Measuring Influence and Topic Dependent Interactions in Social Media Networks Based on a Counting Process Modeling Framework

by

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To my parents
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ABSTRACT

Measuring Influence and Topic Dependent Interactions in Social Media Networks Based on a Counting Process Modeling Framework

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Data extracted from social media platforms, such as Twitter, are both large in scale and complex in nature, since they contain both unstructured text, as well as structured data, such as time stamps and interactions between users. Some key questions for such platforms are (i) to determine influential users, in the sense that they generate interactions between members of the platform and (ii) identifying important interactions between nodes in the corresponding user network.

Regarding the first question, common measures used both in the academic literature and by companies that provide analytics services are primarily variants of the popular web-search PageRank algorithm applied to networks that capture connections between users. In this work, we develop a modeling framework using multivariate interacting counting processes to capture the detailed actions that users undertake on such platforms, namely posting original content, reposting and/or mentioning other users’ postings. Based on the proposed model, we also derive a novel influence measure. We discuss estimation of the model parameters through maximum likelihood and establish their asymptotic properties. The proposed model and the accompa-
nying influence measure are illustrated on a data set covering a five year period of the Twitter actions of the members of the US Senate, as well as mainstream news organizations and media personalities.

We then turn our attention to the problem of identifying important interactions both globally and also based on the particular topics under discussion. We modify the previously introduced modeling framework, so that topic dependent interactions can also be identified. We extend our previous algorithm to accommodate the new framework and also establish asymptotic properties of the key model parameters. We illustrate the results on the same Twitter data set.
CHAPTER I

Introduction

1.1 Background and Literature Review

Leading business and non-profit organizations are integrating growing volumes of increasingly complex structured and unstructured data to create big data ecosystems for content distribution, as well as to gain insights for decision making. A recent, substantial area of growth has been online review and social media platforms, which have fundamentally altered the public discourse by providing easy to use forums for the distribution and exchange of news, ideas and opinions. The focus in diverse areas, including marketing, business analytics and social network analysis, is to identify trends and extract patterns in the vast amount of data produced by these platforms, so that more careful targeting of content distribution, propagation of ideas, opinions and products, as well as resource optimization is achieved (Dave, 2015; Probst et al., 2013).

One platform that has become of central importance to both business and non-profit enterprises is Twitter. According to its second quarter 2014 financial results announcement, Twitter had more than half a billion users in July 2014, out of which more than 271 million were active ones (Twitter, 2014). Although Twitter lags behind in terms of active users to Facebook, it is nevertheless perceived by most businesses and non-profit organizations as an integral part of their digital presence (Bulearca
and Bulearca, 2010).

The mechanics of Twitter are as follows: the basic communication unit is the account. The platform allows account users to post messages of at most 140 characters, and thus has been described as the Short Message Service (SMS) of the Internet. As of mid-2014, over half a billion messages were posted on a daily basis. Further, Twitter allows accounts to “follow” other accounts, which means the follower receives notification whenever the followed account posts a new message. Thus, the follow-follower relations serve as a primary channel for content to spread within the social networking platform. Accounts tend to interact with each other over these channels in two directed ways. First, an account can copy or rebroadcast another account’s tweet, which is referred to as a “retweeting”. Second, an account can mention another account within a tweet by referring to their account name with the @ symbol as a prefix. These two actions, retweeting and mentioning, are directed responses from one account to another and thus, provide the mechanisms for online conversation.

The mechanics of Twitter, together with the original messages generated by users, give rise to rich Big Data. Specifically, the content of the message, together with easily searchable key terms or topics that use the # symbol as a prefix, constitute a large corpus of unstructured text. The hashtag function enables searches to identify emerging themes and topics of discussion. In 2014, more than 2.1 billion search queries were generated (Twitter, 2014). Further, the following built-in capability, creates a network for potential information flow and dissemination, while the retweeting and mentioning actions create subnetworks of actual interactions between user accounts.

A key problem in all social networking platforms is that of identifying user influence, since such users are capable of driving action (e.g. steer discussions to particular themes and topics) or provoking interactions amongst other users and thus, are also potentially more valuable to businesses (Trusov et al., 2010). In fact, as argued in SAS (2015), insight from social networking platforms “enhance the customer journey
across all customer touch points - customer care, brand marketing, public and community relations, merchandising and more.” Thus, the ranking of Twitter users based on their influence constitutes both an active research topic and a business opportunity, as manifested by services such as Klout (Klout, 2014) and PeerIndex (PeerIndex, 2014) that market and sell to businesses and other organizations influence scoring metrics. The most standard metric employed is the number of followers an account has. However, a number of studies (Cha et al., 2010; Weng et al., 2010) have concluded that it is not a good indicator, since most followers fail to engage with the messages that have been broadcast. For that reason, the number of retweets an account receives (Kwak et al., 2010) is a better measure of influence. Since we are interested in ranking of users, more sophisticated influence measures based on the popular PageRank (Page et al., 1999) and HITS (Kleinberg, 1999) ranking algorithms, widely used for ranking search results on the Web, have been used (Haveliwala, 2003; Kwak et al., 2010; Weng et al., 2010; Gayo-Avello et al., 2011). However, these algorithms have been developed for and applied to the followers network, which clearly captures the general popularity of users, but not necessarily of their influence. For example, the twenty most followed accounts with a minimum of 25 million followers comprise of entertainers and athletes, the sole exception being President Obama.

1.2 Outline of the Thesis

In Chapter 2, we propose to measure an account/user’s influence on the Twitter social media platform, by taking into consideration both their ability to produce new content by posting messages, and also to generate interactions from other accounts through retweeting and mentioning. To that end, we build a statistical model for an account’s actions and interactions with other accounts. It uses a counting process framework to capture the posting, retweeting and mentioning actions. In addition, based on this model we introduce a novel influence measure that leverages both
the follower network (that captures the potential for posted messages to generate interactions with other users) and the intensity over time of the basic actions involved (posting, retweeting and mentioning).

Chapter 3 considers the problem of identifying important interactions between nodes in the user network. In our proposed framework, as presented in Chapter 2, we still model actions occurring on the nodes as counting processes. However, we allow for a much more flexible parameterization than the one used in the previous chapter. Instead of having two global parameters for each node, reflecting capability to generate responses ($\alpha$) and susceptibility to respond to other nodes’ actions ($\beta$), we allow for independent parameters between every pair of nodes for selected topics. We then define an edge’s importance as the expected “influence” the followed node can borrow from its follower on the other end of the edge, after a unit length of time, with a single action.

Hence, underlying the model in this thesis is the idea that conversations, and in particular the rate of directed activity, between accounts reveal their real-world position and influence. The modeling frameworks of the two chapters are illustrated on a closely knit community, namely that of the members of the United States Senate, the upper legislative house in the bicameral legislative body for the United States. Two senators are democratically elected to represent each state for six year terms. We further augment the set of Twitter accounts analyzed by including selected prominent news organizations (e.g. Financial Times, Washington Post, CNN), as well as popular bloggers (e.g. Nate Silver, Ezra Klein), the accounts of President Obama and the White House, and two influential federal agencies (the US Army and the Federal Reserve Board); for details refer to Section 3.6. Thus, we examine an ecosystem of key participants that influence the political conversation and discourse of the country.

The retweeting and mentions interactions from our data are drawn as directed edges in Figure 1.1. Given this sequence of network snapshots, we identify particular
Figure 1.1: Weekly Twitter (mentions and retweet) network statistic time-series and drawings. The nodes (Twitter accounts) contain democratic senators (blue circles), republican senators (red squares), media (purple triangles), and government agencies (green stars).

senators and news agencies that tend to elicit interactions from other accounts (i.e., have many incoming edges relative to how often they tweet), thus revealing their influence on Twitter. Our results in Section 3.6 further indicate that the proposed approach produces influence measures for the U.S. Senators that correspond more closely with their legislative importance than purely network-based solutions based on the PageRank algorithm.
CHAPTER II

Measuring Influence in Twitter Ecosystems
using a Counting Process Modeling Framework

2.1 Background and Literature Review

There has been a great deal of work on ranking nodes in online social networks by their influence motivated by fundamental questions in marketing, such as how to identify the best set of users to create cascades or viral campaigns. Probst et al. (2013), in an extensive survey article, find that the most common measures to quantify the influence of a certain node are completely based on network topology and fail to account for “further characteristics of influential users” or the actual dynamics on the social network. They identify several papers that propose variations to the core idea of measuring influence with network metrics of the followers network. An illustration of the standard methodology with data similar to ours is Dubois and Gaffney (2014), where Canadian political communities on Twitter are explored using degree, clustering coefficient, and other network metrics calculated from the followers network to identify “opinion leaders”, i.e., accounts that steer online conversations.

To create a more nuanced influence measure that addresses the challenges highlighted by Probst et al. (2013) and references therein, researchers have begun to utilize the content of the communication like the underlying topic or theme of conversation,
which allows for more realistic models, since some individuals are authoritative or receptive to others only along certain topical dimensions. As such, a number of recent works have extended the classical network topology measures to account for topic of conversation. Haveliwala (2003) and Weng et al. (2010) take into account topic similarity of the actual messages and the social link (followers network) structure via modified PageRank algorithms that are applied to the followers network. Barbieri et al. (2013) propose a similar idea for the related problem of identifying the optimal choice of initial users for inducing cascades. The model we propose relates to these previous works by also separating behavior according to the topic of conversation. Our contribution lies in measuring influence with actual conversation dynamics by combining the mentions and retweets along different topics with the followers link structure.

Our approach extends recent work in the Statistics community, which uses counting processes to combine conversation dynamics (mentions and retweets) with the followers network structure. In this stream of literature, the hazard rate represents a measure of influence and typically quantifies the effect of a message from one node on each of its followers (Gomez-Rodriguez et al., 2013; Du et al., 2012). Thus, as in (Fleming and Harrington, 2013), the interactions between nodes are modeled as independent counting processes. The model posited in this work exhibits certain key differences, as illustrated in Figure 2.1, because the hazard rate of a node to retweet or mention is a function of the cumulative effect of tweets from its followers. The use of interacting counting processes is an important modeling nuance, since it allows for more realistic account behavior. For instance, accounts that are very popular and receive many tweets on the same topic within a short period of time usually respond once both out of convenience and to avoid spamming their followers. Thus, the model we posit should result in more accurate influence measures for Twitter ecosystems like the US Senate that we investigate in Section 7.
Figure 2.1: Solid lines in panel (a) represent edges in the followers network. Panel (b) illustrates the proposed model, where node $d$ decides to retweet or mention by the cumulative effect of the three tweets from nodes $a$, $b$, and $c$. Panel (c) illustrates the standard counting process model on interactions between nodes as introduced in Fleming and Harrington (2013). Instead of considering the cumulative effect of the three tweets, node $d$ makes a decision on whether to respond (retweet or mention) three separate times.

The remainder of this chapter is organized as follows: in Section 2.2, we review recent literature on measuring influence in online social networks. In Section 2.3, we introduce the modeling framework and the proposed influence measure. Section 2.4 presents the algorithm to obtain the model parameter estimates, as well as establish their statistical properties and those of the influence measure in Section 2.5. The performance of the model is evaluated on synthetic data sets in Section 2.6, while the US Senate application is presented in Section 2.7. Finally, some concluding remarks are drawn in Section 2.8.

### 2.2 The model and the influence measure

We start our presentation by defining some key quantities for future developments. Let $G = (V, L)$ denote the followers network, where $V$ corresponds to the set of nodes of all the Twitter accounts under consideration and $L = \{L_{i,j}, 1 \leq i \neq j \leq n\}$ the edge set between them and captures whether an account follows another account. Note that the network is bidirectional in nature and not necessarily symmetric, since account $i$ may follow account $j$, but not vice versa. In principle, $L$ can be dynamically...
evolving, but in this work we consider $L$ to be static and not changing over time. As explained in the introductory section, in the Twitter platform, accounts (nodes) can undertake the following three actions: post a new message, retweet a message posted by another account that they follow and finally mention another account that they follow. Further, the vast majority of messages posted, retweeted or mentioned have key terms (with a # prefix) that identify the topic(s) that are discussed.

Next, we define the following two key counting processes. Let $N_j(t, l)$ denote the total number of retweets and mentions that account $j$ generates on topic $l$ by time $t$ and let $A_j(t, l)$ denote the total number of posted messages by account $j$ on topic $l$ by time $t$. Define $\alpha_j$ to be a parameter that captures the long-term capability of account $j$ to generate responses by other accounts from the content posted, and $\beta_j$ a parameter that captures the long term susceptibility of account $j$ to respond (retweet/mention) to the postings of the accounts it follows. In this thesis, we mainly focus on $N_j(t, l)$ since it reflects the interactions between accounts while $A_j(t, l)$ is frequently related to accounts’ own habit of posting. We model $\{N_j(t, l)\}_{i=1}^n$ as a set of counting processes through their hazard rates, using a version of Cox (Cox, 1972) proportional hazard model; specifically, the hazard rate $\lambda_{j,l}(t)$ of process $N_j(t, l)$ is given by

$$
\lambda_{j,l}(t) = \lambda_{0,l}(t) \exp \left( \sum_{i \neq j} L_{ij}(\alpha_i + \beta_j) \log(M_i(t, l) + 1) \right),
$$

where

$$
M_i(t, l) = (N_i(t, l) + A_i(t, l))I(N_i(t, l) + A_i(t, l) \leq F) + F \cdot I(N_i(t, l) + A_i(t, l) > F).
$$

$A_j(t, l) + N_j(t, l)$ is the total number of posting, retweets and mentions for account $j$ on topic $l$ by time $t$. And we consider the effect of seeing actions from account $i$ can get saturated when the total number of actions reaches the constrain, $F$. We assume that the parameters $\alpha_i, \beta_i \in (-\infty, \infty)$, since accounts and their users may be
positively or negatively inclined towards other accounts, as well as being more keen in joining specific conversations or passively retweeting messages from favorite accounts.

The nonparametric baseline component $\lambda_{0,l}(t)$ is time varying. In general, we would expect this baseline to be small for large times $t$, since topics in social media platforms have a high churn rate; they become "hot" and generate a lot of action over short time scales and after awhile it stops being discussed (Kwak et al., 2010). The model posits that account $j$ interacts with other accounts at a baseline level $\lambda_{0,l}(t)$, modulated by its ability to generate responses by accounts in its followers network, as well as its own susceptibility to respond to accounts it follows postings and rebroadcasting of messages. Note that we model the retweet-mention process $N_j(t,l)$, since it reflects interactions between nodes and use the total activity process $M_j(t,l)$ as a covariate.

