TIGHT CLOSURE, F-PURITY, AND VARIETIES OF NEARLY COMMUTING MATRICES

by

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To my parents, Balkhash and Gulzhamal, and my sister, Anara

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CHAPTER I

Introduction

As the history of mathematics shows, working in rings where p is a prime integer and the p^{th} multiple of any number is zero often has strong advantages. These are called rings of positive characteristic p. At the end of the 1980s Mel Hochster and Craig Huneke in a series of papers, [HH89a], [HH89b], [HH90], introduced the notion of the tight closure of ideals and modules over such rings. Over time this has proved to be a very powerful technique for attacking problems in both commutative algebra and algebraic geometry, especially in the theory of singularities. Using the Frobenius endomorphism, notions of F-regularity, F-purity, F-rationality and etc. have been defined. In addition, many classes of rings, such as determinantal rings [HH94b], were proved to posses these properties. Significant progress in solving various problems has been made since then by using these tools. However, there are still questions which naturally appeared with the birth of this new theory, many of which are still open. Some of the major questions have been settled.

Question I.1. Let R be a Noetherian commutative ring with identity and of prime characteristic $p > 0$. Let I be an ideal of R and let I^* be its tight closure in R. Is it true that for any prime ideal $P \subseteq R$, $I^* R_P = (IR_P)^*$?

The question was open until 2010, when H.Brenner and P.Monsky found a counterexample, [BM10]. However, it is known that the above question has an affirmative answer for very special class of ideals, parameter ideals, [Smi94].

Other natural conjectures that were stated are the following.

Conjecture I.2. *Let R be a ring in which every ideal is tightly closed (such rings are called weakly F-regular). Then for every multiplicative system $W \subseteq R$, $W^{-1}R$ is weakly F-regular.*

Conjecture I.3. *Let R be a ring such that it splits from its every module-finite ring extension. Then R is weakly F-regular.*

The latter conjecture implies the former which was answered positively for $\mathbb{Q}$-Gorenstein rings (hence for local rings of dimension at most 2), [Sin99], and for graded rings of any dimension, [LS99].

The first part of this thesis concentrates on finding criteria for weak F-regularity that may be used to attack the above questions.

To this end we define the notion of quasi-parameter ideals for Cohen-Macaulay domains for which there exists a pure height one ideal isomorphic to the canonical module.

Definition I.4. If J is a proper canonical ideal (isomorphic to the canonical module ω_R) in a Cohen-Macaulay local ring R of dimension d, and $x_1, \dots, x_i$ are part of a system of parameters for R whose images form part of a system of parameters in R/J, then we call $(x_1, \dots, x_i) + J$ a *quasi-parameter ideal* for J. All the $(x_1^t, \dots, x_i^t) + J$ are also quasi-parameter ideals for J. If $i = d - 1$, then we say that $(x_1, \dots, x_{d-1}) + J$ is a *full quasi-parameter ideal* for J. If R is a Gorenstein local ring, then J is a principal ideal and all the quasi-parameter ideals are simply parameter ideals.

It is an open question whether localization commutes with tight closure of quasi-parameter ideals. However, we have that a positive answer to this question will imply an affirmative answer to Conjecture I.2, see Theorem III.9.

We also use the above notion to give a sufficient condition for R to be weakly F-regular.

Lemma I.5. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay local domain of Krull dimension d with a canonical module ω. Let J denote a pure height one ideal of R such that $J \cong \omega$. If for one choice of a canonical ideal J, every full quasi-parameter ideal for J is tightly closed in R, then R is weakly F-regular.*

In addition, we study a notion of a test ideal relative to an ideal J defined as follows.

Definition I.6. Let R be any Noetherian ring of characteristic $p > 0$ and J be an ideal of R. Then the *J-test ideal* of R is $\tau_J(R) = \bigcap_{J \subseteq I}(I :_R I^*)$.

Moreover, we define the notion of a finitistically amenable ring, see Definition III.22, and we prove that for a finitistically amenable Cohen-Macaulay normal complete local domain there exists a choice of an ideal $J \cong \omega_R$ so that this notion of test ideal coincides with the classical one, i.e., $\tau_J(R) = \tau(R)$, see Corollary III.37.

In a related direction we prove:

Theorem I.7. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay complete local domain with the canonical module ω_R. If $u \in 0^{*R}_{E_R(K)}$, then there exists a choice of an ideal $J \cong \omega_R$ such that $u \in 0^{*R}_{E_{R/J}(K)}$ over R.*

The next part of this thesis focuses on algebraic sets of nearly commuting matrices and their irreducible components. The area of mathematics which studies matrices and related problems is one of the oldest, and since its inception one of the natural questions one could ask was about when two matrices commute or almost commute in a certain given sense. However, such questions have proved to be among the hardest, and many of them are still open. Here, I want to list some of the related open questions.

Let $X = (x_{ij})$ and $Y = (y_{ij})$ be $n \times n$ square matrices with indeterminate entries over a field K. Let $R = K[X, Y]$ be the polynomial ring in the entries of X and Y and let I be the ideal of R generated by the off-diagonal entries of the commutator matrix $C = XY - YX$. Let J be the ideal generated by the entries of C and let P be its radical, which is known to be prime, see [Ger61], [MT55].

Conjecture I.8.

(a) I is a radical ideal.

(b) R/J is Cohen-Macaulay.

(c) $J = P$ is a prime ideal.

(d) R/J is F-pure.

(e) R/J is F-rational.

(f) R/J is F-regular.

Part (a) is known to be true when K has characteristic 0, [You11], and in the case $n = 2$ when characteristic of the field is positive prime. Parts (a),(b) and (c) are known to be true for $n \leqslant 3$, [Tho85]. We prove part (a) in all characteristics for all positive integers n, see Theorem IV.33.

Moreover, it is known that $\mathrm{Rad}(I) = P \bigcap Q$, where Q is a prime ideal and $\mathbb{V}(Q)$ is an irreducible component of $\mathbb{V}(I)$, see [You11]. The variety defined by Q is called the *skew-component* of I. It was not known if $\mathbb{V}(P + Q)$ is irreducible. We prove that $\mathrm{Rad}(P + Q)$ is prime in all characteristics and for all n. Moreover, we show that $P + Q$ is prime when $n = 3$.

Using Fedder's criterion, Lemma IV.12, we prove the following result.

Theorem I.9. *When $n = 3$, R/I, R/P, R/Q and $R/(P+Q)$ are F-pure.*

The proof that R/I is F-pure in the case of 3 by 3 matrices utilizes the fact that I is generated by a regular sequence. Let U be the product of the off-diagonal entries of the commutator matrix C, i.e., of the generators of I. We explicitly find a monomial term μ of U such that μ^{p-1} is a nonzero monomial term of U^{p-1} and $\mu^{p-1} \notin \mathfrak{m}^{[p]}$ for all values of p. It turns out to be a rather nice term, since its coefficient is always 1 modulo p. However, not all terms of U^{p-1} have such a "uniform" behavior. In fact, there are terms which do not work for all p but seem to work for an infinite number of values of p, see Appendix 4.7. Another interesting fact, in characteristic 2, U has 24,846 nonzero terms and only 108 can be used to prove F-purity of R/I. In characteristic 3, U has 12,229,308 terms in total while only 23,823 are useful for us and only 162 are such that every variable in its support has exponent 2.

Furthermore, by applying known results of [Tho85] and [PS74], see Definition IV.7 and Theorem IV.9, we obtain that P and Q are linked via I. Hence we have the following theorem.

Theorem I.10. *R/P is Cohen-Macaulay if and only if R/Q is Cohen-Macaulay, in which case $R/(P+Q)$ is Gorenstein. In particular, when $n = 3$, R/P and R/Q are Cohen-Macaulay domains and $R/(P+Q)$ is a Gorenstein domain of dimension $\dim R/P - 1$.*

Moreover, we have the following conjecture.

Conjecture I.11. *R/P, R/Q and $R/(P+Q)$ are F-regular.*

We also show that to prove the conjecture it is sufficient to have that $R/(P+Q)$ is F-rational. This is done by applying the result of F.Enescu, Lemma II.46, and the fact that F-rationality and F-regularity are equivalent for Gorenstein rings.

We finish the chapter by stating conjectures for R/I which we have developed while working on the subject.

The last part of the thesis looks into the theory of algebras with straightening law and some interesting observations are made. Let $A = B[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates over a Noetherian ring B. Let I be an ideal of A. It is well known that if I is generated by monomials, then one can exhibit a monomial B-basis for the quotient A/I and this makes its study more approachable. However, if I is not a monomial ideal, it can be quite a hard problem to understand the quotient. Thus the purpose of the theory of algebras with straightening law (ASL) is to give a (non-monomial) B-basis for A which shares many of the properties of the classical monomial basis and under certain hypotheses one can have that the quotient inherits these properties. One of the important features of such a basis is that we have a partial ordering on its elements.

We have proved that given a basis Π for an ASL and $\xi \in \Pi$ we can replace ξ in the basis by a B-linear combination of the elements of Π which form a chain in Π ending in ξ. Then the new set is again a basis for A as an ASL. That is, we have a B-linear change of basis for the algebra with the induced partial ordering and which preserves the property of being an ASL.

As an application to our result, we prove that the off-diagonal (respectively all) entries of the commutator matrix $XY - YX$, as in Chapter 3, are part of the basis for $R = K[X, Y]$ as an ASL. However, we do not know if the quotient is an ASL. If this was the case, we could try to approach Conjecture I.8 (a), (b) with the methods of the theory of algebras with straightening law. In fact, part (a) would follow immediately.

Let us define the set-up in which we shall work in the coming chapters. All rings will be assumed commutative Noetherian with identity. The notation $(R, \mathfrak{m}, K)$ shall mean that R is a local ring with the unique maximal ideal $\mathfrak{m}$ and the residue field $K = R/\mathfrak{m}$. The $\mathfrak{m}$-adic completion of R is denoted by $\widehat{R}$. By dimension of the ring we shall mean its Krull dimension. Given an ideal I of R, by $\mathbb{V}(I)$ we denote the algebraic set defined by the ideal I. In particular, if R is a finitely generated affine K-algebra, then we can think of $\mathbb{V}(I)$ as the set of common zeros of the generators of I. Also, if $f \in R$, then $\mathbb{V}(I)_f$ represents the set of common zeros of elements of I on which f does not vanish.

CHAPTER II

Introduction and preliminaries

The goal of this chapter is to define the necessary prerequisite material which is used in the thesis. This includes the notions of tight and plus closure of ideals and local cohomology theory.

2.1 Tight closure

In this section we discuss the notion of tight closure of ideals and modules. Throughout this section, unless otherwise stated, let R be a Noetherian ring of positive prime characteristic p and let I be an ideal of R. Let q denote a variable power of p and $I^{[q]}$ the ideal generated by qth powers of elements in I. Let also R° denote the set of all elements of R not in any of the minimal primes of the ring.

Definition II.1. An element $r \in R$ is in the *tight closure* I^* of I if there exists $c \in R^\circ$ such that $cr^q \in I^{[q]}$ for all q sufficiently large.

The following proposition shows that this is in fact a closure operation:

Proposition II.2. *Let $I \subseteq J$ be ideals of R. Then*

(1) $I \subseteq I^* \subseteq \operatorname{Rad} I$.

(2) $(I^*)^* = I^*$.

(3) $I^* \subseteq J^*$*.*

For more of the basic properties of tight closure see [HH90] Proposition 4.1.

Once a closure operation is defined, it is natural to start asking questions about rings with the property that every ideal (in a certain class) is closed and/or if the closure operation commutes with localization. The following definitions were introduced to address them for tight closure.

Definition II.3. A ring R is said to be *weakly F-regular* if every ideal of R is tightly closed.

Definition II.4. A ring R is *F-regular* if every localization of R is weakly F-regular.

Definition II.5. A local ring R is *F-rational* if every ideal generated by a system of parameters is tightly closed. A ring is called F-rational if every localization at a maximal ideal is F-rational.

Clearly, F-regularity implies weak F-regularity and the latter implies F-rationality. One of the nicest examples of F-regular rings, as one might expect, are regular rings, see [HH90] Theorem 4.6.

Theorem II.6 ([HH90])**.** *Let $(R, \mathfrak{m}, K)$ be a local excellent ring or a homomorphic image of a Gorenstein ring. If R is F-rational then it is Cohen-Macaulay and normal.*

Sometimes making a problem more general makes its solution more approachable. This works quite well in the tight closure theory when one extends the theory to modules. Let us now define the relevant notions.

In characteristic p there is a natural ring endomorphism $F_R : R \to R$ defined by $r \to r^p$. It is called the *Frobenius endomorphism*. We shall omit the subscript R in the notation F_R whenever it is clear form the context over which ring we take the Frobenius. When R is reduced, the map is isomorphic to $R^p \to R$ and $R \to R^{1/p}$, where $R^p = \{r^p \,|\, r \in R\}$ and $R^{1/p} = \{r^{1/p} \,|\, r \in R\}$. Naturally, we can take e-fold iterated Frobenius endomorphism F^e and we can get an identification $F^e(R) = R$. Although, R and $F^e(R)$ are isomorphic as rings, they are not isomorphic as R-modules. If M is an R-module, then via a base change, $F^e(M) = F^e(R) \otimes_R M$ is a left $F^e(R)$ module with $r \cdot (r' \otimes m) = r^{p^e} r' \otimes m = r' \otimes rm$.

Let $N \subseteq M$ be finitely generated R-modules. Let $q = p^e$. Denote by $N^{[q]}$ the image of the map $N \otimes F^e(R) \to M \otimes F^e(R)$. For every element $u \in N$, its image in $N^{[q]}$ we shall denote by u^q. In fact, $N^{[q]}$ is the R-span of all such u^q. We also have that $F^e(M/N) \cong F^e(M)/N^{[q]}$ and, in particular, $F_R(R/I) = R/I^{[q]}$.

Definition II.7. An element $u \in M$ is in the *tight closure* N^*_M of N in M if there exists $c \in R^\circ$ such that $cu^q \in N^{[q]}$ for all q sufficiently large.

We shall omit the subscript M in the notation N^*_M, whenever it is clear from the context what the ambient module is.

Tight closure of modules extends the corresponding notion for ideals and has basic properties similar to those of the ideals.

Proposition II.8. *Let $N \subseteq M \subseteq Q$ be finitely generated modules over R. Then*

(1) $N \subseteq N^*_M$.

(2) N^*_M *is an R-module.*

(3) $(N^*_M)^*_M = N^*_M$.

(4) $N^*_M \subseteq N^*_Q$ *and* $N^*_Q \subseteq M^*_Q$.

(5) $u \in N^*_M$ *in M if and only if its image* $\bar{u} \in 0^*_{M/N}$ *in* M/N.

(6) $I^* N^*_M \subseteq (IN)^*_M$.

The next theorem allows us to reduce to the local case when studying weak F-regularity and shows that if every ideal of R is tightly closed, then so are all submodules of finitely generated R-modules. Moreover, in the local case the tight closure of ideals is completely determined by those primary to the maximal ideal.

Theorem II.9 ([HH90]). *The following are equivalent for a Noetherian ring R of prime characteristic $p > 0$*

(1) R is weakly F-regular.

(2) Every submodule of a finitely generated R-module is tightly closed.

(3) For every maximal ideal $m \subseteq R$, R_m is weakly F-regular.

(4) Every ideal primary to a maximal ideal is tightly closed.

(5) For every maximal ideal $m \subseteq R$ and $\{I_t\}$ a descending sequence of irreducible m-primary ideals cofinal with the powers of m, $I_t^* = I_t$ in R_m for all t.

The following notion of a test ideal is central to the study of tight closure of ideals.

Definition II.10. The *test ideal* of R is defined to be $\tau(R) = \bigcap_{N \subseteq M}(N :_R N_M^*)$ where the intersection is taken over all finitely generated R-modules $N \subseteq M$.

Theorem II.11 ([HH90] Proposition 8.3 (f)). *Let $(R, \mathfrak{m}, K)$ be a reduced local ring. Then $\tau(R) = \bigcap_{I \subseteq R}(I : I^*)$, where the intersection is taken over all ideals of R or equivalently over all irreducible $\mathfrak{m}$-primary ideals.*

Lemma II.12 ([Nag72] Theorem 18.1). *Let $A \to B$ be a flat ring homomorphism. Let I and J be ideals of A such that J is finitely generated. Then $(I :_R J)S = IS :_S JS$.*

Theorem II.13 (Persistence [HH94a] Theorem 6.24). *Let $A \to B$ be a homomorphism of Noetherian rings of prime characteristic $p > 0$. Let $N \subseteq M$ be finitely* *generated A-modules and let $w \in M$ be an element of M in N^*. Suppose also that B is a complete local ring. Then $1 \otimes w$ is in the tight closure of the image of $B \otimes N$ in $B \otimes M$.*

Definition II.14. A map of R-modules $S \to T$ is said to be *pure* if for every R-module M, the induced map $M \otimes S \to M \otimes T$ is injective.

The following notion was first defined in the works of M. Hochster and J.Roberts, [HR74], [HR76] and was extensively studied by Fedder and Watanabe, see [Fed97] and [FW89].

Definition II.15. A ring R is *F-pure* if the Frobenius endomorphism F is pure.

Remark II.16. If R is F-pure, then $F : R \to R$ is injective and hence R is necessarily reduced.

The Frobenius endomorphism can be used to define another closure operation, namely the Frobenius closure.

Definition II.17. An element $u \in R$ is in I^F, the *Frobenius closure* of an ideal I, if there exists q such that $u^{[q]} \in I^{[q]}$.

It is indeed a closure operation, and properties similar to those in Proposition II.2 can be easily verified. The following are obvious but important properties of F-pure rings.

Theorem II.18. *Suppose R is F-pure. Then every ideal of R is Frobenius closed.*

Theorem II.19. *A weakly F-regular ring is F-pure.*

Remark II.20. There are rings which are F-pure but not F-rational and vice versa, [Wat88], [Wat91].

2.2 Plus-closure

In this section we look into the notion of plus-closure, nice properties of which we shall utilize in order to attack the problem on the localization of weakly F-regular rings.

Definition II.21. Let R be an integral domain. Then by R^+ we shall denote the integral closure of R in the algebraic closure of its fraction field, it is called the *absolute integral closure* of R.

Definition II.22. Let I be an ideal of an integral domain R. Then $I^+ = IR^+ \bigcap R$ is the *plus-closure* of I.

Unlike the tight closure of ideals, the operation of plus-closure commutes with localization.

Theorem II.23 ([HH92] Lemma 6.5)**.** *Let U be any multiplicative system in a commutative ring R. Then $I^+(U^{-1}R) = (U^{-1}(IR))^+$.*

However, the following result shows that for parameter ideals tight closure does commute with localization.

Definition II.24. Elements $x_1, \ldots, x_n \in R$ are called *parameters*, if their images form a part of a system of parameters in every local ring R_P of R such that $x_1, \ldots, x_n \in P$.

Definition II.25. An ideal $I \subseteq R$ is called a *parameter ideal* if it is generated locally (at each maximal ideal) by parameters.

Theorem II.26 ([Smi94] Theorem 5.1)**.** *Let R be a locally excellent Noetherian domain of characteristic $p > 0$. Let I be a parameter ideal. Then $I^* = I^+$.*

Here is a very interesting characterization of rings where this closure operation is equal to the identity closure.

Theorem II.27 ([Hoc73]). *Let R be a Noetherian ring of positive characteristic p. Then every ideal I of R is plus-closed if and only if R splits from its every module-finite extension.*

Rings as in the above theorem are called *splinters*.

We shall return to the defined closure operations once we give preliminaries for the machinery we shall need in order to attack our problem.

