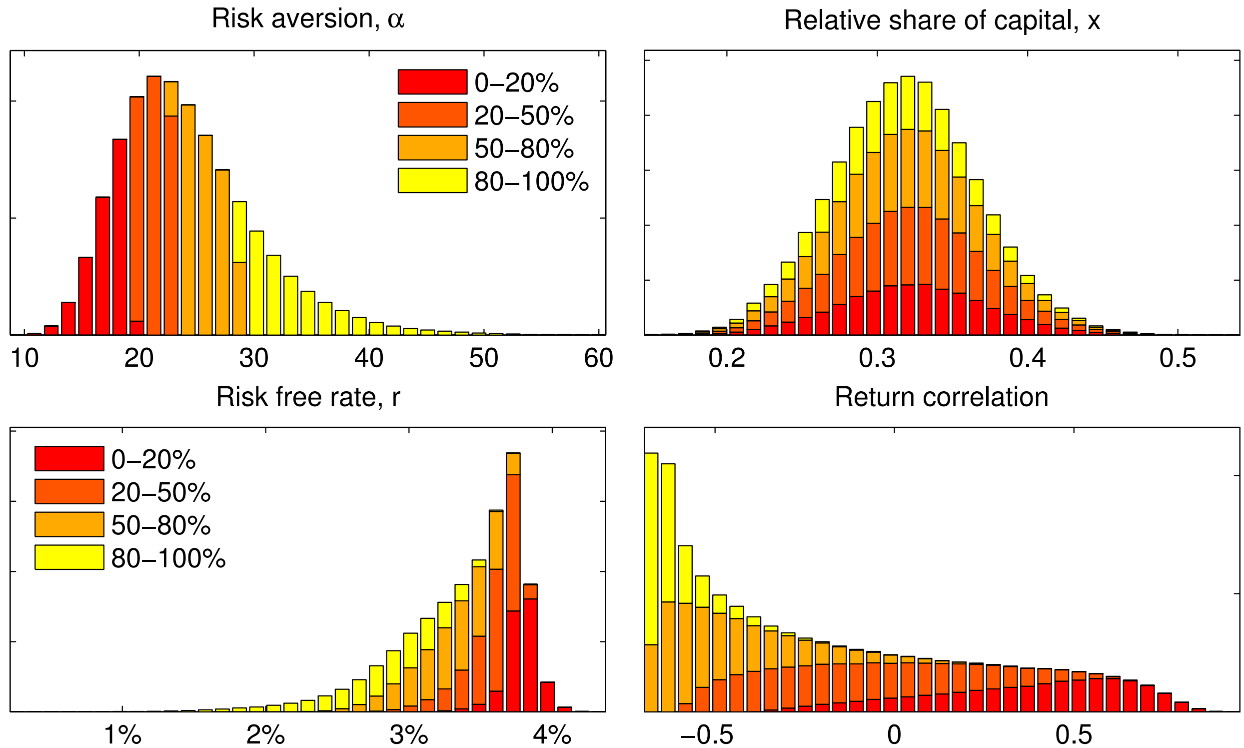
4.3 Results

4.3.1 Conditional distributions of major moments

To construct conditional distributions, I perform Monte-Carlo simulations starting from a steady state (unconditional means of state variables) and show histograms of variables of interest. I show histograms for two reasons. First, they give a clear understanding of where the economy "lives" in the state space and what is the distribution of variables of interests. Second, because they are conditional histograms, we can gain insights about how the level of risk aversion affects real and financial variables in the model. For example, we will see risk premium on risky technology increases and risk premium on riskless technology falls as risk aversion rises.

Figure 2 depicts conditional distribution of state variables, interest rate, and return correlation of the economy. Disregarding the colors, each plot shows a histogram of a respective variable. Different colors within each bar of a histogram show what fraction of observations that contributed to the bar had risk aversion within a certain percentile. For example, yellow (bright) depicts observations with risk aversion being in top 20 percentile, whereas red (dark) depicts observations with low risk aversion in bottom 20 percentile. Because the sort is made on risk aversion, the first histogram that shows the distribution of the risk-aversion parameter naturally changes color from dark red to bright yellow as we move from left to

Figure 2: Conditional distributions of state variables. interest rate, and correlation

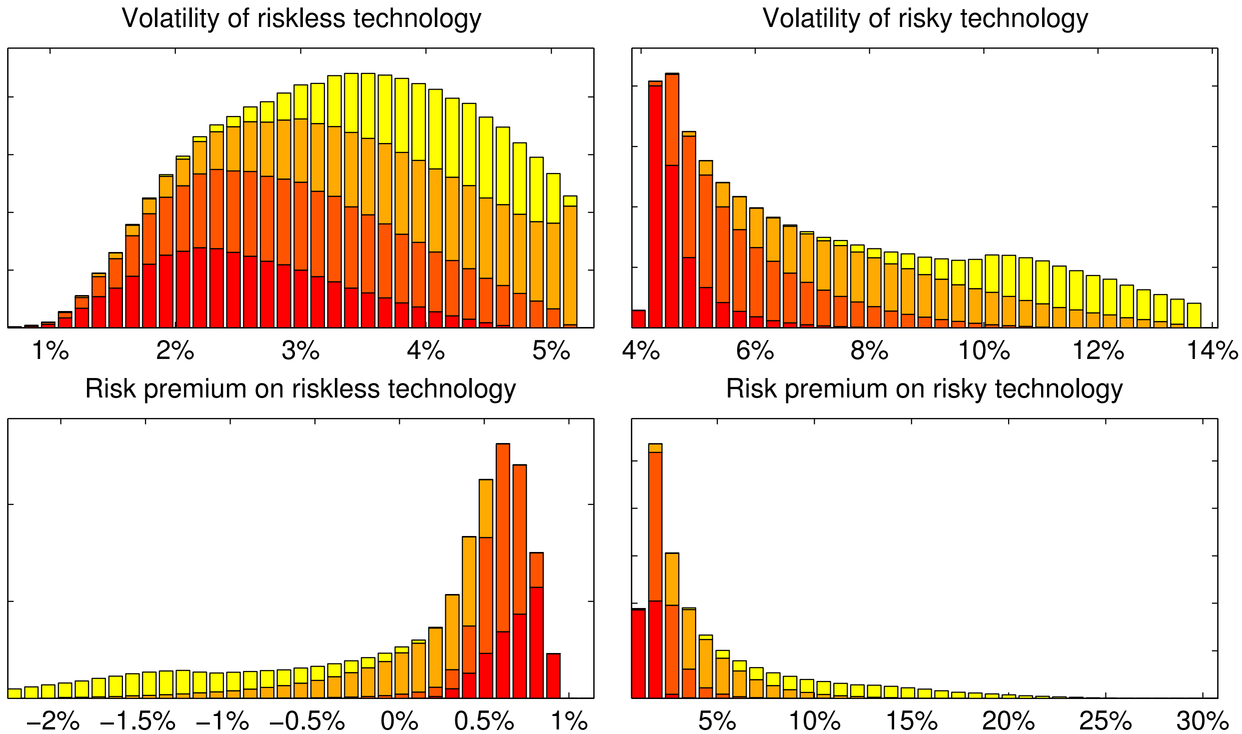


Notes: Conditional distributions of state variables, interest rate, and correlation in the simulated economy. Disregarding the colors, each plot shows a histogram of a respective variable. Different colors within each bar of a histogram show what fraction of observations that contributed to the bar had risk aversion within a certain percentile. For example, yellow (bright) depicts observations with risk aversion being in top 20 percentile. Because the sort is made on risk aversion, the first histogram that shows the distribution of the risk-aversion parameter naturally changes color from dark red to bright yellow as we move from left to right (overlapping colors within one bar in the risk-aversion plot are due to the discrete width of bars).

right (overlapping colors within one bar in the risk-aversion plot are due to the discrete width of bars). The second plot shows the distribution of a state variable x. We can see the risky technology has a mean of about 32% of total capital.[7] Unlike Cochrane et al. (2008), the model therefore produces a stationary distribution of capital with both technologies co-existing in equilibrium. The bottom-left figure shows real interest rate varies from around 1% to 4% annually. It is high when risk aversion is low, and low when risk aversion is high, consistent with the empirical evidence. The bottom-right histogram depicts the distribution

[7]Lustig et al. (2008) find the return on total wealth behaves much more like a long-term bond rather than the stock market. This finding is consistent with a relatively low share of risky capital in the total wealth portfolio.

Figure 3: Conditional distributions of financial variables

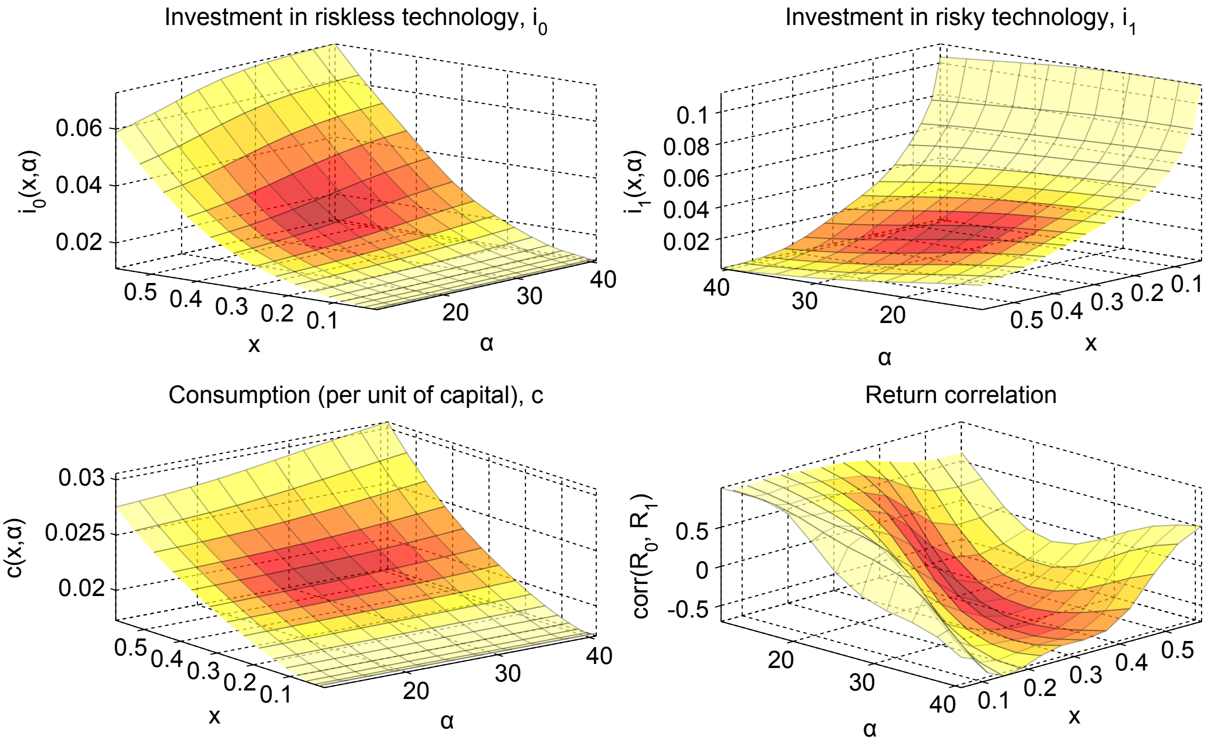


Notes: Conditional distributions of financial variables in the simulated economy. Disregarding the colors, each plot shows a histogram of a respective variable. Different colors within each bar of a histogram show what fraction of observations that contributed to the bar had risk aversion within a certain percentile. For example, yellow (bright) depicts observations with risk aversion being in top 20 percentile.

of correlations between returns on risky and riskless technologies. It is highly time-varying, changing signs, and is high and positive when risk aversion is low, and low and negative when risk aversion is high.

Figure 3 shows distributions of volatility and risk premia on riskless and risky technologies. The risky technology is more volatile due to cash-flow shocks. Both technologies are more volatile when risk aversion is high. Sharpe ratios (not shown), however, go in opposite directions: the Sharpe ratio on risky technology is high when risk aversion is high, and is low and negative on riskless technology. The risk premium on riskless technology varies from around -2% to about $+1\%$ annually. It is negative when risk aversion is high and positive otherwise. The risk premium on risky technology varies from 0% to about 25% annually. It is high when risk aversion is high and low otherwise.

Figure 4: Policy and transition functions



Notes: Policy and transition function are shown as functions of two state variables, x and α. Warmer colors depict regions of the state space with higher probability density.

4.3.2 Policy functions

I now discuss policy and transition functions as functions of two state variables, x and α. Policy functions for consumption and investment in the two technologies are shown in Figure 4. Warmer colors depict regions of the state space with higher probability density.

Investment in the riskless technology tends to increase when the share of risky capital x becomes large, as the economy tries to sustain the stationary distribution of x. Similarly, investment in the risky technology increases as x falls. These two forces generate a strong drift toward the mean value of x (conditional on α) and are key for the existence of equilibrium in which two technologies survive. Cochrane et al. (2008) lack such a force in modeling two technologies as trees and thus allowing for no endogenous investment. As a result, their economy is not stationary and one tree always dominates the other. Stationarity in my model is achieved precisely through endogenous capital reallocation (and high risk aversion, as opposed to log, which makes mean-reverting forces even stronger).

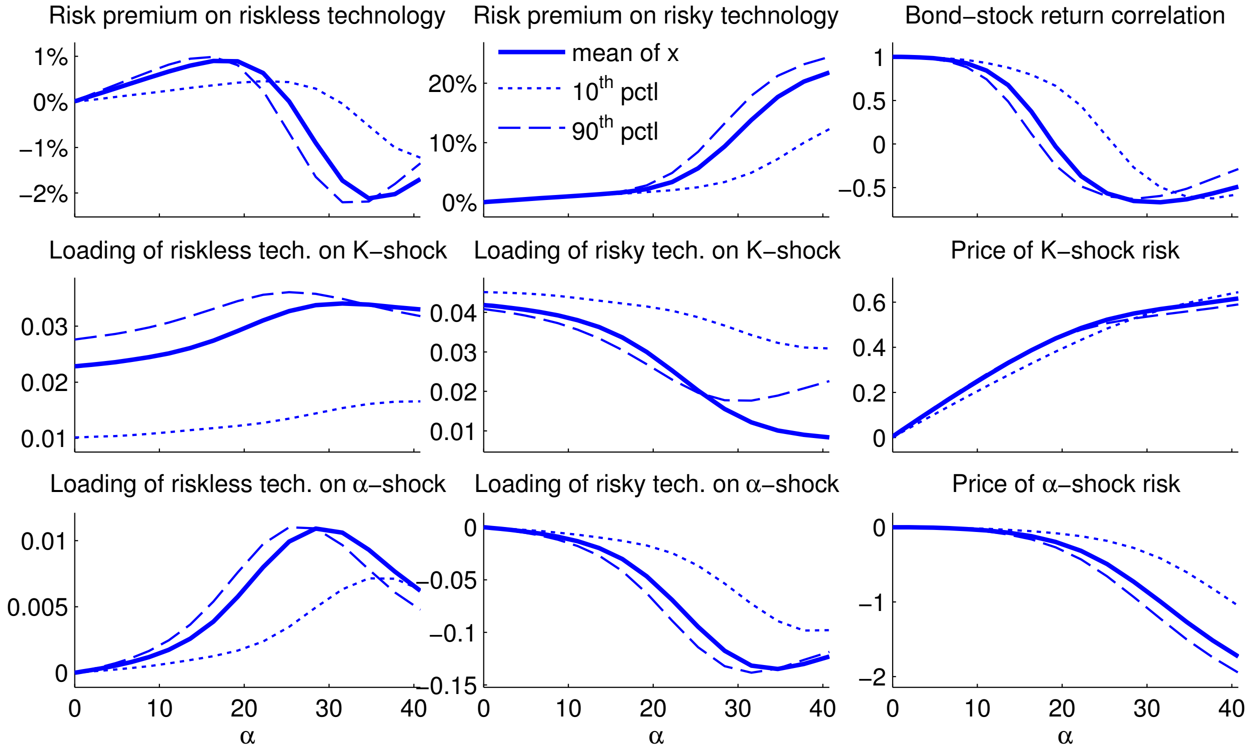
Investment in the riskless technology increases in risk aversion α while investment in the risky technology falls in α. This pattern of investment is a manifestation of the flight-to-safety mechanism. When risk aversion is high, people tend to reallocate from risky to riskless assets, driving up the investment and prices on riskless technology.

Consumption is increasing in the share of risky technology x because of higher aggregate productivity when the more productive technology is larger. Consumption is also increasing in risk aversion α, rising slightly as risk aversion increases. With the EIS greater than one, agents smooth out their consumption in time and consume relatively more per unit of capital when risk aversion increases, contributing to a less volatile consumption path. This fact and the aggregate resource constraint imply the aggregate investment per unit of capital must fall as risk aversion rises. When the EIS is fixed at 1 (stylized model), consumption is completely flat in risk aversion.