To complete the modeling framework, denote the set of topics in the data as $\{1, \ldots, \Gamma\}$. Further, let $T^t_j = \{T^t_{j,1}, \ldots, T^t_{j,n^t_j}\}$, $t = 1, \ldots, n^t_j$, denote the set of time points that account $j$ took action (post, retweet, mention) on topic $l$, until our end of observation time point $t_0$. Finally, for identification purposes, we require one member of the parameter vector $\Omega = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ to be set to a fixed value, and without loss of generality we set $\alpha_1 = 0$. Following, Andersen and Gill (1982), we employ a partial-likelihood function to obtain estimates of $\Omega$. Specifically, we treat the baseline $\lambda_{0,l}(t)$ as a nuisance parameter and decomposing the full-likelihood to obtain

$$PL(t) = \prod_{1 \leq t \leq \Gamma} \left( \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n^t_j, T^t_{j,k} \leq t} \frac{\lambda_j(T^t_{j,k})}{\sum_{1 \leq i \leq n} \lambda_i(T^t_{j,k})} \right)$$

Plugging the exact form of the hazard rate from (2.1) into the partial-likelihood function (PL), we get:

$$PL(t) = \prod_{1 \leq t \leq \Gamma} \left( \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n_j, T^t_{j,k} \leq t} \frac{\exp \left( \sum_{i \neq j} L_{ij}(\alpha_i + \beta_j) \log(M_i(T^t_{j,k}, l) + 1) \right)}{\sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui}(\alpha_u + \beta_i) \log(M_u(T^t_{j,k}, l) + 1) \right)} \right)$$

(2.2)
2.2.1 The Influence Measure

Leveraging the structure of the model, we propose to measure an account’s (node) influence as the total hazard rate change it will bring to its followers. Specifically, for an account \( j \) its relative hazard rate (ignoring the baseline) at time \( t \) is given by:

\[
H_j = \exp \left( \sum_{k \neq j} \log(M_k(t,l) + 1) L_{kj} (\alpha_k + \beta_j) \right).
\]

Further, the contribution of node \( i \) is

\[
H_j^{(i)} = \exp \left( \log(M_i(t,l) + 1) L_{ij} (\alpha_i + \beta_j) \right).
\]

Then, after some algebra we obtain that the total hazard rate change \( i \) brings to its followers can be written as:

\[
TH^{(i)} = \sum_{j \neq i} L_{ij} \cdot \exp \left( \log(M_i(t,l) + 1) (\alpha_i + \beta_j) \right). \tag{2.3}
\]

Since \( M_i(t,l) \) is a random value, we approximate it by its observed average value, \( \bar{M}_i \), calculated from the data over all topics and time points. Hence, the influence measure becomes

\[
\tilde{TH}^{(i)} = \sum_{j \neq i} L_{ij} \cdot \exp \left( \log(\bar{M}_i + 1) (\alpha_i + \beta_j) \right). \tag{2.4}
\]

Finally, we express it in a log-scale, so as to linearize the scale and make it compatible with the range of values of the response and susceptibility parameters \( \alpha \) and \( \beta \):

\[
\Xi^{(i)} = \log \left[ \sum_{j \neq i} L_{ij} \cdot \exp \left( \log(\bar{M}_i + 1) (\alpha_i + \beta_j) \right) \right]. \tag{2.5}
\]

In real application, we estimate \( \Xi^{(i)} \) by using the estimated \( \hat{\alpha_i} \) and \( \hat{\beta_j} \) values.
2.3 Computation and Inference

Next, we present a Newton-type algorithm for computing the parameter estimates $\Omega$. The logarithm of the partial likelihood function (3.3) is given by

$$LL(t) = \log(PL(t))$$

$$= \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n} \sum_{i \neq j} L_{ij}(\alpha_i + \beta_j) \log(M_i(T_{j,k}^l, l) + 1) \right\}$$

$$- \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n} \log \left[ \sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui}(\alpha_u + \beta_i) \log(M_u(T_{j,k}^l, l) + 1) \right) \right]$$

(2.6)

The objective function corresponds to $LL(t_0)$, which considers all events $k$ in its equation (3.11). For the sake of notation simplicity, we will use $LL$ to represent $LL(t_0)$ in the rest of the paper. Due to its smoothness, we employ Newton’s algorithm that uses the gradient and the Hessian of $LL$. The detailed expressions for the gradient vector $G \equiv \nabla_{\Omega} LL$ and the Hessian $H \equiv \nabla_{\Omega} \nabla_{\Omega}(LL)$ are given in the Appendix.

Algorithm 1 Estimating the parameters by Newton’s algorithm
1: Initialize the vector $\Omega$ value by $\alpha_1 = \ldots = \alpha_n = \beta_1 \ldots = \beta_n = 0$
2: Define $s$ as a positive thresholding constant for the minimum step size
3: while $t > s$ do
4: Calculate $G$ by using (2.11) and (2.12)
5: Calculate $H$ by using (2.13) to (2.18)
6: Find the optimum positive $\tau$ value such that $\Omega - \tau \cdot H^{-1}G$ will maximize the log-partial-likelihood (3.11)
7: Update $\Omega \leftarrow \Omega - \tau \cdot H^{-1}G$.
8: In the updated $\Omega$, set $\alpha_1 = 0$.
9: end while
10: return $\Omega$

To speed up calculations, we take advantage of the structure of the problem, as explained in detail in the Appendix.

The steps of the optimization are given in Algorithm 1. As stated in the algorithm,
s is a positive constant to judge the convergence of the the Newton’s algorithm. The computational complexity of this algorithm is dominated by the computation of $H$. Denote by $m_n = \max_{1 \leq j \leq n} \{n_j\}$. Based on (2.11) and (2.12), it costs $O(\Gamma nm_n)$ operations to calculate an entry of $G$. Further, since $G$ is of dimension $2n$, it takes $O(\Gamma n^3 m_n)$ to obtain the entire $G$ vector. Analogously, based on (2.13) to (2.18), it costs $O(\Gamma nm_n)$ operations to calculate an entry of $H$, if proper book-keeping is used on the results obtained for the gradient $G$. Further, since $H$ is of dimension $n^2$, it takes $O(\Gamma n^3 m_n)$ to obtain the entire $H$ matrix. Hence, the overall time complexity for each iteration of the algorithm is of the order $O(\max\{\Gamma n^3 m_n\})$. The time complexity for the whole algorithm is then $O(\max\{\Gamma n^3 m_n R\})$, where $R$ is the number of repetitions needed for the algorithm to converge, which depends on the threshold $s$. Empirically, with $s = 10^{-3}$, in our simulations in Section 6 and real data analysis in Section 7, we found the algorithm generally converges in no more than 10 repetitions.

### 2.4 Properties of the $\hat{\Omega}$ estimates

Next, we establish that the estimator $\hat{\Omega}$ which maximizes (3.11) will converge to the true parameter $\Omega$ in probability under certain regularity conditions.

**Theorem 1.** Conditions:

A. (Bounded hazard rate) $C_0 \leq \lambda_{0,l}(t) \leq C_1$ for $0 \leq t \leq t_0$ and $1 \leq l \leq \Gamma$,

B. (Bounded parameters) $\max_{1 \leq i,j \leq n} \{|\alpha_i|, |\beta_j|\} \leq C_2$,

C. (Limited posting frequencies)

\[
\begin{align*}
P (A_j(t+h,l) - A_j(t,l) \geq 1) & \leq C_3 \cdot h, \\
P (N_j(t+h,l) - N_j(t,l) \geq 1) & \leq C_3 \cdot h,
\end{align*}
\]

(2.7) when $t, h \geq 0, t + h \leq t_0$.

D. (Positive definite limit of Hessian) Let $\Omega'$ be any choosable parameter vector
satisfying (B). For large enough $\Gamma$ and some $C_4$, we have the holding condition to hold on the smallest eigenvalue of $-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)$, at $\Omega' = \Omega, t = t_0$,

$$P(\lambda_{\min}(-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)) |_{\Omega' = \Omega, t = t_0} > C_4) \to 1,$$  as $\Gamma \to \infty$,  

where

$$LT(\Omega, t) \equiv \Gamma^{-1} \left\{- \sum_{l=1}^{\Gamma} \lambda_{0,l}(u) \log \left\{ \sum_j \exp \left( \sum_{i \neq j} L_{ij}(\alpha'_i + \beta'_j) \log(M_i(t, l) + 1) \right) \right\} \right\}$$  

(2.8)

In the four Conditions A, B, C and D above, $C_0$, $C_1$, $C_2$, $C_3$ and $C_4$ are all positive constants. Under these conditions, we will have:

$$\hat{\Omega} \to_P \Omega \text{ as } \Gamma \to \infty.$$  

The detailed proof is given in Section 2.8.4.

When we have some information on the boundaries of the baseline hazard rate and parameter values, Condition A and B of Theorem 1 can be straightforwardly verified. We have the following Lemma 1 to show one example of counting processes in which Condition C naturally holds. It is however quite difficult to derive conditions under which Condition D will hold. As shown in Section 6, we propose to verify it empirically.

Lemma 1 When both $A_j(t, l)$ and $N_j(t, l)$ are both poisson processes and the hazard rate of $A_j(t, l)$ is smaller than a constant $K$, Condition C in Theorem 1 is satisfied.

The detailed proof is also presented in Section 2.8.3.
Based on Theorem 1, by leveraging the properties of continuous functions, we can establish the consistency of the proposed influence measure.

**Proposition 1.** Let \(\Xi(t) = (\Xi_1(t), \ldots, \Xi_n(t))\) denote the \(n\)-dimensional vector of influence measures at time \(t\). Further, denote by \(\hat{\Xi}(t) = (\hat{\Xi}_1(t), \ldots, \hat{\Xi}_n(t))\) their empirical estimates. Under the conditions of Theorem 1, we have that

\[
\|\hat{\Xi}(t) - \Xi(t)\| \to P 0 \tag{2.9}
\]

for any \(t \geq 0\).

Based on Theorem 1, the proof of the proposition is straightforward, since each element of the vector \(\hat{\Xi}\) is a continuous function of \(\hat{\Omega}\).

### 2.5 Performance evaluation

In this section, we evaluate the proposed model and influence measure on synthetic data. We start by outlining the data generation mechanism.

**Step 1:** Building the followers network \(L\).

The tasks employed for step 1 are presented next.

- First, for each node \(i\), generate \(K_1(i)\) from a uniform distribution on the integers \(\{1, \ldots, K\}\), where \(K = \lfloor n/2 \rfloor\) and \(\lfloor \cdot \rfloor\) is the floor function that returns the maximum integer not larger than the value inside.

- Generate \(F_1(i)\) for node \(i\) by randomly sampling \(K_1(i)\) users from \(\{1, \ldots, n\} \setminus \{i\}\). If \(k \in F_1(i)\), let \(L_{ik} = 1, 1 \leq i \leq n\).

- For each node \(j\), sample \(K_2(j)\) uniformly from the set \(\{1, \ldots, K\}\). Generate \(F_2(i)\) for node \(j\) by randomly sampling \(K_2(j)\) users from \(\{1, \ldots, n\} \setminus \{j\}\). If \(k \in F_2(j)\), let \(L_{kj} = 1, 1 \leq j \leq n\).
At the end of this procedure, every node in the network has at least one follower and at least an account that it follows.

**Step 2.** Generate the post, retweets and mentions sequences.

Since the baseline hazard rate \( \lambda_{0,l}(t) \) always gets canceled out within the partial-likelihood function (3.3), we select \( \lambda_{0,l}(t) \) as \( \lambda_{0,l}(t) = a \), whenever \( 0 \leq t \leq t_0 \) and \( \lambda_{0,l}(t) = 0 \) when \( t > t_0 \), where \( a \) is a small positive constant.

We then generate actions on this network with Algorithm 2 below for each topic \( l \in \Gamma_1 \) or \( \Gamma_2 \). In this algorithm, we first let each node send out a number of tweets with distribution Binomial(\( J, p \)) at \( t = 0 \). Then we generate the retweets and mentions in the standard survival analysis way, by using the hazard rate (2.1), as in the algorithm below.

**Algorithm 2 Generate Group A actions**

1: Initialize Indicator which is the sequence to record the nodes that have mentioned or retweeted as an empty sequence.
2: Initial \( t = 0 \). Let each node has a tweet with probability \( p \).
3: Let each node send out tweets from Binomial(\( J, p \)).
4: while \( t < t_0 \) (stopping time for all topics) do
5: Generate survival time for each node with its hazard rate (2.1)
6: Find node \( i \) with the shortest time \( t_s \).
7: if \( t + t_s < t_0 \) then
8: Update \( t \) to be \( t + t_s \). Record the node that has done this retweet or mention.
9: end if
10: if \( t + t_s > t_0 \) then
11: Break
12: end if
13: end while
14: return Indicator

We first illustrate the performance of the Newton estimation algorithm, on a random network of varying size. We set the parameter \( a = 0.5 \) for the baseline hazard rate and choose a time horizon of \( t_0 = 7 \), to emulate a week’s worth of data. We also select the parameters \( \Omega \) uniformly at random in the interval \([-0.3, 0.3]\).

Due to the bounded baseline hazard rate and simulated parameters, and since the retweets and mentions are generated as Poisson, Condition A, B, C of Theo-
rem 1 have been satisfied. Then we empirically “check” Condition D. With a large \( \Gamma = 1000 \), network size \( n = 10, 50 \), we repeated Step 1 and 2 for 20 times to simulate the network and actions. In each repetition, the square root of the smallest eigenvalue of \( \lambda_{\text{min}} \left( -\Gamma \nabla_{\Omega'} \nabla_{\Omega'} L T (\Omega', t) \right) \mid_{\Omega' = \Omega, t = t_0} \) is computed. The results are plotted in Figure 2.2.

![Figure 2.2: Diagnostics for Condition D with the simulated data: \( [\lambda_{\text{min}} \left( -\Gamma \nabla_{\Omega'} \nabla_{\Omega'} L T (\Omega', t) \right) \mid_{\Omega' = \Omega, t = t_0}]^{1/2} \) at \( \Gamma = 1000 \), \( n = 10 \) (left) and \( n = 50 \) (right). Due to large variations, the square root of the smallest eigenvalues is shown for better visualization.](image)

In the plot, it can be seen that smallest eigenvalues of

\[
[\lambda_{\text{min}} \left( -\Gamma \nabla_{\Omega'} \nabla_{\Omega'} L T (\Omega', t) \right) \mid_{\Omega' = \Omega, t = t_0}]^{1/2}
\]

are generally large and greater than 0.5.