2.3 Local cohomology

The purpose of this section is to define necessary notions from the theory of local cohomology that are going to be used throughout this thesis. For a more detailed treatment of the material and the proofs of statements see [BH98].

In this section let R denote a Noetherian commutative ring with identity without any restrictions on the characteristic.

2.3.1 Local cohomology modules

Let I be an ideal of R and M be any R-module. Then we have natural surjective maps $R/I^{t+1} \to R/I^t$ for all $t \in \mathbb{N}$. Hence, we get induced maps

$$\mathrm{Ext}_i^R(R/I^{t+1}, M) \to \mathrm{Ext}_i^R(R/I^t, M)$$

for all $i \in \mathbb{N}$, where $\mathrm{Ext}(\_\_, M)$ is the right derived functor of $\mathrm{Hom}_R(\_\_, M)$.

Definition II.28. $\mathrm{H}_I^i(M) = \varinjlim_t \mathrm{Ext}_i^R(R/I^t, M)$ is the ith *local cohomology module* of M with support at I.

The following proposition summarizes some of main properties of the local cohomology modules that we are going to use.

Proposition II.29. *The following is true for all R-modules M and for all $i \in \mathbb{N}$*

(1) Let I and J be ideals of R with the same radical, then $H_I^i(M) = H_J^i(M)$.

(2) Every element of $H_I^i(M)$ is killed by a power of I.

(3) $H_I^0(M) = \bigcup_{t \geqslant 1} \operatorname{Ann}_M I^t$.

(4) If I is generated by n elements up to radical, then $H_I^i(M) = 0$ for all $i > n$.

(5) If $N \to M$ is a map of R-modules, then there exists an induced map $H_I^i(N) \to H_I^i(M)$.

(6) If $0 \to N \to M \to Q \to 0$ is s short exact sequence of R-modules, then there is a long exact sequence of local cohomology

$$\ldots \to H_I^i(Q) \to H_I^{i+1}(N) \to H_I^{i+1}(M) \to H_I^{i+1}(Q) \to \ldots$$

Local cohomology modules with support at a maximal ideal are of particular interest due to their properties which allow us to reduce our study to a local and/or complete case.

Proposition II.30. *Let m denote a maximal ideal of R and let M be an R-module*

(1) $H_m^i(M) \cong H_{mR_m}^i(M_m)$.

(2) If $(R, \mathfrak{m}, K)$ is a local ring, then $H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}\widehat{R}}^i(M \otimes_R \widehat{R})$, where $\widehat{R}$ is the $\mathfrak{m}$-adic compeletion of R.

When a ring R is Cohen-Macaulay, the local cohomology modules have especially nice properties, which in fact characterize the property of being a Cohen-Macaulay local ring.

Theorem II.31. *Let $(R, \mathfrak{m}, K)$ be a local Cohen-Macaulay ring of Krull dimension d. Then $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq d$ and is nonzero for $i = d$. Moreover, let $x_1, \ldots, x_d$ be a system of parameters for R and $I_t = (x_1^t, \ldots, x_d^t)R$ for all positive integers t. Then $H^d_{\mathfrak{m}}(M) \cong \varinjlim_t R/I_t$ where the direct limit system is injective and the maps $R/I_t \to R/I_{t+1}$ are defined by multiplication by $x_1 \ldots x_d$.*

2.3.2 Injective hulls and Matlis duality

Definition II.32. Let $\phi : N \to M$ be a homomorphism of finitely generated R-modules. It is said to be *essential* if one of the following three equivalent conditions holds

(a) Every nonzero submodule of M has a nonzero intersection with $\phi(N)$.

(b) Every element of M has a nonzero multiple in $\phi(N)$.

(c) For every R-module homomorphism $\theta : M \to Q$, if $\theta \circ \phi$ is injective then ϕ is injective.

Definition II.33. Let $(R, \mathfrak{m}, K)$ be a local ring and let M be an R-module. Then the *socle*, $\operatorname{Soc} M$, of M is $\operatorname{Ann}_M \mathfrak{m}$.

Example. $\operatorname{Soc} M \subseteq M$ is an essential extension.

Below are the basic important properties of essential extensions

Proposition II.34. *Let M, N and Q be R-modules.*

(1) The identity map on M is an essential extension.

(2) If $N \subseteq M \subseteq Q$, then $N \subseteq Q$ is essential if and only if $N \subseteq M$ and $M \subseteq Q$ are essential.

(3) If $N \subseteq M$, then there exists a maximal submodule N' such that $N \subseteq N'$ is essential.

One can prove that every module has a maximal essential extension, which is in fact an injective module.

Theorem II.35. *Let M be an R-module. Then there exists a maximal essential extension of M , and it is unique up to non-canonical isomorphism. Moreover, it is injective as an R-module.*

Such an extension is called an *injective hull* of M and denoted by $E_R(M)$.

Injective hulls of prime cyclic modules R/P are the building blocks for injective modules, more precisely we have the following proposition.

Proposition II.36. *Let E be an injective R-module. Then $E = \oplus E_R(R/P)$ for varying prime ideals P of R. And all such $E_R(R/P)$ are indecomposable.*

It is natural in commutative algebra to reduce our study to the local case, and injective hulls of prime cyclic modules behave quite well in this respect.

Lemma II.37. *For every prime ideal P of R:*

(a) $E_R(R/P) \cong E_{R_P}(R_P/PR_P)$.

(b) $\operatorname{Ass} E_R(R/P) = \{P\}$ *and every element of the injective hull is killed by a power of P.*

Therefore, an injective hull of a residue field of a local ring is of particular interest. So for the rest of the section let us assume that $(R, \mathfrak{m}, K)$ is a local ring and let $E = E_R(K)$. The following theorem summarizes important properties of E.

Theorem II.38. *Let $(R, \mathfrak{m}, K)$ be a local ring.*

(1) $E_R(K) \cong E_{\hat{R}}(K)$.

(2) If $S = R/I$ for an ideal $I \subseteq R$, then $E_S(K) \cong \operatorname{Ann}_E I$.

(3) $\widehat{R} \cong \mathrm{Hom}_R(E, E)$.

(4) E *has DCC as an* R*-module.*

(5) Let $\{I_t\}$ *be a descending chain of irreducible ideals of* R *cofinal with the powers of* $\mathfrak{m}$*. Then* $E \cong \varinjlim_t R/I_t$*, where the maps* $R/I_t \to R/I_{t+1}$ *are injective.*

(6) If R *is Gorenstein of dimension* d*,* $H^d_{\mathfrak{m}}(R) \cong E$.

The concept of Matlis duality was first defined by E.Matlis in the 1950s and has proved to be a very powerful tool in studying local cohomology modules.

Definition II.39. Let M be an R-module. Then the *Matlis dual* $M^\vee$ of M is $\mathrm{Hom}_R(M, E)$.

Remark II.40. $\mathrm{Hom}_R(\_\_, E)$ is an exact covariant functor.

Theorem II.41 (Matlis duality)**.** *Let* $(R, \mathfrak{m}, K)$ *be a complete local ring and* M *be an* R*-module.*

(1) If M *has ACC, then* $M^\vee$ *has DCC. If* M *has DCC, then* $M^\vee$ *has ACC. In either case, the obvious map* $M \to M^{\vee\vee}$ *is an isomorphism.*

(2) As a functor from the set of R*-modules to itself,* $\_\_^\vee$ *gives an anti-equivalence of categories of modules with* ACC *and* DCC*.*

2.3.3 Canonical modules

Definition II.42. Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay ring of dimension d. Then a finitely generated module ω is called a *canonical module* of R if $\omega^\vee \cong \mathrm{H}^d_{\mathfrak{m}}(R)$.

Remark II.43. There may not exist a canonical module for an arbitrary ring R and there are such examples. However, if the ring is complete, there is always one, namely the Matlis dual of $\mathrm{H}^d_{\mathfrak{m}}(R)$.

Here are some useful properties of the canonical modules.

Theorem II.44. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay ring of dimension d.*

(1) *A finitely generated R-modules ω is a canonical module for R if and only if $\widehat{\omega}$ is a canonical module for $\widehat{R}$.*

(2) *If ω and ω' are canonical modules for R, then $\omega \cong \omega'$.*

(3) *A canonical module ω_R for R is a torsion-free Cohen-Macaulay module of dimension d.*

(4) *R is Gorenstein if and only if $\omega_R \cong R$.*

(5) *For every prime ideal P of R, if ω is a canonical module, then ω_P is a canonical module for R_P.*

(6) *If R is a domain which has a canonical module, then there exists a pure height one ideal $J \subseteq R$ such that the canonical module for R is isomorphic to J as an R-module.*

Moreover, in the situation of the above theorem in part (6), we have that the quotient R/J has particularly nice properties.

Lemma II.45 ([BH98] Proposition 3.3.18)**.** *Let R be a Cohen-Macaulay ring with a canonical module isomorphic to an ideal J. Then R/J is a Gorenstein ring of dimension $d - 1$.* □

Furthermore, it is known that sometimes good properties of R/J can be lifted back to R, see [Ene03], [DSNB].

Lemma II.46 ([Ene03] Corollary 2.9)**.** *Let $(R, \mathfrak{m}, K)$ be an F-finite Cohen-Macaulay local domain such that there exists an ideal $J \subseteq R$ isomorphic to the canonical module of R. Suppose that R/J is F-rational. Then R is F-regular.* □

CHAPTER III

Tight closure and localization problems

3.1 Quasi-parameter ideals

In this section we define the notion of quasi-parameter ideals and then use it to give a sufficient condition for weak F-regularity.

First, let us define the set-up in which we shall work throughout this section.

Assumption III.1. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay local domain of Krull dimension d with a canonical module ω. Let J denote a pure height one ideal of R such that $J \cong \omega$.*

Notation III.2. Let $x_1, \ldots, x_{d-1} \in R$ be part of a system of parameters in R that also form a system of parameters for R/J, and let $x_d \in J$ be such that $\{x_1, \ldots, x_d\}$ is a system of parameters for R. Let $I_t = (x_1^t, \ldots, x_{d-1}^t)R + x_d^t J \subseteq R$ and let $\mathfrak{A}_t = (x_1^t, \ldots, x_{d-1}^t)R + J$. Notice that $x_d^t \mathfrak{A}_t \subset I_t \subset \mathfrak{A}_t$.

Next we give the following definition of quasi-parameter ideals.

Definition III.3. Let R be as in Assumption III.1 and let $x_1, \ldots, x_i$ be part of a system of parameters for R whose images form part of a system of parameters in R/J, then we call $(x_1, \ldots, x_i) + J$ a *quasi-parameter ideal* for J. All the $(x_1^t, \ldots, x_i^t) + J$ are also quasi-parameter ideals for J. If $i = d - 1$, then we say that $(x_1, \ldots, x_{d-1}) + J$

is a *full quasi-parameter ideal* for J. If R is a Gorenstein local ring, then J is a principal ideal and all the quasi-parameter ideals are simply parameter ideals.

To make use of our definition we shall study the properties of quasi-parameter ideals $\mathfrak{A}_t$ in connection with those of I_t.

Lemma III.4. *Let R be as in Assumption III.1 and let I_t be ideals defined as in Notation III.2. Then the sequence of ideals $I_t = (x_1^t, \dots, x_{d-1}^t)R + x_d^t J \subseteq R$ is a descending sequence of irreducible ideals cofinal with the powers of the maximal ideal $\mathfrak{m}$ of R.*

Proof. Clearly, $I_t \subseteq \mathfrak{m}^t$ are all $\mathfrak{m}$-primary, so for every t there is a large enough power of $\mathfrak{m}$ contained in I_t . Moreover, for every $t > 0$, R/I_t are Gorenstein 0-dimensional, hence the irreducibility of I_t. $\square$

Remark III.5. The socle of R/I_t is one dimensional. Let u_t be its generator. Hence, $Ku_t \subseteq R/I_t$ is an essential extension and every nonzero ideal of R/I_t contains u_t.

Remark III.6. Since $x_1^t, \dots, x_d^t$ is a system of parameters for R, we have injective maps $R/I_t \to R/I_{t+1}$ defined by multiplication by $x_1 \dots x_d$. Moreover, if u is a socle generator for R/I_1, then $(x_1 \dots x_d)^{t-1}u$ is a socle generator for R/I_t.

Thus if one wants to prove that R is a weakly F-regular ring, it is necessary and sufficient to show that $I_t^* = I_t$ for all $t > 0$, see Theorem II.9. Moreover, the following lemma allows us to reduce the problem to another family of ideals, namely quasi-parameter ideals $\mathfrak{A}_t$, which is relatively easier to handle.

Lemma III.7. *Let R be as in Assumption III.1 and let I_t and $\mathfrak{A}_t$ be ideals as in Notation III.2. If all of the ideals $\mathfrak{A}_t = (x_1^t, \dots, x_{d-1}^t)R + J$ are tightly closed in R then so are all the I_t for all $t > 0$.*

Proof. Since $x_d^t \mathfrak{A}_t \subseteq I_t$, we have a well defined homomorphism of R modules

$$\theta : R/\mathfrak{A}_t \longrightarrow R/I_t$$

given by multiplication by x_d^t. In fact, it is injective. Let $z \in R$ and suppose $x_d^t z \in I_t$. Then there is an element $j \in J$ such that $(z - j)x_d^t \in (x_1^t, \ldots, x_{d-1}^t)$. Since $x_1, \ldots, x_d$ form a regular sequence in R, $z \in (x_1^t, \ldots, x_{d-1}^t) + J = \mathfrak{A}_t$.

Let u be a socle generator in $R/((x_1, \ldots, x_{d-1}) + J)$. Then $u_t = (x_1 \cdots x_{d-1})^{t-1} u$ is a socle generator in $R/((x_1^t, \ldots, x_{d-1}^t) + J)$. We claim that $(x_1 \ldots x_{d-1})^{t-1} x_d^t u$ is a socle generator in R/I_t. For every element $r \in \mathfrak{m}$, we have that

$$r(x_1 \ldots x_{d-1})^{t-1} x_d^t u = r\theta((x_1 \ldots x_{d-1})^{t-1} u) = \theta(r(x_1 \ldots x_{d-1})^{t-1} u) = 0$$

and, since θ is injective, $(x_1 \ldots x_{d-1})^{t-1} x_d^t u \notin I_t$.

Since R/I_t is a 0-dimensional Gorenstein ring, to prove the lemma it is sufficient to show that $(x_1 \ldots x_{d-1})^{t-1} x_d^t u \notin ((x_1^t, \ldots, x_{d-1}^t)R + x_d^t J)^*$, see Remark III.5. Suppose to the contrary that $(x_1 \ldots x_{d-1})^{t-1} x_d^t u \in ((x_1^t, \ldots, x_{d-1}^t)R + x_d^t J)^*$, then there exists $c \in R^\circ$ such that for all $q \gg 0$

$$c(x_1 \ldots x_{d-1})^{qt-q} x_d^{qt} u^q \in (x_1^{tq}, \ldots, x_{d-1}^{qt})R + x_d^{qt} J^{[q]},$$

$$(c(x_1 \ldots x_{d-1})^{qt-q} u^q - j_q) x_d^{qt} \in (x_1^{tq}, \ldots, x_{d-1}^{qt})R,$$

for some $j_q \in J^{[q]}$.

Since $x_1^{qt}, \ldots, x_d^{qt}$ form a regular sequence in R,

$$cu_t^q = c(x_1 \ldots x_{d-1})^{qt-q} u^q \in (x_1^{tq}, \ldots, x_{d-1}^{qt})R + J^{[q]}.$$

Hence, $u_t \in ((x_1^t, \ldots, x_{d-1}^t)R + J)^* = (x_1^t, \ldots, x_{d-1}^t)R + J.$

This is a contradiction with the fact that u_t is a socle generator modulo $\mathfrak{A}_t$. □

The next result is an immediate consequence of the previous lemma.

Theorem III.8. *Let R be as in Assumption III.1 and let $\{\mathfrak{A}_t\}_t$ be full quasi-parameter ideals defined as in Notation III.2. If every full quasi-parameter ideal for J, i.e., every $\mathfrak{A}_t$, is tightly closed in R then R is weakly F-regular.*

In fact, we conclude a stronger statement.

Theorem III.9. *Let R be as in Assumption III.1. Suppose that for every quasi-parameter ideal of R, tight closure commutes with localization. Suppose also that R is weakly F-regular. Then R is F-regular.*

Proof. Let Q be a prime ideal of R of height h. If $J \nsubseteq Q$, then $JR_Q = R_Q$ and R_Q is Gorenstein. Choose $x_1, \ldots, x_h \in Q$ part of a system of parameters in R such that their images form a system of parameters in R_Q. By Theorem II.26, $(x_1, \ldots, x_h)^* R_Q = ((x_1, \ldots, x_h)R_Q)^*$. Hence, R_Q is F-rational and, therefore, is F-regular. Now assume that $J \subseteq Q$. Choose $x_1, \ldots, x_{h-1} \in Q$ part of a system of parameters for R whose image form part of a system of parameters in R_Q and R/J. Then $(x_1^t, \ldots, x_{h-1}^t) + J$ are quasi-parameter ideals for J. Hence $((x_1^t, \ldots, x_{h-1}^t) + J)^* R_Q = ((x_1^t, \ldots, x_{h-1}^t) + J)R_Q)^*$. Finally, by Theorem III.8, R_Q is weakly F-regular. $\square$

We would have that tight closure of quasi-parameter ideals commutes with localization if tight closure agrees with plus-closure for quasi-parameter ideals. Thus we have a natural question.

Question III.10. *Let R be as in Assumption III.1. Is it true that for all quasi-parameter ideals of R tight closure agrees with plus-closure?*

3.2 Comparison of $0^*_{E_R(K)}$ and $0^*_{E_{R/J}(K)}$

For a local ring $(R, \mathfrak{m}, K)$, properties of $0^*_{E_R(K)}$ play an important role in the study of localization problems of tight closure. In this section we use the notion of quasi-parameter ideals to reduce the study of 0^{*R} in $E_R(K)$ to 0^{*R} in $E_{R/J}(K)$, where J shall denote an ideal of R isomorphic to the canonical module.

First, we define the hypothesis on the ring R which we shall assume throughout the section.

Assumption III.11. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macaulay excellent local domain of Krull dimension d with the canonical module ω_R and let $J \subset R$ be a proper ideal of R isomorphic to ω_R. Let E denote the injective hull $E_R(K)$. Recall that J has pure height one in R and R/J is a Gorenstein ring of dimension $d-1$.*

We shall also use the following notation.

Notation III.12. Let $x_1, \ldots, x_{d-1} \in \mathfrak{m}$ be part of a system of parameters in R so that their images form a system of parameters for R/J. Let $x_d \in J$ be such that $\{x_1, \ldots, x_{d-1}, x_d\}$ is a full system of parameters for R. Let $I_t = (x_1^t, \ldots, x_{d-1}^t) + x_d^t J$ and $\mathfrak{A}_t = (x_1^t, \ldots, x_{d-1}^t) + J$ for all positive integers t.

Important! Unless otherwise stated, all the tight closure operations performed in this section are over the ring R.