Finally, the correlation between returns on risky and riskless technology initially falls steeply as risk aversion increases (and flattens out later). It is highly positive for low risk aversion and becomes negative for higher values of risk aversion, consistent with Proposition 6.3. Correlation also decreases in x when α is low and increases in x for high levels of α. When risk aversion α is low, the flight-to-safety mechanism is weak, especially when we have relatively little risky capital. When risk aversion becomes high, however, flight-to-safety becomes the dominant mechanism. As a result, when x is high, the fraction of low-risk technology is low and thus the price impact on the technology in response to capital shocks is magnified, causing the comovement of returns on two technologies increase, and thus making the technology diversification mechanism stronger. Higher x also results in a smaller discount-rate effect on the technology and weaker dampening of the cash-flow effect (will be discussed later), further increasing the comovement of two technologies.

Figure 5 shows the excess returns, loadings of two technologies on two shocks, and prices of risk as functions of risk aversion α for fixed values of x. Three different levels of x

Figure 5: Impact of risk aversion



Notes: The figure shows the excess returns, loadings of two technologies on two shocks, and prices of risk as functions of risk aversion α for fixed values of x. Three different levels of x are considered, each corresponding to a different line: the solid line corresponds to the unconditional mean value of x, the dotted line corresponds to 10^{th} percentile, and the dashed line corresponds to the 90^{th} percentile. The horizontal axis shows the level of risk aversion, α, ranging from 0 to 80 in each plot. The vertical axis depicts the value of each variable, named in titles of each plot.

are considered, each corresponding to a different line: the solid line corresponds to the unconditional mean value of x, the dotted line corresponds to 10^{th} percentile, and the dashed line corresponds to the 90^{th} percentile. The horizontal axis shows the level of risk aversion, α, ranging from 0 to 40 in each plot. The risk premium on the riskless technology displays a complex non-monotonic pattern. It starts from zero (at zero risk aversion, all risk premia and risk prices have to be zero), increases mildly to positive values as risk aversion remains relatively low, and then falls to the negative territory. It starts to rebound for very high levels of risk aversion, but at these levels, the level of risky capital is typically smaller (both due to endogenous investment and negative correlation of shocks) and the economy is much better characterized by the dotted line on the plot, which is fairly flat at high levels of α.

An initially rising and then falling bond risk premium is reminiscent of initially high and then low correlation of bond and stock returns, as can be seen on the top-right plot. The excess return on the risky technology monotonically increases in risk aversion for all values of x, consistent with Proposition 6.4. This result is key to using risk premium on stocks as an empirical proxy for unobserved risk aversion in the empirical section of the paper.

Prices of capital risk and risk-aversion risk are monotone in risk aversion. The price of capital risk is increasing and always positive, whereas the price of risk-aversion risk is decreasing and negative. Finally, loadings on the risk-aversion shock are non-monotone, but always positive for riskless and always negative for risky technologies, consistent with the results of Proposition 6.1 and the flight-to-safety mechanism. Likewise, loadings on the capital shock are always positive for both technologies, consistent with Proposition 6.2 and the technology diversification mechanism. Whereas the loadings of two technologies on a risk-aversion shock mostly increase with risk aversion in absolute value and are of different signs, the loading of risky technology on capital shock falls in the level of risk aversion, because, in response to a capital shock, discount rates on the risky asset move in a way that dampens the cash-flow effect on the asset. This dampening becomes stronger as risk aversion rises, leading to a relatively weaker comovement of bond and stock returns. The weaker comovement contributes to the technology-diversification effect becoming relatively weaker than flight-to-safety for high levels of risk aversion. As a result, the correlation between bond and stock returns becomes more negative as risk aversion rises.

4.3.3 Impulse response functions

I now analyze the dynamics of the model and response to shocks over time. For each variable of interest, I construct an impulse response by hitting an economy with a contemporaneous shock (1 s.d. in magnitude) and performing Monte-Carlo simulations of the economy 20 years forward to trace the impact of the shock. I then average all Monte-Carlo trajectories to compute the expected conditional response to a shock at the unconditional mean values of state variables conditional on a shock hitting at $t = 0$. I perform similar calculations for the economy with no shock at $t = 0$ to compute the trajectory of the economy that was not hit by the shock. The relative difference of two defines a non-linear impulse response. Formally, for each variable of interest V, I compute an impulse response from $t = 0$ to T as

$$IR_{t\to T}\left(\boldsymbol{X_t}\right) = \frac{\mathbb{E}_t\left[V_T\middle|\,\boldsymbol{X_t},\ Z_{n,t}=1\right] - \mathbb{E}_t\left[V_T\middle|\,\boldsymbol{X_t},\ Z_{n,t}=0\right]}{\mathbb{E}_t\left[V_T\middle|\,\boldsymbol{X_t},\ Z_{n,t}=0\right]}, \tag{11}$$

where $\boldsymbol{X_t}$ is a vector of state variables and $Z_{n,t}$ denotes a realization of a shock at $t = 0$.

Because the model is non-linear, these impulse responses cannot be calculated by zeroing out future shocks, because in non-linear models, future shocks interact with future values of state variables and these effects are important for studying the dynamics.

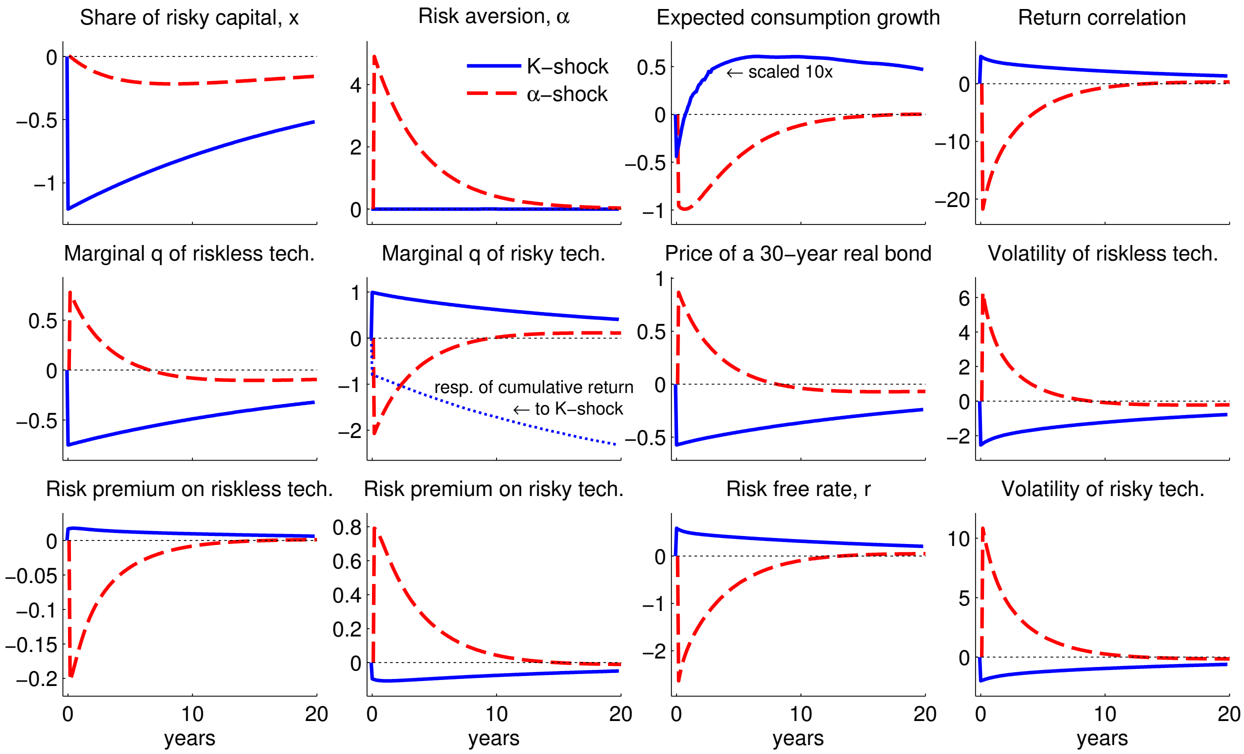
Figure 6 shows the responses of state variables and key quantities and prices of the model in response to two shocks. A solid blue line shows a response to the capital shock. The dashed red line shows an impulse response to a pure risk-aversion shock. I shock the economy at its mean values of state[8] variables and track responses forward for 20 years.

Let's first focus on the risk aversion shock – the dashed red line. A positive risk aversion shock has no contemporaneous effect on share of risky capital x, with small negative impact in the following 20 years. The shock, however, has a strong effect on instantaneous returns of two technologies: marginal q on riskless technology rises on impact, producing positive realized returns while marginal q of risky technology falls, delivering negative realized returns. Instantaneous risk premia on two technologies go in the opposite direction: they fall for riskless technology and rise for risky technology (two bottom-left plots). This effect is a manifestation of the flight-to-safety phenomena. Finally, both volatilities increase on impact, and the risk-free rate and correlation of returns fall.

The blue solid line depicts the response to a capital shock. Risk aversion does not respond on impact. Marginal q of the riskless technology falls, whereas that of the risky

[8] I also plotted impulse responses at many other points of the state space. Qualitatively, all of them look similar to Figure 6.

Figure 6: Impulse responses



Notes: Each figure shows impulse responses (in percent) to a 1 s.d. shock that hits the economy at its stochastic steady state (mean value of state variables). Responses to two orthogonal shocks are analyzed. A solid blue line shows a response to the capital shock. The dashed red line shows an impulse response to the pure risk-aversion shock. I study shocks over 20 years after a shock hits. Excess returns on risky and riskless technologies (first two plots in the bottom row) are shown in absolute deviations from their unconditional values. The response of expected consumption growth to the capital shock is scaled by a factor of 10. The additional line on the marginal q of riskless technology plot shows the cumulative returns on the technology after the capital shock hits.

technology rises. The loss of capital (cash-flow effect) negatively affects contemporaneous returns on risky technology, however. To illustrate this effect, I plot an additional line in the figure for marginal q of risky technology that shows the cumulative return on the technology. It goes down on impact as a result of the loss of capital. Returns on both technologies therefore co-move in response to a capital shock – the technology diversification mechanism. The figure also shows that in response to capital shock, discount rates on the risky asset move in a way that dampens the cash-flow effect on the asset (marginal q of risky technology goes up, whereas the total return falls). This effect makes the flight-to-quality mechanism dominate the technology diversification mechanism at higher levels of risk aversion and therefore achieving the changing sign of correlation of bond and stock returns.

Unlike the risk-aversion shock, the capital shock causes the risk-free rate to rise, both volatilities to fall, and correlation to rise. Expected consumption growth at short horizons is reduced in response to a negative shock but becomes positive quickly and stays positive for more than 20 years. This persistently positive expected consumption growth leads to a negative covariance of contemporaneous consumption growth and an infinite sum of all future expected consumption growth. Coupled with Epstein-Zin preferences, which make agents care about long-run future consumption growth, it tends to generate positive real bond risk premia at long horizons.

I verify impulse responses of a 30-year zero-net-supply default-free real bond that I price in the model using an SDF, are similar to impulse responses of the riskless technology's price (Tobin's q). Two plots in the middle row (first and third) show both respond similarly to either of the shocks. Similar responses to two shocks imply that the dynamics of returns on two assets should be similar. I also verified the similarity of responses at different points of the state space. Additionally, I find that the correlation between returns on the riskless technology and a 30-year real bond (which I price numerically) in the simulated data is above 95%. These facts confirm the conjecture that I can use an analytic solution for returns on the riskless technology as a good approximation for returns on actual long-term bonds, which can be priced only numerically.

Impulse responses of risk premia in the bottom row show another interesting property of the model. We can see that in response to shocks, instantaneous risk premia react significantly on impact as the quantity of capital is fixed in the short run and prices adjust. Responses of expected instantaneous risk premia at longer horizons, however, converge to zero, because the capital is flexible in the long run and quantities adjust. This pattern suggests that the model acts like the "two trees" model of Cochrane et al. (2008) in the short run but behaves as a CIR-type model (Cox et al., 1985) in the long run due to its stationarity.

Finally, impulse responses of x and α show two variables have different persistence. Share of risky capital x is highly persistent, with a half life of around 20 years. This variable drives endogenous low-frequency variation in the model. Risk aversion α, on the other hand, is much less persistent and drives mostly higher-frequency variation.

To study the effects of state dependence on impulse responses, I also looked at responses to shocks away from the stochastic steady state. Qualitatively, the results seem similar to those in Figure 6.

4.3.4 Risk Premia Decomposition

Theorem 4.1. *The model has the following two-state variable ICAPM representation*

$$\boldsymbol{\mu_R} - r \quad = \quad \alpha \times cov\left(dR, dR_{TW}\right) + \lambda_\alpha \times cov\left(dR, d\alpha\right) + \lambda_x \times cov\left(dR, dx\right),$$

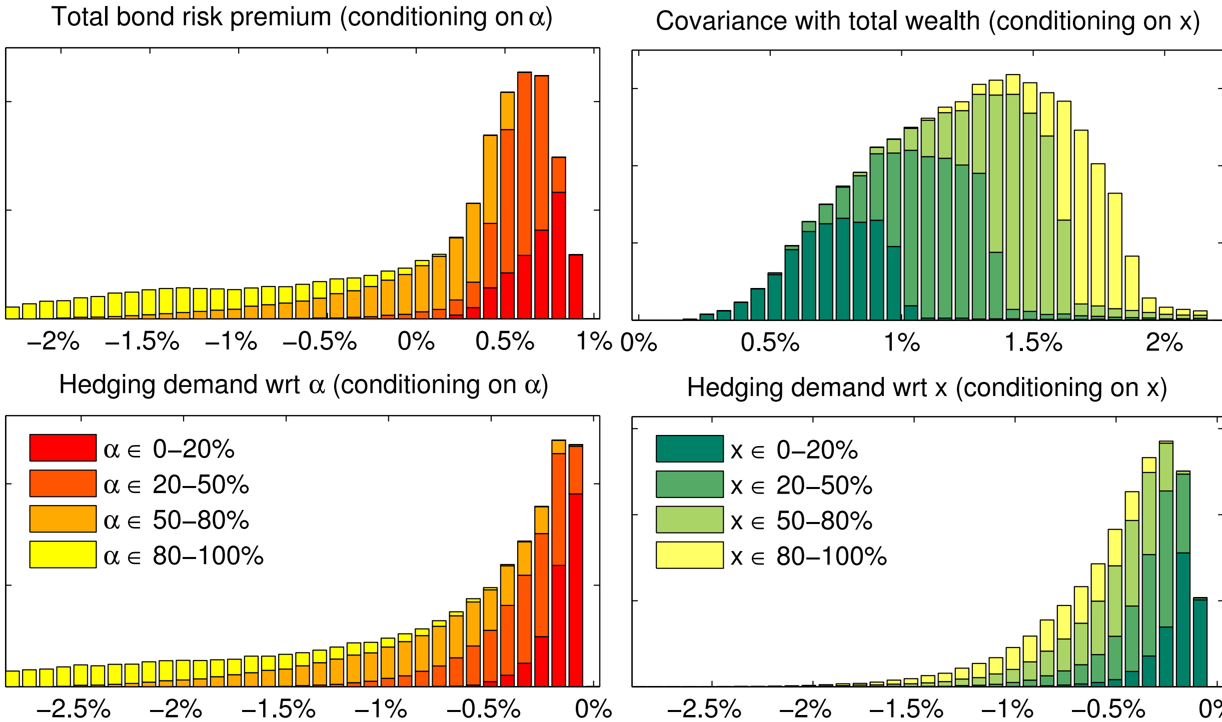
where $\lambda_\alpha = (\alpha - 1)\frac{J_\alpha}{J}$, $\lambda_x = (\alpha - 1)\frac{J_x}{J}$, $\boldsymbol{\mu_R}$ *is a vector of conditional expected returns on two assets, and* R_{TW} *is the return on the total wealth portfolio.*

Proof. Refer to section 6.5. □

Theorem 4.1 can be used to decompose the conditional bond risk premium into the three components. Figure 7 performs such a decomposition for the riskless technology. The left-top figure shows a conditional histogram of the total risk premium on the technology with no capital risk. The other three figures show conditional histograms of three components of the risk premium: a component due to covariance with the total wealth portfolio (CAPM term) and two ICAPM hedging demands. Two plots on the left are conditional on the level of risk aversion α (each color within a bar shows the fraction of observation that had a simulated value of risk aversion within some percentile). Two plots on the right are conditional on the share of capital in risky technology x (each color within a bar shows the fraction of observation that had a simulated value of x within some percentile).