Then as we have verified all conditions are satisfied with network sizes \( n = 10, 50 \) and \( \Gamma = 1000 \), we plot in Figure 2.3 the mean squared error of the parameter and influence estimates \( \| \hat{\Omega} - \Omega \| / \sqrt{2n - 1} \) and \( \| \hat{\Xi} - \Xi \| / \sqrt{n} \) to check the performance of our estimation algorithm, where \( \| \cdot \| \) corresponds to the \( \ell_2 \) norm of a vector. The results are based on 20 replicates of the underlying followers networks, as well as the actions (postings, retweets and mentions) data.

It can be seen that the quality of the estimates improves as a function of the number \( \Gamma \) of topics discussed, while it deteriorates as a function of the number of
nodes in the followers network $L$. Another way to look at the quality of the estimates, is to examine the relative error of the parameter and influence estimates, given by $\frac{\|\hat{\Omega} - \Omega\|}{\|\Omega\|}$ and $\frac{\|\hat{\Xi} - \Xi\|}{\|\Xi\|}$.

It can be seen in the following Figure that especially the influence measure which is of prime interest in applications, exhibits a small (less than 10%) relative error rate.

Next, we use a size 10 network, specially constructed to gain insight into the workings of the proposed influence measure. The settings for the data generation are as follows: $\Gamma = 500$, $\alpha_1 = 0$, $\alpha_2 = -2$, $\alpha_3 = \cdots = \alpha_{10} = 0.2$ and $\beta_1 = \cdots = \beta_{10} = 0$. Finally, the topology of the followers networks is given in Figure 2.5.
Since $\alpha_2 = -2$, node 2 is an "unpopular" one and hence can hardly generate any retweets and mentions of its postings. On the other hand, all nodes have approximately an equal number of followers, which suggests that their ranking according to the PageRank metric (or many other popular ones based on that network like Haveliwala (2003) and Weng et al. (2010)) will be approximately similar. The results based on a single realization of the user actions data generation process is shown in Figure 2.6. It can be seen that relying on the followers network structure gives a false impression, while the proposed influence measure that incorporates the actions of the accounts provides a more insightful picture.

2.6 Identifying Influential Senators

Tweets and follower lists are collected using Twitter’s API and consist of approximately 200,000 tweets and 4671 follower links within the set of 120 accounts from April 2009 to July 2014. The retweeting and mentions interactions are drawn in Fig-
Figure 2.6: Proposed influence VS PageRank Influence, in a plot with "unpopular" node.

Figure 1.1, where accounts are registered to 55 Democratic politicians (U.S. Senators and the President of the U.S.), 46 Republican Senators, 2 government organizations (U.S. Army and the Federal Reserve Board), and 16 media outlets, including newspapers (Financial Times, Washington Post, New York Times, Huffington Post), television networks (MSNBC, Fox News, CNN, CSPAN), reporters (Nate Silver (538), Ezra Klein) and television hosts (Bill O’Reilly, Sean Hannity). The figure shows some periods of increased activity, as in the months surrounding the inauguration of President Obama (January 2013), the debate on raising the debt ceiling of the US government and its temporary suspension around April 2013 and the summer of 2014 (soccer World Cup). Note that the sudden increase during the summer of 2014 may be an artifact of rate limiting data acquisition. Specifically, Twitter’s API allows access to only the past 3000 tweets for any account. As a consequence, for extremely high volume users, like newspapers and television networks, our data traces their Twitter usage for months. For the least active users in our data, 3000 tweets dates back multiple years.

An inspection of actual tweets in Table 2.1 shows, consistent with Golbeck et al. (2010), that senators tend to retweet and mention as a means of self or legislative promotion. In fact, we see a number of references to legislative activity, such as
calls for gun reform, carbon emissions, and references to actual bills on overtime pay, domestic violence protections, among others. Senators often cite news coverage by retweeting or mentioning news media accounts that support their political agenda, which would suggest that the media outlets collectively have enormous influence. This also suggests that Twitter is utilized by senators as part of a larger strategy to build and coalesce public support in order to pass bills through congress.

To test these hypotheses and also rigorously compare the proposed influence measure to PageRank applied to the followers networks (which constitutes the backbone of many ranking algorithms of Twitter accounts), we perform a regression analysis to assess how well each measure explains legislative leadership in Congress. Our response variable is the leadership score, published by www.govtrack.us (GovTrack.us, 2014). GovTrack creates the leadership score by applying the PageRank algorithm to the adjacency matrix of bill cosponsorship data. Thus, the leadership score for each senator is a number between 0 and 1, where higher values denote greater legislative leadership. The regression model we are interested in is

$$\text{Leadership} = \beta \text{Influence} + \Theta \text{Controls},$$

(2.10)

where Influence contains the proposed measure and/or PageRank, and Controls includes party affiliation, gender, age, and number of years in the senate. Seniority endows a number of benefits including preferential assignment to committees. Thus, these control variables likely associate strongly with legislative leadership.

To estimate the proposed influence measure, the data is organized into weekly intervals after using the follow-follower relations to construct the adjacency matrix L. In Twitter it is common to use “hashtags” or the # symbol followed by a user-specified category to identify context, which, as mentioned in Section 1 can be used as an indicator of different conversations. However, we find that senators do not
Table 2.1: Actual tweets mentioning or retweeting the most influential accounts over from May 15, 2014 to July 3, 2014.

<table>
<thead>
<tr>
<th>Date</th>
<th>Account</th>
<th>Tweet</th>
</tr>
</thead>
<tbody>
<tr>
<td>05/19/2014</td>
<td>Menendez</td>
<td>“.@SenBlumenthal &amp; in #NJ the avg student loan debt is over $29K. It’s unacceptable! #GameofLoans <a href="http://t.co/hUJMSeJbfd%E2%80%9D">http://t.co/hUJMSeJbfd”</a></td>
</tr>
<tr>
<td>05/23/2014</td>
<td>Cornyn</td>
<td>“RT @nytimes: Former Defense Secretary Gates Is Elected President of the Boy Scouts <a href="http://t.co/C7STUSVIP3%E2%80%9D">http://t.co/C7STUSVIP3”</a></td>
</tr>
<tr>
<td>05/27/2014</td>
<td>Blumenthal</td>
<td>“RT @msnbc: @SenBlumenthal calls for reviving gun reform debate after mass shooting near Santa Barbara: <a href="http://t.co/7sqfI1AFy%E2%80%9D">http://t.co/7sqfI1AFy”</a></td>
</tr>
<tr>
<td>06/02/2014</td>
<td>Markey</td>
<td>“RT @washingtonpost: A huge majority of Americans support regulating carbon from power plants <a href="http://t.co/lj6ieL5D1Y">http://t.co/lj6ieL5D1Y</a> <a href="http://t.co/2CA63hTqmm%E2%80%9D">http://t.co/2CA63hTqmm”</a></td>
</tr>
<tr>
<td>06/17/2014</td>
<td>Markey</td>
<td>“Proud to intro new bill w @SenBlumenthal 2 protect domestic violence victims from #gunviolence <a href="http://t.co/MsgK40oLiT">http://t.co/MsgK40oLiT</a> <a href="http://t.co/ynEHrEbh2x%E2%80%9D">http://t.co/ynEHrEbh2x”</a></td>
</tr>
<tr>
<td>06/20/2014</td>
<td>Blumenthal</td>
<td>“Proud to stand w/ @CoryBooker &amp; others on enhancing rules to reduce truck driver fatigue. Their safety &amp; safety of others is paramount. -RB”</td>
</tr>
<tr>
<td>06/20/2014</td>
<td>Markey</td>
<td>“Proud to support our workers and this commonsense bill w @SenatorHarkin Keeping Track: Overtime Pay, via @nytimes <a href="http://t.co/TnAS96Hro5%E2%80%9D">http://t.co/TnAS96Hro5”</a></td>
</tr>
<tr>
<td>06/25/2014</td>
<td>Durbin</td>
<td>“Watch now: @OfficialCBC @HispanicCaucus @CAPAC @USProgressives @SenatorCardin on racial profiling #MoreThanAProfile <a href="http://t.co/ZX0Eu65dg%E2%80%9D">http://t.co/ZX0Eu65dg”</a></td>
</tr>
<tr>
<td>06/25/2014</td>
<td>Cardin</td>
<td>“RT @TheTRCP: Thank you @SenatorCardin for standing with sportsmen today for #CleanWater #protectcleanwater”</td>
</tr>
<tr>
<td>06/27/2014</td>
<td>Markey</td>
<td>“Thanks @alfranken @CoryBooker @amyklobuchar @SenBlumenthal for joining me in support of community #broadband <a href="http://t.co/O8Px2MzrCg%E2%80%9D">http://t.co/O8Px2MzrCg”</a></td>
</tr>
<tr>
<td>06/27/2014</td>
<td>Menendez</td>
<td>“ Took my first #selfie at #NJs @ALJBS! Hope @CoryBooker is proud of his NJ Sen colleague. <a href="http://t.co/FrEJonUy9d">http://t.co/FrEJonUy9d</a></td>
</tr>
<tr>
<td>06/28/2014</td>
<td>Booker</td>
<td>“Thanks Adam RT @Alisaacs7 Props to @CoryBooker and @SenRandPaul for their bipartisanship in introducing their amendment #MedicalMarijuana”</td>
</tr>
</tbody>
</table>
utilize hashtags often. To overcome this challenge, we follow previous works on Twitter (Hong and Davison, 2010; Ramage et al., 2010) by applying probabilistic topic modeling, which was first introduced in Blei et al. (2003). Extensive work in computer science and applied statistics has led to fast algorithms capable of analyzing extremely big text archives. Due to space constraints, for statistical and algorithmic details on the topic model, see Blei (2012); Blei and Lafferty (2007) and references therein.

Topic modeling applied to the data results in a soft clustering of tweets into 10 groups (topics), which is appropriate since a single tweet could touch on multiple issues. Thus, tweets are assigned to topics that had at least 0.25 probability. Given the fast moving landscape of social media, based on the original clustered 10 groups, new topics are assigned each week, leading to 2770 topics in total for the entire data set. After preprocessing, we apply Algorithm 3 to estimate the $\alpha$ and $\beta$ parameters for every account using all data. The final influence measure is constructed by computing the influence measure vector $\hat{\Xi}$ over different time intervals to study how influence evolved; i.e. $\hat{\Xi}$ was computed by using the average of $M_i(\mathcal{T}_m, l)$ over all time points in $\mathcal{T}_m$ and topics, where $\mathcal{T}_m$ denotes the $m$–th time interval of interest.

The first time interval $\mathcal{T}_1$ we investigate is May 15, 2014 - July 3, 2014, which captures the most active period in our data and also represents a period when rate limiting is not a concern, i.e., the data for even high volume users extends this far. During this time several major events occurred worldwide, including the soccer World Cup, debate on immigration reform, and the Islamic State in Iraq and the Levant (also known as the ISIS or ISIL) began an offensive in northern Iraq. Table 2.2 shows the top ten most influential accounts under the proposed method and PageRank (Page et al., 1999) calculated from the followers network. Both methods estimate that the Financial Times is the most influential Twitter account, and in general find that the media has an enormous influence that facilitates online conversation between
politicians. We see from Figure 2.7 that these top accounts were actively retweeted and mentioned throughout this period.

Next, we estimate the regression model in Equation 3.15. We note that Senators Baucus, Kerry, Cowan, Lautenberg, and Chiesa are scored by govtrack.us, but are not in our analysis. Max Baucus and John Kerry are left out, because they vacated their Senate seats to become, respectively, Ambassador to China and U.S. Secretary of State. Mo Cowan succeeded Kerry and was senator from February 1, 2013 to July 16, 2013 until a special election could be held. Cowan chose not to run in the election. Likewise, due to the death of Senator Frank Lautenberg, Jeffrey Chiesa was appointed by Governor Chris Christie to be the junior senator from New Jersey from
Table 2.2: Top ten rankings according to the proposed model and PageRank from May 15, 2014 - July 3, 2014.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Proposed Measure</th>
<th>PageRank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Financial Times</td>
<td>Financial Times</td>
</tr>
<tr>
<td>2</td>
<td>Washington Post</td>
<td>U.S. Army</td>
</tr>
<tr>
<td>3</td>
<td>NYTimes</td>
<td>CNN</td>
</tr>
<tr>
<td>4</td>
<td>MSNBC</td>
<td>Barack Obama</td>
</tr>
<tr>
<td>5</td>
<td>Ezra Klein</td>
<td>CSPAN</td>
</tr>
<tr>
<td>6</td>
<td>Fox News</td>
<td>New York Times</td>
</tr>
<tr>
<td>7</td>
<td>Cory Booker</td>
<td>Washington Post</td>
</tr>
<tr>
<td>8</td>
<td>Ben Cardin</td>
<td>Cory Booker</td>
</tr>
<tr>
<td>9</td>
<td>Nate Silver (538)</td>
<td>MSNBC</td>
</tr>
<tr>
<td>10</td>
<td>Richard Blumenthal</td>
<td>Wall Street Journal</td>
</tr>
</tbody>
</table>

Table 2.3: Estimated R-squared values for different regression models, where the proposed measure and/or PageRank is included in the set of independent variables and the influence is computed for the entire data sample. We consistently find that the proposed measure is a better indicator of legislative importance.

<table>
<thead>
<tr>
<th>Response</th>
<th>Proposed Measure</th>
<th>PageRank</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>leadership</td>
<td>V</td>
<td>V</td>
<td>0.311</td>
</tr>
<tr>
<td>leadership</td>
<td>V</td>
<td>V</td>
<td>0.276</td>
</tr>
<tr>
<td>log($\frac{leadership}{1-leadership}$)</td>
<td>V</td>
<td>V</td>
<td>0.114</td>
</tr>
<tr>
<td>log($\frac{leadership}{1-leadership}$)</td>
<td>V</td>
<td>V</td>
<td>0.098</td>
</tr>
<tr>
<td>log($\frac{leadership}{1-leadership}$)</td>
<td>V</td>
<td>V</td>
<td>0.114</td>
</tr>
</tbody>
</table>
Table 2.4: Regression estimates, where the response variable is the raw leadership scores from GovTrack.us and influence is computed for the entire data sample. $R^2 = 0.311; F = 8.228$ on 5 and 92 DF (p-value: 0.000)

| Variable       | Estimate | Std. Error | $t$ value | $P(>|t|)$ |
|----------------|----------|------------|-----------|-----------|
| Intercept      | -0.086   | 0.232      | -0.368    | 0.714     |
| Proposed Influence | 0.062    | 0.028      | 2.241     | 0.027     |
| Republican     | -0.154   | 0.039      | -3.945    | 0.000     |
| Age            | 0.002    | 0.003      | 0.923     | 0.359     |
| Years in Senate| 0.007    | 0.003      | 2.518     | 0.014     |
| Male           | 0.020    | 0.050      | 0.395     | 0.694     |

June 6, 2013 to October 31, 2013. He declined to run in the special election and thus, is also not included in the analysis.