Lemma III.13. *Let R be as in Assumption III.11 and let I_t and $\mathfrak{A}_t$ be ideals defined as in Notation III.12. Then $x_t^d \mathfrak{A}_t \subset I_t \subset \mathfrak{A}_t \subset I_t : x_t^d$. Moreover, the following are well defined injective homomorphisms of R-modules*

(1)

$$R/\mathfrak{A}_t \xrightarrow{\cdot x_d^t} R/I_t$$

(2)

$$\mathfrak{A}_t^*/\mathfrak{A}_t \xrightarrow{\cdot x_d^t} I_t^*/I_t$$

(3)

$$R/I_t \xrightarrow{\cdot(x_1 \ldots x_d)} R/I_{t+1}$$

(4)

$$I_t^*/I_t \xrightarrow{\cdot(x_1 \ldots x_d)} I_{t+1}^*/I_{t+1}$$

(5)

$$R/\mathfrak{A}_t \xrightarrow{\cdot(x_1 \ldots x_{d-1})} R/\mathfrak{A}_{t+1}$$

(6)

$$\mathfrak{A}_t^*/\mathfrak{A}_t \xrightarrow{\cdot(x_1 \ldots x_{d-1})} \mathfrak{A}_{t+1}^*/\mathfrak{A}_{t+1}$$

Proof.

(1) Since $x_d^t \mathfrak{A}_t \subseteq I_t$, we have a well defined homomorphism of R modules $R/\mathfrak{A}_t \longrightarrow R/I_t$ given by multiplication by x_d^t. Let $z \in R$ and suppose $x_d^t z \in I_t$. Then there is an element $j \in J$ such that $(z-j)x_d^t \in (x_1^t, \ldots, x_{d-1}^t)$. Since $x_1, \ldots, x_d$ form a regular sequence in R, $z \in (x_1^t, \ldots, x_{d-1}^t) + J = \mathfrak{A}_t$.

(2) Let $z \in \mathfrak{A}_t^*$, then there exists $c \in R^\circ$ so that $cz^q \in \mathfrak{A}_t^{[q]} = (x_1^{tq}, \ldots, x_{d-1}^{tq}) + J^{[q]}$ for all $q \gg 0$. Therefore, $c(x_d^t z)^q \in (x_1^{tq}, \ldots, x_{d-1}^{tq}) + x_d^{tq} J^{[q]} = I_t^{[q]}$. Thus $z \in I_t^*$.

(3) Since $(x_1 \ldots x_d)I_t \subseteq I_{t+1}$, we have a well defined homomorphism of the quotients. Let $z \in R$ and suppose that $(x_1 \ldots x_d)z \in I_{t+1} = (x_1^{t+1}, \ldots, x_{d-1}^{t+1}) + x_d^{t+1} J$. Then there exist an element j of J such that $(x_1 \ldots x_{d-1} z - x_d^t j)x_d \in I_{t+1} = (x_1^{t+1}, \ldots, x_{d-1}^{t+1})$. Recall that $x_1, \ldots, x_d$ form a regular sequence on R. Therefore, $(x_1 \ldots x_{d-1})z \in (x_1^{t+1}, \ldots, x_{d-1}^{t+1}) + x_d^t J$. Working modulo $x_d^t J$, we get that $(\bar{x}_1 \ldots \bar{x}_{d-1})\bar{z} \in (\bar{x}_1^{t+1}, \ldots, \bar{x}_{d-1}^{t+1})$. Hence, $\bar{z} \in (\bar{x}_1^t, \ldots, \bar{x}_{d-1}^t)$ and $z \in I_t$.

(4) Let $z \in I_t^*$, then there exists $c \in R^\circ$ such that $cz^q \in I_t^{[q]} = (x_1^{tq}, \ldots, x_{d-1}^{tq}) + x_d^{tq} J^{[q]}$ for all q sufficiently large. Hence $c(x_1 \ldots x_d z)^q \in (x_1^{(t+1)q}, \ldots, x_{d-1}^{(t+1)q}) + x_d^{(t+1)q} J^{[q]} = I_{t+1}^{[q]}$ and $x_1 \ldots x_d z \in I_{t+1}^*$.

(5) Use the facts that R/J is Gorenstein and $x_1, \ldots, x_{d-1}$ form a regular sequence on R/J.

(6) Let $z \in \mathfrak{A}_t^*$, then there exists $c \in R^\circ$ such that $cz^q \in \mathfrak{A}_t^{[q]} = (x_1^{tq}, \ldots, x_{d-1}^{tq}) + J^{[q]}$ for all q sufficiently large. Hence $c(x_1 \ldots x_{d-1} z)^q \in (x_1^{(t+1)q}, \ldots, x_{d-1}^{(t+1)q}) + J^{[q]} = \mathfrak{A}_{t+1}^{[q]}$ and $x_1 \ldots x_{d-1} z \in \mathfrak{A}_{t+1}^*$. □

Lemma III.14. *Let R be as in Assumption III.11 and let I_t and $\mathfrak{A}_t$ be ideals defined as in Notation III.12. The following two diagrams of R-modules and injective R-modules homomorphisms commute*

$$\begin{tikzcd}
R/\mathfrak{A}_t \arrow[r, "\cdot x_d^t"] \arrow[d, "\cdot(x_1 \cdot \ldots \cdot x_{d-1})"'] & R/I_t \arrow[d, "\cdot(x_1 \ldots x_d)"] \\
R/\mathfrak{A}_{t+1} \arrow[r, "\cdot x_d^{t+1}"'] & R/I_{t+1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{A}_t^*/\mathfrak{A}_t \arrow[r, "\cdot x_d^t"] \arrow[d, "\cdot(x_1 \ldots x_{d-1})"'] & I_t^*/I_t \arrow[d, "\cdot(x_1 \ldots x_d)"] \\
\mathfrak{A}_{t+1}^*/\mathfrak{A}_{t+1} \arrow[r, "\cdot x_d^{t+1}"'] & I_{t+1}^*/I_{t+1}
\end{tikzcd}$$

and induce a commutative diagram of direct limits

$$\begin{tikzcd}
\varinjlim_t \mathfrak{A}_t^*/\mathfrak{A}_t \arrow[r] \arrow[d] & \varinjlim_t I_t^*/I_t \arrow[d] \\
H_{\mathfrak{m}}^{d-1}(R/J) \cong \varinjlim_t R/\mathfrak{A}_t \arrow[r] & E_R(K) \cong \varinjlim_t R/I_t
\end{tikzcd}$$

Proof. The proof is clear from the previous lemma. □

Notation III.15. Let R be as in Assumption III.11 and let I_t and $\mathfrak{A}_t$ be ideals defined as in Notation III.12. Let $W_J = \varinjlim_t \mathfrak{A}_t^*/\mathfrak{A}_t$ and $W = \varinjlim_t I_t^*/I_t$.

Remark III.16. $W_J \cong 0^{*Rfg}_{E_{R/J)(K)}}$ and $W \cong 0^{*Rfg}_{E}$.

Proof. First observe that $H^{d-1}_{\mathfrak{m}}(R/J) \cong E_{R/J}(K)$, since R/J is Gorenstein. It is sufficient to show that $0^{*Rfg}_{E} \subseteq W$. Let $z \in 0^{*Rfg}_{E}$, then there exists a finitely generated submodule $N \subseteq E$ such that $z \in 0^*_N$. Moreover, since the direct limit system defining E is injective, $N \subseteq R/I_t$ for $t \gg 0$. Hence $z \in 0*_{R/I_t}$. Finally by properties of tight closure $z \in I_t^*/I_t$. The proof for W_J is similar. □

The next theorem is due to J. Stubbs and it will help us to compare the tight closure of 0 in E and $E_{R/J}(K)$.

Theorem III.17 ([Stu08]). *Let $(R, \mathfrak{m}, K)$ be a complete local ring and let V be an Artinian R-module. Let M be the Matlis dual of V and let I define the non-finite injective dimension locus in M. If $u \in 0^*_V$, then $u \in 0^*_{\mathrm{Ann}_V I^t}$ for all t sufficiently large.*

We shall use the injective hull E as the Artinian module in Stubbs's theorem. Its Matlis dual is R. When the ring R is not Gorenstein, it does not have a finite injective dimension over itself. For any prime ideal $P \subseteq R$, R_P is Gorenstein if and only if it has a finite injective dimension over itself if and only if $JR_P \cong R_P$ for all choices of J isomorphic to the canonical module. Moreover, for all $r \in J$, we have that R_r is Gorenstein. Hence J is contained in the ideal defining the non-finite injective dimension locus for R.

Theorem III.18. *Let $(R, \mathfrak{m}, K)$ be a Cohen-Macalay complete local domain with the canonical module ω_R. If $u \in 0^{*R}_{E_R(K)}$, then there exists a choice of an ideal $J_u \cong \omega_R$*

such that $u \in 0^{*R}_{E_{R/J_u}(K)}$ *over* R.

Proof. Let E denote the injective hull $E_R(K)$.

Let I denote the ideal defining the non-finite injective dimension locus for R. Then for any proper ideal $J \cong \omega_R$, we have that $J \subseteq I$. Let J be any ideal of R isomorphic to ω_R. By Stubbs's theorem, there exists a natural number n such that if $u \in 0^*_E$, then $u \in 0^*_{\operatorname{Ann}_E I^n}$. Moreover, $0^*_{\operatorname{Ann}_E I^n} \subseteq 0^*_{\operatorname{Ann}_E J^n}$. Let a be any nonzero element in J^n. Then, since $aJ \subseteq J^n$, we have that $u \in 0^*_{\operatorname{Ann}_E aJ}$. Recall that $E_{R/J}(K) \cong \operatorname{Ann}_E J$. Therefore, $u \in 0^*_{E_{R/(aJ)}(K)}$. Since $aJ \cong J \cong \omega_R$, we have the result. □

It is known that E has DCC as an R-module. If, in addition, 0^*_E is of finite length, then we have the following corollary.

Corollary III.19. *Let* 0^*_E *be a module of finite length. Then* $0^*_E \cong 0^*_{E_{R/J}(K)}$ *for some* $J \cong \omega$.

Proof. Let $u_1, \ldots, u_m$ be a generating set of 0^*_E as an R-module. Then there exist natural numbers $n_1, \ldots, n_m$ such that $u_i \in 0^*_{\operatorname{Ann}_E I^{n_i}}$ for all $1 \leqslant i \leqslant m$. Take $n = \max\{n_i \mid 1 \leqslant i \leqslant m\}$. Then we have that $0^*_E \subseteq 0^*_{\operatorname{Ann}_E I^n}$. Finally, let $a \in J^n$ for any $J \cong \omega_R$. Then $0^*_E \subseteq 0^*_{E_{R/(aJ)}(K)}$. Hence the result. □

We always have that $0^{*R}_{E_{R/J}(K)} \subseteq 0^{*R}_E \subseteq E$ for all choices of $J \cong \omega_R$. Therefore, we may form $\sum_{J \cong \omega_R} 0^{*R}_{E_{R/J}(K)}$ inside E. Denote $\sum_{J \cong \omega_R} 0^{*R}_{E_{R/J}(K)}$ by O^*_E. Thus we have that $O^*_E \subseteq 0^*_E$. By applying the above theorem we get that $O^*_E = 0^*_E$.

Furthermore, we are motivated to give the following definition.

Definition III.20. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of positive prime characteristic p. Let I be the defining ideal of the non-finite injective dimension locus of R and let E denote the injective hull of K over R. We say that R is *amenable* if 0^*_E is an R-module of finite length.

Remark III.21. Corollary III.19 holds for an amenable Cohen-Macaulay complete local domain.

It is a natural question if Stubbs's theorem has analogue for finitistic tight closure. This leads us to the following definition.

Definition III.22. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring of positive prime characteristic p. Let I be the defining ideal of the non-finite injective locus of R and let E denote the injective hull of K over R. Then we say that R is *finitistically amenable* if $0_E^{*fg} \cong 0^{*fg}_{\operatorname{Ann}_E I^t}$ for all t sufficiently large and 0_E^* has finite length as an R-module.

Immediately, we get a result whose proof is identical to that of the Theorem III.18.

Theorem III.23. *Let $(R, \mathfrak{m}, K)$ be a finitistically amenable Cohen-Macaulay complete local domain with the canonical module ω_R. Then there exists a choice of an ideal $J \subset R$ isomorphic to ω_R so that $0^{*Rfg}_{E_R(K)} \cong 0^{*Rfg}_{E_{R/J}(K)}$.* □

Therefore, the modules we have defined earlier, W and W_J, are in fact the same for some choice of J when the ring R has nice properties.

Corollary III.24. *Let R be a finitistically amenable ring with the hypotheses defined as in Assumption III.1 and in Notation III.2. Then $W \cong W_J$ for some choice of $J \cong \omega_R$.*

Proof. Recall that $W \cong 0_E^{*fg}$ and $W_J \cong 0^{*fg}_{E_{R/J}(K)}$, see Remark III.16. Then apply Theorem III.23. □

We have that R is weakly F-regular if and only if all the ideals $\{I_t\}$ are tightly closed in R, and the fact that all the ideals $\{\mathfrak{A}_t\}$ are tightly closed is a sufficient condition. Hence we have the following result.

Corollary III.25. *Let R be as in Assumption III.1. Then $W = 0$ if and only if $W_J = 0$ for all choices of J.* $\square$

3.3 The J-test ideal

In this section we develop a notion of the J-test ideal which arises naturally in the course of working on the Conjecture I.2 stated in Chapter I. Let us define the condition on a ring R which we use in this section.

Assumption III.26. *Let $(R, \mathfrak{m}, K)$ be Cohen-Macaulay complete local normal domain of Krull dimension d with the canonical module ω_R. Let J be an ideal of R isomorphic to ω_R.*

Let $x_1, \ldots, x_{d-1} \in R$ be part of a system of parameters in R that also form a system of parameters for R/J, and let $x_d \in J$ be such that $\{x_1, \ldots, x_d\}$ is a system of parameters for R. Let $\mathfrak{A}_t = (x_1^t, \ldots, x_{d-1}^t)R + J$ be full quasi-parameter ideals for J. Then we have that $R/\mathfrak{A}_t$ is a Gorenstein ring for all t, as well as R/J.

As before, let $W_J = \varinjlim_t \mathfrak{A}_t^* / \mathfrak{A}_t$.

Throughout this section the operation of tight closure is performed over R.

Definition III.27. Let R be any Noetherian ring of characteristic $p > 0$. Let $\mathcal{B}$ be an ideal of R. Then the $\mathcal{B}$-*test ideal* of R is $\tau_{\mathcal{B}}(R) = \bigcap_{\mathcal{B} \subseteq I} (I :_R I^*)$, where the intersection is taken over all ideals of R containing $\mathcal{B}$.

We are primarily interested in the case when R is a ring as in Assumption III.26 and $J \subset R$ is an ideal isomorphic to the canonical module ω_R.

The next result shows that it is sufficient to consider quasi-parameter ideals for J in order to define $\tau_J(R)$.

Lemma III.28. *Let R be a ring and $\{\mathfrak{A}_t\}$ be a sequence of full quasi-parameter ideals defined as in Assumption III.26. Then $\tau_J(R) = \bigcap_t (\mathfrak{A}_t : \mathfrak{A}_t^*)$.*

Proof. Clearly, $\tau_J(R) \subseteq \bigcap_t (\mathfrak{A}_t : \mathfrak{A}_t^*)$. Suppose that there exists $c \in \bigcap_t (\mathfrak{A}_t : \mathfrak{A}_t^*) - \tau_J(R)$, hence, there exist an ideal $\mathfrak{A}$ of R and an element $u \in \mathfrak{A}^* - \mathfrak{A}$, so that $cu \notin \mathfrak{A}$. Choose an ideal $\mathfrak{B}$ of R maximal with respect to the property $\mathfrak{B} \supseteq \mathfrak{A}$ and $cu \notin \mathfrak{B}$. Then $R/\mathfrak{B}$ is a module of finite length and the image $\bar{u}$ of u belongs to every nonzero ideal of $R/\mathfrak{B}$ and spans its socle. Therefore, $R/\mathfrak{B}\,\bar{u} \cong K\bar{u} \hookrightarrow R/\mathfrak{B}$ is an essential extension of R-modules. Then $R/\mathfrak{B} \subseteq E_{R/J}(K) \cong \bigcup_t R/\mathfrak{A}_t$. Finally, we get that $R/\mathfrak{B} \hookrightarrow R/\mathfrak{A}_t$ for all $t \gg 0$ and $\bar{u}$ corresponds to a socle element of $R/\mathfrak{A}_t$. Thus $u \in \mathfrak{A}_t^*$ and $cu \in \mathfrak{A}_t$ for all $t \gg 0$, which contradicts our choice of $\mathfrak{B}$. □

Next we shall relate the notions of W_J and $\tau_J(R)$. In fact, we show that the J-test ideal is the annihilator of W_J.

Consider $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J) \cong \varinjlim_t R/((x_1^t, \dots, x_{d-1}^t) + J) = \varinjlim_t R/\mathfrak{A}_t$ which is an injective direct limit system and is defined by multiplication by $x_1 \dots x_{d-1}$. Every element $\eta \in \varinjlim_t R/\mathfrak{A}_t$ can be thought as an equivalence class $[z + \mathfrak{A}_t]$ for some $z \in R$, where $[z + \mathfrak{A}_t] = [w + \mathfrak{A}_s]$ for $s \geqslant t$ if and only if $(x_1 \dots x_{d-1})^{s-t} z - w \in \mathfrak{A}_s$. Moreover, since R/J is Gorenstein, $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J) \cong E_{R/J}(K)$.

Recall that $W_J = \varinjlim_t \mathfrak{A}_t^* / \mathfrak{A}_t$ is a submodule of $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)$. It is also an injective direct limit system defined by multiplication by $x_1 \dots x_{d-1}$, see Lemma III.13. Similarly, every element $\eta \in W_J$ can be represented by its equivalence class $[z + \mathfrak{A}_t]$, for some $z \in \mathfrak{A}_t^*$.

We do not have a natural action of the Frobenius endomorphism F_R on $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)$. However, we can define an endomorphism which has properties very similar to that of the Frobenius.

Lemma III.29. *Let R be as in Assumption III.26. Let $\Psi : H^{d-1}_{\mathfrak{m}}(R/J) \to H^{d-1}_{\mathfrak{m}}(R/J)$ be such that if $\eta = [z + \mathfrak{A}_t] \in H^{d-1}_{\mathfrak{m}}(R/J)$, then $\Psi(\eta) = [z^p + \mathfrak{A}_{pt}]$. Then*

(a) Ψ is a well-defined endomorphism of $H^{d-1}_{\mathfrak{m}}(R/J)$.

(b) W_J is Ψ-stable submodule of $H^{d-1}_{\mathfrak{m}}(R/J)$, i.e., $\Psi(W_J) \subseteq W_J$.

Proof.

(a) Suppose that $[z + \mathfrak{A}_t] = [w + \mathfrak{A}_s] \in \mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)$ with $s \geqslant t$. Then

$$(x_1 \dots x_{d-1})^{s-t} z - w \in \mathfrak{A}_s \,.$$

Hence

$$((x_1 \dots x_{d-1})^{s-t} z - w)^p \in \mathfrak{A}_s^{[p]} \subseteq \mathfrak{A}_{ps} \,.$$

Thus

$$[z^p + \mathfrak{A}_{pt}] = [w^p + \mathfrak{A}_{ps}].$$

(b) For every $\eta = [z + \mathfrak{A}_t] \in W_J$ there exists $c \in R^{\circ}$ such that $cz^q \in \mathfrak{A}_t^{[q]} \subseteq \mathfrak{A}_{qt}$ for all $q \gg 0$. Equivalently, $c(z^p)^q \in \mathfrak{A}_{pt}^{[q]}$ for all q sufficiently large. Thus, $z^p \in \mathfrak{A}_{pt}^*$ and $F(\eta) = [z^p + \mathfrak{A}_{pt}] \in W_J$. □

Similar to the notion of an F-stable ideal in [Smi94], we define Ψ-ideals as follows.