The covariance with total wealth portfolio is mostly positive and generates most of the

Figure 7: ICAPM decomposition of bond risk premium

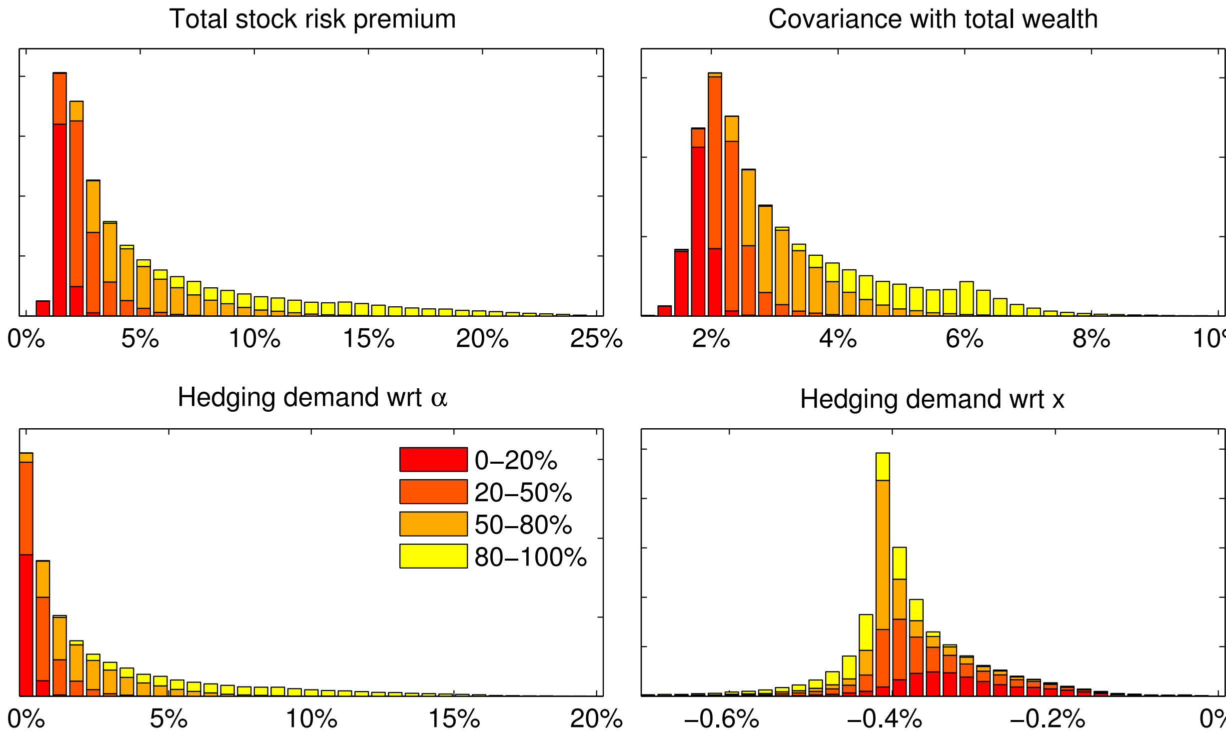


Notes: Left-top figure shows a conditional histogram of the total risk premium on the technology with no capital risk. The other three figures show conditional histograms of three components of the risk premium: a component due to covariance with the total wealth portfolio (CAPM term) and two ICAPM hedging demands. Two plots on the left are conditioned on the level of risk aversion α (each color within a bar shows the fraction of observation that had a simulated value of risk aversion within some percentile). Two plots on the right are conditioned on the share of capital in risky technology x (each color within a bar shows the fraction of observation that had a simulated value of x within some percentile).

positive risk premium. This component is highly correlated with the share of risky capital x (as can be seen from its conditional distribution), whereas the correlation with risk aversion α is low. Most of the technology-diversification effect therefore manifests in the CAPM component. The hedging demand with respect to risk aversion, on the other hand, generates most of the negative risk premium on the riskless technology and is highly correlated with risk aversion, which is a manifestation of the flight-to-safety mechanism. The hedging demand with respect to state variable x partially offsets the risk premium generated by both the CAPM term and the other hedging demand term (with respect to α), but not all of it, because it is weaker.

Figure 8 shows the same decomposition for stock risk premia. It is primarily driven by the

Figure 8: ICAPM decomposition of stock risk premium

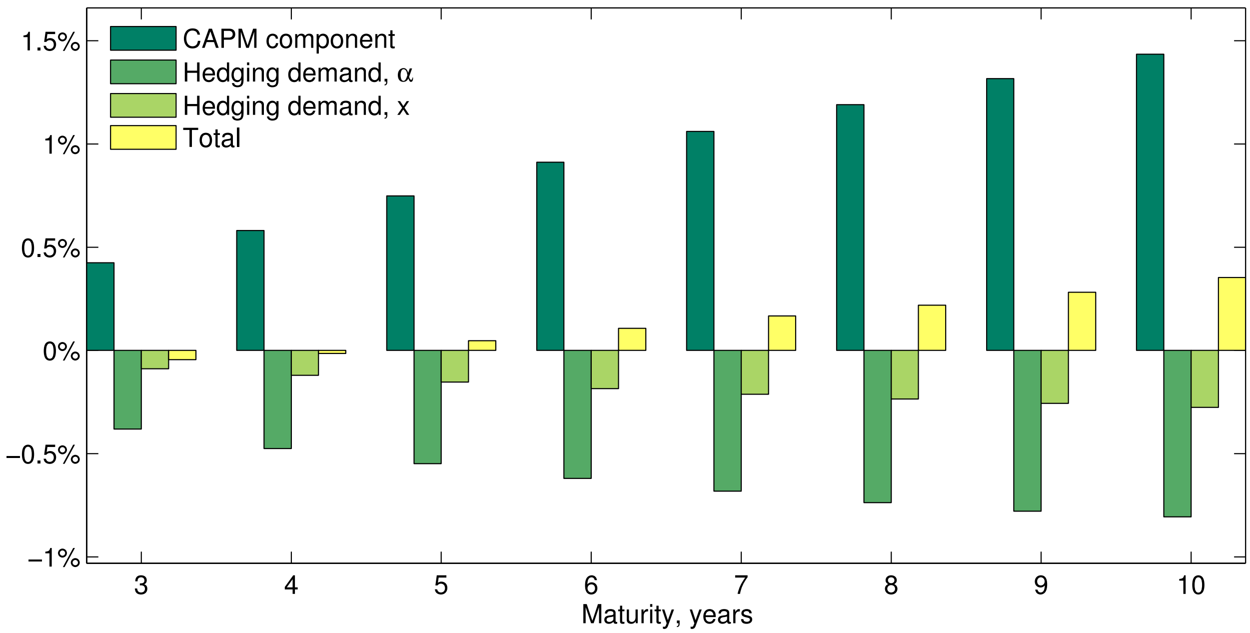


Notes: Left-top figure shows a conditional histogram of the total risk premium on the risky technology. The other three figures show conditional histograms of three components of the risk premium: a component due to covariance with the total wealth portfolio (CAPM term) and two ICAPM hedging demands. Different colors within each bar of a histogram show what fraction of observations that contributed to the bar had risk aversion within a certain percentile.

first two components, whereas the contribution of the hedging demand with respect to x is small. Hedging demand with respect to α produces most of the variation of stock risk premia and is the single most important component. Kozak and Santosh (2015) show an empirical factor that proxies for this component captures most of the cross-sectional variation in stock returns.

The value of the ICAPM decomposition is in showing how the two mechanisms work dynamically, how they relate to classical portfolio theory, and how risk premia on different assets are related at each point in time. Figure 7 and Figure 8, for example, suggest bond and stock risk premia tend to be negatively correlated (both have low correlation with x; therefore, dependence on α reveals the sign of correlation) and this negative correlation is due to flight-to-safety mechanism (hedging demand w.r.t. α). Theorem 4.1 also illustrates

Figure 9: ICAPM decomposition of bond risk premium by maturity



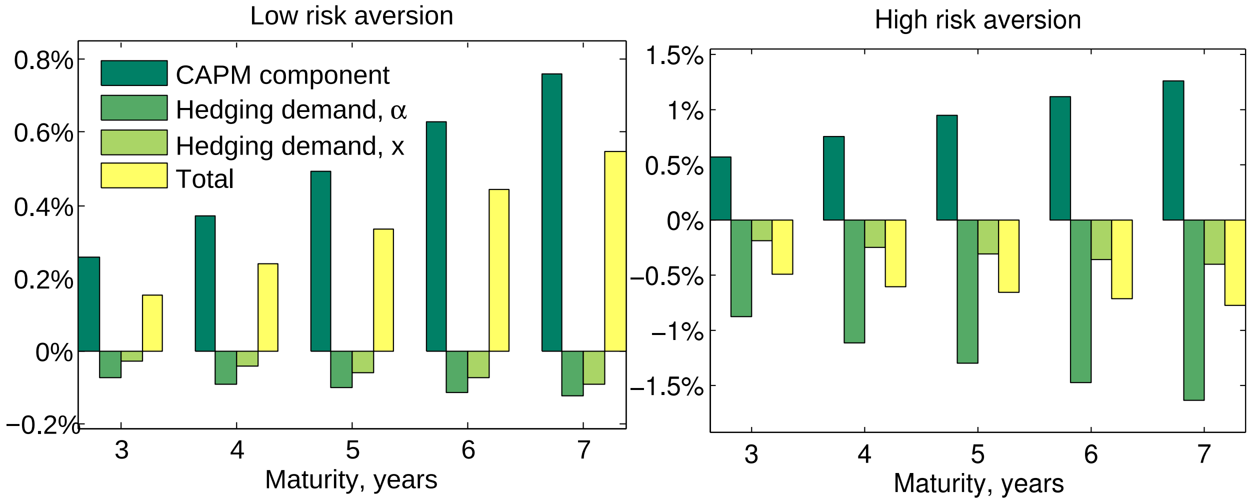
Notes: Decomposition of mean bond risk premia by maturity. Yellow bars show total bond risk premia for each maturity from 1 to 10 years. Green bars (appropriately labeled) show contributions of each of the three ICAPM components to the total bond risk premium of any given maturity.

the hedging demands with respect to risk aversion α is big and responsible for most of the negative risk premium on bonds, and thus cannot be ignored. In fact, the hedging demand is fully driving the flight-to-safety mechanism of this paper, which is similar in magnitude to the technology diversification mechanism.

Figure 9 shows the decomposition of average excess returns on real default-free zero-coupon bonds for maturities from three to 10 years. Each bond is priced by explicitly calculating the expectation of the SDF at any given maturity. The plot shows the total excess return on a bond increases with maturity. Similarly, both the CAPM component of the total excess return and hedging demands increase with maturity. The CAPM component is always positive, whereas the hedging demand with respect to risk aversion is always negative. The hedging demand with respect to x is negative and relatively small compared to the other two components.

Decomposition answers the question of what sources of risk compensation are embedded

Figure 10: ICAPM decomposition of conditional bond risk premium by maturity



Notes: Decomposition of the mean bond risk premia by maturity conditional on risk aversion being in the bottom 20^{th} percentile (left panel) and top 20^{th} percentile (right panel). Yellow bars show total bond risk premia for each maturity from 1 to 7 years. Green bars (appropriately labeled) show contributions of each of the three ICAPM components to the total bond risk premium of any given maturity.

in bond risk premia at various maturities. It shows two major components are the covariance with total wealth, which is primarily driven by the technology diversification mechanism, and the hedging demand with respect to risk aversion, which is reminiscent of the flight-to-safety mechanism. Two mechanisms require compensation of different signs. Covariance with the total wealth is a positively priced risk (as predicted by the CAPM), and the hedging demand is a hedge (because bonds tend to increase in price in bad times) and thus is negatively priced.

In section 5, section 5.3, I construct an empirical counterpart to decomposition in Figure 9 using the time-series of US government treasuries (nominal). I decompose empirical excess returns on bonds of different maturities into two components: covariance with the return on the total wealth portfolio and covariance with stock risk premium (proxy for risk aversion). The resulting empirical decomposition in Figure 13 closely mirrors the the unconditional decomposition in section 5.

The model is particularly useful for analyzing *conditional* risk premia and their components. Left and right panels of Figure 10 depict decompositions of excess returns conditional on low risk aversion (bottom 20^{th} percentile) and high risk aversion (top 20^{th} percentile) , respectively. They show that when risk aversion is high, the flight-to-safety mechanism is strong and dominates technology diversification at all maturities, which results in negative conditional bond risk premia. Similarly, when risk aversion is low, the technology diversification mechanism dominates, and conditional bond risk premia tend to be high.

5 Empirical Evidence

In this section I provide empirical evidence to justify the mechanisms of the model.

5.1 Risk aversion

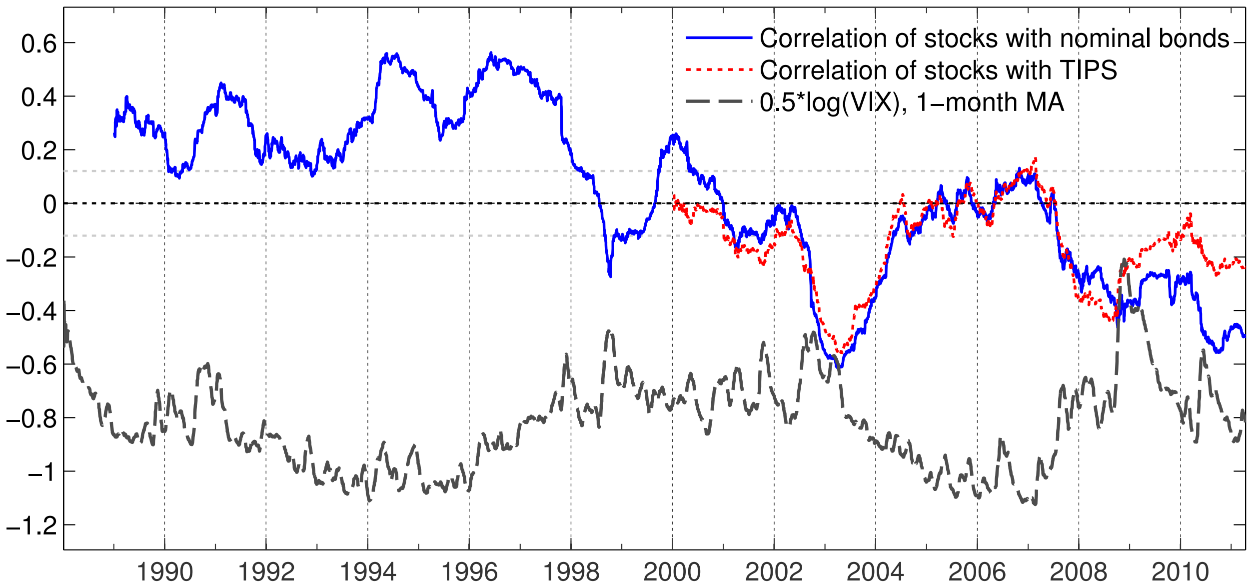
The model predicts that risk aversion is the main driver of variation in financial variables. In particular, we saw in Figure 2 and Figure 3 that when risk aversion is high in the model, correlation between bond and stock returns is low and stock volatility is high. It is informative to look at observable financial variables and see whether they behave in a way consistent with model's predictions.

In Figure 11 I show that bond-stock correlation and VIX are indeed highly negatively correlated in the data. The solid gray line shows the rescaled value of the smoothed VIX (1 month MA). The blue and the red lines show rolling 1-year correlations between daily stock excess returns and 10-year bond excess returns (nominal and real, respectively).

Additionally, I estimate a time-series regression $R_{B,t} = \alpha + \beta R_{M,t} + \delta R_{M,t} \times \text{VIX}_t + \varepsilon_{t+1}$ at the daily horizon, where R_B and R_M are returns on a 10-year nominal government bond and the stock market index, respectively, and test $\delta = 0$. I find that $\delta = -.027$ with a t-statistics of -4.44. Therefore, times when VIX is high are also times when correlation between bond and stock returns is relatively low.

Risk aversion exhibits high-frequency movements in the model and in the data. In the calibration of the model that I consider, risk aversion has relatively low persistence and

Figure 11: Bond-stock correlation and VIX



Notes: The solid gray line shows the rescaled value of the smoothed VIX (1 month MA). The blue and the red lines show rolling 1-year correlations between daily stock excess returns and 10-year bond excess returns (nominal and real, respectively).