Since the leadership score provided by GovTrack are between 0 and 1, we estimate two models. One model uses the raw leadership scores, and another uses $\log\left(\frac{\text{leadership}}{1-\text{leadership}}\right)$ for the response variable. In both cases, as shown in Table 2.3, we consistently find that the proposed influence measure explains more variation in leadership and when both the proposed and PageRank influence measures are included as independent variables, PageRank does not provide additional explanatory power. Tables 2.4 and 2.5 show a significant positive coefficient for the proposed influence measure, meaning that senators who are more influential in Twitter by successfully steering conversation of their colleagues onto particular topics, tend to be more influential in real life in passing legislation. These results are consistent across different time intervals. For instance, in the Appendix we present similar results, where influence is calculated from January 1, 2013 to March 1, 2013 corresponding to sequestration and also from November 1, 2012 to January 31, 2013 corresponding to the president’s reelection and subsequent inauguration.
Table 2.5: Regression estimates, where the response variable is $\log(\frac{leadership}{1-leadership})$, where leadership is from GovTrack.us and influence is computed for the entire data sample. $R^2 = 0.114; F = 2.334$ on 5 and 92 DF (p-value: 0.048)

| Variable        | Estimate | Std. Error | t value | P($>|t|)$ |
|----------------|----------|------------|---------|----------|
| Intercept       | -3.590   | 2.604      | -1.379  | 0.171    |
| Proposed Influence | 0.437    | 0.308      | 1.416   | 0.160    |
| Republican      | -1.112   | 0.438      | -2.538  | 0.013    |
| Age             | 0.009    | 0.029      | 0.323   | 0.747    |
| Years in Senate | 0.034    | 0.032      | 1.063   | 0.290    |
| Male            | 0.470    | 0.563      | 0.834   | 0.407    |

2.7 Discussion

The goal in this paper was to characterize the influence of users in a large scale social media platform when given information about the detailed actions users take on it. Our comprehensive analysis of the ecosystem comprising of US Senators and influential government agency and media related accounts demonstrated that conversations, and in particular the rate of directed activity, between accounts are correlated with their real-world position and influence. We expect similar conclusions to hold broadly for other types of directed interaction data when the nodes form a clearly defined ecosystem or closely knit social group/community.

The proposed approach only utilizes network information (e.g. followers network), plus time stamps of actions (e.g. retweets and mentions), thus allowing to process a large volume of data. However, it does not consider the tone of the message (positive, negative or neutral), a topic addressed in Taddy (2013), where the goal is to understand how messages related to a specific topic are perceived by other users. Since in that approach the message content needs to be analyzed - a computationally demanding task - Taddy (2013) develops efficient sampling designs for that task. It is of interest though to combine such sampling ideas with the current approach in order to be able to address user influence issues in very large ecosystems comprising of millions of users.
The modeling and statistical inference issues, associated with large scale data obtained from these social media platforms are different from those in the related literature on network community detection (Kolaczyk, 2009; Fienberg, 2012; Salter-Townshend et al., 2012), where the goal is to identify relatively dense groups of nodes (users), even though the underlying data (observed adjacency matrices) are the same. Relative to other recent work on modeling directed networks, as in Perry and Wolfe (2013), our study has important modeling differences motivated by the online social media platform domain. For instance, our approach incorporates the fundamental differences between actions like retweeting, mentioning, and posting. As a consequence, our final influence measure, which sums all possible influences from the social network, is able to outperform traditional topology driven approaches like PageRank (Page et al., 1999). Perhaps most importantly, given the massive volumes of data generated by platforms like Twitter, we presented a fast estimation algorithm and established statistical properties for the model estimates and those of the final influence measure.
2.8 Estimation Algorithm and Proofs

2.8.1 Expressions for the gradient vector and Hessian matrix of the $LL$ function

Some rather straightforward algebra yields the following expressions for the elements of the gradient vector $G \equiv \nabla_\Omega LL$:

$$
\frac{\partial LL}{\partial \alpha_i} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq j \leq n, j \neq i} \sum_{1 \leq k \leq n_j} L_{ij} \log(M_i(T_{j,k}^l, l) + 1) \\
- \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \frac{\sum_{v \neq i} L_{iv} \log(M_i(T_{j,k}^l, l) + 1)}{\sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1) \right)} \right\}
$$

(2.11)

for $2 \leq i \leq n$, and

$$
\frac{\partial LL}{\partial \beta_j} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq k \leq n_j} \sum_{i \neq j} L_{ij} M_i(T_{j,k}^l, l) \\
- \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s} \frac{\left( \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k}^l, l) + 1) \right)}{\sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right)} \right\}
$$

(2.12)

for $1 \leq j \leq n$.

Next, we obtain the necessary expressions for the Hessian matrix $H(\Omega \llbracket L \rrbracket)$. We start by computing the sub-matrix of $H$ that includes the second partial derivatives
of $LL$ with respect to the $\alpha$ parameters and obtain

\[
\frac{\partial^2 LL}{\partial \alpha_i^2} = 
\sum_{1 \leq l \leq \Gamma} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \sum_{1 \leq l \leq \Gamma} \frac{\sum_{v \neq i} L_{iv} \log(M_i^2(T_{j,k}^l, l) + 1)}{\exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right)} \right. \\
\cdot \exp\left(\sum_{u \neq v} L_{uv}(T_{j,k}^l) (\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \left[ \sum_{1 \leq l \leq \Gamma} \sum_{v \neq i} L_{iv} \log(M_i(T_{j,k}^l, l) + 1) \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \right]^2 \\
\cdot \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \\
\left. \left[ \sum_{1 \leq v \leq n} \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \right]^2 \right\}
\]

(2.13)

When $i \neq q$, we similarly have

\[
\frac{\partial^2 LL}{\partial \alpha_i \partial \alpha_q} = 
\sum_{1 \leq l \leq \Gamma} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \sum_{1 \leq l \leq \Gamma} \frac{\sum_{v \neq i,q} L_{iv} \log(M_i(T_{j,k}^l, l) + 1)L_{qv} \log(M_q(T_{j,k}^l, l) + 1)}{\exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right)} \right. \\
\cdot \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \left[ \sum_{1 \leq l \leq \Gamma} \sum_{v \neq i,q} L_{iv} \log(M_i(T_{j,k}^l, l) + 1) \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \right]^2 \\
\cdot \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \\
\left. \left[ \sum_{1 \leq v \leq n} \exp\left(\sum_{u \neq v} L_{uv}(\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1)\right) \right]^2 \right\}
\]

(2.14)

Next, we obtain the sub-matrix of $H$ that includes the second partial derivatives of
\( LL \) with respect to the \( \beta \) parameters and get

\[
\frac{\partial^2 LL}{\partial \beta_j^2} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_i} \left[ \left( \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k}^l, l) + 1) \right) \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \right]^2 \right. \\
- \left. \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_i} \left( \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k}^l, l) + 1) \right)^2 \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right) \right\}
\]

for \( 1 \leq j \leq n \). When \( j \neq q \), we can similarly have

\[
\frac{\partial^2 LL}{\partial \beta_j \partial \beta_q} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_i} \left[ \left( \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k}^l, l) + 1) \right) \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \right]^2 \right. \\
\cdot \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \left. \left( \sum_{q \neq j} L_{uq} \log(M_u(T_{s,k}^l, l) + 1) \right) \exp \left( \sum_{u \neq q} L_{uj} (\alpha_u + \beta_q) \log(M_u(T_{s,k}^l, l) + 1) \right) \right\}
\]

Finally, we provide expressions for the cross-partial

\[
\frac{\partial^2 LL}{\partial \alpha_i \partial \beta_i} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_i} \left[ \left( \sum_{u \neq i} L_{ui} \log(M_u(T_{s,k}^l, l) + 1) \right) \exp \left( \sum_{u \neq i} L_{ui} (\alpha_u + \beta_i) \log(M_u(T_{s,k}^l, l) + 1) \right) \right]^2 \right. \\
\cdot \exp \left( \sum_{u \neq i} L_{ui} (\alpha_u + \beta_i) \log(M_u(T_{s,k}^l, l) + 1) \right) \left. \left( \sum_{v \neq i} L_{iv} \log(M_i(T_{s,k}^l, l) + 1) \right) \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right) \right\}
\]

When \( i \neq j \),

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\[
\frac{\partial^2 LL}{\partial \alpha_i \partial \beta_j} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n'_s} \left[ \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right) \right] \right. \\
\left. \cdot \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \right\} \\
\cdot \sum_{v \neq i} L_{iv} \log(M_i(T_{s,k}^l, l) + 1) \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right) \\
- \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n'_s} \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{s,k}^l, l) + 1) \right) \\
\left. \cdot \exp \left( \sum_{u \neq j} L_{uj} (\alpha_u + \beta_j) \log(M_u(T_{s,k}^l, l) + 1) \right) \right\}
\]

(2.18)

2.8.2 Implementation Issues

As outlined above, the maximum likelihood estimator is obtained by Newton’s algorithm and detailed expressions for the respective gradient and Hessian are given in Section 9.1. However, the structure of the problem allows us to precompute and store several quantities for repeated use, thus saving on computational time in practice. Note that the data containing the actions are stored according to their time stamps. We start by computing four groups of quantities introduced by an action, labeled respectively by indices \(j, l, k\) and possibly some other parameters, where \(j\) indicates the node that takes the activity, \(l\) is the topic label and \(k\) represents the relative sequence number of the action, in all the actions that node \(j\) has taken under topic \(l\).

First, we define

\[
E_{j,v,k,l} = \exp \left( \sum_{u \neq v} L_{uv} (\alpha_u + \beta_v) \log(M_u(T_{j,k}^l, l) + 1) \right).
\]
Then, we compute
\[ SE_{j,k,l} = \sum_{1 \leq v \leq n} E_{j,v,k,l}, \]
and
\[ ME_{j,i,k,l} = \sum_{v \neq i} L_{iv} \log(M_i(T_{j,k,l}^l) + 1)E_{j,v,k}. \]

Also, we have
\[ LM_{j,s,k,l} = \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k,l}^l) + 1). \]

Then, based on the precomputed components values, the elements of the gradient vector \( G \equiv \nabla_\Omega LL \) are obtained as follows:
\[
\frac{\partial LL}{\partial \alpha_i} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq j \leq n, j \neq i} \sum_{1 \leq k \leq n_j} L_{ij} \log(M_i(T_{j,k,l}^l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \frac{ME_{j,i,k,l}}{SE_{j,k,l}} \right\}
\]
for \( 2 \leq i \leq n \), and
\[
\frac{\partial LL}{\partial \beta_j} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq k \leq n_j} \sum_{i \neq j} L_{ij} M_i(T_{j,k,l}^l) - \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s} \frac{LM_{j,s,k,l}E_{j,s,k,l}}{SE_{s,k,l}} \right\}
\]
for \( 1 \leq j \leq n \).

Regarding the Hessian, based on the four precomputed groups of quantities, we start by computing the sub-matrix of \( H \) that includes the second partial derivatives of \( LL \) with respect to the \( \alpha \) parameters. We get
\[
\frac{\partial^2 LL}{\partial \alpha_i^2} = \sum_{1 \leq l \leq \Gamma} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \frac{\sum_{v \neq i} L_{iv} \log(M_i^2(T_{j,k,l}^l) + 1)E_{j,v,k,l}}{SE_{j,k,l}} + \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \frac{(ME_{j,i,k,l})^2}{(SE_{j,k,l})^2} \right\}
\]
When $i \neq q$, we similarly have

$$\frac{\partial^2 LL}{\partial \alpha_i \partial \alpha_q} = \sum_{1 \leq l \leq \Gamma} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \sum_{v \neq i,q} L_{iv} \log(M_i(T_{j,k}^l, l) + 1) L_{qv} \log(M_q(T_{j,k}^l, l) + 1) \frac{E_{j,v,k,l}}{SE_{j,k,l}} \right. $$

$$\left. + \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \frac{ME_{j,i,k,l}ME_{j,q,k,l}}{(SE_{j,k,l})^2} \right\}$$

Next, we obtain the sub-matrix of $H$ that includes the second partial derivatives of $LL$ with respect to the $\beta$ parameters and get

$$\frac{\partial^2 LL}{\partial \beta_j^2} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{(LM_{j,s,k,l}E_{j,s,k,l})^2}{(SE_{s,k,l})^2} - \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{(LM_{j,s,k,l})^2 E_{j,s,k,l}}{SE_{s,k,l}} \right\}$$

for $1 \leq j \leq n$. When $j \neq q$, we can similarly have

$$\frac{\partial^2 LL}{\partial \beta_j \partial \beta_q} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{LM_{s,i,k,l} \cdot E_{s,i,k,l} \cdot LM_{q,s,k,l} \cdot E_{q,s,k,l}}{(SE_{s,k,l})^2} \right\}$$

Finally, we provide expressions for the cross-partials

$$\frac{\partial^2 LL}{\partial \alpha_i \partial \beta_j} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{LM_{j,s,k,l} \cdot E_{s,i,k,l} \cdot ME_{j,i,k,l}}{(SE_{s,k,l})^2} \right\}$$

When $i \neq j$,

$$\frac{\partial^2 LL}{\partial \alpha_i \partial \beta_j} = \sum_{1 \leq l \leq \Gamma} \left\{ \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{LM_{j,s,k,l} \cdot E_{s,j,k,l} \cdot ME_{s,i,k,l}}{(SE_{s,k,l})^2} \right. $$

$$\left. - \sum_{1 \leq s \leq n} \sum_{1 \leq k \leq n_s^l} \frac{LM_{j,s,k,l} \cdot L_{ij} \log(M_i(T_{j,k}^l, l) + 1) \cdot E_{s,j,k,l}}{SE_{s,k,l}} \right\}$$
2.8.3 Proof of Lemma 1