Definition III.30. For a ring R as in Assumption III.26, let $\mathcal{I} \subseteq R$ be an ideal. Then we say that it is a Ψ-*ideal* if $\mathrm{Ann}_{\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)} \mathcal{I}$ is a Ψ-stable submodule of $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)$.

We also get a characterization of Ψ-ideals analogous to the Lemma 4.6 in [Smi94].

Lemma III.31. *Let R be a ring and $\{\mathfrak{A}_t\}$ be a sequence of quasi-parameter ideals defined as in Assumption III.26. Let $\mathcal{I}$ be an ideal of R. Then $\mathcal{I}$ is a Ψ-ideal if and only if for any $z \in R$ if $\mathcal{I}z \subseteq \mathfrak{A}_1$, then $\mathcal{I}z^p \subseteq \mathfrak{A}_p$.*

Proof. For an element $\eta = [z + \mathfrak{A}_1]$ in $\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)$, $\mathcal{I}\eta = [\mathcal{I}z + \mathfrak{A}_1] = 0$ if and only if $\mathcal{I}z \subseteq \mathfrak{A}_1$. Hence, $\eta \in \mathrm{Ann}_{\mathrm{H}^{d-1}_{\mathfrak{m}}(R/J)}\mathcal{I}$. Then $\mathcal{I}$ is a Ψ-ideal if and only if $\Psi(\eta) = [z^p + \mathfrak{A}_p]$ is killed by $\mathcal{I}$. □

The next lemma is analogous to Proposition 4.8 in [Smi94].

Lemma III.32. *Let R be a ring and $\{\mathfrak{A}_t\}$ be a sequence of quasi-parameter ideals defined as in Assumption III.26. If $\mathcal{I}$ is a Ψ-ideal of R, then for any prime ideal P of R, $\mathcal{I}R_P$ is a Ψ-ideal of R_P.*

Proof. The proof is similar to that of Proposition 4.8 in [Smi94]. We use the criterion from the previous lemma to prove the statement.

First let P be a prime ideal of R of height h such that $J \nsubseteq P$. Then $JR_P = R_P$. $\mathcal{I}R_P$ is a Ψ-ideal if $\mathrm{Ann}_{\mathrm{H}^{h-1}_{PR_P}(R_P/(JR_P))}\mathcal{I}$ is Ψ- stable in $\mathrm{H}^{h-1}_{PR_P}(R_P/(JR_P))$. However, $\mathrm{H}^{h-1}_{PR_P}(R_P/(JR_P)) = 0$.

Let P be a prime ideal of R of height h such that $J \subseteq P$. Let $x_1, \ldots, x_{d-1} \in R$ be such that they form a part of a system of parameter ideals for R and a system of parameters for R/J and such that images of $x_1, \ldots, x_{h-1} \in P$ form part of a system of parameters for R_P and a system of parameters for R_P/JR_P. Let $\mathcal{B} = (x_1, \ldots, x_{h-1}) + J$. Then $\mathcal{B}R_P$ is a quasi-parameter ideal of R_P.

Let $z/1 \in R_P$ such that $\mathcal{I}R_P z \subseteq \mathcal{B}R_P$. Then since R is a domain, there exists an element $f \in R - P$ such that $f\mathcal{I}z \subseteq \mathcal{B}$. Hence

$$f\mathcal{I}z \subseteq (x_1, \ldots, x_{h-1}, x_h^N, \ldots, x_{d-1}^N) + J$$

for all $N \geqslant 0$. Since $\mathcal{I}$ is a Ψ-ideal, by Lemma III.31 we have that

$$\mathcal{I}(fz)^p \subseteq (x_1^p, \ldots, x_{h-1}^p, x_h^{Np}, \ldots, x_{d-1}^{Np}) + J.$$

Therefore,

$$\mathcal{I}(fz)^p \subseteq \bigcap_{N=0}^{\infty}((x_1^p, \ldots, x_{h-1}^p, x_h^{Np}, \ldots, x_{d-1}^{Np}) + J).$$

Finally, we have that

$$\mathcal{I}(fz)^p \subseteq (x_1^p, \ldots, x_{h-1}^p) + J.$$

Hence apply Lemma III.31 again to get the desired result. □

Remark III.33. As a submodule of $\mathrm{H}_{\mathfrak{m}}^{d-1}(R/J)$, W_J has DCC over R.

Consider the Matlis dual of W_J.

$$W_J^\vee = \mathrm{Hom}_R(W_J, E) = \mathrm{Hom}(\varinjlim_t \mathfrak{A}_t^* / \mathfrak{A}_t, E) = \varprojlim_t (\mathfrak{A}_t^* / \mathfrak{A}_t)^\vee = \varprojlim_t R/(\mathfrak{A}_t :_R \mathfrak{A}_t^*).$$

Here we used the facts that if $M \subseteq E$, then $M^\vee \cong R/\operatorname{Ann}_R M$ and $R/\mathfrak{A}_t \subseteq E$.

Moreover, since the direct limit system defining W_J is injective, the inverse limit system of $W_J^\vee$ is surjective and hence $\mathfrak{A}_{t+1} : \mathfrak{A}_{t+1}^* \subseteq \mathfrak{A}_t : \mathfrak{A}_t^*$ for all $t > 0$. Therefore,

$$W_J^\vee = R/\bigcap_{t>0}(\mathfrak{A}_t :_R \mathfrak{A}_t^*).$$

Thus we have the following lemma.

Lemma III.34. *Let R be as in Assumption III.26. Then $W_J^\vee = R/\tau_J(R)$.*

Corollary III.35. *Let R be a ring defined as in Assumption III.26. If every ideal of R containing J is tightly closed then R is weakly F-regular.*

The next result is a weaker version of similar statements in [Ene03] and [DSNB], see Lemma II.46.

Corollary III.36. *Let R be a ring defined as in Assumption III.26. If R/J is weakly F-regular, then so is R.*

Proof. Observe first that, by persistence, $\tau(R/J) \subseteq \tau_J(R)(R/J)$, then the result is immediate. □

Moreover, combining with the results we obtained in the previous sections, we have the following theorem.

Theorem III.37. *Let R be as in Assumption III.26. Suppose also that R is finitistically amenable. Then there exists a choice of J such that $\tau_J(R) = \tau(R)$.*

Proof. Use the facts that $W \cong W_J$ for some choice of J and the test ideals are annihilators of W and W_J respectively. □

Our next goal is to show that if we assume that R is locally F-regular in addition to hypotheses in Assumption III.26, then W_J has finite length as an R-module.

The following result shows that if the tight closure of quasi-parameter ideals commutes with localization, then the J-test ideal commutes with localization.

Lemma III.38. *Let R be a ring as in Assumption III.26. Let P be a prime ideal of height h and $J \subseteq P$. Choose $x_1, \ldots, x_{h-1} \in P$ so that they form a system of parameters for R_P/JR_P and a part of system of parameters in R and in R/J. Suppose that for all prime ideals $P \neq \mathfrak{m}$ and parts of a system of parameters as above, the following holds for all $t > 0$*

$$((x_1^t, \ldots, x_{h-1}^t) + J)^* R_P = ((x_1^t, \ldots, x_{h-1}^t) + JR_P)^{*R/P}.$$

Then $\tau_J(R)R_Q = \tau_{JR_Q}(R_Q)$ for every prime ideal Q of R.

Proof. First assume that $J \nsubseteq Q$, then $JR_Q = R_Q$ and $\tau_{JR_Q}(R_Q) = R_Q$. On the other hand, $J \subseteq \tau_J(R)$, so $\tau_J(R)R_Q \supseteq JR_Q = R_Q$.

Now consider the case when $J \subseteq Q$. Choose $x_1, \ldots, x_{h-1} \in Q$ as in the statement of the lemma, where $\operatorname{ht} Q = h$. Let $\mathfrak{B}_t = (x_1^t, \ldots, x_{h-1}^t) + J$ be ideals of R. We have that $\tau_{JR_Q}(R_Q) = \bigcap_t(\mathfrak{B}_t R_Q :_{R_Q} (\mathfrak{B}_t R_Q)^*) = \bigcap_t(\mathfrak{B}_t R_Q :_{R_Q} \mathfrak{B}_t^* R_Q) = \bigcap_t[(\mathfrak{B}_t :_R \mathfrak{B}_t^*)R_Q]$. Let $c \in \tau_J(R) = \bigcap_{I \supseteq J}(I : I^*)$, then $c\mathfrak{B}_t^* \subseteq \mathfrak{B}_t$ for all t. Therefore, $c/1 \in (\mathfrak{B}_t :_R \mathfrak{B}_t^*)R_Q$ for all t and $c/1 \in \tau_{JR_Q}(R_Q)$. So, $\tau_J(R)R_Q \subseteq \tau_{JR_Q}(R_Q)$.

Now let us prove the opposite inclusion. Recall that $W_J^\vee = R/\tau_J(R)$, hence $\tau_J(R) = \operatorname{Ann}_R W^\vee = \operatorname{Ann}_R W$. Therefore,

$$\tau_J(R)R_Q = (\operatorname{Ann}_R W^\vee) \otimes_R R_Q =$$

$$= \operatorname{Ann}_{R_Q}(W^\vee \otimes_R R_Q) =$$

$$= \operatorname{Ann}_{R_Q}(W^\vee \otimes_R R_Q)^{\vee^Q} =$$

$$= \operatorname{Ann}_{R_Q}(\operatorname{Hom}_{R_Q}(W^\vee \otimes_R R_Q, E_{R_Q}(R_Q/QR_Q))) =$$

$$= \operatorname{Ann}_{R_Q}(\operatorname{Hom}_R(W^\vee, E_R(R/Q)) \otimes_R R_Q) =$$

$$= \operatorname{Ann}_{R_Q}(\operatorname{Hom}_R(R/\tau_J(R), E(R/Q)) \otimes_R R_Q) =$$

$$= \operatorname{Ann}_{R_Q} \operatorname{Ann}_{E(R/Q)}(\tau_J(R)R_Q).$$

We also have that $\tau_{JR_Q}(R_Q) = \operatorname{Ann}_{R_Q} \varinjlim_t \frac{(\mathfrak{B}_t R_Q)^*}{\mathfrak{B}_t R_Q} = \operatorname{Ann}_{R_Q} \varinjlim_t \frac{\mathfrak{B}_t^* R_Q}{\mathfrak{B}_t R_Q}$.

To show that $\tau_{JR_Q}(R_Q) \subseteq \tau_J(R)R_Q$ it is equivalent to prove that

$$\operatorname{Ann}_{R_Q} \varinjlim_t \frac{\mathfrak{B}_t^* R_Q}{\mathfrak{B}_t R_Q} \subseteq \operatorname{Ann}_{R_Q} \operatorname{Ann}_{E(R/Q)}(\tau_J(R)R_Q)$$

and therefore, since $E_{R_Q/JR_Q}(R_Q/QR_Q) \subseteq E_{R_Q}(R_Q/QR_Q)$, it is sufficient to prove that

$$\varinjlim_t \frac{\mathfrak{B}_t^* R_Q}{\mathfrak{B}_t R_Q} \supseteq \operatorname{Ann}_{E_{R_Q/JR_Q}(R_Q/QR_Q)}(\tau_J(R)R_Q).$$

For simplicity of notation, let E' denote $E_{R_Q/JR_Q}(R_Q/QR_Q)$. Let $c/1 \in \tau_J(R)R_Q$ and $\eta \in \operatorname{Ann}_{E'}(\tau_J(R)R_Q) \subseteq E' \cong H^{h-1}_{QR_Q}(R_Q/JR_Q) \cong \varinjlim_t R_Q/\mathfrak{B}_t R_Q$. We know that W_J is a Ψ-stable submodule of $H^{d-1}_{\mathfrak{m}}(R/J)$, hence $\operatorname{Ann}_R W_J = \tau_J(R)$ is a Ψ-ideal of R by definition. Based on Lemma III.32, $\tau_J(R)R_Q$ is a Ψ-ideal of R_Q, hence we have $\mathrm{Ann}_{E'}(\tau_J(R)R_Q)$ is an F-stable submodule of E' and $F^e(\eta) \in \mathrm{Ann}_{E'}(\tau_J(R)R_Q)$. Finally, we get that $(c/1)F^e(\eta) = 0$ in E' for all $e > 0$. Suppose that $\eta = [z/1+\mathfrak{B}_t\, R_Q]$ with $z \in R$. Then $(c/1)F^e(\eta) = [cz^q/1 + \mathfrak{B}_{tq}\, R_Q] = 0$, so $cz^q \in \mathfrak{B}_t^{[q]}$ for all $q = p^e > 0$. Since $c \neq 0$, $c \in R^\circ$ and $z/1 \in \mathfrak{B}_t^*\, R_Q$. Thus, $\eta \in \varinjlim_t \frac{\mathfrak{B}_t^*\, R_Q}{\mathfrak{B}_t\, R_Q}$. □

Theorem III.39. *Let R be a ring as in Assumption III.26. Suppose that for every prime ideal $P \neq \mathfrak{m}$ of R, the local ring R_P is weakly F-regular. Then W_J has finite length as an R-module.*

Proof. Since R is complete, it suffices to show that the Matlis dual $W_J^\vee$ is a module with DCC over R.

Since W_J is a module with DCC over R, its dual satisfies ACC. Therefore, to prove the theorem It is sufficient to show that $W_J^\vee$ is supported only at the maximal ideal $\mathfrak{m}$. Let $P \subseteq R$ be a prime ideal of height h containing J and choose $x_1, \ldots, x_{h-1} \in P$ so that $(x_1, \ldots, x_{h-1}) + J$ is a quasi-parameter ideal for J. Then by persistence of tight closure we have that $((x_1^t, \ldots, x_{h-1}^t) + J)^* R_P \subseteq ((x_1^t, \ldots, x_{h-1}^t) + JR_P)^{*R/P}$. By our assumption, R_P is weakly F-regular, so $((x_1^t, \ldots, x_{h-1}^t) + JR_P)^* R/P = ((x_1^t, \ldots, x_{h-1}^t) + J)R_P$, hence $((x_1^t, \ldots, x_{h-1}^t) + J)^* R_P = ((x_1^t, \ldots, x_{h-1}^t) + JR_P)^{*R/P}$. Therefore, by the above lemma $\tau_J(R)R_Q = \tau_{JR_Q}(R_Q)$ for all prime ideals Q of R. Moreover, $\tau_J(R)R_Q = R_Q$ for all $Q \neq \mathfrak{m}$. Since $W_J^\vee = R/\tau_J(R)$, we have that $(W_J^\vee)_Q = R_Q/\tau_{JR_Q}(R_Q) = 0$ for for all $Q \neq \mathfrak{m}$. □

Corollary III.40. *Let $(R, \mathfrak{m}, K)$ be a complete Cohen-Macaulay local ring. Suppose that for every prime ideal $P \neq \mathfrak{m}$ of R, the local ring R_P is weakly F-regular. Then the J-test ideal $\tau_J(R)$ is $\mathfrak{m}$-primary. Moreover, if R is finitistically amenable, then the test ideal of R is $\mathfrak{m}$-primary.*

Proof. We have that for all $J \cong \omega_R$, $\tau_J(R) = \operatorname{Ann}_R W_J$ and $\tau(R) = \operatorname{Ann}_R W$. Moreover, $W_J \cong W$ for some choice of J in case R is amenable. $\square$

3.4 Big ideals

In the theory of tight closure we study rings which have the property that every ideal is tightly closed and it is equivalent to the property that every submodule of a finitely generated module is tightly closed. Then one might ask the following question. Can one characterize finitely generated modules such that all submodules are tightly closed? We shall see that this reduces, in a sense, to the study of ideals I of R such that every submodule of R/I is tightly closed over R, which we refer to as *big ideals*.

Theorem III.41. *Let R be a Noetherian commutative ring of positive prime characteristic p. Let M be a finitely generated R-module. Suppose that for every submodule N of M, $N^*_M = N$. Then every ideal of $R/\operatorname{Ann}_R M$ is tightly closed over R.*

Proof. Throughout this proof all tight closure operations are taken over the ring R.

First observe that if M_1 and M_2 are finitely generated R-modules with the property that every submodule is tightly closed then $M_1 \oplus M_2$ has this property as well. Let $N \subseteq M_1 \oplus M_2$ be any submodule. We have a short exact sequence

$$0 \to M_1 + N \to M_1 \oplus M_2 \to M_2/(N \bigcap M_2) \to 0.$$

Hence $M_1 + N$ is tightly closed in $M_1 \oplus M_2$. Next, we have a short exact sequence

$$0 \to N \to M_1 + N \to M_1/(N \bigcap M_1) \to 0.$$

Therefore, N is tightly closed in $M_1 + N$ and hence in $M_1 \oplus M_2$.

Now let M be a finitely generated R-module with the property as in the statement of the lemma. Let $u_1, \ldots, u_k$ be generators of M. Then we have a short exact sequence

$$0 \to \operatorname{Ann}_M R \to R \to M^{\oplus k} \to 0$$

such that

$$r \to ru_1 \oplus \ldots ru_k$$

Then $R/\operatorname{Ann}_M R \hookrightarrow M^{\oplus}$ and every R-submodule of $R/\operatorname{Ann}_M R$ is tightly closed over R. $\square$

Therefore, we can focus on rings R/I in which every ideal is tightly closed over R.

The next natural question one might ask: if we have a quotient ring with the property as above, is R/I weakly F-regular? The answer is no. Take $R = K[x, y, z]$ the polynomial ring over a field K. Let $I = (x^3+y^3+z^3)$ be an ideal of R. Then every ideal of R/I is tightly closed since R is regular, but R/I is not weakly F-regular.

However, we might ask this question. Let I be an ideal of R such that every ideal of R/I is tightly closed. Does there exist an ideal J of R contained in I such that R/J is weakly F-regular? In the above example, we have that $J = (0)$ works.

Remark III.42. If I has the property that every ideal of R/I is tightly closed over R, then every ideal in R containing I is tightly closed in R.

Remark III.43. The question can be positively answered for a Cohen-Macaulay complete local normal domain. If $I \cong \omega_R$, then $J = (0)$, see Corollary III.36.

Corollary III.36, originally proved by F. Enescu in a more general setting and generalized even further by A. De Stefani and L. Nunez-Betancourt, leads to a question:

for an F-regular ring R, does there exist an ideal $J \cong \omega_R$ such that R/J is F-regular. Unfortunately, the answer is no, as an example in [DSNB] shows.

CHAPTER IV

Nearly commuting and commuting matrices

4.1 Introduction and preliminaries

In this chapter we study algebraic sets of pairs of matrices such that their commutator is either nonzero diagonal or zero. We also consider some other related algebraic sets. Let us define relevant notions.

Let $X = (x_{ij})_{1\leqslant i,j\leqslant n}$ and $Y = (y_{ij})_{1\leqslant i,j\leqslant n}$ be $n \times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1\leqslant i,j\leqslant n}$ and let I denote the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and J denote the ideal generated by the entries of $XY - YX$. The ideal I defines an algebraic set of pairs of matrices with nonzero diagonal commutator and is called *an algebraic set of nearly commuting matrices.*

Let u_{ij} denote the (i, j)th entry of the matrix $XY - YX$. Then $I = (u_{ij} \,|\, 1 \leqslant i \neq j \leqslant n)$ and $J = (u_{ij} \,|\, 1 \leqslant i,\, j \leqslant n)$.

Theorem IV.1 ([Ger61])**.** *The algebraic set of commuting matrices is irreducible, i.e., it is a variety. Equivalently,* Rad(J) *is prime.*

The following result is due to Knutson [Knu05], when the characteristic of the field is 0, and to H.Young [You11] in all characteristics.