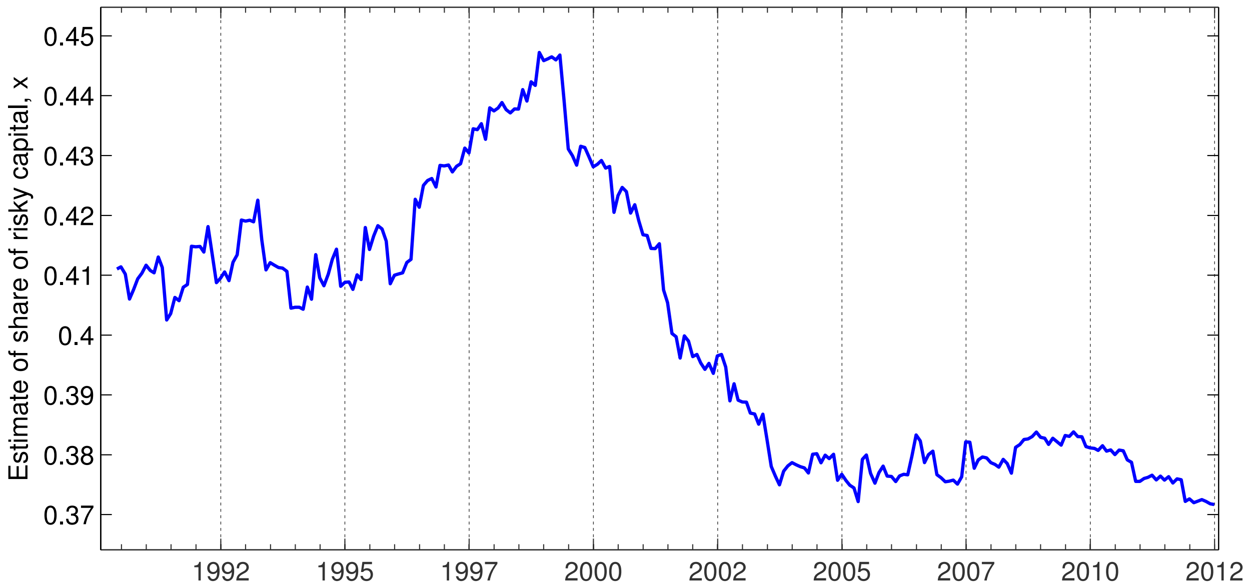
drives a lot of high-frequency dynamics.

5.2 Share of risky capital

The share of risky capital x drives low-frequency variation in the model. Recall that the mechanism behind this variable, technology diversification, was responsible for comovement of returns of bonds and stocks and positive real term premium. We would therefore expect x to be high before 2000s – the period when correlation between bond and stock returns was positive (see Figure 1).

I use high-risk and low-risk industries to construct an empirical counterpart to the dynamics of the state variable x in the data. I define low-risk industries as non-durables and utilities, and high-risk industries as durables and manufacturing. Total capital is defined as the sum of net PPE (property, plant, and equipment), non-tangible assets net of amortization, and goodwill. With these definitions of low- and high-risk capital, I construct an empirical estimate of x as total capital in risky companies divided by the sum of two types

Figure 12: Estimate of share of risky capital x



Notes: Estimate of the dynamics of the state variable x in the data. Low-risk (non-durables and utilities) and high-risk (durables and manufacturing) industries are used to proxy for the two types of capital in the model. Total capital is defined as the sum of net PPE (property, plant, and equipment), non-tangible assets net of amortization, and goodwill. The estimate of x is defined as total capital in high-risk industries divided by the sum of two types of capital.

of capital. The corresponding dynamics are shown in Figure 12. Obviously, this measure cannot serve as an estimate of the level of x because of an arbitrary split of quantity of capital into low and high risk, but it can give us a sense of how x might be evolving in the economy. Figure 12, in particular, shows x was high and increasing up until 2000 and fell steeply afterward.

The dynamics of x reproduced from capital of high- and low-risk companies suggests the state variable x should be persistent and drive low-frequency variation in the model and in the data. I calibrated the model in a way that makes the process for x highly persistent (endogenously), resulting in long swings of capital, investment, and correlations in the model. In the data, the bond-stock correlations were high on average before 2000 and then fell significantly. This evidence is in line with the estimate of x dynamics, which also fell significantly after 2000. Low x coupled with high risk aversion contributed to negative average correlation of bond and stock returns in the latter part of the sample. State variable

x therefore tends to drive low-frequency dynamics in the model and in the data, whereas α is mostly responsible for high-frequency variation.

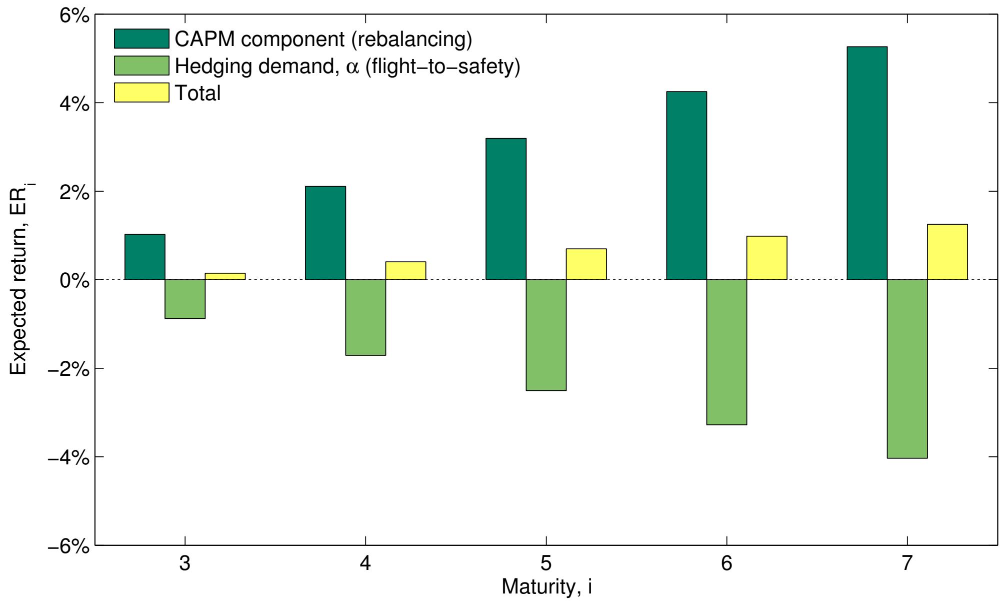
5.3 Bond risk premia decomposition

I now turn to analyze the composition and dynamics of bond and stock risk premia the model implies. In section 4.3.4 I showed that bond and stock risk premia are primarily driven by two components: technology-diversification risk and flight-to-safety hedging. I showed decomposition of bond risk premia on these two components by maturity, as implied by the model. The model also suggests that assets with different amounts of cash-flow risk load differentially on the underlying factors and the pattern of loadings determines the risk premia. I now test these predictions empirically.

The exercise I perform here closely follows Kozak and Santosh (2015). In particular, in that paper, they find that under some assumptions, a related ICAPM specification similar to the one in Theorem 4.1 can be conditioned down to a model with three factors: excess return on the stock market portfolio, excess return on a long-term bond, and future realized returns on the market portfolio in the following year. The first two factors are components of the total wealth portfolio and follow naturally from my derivation of the ICAPM. The latter factor is an empirical unbiased proxy for market expectations of future stock returns (stock risk premium) and proxies for unobserved risk aversion. This specification admittedly ignores time-variation in the share of risky asset, x, which I found to be smaller and to operate at a much lower frequency (see section 4.3.4). Santos and Veronesi (2006) finds the share of labor income in the total wealth portfolio does have an explanatory power for the cross-section of stock returns.

Kozak and Santosh (2015) estimate the following three-factor ICAPM: $E\left[R_i^e\right] = C_{i,M}\delta_M + C_{i,\lambda}\delta_\lambda + C_{i,B}\delta_B$, where $C_{i,n}$ are covariances of test asset returns with excess stock market returns $Cov\left[rx_{t+1}^{(i)},\ rx_{M,t+1}\right]$, future stock realized returns $Cov\left[rx_{t+1}^{(i)},\ E_t\left(\Sigma_i^k rx_{M,t+i}\right)\right]$, and long-term bond excess returns $Cov\left[rx_{t+1}^{(i)}, \left(E_{t+1} - E_t\right) rx_{t+1}^{(LT)}\right]$, respectively. δs give the cor-

Figure 13: Decomposition of Bond Risk Premia



Notes: Decomposition of bond risk premia into the technology-diversification component (covariance with total wealth) and the flight-to-safety component (hedging demand with respect to the risk aversion). The other hedging demand is small. The covariance with the total wealth is entirely driven by the covariance with returns on a long-term bond. Covariance with the stock market return is small. The horizontal axis shows the maturity of each bond, from 3 to 7 years. The vertical axis shows the fraction of expected returns that each component contributes.

responding prices of risk. I use this empirical specification to decompose the expected excess return on the various bonds in Figure 13. The premium due to market risk, $C_{i,M}$, is excluded because it is negligible for bonds. Bonds earn a large premium for loading on the total wealth risk (technology-diversification mechanism), whereas they command a large negative premium for loading on the expected return factor. This pattern is consistent with the flight-to-safety interpretation where investors' appetite for risk falls and they attempt to rebalance their portfolios toward safer securities. Because everyone cannot rebalance in this way at the same time, prices adjust instead of quantities. The prices of "risky" assets fall relative to the prices of "safer" assets.

Therefore, real bonds are good hedges against the stock market when risk aversion is high (consistent with the flight-to-safety mechanism), which results in negative risk premia on bonds. At the same time, real bonds load on the risk of the total wealth portfolio (reminiscent of the technology-diversification mechanism), which contributes positively to their risk premia. Relative magnitudes of each of these two contributions determine the sign of bond risk premium. Risk premia on assets with different amounts of cash-flow risk are determined by the assets' respective loadings on the two risk factors (technology diversification and flight-to-safety).

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6 Appendix

6.1 Portfolio Problem

Assume complete markets. A representative investor in this economy maximizes his utility over consumption,

$$J_t = \mathbb{E}_t \left(\int_t^T \left[f\left(C_\tau, J_\tau\right) + \frac{1}{2} A\left(J_\tau\right) \parallel J_{\boldsymbol{X}}\left(\boldsymbol{X_\tau}, \tau\right) \sigma_{\boldsymbol{X}}\left(\boldsymbol{X_\tau}, C_\tau, \tau\right) \parallel^2 \right] d\tau \right), \tag{12}$$

subject to total wealth constraint

$$dW_t = \left[W_t \boldsymbol{\theta_t'} \boldsymbol{\lambda_t} + W_t r_t - C_t \right] dt + W_t \boldsymbol{\theta_t'} \boldsymbol{\sigma_R} d\boldsymbol{Z},$$

where $\boldsymbol{X}(\alpha, \boldsymbol{\lambda})$ is a vector of aggregate state variables that are taken by agent as given and evolve according to

$$d\boldsymbol{X} = \mu_{\boldsymbol{X}} dt + \boldsymbol{\sigma_X} d\boldsymbol{Z},$$

and the vector of prices

$$d\boldsymbol{S_t} = \left(\boldsymbol{S_t} \cdot \left[r_t + \boldsymbol{\lambda_t} \right] - \boldsymbol{D_t} \right) dt + \boldsymbol{S_t} \cdot \boldsymbol{\sigma_R} d\boldsymbol{Z}.$$

The flow utility function can be expressed as

$$U\left(C_\tau\right) = f\left(C_\tau, J_\tau\right) - \frac{1}{2} \frac{\alpha_t}{J_t} \| J_W W_t \boldsymbol{\theta_t'} \boldsymbol{\sigma_R} + J_X \boldsymbol{\sigma_X} \|^2.$$

The first-order conditions are

$$\begin{aligned} f_C &= J_W \\ 0 &= -\frac{\alpha}{J} \boldsymbol{\sigma_J} \boldsymbol{\sigma_R'} + \boldsymbol{\lambda} + \frac{J_{WW}}{J_W} W \boldsymbol{\sigma_R} \boldsymbol{\sigma_R'} \boldsymbol{\theta} + \boldsymbol{\sigma_R} \boldsymbol{\sigma_X'} \frac{J_{W\boldsymbol{X}}}{J_W}. \end{aligned} \tag{13}$$

6.2 SDF

6.2.1 Duffie-Epstein aggregators and the SDF when α is constant

Ordinally equivalent aggregator Define the change of variables

$$\begin{aligned} \chi(J) \equiv \bar{J} &= \frac{1}{1-\alpha} J^{1-\alpha} \\ \chi'(J) &= J^{-\alpha} \\ \chi''(J) &= -\alpha J^{-\alpha-1}. \end{aligned} \tag{14}$$

Duffie and Epstein (1992b) call two aggregators (f, A) and $\left(\bar{f}, \bar{A}\right)$ ordinally equivalent, if there is a change of variable χ such that the following two conditions hold:

$$\begin{aligned} f(C, J) &= \frac{\bar{f}(C, \chi(J))}{\chi'(J)} \\ A(J) &= \chi'(J) \bar{A}(\chi(J)) + \frac{\chi''(J)}{\chi'(J)}. \end{aligned}$$

We can now find an ordinally equivalent aggregator $\left(\bar{f},\bar{A}\right)$ produced by the change of variables in Eq. 14:

$$\begin{aligned}\bar{f}\left(C,\chi\left(J\right)\right) &= \frac{\phi}{\rho}\frac{C^{\rho}-\left(\left(1-\alpha\right)\bar{J}\right)^{\frac{\rho}{1-\alpha}}}{\left(\left(1-\alpha\right)\bar{J}\right)^{\frac{\rho}{1-\alpha}-1}} \\ \frac{-\alpha}{J} &= \frac{\chi''\left(J\right)}{\chi'\left(J\right)} \qquad \Longrightarrow \qquad \bar{A}=0.\end{aligned} \tag{15}$$

Therefore two aggregators (f,A) and $\left(\bar{f},0\right)$ are ordinally equivalent with a change of variables $\chi\left(J\right)$ defined in Eq. 14. Furthermore, since $\bar{A}=0$, the aggregator $\left(\bar{f},0\right)$ is a *normalized* aggregator with $\bar{f}\left(C,\bar{J}\right)$ given by Eq. 15.

We can now use the normalized aggregator to derive the SDF.

SDF The SDF is given by (Duffie and Epstein, 1992b):

$$\frac{d\Lambda}{\Lambda} \equiv \bar{f}_V\left(C,\bar{J}\right)dt+\frac{d\bar{f}_C\left(C,\bar{J}\right)}{\bar{f}_C\left(C,\bar{J}\right)}.$$

The loading on shocks, $\mathcal{L}\left(\frac{d\Lambda}{\Lambda}\right)$ is given by

$$\begin{aligned}\mathcal{L}\left(\frac{d\Lambda}{\Lambda}\right) &= \mathcal{L}\left(dln\bar{f}_C\left(C,\bar{J}\right)\right) \\ &= \mathcal{L}\left(dlnf_C\left(C,J\right)-\alpha dlnJ\right).\end{aligned}$$

Note this expression is a special case of the equivalent one in section 6.2.2 when α is constant.

6.2.2 SDF when α is time-varying

To find the SDF, I solve the portfolio problem. I assume that agents take the process for α as given exogenously.

Guess that the SDF is of the form

$$\begin{aligned}d\Lambda &= -r\left(\boldsymbol{X}\right)\Lambda dt+\boldsymbol{\sigma_{\Lambda}}d\boldsymbol{Z} \\ \boldsymbol{\sigma_{\Lambda}} &= \Lambda\times\mathcal{L}\left[dlnf_C-\alpha_t dlnJ\right],\end{aligned}$$

where $\mathcal{L}\left(\cdot\right)$ denotes the vector of loading on the shocks. Substitute the FOC to get

$$\begin{aligned}\boldsymbol{\sigma_{\Lambda}} &= \Lambda\times\mathcal{L}\left[\frac{J_{WW}}{J_W}dW+\frac{J_{W\boldsymbol{X}}}{J_W}d\boldsymbol{X}-\frac{\alpha}{J}\boldsymbol{\sigma_J}d\boldsymbol{Z}\right] \\ \boldsymbol{\sigma_{\Lambda}} &= \Lambda\frac{J_{WW}}{J_W}W_t\boldsymbol{\theta_t'}\boldsymbol{\sigma_R}+\Lambda\frac{J_{W\boldsymbol{X}}}{J_W}\boldsymbol{\sigma_X}-\Lambda\frac{\alpha}{J}\boldsymbol{\sigma_J} \\ \boldsymbol{\sigma_R}\boldsymbol{\sigma_{\Lambda}'} &= -\boldsymbol{\lambda}\Lambda.\end{aligned}$$

Define $\boldsymbol{Y_t} = \boldsymbol{S_t}\Lambda_t$. Net of dividends, it should be a martingale,

$$\begin{aligned} \boldsymbol{\mu_Y} &= -\Lambda_t \boldsymbol{D_t} \\ -\Lambda_t \boldsymbol{S_t} r + \Lambda_t \left(\boldsymbol{S_t} \cdot [r + \boldsymbol{\lambda_t}] - D\right) + \boldsymbol{S_t}\boldsymbol{\sigma_R}\boldsymbol{\sigma}_{\boldsymbol{\Lambda}}^{'} &= -\Lambda_t \boldsymbol{D_t} \\ -\boldsymbol{\lambda_t}\Lambda_t &= \boldsymbol{\sigma_R}\boldsymbol{\sigma}_{\boldsymbol{\Lambda}}^{'}. \end{aligned}$$

Hence the guess was indeed correct. Therefore, the SDF is given by

$$\frac{d\Lambda}{\Lambda} = -r\left(\boldsymbol{X}\right)dt + \mathcal{L}\left[dln f_C - \alpha_t dln J\right] d\boldsymbol{Z}.$$

Note that if α is constant, Λ integrates to the usual expression, $\Lambda = \text{const} \times f_C J^{-\alpha}$ which is the same as we get when using normalized aggregator (see section 6.2.1). If α is not constant, however, the expression above does not integrate easily.