First, under the condition that \( N_j(t, l) \) is a Poisson Process, we have

\[
P(N_i(t + h, l) - N_i(t, l) = k) = \frac{(\mu_i(t, h, l))^k}{k!} \exp(-\mu_i(t, h, l)),
\]

where \( \mu_i(t, h, l) = \int_t^{t+h} \lambda_i(u, l) du \). Since

\[
|\lambda_i(t, l)| = \left| \lambda_{0,l}(t) \exp \left( \sum_{k,k \neq i} L_{ki}(t)(\alpha'_i + \beta'_j) \log(M_i(t, l) + 1) \right) \right| \leq C_1 \exp \left( 2nC_2 \log(F + 1) \right),
\]

we have

\[
\mu_i(t, h, l) \leq C_1 n \exp \left( 2nC_2 \log(F + 1) \right) h.
\]

Then,

\[
P(N_i(t + h, l) - N_i(t, l) \geq 1) = \sum_{k=1}^{\infty} \frac{(\mu_i(t, h, l))^k}{k!} \exp(-\mu_i(t, h, l))
\]

\[
= \exp(-\mu_i(t, h, l)) \mu_i(t, h, l) \sum_{k=1}^{\infty} \frac{(\mu_i(t, h, l))^{k-1}}{(k-1)!} \leq \exp(-\mu_i(t, h, l)) \mu_i(t, h, l) \sum_{k=1}^{\infty} \frac{(\mu_i(t, h, l))^{k-1}}{(k-1)!} = \mu_i(t, h, l)
\]

\[
\leq C_1 \exp \left( 2nC_2 \log(F + 1) \right) h.
\]

Similarly, we can show

\[
P(A_i(t + h, l) - A_i(t, l) \geq 1) \leq Kh.
\]

If we let \( C_3 = \max\{K, C_1 \exp \left( 2nC_2 \log(F + 1) \right) \} \), Condition C in Theorem 1 has been satisfied.
2.8.4 Proof of Theorem 1

Before we start the actual proof, to simplify the proof of Theorem 1, we first define some notations:

\[ E_l(t, \Omega') = \sum_{j=1}^{n} \lambda_{0,l}(t) \exp \left( \sum_{i,i \neq j} L_{ij}(t)(\alpha_i' + \beta_j') \log(M_i(t,l) + 1) \right) \]

\[ \Phi_j' = (\phi_{1j}', \ldots, \phi_{nj}') := (\alpha_1' + \beta_j', \ldots, \alpha_n' + \beta_j')' \]

\[ E_{ij}^{(1)}(t, \Omega') = \left( \frac{\partial E_l(t, \Omega')}{\partial \phi_{1j}'}, \ldots, \frac{\partial E_l(t, \Omega')}{\partial \phi_{nj}'} \right) \]

\[ E_{ij}^{(2)}(t, \Omega') = \left( \frac{\partial^2 E_l(t, \Omega')}{\partial \phi_{ij}' \partial \phi_{kj}'} \right)_{1 \leq i,k \leq n} \]

To prove Theorem 1, we also need the following two lemmas.

**Lemma 2.** When Conditions A, B, C of Theorem 1 hold, if we define

\[ e_l(t, \Omega') = E[E_l(t, \Omega')] = \sum_{j} \lambda_{0,l}(t) E \left[ \exp \left( \sum_{i,i \neq j} L_{ij}(t)(\alpha_i' + \beta_j') \log(M_i(t,l) + 1) \right) \right], \]

we will have

\[ \sup_{t \in [0,t_0], |\alpha_i| \leq C_2, |\beta_j| \leq C_2} \Gamma^{-1} \sum_{l=1}^{\Gamma} \left| E_l(t, \Omega') - e_l(t, \Omega') \right| \rightarrow_p 0. \quad (2.20) \]

\[ \sup_{t \in [0,t_0], |\alpha_i| \leq C_2, |\beta_j| \leq C_2} \Gamma^{-1} \sum_{j=1}^{n} \left\| \sum_{l=1}^{\Gamma} \left[ E_{ij}^{(k)}(t, \Omega') - e_{ij}^{(k)}(t, \Omega') \right] \right\|_{\infty} \rightarrow_p 0. \quad (2.21) \]

where \( k = 1, 2, \| \cdot \|_\infty \) gives the largest absolute value of entries of a vector (matrix) and \( e_{ij}^{(1)}(t, \Omega') \) and \( e_{ij}^{(k)}(t, \Omega') \) are defined by

\[ e_{ij}^{(1)}(t, \Omega') = \left( \frac{\partial e_l(t, \Omega')}{\partial \phi_{1j}'}, \ldots, \frac{\partial e_l(t, \Omega')}{\partial \phi_{nj}'} \right) \]

\[ e_{ij}^{(2)}(t, \Omega') = \left( \frac{\partial^2 e_l(t, \Omega')}{\partial \phi_{ij}' \partial \phi_{kj}'} \right)_{1 \leq i,k \leq n} \]
Lemma 3. When Conditions A, B, C of Theorem 1 hold, following the definitions of \( e_l(t, \Omega') \), \( e_{ij}^{(1)}(t, \Omega') \) and \( e_{ij}^{(2)}(t, \Omega') \) in Lemma 2, we have:

1. \( e_l(t, \Omega') \), \( e_{ij}^{(1)}(t, \Omega') \) and \( e_{ij}^{(2)}(t, \Omega') \) are continuous function of \( \Omega' \) and \( t \). Since \( \Omega' \) and \( t \) can only be selected from compact sets, they are automatically uniform continuous.

2. \( e_l(t, \Omega') \), \( e_{ij}^{(1)}(t, \Omega') \) and \( e_{ij}^{(2)}(t, \Omega') \) are bounded on the selectable sets of \( \alpha_i, \beta_j' \in [-C_2, C_2] \) and \( t \in [0, t_0] \).

3. \( e_l(t, \Omega') \) is bounded away from zero.

Proof of Lemma 2

First, we focus on the proof of (2.20). Given \( \epsilon_0 > 0, h > 0 \) since the choosable set for \( \Omega', [-C_2, C_2]^n \) and \( [0, t_0] \) are all bounded compact, we can have \( [-C_2, C_2]^n \times [0, t_0] \) to be covered by a number of \( n_{\epsilon_0, h} \) open sets \( OS_i \equiv \{ \Omega', t : \|\Omega' - \Omega_i\|_\infty < \epsilon_0, |t - t_i| < h, 1 \leq i \leq n_{\epsilon_0, h} \} \).

Then for any \( \Omega', t \) we can always find it to fall in a certain, say \( OS_i \). Now we can have

\[
\sup_{t \in [0, t_0], |\alpha| \leq C_2, |\beta_j| \leq C_2} \Gamma^{-1} \left| \sum_{l=1}^{r} [E_l(t, \Omega') - e_l(t, \Omega')] \right| \\
= \max_{1 \leq i \leq n_{\epsilon_0, h}} \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{r} [E_l(t, \Omega') - e_l(t, \Omega')] \right| \\
\leq \max_{1 \leq i \leq n_{\epsilon_0, h}} \left( \Gamma^{-1} \left| \sum_{l=1}^{r} [E_l(t_i, \Omega_i) - e_l(t_i, \Omega_i)] \right| + \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{r} [E_l(t, \Omega') - E_l(t_i, \Omega_i)] \right| \right) \\
+ \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{r} [e_l(t, \Omega') - e_l(t_i, \Omega_i)] \right|
\]

The rest of the proof is organized as follows. In Step 1, we bound the first term in the last inequality above. In Step 2 and 3, we try to find appropriate \( \epsilon_0 \) and \( h \) values to bound the second term, respectively. The results from Step 2 and 3 are combined together, and the third term is bounded in Step 4. In Step 5, we show (2.21).
Step 1 First, we show that for each selectable $\Omega', t, \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} [E_l(t, \Omega') - e_l(t, \Omega')] \right| = o_P(1)$. Since $\lambda_{0,l}(t) < C_1$ and $M_i(t,l)$ values are not larger than $F$, by looking at each term in $E_j(t, \Omega')$, we have

\[
\sup_{1 \leq l \leq \Gamma, 0 \leq t \leq t_0} E \left[ \lambda_{0,l}(t) \exp \left( \sum_{1 \leq i \leq n, i \neq j} (\alpha_i + \beta_j) \log(M_i(t,l) + 1) \right) \right]^2 \leq (C_1)^2 E \left[ \exp \left( 2 \sum_{1 \leq i \leq n, i \neq j} (\alpha_i + \beta_j) \log(M_i(t,l) + 1) \right) \right] \\
\leq (C_1)^2 E \left[ \exp \left( 4C_2 \sum_{1 \leq i \leq n, i \neq j} \log(F + 1) \right) \right] \\
\leq (C_1)^2 \exp (4nC_2 \log(F + 1))
\]

Let $C_5 = (C_1)^2 \exp (4nC_2 \log(F + 1))$. Then,

\[
\text{Var} [E_l(t, \Omega')] = \text{Var} \left[ \sum_{j=1}^{n} \lambda_{0,l}(t) \exp \left( \sum_{1 \leq i \leq n, i \neq j} (\alpha_i + \beta_j) \log(M_i(t,l) + 1) \right) \right] \\
\leq n \sum_{j=1}^{n} E \left[ \lambda_{0,l}(t) \exp \left( \sum_{1 \leq i \leq n, i \neq j} (\alpha_i + \beta_j) \log(M_i(t,l) + 1) \right) \right]^2 \\
< n^2 C_5,
\]

Now, due to the independency between topics, for any $\epsilon > 0$,

\[
P \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} [E_l(t, \Omega') - e_l(t, \Omega')] \right| > \epsilon \right) \leq \frac{\sum_{l=1}^{\Gamma} \text{Var}(E_l(t, \Omega'))}{\Gamma^2 \epsilon^2} \\
< \frac{n^2 C_5}{\epsilon^2 \Gamma}
\]
Step 2. Similarly to the derivation in Step 1,

\[
\Gamma^{-1} \sum_{l=1}^{r} \frac{\partial E_l(t, \Omega')}{\partial \alpha'_i} = \Gamma^{-1} \sum_{l=1}^{r} \sum_{j=1}^{n} \frac{\partial E_l(t, \Omega')}{\partial \phi'_{ij}} = \Gamma^{-1} \sum_{l=1}^{r} \sum_{j=1}^{n} \lambda_{0,j}(t) \log(M_i(t, l) + 1) \exp \left( \sum_{i,i \neq j} L_{ij}(t)(\alpha'_i + \beta'_j) \log(M_i(t, l) + 1) \right) \\
\leq \Gamma^{-1} \log(F + 1) \sum_{l=1}^{r} \sum_{j=1}^{n} \lambda_{0,j}(t) \exp \left( \sum_{i,i \neq j} L_{ij}(t)(\alpha'_i + \beta'_j) \log(M_i(t, l) + 1) \right) \\
\leq C_1 \log(F + 1) n \exp (2nC_2 \log(F + 1)),
\]

and similarly we can show,

\[
\left| \Gamma^{-1} \sum_{l=1}^{r} \frac{\partial E_l(t, \Omega')}{\partial \phi'_{ij}} \right| \leq C_1 \log(F + 1) n \exp (2nC_2 \log(F + 1))
\]

are bounded by a constant. Let \( C_6 = C_1 \log(F + 1) n \exp (2nC_2 \log(F + 1)) \), then

\[
\left\| \Gamma^{-1} \sum_{l=1}^{r} E_l(t, \Omega') - \Gamma^{-1} \sum_{l=1}^{r} E_l(t, \Omega'') \right\| \leq C_6 \| \Omega' - \Omega'' \|_2.
\]

Step 3. In this step, we try to find the appropriate \( h \). For any \( t \in [0, t_0] \) and \( \Omega' \) satisfying Condition B in Theorem 1, let \( E_{l,M}^{(1)}(t, \Omega') = \frac{\partial E_l(t, \Omega')}{\partial M_i(t, l)} \). For any \( h > 0 \) such that \( t + h \in [0, t_0] \), we can then expand \( E_{l,M}(t + h, \Omega') \) at \( t \), at the first order with a continuous derivative as in equation below:

\[
E_{l,M}(t+h, \Omega') = E_{l,M}(t, \Omega') + \sum_{i=1}^{n} E_{l,i,M}^{(1)}(t+\theta h, \Omega') \cdot (M_i(t+h, \Omega') - M_i(t, \Omega')) \quad (2.22)
\]

Similar to our previous derivation, we can show that there exists a constant \( C_7 \)
that $\sup_{t,i,t,\Omega'}|E^{(1)}_{t,i,M}(t + \theta h, \Omega')| \leq C_7$. By (2.22), we then have

$$|E_{i,M}(t + h, \Omega') - E_{i,M}(t, \Omega')| \leq \sum_{i=1}^{n} C_7 \cdot (M_i(t + h, \Omega') - M_i(t, \Omega')) \leq nC_7F \quad (2.23)$$

Recall our definition of $M(t,l)$ that

$$M_i(t,l) = (N_i(t,l) + A_i(t,l))I(N_i(t,l) + A_i(t,l) \leq F) + F \cdot I(N_i(t,l) + A_i(t,l) > F)$$

We have

$$P(M_i(t + h, l) - M_i(t, l) \geq 1) \leq P(N_i(t + h, l) - N_i(t, l) \geq 1) + P(A_j(t + h, l) - A_j(t, l) \geq 1) \leq C_8h,$$

where $C_8 = 2C_3$. Then

$$P \left( \max_{1 \leq i \leq n} [M_i(t + h, l) - M_i(t, l)] \geq 1 \right) \leq \sum_{i=1}^{n} P(M_i(t + h, l) - M_i(t, l) \geq 1) \leq nC_8h. \quad (2.24)$$

Now, we look back at $\Gamma^{-1} \left| \sum_{t=1}^{\Gamma} [E_i(t + h, \Omega') - E_i(t, \Omega')] \right|$. If we want this value to be larger than $\epsilon > 0$, by (2.23), we need at least $K_{\Gamma} \equiv \lfloor \epsilon nC_7F \rfloor$ of the term $E_i(t + h, \Omega') - E_i(t, \Omega')$ to be non-zero, i.e. $\max_{1 \leq i \leq n}[M_i(t + h, l) - M_i(t, l)]$ to be non-zero.