Theorem IV.2 ([Knu05], [You11])**.** *The algebraic set of nearly commuting matrices* *is a complete intersection, with the variety of commuting matrices as one of its irreducible components.*

Theorem IV.3 ([Knu05], [You11])**.** *When K has characteristic zero, I is a radical ideal.*

A. Knutson in his paper [Knu05] conjectured that $\mathbb{V}(I)$ has only two irreducible components and its was proved in all characteristics by H.Young in his thesis, [You11].

Theorem IV.4 ([You11])**.** *If $n \geqslant 2$, the algebraic set of nearly commuting matrices has two irreducible components, one of which is the variety of commuting matrices and the other is the so-called skew component. That is, I has two minimal primes, one of which is* $\mathrm{Rad}(J)$*.*

Let $P = \mathrm{Rad}(J)$ and let Q denote the other minimal prime of I, i.e., $\mathrm{Rad}(I) = P \bigcap Q$.

The following conjecture was made in 1982 by M. Artin and M. Hochster.

Conjecture IV.5. *J is reduced, i.e., $J = P$.*

It was answered positively by Mary Thompson in her thesis in the case of 3×3 matrices.

Theorem IV.6 ([Tho85])**.** *R/J is a Cohen-Macaulay domain when $n = 3$.*

The following known results will show that the properties of varieties defined by P and Q are closely related.

Definition IV.7 ([HU87])**.** Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals in a Cohen-Macaulay ring A. Then $\mathcal{I}$ and $\mathcal{J}$ are said to be *linked*, if there is a regular sequence $\alpha = \alpha_1, \ldots, \alpha_k$ in $\mathcal{I} \bigcap \mathcal{J}$ such that $\alpha : \mathcal{I} = \mathcal{J}$ and $\alpha : \mathcal{J} = \mathcal{I}$.

Lemma IV.8. *Let A be a Noetherian Cohen-Macaulay ring and let $\mathfrak{I}$ be a radical ideal of A generated by a regular sequence and with the primary decomposition $\mathfrak{I} = P \bigcap Q$. Then $\mathfrak{I} : P = Q$ and $\mathfrak{I} : Q = P$, i.e., P and Q are linked.*

Proof. Clearly, $PQ \subseteq \mathfrak{I}$. Hence we need only to show that $\mathfrak{I} : P \subseteq Q$ and $\mathfrak{I} : Q \subseteq P$. Let $z \in \mathfrak{I} : P$, then $Pz \in \mathfrak{I} \subseteq Q$. Then since $P \nsubseteq Q$, $z \in Q$. $\mathfrak{I} : Q \subseteq P$ can be proved similarly. □

For more properties of linkage see [HU87].

The next theorem is due to C. Peskine and L. Szpiro and it shows that quotients by ideals which are linked share certain nice properties. We shall make use of this result later in the chapter when we discuss F-regularity of the variety of commuting matrices and the skew-component.

Theorem IV.9 ([PS74])**.** *In the situation of the above lemma, A/P is Cohen-Macaulay if and only if A/Q is Cohen-Macaulay, in which case the corresponding canonical modules are $\omega_{A/P} \cong (P+Q)/P$ and $\omega_{A/Q} \cong (P+Q)/Q$. Hence, $A/(P+Q)$ is Gorenstein of dimension* $\dim A - 1$.

Now let us go back to algebraic sets of nearly commuting matrices and their irreducible components. First, let us take a look at what we have when $n = 1, 2$.

When $n = 1$, everything is trivial. More precisely, $I = P = Q = K[x_{11}, y_{11}]$.

When $n = 2$, without loss of generality we may replace X and Y by $X - x_{22}\mathrm{I}_n$ and $Y - y_{22}\mathrm{I}_n$ respectively. Here I_n is the identity matrix of size n. Denote $x'_{11} = x_{11} - x_{22}$, $y'_{11} = y_{11} - y_{22}$. Then the generators of I are 2 by 2 minors

$$u_{12} = \begin{vmatrix} x'_{11} & x_{12} \\ y'_{11} & y_{12} \end{vmatrix}, \quad u_{21} = -\begin{vmatrix} x'_{11} & x_{21} \\ y'_{11} & y_{21} \end{vmatrix}.$$

The diagonal entries of $XY - YX$ are

$$u_{11} = \begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix}, \quad u_{22} = -\begin{vmatrix} x_{12} & x_{21} \\ y_{12} & y_{21} \end{vmatrix}.$$

Then J is the ideal generated by size 2 minors of $\begin{bmatrix} x'_{11} & x_{12} & x_{21} \\ y'_{11} & y_{12} & y_{21} \end{bmatrix}$ and therefore, $J = P$ is prime. We also have that $Q = (x'_{11}, y'_{11})$. Moreover, $I = P \bigcap Q$ is radical and $P + Q = x_{12}y_{21} - x_{21}y_{12}$ is prime.

We have that

$$(u_{12}u_{21})^{p-1} = (x'_{11}y_{12} - x_{12}y'_{11})^{p-1}(x'_{11}y_{21} - x_{21}y'_{11})^{p-1} =$$

$$\sum_{\alpha=0}^{p-1}\sum_{\beta=0}^{p-1}(-1)^{\alpha+\beta}\binom{p-1}{\alpha}\binom{p-1}{\beta}(x'_{11})^{\alpha+\beta}(y'_{11})^{2(p-1)-\alpha-\beta}x_{12}^{p-1-\alpha}y_{12}^{\alpha}x_{21}^{p-1-\beta}y_{21}^{\beta}.$$

Therefore, $(u_{12}u_{21})^{p-1}$ has a monomial term $(x'_{11}y'_{11}x_{12}y_{21})^{p-1}$ with coefficient $(-1)^{p-1}$. Since $I^{[p]} : I = (u_{12}u_{21})^{p-1} + I^{[p]}$, R/I is F-pure, see Fedder's criterion Lemma IV.12. Furthermore, determinantal rings R/P, R/Q, $R/(P+Q)$ are F-regular, see [HH94b].

Therefore, for the rest of the chapter we shall use the following notations.

Notation IV.10. Let $n \geqslant 3$ be an integer. Let $X = (x_{ij})_{1 \leqslant i,j \leqslant n}$ and $Y = (y_{ij})_{1 \leqslant i,j \leqslant n}$ be $n \times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1 \leqslant i,j \leqslant n}$ and let I denote the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and J denote the ideal generated by the entries of $XY - YX$. Let P denote the radical of J and Q be the other minimal prime of $\mathrm{Rad}(I)$.

4.2 F-purity

In this section we show that the coordinate ring of the algebraic set of pairs of matrices with a diagonal commutator is F-pure in the case of 3 by 3 matrices.

Moreover, we also prove that it implies the corresponding fact for its irreducible components, the variety of commuting matrices and the skew-component, and their intersection.

First we state two lemmas due to R. Fedder and this includes a criterion for F-purity for finitely generated K-algebras and which has a particularly convenient form for complete intersections.

Lemma IV.11 (Fedder [Fed97])**.** *Let S be a regular local ring or a polynomial ring over a field. If S has characteristic $p > 0$ and I is an unmixed proper ideal (homogeneous in the polynomial case) with the primary decomposition $I = \bigcap_{i=1}^{n} \mathfrak{A}_i$, then $I^{[p]} : I = \bigcap_{i=1}^{n}(\mathfrak{A}^{[p]} : \mathfrak{A})$.*

Lemma IV.12 (Fedder's criterion [Fed97])**.** *Let $(S, \mathfrak{m})$ be a regular local ring or a polynomial ring over a field and its homogeneous maximal ideal. If S has characteristic $p > 0$ and I is a proper ideal (homogeneous in the polynomial case), then S/I is F-pure if and only if $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$.*

Next result is a straightforward consequence of the above two lemmas. It will prove to be quite useful for us.

Lemma IV.13. *Let S be a regular local ring or a polynomial ring over a field. Suppose that S has characteristic $p > 0$ and I is an ideal of S (homogeneous in the polynomial case). Suppose also that S/I is F-pure and $I = \bigcap_{i=1}^{n} \mathfrak{A}_i$ is the primary decomposition. Then $S/(\mathfrak{A}_{i_1} + \ldots \mathfrak{A}_{i_m})$ is F-pure for all $1 \leqslant i_1 < ... < i_m \leqslant n$ and for all $1 \leqslant m \leqslant n$.*

Proof. Observe first that $(\mathfrak{A}_{i_1} + \ldots \mathfrak{A}_{i_m})^{[p]} :_S (\mathfrak{A}_{i_1} + \ldots \mathfrak{A}_{i_m}) \supseteq \bigcap_{j=1}^{m}(\mathfrak{A}_{i_j}^{[p]} : \mathfrak{A}_{i_j}) \supseteq \bigcap_{i=1}^{n}(\mathfrak{A}_i^{[p]} : \mathfrak{A}_i) = (I^{[p]} : I)$. The rest is immediate from Lemma IV.11 and Lemma IV.12.

□

The above lemma is closely related to results on compatibly split ideals, cf [ST12]. Immediately we get the corresponding result for our algebraic set.

Corollary IV.14. *Suppose that the coordinate ring of the algebraic set of nearly commuting matrices R/I is F-pure. Then R/P, R/Q and $R/(P+Q)$ are F-pure.*

Next, we use Fedder's criterion to show F-purity of R/I in case when $n = 3$.

Theorem IV.15. *Let K be a field of characteristic $p > 0$ and let $n = 3$. Let R be a ring as in Notation IV.10. Then R/I is F-pure.*

Proof. Recall that I is generated by a regular sequence $\{u_{ij} | 1 \leqslant i \neq j \leqslant n\}$. Therefore, $I^{[p]} : I = (\prod_{1\leqslant i\neq j\leqslant n} u_{ij}^{p-1})R + I^{[p]}$. Thus by Fedder's criterion it is sufficient to prove that $\prod_{1\leqslant i\neq j\leqslant n} u_{ij}^{p-1} \notin \mathfrak{m}^{[p]}$. We show this by proving the following claim.

Claim. If $\mu = x_{12}x_{13}x_{21}x_{23}x_{31}x_{33}y_{11}y_{12}y_{23}y_{31}y_{32}y_{33}$, then μ^{p-1} is a monomial term of $\prod_{1\leqslant i\neq j\leqslant 3} u_{ij}^{p-1}$ with a nonzero coefficient modulo p.

Proof. We compute the coefficient of μ^{p-1}. It can be obtained by choosing a monomial from every u_{ij} in the following way:

$u_{12}: \quad (-x_{12}y_{11})^{\alpha_1}(x_{13}y_{32})^{\beta_1}$

$u_{13}: \quad (-x_{23}y_{12})^{\alpha_2}(x_{12}y_{23})^{\beta_2}(-x_{13}y_{11})^{\gamma_2}(x_{13}y_{33})^{\delta_2}$

$u_{21}: \quad (-x_{31}y_{23})^{\alpha_3}(x_{21}y_{11})^{\beta_3}(x_{23}y_{31})^{\gamma_3}$

$u_{23}: \quad (x_{23}y_{33})^{\alpha_4}(-x_{33}y_{23})^{\beta_4}$

$u_{31}: \quad (-x_{21}y_{32})^{\alpha_5}(x_{33}y_{31})^{\beta_5}(-x_{31}y_{33})^{\gamma_5}(x_{31}y_{11})^{\delta_5}$

$u_{32}: \quad (x_{31}y_{12})^{\alpha_6}(-x_{12}y_{31})^{\beta_6}(x_{33}y_{32})^{\gamma_6}$

Then the exponents A_{st} and B_{st} of each x_{st} and y_{st} respectively are

$$A_{12} = \alpha_1 + \beta_2 + \beta_6 \qquad A_{23} = \alpha_2 + \gamma_3 + \alpha_4$$

$$A_{13} = \beta_1 + \gamma_2 + \delta_2 \qquad A_{31} = \alpha_3 + \gamma_5 + \delta_5 + \alpha_6$$

$$A_{21} = \beta_3 + \alpha_5 \qquad A_{33} = \beta_4 + \beta_5 + \gamma_6$$

$B_{11} = \alpha_1 + \gamma_2 + \beta_3 + \delta_5$

$B_{12} = \alpha_2 + \alpha_6$

$B_{23} = \beta_2 + \alpha_3 + \beta_4$

$B_{31} = \gamma_3 + \beta_5 + \beta_6$

$B_{32} = \beta_1 + \alpha_5 + \gamma_6$

$B_{33} = \delta_2 + \alpha_4 + \gamma_5$

In addition, denote

$C_{12} = \alpha_1 + \beta_1,$

$C_{13} = \alpha_2 + \beta_2 + \gamma_2 + \delta_2,$

$C_{21} = \alpha_3 + \beta_3 + \gamma_3,$

$C_{23} = \alpha_4 + \beta_4,$

$C_{31} = \alpha_5 + \beta_5 + \gamma_5 + \delta_5,$

$C_{32} = \alpha_6 + \beta_6 + \gamma_6.$

Our goal is to find all nonnegative integer tuples $\alpha = (\alpha_1, \ldots, \alpha_6)$, $\beta = (\beta_1, \ldots, \beta_6)$, $\gamma = (\gamma_2, \gamma_3, \gamma_5, \gamma_6)$, $\delta = (\delta_5, \delta_6)$ such that $A_{st} = p - 1$, $B_{st} = p - 1$ for all $1 \leqslant s, t \leqslant 3$ and $C_{ij} = p - 1$ for all $1 \leqslant i \neq j \leqslant 3$.

Notice that the linear system does not have a nonzero determinant: the sum of the first 12 equations is twice the sum of the rest 6 equations. Therefore, there is not a unique solution.

The above linear system can be solved using standard methods from linear algebra and has the following solution

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a & b & d & p-1-b-a+d & a & p-1-b \\ p-1-a & p-1-a-b & p-1-a & a+b-d & p-1-a-b+d & b \\ - & a-c & a-d & - & b+a-d-c & 0 \\ - & c & - & - & c-a & - \end{bmatrix}$$

where column vector $[\alpha, \beta, \gamma, \delta]$ represents the matrix of solutions and a, b, c, d are elements of the field K.

Since we look for nonnegative integer solutions we must have that $a = c$ and a, $b \geqslant d$ and $a + b \leqslant p - 1$. Hence we have that

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a & b & d & p-1-b-a+d & a & p-1-b \\ p-1-a & p-1-a-b & p-1-a & a+b-d & p-1-a-b+d & b \\ - & 0 & a-d & - & b-d & 0 \\ - & a & - & - & 0 & - \end{bmatrix}$$

Therefore, the coefficient of μ^{p-1} is the sum of expressions of the form

$$(-1)^{\alpha_1+\alpha_2+\gamma_2+\alpha_3+\beta_4+\alpha_5+\gamma_5+\beta_6}((p-1)!)^6/(\alpha_1!\ldots\alpha_6!\beta_1!\ldots\beta_6!\gamma_2!\gamma_3!\gamma_5!\gamma_6!\delta_5!\delta_6!)$$

where $\alpha = (\alpha_1, \ldots, \alpha_6)$, $\beta = (\beta_1, \ldots, \beta_6), \gamma = (\gamma_2, \gamma_3, \gamma_5, \gamma_6)$, $\delta = (\delta_5, \delta_6)$ run over all solutions of the linear system above. That is,

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geqslant d,\, a+b\leqslant p-1} (-1)^{a-d} \binom{p-1}{a}^2 \binom{p-1}{b}^2 \binom{p-1}{a+b-d}^2 \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

which modulo p is equivalent to

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geqslant d,\, a+b\leqslant p-1} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

It is also can be written as

$$\sum_{d=0}^{(p-1)/2} \sum_{a,b\geqslant d,\, a+b\leqslant p-1} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{a-d \quad b-d \quad d}$$

or

$$\sum_{b=0}^{p-1} \sum_{d=0}^{b} \sum_{a=d}^{p-1-b} (-1)^{a-d} \binom{p-1-b}{a} \binom{a+b-d}{b} \binom{b}{d}$$

The following lemma shall show that the above expression is equal to 1 for all values of p. In fact, for this purpose p does not have to be prime.

Lemma IV.16. *Let* $C_m = \sum_{b=0}^{m} \sum_{d=0}^{b} \sum_{a=d}^{m-b} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d}$. *Then* $C_m = 1$ *for all* $m \geqslant 1$.

Proof. We shall prove a stronger statement.

Claim. Let $B_{m,b} = \sum_{d=0}^{b} \sum_{a=d}^{m-b} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d}$. Then for all $m \geqslant 1$

$$B_{m,b} = \begin{cases} 0 & \text{if } 0 \leqslant b \leqslant m-1; \\ 1 & \text{if } b = m. \end{cases}$$

Proof. First observe that $B_{m,m} = \sum_{d=0}^{m} \sum_{a=d}^{0} (-1)^{a-d} \binom{m-b}{a} \binom{a+b-d}{b} \binom{b}{d} = 1$ and $B_{m,0} = \sum_{a=0}^{m} (-1)^a \binom{m}{a} = 0$. Hence we may assume that $0 < b < m$.

Let $A_{m,b,d} = \sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d}{b}$, then $B_{m,b} = \sum_{d=0}^{b} (-1)^d \binom{b}{d} A_{m,b,d}$. Consider the difference

$$A_{m,b,d} - A_{m,b,d+1} =$$

$$\sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d}{b} - \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b} =$$

$$(-1)^d \binom{m-b}{d} + \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} (\binom{a+b-d}{b} - \binom{a+b-d-1}{b}) =$$

Using Pascal's identity, we get

$$(-1)^d \binom{m-b}{d} + \sum_{a=d+1}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b-1} =$$

$$\sum_{a=d}^{m-b} (-1)^a \binom{m-b}{a} \binom{a+b-d-1}{b-1} =$$

$$\sum_{a=d}^{m-1-(b-1)} (-1)^a \binom{m-1-(b-1)}{a} \binom{a+(b-1)-d}{b-1}.$$

Thus we have that

$$A_{m,b,d} - A_{m,b,d+1} = A_{m-1,b-1,d} \text{ for all } m-1 \geqslant b \geqslant d+1 \text{ and } d \geqslant 0.$$

Therefore,

$$B_{m-1,b-1} = \sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} A_{m-1,b-1,d} = \sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} (A_{m,b,d} - A_{m,b,d+1}) =$$

$$\sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} A_{m,b,d} - \sum_{d=0}^{b-1} (-1)^d \binom{b-1}{d} A_{m,b,d+1} =$$

$$\sum_{d=0}^{b-1}(-1)^d\binom{b}{d}\frac{b-d}{b}A_{m,b,d}-\sum_{d=1}^{b}(-1)^{d-1}\binom{b-1}{d-1}A_{m,b,d}=$$

$$\sum_{d=0}^{b-1}(-1)^d\binom{b}{d}\frac{b-d}{b}A_{m,b,d}+\sum_{d=1}^{b}(-1)^{d}\binom{b}{d}\frac{d}{b}A_{m,b,d}=$$

$$\sum_{d=1}^{b-1}(-1)^d\binom{b}{d}A_{m,b,d}+A_{m,b,0}+(-1)^bA_{m,b,b}=$$

$$\sum_{d=0}^{b}(-1)^d\binom{b}{d}A_{m,b,d}=B_{m,b}.$$

Thus we have that $B_{m-1,b-1}=B_{m,b}$ for all $m\geqslant 1$ and $m-1\geqslant b\geqslant 1$.

In case $m=1$, we only have $B_{1,0}=0$. Finally, use induction on m to conclude that $B_{m,b}=0$ for all $m\geqslant 1$ and $m-1\geqslant b$.