6.3 Planner's Problem

6.3.1 Derivation of the PDE

Planner chooses investment and consumption in order to maximize agent's lifetime utility in Eq. 3.

HJB equation Define the flow utility $U\left(C_\tau\right)$ as by $U\left(C_\tau\right) = f\left(C_\tau, J_\tau\right) + \frac{1}{2}A\left(J_\tau\right)\boldsymbol{\sigma_{J,\tau}}\boldsymbol{\sigma}_{\boldsymbol{J,\tau}}^{'}$ where $\boldsymbol{\sigma_J} = J_1 K_1 \boldsymbol{\sigma_K} + J_\alpha \boldsymbol{\sigma_\alpha}$, $J = J(K_0, K_1, \alpha; t)$ is a continuation value, and J_n denotes a derivative of J with respect to K_n. The Hamilton-Jacobi-Bellman (HJB) equation for the planner's problem is given by

$$\begin{aligned} 0 &= \sup_{\{i_0, i_1\}_t} \Big\{ U\left(C_t\right) + J_0 \mathbb{E}\left(dK_0\right)/dt + J_1 \mathbb{E}\left(dK_1\right)/dt + J_\alpha \mathbb{E}\left(d\alpha\right)/dt \\ &\quad + \frac{1}{2} J_{11} \mathbb{E}\left(dK_1^2\right)/dt + \frac{1}{2} J_{\alpha\alpha} \mathbb{E}\left(d\alpha^2\right)/dt + J_{1\alpha}\mathbb{E}\left(dK_1 d\alpha\right)/dt \Big\}. \end{aligned}$$

Due to homogeneity, we can guess that solution is linear in K:

$$J(K_0, K_1, \alpha; t) \equiv J\left(K_0, K_1, \alpha\right) = K \times F\left(x, \alpha\right).$$

Value function guess Find a solution of the form

$$\begin{aligned} J(K_0, K_1, \alpha; t) &\equiv J\left(K_0, K_1, \alpha\right) = K \times F\left(x, \alpha\right) \\ K J_0/J &= 1 - \frac{F_x}{F} x \\ K J_1/J &= 1 + \frac{F_x}{F}\left(1 - x\right) \\ K^2 J_{11}/J &= \frac{F_{xx}}{F}\left(1 - x\right)^2 \\ J_\alpha/J &= \frac{F_\alpha}{F} \\ J_{\alpha\alpha}/J &= \frac{F_{\alpha\alpha}}{F} \\ K J_{1\alpha}/J &= \frac{F_\alpha}{F} + \frac{F_{x\alpha}}{F}\left(1 - x\right). \end{aligned}$$

Volatility of the value function $\boldsymbol{\sigma_J}$ is given by

$$\begin{aligned}
J &= K \times F(x,\alpha) \\
\boldsymbol{\sigma_J} &= F(x,\alpha) K x \boldsymbol{\sigma_K} + K \times sd(dF) \\
\boldsymbol{\sigma_J} &= J x \boldsymbol{\sigma_K} + J \frac{1}{F} \times sd(F_x dx + F_\alpha d\alpha) \\
\boldsymbol{\sigma_J} &= J x \boldsymbol{\sigma_K} + J \frac{1}{F} \left[F_x x (1-x) \boldsymbol{\sigma_K} + F_\alpha \alpha \boldsymbol{\sigma_\alpha}\right] \\
\boldsymbol{\sigma_J} &= J \left[\left(1 + \frac{F_x}{F}(1-x)\right) x \boldsymbol{\sigma_K} + \frac{F_\alpha}{F} \alpha \boldsymbol{\sigma_\alpha}\right] \\
&\equiv J \boldsymbol{\sigma_F}.
\end{aligned}$$

Solution to the problem above is given by a system of one second-order PDE in two state variables, two first-order conditions for optimal investment, and the aggregate budget constraint

$$\begin{aligned}
&\frac{\delta}{\rho}\left[\left(\frac{c(x,\alpha)}{F(x,\alpha)}\right)^\rho - 1\right] - \frac{1}{2}\alpha \|\boldsymbol{\sigma_F}\|^2 + (1-x)\left(1 - \frac{F_x}{F}x\right)\phi_0(i_0) + x\left(1 + \frac{F_x}{F}(1-x)\right)\phi_1(i_1) \\
&\quad + \frac{F_\alpha}{F}\phi(\bar{\alpha} - \alpha) + \frac{1}{2}\frac{F_{xx}}{F}(1-x)^2 x^2 \varsigma_K^2 + \frac{1}{2}\frac{F_{\alpha\alpha}}{F}\alpha^2 \|\boldsymbol{\sigma_\alpha}\|^2 + \left[\frac{F_\alpha}{F} + \frac{F_{x\alpha}}{F}(1-x)\right] x\alpha\lambda\varsigma_K^2 = 0
\end{aligned}$$

$$\begin{aligned}
\delta\left(\frac{c}{F}\right)^{\rho-1} &= (F - F_x x)\,\phi_0'(i_0) \\
\delta\left(\frac{c}{F}\right)^{\rho-1} &= (F + F_x(1-x))\,\phi_1'(i_1) \\
c &= (A_0 - i_0)(1-x) + (A_1 - i_1)x,
\end{aligned}$$

that have to be solved jointly.

6.3.2 Asset Prices

Marginal q of technologies are

$$\begin{aligned}
q_0 &= \frac{1}{\delta}\left[1 - \omega_x(x,\alpha) x\right] c \\
q_1 &= \frac{1}{\delta}\left[1 + \omega_x(x,\alpha)(1-x)\right] c
\end{aligned} \tag{16}$$

where $\omega_x(x,\alpha) = \frac{F_x}{F}$, and $\omega_\alpha(x,\alpha) = \frac{F_\alpha}{F}$.

Realized returns on two assets are

$$dR_n = \frac{A_n - i_n}{q_n}dt + \frac{dq_n}{q_n} + \frac{dK_n}{K_n} + \left\langle \frac{dq_n}{q_n}, \frac{dK_n}{K_n} \right\rangle.$$

The loadings of returns on shocks are given by $\mathcal{L}(dR_n) = \mathcal{L}\left(\frac{dK_n}{K_n} + \frac{dq_n}{q_n}\right)$ and the excess returns on two technologies, therefore, are

$$RX_n = \Big(\bar{A}(1-x)x\boldsymbol{\sigma_K} + \alpha x \boldsymbol{\sigma_K} + (\alpha-1)\omega_x(x,\alpha)x(1-x)\boldsymbol{\sigma_K} + (\alpha-1)\omega_\alpha(x,\alpha)\alpha\boldsymbol{\sigma_\alpha}\Big)\Big(l_{n,k} + l_{n,x}x(1-x)\boldsymbol{\sigma_K} + l_{n,\alpha}\alpha\boldsymbol{\sigma_\alpha}\Big)^\top, \tag{17}$$

where $l_{n,x}(x,\alpha) \equiv \frac{q_{n,x}}{q_n}$ and $l_{n,\alpha}(x,\alpha) \equiv \frac{q_{n,\alpha}}{q_n}$ are the loadings of $\frac{dq}{q}$ on shocks (price effects), $l_{0,k} = (0,\, 0)$ and $l_{1,k} = \boldsymbol{\sigma_K}$ are the loadings of two technologies on capital shocks in their respective capital-accumulation processes ("cash-flow" effects). The first term of the product above shows the loadings of the SDF on shocks (prices of risk) and is derived using Theorem 3.2 in section 6.4.3.

The key to characterizing the excess returns and understanding the dynamics of prices is to describe how the functions $\omega_x(x,\alpha)$, $\omega_\alpha(x,\alpha)$, $l_{n,x}(x,\alpha)$, and $l_{n,\alpha}(x,\alpha)$ look like. Although obtaining closed-form solutions for the unknown functions is impossible, Theorem 6.2 below and small-noise expansions in section 6.4.2 deliver easy-to-analyze expressions that can be used to better understand underlying mechanisms of the model.

Expected return on risky asset Adjustment costs are given by

$$\begin{aligned}\phi(i) &= \xi ln\left(1+\frac{i}{\theta}\right)\\ \phi'(i) &= \frac{\xi}{\theta+i}.\end{aligned}$$

Marginal q evolves as follows

$$\begin{aligned}dq &= d\left[\frac{\theta+i}{\xi}\right] = \frac{di}{\xi}\\ \frac{dq}{q} &= (\theta+i)^{-1}\mu_i dt + (\theta+i)^{-1}\sigma_i dZ,\end{aligned}$$

where

$$\begin{aligned}\mu_i &= i_x\mu_x + i_g\mu_g + \frac{1}{2}i_{xx}\sigma_x^2 + \frac{1}{2}i_{gg}\sigma_g^2 + i_{xg}\sigma_x\sigma_g\\ \sigma_i &= i_x\sigma_x + i_g\sigma_g,\end{aligned}$$

and

$$\begin{aligned}\mu_x &= x(1-x)\left[\phi_1(i_1) - \phi_0(i_0) - x\sigma_K^2\right]\\ \sigma_x &= x(1-x)\sigma_K.\end{aligned}$$

Expected return on risky asset,

$$\frac{1}{dt}\mathbb{E}(dR_n) = \frac{A_n - i_n}{q_n} + (\theta+i)^{-1}\mu_i + \phi_n(i_n) + (\theta+i)^{-1}\sigma_i\sigma_{K_n}.$$

Interest rate and excess return Using the previous two results, we can express the interest rate as

$$r = \frac{1}{dt}\mathbb{E}\left(dR\right) - rx,$$

where rx is the excess return and is given by

$$\begin{aligned} rx &= -\mathbb{E}\left[\frac{d\Lambda}{\Lambda}\frac{dP}{P}\right] \\ &= -\frac{1}{dt}\left\langle dlnf_C - \alpha_t dlnJ,\ \theta q\sigma_i + \sigma_{K_n}\right\rangle. \end{aligned}$$

6.3.3 Boundary conditions

$x = 0$ **boundary** FOC for i_0 gives

$$F\phi_0^{'}\left(i_0\right) = \delta\left(\frac{c}{F}\right)^{\rho-1},$$

where $c = A_0 - i_0$. With only one shock, the economy is completely riskless, so risk aversion does not matter, $F\left(\alpha\right) \equiv F$. Solve for a constant F:

$$F = \left(\frac{\delta}{\phi_0^{'}\left(i_0\right)}\left(A_0 - i_0\right)^{\rho-1}\right)^{\frac{1}{\rho}}.$$

Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \frac{\delta}{\rho}\left[\left(\frac{c}{F}\right)^{\rho} - 1\right] + \phi_0\left(i_0\right) &= 0 \\ \frac{\phi_0^{'}\left(i_0\right)}{\rho}\left(A_0 - i_0\right) - \frac{\delta}{\rho} + \phi_0\left(i_0\right) &= 0. \end{aligned}$$

This gives $i_0\left(0,\alpha\right) = const$. Set $i_1\left(0,\alpha\right) = 0$.

$x = 1$ **boundary** This is the usual problem with one risky technology. Solve a corresponding ODE for $i_1\left(1,\alpha\right)$.

6.4 Stylized model

Unfortunately, the planner's problem does not have an analytic solution in general. To better understand the mechanisms at work, specializing the general model above to the case when EIS is equal to 1 and installation function $\phi_n\left(\cdot\right)$ is the same for two technologies and takes a log form, is therefore useful. After the discussion of the stylized model, I will solve numerically the *full* model, free of the next two assumptions.

Assumption 1. *Elasticity of intertemporal substitution (EIS) is equal to unity,* $\psi = 1$.

Assumption 2. *Installation function $\phi(i_n)$ is the same for two technologies and takes the log form,*

$$\phi_n(i_n) = \xi \times ln\left(1 + \frac{i_n}{\xi}\right). \tag{18}$$

The functional form in Assumption 2 is concave, ensuring high levels of investment or disinvestment is costly. It has a slope equal to one at $i_n = 0$, i.e., no adjustment costs on the margin at zero investment. I also set depreciation equal to zero. A corresponding adjustment cost function is $\varphi(i) = i - \xi ln\left(1 + \frac{i_n}{\xi}\right)$, which is a convex function.

Appendix, section 6.4.3 specializes the solution in Theorem 3.1 to the case when Assumption 1 and Assumption 2 hold.

6.4.1 Aggregates

Assumption 1 and Assumption 2 deliver the following result.

Theorem 6.1. *Aggregates in the stylized economy are given by*

$$\begin{aligned} q_t &= \frac{A_t + \xi}{\xi + \delta} \\ i_t &= \frac{A_t - \delta}{\xi + \delta} \\ c_t &= \delta\frac{A_t + \xi}{\delta + \xi}, \end{aligned}$$

where $q_t = (1 - x_t)\, q_{0,t} + x_t q_{1,t}$ is the aggregate Tobin's q, $i_t = (1 - x_t)\, i_{0,t} + x_t i_{1,t}$ is the aggregate investment per unit of capital, c_t is the aggregate consumption per unit of capital, and $A_t = (1 - x_t)\, A_0 + x_t A_1$ is the aggregate productivity.

Proof. Using the fact that $q_n = \frac{1}{\phi'(i_n)} = \frac{\xi}{\xi + i_n}$ and expressions for investment in Eq. 23, we get $q = (1 - x)\, q_0 + x q_1 = \frac{1}{\delta} c$. Investment is $i = (1 - x)\, i_0 + x i_1 = \frac{\xi}{\delta} c - \xi = \xi q - \xi$. Using the resource constraint, we know $i = A - c = A - \delta q$. Equalizing both expressions gives $q = \frac{A + \xi}{\xi + \delta}$. □

Corollary 6.1. *When elasticity of intertemporal substitution is equal to one, the aggregate consumption-to-wealth ratio is constant, $\frac{C}{W} \equiv \frac{c}{q} = \delta$.*

The aggregates therefore vary in time *only* because the aggregate productivity, A_t, is time-varying, $A_t = (1 - x_t)\, A_0 + x_t A_1$. In fact, if technologies were identical and thus $A_0 = A_1$, all aggregates relative to capital would be constant, and the only reason the levels of aggregate variables vary is because the level of capital varies. This implication and Corollary 6.1, however, may be viewed as benefits, because they allow us to consider the pricing implications for two technologies *independently* of those of aggregate economy and to grasp additional economic insights about the underlying mechanisms of the model. I will later solve the model numerically in section 4, without relying on Assumption 1 and Assumption 2, and analyze the solution to better understand how both aggregate and relative pricing mechanisms interact.

The Appendix, section 6.4.3 shows derivations of expressions for market risk premium and risk-free rate in the economy. Finally, the following result is useful for future analysis.

Definition 6.1. A *point of equal investment* (PEQ) of the economy in section 6.4.1 is an equilibrium in which the level of investment in risky and riskless technologies are equal for a given value of risk aversion α, $i_0(x^*(\alpha),\alpha) = i_1(x^*(\alpha),\alpha)$, where $x^* = x^*(\alpha)$ denotes a share of risky capital at a PEQ. Two technologies grow at the same rate at a PEQ.

Theorem 6.2. *Derivative of the value function $\omega_x(x,\alpha) = \frac{F_x}{F}$ is zero at a PEQ, above zero as x approaches a PEQ from below, $x \nearrow x^*(\alpha)$, and below zero for $x \searrow x^*(\alpha)$.*

Proof. When investments are equal, $1 - x^*\frac{F_x^*}{F^*} = 1 + (1-x^*)\frac{F_x^*}{F^*}$, which implies $F_x(x^*(\alpha),\alpha) = 0$ and thus $\omega_x(x,\alpha) = 0$. Next, $\frac{\partial}{\partial x}\left[\frac{F_x}{F}\right]\Big|_{x=x*} = \frac{F_{xx}^*}{F^*}$. Because no reallocation of capital optimally takes place, the value function must be maximized with respect to x, implying $F_{xx}(x^*(\alpha),\alpha) < 0$. □

Although obtaining closed-form solutions for asset prices is impossible in general, the small-noise expansions in the following section deliver easy-to-analyze analytic approximations that can be used to better understand underlying mechanisms of the model.