Then, when $0 < h = \frac{\epsilon}{2nC_8C_7F}$, let $t_1, t_2$ be (arbitrary) time points in $[t, t + h]$ and $\hat{t}_1(t, h, \Omega') \hat{t}_2(t, h, \Omega')$ to be the pair of values that maximize $\left| \sum_{t=1}^{\Gamma} [E_i(t_1, \Omega') - E_i(t_2, \Omega')] \right|$. We want to mention that this pair of maximizers always exists since on any sample path, there are only a finite number of possible combinations of the $M_{i,t,l}$ values. From the combination that maximizes the absolute difference, we can find the corresponding $\hat{t}_1(t, h, \Omega')$ and $\hat{t}_2(t, h, \Omega')$. Then, noticing that (2.23) holds for any $\Omega'$, by
Since for any $S$ covered by the series of sets, $(2.24)$ and the fact that $M_i(t, l)$ is non-decreasing in $t$,

$$
P \left( \Gamma^{-1} \sup_{t_1, t_2 \in [t, t+h], \Omega'} \left| \sum_{l=1}^{\Gamma} [E_l(t_1, \Omega') - E_l(t_2, \Omega')] \right| > \epsilon \right) \\
= P \left( \Gamma^{-1} \sup_{\Omega'} \left| \sum_{l=1}^{\Gamma} [E_l(t_1, t, \Omega') - E_l(t_2, t, \Omega')] \right| > \epsilon \right) \\
\leq \sum_{K=K_1}^{\Gamma} \sum_{V_K \subset \{1, \ldots, \Gamma\}} \prod_{i \in V_K} P(\max_{1 \leq i \leq n} [M_i(t_1, h, \Omega') + h, l) - M_i(t_2, h, \Omega'), l] \geq 1) \\
\leq \sum_{K=K_1}^{\Gamma} \sum_{V_K \subset \{1, \ldots, \Gamma\}} \prod_{i \in V_K} P(\max_{1 \leq i \leq n} [M_i(t + h, l) - M_i(t, l)] \geq 1) \text{ (by the nondecreasing property)} \\
\leq \sum_{K=K_1}^{\Gamma} \left( \frac{\Gamma}{K} \right) (nC_8h)^K (1 - nC_8h)^{\Gamma - K} \\
\equiv P_{0, \Gamma, \epsilon}
$$

Let $t_i = \frac{i}{h}$, $0 \leq i \leq \left\lfloor \frac{2t_0}{h} \right\rfloor$. The total time interval $[0, t_0]$ can then be covered by the series of sets, $S_i = [t_i, t_{i+1}], 1 \leq i \leq \left\lfloor \frac{2t_0}{h} \right\rfloor - 1, S_{\left\lfloor \frac{2t_0}{h} \right\rfloor} = \left[ t_{\left\lfloor \frac{2t_0}{h} \right\rfloor}, t_0 \right]$. Since for any $|t_1 - t_2| < h$, the two time points must be contained in the union of two subsequent $S_i$s, we have

$$
P \left( \Gamma^{-1} \sup_{|t_1 - t_2| < h, \Omega'} \left| \sum_{l=1}^{\Gamma} [E_l(t_1, \Omega') - E_l(t_2, \Omega')] \right| > \epsilon \right) \\
= P \left( \Gamma^{-1} \max_{0 \leq i \leq \left\lfloor \frac{2t_0}{h} \right\rfloor - 1} \sup_{t_1, t_2 \in S_i \cup S_{i+1}, \Omega'} \left| \sum_{l=1}^{\Gamma} [E_l(t_1, \Omega') - E_l(t_2, \Omega')] \right| > \epsilon \right) \\
\leq \sum_{i=0}^{\left\lfloor \frac{2t_0}{h} \right\rfloor - 1} P \left( \Gamma^{-1} \sup_{t_1, t_2 \in S_i \cup S_{i+1}, \Omega'} \left| \sum_{l=1}^{\Gamma} [E_l(t_1, \Omega') - E_l(t_2, \Omega')] \right| > \epsilon \right) \\
\leq \left\lfloor \frac{2t_0}{h} \right\rfloor \sum_{K=K_1}^{\Gamma} \left( \frac{\Gamma}{K} \right) (nC_8h)^K (1 - nC_8h)^{\Gamma - K} \equiv P_{\Gamma, \epsilon}
$$

$P_{\Gamma, \epsilon}$ can be viewed as

$$
\left\lfloor \frac{2t_0}{h} \right\rfloor P \left( \sum_{i=1}^{\Gamma} Y_i \geq K_{\Gamma} \right),
$$
where $Y_i, 1 \leq i \leq \Gamma$ are i.i.d random variables with Binomial$(1, nC_8h)$. Since $\frac{K_7}{\Gamma} = 2nC_8h$, and $h$ does no depends on $\Gamma$, by the law of large numbers, we can show that $P_{\Gamma, \epsilon}$ converges to zero as $\Gamma \to \infty$.

**Step 4.** Actually what we have shown in Step 2 and 3 is stronger than what we need. In this step, we show that (2.20) holds.

For any $\epsilon > 0$, for given $\Omega''$ and $t, \in [0, t_0]$, from Step 2 and 3, we can see that for any $\Omega', \Omega''$ and $t, t'$ that satisfy $\|\Omega'' - \Omega'\|_\infty < \frac{\epsilon}{C_6}, |t'' - t| < h = \frac{\epsilon}{2n^2C_8C_7F}$, we have

$$P \left( \Gamma^{-1} \sup_{\Omega', \Omega'', t, t'} \left| \sum_{l=1}^{r} [E_l(t', \Omega'') - E_l(t, \Omega')] \right| > 2\epsilon \right)$$

$$\leq P \left( \Gamma^{-1} \sup_{t, t', \Omega'} \left| \sum_{l=1}^{r} [E_l(t', \Omega') - E_l(t, \Omega')] \right| + \Gamma^{-1} \sup_{t, t', \Omega', \Omega''} \left| \sum_{l=1}^{r} [E_l(t', \Omega'') - E_l(t', \Omega')] \right| > 2\epsilon \right)$$

$$\leq P \left( \Gamma^{-1} \sup_{t, t', \Omega'} \left| \sum_{l=1}^{r} [E_l(t', \Omega') - E_l(t, \Omega')] \right| > \epsilon \right) + P \left( \Gamma^{-1} \sup_{t, t', \Omega', \Omega''} \left| \sum_{l=1}^{r} [E_l(t', \Omega'') - E_l(t', \Omega')] \right| > \epsilon \right)$$

$$\leq P_{\Gamma, \epsilon} + 0 = P_{\Gamma, \epsilon} \to 0, \text{ as } \Gamma \to \infty.$$

Then, due to the pointwise convergence in probability as proved in Step 1, for any $\Omega''$ and $t'$ that satisfy $\|\Omega'' - \Omega'\|_\infty < \frac{\epsilon}{C_6}, |t'' - t| < h = \frac{\epsilon}{2n^2C_8C_7F}$, we also have

$$\Gamma^{-1} \sup_{\Omega', t'} \left| \sum_{l=1}^{r} [e_l(t', \Omega'') - e_l(t, \Omega')] \right| < 2\epsilon$$

Since the choosable set for $\Omega', [-C_2, C_2]^n$ and $[0, t_0]$ are all bounded compact, we can have $[-C_2, C_2]^n \times [0, t_0]$ to be covered by a number of $n_\epsilon$ open sets $OS_i \equiv \{\Omega', t : \|\Omega' - \Omega_i\|_\infty < \frac{\epsilon}{C_6}, |t - t_i| < h = \frac{\epsilon}{2n^2C_8C_7F} \}, 1 \leq i \leq n_\epsilon$.

Then for any $\Omega', t$ we can always find it to fall in a certain, say $OS_i$. Now we can
have

\[
\sup_{t \in [0,T], |\alpha_i| \leq C_2, |\beta_j| \leq C_2} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - e_l(t, \Omega') \right] \right|
\]

\[
= \max_{1 \leq i \leq n_r} \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - e_l(t, \Omega') \right] \right|
\]

\[
\leq \max_{1 \leq i \leq n_r} \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t_i, \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right| + \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - E_l(t_i, \Omega_{i}) \right] \right| \right)
\]

\[
+ \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ e_l(t', \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right|
\]

\[
\leq \max_{1 \leq i \leq n_r} \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t_i, \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right| + \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - E_l(t_i, \Omega_{i}) \right] \right| + 2\epsilon \right)
\]

Then, combing what we have proved in Step 1,

\[
P \left( \max_{1 \leq i \leq n_r} \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t_i, \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right| \right) > 2\epsilon \right)
\]

\[
\leq P \left( \max_{1 \leq i \leq n_r} \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t_i, \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right| \right) > 2\epsilon \right)
\]

\[
+ P \left( \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - E_l(t_i, \Omega_{i}) \right] \right| \right) > 2\epsilon \right)
\]

\[
\leq \sum_{i=1}^{n_r} P \left( \left( \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t_i, \Omega_{i}) - e_l(t_i, \Omega_{i}) \right] \right| \right) > 2\epsilon \right)
\]

\[
+ \sum_{i=1}^{n_r} P \left( \left( \sup_{(\Omega', t) \in OS_i} \Gamma^{-1} \left| \sum_{l=1}^{\Gamma} \left[ E_l(t, \Omega') - E_l(t_i, \Omega_{i}) \right] \right| \right) > 2\epsilon \right)
\]

\[
\leq \frac{n_r n^2 C_5}{4\epsilon^2 \Gamma} + n_r P_{\Gamma, \epsilon} \rightarrow 0, \text{ as } \Gamma \rightarrow \infty.
\]

We have proved (2.20).

**Step 5.** In this step, we show that (2.21) holds.

Since \( E_l(t, \Omega'), 1 \leq l \leq \Gamma \) are independent, \( \Gamma^{-1} \sum_{l=1}^{\Gamma} |E_l(t, \Omega') - e_l(t, \Omega')| \rightarrow_p 0 \) is equivalent to \( \Gamma^{-1} \sum_{l=1}^{\Gamma} |E_l(t, \Omega') - e_l(t, \Omega')| \rightarrow 0 \), a.e. Since \( \Gamma^{-1} \sum_{l=1}^{\Gamma} E_l(t, \Omega') \) has bounded continuous second order derivatives, we have \( \sum_{j=1}^{n} \Gamma^{-1} \sum_{l=1}^{\Gamma} [E_{ij}^{(l)}(t, \Omega') - \)
Similarly, since $E_l(t, \Omega')$ has bounded continuous third order derivatives, we have $\sum_{j=1}^n \|\Gamma^{-1} \sum_{i=1}^{\Gamma} [E^{(2)}_{ij}(t, \Omega') - e^{(2)}_{ij}(t, \Omega')]\|_\infty \to 0$, a.e. Then, similar to the derivations in Step 2, 3 and 4, we can show all the entries of $\Gamma^{-1} \sum_{i=1}^{\Gamma} E^{(1)}_{ij}(t, \Omega')$ and $\sum_{i=1}^{\Gamma} E^{(2)}_{ij}(t, \Omega')$ have similar properties in $\Omega'$ and $t$. Then similar to Step 4, we can show (2.21).

**Proof of Lemma 3**

Following Step 5 in the Proof of Lemma 1, considering the existence of all the third order derivatives, it becomes obvious that $e_i(t, \Omega'), e^{(1)}_{ij}(t, \Omega')$ and $e^{(2)}_{ij}(t, \Omega')$ are continuous in $\Omega'$ and $t$. Since the selectable sets of $\alpha_i, \beta'_j \in [-C_2, C_2]$ and $t \in [0, t_0]$ are all compact, $e_j(t, \Omega')$, $e^{(1)}_j(t, \Omega')$ and $e^{(2)}_j(t, \Omega')$ are also bounded. Actually, an actual bound can be got following our argument in Step 1 of Lemma proof. At last, since

\[
E_l(t, \Omega') = \sum_{j=1}^n \lambda_{j,l}(t) \\
= \sum_{j=1}^n \lambda_{0,l}(t) \exp \left( \sum_{i,i \neq j} L_{ij}(t) (\alpha_i' + \beta'_j) \log(M_i(t,l) + 1) \right) \\
\geq \sum_{j=1}^n C_0 \exp \left( -2nC_2 \log(F + 1) \right) \\
= nC_0 \exp \left( -2nC_2 \log(F + 1) \right)
\]

Then, by the properties of almost sure convergence, we can also get

\[
e_j(t, \Omega') \geq nC_0 \exp \left( -2nC_2 \log(F + 1) \right).
\]

Lemma 3 has been proved.

To prove Theorem 1, we also need the following w lemmas, which are originally the Theorem II.1 and Corollary II.2 of (Andersen and Gill, 1982).

**Lemma 4.** Let $E$ be an open convex subset of $\mathbb{R}^p$ and let $F_1, F_2, \ldots$, be a sequence of random concave functions on $E$ such that for any $x \in E$, $F_\Gamma(x) \to P f(x)$ as $n \to \infty$. 

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where $f$ is some non-random function on $E$. If $f$ is also concave, then for all compact $A \subseteq E$,

$$\sup_{x \in A} |F_\Gamma(x) - f(x)| \to_P 0, \text{ as } \Gamma \to \infty$$

**Lemma 5.** Suppose $f$ has a unique maximum at $\hat{x} \in E$. Let $\hat{x}_\Gamma$ maximize $F_\Gamma$. Then under the condition of Lemma 4, $\hat{x}_\Gamma \to \hat{x}$ as $n \to \infty$.

**Proof of Theorem 1**

We prove the Theorem by combining the results in Lemma 1 and Lemma 2, in the following 3 steps.

**Step 1**

In this step, we first analyze some properties of the counting processes and represent the log-likelihood function by using integrals of counting processes as a preparation. We notice that by stating that $\lambda_j, l(t)$ is the hazard rate of $N_j(t, l)$, we actually have that the processes $K_j(t, l)$ defined by

$$K_j(t, l) = N_j(t, l) - \int_0^t \lambda_j, l(t) \, du$$

$$= N_j(t, l) - \int_0^t \lambda_0, l(t) \exp \left( \sum_{i,i \neq j} L_{ij}(u)(\alpha_i + \beta_j) \log(M_i(u, l) + 1) \right) \, du \tag{2.26}$$

$j = 1, \ldots, n, t \in [0, t_0]$, are local martingales on the time interval $[0, t_0]$. As a consequence, they are in fact local square integrable martingales, with

$$\langle K_j(\cdot, l), K_j(\cdot, l) \rangle(t) = \int_0^t \lambda_j(u, l) \, du, \langle K_i(\cdot, l_1), K_j(\cdot, l_2) \rangle = 0, i \neq j \text{ or } l_1 \neq l_2, \tag{2.27}$$

i.e. $K_i(t, l_1)$ and $K_j(t, l_2)$ are orthogonal when $i \neq j$ or $l_1 \neq l_2$. 