Thus, $C_m=\sum_{b=0}^{m}B_{m,b}=1$. □

Finally, we complete the proof of Theorem IV.15. We have that $\prod_{1\leqslant i\neq j\leqslant n}u_{ij}^{p-1}\notin\mathfrak{m}^{[p]}$ and R/I is F-pure when $n=3$. □

Corollary IV.17. *Let R be a ring as in Notation IV.10. When $n=3$, $R/P, R/Q$ and $R/(P+Q)$ are F-pure Cohen-Macaulay rings and $R/(P+Q)$ is Gorenstein.*

Corollary IV.18. *Let R be a ring as in Notation IV.10. Then $P+Q$ is radical when $n=3$.*

Remark IV.19. We shall prove in the next section that the radical of $P+Q$ is prime, which will imply that $P+Q$ is prime when $n=3$. In particular, we shall have that $R/(P+Q)$ is a domain when $n=3$.

4.3 Irreducibility of $P+Q$

In this section we prove that the intersection of the variety of commuting matrices and the skew-component is irreducible. But first we need to define some notions.

Definition IV.20. Let X be an n by n matrix of indeterminates. Then $D(X)$ is an n by n matrix whose ith column is defined by the diagonal entries of X^{i-1} numbered from upper left corner to lower right corner. Let $\mathcal{P}(X)$ denote the determinant of $D(X)$.

Theorem IV.21 ([You11])**.** *$\mathcal{P}(X)$ is an irreducible polynomial.*

Remark IV.22. $\mathcal{P}(X) = \mathcal{P}(X - aI)$, where $a \in K$ and $I \in M_n(K)$ is the identity matrix.

The next two lemmas are due to H. Young. They give us the connection between the variety defined by $\mathcal{P}(X)$ and the algebraic set of nearly commuting matrices.

Lemma IV.23 ([You11])**.** *Given an $n \times n$ matrix A, if there exists a matrix B such that $[A, B]$ is a non-zero diagonal matrix, then $\mathcal{P}(A) = 0$.*

Lemma IV.24 ([You11])**.** *There is a dense open set $\mathcal{U}$ in the variety defined by $\mathcal{P}(X)$ where for every point A in $\mathcal{U}$, there exists a matrix B such that $[A, B]$ is a nonzero diagonal matrix.* □

The following notion of a discriminant is of significant importance in matrix theory. We use it in this section in order to reduce our study to the case when commuting matrices have a particularly simple characterization.

Definition IV.25. Let $A \in M_n(K)$. Then the *discriminant* $\Delta(A)$ of A is the discriminant of its characteristic polynomial. That is, if K contains all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, then $\Delta(A) = \prod_{1 \leqslant i < j \leqslant n} (\lambda_i - \lambda_j)^2$.

Fact. Let $A \in M_n(K)$ be a matrix such that $\Delta(A) \neq 0$, or equivalently, A has distinct eigenvalues. Then a matrix B commutes with A if and only if B is a polynomial in A of degree at most $n - 1$, see Theorem 3.2.4.2 [HJ85].

Remark IV.26. $\mathcal{P}(X)$ is an irreducible polynomial of degree $n(n-1)/2$ and $\Delta(X)$ is a polynomial of degree $n(n-1)$. Moreover, when $n \geqslant 3$, $\mathcal{P}(X)$ does not divide $\Delta(X)$. This can be proved by showing that there exists a matrix A with the property that $\mathcal{P}(A) = 0$ while $\Delta(A) \neq 0$. For example, for this purpose one can use the following matrices.

$$E_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \text{ if } p \nmid n, \text{ and } \widetilde{E}_n = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{n-1} \end{array}\right], \text{ otherwise.}$$

The characteristic polynomials are $x^n - 1$ for E_n and $x(x^{n-1} - 1)$ for $\widetilde{E}_n$.

Our next goal is to show that $\Delta(X)$ is not in any of the minimal primes of $P + Q$. We do it by proving that the dimension of $R/(P + Q)$ drops when we kill $\Delta(X)$.

Now lets us define the set-up which we need to state and prove our next result.

Let m be an integer such that $m \leqslant n$. Fix a partition $(h_1, \dots, h_m)$ of n, that is, choose positive integers $h_1, \dots, h_m$ such that $h_1 + \dots + h_m = n$. Let J_i be an upper triangular Jordan form of a nilpotent matrix of size h_i. For each h_i there are finitely many choices of J_i. Let $J = (J_1, \dots, J_m)$ and let I_i denote the identity matrix of size h_i.

For any m-tuple $\underline{\lambda} = \lambda_1, \dots, \lambda_m$ of distinct elements of K, let $J(\underline{\lambda}) = J(\lambda_1, \dots, \lambda_m)$ be a matrix such that for all $1 \leqslant i \leqslant m$, the blocks $\lambda_i I_i + J_i$ are on the main diagonal. That is, $J(\underline{\lambda})$ is the direct sum of matrices $\lambda_i I_i + J_i$.

Let $\Lambda = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{A}^m \mid \lambda_i \neq \lambda_j \text{ for all } 1 \leqslant i \neq j \leqslant m\}$. It is an open subset of $\mathbb{A}^m$ and therefore is irreducible and has dimension m. Let

$$W_J = \{A \in M_n(K) \mid\ A \text{ is similar to some } J(\lambda_1, \ldots, \lambda_m) \text{ with } \lambda_1, \ldots, \lambda_m \in K \text{ distinct}\}.$$

Let c_J denote the dimension of the set of matrices that commute with $J(\underline{\lambda})$, for some $\underline{\lambda}$. This number is independent of the choice of $\underline{\lambda}$, since $J(\underline{\lambda})$ commutes with a matrix A if and only if A is a direct sum of matrices A_i such that each A_i has size h_i and A_i commutes with J_i. Moreover, c_J is the dimension of the set of invertible matrices that commute with $J(\underline{\lambda})$, for some $\underline{\lambda}$.

Lemma IV.27. *The dimension of W_J is $n^2 - c_J + m$.*

Proof. Define a surjective map of algebraic sets

$$GL_n(K) \times \Lambda \to W_J$$

such that

$$(U, \lambda_1, \ldots, \lambda_m) \to U^{-1} J(\lambda_1, \ldots, \lambda_m) U.$$

Fix $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Then

$$\theta^{-1}(J(\underline{\lambda})) = \{U \in GL_n(K) \mid U^{-1} J(\underline{\lambda}) U = J(\underline{\lambda})\} = \{U \in GL_n(K) \mid J(\underline{\lambda}) U = U J(\underline{\lambda})\},$$

that is, it is the set of all invertible matrices commuting with $J(\underline{\lambda})$.

Let $M = U^{-1} J(\underline{\lambda}) U \in W_J$ and let $V \in \theta^{-1}(M)$. Then $U^{-1} J(\underline{\lambda}) U = V^{-1} J(\underline{\lambda}) V$ and $J(\underline{\lambda}) = (UV^{-1})^{-1} J(\underline{\lambda})(UV^{-1})$. Hence, $V \in \theta^{-1}(J(\underline{\lambda}))U$. Therefore, $\theta^{-1}(J(\underline{\lambda}))$ and $\theta^{-1}(M)$ have the same dimension. Since the dimension of W_J is the dimension of $GL_n(K) \times \Lambda$ minus the dimension of a generic fiber $\theta^{-1}(J(\underline{\lambda}))$, we have that the dimension of W_J is $n^2 - c_J + m$.

Moreover, the set of pairs of matrices $(A, B) \in M_n(K) \times M_n(K)$ such that A and B commute has dimension $(n^2 - c_J + m) + c_J = n^2 + m \leqslant n^2 + n$. □

Now we are ready to prove the following lemma.

Lemma IV.28. *Let $X = (x_{ij})$ be an n by n matrix of indeterminates over a field K. Let $S = K[X]$ and let $\mathcal{P} = \det D(X)$. Then the discriminant $\Delta(X)$ of X is not in any of the minimal primes of the ideal $(\mathcal{P})R$.*

Proof. We prove the lemma by showing that the dimension of $R/(P+Q)$ drops when we kill $\Delta(X)$. This is done by proving the claim below.

Claim. The dimension of the set $W = \{(A, B) \in M_n(K) \times M_n(K) | \, [A, B] = 0, \, \Delta(A) = 0, \, \mathcal{P}(A) = 0, \, \mathcal{P}(B) = 0\}$ is at most $n^2 + n - 2$.

Proof. Let $V = \{(A, B) \in M_n(K) \times M_n(K) | \, [A, B] = 0, \, \Delta(A) = 0\}$ and $V_m = \{(A, B) \in V | \ A$ has m distinct eigenvalues $\}$. Then we have that $\dim V_m = n^2 + m$ and $V = \bigcup_{m=1}^{n-1} V_m$. Therefore, $\dim V \leqslant n^2 + n - 1$. Notice that since $\Delta(A) = 0$, $m \leqslant n - 1$.

Similarly, let $W_m = \{(A, B) \in W | \ A$ has m distinct eigenvalues $\}$. Then $W = \bigcup_{m=1}^{n-1} W_m$. For each value of m, $W_m \subseteq V_m$. Therefore, the dimension of W is at most $n^2 + n - 1$. Moreover, W is a closed subset of V defined by the vanishing of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. To prove the lemma we need to show that $\dim W$ cannot be $n^2 - n - 1$. We do this by showing that $W \neq V$. In other words, we show that there are pairs of matrices $(A, B) \in V$ but not in W, i.e., either $\mathcal{P}(A) \neq 0$ or $\mathcal{P}(B) \neq 0$.

Let $A \in M_n(K)$ be a matrix with distinct eigenvalues $\lambda = \lambda_1 = \lambda_2, \lambda_3, \ldots, \lambda_n$. Then A is similar to a Jordan matrix in two possible forms.

Case 1. A is similar to $J = \begin{bmatrix} \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & \dots & 0 \\ \dots & & \dots & & \dots & \\ 0 & 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$

Take $B = \mathrm{diag}(a_1, \dots, a_n)$ be a diagonal matrix with distinct entries on the diagonal. Then $[A, B] = 0$ and $\mathcal{P}(B) = \prod_{1 \leqslant i < j \leqslant n}(a_i - a_j) \neq 0$. Hence $\dim W \leqslant n^2 + n - 2$.

Case 2. A is similar to $J = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 0 & \dots & 0 \\ \dots & & \dots & & \dots & \\ 0 & 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$

Write $J = \left[\begin{array}{c|c} J_0 & 0 \\ \hline 0 & J_1 \end{array}\right]$, where $J_0 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and $J_1 = \begin{bmatrix} \lambda_4 & 0 & \dots & 0 \\ 0 & \lambda_5 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$.

Take an n by n block-diagonal matrix $U = \left[\begin{array}{c|c} U_0 & 0 \\ \hline 0 & U_1 \end{array}\right]$ such that $U_0 = \begin{bmatrix} 0 & 0 & a \\ b & c & 0 \\ 0 & d & e \end{bmatrix}$ and $U_1 \in M_{n-3}(K)$ are invertible matrices with $abcde \neq 0$.

Then $U^{-1} = \left[\begin{array}{c|c} U_0^{-1} & 0 \\ \hline 0 & U_1^{-1} \end{array}\right]$ and $U^{-1}JU = \left[\begin{array}{c|c} U_0^{-1}J_0U_0 & 0 \\ \hline 0 & J_1 \end{array}\right]$.

Our goal is to show that $\mathcal{P}(U^{-1}JU) \neq 0$. First, we shall prove it for the case of 3 by 3 matrices, i.e., for $U_0^{-1}J_0U_0$.

Observe that $\mathcal{P}(U_0^{-1}J_0U_0) = \mathcal{P}(U_0^{-1}J_0U_0 - \lambda I) = \mathcal{P}(U_0^{-1}(J_0 - \lambda I)U_0)$.

Denote $M = U^{-1}(J - \lambda I)U$ and $M_0 = U_0^{-1}(J_0 - \lambda I)U_0$.

We have that

$$U_0^{-1} = 1/(abd)\begin{bmatrix} ce & ad & -ac \\ -be & 0 & ab \\ bd & 0 & 0 \end{bmatrix}$$

and

$$M_0 = U_0^{-1}(J_0 - \lambda I)U = 1/(abd)\begin{bmatrix} bce & ce^2 - acd(\lambda_3 - \lambda) & -ace(\lambda_3 - \lambda) \\ -b^2e & -bce + abd(\lambda_3 - \lambda) & abe(\lambda_3 - \lambda) \\ b^2d & bcd & 0 \end{bmatrix}.$$

Moreover,

$$M_0^2 = 1/(abd)^2\begin{bmatrix} 0 & -a^2bcd^2(\lambda_3 - \lambda)^2 & -a^2bcde(\lambda_3 - \lambda)^2 \\ 0 & (abd(\lambda_3 - \lambda))^2 & a^2b^2de(\lambda_3 - \lambda)^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, the diagonal $\mathrm{diag}(M_0^i) = (0, (\lambda_3 - \lambda)^i, 0)$ for all $i \geqslant 2$. Then

$$\mathcal{P}(M_0) = \det\begin{bmatrix} 1 & bce & 0 \\ 1 & -bce + (\lambda_3 - \lambda) & (\lambda_3 - \lambda)^2 \\ 1 & 0 & 0 \end{bmatrix} = bce(\lambda_3 - \lambda)^2 \neq 0.$$

Finally,

$$\mathcal{P}(M) = \det \begin{bmatrix} 1 & bce & 0 & 0 & \dots & 0 \\ 1 & -bce + \lambda_3 - \lambda & (\lambda_3 - \lambda)^2 & (\lambda_3 - \lambda)^3 & \dots & (\lambda_3 - \lambda)^{n-1} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & \lambda_4 - \lambda & (\lambda_4 - \lambda)^2 & (\lambda_4 - \lambda)^3 & \dots & (\lambda_4 - \lambda)^{n-1} \\ \dots & & \dots & & \dots & \\ 1 & \lambda_n - \lambda & (\lambda_n - \lambda)^2 & (\lambda_n - \lambda)^3 & \dots & (\lambda_n - \lambda)^{n-1} \end{bmatrix} =$$

Subtract the third row from every other row

$$= \det \begin{bmatrix} 0 & bce & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 - \lambda & (\lambda_3 - \lambda)^2 & (\lambda_3 - \lambda)^3 & \dots & (\lambda_3 - \lambda)^{n-1} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_4 - \lambda & (\lambda_4 - \lambda)^2 & (\lambda_4 - \lambda)^3 & \dots & (\lambda_4 - \lambda)^{n-1} \\ \dots & & \dots & & \dots & \\ 0 & \lambda_n - \lambda & (\lambda_n - \lambda)^2 & (\lambda_n - \lambda)^3 & \dots & (\lambda_n - \lambda)^{n-1} \end{bmatrix} =$$

Subtract from the ith row $(\lambda_i - \lambda)/bce$ times the first row, for all $i \neq 1, 3$:

$$= \det \begin{bmatrix} 0 & bce & 0 & 0 & \dots & 0 \\ 0 & 0 & (\lambda_3 - \lambda)^2 & (\lambda_3 - \lambda)^3 & \dots & (\lambda_3 - \lambda)^{n-1} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & (\lambda_4 - \lambda)^2 & (\lambda_4 - \lambda)^3 & \dots & (\lambda_4 - \lambda)^{n-1} \\ \dots & & \dots & & \dots & \\ 0 & 0 & (\lambda_n - \lambda)^2 & (\lambda_n - \lambda)^3 & \dots & (\lambda_n - \lambda)^{n-1} \end{bmatrix} =$$

$$= bce \det \begin{bmatrix} (\lambda_3-\lambda)^2 & \dots & (\lambda_3-\lambda)^{n-1} \\ (\lambda_4-\lambda)^2 & \dots & (\lambda_4-\lambda)^{n-1} \\ & \dots & \\ (\lambda_n-\lambda)^2 & \dots & (\lambda_n-\lambda)^{n-1} \end{bmatrix} =$$

$$= bce(\lambda_3-\lambda)^2(\lambda_4-\lambda)^2\cdots(\lambda_n-\lambda)^2 \det \begin{bmatrix} 1 & \lambda_3-\lambda & \dots & (\lambda_3-\lambda)^{n-3} \\ 1 & \lambda_4-\lambda & \dots & (\lambda_4-\lambda)^{n-3} \\ & & \dots & \\ 1 & \lambda_n-\lambda & \dots & (\lambda_n-\lambda)^{n-3} \end{bmatrix} =$$

$$= bce\prod_{i=3}^{n}(\lambda_i-\lambda)^2 \prod_{3\leqslant i<j\leqslant n}(\lambda_i-\lambda_j).$$

The final expression for the determinant is nonzero. $\square$

Thus we have that $\Delta(X)$ is not in any minimal primes of $P+Q$. $\square$

Next we observe that $P+Q$ has no minimal primes of height larger than one over P and Q. First we need the following theorem due to Hartshorne.

Theorem IV.29 ([Har62] Proposition 2.1)**.** *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. If* $\operatorname{Spec}(A)-\{\mathfrak{m}\}$ *is disconnected, then the depth of A is at most 1.*

Lemma IV.30. *Let P and Q be ideals as in Notation IV.10. Then every minimal prime of $P+Q$ has height n^2-n+1.*

Proof. Suppose that there exists a minimal prime ideal T of $P+Q$ of height at least $\operatorname{ht}(I)+2$. Localize at T. Then $(P+Q)(R/I)_T$ is $T(R/I)_T$-primary. Moreover, $\mathbb{V}(P)$ and $\mathbb{V}(Q)$ are disjoint on the punctured spectrum $\operatorname{Spec}((R/I)_T)-\{T(R/I)_T\}$. However, the above theorem shows that this is not possible. $\square$

Now we prove that $P + Q$ has only one minimal prime.

Theorem IV.31. *Let P and Q be as in Notation IV.10. Then $\mathbb{V}(P + Q)$ is irreducible, i.e.,* $\mathrm{Rad}(P + Q)$ *is prime.*

Proof. Let $\mathcal{U}$ be a dense open subset in the algebraic set defined by $\mathcal{P}(X)$ as in Lemma IV.24. Let $A \in M_n(K)$ be such that $\mathcal{P}(A) = 0$. Suppose that $A \in \mathcal{U}$. Then by Lemma IV.24 there exists a matrix B such that (A, B) is in the skew-component of the algebraic set of nearly commuting matrices, that is $(A, B) \in \mathbb{V}(Q)$. Let $K[t]$ be a polynomial ring in one independent variable t. Fix any $f \in K[t]$. Then $(A, cB + f(A)) \in \mathbb{V}(Q)$ for all $c \in K - \{0\}$. Since Q defines a closed set, we must have that $(A, f(A)) \in \mathbb{V}(Q)$, i.e., when $c = 0$ as well. Since U is a dense subset in $\mathbb{V}(\mathcal{P}(X))$, $(A, f(A)) \in \mathbb{V}(Q)$ for all $A \in \mathbb{V}(\mathcal{P}(X))$. Recall that f was an arbitrary element of $K[t]$.

Now assume also that $\Delta(A) \neq 0$. Then every matrix B that commutes with A is a polynomial in A of degree at most $n - 1$. Thus

$$\mathbb{V}(P)_{\Delta(X)} = \{(A, f(A)) \mid \Delta(A) \neq 0 \text{ and } f \text{ is a polynomial of degree at most } n - 1\}.$$

Moreover, since $\mathbb{V}(P + Q) \subset \mathbb{V}(P)$, every element of $\mathbb{V}(P + Q)_{\Delta(X)}$ is of the form $(A, f(A))$, where $\mathcal{P}(A) = 0$ and f is a polynomial of degree at most $n - 1$.