6.4.2 Small-noise expansions

Define a perturbation parameter $\epsilon \in [0,1]$ such that when $\epsilon = 0$, the economy is along its deterministic trajectory, and when $\epsilon = 1$, the economy corresponds to the economy of interest. I perturb the process for evolution of risky capital, productivity of risky technology, and risk aversion as follows:

$$dK_1 = \phi_1(i_1)K_1 dt + K_1\sqrt{\epsilon}\boldsymbol{\sigma_K} d\boldsymbol{Z} \tag{19}$$
$$d\alpha = \phi(\bar{\alpha}-\alpha)dt + \sqrt{\epsilon}\boldsymbol{\sigma_\alpha} d\boldsymbol{Z} \tag{20}$$
$$A_1(\epsilon) = (1-\epsilon)A_0 + \epsilon A_1. \tag{21}$$

Assumption 1, Assumption 2, and Corollary 6.2 allow us to characterize the solution of the stylized model in terms of a single PDE in Eq. 23. Moreover, at $\epsilon = 0$, all derivatives of the value function F are zero (in a deterministic steady state, risk aversion has no impact; x is indeterminate because two technologies are riskless and have the same productivity). These two facts simplify computations of expansions around the deterministic path substantially and make computing analytical small-noise expansions feasible.

The perturbation of the productivity of riskless technology in Eq. 21 is needed because in the steady state in which productivities of two technology are not equal and no uncertainty is present, a technology with lower productivity will be completely dominated by the other technology and thus x^* will be either 0 or 1. Such a steady state might be a very bad point of expansion. Instead of expanding around it, I will seek for expansions around some deterministic path on which the productivities of two technologies are equal.

I therefore proceed with perturbations in three different directions as defined by the system of equations (19) – (21). An advantage of perturbing around a non-stochastic path is that all expansions of interest can be computed analytically. I first parametrize the value function by a perturbation parameter ϵ, $F(x,\alpha;\epsilon)$. In my notation, $F(x,\alpha;0)$ corresponds to a deterministic path with two equal productivities $A_0 = A_1(\epsilon)|_{\epsilon=0}$, whereas $F(x,\alpha;1)$ corresponds to the value function in the economy of interest. I look for a first-order expansion[9] of $F(x,\alpha;\epsilon)$ as a power

[9]When $\epsilon \neq 0$, Eq. 23 is a second-order partial differential equation, but when $\epsilon = 0$, it reduces to a first-order differential equation. This reduction in order induces a so-called "singular perturbation" to the problem, which is often associated with substantial complications. However, as argued by Judd (1998) and

series in ϵ:

$$F(x,\alpha;\epsilon) = F(x,\alpha;0) + F_\epsilon(x,\alpha;0)\,\epsilon + o\left(\epsilon^2\right),$$

where $F(x,\alpha;0)$ gives the value of F at $\epsilon = 0$ and $F_\epsilon(x,\alpha;\epsilon)$ gives the derivative at $\epsilon = 0$. This derivative, as well as higher-order derivatives, can be computed analytically, which delivers additional insights and characterizations of the behavior around the steady state. Moreover, Judd (1998) emphasizes that whereas the low-order expansions describe the behavior only locally, as we increase the order of expansions, a solution becomes global within the radius of convergence.

Small-noise expansions employed in my paper are closely related to expansions in control-theory literature, namely, Fleming (1971), Fleming and Yang (1994), and James and Campi (1996). Anderson et al. (2012) and Kogan and Uppal (2001) have analyzed a similar type of expansions. These expansions differ from the one typically used in economic literature where expansion takes place in the shock standard deviation and around some deterministic steady state (imposing steady-state values of variables). I expand with respect to the shock variance and around common productivity A_0 without imposing steady-state values. The expansion therefore is around some deterministic *trajectory*, rather than a steady state. Moreover, because I am expanding with respect to the shock variance, my first-order expansions correspond to a second-order expansions used in economic literature and my second-order expansions correspond to the fourth-order expansions in the literature (see Anderson et al., 2012).

I now proceed with summarizing some analytical results of first-order[10] small-noise expansions. All the major mechanisms of the model are operable in these expansions. Note analytical expansions of *any* order can be derived with this method. Higher-order expansions, however, prove to be difficult to analyze and understand, while providing no more of economic intuition.

Theorem 6.3. *The value function $F(x,\alpha;\epsilon)$ can be expressed as*

$$F(x,\alpha;\epsilon) \quad = \quad f_0 + \left[\zeta_A x\bar{A} - \frac{1}{2}(\zeta_0 + \zeta_\alpha\alpha)\,x^2\varsigma_K^2\right] f_0\epsilon + o\left(\epsilon^2\right), \tag{22}$$

where $f_0 \equiv F(x,\alpha;0)$ is the value function evaluated at $\epsilon = 0$, $\bar{A}$ is defined as $\bar{A} = \frac{A_1 - A_0}{A_0+\xi}$; $\zeta_A = \frac{\delta+\xi}{\delta}$, $\zeta_0 = \frac{\phi\bar{\alpha}}{\delta(\delta+\phi)}$, and $\zeta_\alpha = \frac{1}{\delta+\phi}$ are constants. Small-noise expansion of the value function around a deterministic trajectory is therefore linear in α and quadratic in x.

Proof. Refer to section 6.5 for further details. □

To understand the pricing implications for two technologies in the model, I analyze small-noise expansions for Tobin's q's of these technologies (which can be used to infer investment decision rules in a straightforward fashion).

Theorem 6.4. *Tobin q's of riskless and risky technologies are given by*

$$\begin{aligned}
q_0(x,\alpha;\epsilon) \quad &= \quad \zeta_0^q + \zeta_0^q\left[-\frac{\xi}{\delta}x\bar{A} + (\zeta_0+\zeta_\alpha\alpha)\,x^2\varsigma_K^2\right]\epsilon + o\left(\epsilon^2\right)\\
q_1(x,\alpha;\epsilon) \quad &= \quad \zeta_0^q + \zeta_0^q\left[\left(1 + (1-x)\frac{\xi}{\delta}\right)\bar{A} - (\zeta_0+\zeta_\alpha\alpha)\,x(1-x)\,\varsigma_K^2\right]\epsilon + o\left(\epsilon^2\right),
\end{aligned}$$

formally shown by Fleming (1971), the remarkable feature of stochastic control problems is that perturbation ϵ can be analyzed as a regular perturbation when it enters as a square root above.

[10] It is often argued that at least second-order perturbations are needed to generate non-zero risk premium, and at least third-order to generate time-variation in risk premium. Because I perturb variance, first order perturbations of a system in Eq. 20 does generate non-zero risk premia, which also time-vary, due to an exogenous specification of risk aversion.

where ζ_0, ζ_α, $\zeta_0^q = \frac{A_0+\xi}{\delta+\xi}$ are constants. Expansions of Tobin q's around a deterministic trajectory are therefore linear in α and quadratic in x.

Proof. Refer to section 6.5 for further details. □

An important feature of small-noise expansions employed in Theorem 6.3 and Theorem 6.4 is that the value function and prices directly depend on two state variables. We can therefore analyze how shocks to state variables affect each quantity.

Note prices depend on risk aversion α. This dependence is an important feature of the model that is *not* present in a single technology model with time-varying risk aversion. In such a model, prices of risk depend on α, but the quantity of risk is constant. In the two-technology model, *both* price and quantity of risk are time-varying for each technology, producing potentially significant variation in price and risk premia of underlying technologies and generating interesting joint dynamics.

I use the insights provided by small-noise expansions to establish Propositions 6.1 – 6.4 below.

Proposition 6.1. *Around a non-stochastic trajectory of the model in section 6.4, shocks to risk aversion that are orthogonal to shocks to the capital-accumulation process move returns on risky and riskless technologies in opposite directions.*

Proof. Direct shocks to risk aversion that are uncorrelated with the shock to capital accumulation of risky technology, have only price impacts on the two technologies (via changes in q's). Expressions for prices in Theorem 6.4 imply that the price of the riskless technology increases in α, while the price of the risky technology falls in α. Therefore, a positive shock to α (unexpected increase in risk aversion) increases the contemporaneous return of the riskless technology and lowers the contemporaneous return of the risky technology.

□

Proposition 6.2. *Around a non-stochastic trajectory of the model in section 6.4 and for values of x in $\left[\frac{1}{2}\frac{\xi}{\xi+\delta}x^*, 1\right]$, shocks to the risky capital-accumulation process that are orthogonal to shocks to risk aversion move returns on risky and riskless technologies in the same direction. $x^* = x^*(\alpha)$ denotes the value of x at the PEQ.*

Proof (sketch). Intuitively, returns on risky technology respond positively to a capital shock. This response is due to the direct loading on the capital shock, which dominates the pricing effect for a broad range of parameter values. In section 6.5 I show the returns on the riskless technology respond positively to a capital shock when the share of risky capital x is in the range $\left[\frac{1}{2}\frac{\xi}{\xi+\delta}x^*, 1\right]$, where $x^* = x^*(\alpha)$ is the value of x at the PEQ. The rebalancing mechanism drives this positive response. Both returns therefore move in the same direction. Refer to section 6.5 for a more formal argument. □

Proposition 6.3. *Around a non-stochastic trajectory of the model in section 6.4, one can find a calibration of the model that produces a positive correlation between returns on riskless and risky technologies for low levels of risk aversion and negative correlation when risk aversion is high.*

Proof (sketch). Proposition 6.1 and Proposition 6.2 establish that two shocks produce different signs of correlation between bond and stock returns. Additionally, in response to a capital shock, discount rates on the risky asset move in a way that dampens the cash-flow effect on the asset. This dampening becomes stronger as risk aversion rises, leading to a relatively weaker comovement of bond and stock returns. At the same time, an increase in risk aversion leads to a stronger "decoupling" of bond and stock returns. As a result, the flight-to-quality mechanism starts dominating the rebalancing mechanism at higher levels of risk aversion. We can therefore select the variance of risk-aversion shocks and parameter λ such that the capital effects are stronger for low levels of α and get dominated for high levels of α and hence the proposition holds. Refer to section 6.5 for a more formal argument. □

Proposition 6.4. *When two shocks are uncorrelated, the risk premium on risky technology is monotonically increasing in risk aversion for* $x \in \left[0, \min\left(x^* + \frac{\xi}{\xi+\delta}\frac{1}{A}x^*, \frac{1}{2}\right)\right]$, *where* $x^* = x^*(\alpha)$ *denotes the value of x at the PEQ.*

Proof. Refer to section 6.5 for more details. □

The main purpose of small-noise expansions and the propositions in this section was to analytically characterize the main mechanisms at work and lay the ground for empirical hypotheses I develop in section 5. Although small-noise expansions are valid only sufficiently close to a deterministic trajectory, I verify numerically that higher-order expansions do not overturn the main qualitative results obtained in the section and that the approximation is sufficiently good on the parts of a state space that are visited in equilibrium. Furthermore, I relax Assumption 1 and Assumption 2 in section 4 and verify the qualitative results of this section are robust to such modification.

6.4.3 Stylized Model: Details

Corollary 6.2. *When Assumption 1 and Assumption 2 hold, the solution to equation in Theorem 3.1 specializes to*

$$\begin{aligned}\delta\Big[ln(c) - ln(F)\Big] - \frac{1}{2}\alpha\left\|\boldsymbol{\sigma_F}\right\|^2 + (1-x)\left(1 - \frac{F_x}{F}x\right)\phi_0(i_0) + x\left(1 + \frac{F_x}{F}(1-x)\right)\phi_1(i_1) \\ +\frac{F_\alpha}{F}\phi(\bar{\alpha}-\alpha) + \frac{1}{2}\frac{F_{\alpha\alpha}}{F}\alpha\left\|\boldsymbol{\sigma_\alpha}\right\|^2 + \frac{1}{2}\frac{F_{xx}}{F}(1-x)^2x^2\varsigma_K^2 + \left[\frac{F_\alpha}{F} + \frac{F_{x\alpha}}{F}(1-x)\right]x\alpha\lambda\varsigma_K^2 = 0,\end{aligned} \tag{23}$$

with investments determined by the first-order conditions,

$$i_0 + \xi = \frac{\xi}{\delta}\left(1 - x\frac{F_x}{F}\right)c \tag{24}$$

$$i_1 + \xi = \frac{\xi}{\delta}\left(1 + (1-x)\frac{F_x}{F}\right)c, \tag{25}$$

and the aggregate resource constraint

$$c = (A_0 - i_0)(1-x) + (A_1 - i_1)x.$$

Expected returns on the market portfolio Expected returns on the market portfolio can be readily analyzed in the stylized model,

$$\begin{aligned}\mathbb{E}(dR_M) &= \frac{A-i}{q}dt + \mathbb{E}\frac{dq}{q} + \mathbb{E}\frac{dK}{K} + \left\langle\frac{dq}{q}, \frac{dK}{K}\right\rangle \\ &= \underbrace{\delta + \phi(i_1)}_{\text{1-sector model's }\mathbb{E}R} + \underbrace{\bar{A}\left[\mu_x + (1-x)x^2\varsigma_K^2\right]}_{\mathbb{E}R\text{ due to time-varying productivity}},\end{aligned}$$

where $\bar{A} \equiv \frac{A_1 - A_0}{A + \xi}$. The first two terms correspond to a solution of one sector model, as was shown in Example 6.1. It comprises of two components: time discounting δ and expected growth of capital, $\phi(x)$. The last term is new and reflects the presence of time-varying productivity. If two technologies were identical, the term would vanish. Otherwise, it adjusts for growth in the share of risky technology x, as the economy endogenously reallocates towards its optimal mix of riskless and risky capital via investment.

Risk prices Next, I derive the expression for an SDF of the model. Appendix, section 6.2.2 shows that risk prices are given by

$$\begin{aligned} -\mathcal{L}\left(\frac{d\Lambda}{\Lambda}\right) &= -\mathcal{L}\left[dlnf_C - \alpha_t dlnJ\right] = \alpha dlnK + dln\tilde{A} + (\alpha - 1)\, dlnF \\ &= \underbrace{\alpha x \boldsymbol{\sigma_K}}_{①} + \underbrace{\bar{A}\,(1-x)\, x \boldsymbol{\sigma_K}}_{②} + \underbrace{(\alpha-1)\,\omega_x(x,\alpha)\, x\,(1-x)\,\boldsymbol{\sigma_K}}_{③} + \underbrace{(\alpha-1)\,\omega_\alpha(x,\alpha)\,\alpha\boldsymbol{\sigma_\alpha}}_{④}, \end{aligned} \tag{26}$$

where $\tilde{A} = A + \xi$, $\omega_x(x,\alpha) = \frac{F_x}{F}$, and $\omega_\alpha(x,\alpha) = \frac{F_\alpha}{F}$.

The first term is the usual risk price which is also present in a model with one sector. It can be easily seen by setting $x = 1$, $F_x = 0$. In this case the last three terms drop out (when risk aversion is constant) and we end up with an expression for the price of capital risk in one sector economy with constant risk aversion, $rp = \alpha\sigma_K$. See Appendix, section 6.4.4 for more details. The second term is a new risk price due to changing productivity. Productivity falls on capital shocks and thus the price of risk is positive. This term is present only due to time-varying productivity; in case when $A_0 = A_1$, it disappears.

The third piece reflects the presence of two distinct technologies. It is present even when productivities of two technologies are equal. It depends on an unknown function $\omega_x(x,\alpha)$. One can show that $\omega_x(x,\alpha) = 0$ at the point where investments in two technologies are equal (PEQ; see the definition below), $\omega_x(x,\alpha) > 0$ when x is below this point, and $\omega_x(x,\alpha) < 0$ when x is above the PEQ. It is therefore positive when we have "too little" capital, and negative otherwise. Risk prices therefore tend to fall as x falls due to direct effect of x, but tend to rise due to an increase in $\omega_x(x,\alpha)$. It's not clear which effect dominates.

When risk aversion is time-varying, the last component reflects the appropriate price of risk due to this variation.

Example 6.1. Aggregates in one-sector economy.