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Let $TN(t, l) = \sum_i N_i(t, l)$, also based on the definition of $N_j(t, l)$, we have

$$LL(\Omega', t) = \sum_{l=1}^{r} \left[ \sum_{j=1}^{n} \int_0^t \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j') \log(M_i(u, l) + 1) dN_i(u, l) - \int_0^t \log \left( \sum_{j=1}^{n} \exp \left( \sum_{i, i \neq j} L_{ij}(u) (\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right) \right) dTN(t, l) \right]$$

(2.28)

Now based on (2.28), we consider the process

$$X(\Omega', t) = \Gamma^{-1}(LL(\Omega', t) - LL(\Omega, t))$$

$$= \Gamma^{-1} \left\{ \sum_{l=1}^{r} \int_0^t \sum_{j=1}^{n} \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j' - \alpha_i - \beta_j) \log(M_i(u, l) + 1) dN_j(u, l) - \int_0^t \log \left( \frac{\sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(u) (\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right) \right) \right\} dTN(u, l)$$

(2.29)

where recall that $\Gamma$ denotes the number of topics under consideration.

**Step 2**

By definition, $\hat{\Omega}$ maximizes $X(\Omega', t)$ defined in (2.29). In this step, we find another easier to analyze function to approximate $X(\Omega', t)$. Notice that if we replace $dN_j(t, l)$
with the hazard rates of $N_j(t, l)$, $\lambda_{j,t}(t)$ in (2.29), we can get

$$R(\Omega', t) = \Gamma^{-1} \left\{ \sum_{l=1}^{r} \int_{0}^{t} \lambda_{0,l}(u) \sum_{j=1}^{n} \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j' - \alpha_i - \beta_j) \log(M_i(u, l) + 1) \right.$$ 

$$\cdot \exp \left( \sum_{i \neq j} L_{ij}(t)(\alpha_i + \beta_j) \log(M_i(u, l) + 1) \right) \ du$$

$$- \int_{0}^{t} \lambda_{0,l}(u) \log \left\{ \sum_{j=1}^{n} \exp \left( \sum_{i \neq j} L_{ij}(\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right) \right\} \ du$$

$$\cdot \left( \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i + \beta_j) \log(M_i(u, l) + 1) \right) \right) \ du \right\}$$

$$= \int_{0}^{t} LT(\Omega', u)du.$$

(2.30)

For each $\Omega'$, $X(\Omega', t) - R(\Omega', \cdot)$ can be written as sums of $K_j(t, l)$ defined in (2.26).

By Theorem 2.4.3 in (Fleming and Harrington, 2013), we have

$$< X(\Omega', t) - R(\Omega', t), X(\Omega', t) - R(\Omega', t) >= B(\Omega', t),$$

where

$$B(\Omega', t) = \Gamma^{-2} \sum_{l=1}^{r} \int_{0}^{t} S(u, l)\lambda_{0,l}(u)du,$$

where $S(u, l)$ is given by

$$S(u, l) = \sum_{j=1}^{n} \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j' - \alpha_i - \beta_j) \log(M_i(u, l) + 1) \right.$$

$$- \log \left\{ \sum_{j=1}^{n} \exp \left( \sum_{i, i \neq j} L_{ij}(\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right) \right\} \right\}^{2}$$

47
Let
\[ Q_l(\Omega', t) = \int_0^t S(u, l)\lambda_{0,l}(u)du. \]

Then \( Q_l, l = 1, \ldots, \Gamma \) are independent. We can write \( B(\Omega', t) \) as
\[ B(\Omega', t) = \Gamma^{-2} \sum_{l=1}^\Gamma Q_l(\Omega', t) \]

Similar to the proof of Lemma 2, since \( \alpha_i', \beta_j', \alpha_i, \beta_j, M_i(u, l), 1 \leq i, j \leq n, u \in [0, t_0] \), we can find a constant \( C_8 \), such that \( E[(Q_l(\Omega', t))^2] < C_8 \). Then we will have
\[ \left| \Gamma B(\Omega', t) - \Gamma^{-1} \sum_{l=1}^\Gamma E[Q_l(\Omega', t)] \right| \to P 0. \]

Therefore by the inequality of Lenglart (I.2), we see that \( X(\Omega', t) \) should converge in probability to the same limit as \( R(\Omega', t) \) for each \( \Omega' \), when at least one of they converges.

**Step 3**

In this step, we show \( R(\Omega', t) \) converges and analyze the limit function. Note that by using the notations in (3.24), \( R(\Omega', t) \) can be simplified to
\[ R(\Omega', t) = \int_0^t \Gamma^{-1} \sum_{l=1}^\Gamma \left[ \sum_{j=1}^n (\Phi_j' - \Phi_j) E_{ij}(u, \Omega) - \log \left\{ \frac{E_l(u, \Omega')}{E_l(u, \Omega)} \right\} E_l(u, \Omega) \right] du \]

It follows that by our assumption, \( \lambda_{0,l} < C_1 \), and Lemma 1 and Lemma 2, for each \( \Omega' \), as \( \Gamma \to \infty \),
\[ |R(\Omega', t_0) - P(\Omega', t_0)| \to P 0, \]
where

\[ P(\Omega', t_0) = \int_0^{t_0} \Gamma^{-1} \sum_{l=1}^{\Gamma} \left[ \sum_{j=1}^{n} (\Phi_j' - \Phi_{j'}) e_{ij}'(u, \Omega) - \log \left\{ \frac{e_l(u, \Omega)}{e_l(u, \Omega')} \right\} e_l(u, \Omega) \right] du \]

(2.31)

Following our argument in Step 3, equivalently we should have

\[ |X(\Omega', t_0) - P(\Omega', t_0)| \to P_0, \]

(2.32)

By Lemma 4, since \( X(\Omega', t_0) \) are concave (as will be shown in Lemma 5 below) for all \( \Gamma \), we have convergence in (2.32) is equivalent to

\[ \sup_{\Omega'} |X(\Omega', t_0) - P(\Omega', t_0)| \to P_0 \]

Also as shown in Lemma 4 below, \( P(\Omega', t_0) \) is concave and uniquely maximized at \( \Omega' = \Omega \). Since by definition, \( \hat{\Omega} \) maximizes \( X(\Omega', t) \), then by Lemma 5, \( \hat{\Omega} \to \Omega \).

**Lemma 6.** Under the Conditions A, B, C,D of Theorem 1, the \( P(\Omega', t_0) \) defined in (2.31) is concave and uniquely maximized at \( \Omega' = \Omega \).

**Lemma 7.** \( LT(\Omega', t) \) and \( X(\Omega', t) \) are both concave in \( \Omega' \).

**Proof of Lemma 6:**

We establish the concavity of \( P(\Omega', t_0) \) and its unique maximizer based on the evaluation of its first and second derivative of \( P_1(\Omega', t_0) \) to show its convexity. By Lemma 2, we may evaluate the first and second derivatives of \( P(\Omega', t_0) \) inside the integral(cf. Bartle, 1966, Corollary 5.9). We compute the first derivatives as

\[ \frac{\partial P(\Omega', t_0)}{\partial \beta_j'} = \int_0^{t_0} \Gamma^{-1} \sum_{l=1}^{\Gamma} \left[ \sum_{j=1}^{n} I_n' e_{ij}'(u, \Omega) - I_n' e_{ij}'(u, \Omega') \right] e_l(u, \Omega) du \]
and

\[
\frac{\partial P(\Omega', t_0)}{\partial \alpha'_i} = \int_0^{t_0} \Gamma^{-1} \sum_{l=1}^{\Gamma} \left[ \sum_{j=1}^n G'_i e_{ij}^{(1)}(u, \Omega) - (\alpha'_1, \ldots, \alpha'_n)' \sum_j G'_i e_{ij}^{(1)}(u, \Omega) e_l(u, \Omega) \right] du
\]

(2.33)

where \( I_n \) is the \( n \times n \)-dimensional diagonal matrix and \( G_i \) is a \( n \)-dimensional vector with all zeros except one on the \( i \)-th entry.

Note that the above partial derivatives are all zero at \( \Omega' = \Omega \). Further, the Hessian matrix of \( P(\Omega', t_0) \) can be written as

\[
\nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0) = -T' DT,
\]

(2.34)

where \( T = \left( \frac{\partial \phi'_1}{\partial \alpha'_2}, \ldots, \frac{\partial \phi'_n}{\partial \alpha'_2}, \ldots, \frac{\partial \phi'_1}{\partial \alpha'_n}, \frac{\partial \phi'_2}{\partial \beta'_1}, \ldots, \frac{\partial \phi'_n}{\partial \beta'_1}, \ldots, \frac{\partial \phi'_1}{\partial \beta'_n}, \frac{\partial \phi'_2}{\partial \beta'_n}, \ldots, \frac{\partial \phi'_n}{\partial \beta'_n} \right)' \)

is a matrix of zeros and ones, which describes the linear combination relationship between \( \phi'_{ij} \) and \( \alpha'_i + \beta'_j \). Matrix \( D \) is a block diagonal matrix of dimension \( n^2 \times n^2 \), with a number of \( n \) block matrix of size \( n \times n \), \( D_j, 1 \leq j \leq n \), on the diagonal, where

\[
D_j = \int_0^{t_0} \Gamma^{-1} \sum_{l=1}^{\Gamma} T' \left[ e_{lj}^{(2)}(u, \Omega) + e_{lj}^{(1)}(u, \Omega) \otimes 2 e_l(u, \Omega) \right] du,
\]

where "\( \otimes 2 \)" denotes the outer product of a vector. Then the entries of \( \nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0) \)

will be linear combinations of \( e_l \) and \( e^{(k)}_{lj} \), \( k = 1, 2, 1 \leq l \leq \Gamma, 1 \leq j \leq n \).

The exact form of \( \nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0) \) in (2.34) may look intimidating. But from the definition of \( LT(\Omega', t) \) in (2.8) we can see \( LT(\Omega', t) \) and \( P(\Omega', t) \) in (2.30) have exactly the same Hessian matrix. Then, again Lemma 1 and 2 implies as \( \Gamma \rightarrow \infty \),

\[
\left\| \int_0^{t_0} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t) dt - \nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0) \right\|_\infty \rightarrow 0.
\]

(2.35)
By Condition D of Theorem 1, we have

$$\lim_{\Gamma \to \infty} P (\lambda_{\min} (-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t_0)|_{\Omega' = \Omega}) > C_4) = 1$$

By (2.25) we have proved and the continuity of the function \(\lambda_{\min}(\cdot)\), we can find a constant \(\delta\) (not depending on \(\Gamma\) and \(\Omega\)), such that

$$\lim_{\Gamma \to \infty} P \left( \min_{t \in [t_0 - \delta, t_0]} \lambda_{\min} (-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega}) > \frac{C_4}{2} \right) = 1 \quad (2.36)$$

We can write

$$\int_{t_0}^{t_0 - \delta} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt = \int_{t_0}^{t_0 - \delta} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt + \int_{0}^{t_0 - \delta} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt \quad (2.37)$$

By the definition of integration and the fact that the first integration on the right hand side of the equation above exists, from (2.36), we should also have

$$\lim_{\Gamma \to \infty} P \left( \lambda_{\min} \left( -\int_{t_0 - \delta}^{t_0} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt \right) > \frac{C_4 \delta}{2} \right) = 1 \quad (2.38)$$

Also, since \(LT(\Omega', t)\) is concave (as shown in Lemma 5), also by the definition of integration and the fact that the second integration on the right hand side of (2.37) exists, we have the following Hessian matrix

$$\int_{t_0 - \delta}^{t_0} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt$$

to be at least semi-positive definite. Based on (2.38) Actually, we have already shown that

$$\lim_{\Gamma \to \infty} P \left( \lambda_{\min} \left( -\int_{0}^{t_0} \nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)|_{\Omega' = \Omega} dt \right) > \frac{C_4 \delta}{2} \right) = 1$$
Therefore, combining the result in (2.35), we have

\[
\lambda_{\min} \left( -\int_0^{t_0} \nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0)|_{|\Omega'|=\Omega} \right) \geq \frac{C_4 \delta}{2} \tag{2.39}
\]

Since \( P(\Omega', t_0) \) has zero derivative at \( \Omega' = \Omega \), and by (2.39), \( P(\Omega', t_0) \) is uniquely maximized at \( \Omega \). Lemma 4 has been proved.

**Proof of Lemma 7:**

Considering the definition of \( X(\Omega', t) \) and \( LT(\Omega', t) \) as in (2.29) and (2.8), ignoring the terms linear in \( \Omega' \); we notice that \( X(\Omega', t) \) and \( LT(\Omega', t) \) are both positively weighted sums (with weights independent of \( \Omega' \)) of

\[
LE_l(\Omega', t) \equiv -\log \left[ \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right) \right].
\]

Then, to finish the proof of the lemma, it is equivalent to show the concavity of \( LE_l(\Omega', t) \). Let

\[
SE_l(\Omega', t) = \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha_i' + \beta_j') \log(M_i(u, l) + 1) \right)
\]

For any \( a, b > 0 \) and \( a + b = 1 \) and \( \Omega'_k = (\alpha'_{k,2}, \ldots, \alpha'_{k,n}, \beta'_{k,1}, \ldots, \beta'_{k,n}), k = 1, 2, \ldots, n \).
satisfying Condition B of Theorem 1, by Jensen’s inequality, we have

\[
SE_t(a\Omega'_1 + b\Omega'_2, t) = \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(a\alpha'_{1,i} + a\beta'_{1,j} + b\alpha'_{2,i} + b\beta'_{2,j}) \log(M_i(u, l) + 1) \right)
\]

\[
\leq \left( \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha'_{1,i} + \beta'_{1,j}) \log(M_i(u, l) + 1) \right) \right)^a \cdot \left( \sum_{j=1}^{n} \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij}(\alpha'_{2,i} + \beta'_{2,j}) \log(M_i(u, l) + 1) \right) \right)^b
\]

\[
= [SE_t(\Omega'_1, t)]^a [SE_t(\Omega'_2, t)]^b
\]

And the above inequality is just equivalent to

\[
- \log(SE_t(a\Omega'_1 + b\Omega'_2, t)) \geq -a \log(SE_t(\Omega'_1, t)) - b \log(SE_t(\Omega'_2, t)).
\]

Lemma 5 has been proved.

### 2.9 Additional Senator Results

Table 2.6 shows the top ten most influential accounts under the proposed method for different time periods. We see consistent results with the findings from summer 2014. Important newspapers like the Financial Times and Washington Post still appear in the top ten when utilizing the full data. Other prominent accounts include senators that have leadership positions, like Harry Reid (Senate Majority Leader) and several others with high profile committee chairmanships or ranking appointments.