Identify polynomials $f \in K[t]$ of degree at most $n - 1$ with $\mathbb{A}^n$. Then we can consider a map

$$\mathbb{V}(\mathcal{P}(X)) \times \mathbb{A}^n \to \mathbb{V}(P + Q)_{\Delta(X)}$$

such that

$$(A, f) \to (A, f(A)).$$

Moreover, this map is a bijective morphism. Therefore, $\mathbb{V}(P + Q)_{\Delta(X)}$ is irreducible. If $\mathbb{V}(P + Q)$ is not irreducible, then its nontrivial irreducible decomposition will give us a nontrivial irreducible decomposition of $\mathbb{V}(P+Q)_{\Delta(X)}$. Thus the result.

$\square$

Corollary IV.32. *Let P and Q be as in Notation IV.10. Then, when $n = 3$, $P+Q$ is prime.* $\square$

4.4 The ideal of nearly commuting matrices is a radical ideal

In this section we prove that I is a radical ideal in all characteristics. We know that $\mathrm{Rad}(I) = P \bigcap Q$. To prove the result it is sufficient to show that I becomes prime or radical once we localize at P or Q.

Theorem IV.33. *The defining ideal of the algebraic set of nearly commuting matrices is radical.*

Proof. For simplicity of notation, let $\mathcal{P}$ denote $\mathcal{P}(X)$.

We have that $K[X] \bigcap P = (0)$, since otherwise every $f \in K[X] \bigcap P$ must vanish when we set $X = Y$. Therefore, $W = K[X] - \{0\}$ is disjoint from P and hence from I. Localize at P. Then we have an injective homomorphism of $K[X,Y]/I$-modules

$$(K[X,Y]/I)_P \hookrightarrow (K(X)[Y]/I)_P \cong (L[Y]/I)_P,$$

where $L = K(X)$ and now I is an ideal generated by $n^2 - n$ linear equations in the entries of Y with coefficients in L. We can always choose at least n variables y_{ij}, $(i,j) \in \Lambda$, and write the rest of them as L-linear combinations of the chosen ones. Thus $(K[X,Y]/I)_P \hookrightarrow L[y_{ij}]_{(i,j)\in\Lambda}$ and $IK[X,Y]_P$ is prime.

Next observe that $K[X] \bigcap Q = (\mathcal{P})$. Clearly, $\mathcal{P} \subseteq Q$. To prove the other direction, let $f \in K[X] \bigcap Q$ be nonzero. Then by Lemma IV.23, $f \in (\mathcal{P})$. In other words, for all $A \in M_n(K)$ such that $A \in \mathbb{V}(Q)$ and such that there exists a matrix B with the property that $[A, B]$ is nonzero diagonal, then $\mathcal{P}(A) = 0$.

Therefore, we have an injective homomorphism of $K[X, Y]/I$-modules

$$(K[X, Y]/I)_Q \hookrightarrow (V[Y]/I)_Q,$$

where $V = K[X]_{(\mathcal{P})}$ is a discrete valuation domain. Then generators of I become linear polynomials in the entries of Y with coefficients in V. Let $\mathcal{B}$ be the matrix of coefficients of this linear system such that its rows are indexed by (i, j) for $1 \leqslant i \neq j \leqslant n$ and columns are indexed by (h, k) for all $1 \leqslant h, k \leqslant n$. Then $\mathcal{B}$ has an entry x_{ih} in the $(i, h), (h, k)$ spot, has an entry $-x_{kj}$ in the $(i, j), (i, k)$ spot, and zero everywhere else. Let $y_1, \ldots, y_{n^2}$ denote the entries of Y such that $y_{(i-1)n+j} = y_{ij}$. In $V[Y]$, I is generated by the entries of the matrix

$$\mathcal{B} \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_{n^2} \end{bmatrix}.$$

By doing elementary row operations over V, we can transform $\mathcal{B}$ into a diagonal matrix $\mathcal{C}$. This gives new generators of I. To prove that $IV[Y]$ is radical, it is sufficient to show that the diagonal entries in $\mathcal{C}$ have order at most one in V. To this end it reduces to show that $\mathcal{C}$ has rank $n^2 - n$ and the ideal generated by the minors of $\mathcal{C}$ of size $n^2 - n$ cannot be contained in $\mathcal{P}^2 V$. But then it is sufficient to prove this for the original matrix $\mathcal{B}$. Hence it suffices to show:

Claim.

(1) The submatrix $\mathcal{B}_0$ of $\mathcal{B}$ obtained from the first $n^2 - n$ columns has nonzero determinant in V.

(2) The determinant of $\mathcal{B}_0$ is in $(\mathcal{P}) - (\mathcal{P}^2)$.

Proof.

(1) It is sufficient to prove the first part of the claim over $K(X) = \mathrm{frac}(V)$, i.e., after we invert $\mathcal{P}$. Hence in S, since X and Y nearly commute, they must commute, see Lemma IV.23. Moreover, X is a generic matrix, hence its discriminant is nonzero and is not divisible by $\mathcal{P}$. Thus X has distinct eigenvalues and Y is a polynomial in X of degree at most $n-1$. Write $\mathcal{B} = [\mathcal{B}_0|\mathcal{B}_1]$, then our equations become

$$\mathcal{B}_0 \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n^2-n} \end{bmatrix} + \mathcal{B}_1 \begin{bmatrix} y_{n^2-n+1} \\ y_{n^2-n+2} \\ \dots \\ y_{n^2} \end{bmatrix} = 0.$$

Notice that $\mathcal{B}_0$ is invertible if and only if for every choice of the values for $[y_{n^2-n+1}, \dots, y_{n^2}]$ there is a unique solution for the above equation.

Furthermore, the bottom rows of X^0, $X, \dots, X^{n-1}$ are linearly independent for a generic matrix X. This is true because it even holds for the permutation matrix

$$E = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \dots & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

for which the bottom rows of $E^0, E, \dots, E^{n-1}$ are the standard basis vectors e_i for $1 \leqslant i \leqslant n$.

Thus, given any bottom row ρ of Y, there exist $\alpha_0, \dots, \alpha_{n-1} \in K(X)$ such that ρ equals the bottom row of $\alpha_0 + \alpha_1 X + \dots + \alpha_{n-1} X^{n-1}$. That is, such Y is uniquely determined by the entries of its bottom row. Therefore, $\mathcal{B}_0$ is invertible in $K(X)$.

(2) First, let us show that $\det \mathcal{B}_0 \in \mathcal{P}$. For any matrix A in an open dense subset defined by $\mathcal{P}$, there exists a matrix A' such that the commutator $[A, A']$ is a nonzero diagonal matrix, see Lemma IV.24 . Hence, for all $c \in K - \{0\}$ and for all $f \in K[X]$ polynomials of degree at most $n-1$, $(A, cA' + f(A)) \in I$. Therefore, the space of solutions of $\mathcal{B} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n^2} \end{bmatrix} = 0$ has dimension $n+1$, which is a contradiction since we showed that it must be n. Therefore, the minors of $\mathcal{B}$ must vanish whenever $\mathcal{P}$ vanishes.

Now let us put grading on the entries of X and Y. Let deg $x_{ij} =$ deg $y_{ij} = i - j$. Then their products XY and YX and sums have this property as well: deg $(XY)_{ij} = i-j$ and deg $(X+Y)_{ij} = i-j$. Therefore, so does the commutator matrix $XY - YX$. In fact, any polynomial in X and Y has this property. Notice that the diagonal entries have degree 0, thus $\mathcal{P}$ has degree 0. However, this is not the case for the determinant of the matrix $\mathcal{B}_0$. The nonzero entry corresponding to $(i, j), (h, k)$ has degree $i - j + h - k$. Therefore, if a product of the entries is a nonzero term of the determinant of $\mathcal{B}_0$, then its degree is $\sum_{1 \leqslant i \neq j \leqslant n} \sum_{1 \leqslant h \leqslant n} \sum_{1 \leqslant k < n} (i - j + h - k) = \sum_{1 \leqslant i \neq j \leqslant n} (i-j) + \sum_{1 \leqslant h,k \leqslant n} (h-k) - \sum_{1 \leqslant h \leqslant n} (h-n) = 0 + 0 - (1-n+2-n+\dots(-1)) = n(n-1)/2 \neq 0$ for all $n \geqslant 2$. Hence $\mathcal{B}_0$ cannot be a K-scalar multiple of a power of $\mathcal{P}$. That is, when we factor out $\mathcal{P}$ from the minors of $\mathcal{B}$, the remaining expression is not divisible by $\mathcal{P}$. $\square$

Now we are ready to finish our discussion. We have that $\mathcal{C}$ is a diagonal matrix of maximal rank with entries in V and its submatrix $\mathcal{C}_0$ has the determinant in $\mathcal{P} - \mathcal{P}^2$.

More precisely, I is generated by the following equations.

$$\left[\begin{array}{cccccc|ccc} v_{11} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & v_{22} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & \dots & & \dots & & & \dots & \\ 0 & 0 & 0 & \dots & v_{n^2-n-1,n^2-n-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & v_{n^2-n,n^2-n} & 0 & \dots & 0 \end{array}\right] \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{n^2} \end{bmatrix},$$

where only one of the $v_{ij} \in V$ is divisible by $\mathcal{P}$, the rest are units in V. Then $V[Y]/I \cong V[z]/(z\mathcal{P})$ is reduced. Finally, $(R/I)_Q \hookrightarrow (V[Y]/I)_Q$. Hence IR_Q is radical and therefore is prime. $\square$

4.5 F-regularity

In this section we state a conjecture that the variety of commuting matrices $\mathbb{V}(P)$, the skew-component $\mathbb{V}(Q)$ and their intersection $\mathbb{V}(P+Q)$ are F-regular. We show that when R/P is Cohen-Macaulay it is sufficient to have F-regularity of $R/(P+Q)$ in order to prove the conjecture. Furthermore, we discuss various ways to attack the problem.

Conjecture IV.34. *Let R be as in Notation IV.10. Then R/P, R/Q and $R/(P+Q)$ are F-regular.*

The following lemma allows us to reduce the above conjecture to F-regularity of $R/(P+Q)$.

Lemma IV.35. *Let R be a Noetherian local or $\mathbb{N}$-graded ring of prime characteristic $p>0$ and let I be an ideal (homogeneous in the graded case) generated by a regular sequence. Let P and Q be ideals of R of the same height such that P and Q are linked* *via* $I = P \bigcap Q$. *Let* R/P *be Cohen-Macaulay. Suppose that* $R/(P+Q)$ *is* F*-regular (or equivalently,* F*-rational). Then* R/P *and* R/Q *are* F*-regular.*

Proof. By [PS74], R/Q is Cohen-Macaulay and has the canonical module isomorphic to $(P+Q)/Q$. Similarly, the canonical module of R/P is $(P+Q)/P$. Then $R/(P+Q)$ is Gorenstein, hence it is F-rational if and only if it is F-regular.

Recall that a graded ring R is F-regular if and only if $R_{\mathfrak{m}}$ is F-regular, [LS99]. Then $R/(P+Q)$ is F-rational if and only if its localization at the homogeneous maximal ideal is F-rational. Then by applying Lemma II.46 we get that $R/(P+Q)$ is F-rational implies that so are R/P and R/Q. □

Thus if we want to prove that the variety of commuting matrices and the skew component are F-regular, it is sufficient to prove the statement for their intersection. Of course we need to know whether R/P is Cohen-Macaulay.

There are few ways to achieve F-regularity of $R/(P+Q)$. There exists a criterion similar to Fedder's criterion for F-purity, Lemma IV.12. It is due to D. Glassbrenner, [Gla96], who proved the following result.

Theorem IV.36. *Let* $(S, \mathfrak{m})$ *be an* F*-finite regular local ring of positive characteristic* p*. Let* I *be an ideal of* S *and let* $s \in R^{\circ}$ *be such that* R_s *is regular. Then* S/I *is strongly* F*-regular if and only if there exists a positive integer* e *such that* $s(I^{[p^e]} : I) \nsubseteq \mathfrak{m}^{[p^e]}$.

One direction that one might be interested in pursuing is to give a criterion for F-regularity of R/P, R/Q and $R/(P+Q)$ similar to that of Lemma IV.13. That is, we want to have a certain condition on I which will imply F-regularity of our rings. However, one needs to be cautious, since R/I cannot be F-regular itself.

Another possible way to show that $R/(P+Q)$ is F-regular utilizes a result due to M. Hochster and C. Huneke, see [HH94b] Corollary 7.13, which is the following.

Theorem IV.37. *Let R be a finitely generated $\mathbb{N}$-graded Gorenstein K-algebra with homogeneous maximal ideal $\mathfrak{m}$ such that $R_0 = K$. Suppose that $\dim R = d \geqslant 2$. Then the following are equivalent.*

(1) R is F-regular.

(2) Localization of R at any prime except $\mathfrak{m}$ is F-regular, there is an ideal generated by a homogeneous system of parameters such that it is Frobenius closed and $a(R) < 0$, where the a-invariant of R is $\max\{i \mid [H^d_{\mathfrak{m}}(R)]_i \neq 0\}$.

Thus if we want to use the above theorem to prove F-regularity of $R/(P+Q)$ when $n = 3$, then we only need to show that $R/(P+Q)$ is locally F-regular. This is true because we already know that it is F-pure when $n = 3$. As for the a-invariant, we have a result which insures that $a(R/(P+Q)) < 0$. However, we again need to rely on the fact that R/P and R/Q are Cohen-Macaulay.

Lemma IV.38. *Let R be a Noetherian $\mathbb{N}$-graded ring. Let M, M', M'' be finitely generated $\mathbb{Z}$-graded Cohen-Macaulay R-modules such that $\dim M = \dim M'' = \dim M' - 1$. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence and $a(M) < 0$. Then $a(M), a(M'')$ are negative.*

Before we prove the lemma let us state some properties of a-invariants, see [HH94b] Section 7.4.

Lemma IV.39. *Let R be a Noetherian $\mathbb{N}$-graded ring such that $R_0 = K$. Then*

(1) Let S be a polynomial ring $K[x_1, \ldots, x_n]$ and I be a homogeneous ideal of S generated by a regular sequence $f_1, \ldots, f_m$. Let $R = S/I$. Then $a(R) = \sum_{i=1}^{m} deg f_i - \sum_{i=j}^{n} deg\ x_j$.

(2) Let $h(z)$ be a rational function whose expansion is a Hilbert-Poincaré series of R, $\sum_{i=1}^{\infty} \dim_K [R]_i$. Then $a(R) = deg\ h(z)$, where the degree of a rational function

$f(z)/g(z)$ is deg $f(z)$ − deg $g(z)$. Moreover, if R has dimension d, then $h(z) = f(z)/(1-t)^d$, where $f(z) \in \mathbb{Z}[z]$. □

Now we can prove the lemma.

Proof. Let $\dim M' = d$. Since M', M, M'' are Cohen-Macaulay, their Hilbert-Poincaré series are of the form $F(z)/(1-t)^d$, $G(z)/(1-t)^{d-1}$ and $H(z)/(1-t)^{d-1}$ respectively and F, G, H are polynomials with positive integer coefficients.

Since $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, we have that

$$G(z)/(1-t)^{d-1} = F(z)/(1-t)^d + H(z)//(1-t)^{d-1}.$$

Equivalently,

$$G(z)(1-t) = F(z) + H(z)(1-t).$$

By hypothesis, $a(M') < 0$, therefore, $\deg G(z) < d-1$. However, the leading coefficient of $G(z)(1-t)$ is negative. Hence it should come with a contribution from a leading coefficient of $H(z)(1-t)$. Then $\deg G = \deg H < d-1$. Also, $\deg F(z) \leqslant \deg G(z)(1-t) < d$. Thus, $a(M) < 0$ and $a(M'') < 0$. □

Next we apply Lemma IV.38 to our situation.

Lemma IV.40. *Let R be as in Assumption IV.10. Suppose that R/P is Cohen-Macaulay, which is true when $n = 3$. Then $a(R/P), a(R/Q)$ and $a(R/(P+Q)$ are negative.*

Proof. We have the following short exact sequence of R-modules.

$$0 \to R/I \to R/P \oplus R/Q \to R/(P+Q) \to 0$$

Since I is generated by a regular sequence, $a(R/I) = 2(n^2-n) - 2n^2 = -2n < 0$, see Lemma IV.39 (1). By Lemma IV.9, R/Q is Cohen-Macaulay and $R/(P+Q)$

is Gorenstein of dimension $\dim R/P - 1$. Therefore, by Lemma IV.38 we have that $R/P, R/Q$ and $R/(P+Q)$ have negative a-invariant. $\square$

4.6 Conjectures

Here we list the conjectures we have developed in the course of working on the algebraic set of nearly commuting matrices and its irreducible components.

Conjecture IV.41. *R/I is F-pure for all $n \geqslant 4$.*

The above conjecture can be solved by proving the following one.

Conjecture IV.42. *Let $\mu = \frac{\prod_{i=1,j=1}^{n} x_{ij}y_{ij}}{\prod_{i=1}^{n-1} x_{ii}y_{i,n-i+1} \cdot x_{n,n-1} \cdot y_{n-1,1}}$.*

Then μ^{p-1} is a monomial term of $\prod_{1 \leqslant i \neq j \leqslant n} u_{ij}^{p}$ with the coefficient equal to 1 modulo p.

Remark IV.43. The above monomial can be obtained taking the product of all the variables and dividing by the variables according to the following pattern: denote by $\star$ the variable to be divided out.

$$X = \begin{vmatrix} \star & x_{12} & \dots & x_{1,n-2} & x_{1,n-1} & x_{1n} \\ x_{21} & \star & \dots & x_{2,n-2} & x_{2,n-1} & x_{2n} \\ & \dots & & & \dots & \\ x_{n-2,1} & x_{n-2,2} & \dots & \star & x_{n-2,n-1} & x_{n-1,n} \\ x_{n-1,1} & x_{n-1,2} & \dots & x_{n-1,n-2} & \star & x_{n-1,n} \\ x_{n,1} & x_{n,2} & \dots & x_{n,n-2} & \star & x_{n,n} \end{vmatrix},$$

$$Y = \begin{vmatrix} y_{11} & y_{12} & y_{1,3} & \cdots & y_{1,n-1} & \star \\ y_{21} & y_{22} & y_{2,3} & \cdots & \star & y_{2n} \\ & \cdots & & \cdots & & \\ y_{n-2,1} & y_{n-2,2} & \star & \cdots & y_{n-2,n-1} & y_{n-2,n} \\ \star & \star & y_{n-1,3} & \cdots & y_{n-1,n-1} & y_{n-1,n} \\ y_{n,1} & y_{n,2} & y_{n,3} & \cdots & y_{n,n-1} & y_{n,n} \end{vmatrix}$$

Conjecture IV.44. *$R/(P+Q)$ is F-regular for all $n \geqslant 3$.*

Remark IV.45. In the case when $n = 2$ the conjecture is true.

Conjecture IV.46. *Let X be a matrix of indeterminates of size n over a field K. Let $\mathcal{P}(X)$ be the irreducible polynomial as in Definition IV.20. Then $K[X]/\mathcal{P}(X)$ is F-regular.*

Conjecture IV.47. *The following is a regular sequence on R/I and hence a part of a system of parameters on R/J and R/Q.*

$$x_{st} - y_{t,\theta(s,t)},\ x_{1n},\ x_{nn},\ x_{11} - y_{2n}$$

for all $1 \leqslant s, t, \leqslant n$ and where $\theta(s,t) = \begin{cases} (s+t)\text{mod } n, & \text{if } s+t \neq n; \\ n, & \text{if } s+t = n. \end{cases}$

Remark IV.48. The conjecture was verified by using Macaulay2 software when $n = 3, 4$ over $K = \mathbb{Q}$ and in some small prime characteristics.