Consider an economy that features only a risky sector and constant risk aversion α. Eq. 23 can be easily specialized to this case by setting $x = 1$, $F_x = 0$. For a more formal argument, see Appendix, section 6.4.4. In such an economy, all aggregates relative to capital are constant. In particular, $c = \delta\frac{A+\xi}{\delta+\xi}$, $q = \frac{A+\xi}{\xi+\delta}$, and $i = \xi\frac{A-\delta}{\delta+\xi}$. Normalized aggregates in a one-sector and a two-sector economies therefore differ only due to a persistent time-variation in aggregate productivity in a two-sector economy. Appendix, section 6.4.4 also shows the expressions for expected market returns, excess returns, and interest rate in a one-sector economy. In particular, I find expected market return is given by $\frac{1}{dt}\mathbb{E}[dR] = \delta + \phi(i)$, excess returns are $rx = \alpha \times \sigma_K^2$ (when risk-aversion is constant), and the risk-free rate is $r = \delta + \phi(i) - \alpha\sigma_K^2$ (when risk aversion is constant).

Example 6.2. Cox-Ingersoll-Ross (CIR) economy with constant risk aversion.

Consider a model with no adjustment cost, or, equivalently, $\xi = \infty$ in the setup above. When adjusting capital is costless, the economy always supports a constant level of $x \equiv x^* =$

$min\left(\frac{A_1-A_0}{\alpha\sigma_K^2},1\right)$. Value function and all aggregates are therefore constant, $c=\delta$, $i=A-\delta$. Marginal q in such an economy is constant and equal to 1 (on aggregate and for each individual technology). All the risk is thus cash-flow risk, $\frac{1}{dt}\mathbb{E}\left[dR\right]=(A-i)+i=A$, $\frac{1}{dt}\mathbb{E}\left[dR_0\right]=A_0$, $\frac{1}{dt}\mathbb{E}\left[dR_1\right]=A_1$, $rx_M=rx_1=\alpha x^*\sigma_K^2$, $rx_0=0$, and $r=A-\alpha x^*\sigma_K^2$. See Appendix, section 6.4.5 for derivations.

6.4.4 One sector model, $EIS=1$, log adjustment costs, constant α

HJB equation:

$$\begin{aligned}\delta\Big[ln\left(c\right)-ln\left(F\right)\Big]-\frac{1}{2}\alpha\left(\sigma_K+\frac{F_\alpha}{F}\sigma_\alpha\right)\left(\sigma_K+\frac{F_\alpha}{F}\sigma_\alpha\right)^{'}\\+\phi_1\left(i\right)+\frac{F_\alpha}{F}\phi\left(\bar{\alpha}-\alpha\right)+\frac{1}{2}\frac{F_{\alpha\alpha}}{F}\sigma_\alpha\sigma_\alpha^{'}+\frac{F_\alpha}{F}\sigma_K\sigma_\alpha^{'} &= 0.\end{aligned}$$

Resource constraint:

$$c=A-i.$$

Assume log installation function

$$\phi_1\left(i\right)=\zeta+\xi\times ln\left(1+\frac{i}{\xi}\right).$$

FOC:

$$\begin{aligned}\delta\frac{1}{c} &= \xi\frac{1}{\xi+i}\\ c &= \frac{\delta}{\xi}\left(\xi+A-c\right)\\ c &= \frac{A+\xi}{1+\frac{\xi}{\delta}}\end{aligned}$$

$$\begin{aligned}i &= \frac{A+\xi}{1+\frac{\delta}{\xi}}-\xi\\ q &= \frac{1}{\phi'\left(i\right)}=\frac{\xi+i}{\xi}=\frac{A+\xi}{\xi+\delta}.\end{aligned}$$

So all aggregates c, i, q (per unit of capital) are constant.

Assume σ_α is constant.

Risk premium on this technology can be derived by setting all $F_x=0$, $x=1$ in my general two tree model,$rx=\alpha\sigma_K^2$. Expected returns are constant,

$$\frac{1}{dt}\mathbb{E}\left[dR\right]=\frac{A-i}{q}+\mathbb{E}\frac{dK}{K}=\delta+\phi\left(i\right).$$

Risk-free rate is given by

$$r=\frac{1}{dt}\mathbb{E}\left[dR\right]-rx=\delta+\phi\left(i\right)-\alpha\sigma_K^2.$$

So all the variation in risk premium is driven entirely by variation in the risk-free rate.

6.4.5 CIR Model

Assume constant risk aversion.

$$\begin{aligned} q &= \phi'(i) = 1 \\ F - F_x x &= F + F_x (1-x) \\ F_x &= 0 \end{aligned}$$

So value function if flat wrt x, since the capital can be freely reallocated. It means that x is not a state variable any more, rather it is a choice: there exists an optimal value x^* which maximizes utility.

HJB equation becomes

$$\frac{\delta}{\rho}\left[\left(\frac{c}{F}\right)^{\rho} - 1\right] - \frac{1}{2}\alpha x^2\sigma_K^2 + i = 0.$$

FOC:

$$\begin{aligned} \delta\left(\frac{c}{F}\right)^{\rho-1} &= F \\ c &= \left(\delta F^{\rho}\right)^{\frac{1}{\rho-1}} \\ F &= \left[(A-i)^{\rho-1}\frac{1}{\delta}\right]^{\frac{1}{\rho}}. \end{aligned}$$

From HJB,

$$\begin{aligned} i &= \frac{1}{2}\alpha x^2\sigma_K^2 - \frac{\delta}{\rho}\left[\left(\frac{F}{\delta}\right)^{\frac{\rho}{\rho-1}} - 1\right] \\ c &= A - i \\ \left(\delta F^{\rho}\right)^{\frac{1}{\rho-1}} &= A - \frac{1}{2}\alpha x^2\sigma_K^2 - \frac{\delta}{\rho}\left[\left(\frac{F}{\delta}\right)^{\frac{\rho}{\rho-1}} - 1\right] \\ \delta^{\frac{1}{\rho-1}}F^{\frac{\rho}{\rho-1}}\left(1-\frac{1}{\rho}\right) &= A - \frac{1}{2}\alpha x^2\sigma_K^2 + \frac{\delta}{\rho}. \end{aligned}$$

Hence,

$$F = \left[\left((1-x)A_0 + xA_1 - \frac{1}{2}\alpha x^2\sigma_K^2 + \frac{\delta}{\rho}\right)\frac{\rho}{\rho-1}\delta^{-\frac{1}{\rho-1}}\right]^{\frac{\rho-1}{\rho}},$$

or for EIS=1 case,

$$\begin{aligned} F &= exp\left(\frac{1}{\delta}\left[\delta ln\delta - \frac{1}{2}\alpha x^2\sigma_K^2 - \delta + (1-x)A_0 + xA_1\right]\right) \\ c &= \delta \\ i &= A - \delta. \end{aligned}$$

Maximize either of these wrt x:

$$x^* = min\left(\frac{A_1 - A_0}{\alpha\sigma_K^2}, 1\right).$$

Marginal q in such economy is constant and equal 1 (on aggregate and for each individual technology). All the risk is thus cash flow risk, $\frac{1}{dt}\mathbb{E}\left[dR\right]=(A-i)+i=A$, $\frac{1}{dt}\mathbb{E}\left[dR_0\right]=A_0$, $\frac{1}{dt}\mathbb{E}\left[dR_1\right]=A_1$, $rx_M=rx_1=\alpha x^*\sigma_K^2$, $rx_0=0$, and $r=A-\alpha x^*\sigma_K^2$.

6.4.6 Two sector model, $EIS=1$, log adjustment costs, time-varying α

SDF

$$\begin{aligned} f &= \delta J\left(lnC-lnJ\right) \\ f_C &= \delta J\frac{1}{C}=\delta F\frac{1}{c} \\ dlnf_C &= dlnF-dlnc \\ dlnJ &= dlnK+dlnF \end{aligned}$$

$$\begin{aligned} -sd\left(\frac{d\Lambda}{\Lambda}\right) &= dlnc-dlnF+\alpha\left(dlnK+dlnF\right) \\ &= dlnc+\alpha dlnK+\left(\alpha-1\right)dlnF. \end{aligned}$$

6.5 Proofs

Proof of Theorem 3.2. In section 6.2.1 I show simple manipulations of Duffie and Epstein (1992a,b) formulas to get risk prices for the case when the risk aversion α is constant. In section 6.2.2 I derive the expression for market prices of risk in the general case when α is time-varying. □

Proof of Theorem 6.3. I look for a first-order expansion of $F\left(x,\alpha;\epsilon\right)$ as a power series in ϵ:

$$F\left(x,\alpha;\epsilon\right)=F\left(x,\alpha;0\right)+F_\epsilon\left(x,\alpha;0\right)\epsilon+o\left(\epsilon^2\right),$$

where $F\left(x,\alpha;0\right)$ gives the value of F at $\epsilon=0$ and $F_\epsilon\left(x,\alpha;\epsilon\right)$ gives the derivative at $\epsilon=0$. To compute the first-order functional perturbation of Eq. 22, I first evaluate the PDE in Eq. 23 at $\epsilon=0$ to find $F\left(x,\alpha;0\right)=\delta\left(A_0+\xi\right)^{1+\frac{\xi}{\delta}}\left(\delta+\xi\right)^{-1-\frac{\xi}{\delta}}$. Next, I differentiate the PDE with respect to ϵ and drop all terms multiplying ϵ. I do so because these terms will always drop out in computations of all derivatives $\left.\frac{\partial^{k+1}F}{\partial\epsilon\partial x^k}\right|_{\epsilon=0}$ and $\left.\frac{\partial^{k+1}F}{\partial\epsilon\partial\alpha^k}\right|_{\epsilon=0}$ for any $k>0$ and evaluating at $\epsilon=0$. Next, all $\frac{\partial^{k+n}F}{\partial x^n\partial\alpha^k}=0$ for any k and n such that $k+n>0$. The resulting expression is:

$$-\bar{A}(\delta+\xi)x-\frac{1}{2}\alpha x^2\varsigma_K^2-\delta\frac{F_\epsilon}{F}+\phi(\bar{\alpha}-\alpha)\frac{F_{\epsilon x}}{F}=0.$$

Solving for F_ϵ and choosing a family of solutions in the space of real numbers delivers the expression for $F\left(x,\alpha;\epsilon\right)$ in Theorem 6.3. First order Taylor expansion of $\omega_x\left(x,\alpha;\epsilon\right)\equiv\frac{F_x}{F}$ and $\omega_\alpha\left(x,\alpha;\epsilon\right)\equiv\frac{F_\alpha}{F}$ around $\epsilon=0$ gives the expressions for components of risk prices. □

Proof of Theorem 6.4. Eq. 16 gives expressions of marginal q's as function of $\omega_x\left(x,\alpha\right)$ which we have already calculated in Theorem 6.3. Performing Taylor expansion of these expressions around $\epsilon=0$ followed by a similar procedure discussed in the proof of Theorem 6.3 to find $q\left(x,\alpha;0\right)$ and $q_\epsilon\left(x,\alpha;0\right)$ delivers the expressions in the theorem. Loadings $l_{n,x}\left(x,\alpha;\epsilon\right)\equiv\frac{q_{n,x}}{q_n}$, $l_{n,\alpha}\left(x,\alpha;\epsilon\right)\equiv\frac{q_{n,\alpha}}{q_n}$ are further calculated by Taylor-expanding the resulting expressions one more time. □

Proof of Proposition 6.2. Unexpected returns of the riskless technology in response to an orthogonal capital shock are given by (up to the first order in ϵ):

$$\begin{aligned} dR_0 - \mathbb{E}dR_0 &= l_{0,x}x\left(1-x\right)\boldsymbol{\sigma_K}d\boldsymbol{Z} \\ &\underset{\epsilon\to 0}{\simeq} \left\{-\frac{\xi}{\delta}\bar{A} + 2x\left(\zeta_0+\zeta_\alpha\alpha\right)\varsigma_K^2\right\}x\left(1-x\right)\sqrt{\epsilon}\boldsymbol{\sigma_K}d\boldsymbol{Z}. \end{aligned}$$

According to Theorem 6.2, for $x > x^*$ sufficiently close to x^*, $\omega_x = \frac{\delta+\xi}{\delta}\bar{A} - \left(\zeta_0+\zeta_\alpha\alpha\right)x\sigma_K^2 < 0$ and $\omega_x = 0$ at $x = x^*$. Around the first order expansion in ϵ, Theorem 6.3 can be used to find that $F_{xx} = -\left(\zeta_0+\zeta_\alpha\alpha\right)\sigma_K^2 < 0$ does not change sign and the above result therefore holds globally on $x \in [x^*, 1]$ (up to the first-order expansion in ϵ). Plugging this in the formula above implies that the unexpected return on riskless technology is positive when $x > x^*$.

At x^*, $\left(\zeta_0+\zeta_\alpha\alpha\right)x^*\varsigma_K^2 = \frac{\delta+\xi}{\delta}\bar{A}$ producing

$$\mathcal{L}\left(dR_0 - \mathbb{E}dR_0\Big|_{x=x^*}\right) \underset{\epsilon\to 0}{\simeq} \left(2+\frac{\xi}{\delta}\bar{A}\right)x^*\left(1-x^*\right)\varsigma_K,$$

which is positive. Evaluating the unexpected return at $x = k\,x^*$ for any constant $k < 1$ gives

$$\mathcal{L}\left(dR_0 - \mathbb{E}dR_0\Big|_{x=kx^*}\right) \underset{\epsilon\to 0}{\simeq} \left[-\frac{\xi}{\delta}\bar{A} + 2k\left(1+\frac{\xi}{\delta}\right)\bar{A}\right]x^*\left(1-x^*\right)\varsigma_K,$$

which is positive when $k > \frac{1}{2}\frac{\xi}{\xi+\delta}$.

Therefore on the range $x \in [\frac{1}{2}\frac{\xi}{\xi+\delta}x^*,\, 1]$ unexpected returns on the riskless technology respond positively to an orthogonal capital shock.

Unexpected returns of the risky technology in response to an orthogonal capital shock are given by (up to the first order in ϵ):

$$\begin{aligned} dR_1 - \mathbb{E}dR_1 &= \left[1 + l_{1,x}x\left(1-x\right)\right]\boldsymbol{\sigma_K}d\boldsymbol{Z} \\ &\underset{\epsilon\to 0}{\simeq} \left\{1 - \frac{\xi}{\delta}\bar{A}x\left(1-x\right) + \left(2x-1\right)\left(\zeta_0+\zeta_\alpha\alpha\right)x\left(1-x\right)\varsigma_K^2\right\}\sqrt{\epsilon}\boldsymbol{\sigma_K}d\boldsymbol{Z}. \end{aligned}$$

I assume that the quantity in curly brackets is positive for any value of state variables (make it a standalone assumption and solve for required parameter values!). The assumption is fairly innocuous and holds for a wide range of plausible calibrations. It rules out the case when stock prices rise on a negative capital shock. With this assumption, returns on the risky technology react positively to an orthogonal capital shock. □

Proof of Proposition 6.3. Consider the case of uncorrelated shocks, $\lambda = 0$. It is sufficient to show that the proposition holds in this case. Unexpected returns of the riskless technology are given by (up to the first order in ϵ)

$$dR_0 - \mathbb{E}dR_0 = \left[l_{0,x}x\left(1-x\right) + l_{0,\alpha}\alpha\lambda\right]\boldsymbol{\sigma_K}d\boldsymbol{Z} + l_{0,\alpha}\boldsymbol{\sigma_\alpha}d\boldsymbol{Z},$$

and unexpected returns of the risky technology by

$$dR_1 - \mathbb{E}dR_1 = \left[1 + l_{1,x}x\left(1-x\right) + l_{1,\alpha}\alpha\lambda\right]\boldsymbol{\sigma_K}d\boldsymbol{Z} + l_{1,\alpha}\boldsymbol{\sigma_\alpha}d\boldsymbol{Z}.$$

When $\lambda = 0$ and up to a first order expansion in ϵ, covariance of two returns is given by

$$cov_t\left(dR_0,\, dR_1\right) = l_{0,x} x\left(1-x\right)\left[1 + l_{1,x} x\left(1-x\right)\right] \varsigma_K^2 + l_{0,\alpha} l_{1,\alpha} \alpha^2 \varsigma_\alpha^2.$$

The last term is negative and quadratic in α. The first term is positive for $x \in [\frac{1}{2}\frac{\xi}{\xi+\delta}x^*,\, 1]$ and consist of two terms linear in α and one quadratic in α, $2x\left(2x-1\right)\left(\zeta_0 + \zeta_\alpha \alpha\right)^2 x^2\left(1-x\right)^2 \varsigma_K^6$. The quadratic term is decreasing in α for $x < \frac{1}{2}$. For high levels of α quadratic terms dominate and thus the covariance between two returns becomes more negative. It is therefore possible to calibrate a model in such a way that the covariance is positive for low levels of α (when $-l_{0,\alpha} l_{1,\alpha} \alpha^2 \varsigma_\alpha^2$ is relatively small and $l_{0,x} x\left(1-x\right)\left[1 + l_{1,x} x\left(1-x\right)\right] \varsigma_K^2$ is positive and dominates) and negative when α becomes high, as quadratic terms start dominating and $l_{0,x} x\left(1-x\right)\left[1 + l_{1,x} x\left(1-x\right)\right] \varsigma_K^2$ becomes small.