In the case when $n = 3$, this is equivalent to the following identifications of variables in matrices X and Y

$$X = \begin{vmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & x_{22} \\ x_{31} & x_{32} & 0 \end{vmatrix}, \quad Y = \begin{vmatrix} x_{31} & x_{11} & x_{21} \\ x_{22} & x_{32} & x_{12} \\ 0 & x_{22} & 0 \end{vmatrix}.$$

Conjecture IV.49. *Let $Z \subseteq \{u_{ij} \,|\, 1 \leqslant i \neq j \leqslant n\}$ be any subset of cardinality at most $n^2 - n - 1$. Let $\mathcal{I}_Z$ be the ideal of R generated by the elements of Z. Then $R/\mathcal{I}_Z$ is F-regular. In particular, $\mathcal{I}_Z$ is a prime ideal.*

4.7 Appendix

We devote this section to interesting computational observations we have obtained using Macaulay2, a computer algebra system, [GS], while working on the proof of F-purity of the coordinate ring R/I of the algebraic set of nearly commuting matrices of size 3.

Recall that we used Fedder's criterion, Lemma IV.12, to prove F-purity of R/I in the case of 3 by 3 matrices. This was done by exhibiting a nonzero monomial term of $\prod_{1 \leqslant i \neq j \leqslant n} u_{ij}^{p-1}$ such that it is not in $\mathfrak{m}^{[p]}$. The term given in the proof turns out to be a rather nice one as our computations on Macaulay2 show. In particular, it has coefficient 1 modulo p for all values of characteristic p. However, there are terms which appear to work in many but not all characteristics. We want to give one example of such a term.

Let $\mu = x_{11}x_{12}x_{21}x_{22}x_{31}x_{32}y_{11}y_{12}y_{13}y_{21}y_{22}y_{23}$. It is the product of all the entries of X except for the last column and all the entries of Y except for the last row. As in the proof for F-purity given earlier in this chapter, we first look at all possible ways to obtain μ^{p-1} in $\prod_{1 \leqslant i \neq j \leqslant n} u_{ij}^{p-1}$. We then solve a linear system of equations associated with all the choices. Then we have that the coefficient of μ^{p-1} modulo p is equal to

$$\sum_{0\leqslant a+b\leqslant p-1}\sum_{c=0}^{a}\binom{p-1-b}{a}^2\binom{a}{c}^2.$$

Unfortunately, this seems to be zero for many values of p. The table below gives the coefficient of μ^{p-1} in $\prod_{1\leqslant i\neq j\leqslant n} u_{ij}^{p-1}$ for prime values of $p<1000$.

p	coefficient of μ^{p-1}	p	coefficient of μ^{p-1}	p	coefficient of μ^{p-1}
2	0	67	-5	157	0
3	1	71	0	163	4
5	0	73	4	167	0
7	0	79	0	173	0
11	3	83	-8	179	-34
13	0	89	-32	181	0
17	2	97	3	191	0
19	4	101	0	193	-95
23	0	103	0	197	0
29	0	107	36	199	0
31	0	109	0	211	-15
37	0	113	-15	223	0
41	-5	127	0	227	-8
43	14	131	62	229	0
47	0	137	36	233	-32
53	0	139	67	239	0
59	-23	149	0	241	-47
61	0	151	0	251	36

p	coefficient of μ^{p-1}
257	-128
263	0
269	0
271	0
277	0
281	43
283	-82
293	0
307	-72
311	0
313	100
317	0
331	14
337	-141
347	36
349	0
353	-159
359	0
367	0
373	0
379	-72
383	0
389	0

p	coefficient of μ^{p-1}
397	0
401	36
409	75
419	-95
421	0
431	0
433	145
439	0
443	-8
449	-32
457	219
461	0
463	0
467	-34
479	0
487	0
491	-200
499	196
503	0
509	0
521	36
523	-125
541	0

p	coefficient of μ^{p-1}
547	-72
557	0
563	-226
569	57
571	-87
577	2
587	36
593	-269
599	0
601	-288
607	0
613	0
617	283
619	57
631	0
641	-159
643	-72
647	0
653	0
659	324
661	0
673	100
677	0

p	coefficient of μ^{p-1}
683	-285
691	43
701	0
709	0
719	0
727	0
733	0
739	-322
743	0
751	0
757	0
761	-128
769	-285
773	0
787	139

p	coefficient of μ^{p-1}
797	0
809	36
811	-178
821	0
823	0
827	-392
829	0
839	0
853	0
857	345
859	-72
863	0
877	0
881	324
883	4

p	coefficient of μ^{p-1}
887	0
907	100
911	0
919	0
929	129
937	219
941	0
947	-47
953	-142
967	0
971	3
977	36
983	0
991	0
997	0

CHAPTER V

Algebras with straightening law

5.1 Introduction and preliminaries

Definition V.1. Let A be a commutative ring and let R be an A-algebra. Let $\Pi \subseteq R$ be a finite subset with a partial order $\leqslant$, called a poset for short. Then R is a *graded algebra with straightening law* (ASL) over A on Π if the following conditions hold:

(H_0) $R = \oplus_{i \geqslant 0} R_i$ is a graded A-algebra such that $R_0 = A$, Π consists of homogeneous elements of positive degree and generates R as an A-algebra.

(H_1) The products $\xi_1 \dots \xi_m$, where $m \in \mathbb{N}$, $\xi_i \in \Pi$ and $\xi_1 \leqslant \dots \leqslant \xi_m$, are linearly independent. They are called *standard monomials*.

(H_1) (Straightening law) For all incomparable ξ, $\nu \in \Pi$, the product $\xi\nu$ has a representation $\xi\nu = \sum a_\mu \mu$ where $0 \neq a_\mu \in A$ and μ is a standard monomial, satisfying the following condition: every μ contains a factor $\zeta \in \Pi$ such that $\zeta \leqslant \xi$, $\zeta \leqslant \nu$. It is allowed that $\xi\nu = 0$, the sum $\sum a_\mu \mu$ being empty.

Lemma V.2 ([BV88] Proposition 4.1)**.** *Let R be an ASL over A on Π. Then every monomial $\mu = \mu_1 \dots \mu_k$ in elements of Π is an A-linear combination $\sum_\lambda a_\lambda \lambda$ of standard monomials, such that every standard monomial λ on the right hand side* *has a factor $\lambda' \leqslant \mu_1, \ldots, \mu_n$ and $\omega(\mu) \leqslant \omega(\lambda)$ with $\omega(\mu) = \omega(\lambda)$ if and only if μ is standard (and hence $\mu = \lambda$).*

5.2 Change of basis for a graded algebra with straightening law

In this section we shall prove that given a basis for an ASL we can obtain a new one with an induced partial order by a certain linear transformation which preserves the property of being an ASL.

For the rest of the section assume the following hypothesis on R.

Let R be a graded algebra with straightening law on a finite poset $(\Pi, \leqslant)$ over a commutative ring A. Let $\xi_1 < \xi_2 \in \Pi$ be homogeneous elements of the same degree. Let $u = c\xi_1 + \xi_2$, where $c \in A$. Define a partial order $\leqslant$ on $\Sigma = \Pi \bigcup \{u\}$ extending $\leqslant$ on Π such that if $\mu, \nu \in \Pi - \{\xi_2\}$, $\mu < u$ if and only if $\mu < \xi_2$ and $u < \nu$ if and only if $\xi_2 < \nu$. Let $\Pi' = \Sigma - \{\xi_2\} = \Pi \bigcup \{u\} - \{\xi_2\}$ be a poset with the partial order induced from Σ.

Definition V.3. For an element $\xi \in \Sigma$ first define $\alpha(\xi) = |\{\delta \in \Sigma : \xi \leqslant \delta\}|$. Then let $\omega(\xi) = 3^{\alpha(\xi)}$ and $\omega(\xi_1 \cdots \xi_n) = \sum_{i=1}^{n} \omega(\xi_i)$.

Definition V.4. For an element $\xi \in \Sigma$ define $\mathrm{rk}\ \xi = \max\{k \in \mathbb{N} :$ there is a chain $\xi = \xi_k > \xi_{k-1} > \ldots > \xi_1,\ \xi_i \in \Pi\}$. For a subset $\Omega \subseteq \Sigma$ let $\mathrm{rk}\ \Omega = \max\{\mathrm{rk}\ \xi : \xi \in \Omega\}$.

Lemma V.5. *Every monomial $\mu = \mu_1 \ldots \mu_t$ in elements of Π' can be written as an A-linear combination of standard monomials in Π', i.e., $\mu = \sum_\tau a_\tau \tau$, where τ is a standard monomial in Π', and such that τ has a factor $\zeta \leqslant \mu_1, \ldots, \mu_t$.*

Proof. We shall prove a slightly more general statement: every monomial in elements of Σ is an A-linear combination of standard monomials in Π'. If $\mu = \mu_1^t$ and μ_1 is minimal in Σ, then $\mu_1 \neq \xi_2$ and the lemma holds.

Let μ be a monomial in elements of Σ. If u does not divide μ, then it is a monomial in Π. Then by Lemma V.2, it is a linear combination of standard monomials in Π. If none of them have ξ_2 as a factor, then they are also standard monomials in Π'. Otherwise, replace ξ_2 by $u - c\xi_1$. Thus it is sufficient to consider monomials $\mu = \mu_1 \cdots \mu_r u^m \mu_s \cdots \mu_t$ with $\mu_1, \dots \mu_r, \mu_s, \dots, \mu_t \in \Pi' - \{u\}$ and $m \geqslant 1$.

If μ is standard, then we are done. So assume that μ is not standard.

Then

$$\mu = \mu_1 \cdots \mu_r (c\xi_1 + \xi_2)^m \mu_s \cdots \mu_t =$$
$$\sum_{j=0}^{m-1} \binom{m}{j} c_j \mu_1 \cdots \mu_r \xi_1^j \xi_2^{m-j} \mu_s \cdots \mu_t + \mu_1 \cdots \mu_r \xi_2^m \mu_s \cdots \mu_t.$$

Since $\xi_1 < \xi_2$, $\omega(\xi_1) > \omega(\xi_2)$. Therefore, $\omega(\mu) < \omega(\mu_1 \cdots \mu_r \xi_1^j \xi_2^{m-j} \mu_s \cdots \mu_t)$ for all $j < m$ and $\omega(\mu) = \omega(\mu_1 \cdots \mu_r \xi_2^m \mu_s \cdots \mu_t)$. Since μ is not standard, so is not $\mu' = \mu_1 \cdots \mu_r \xi_2^m \mu_s \cdots \mu_t$. Therefore by Lemma V.2, it is a linear combination $\sum a_\lambda \lambda$ of standard monomials in Π with a larger weight such that λ has a factor $\lambda' \in \Pi$ and $\lambda' \leqslant \mu_1, \dots, \mu_r, \xi_2, \mu_s, \dots, \mu_t$. Then by decreasing induction on the values of ω we can write λ as a linear combination of standard monomials in Π' which have a factor $\zeta \in \Pi'$ such that $\zeta \leqslant \lambda'$. Also, by decreasing induction on the values of ω each of the summands $\mu_1 \cdots \mu_r \xi_1^j \xi_2^{m-j} \mu_s \cdots \mu_t = \sum b_\tau \tau$ is linear combination of standard monomials in Π' such that τ has a factor $\tau' \in \Pi'$ and $\tau' \leqslant \mu_1, \dots, \mu_r, \xi_1, \mu_s \dots, \mu_t$. Thus the desired result. □

Theorem V.6. *R is an ASL over A on Π'.*

Proof. The first two conditions (H_0) and (H_1) are immediate from the corresponding properties of R as an ASL on Π. Let us prove the remaining condition (H_2).

Let λ, $\nu \in \Pi'$ be incomparable. By the previous lemma, $\lambda\nu = \sum b_\tau \tau$ where each τ is a standard monomial in elements of Π' and has a factor $\tau' \in \Pi'$ such that

$\tau' \leqslant \lambda, \nu$. $\square$

Corollary V.7. *Let R be a graded algebra with straightening law on a finite poset $(\Pi, \leqslant)$ over a commutative ring A.*

For $n \geqslant 2$ and $m \geqslant 1$ let $\begin{array}{ccccc} \xi_{11} & < & \ldots & < & \xi_{1n} \\ & \ldots & \ldots & \ldots & \\ \xi_{m1} & < & \ldots & < & \xi_{mn} \end{array}$ *be homogeneous elements in Π such that elements in each row have the same degree and either rows are pairwise disjoint or the maximal elements in each row do not occur in other rows.*

Let $u_i = c_{i1}\xi_{i1} + \ldots + c_{i,n-1}\xi_{i,n-1} + \xi_{in}$, where $c_{i1}, \ldots, c_{i,n-1} \in A$ for all $1 \leqslant i \leqslant m$. Define a partial order $\leqslant$ on $\Sigma = \Pi \bigcup \{u_i\}_{i=1}^m$ extending $\leqslant$ on Π such that if $\mu, \nu \in \Pi - \{\xi_{in}\}$, $\mu < u_i$ if and only if $\mu < \xi_{in}$ and $u_i < \nu$ if and only if $\xi_{in} < \nu$. Let $\Pi' = \Sigma - \bigcup\{\xi_{in}\}_{i=1}^m$ be a poset with the partial order induced from Σ. Then R is an ASL on Π' over A.

Proof. Induct on n and m. Case when $n = 2$ and $m = 1$ comes from the Theorem V.6.

$\square$

5.3 Application to algebraic sets of nearly commuting and commuting matrices

Let $X = (x_{ij})_{1 \leqslant i,j \leqslant n}$ and $Y = (y_{ij})_{1 \leqslant i,j \leqslant n}$ be $n \times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1 \leqslant i,j \leqslant n}$. As in Chapter 2 let I be the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and let J be the ideal generated by the entries of $XY - YX$. Let u_{ij} denote the (i, j)th entry of the matrix $XY - YX$. Then $I = (u_{ij} \mid 1 \leqslant i \neq j \leqslant n)$. Moreover, $u_{ij} = \sum_{k=1}^n x_{ik}y_{kj} - \sum_{k=1}^n y_{ik}x_{kj} = \sum_{k=1}^n (x_{ik}y_{kj} - x_{kj}y_{ik}) = \sum_{k=1}^n \det \begin{vmatrix} x_{ik} & x_{kj} \\ y_{ik} & y_{kj} \end{vmatrix}$.

Notice that every minor of the form $\begin{vmatrix} x_{ik} & x_{kj} \\ y_{ik} & y_{kj} \end{vmatrix}$ occurs as one of the summands of u_{ij} for $1 \leqslant i, j \leqslant n$. Moreover, for all $i \neq j$ there are $n^2(n-1)$ of such minors in total. Also, there are $n^2 - n = n(n-1)$ generators u_{ij} of I each being a sum of n minors. Therefore, all the minors that occur as summands of the generators of I are distinct.

Write $R = K\begin{bmatrix} x_{11} & x_{12} & \dots & x_{21} & x_{22} & \dots & x_{nn} \\ y_{11} & y_{12} & \dots & y_{21} & y_{22} & \dots & y_{nn} \end{bmatrix}$, where the $n(k-1)+l$th column is the column vector $[x_{kl}, y_{kl}]$.

Let $Z = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{21} & x_{22} & \dots & x_{nn} & 1 & 0 \\ y_{11} & y_{12} & \dots & y_{21} & y_{22} & \dots & y_{nn} & 0 & 1 \end{bmatrix}$.

Let Π be a set of all 2×2 minors of Z partially ordered in the following way: each minor can be represented by a pair $[k \quad l]$, for some $1 \leqslant k \neq l \leqslant n+2$, and we say that $[k \quad l] \leqslant [s \quad t]$ if and only if $k \leqslant s$ and $l \leqslant t$.

Therefore, each 2 by 2 minor $\begin{vmatrix} x_{kl} & x_{st} \\ y_{kl} & y_{st} \end{vmatrix}$ of Z corresponds to $[n(k-1)+l \quad n(s-1)+t]$.

In particular, $u_{ij} = \sum_{k=1}^{n} \det \begin{vmatrix} x_{ik} & x_{kj} \\ y_{ik} & y_{kj} \end{vmatrix} = \sum_{k=1}^{n} [n(i-1)+k \quad j+n(k-1)]$. Notice that the summands of u_{ij} increase by $[1 \quad n]$ as k increases.

Proposition V.8 ([BV88] Theorem 4.11)**.** *Let $R = K[X, Y]$ be defined as above. Then R is an algebra with straightening law on Π over K.*

For all $1 \leqslant i \neq j \leqslant n, 1 \leqslant k \leqslant n$, let $\xi_{ij}^{(k)} = [n(i-1)+k \quad j+n(k-1)]$, then $u_{ij} = \sum_{k=1}^{n} \xi_{ij}^{(k)}$. Moreover, $\xi_{ij}^{(1)} < \ldots < \xi_{ij}^{(n)}$. Define a partial order $\leqslant$ on $\Sigma = \Pi \bigcup \{u_{ij}\}_{1 \leqslant i \neq j \leqslant n}$ extending $\leqslant$ on Π such that if $\mu, \nu \in \Pi - \{\xi_{ij}^{(n)}\}$, $\mu < u_{ij}$ if and only if $\mu < \xi_{ij}^{(n)}$ and $u_{ij} < \nu$ if and only if $\xi_{ij}^{(n)} < \nu$. Let $\Pi' = \Sigma - \bigcup\{\xi_{ij}^{(n)}\}_{1 \leqslant i \neq j \leqslant n}$ be a poset with the partial order induced from Σ.

Proposition V.9. *The polynomial ring $R = K[X, Y]$ is a graded ASL on Π' over K.*

Proof. First notice that $\xi_{ij}^{(k)} = \xi_{rs}^{(t)}$ if and only if $i = r$, $j = s$, $k = t$, i.e., the sets of summands of u_{ij} are disjoint for distinct pairs of (i, j). Therefore, we may apply Corollary V.7 and get the desired result. □

Now let us consider the elements $\{u_{ii}\}_{i=1}^{n} = \sum_{k=1}^{n} \det \begin{vmatrix} x_{ik} & x_{kj} \\ y_{ik} & y_{kj} \end{vmatrix}$ on the diagonal of $XY - YX$. We do not have disjointness of the sets of summands. Each $\xi_i^{(k)} = \det \begin{vmatrix} x_{ik} & x_{kj} \\ y_{ik} & y_{kj} \end{vmatrix}$ occurs precisely twice: in u_{ii} and with negative sign in u_{kk}. In particular, the largest terms $\xi_i^{(n)}$ in every u_{ii} also occur in u_{nn}. Therefore, if we only consider $\{u_{ii}\}_{1 \leqslant i < n}$, then the largest terms appear only once.

Recall that $tr(XY - YX) = 0$, hence $u_{nn} = -u_{11} - \ldots - u_{n-1,n-1}$ can be omitted from the list of generators of J.

Define a partial order $\leqslant$ on $\Sigma' = \Pi' \bigcup \{u_{ii}\}_{1 \leqslant i < n}$ extending $\leqslant$ on Π' such that if $\mu, \nu \in \Pi' - \{\xi_i^{(n)}\}$, $\mu < u_{ii}$ if and only if $\mu < \xi_i^{(n)}$ and $u_{ii} < \nu$ if and only if $\xi_i^{(n)} < \nu$. Let $\Pi'' = \Sigma' - \bigcup\{\xi_i^{(n)}\}_{1 \leqslant i < n}$ be a poset with the partial order induced from Σ'.

Proposition V.10. *R is a graded ASL on Π'' over K.*

Proof. Follows from Proposition V.9 and Corollary V.7. □

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