□

Proof of Proposition 6.4. When $\lambda = 0$, the unexpected returns of the risky technology are given by

$$dR_1 - \mathbb{E}dR_1 \underset{\epsilon\to 0}{\simeq} \left\{1 - \frac{\xi}{\delta}\bar{A}x\left(1-x\right) + \left(2x-1\right)\left(\zeta_0 + \zeta_\alpha \alpha\right) x\left(1-x\right)\sigma_K^2\right\}\sqrt{\epsilon}\sigma_K d\boldsymbol{Z} + l_{1,\alpha}\alpha\sigma_\alpha d\boldsymbol{Z}.$$

The loading on capital shock therefore increases in α when $x < \frac{1}{2}$. The loading on risk aversion risk decreases in α since $l_{1,\alpha}$ is negative and decreasing in α.

The loading of the SDF on shocks is given by

$$\mathcal{L}\left(-\frac{d\Lambda}{\Lambda}\right) = \bar{A}\left(1-x\right)x\sigma_K + \alpha x \sigma_K + \left(\alpha - 1\right)\omega_x\left(x,\alpha\right)x\left(1-x\right)\boldsymbol{\sigma_K} + \left(\alpha-1\right)\omega_\alpha\left(x,\alpha\right)\alpha\boldsymbol{\sigma_\alpha}.$$

The price of the α-shock risk is therefore negative and decreasing in α (for $\alpha > 1$), since $\omega_\alpha\left(x,\alpha\right)$ is negative. Since both the loading and the price of risk aversion risk are negative and decreasing in α, the risk premium due to the risk aversion always increases in the level of α. The price of the capital risk is given by

$$\begin{aligned}&\left[\bar{A}\left(1-x\right) + \alpha + \left(\alpha-1\right)\omega_x\left(x,\alpha\right)\left(1-x\right) + \left(\alpha-1\right)\omega_\alpha\left(x,\alpha\right)\alpha\lambda\right] x\sigma_K \quad \underset{\epsilon\to 0}{\simeq} \\ &\left[\bar{A}\left(1-x\right) + 1 + \left(\alpha-1\right)\left(1 + \left[\zeta_A\bar{A} - \left(\zeta_0 + \zeta_\alpha\alpha\right)x\sigma_K^2\right]\left(1-x\right)\right)\right] x\sigma_K.\end{aligned}$$

By expressing x as $x = k\,x^*$, the last term can be rewritten as $\left(\alpha-1\right)\left(1 + \left(1-k\right)\frac{\delta+\xi}{\xi}\bar{A}\left(1-x\right)\right)$. When $k < 1 + \frac{\xi}{\xi+\delta}\frac{1}{\bar{A}}$, the price of capital risk is always positive and increasing in α and thus the risk premium due to capital risk is also increasing in α. The overall risk premium then must be monotonically increasing in α.

□

Proof of Theorem 4.1. Plug $\boldsymbol{\sigma_J}$ inside Eq. 13,

$$0 = \boldsymbol{\lambda} + \left(\frac{J_{WW}}{J_W} - \alpha\frac{J_W}{J}\right) W\boldsymbol{\sigma_R}\boldsymbol{\sigma_R'}\boldsymbol{\theta} + \left(\frac{J_{W\boldsymbol{X}}}{J_W} - \alpha\frac{J_{\boldsymbol{X}}}{J}\right)\boldsymbol{\sigma_R}\boldsymbol{\sigma_X'}.$$

The value function is homogeneous in wealth, $J = W \times G(\boldsymbol{X})$. Then,

$$\begin{aligned} J_{WW} &= 0 \\ \frac{WJ_W}{J} &= 1 \\ \frac{J_{W\boldsymbol{X}}}{J_W} &= \frac{J_{\boldsymbol{X}}}{J} = \frac{G'(\boldsymbol{X})}{G(\boldsymbol{X})}. \end{aligned}$$

Plugging this in gives

$$\begin{aligned} \boldsymbol{\lambda} &= \alpha \boldsymbol{\sigma_R}\boldsymbol{\sigma_R'}\boldsymbol{\theta} + (\alpha - 1)\frac{J_{\boldsymbol{X}}}{J}\boldsymbol{\sigma_X}\boldsymbol{\sigma_R'} \\ \boldsymbol{\mu_R} - r &= \alpha \times cov(dR, dR_{TW}) + \eta \times cov(dR, d\boldsymbol{X}), \end{aligned}$$

where $\eta = (\alpha - 1)\frac{J_{\boldsymbol{X}}}{J}$, R_{TW} is return on the total wealth portfolio, $d\boldsymbol{X} = (\alpha, x)'$. □

6.6 Computational Details

6.6.1 Small-noise expansions

I use Mathematica to calculate analytical derivatives required by the small-noise expansions. First order expansions are straightforward to derive as described in section 6.5. Expansions of higher orders can be computed in a similar fashion. I computed expansions up to a third order (analytically) to verify that the directions of the main forces in the first order expansions are not overturn by higher orders. I computed expansions up to the 10^{th} order numerically (using high precision arithmetic) as well.

6.6.2 Projections

I use high order projection methods described in Judd (1998) to solve the general model. In particular, I parametrize the value function and two investment functions as a complete product of 20^{th} order Chebyshev polynomials in two state variables, x and α. Next, I evaluate the system of equations in Theorem 3.1 at 30×30 points on the state space. Points are chosen as Chebyshev's zeroes. I then search for coefficients of three policy functions to minimize the L_1 norm of PDE errors. In practice the algorithm is iterative. I start by fitting low order polynomials on the grid of 30×30 Chebyshev zeroes and iteratively increase the order of the fit until the desired precision is reached. I impose boundary conditions as given in section 6.3.3. The resulting fit is the global solution on the entire state space.

The problem is therefore formulated as a sequence of standard constrained optimization problems with thousands of constraints (one at each grid node) and thousands of unknowns (Chebyshev coefficients). I use the GAMS modeling language together with CONOPT and SNOPT non-linear constrained optimizers to find a solution. Once the solution to the optimization problem is found, I import the results in Matlab to perform 1,000,000 simulations and report the results. Most time-sensitive parts of the code for simulations and impulse responses have been programmed in C++ for faster execution. As a result, it takes about a minute to perform 1,000,000 simulations of the economy on a laptop (2GHZ CPU and 4GB of RAM).

To calibrate the model I estimated many individual calibrations. Each solution takes about $30 - 60$ minutes to compute (depending on the size of the grid and the order of approximation) on a laptop. To facilitate computations of many calibrations, I used the NEOS server, described in Czyzyk et al. (1998).

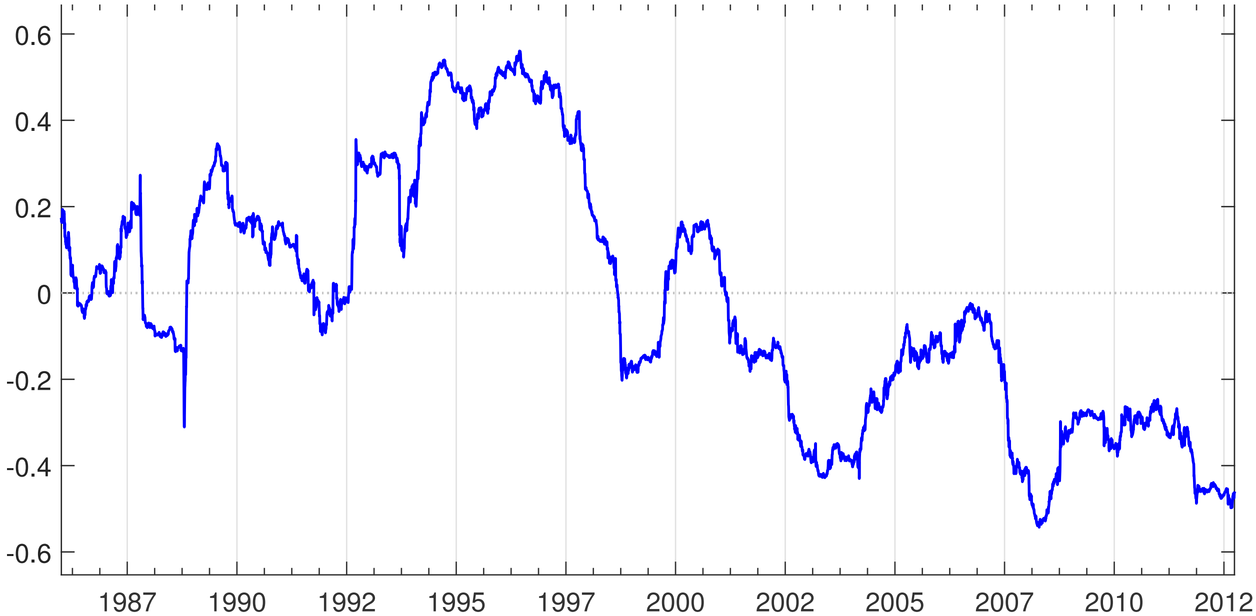
6.6.3 Impulse responses

I compute impulse responses by shocking the economy at its steady state (unconditional means of state variables) and performing Monte-Carlo simulations for the following 20×12 months. I simulate 10,000,000 Monte-Carlo trajectories for each shock and calculate the means of realizations of moments of interest. The simulations were programmed in Matlab with C++ code inserts and executed on Acropolis server at the University of Chicago. Execution time was less than an hour (using 64 parallel threads). The high number of simulations is necessary due to high volatility of the SDF and persistence in the state variables.

6.6.4 Term structure

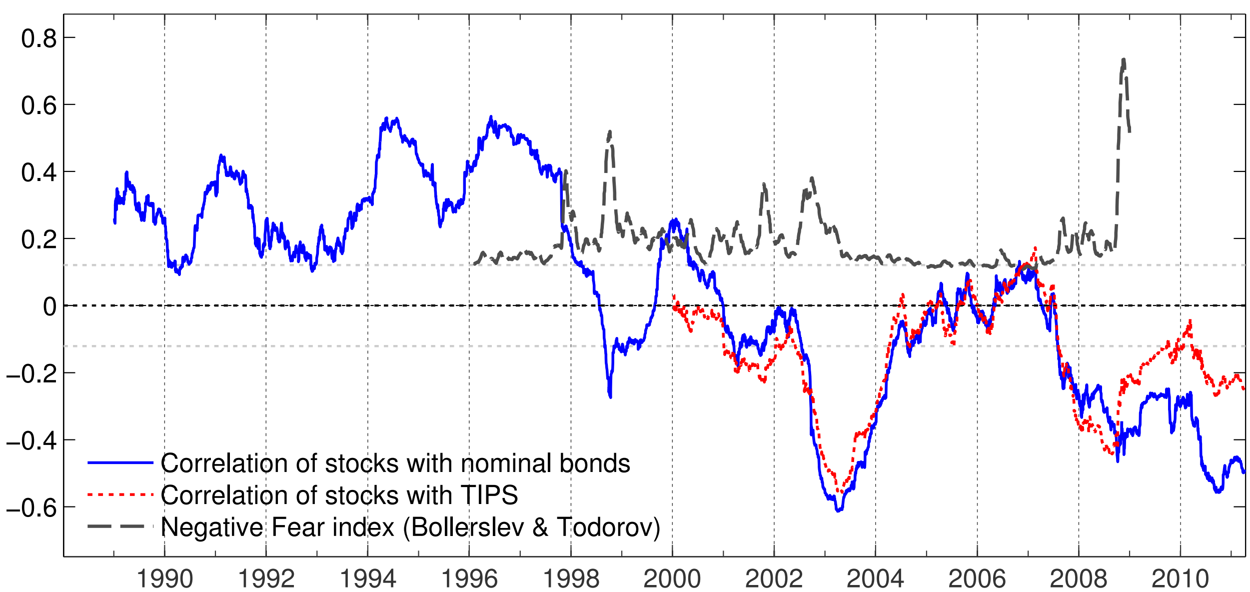
To compute the term structure I calculate conditional expectations of the SDF at each horizon iteratively starting from the end. This requires fitting the price of a bond as function of state variables at each iteration. I do so by evaluating the conditional expectation at each node (Chebyshev zeroes) and then fitting a smooth function of two state variables to these points. The function is constructed as a complete product of two 10 degree Chebyshev polynomials. The fitting requires a search for the coefficients of this function (55 coefficients). I call GAMS within my Matlab code to compute each fit. Finally, I use Gauss-Hermite quadrature to compute the required integrals (conditional expectations).

Figure 14: Correlation between market excess return and 10-year real UK bond return



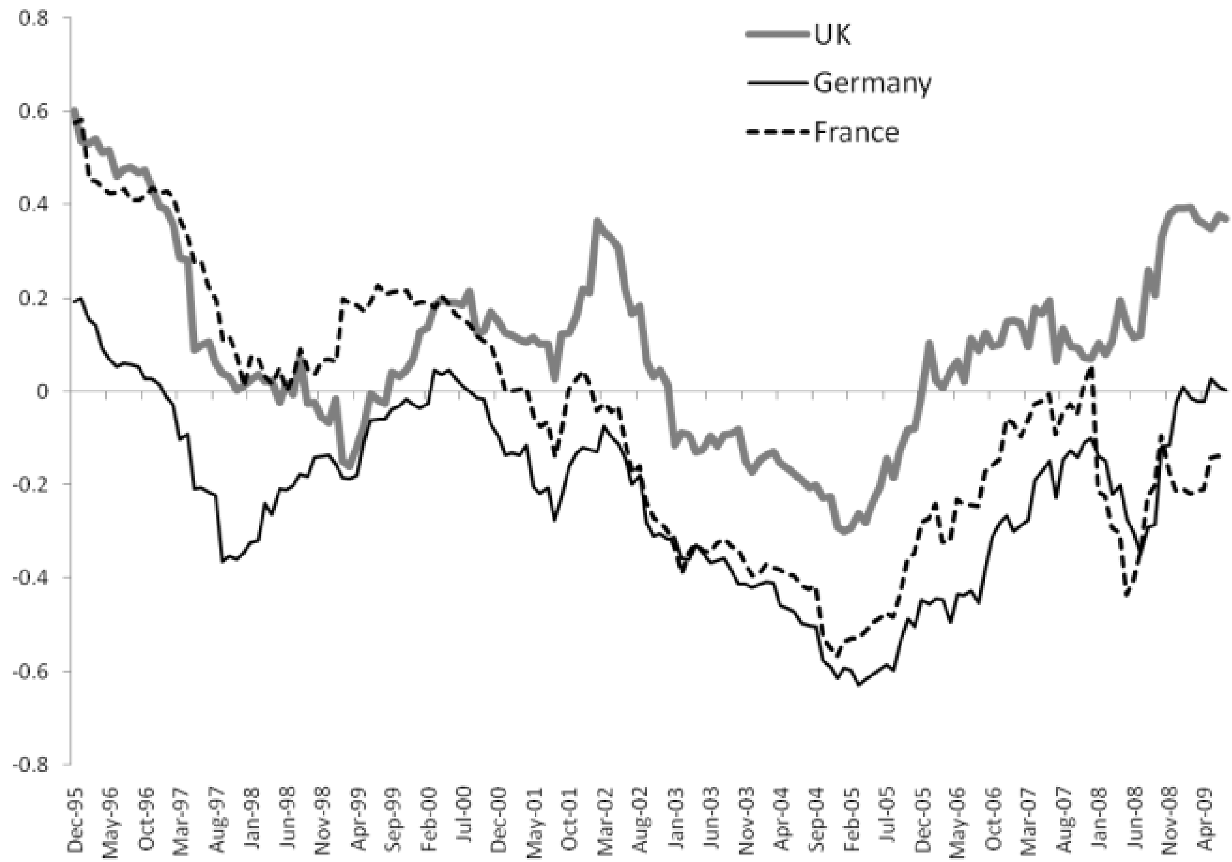
Notes: Rolling 1-year correlation between daily stock excess returns and 10-year UK real bond returns. The dashed gray lines around zero show block-bootstrapped 2 s.d. error bounds for the null of constant zero correlation.

Figure 15: Bond-stock correlation and Investors Fear Index



Notes: The solid gray line shows negative of the rescaled value of Investors Fear Index from Bollerslev and Todorov (2011). The blue and the red lines show rolling 1-year correlations between daily stock excess returns and 10-year bond excess returns (nominal and real, respectively).

Figure 16: Bond-stock correlation across countries



Notes: International evidence. 24 months window. Source: MSCI Barra Research Bulletin, October 2009.