Tables 2.7 and 2.8 show regression results for the sequestration period, and Tables 2.9 and 2.10 show regression results for the inauguration period. The results are consistent with the results presented in the main text. Regressing directly on the leadership scores shows a strongly significant and positive coefficient for the proposed
Table 2.6: Top ten rankings under the proposed model for different time intervals.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Sequestration</th>
<th>2014 Inauguration</th>
<th>Entire Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Leahy</td>
<td>Leahy</td>
<td>Financial Times</td>
</tr>
<tr>
<td>2</td>
<td>Grassley</td>
<td>Grassley</td>
<td>Grassley</td>
</tr>
<tr>
<td>3</td>
<td>Mikulski</td>
<td>Begich</td>
<td>Leahy</td>
</tr>
<tr>
<td>4</td>
<td>Begich</td>
<td>Mikulski</td>
<td>Cruz</td>
</tr>
<tr>
<td>5</td>
<td>Shaheen</td>
<td>Johanns</td>
<td>Washington Post</td>
</tr>
<tr>
<td>6</td>
<td>McCaskill</td>
<td>Reid</td>
<td>Reid</td>
</tr>
<tr>
<td>7</td>
<td>Reid</td>
<td>McCaskill</td>
<td>Begich</td>
</tr>
<tr>
<td>8</td>
<td>Blunt</td>
<td>Graham</td>
<td>Mikulski</td>
</tr>
<tr>
<td>9</td>
<td>Graham</td>
<td>Shaheen</td>
<td>Ezra Klein</td>
</tr>
<tr>
<td>10</td>
<td>Collins</td>
<td>Hagan</td>
<td>Schatz</td>
</tr>
</tbody>
</table>

Table 2.7: Regression estimates, where the response variable is the raw leadership scores from GovTrack.us and influence is computed from January 1, 2013 to March 1, 2013. $R^2 = 0.327$; $F = 8.839$ on 5 and 92 DF (p-value: 0.000)

| Variable         | Estimate | Std. Error | t value | P(>|t|) |
|------------------|----------|------------|---------|--------|
| Intercept        | -0.153   | 0.228      | -0.669  | 0.505  |
| Proposed Influence| 0.074   | 0.028      | 2.689   | 0.009  |
| Republican       | -0.153   | 0.039      | -3.960  | 0.000  |
| Age              | 0.002    | 0.003      | 0.833   | 0.407  |
| Years in Senate  | 0.007    | 0.003      | 2.532   | 0.013  |
| Male             | 0.020    | 0.050      | 0.397   | 0.692  |

influence measure. The regressions with transformed leadership scores show effects are moderately significant.
Table 2.8: Regression estimates, where the response variable is $log(\frac{\text{leadership}}{1-\text{leadership}})$, where leadership is from GovTrack.us and influence is computed from January 1, 2013 to March 1, 2013. $R^2 = 0.119$; $F = 2.466$ on 5 and 92 DF (p-value: 0.038)

| Variable     | Estimate | Std. Error | t value | $P(>|t|)$ |
|--------------|----------|------------|---------|----------|
| Intercept    | -3.925   | 2.574      | -1.525  | 0.131    |
| Proposed Influence | 0.504   | 0.312      | 1.614   | 0.110    |
| Republican   | -1.103   | 0.437      | -2.526  | 0.013    |
| Age          | 0.008    | 0.029      | 0.267   | 0.790    |
| Years in Senate | 0.033   | 0.031      | 1.062   | 0.291    |
| Male         | 0.465    | 0.561      | 0.830   | 0.409    |

Table 2.9: Regression estimates, where the response variable is the raw leadership scores from GovTrack.us and influence is computed from November 1, 2012 to January 31, 2013. $R^2 = 0.328$; $F = 8.839$ on 5 and 92 DF (p-value: 0.000)

| Variable     | Estimate | Std. Error | t value | $P(>|t|)$ |
|--------------|----------|------------|---------|----------|
| Intercept    | -0.132   | 0.220      | -0.597  | 0.552    |
| Proposed Influence | 0.072   | 0.026      | 2.726   | 0.008    |
| Republican   | -0.154   | 0.039      | -3.978  | 0.000    |
| Age          | 0.002    | 0.003      | 0.797   | 0.427    |
| Years in Senate | 0.007   | 0.003      | 2.616   | 0.010    |
| Male         | 0.020    | 0.050      | 0.395   | 0.693    |

Table 2.10: Regression estimates, where the response variable is $log(\frac{\text{leadership}}{1-\text{leadership}})$, where leadership is from GovTrack.us and influence is computed from November 1, 2012 to January 31, 2013. $R^2 = 0.117$; $F = 2.402$ on 5 and 92 DF (p-value: 0.043)

| Variable     | Estimate | Std. Error | t value | $P(>|t|)$ |
|--------------|----------|------------|---------|----------|
| Intercept    | -3.578   | 2.495      | -1.434  | 0.155    |
| Proposed Influence | 0.452   | 0.297      | 1.521   | 0.132    |
| Republican   | -1.105   | 0.437      | -2.527  | 0.013    |
| Age          | 0.007    | 0.029      | 0.255   | 0.800    |
| Years in Senate | 0.035   | 0.031      | 1.113   | 0.269    |
| Male         | 0.460    | 0.561      | 0.819   | 0.415    |
CHAPTER III

Measuring Topic Dependent Edge Importance in Twitter Ecosystems Using a Counting Process Modeling Framework

3.1 Introduction

Over the past two decades, the functional properties of edges in complex networks have gained much attention in literature, (Miritello et al., 2011) (Dorogovtsev and Mendes, 2002) (Newman, 2003). The importance (weights) of edges, a key factor in determining structural properties of a network has also been fairly extensively studied. For example, in (Tong et al., 2012), an edge’s importance is measured by its topology strength in the connection network, while in (Toivonen et al., 2007), it is modeled by the short time probability that a node can send messages to other nodes in the network. In our proposed framework, as presented in the previous chapter, we take into consideration the actions occurring from the nodes (such as posting, retweeting and mentioning) on multiple edges simultaneously. To capture the temporal evolution of actions, we model them as a counting process.

However, we allow for a much more flexible parameterization than the one used in the previous chapter. Instead of having two global parameters for each node, reflecting capability to generate responses ($\alpha$) and susceptibility to respond to other
nodes’ actions ($\beta$), we allow for independent parameters between every pair of nodes for \textit{selected topics}. With this extension we are trying to incorporate the heterogeneous nature of topics discussed and the fact that for specific topics, selected nodes have either greater susceptibility due to their particular interest, or greater capability for generating responses, due to their perceived expertise. For example, if we considered a general Twitter network, it is reasonable to assume that sport fans would engage in a different manner when the topic under discussion involves sports (and even more so, if it involves their favorite sport or favorite sport team) or the node posting is a sports-writer or an athlete and hence it is generally perceived that (s)he has expertise or additional information.

The remainder of the chapter is organized as follows: in Section 3.2, we introduce the counting process modeling framework and the proposed edge importance measure. Section 3.3 presents the computational algorithm we use to obtain the parameter estimates, as well as establish their statistical properties and those of the influence measure in Section 3.4. In Section 3.5, we use simulation studies to evaluate the performance of the model, while the US Senate application is presented in Section 3.6. Finally, a short summary is given in Section 3.7.

3.2 The model and the influence measure

Leveraging the model developed in the previous chapter, we have similar definitions of some key quantities for the network under consideration. We continue to represent the followers network as $G = (V, L)$, where $V$ corresponds to the set of nodes of all the Twitter accounts under consideration and $L = \{L_{i,j}, 1 \leq i \neq j \leq n\}$ the edge set between them. This network establishes \textit{potential channels of communication} between accounts, since if an account follows another, then they can actively interact. In principle, $L$ can be dynamically evolving, but in this work we continue to consider $L$ to be static and not changing over time, which is a reasonable assumption.
for periods of time extending to months. We continue to consider the following three actions: posting a new message, retweeting a message posted by another account that they follow and finally mentioning another account that they follow in a new posted message.

Next, we define the following two key counting processes. Still we use $N_j(t)$ to denote the total number of retweets and mentions that account $j$ generates on topic $l$ by time $t$ and let $A_j(t, l)$ denote the total number of posted messages by account $j$ on topic $l$ by time $t$. Denote the set of topics discussed by $\Gamma$. In the model presented in the previous chapter, all topics were treated identically; thus, the cardinality of the set $\Gamma$ corresponds to the total “sample size”. However, in practice different topics elicit different responses and/or interactions for different sets of users, as briefly discussed in the introductory section. In principle, the set $\Gamma$ can be partitioned into several groups. However, for the sake of simplicity we consider two groups; namely, $\Gamma = \Gamma_1 \cup \Gamma_2$. We assume that an account has different long-term interaction parameters with its followers on all topic sets. On topic set $\Gamma_1$, we use $\alpha_{ij}$ to denote account $i$’s interactions with its followers $j \in L$, while on set $\Gamma_2$, we assume the interaction to be $\alpha_{ij} + \delta_i$. The $\delta_i$ parameter captures the differential capability of account $i$ to elicit responses from all the other accounts on the followers’ network for the subset of topics in $\Gamma_2$.

We continue to adopt the previously described modeling strategy and model $\{N_j(t, l)\}_{i=1}^n$ as a set of counting processes through their hazard rates, using a version of Cox (Andersen and Gill, 1982) proportional hazard model; specifically, the hazard rate $\lambda_{j,l}(t)$ of process $N_j(t, l)$ is given by

$$\lambda_{j,l}(t) = \lambda_{0,l}(t) \exp \left( \sum_{i \neq j} L_{ij}(\alpha_{ij}) \log(M_i(t, l) + 1) \right), \quad (3.1)$$
when \( l \in \Gamma_1 \) and
\[
\lambda_{j,l}(t) = \lambda_{0,l}(t) \exp \left( \sum_{i \neq j} L_{ij}(\alpha_{ij} + \delta_i) \log (M_i(t,l) + 1) \right),
\] (3.2)
when \( l \in \Gamma_2 \), where
\[
M_i(t,l) = (N_i(t,l) + A_i(t,l))I(N_i(t,l) + A_i(t,l) \leq F) + F \cdot I(N_i(t,l) + A_i(t,l) > F).
\]
\( A_j(t,l) + N_j(t,l) \) is the total number of postings, retweets and mentions for account \( j \) on topic \( l \) by time \( t \). We also consider the effect of seeing actions from account \( i \) can get saturated when the total number of actions reaches the constraint, \( F \). We assume that the parameters \( \alpha_{ij}, \delta_i \in (-\infty, \infty) \), since accounts and their users may be positively or negatively inclined towards other accounts, and the accounts’ capability can vary in a differential manner between the two topic sets. The nonparametric baseline component \( \lambda_{0,l}(t) \) is time varying. In general, due to limited time span, we assume all the observations are made within \([0, t_0]\), since as observed in real data sets, interest in a particular topic wanes fairly quickly. The model posits that account \( j \) interacts with other accounts at a baseline level \( \lambda_{0,l}(t) \), modulated by its (different) ability to generate responses from its followers (in the two topic sets). Note that we continue to model the retweet-mention process \( N_j(t,l) \), since it reflects interactions between nodes and use the total effective activity process \( M_j(t,l) \) as a covariate.

To complete the modeling framework, let \( T^l_j = \{ T^l_{j,1}, \ldots, T^l_{j,n^l_j} \}, t = 1, \ldots, n^l_j \), denote the set of time points that account \( j \) took action (post, retweet, mention) on topic \( l \), until the last observation time point \( t_0 \). Finally, for identification purposes, we require \( \alpha_{ij} = 0 \) when \( L_{ij} = 0 \), i.e., when account \( j \) does not follow \( i \). Use \( \Psi = (\alpha_{ij})_{1 \leq i,j \leq n} \) to denote the \( \alpha \) matrix and \( \Delta \) to represent \( (\delta_1, \ldots, \delta_n) \). Then the parameters of interest are \( \Omega = \{ \Psi \cdot L, \Delta \} \), where “.” denotes the point-wise multiplication of two matrices. As before and following, Andersen and Gill (1982), we
employ a partial-likelihood function to obtain estimates of $\Omega$. Specifically, we treat the baseline $\lambda_{0,l}(t)$ as a nuisance parameter and decomposing the full-likelihood to obtain

$$PL_1(t) = \prod_{1 \leq l \leq \Gamma} \left( \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n_j, T_{j,k}^l \leq t} \frac{\lambda_j(T_{j,k}^l)}{\sum_{1 \leq i \leq n} \lambda_i(T_{j,k}^l)} \right)$$

Plugging the exact form of the hazard rate from (3.1) and (3.2) into the partial-likelihood function (PL), we get:

$$PL_1(t) = \prod_{l \in \Gamma_1} \left( \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n_j, T_{j,k}^l \leq t} \frac{\exp \left( \sum_{i \neq j} L_{ij} \alpha_{ij} \log \left( M_i(T_{j,k}^l, l) + 1 \right) \right)}{\sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} \alpha_{ui} \log \left( M_u(T_{j,k}^l, l) + 1 \right) \right)} \right)$$

$$+ \prod_{l \in \Gamma_2} \left( \prod_{1 \leq j \leq n} \prod_{1 \leq k \leq n_j, T_{j,k}^l \leq t} \frac{\exp \left( \sum_{i \neq j} L_{ij} (\alpha_{ij} + \delta_i) \log \left( M_i(T_{j,k}^l, l) + 1 \right) \right)}{\sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} (\alpha_{ui} + \delta_u) \log \left( M_u(T_{j,k}^l, l) + 1 \right) \right)} \right)$$

(3.3)

### 3.2.1 The Edge Importance Measure

Next, we define the edge importance measure, leveraging the structure of the new model. Specifically, for an edge $(i, j)$ with $L_{ij} = 1$, when $\lambda_{0,l}(t) = 1$, and $M_i(t, l) = 1$, other $M_k(t, l) = 0, t \in [0, t_0], k \neq i, j$, account $j$’s hazard rate within unit time $[0, 1]$ on topic set $\Gamma_1$ is given by

$$H_j^{(1)} = \exp \left( \log(2) \alpha_{ij} \right),$$

and its hazard rate on topic set $\Gamma_2$ is

$$H_j^{(2)} = \exp \left( \log(2) (\alpha_{ij} + \delta_i) \right).$$
From the properties of the exponential distribution, the probability that account $j$ will retweet $i$, within $[0, 1]$ on topic set $\Gamma_1$ is

$$P_{ij}^{(1)} = 1 - \exp \left[-\exp \left(\log(2)\alpha_{ij}\right)\right],$$

and the probability that account $j$ will retweet $i$, within $[0, 1]$ on topic set $\Gamma_2$ is

$$P_{ij}^{(2)} = 1 - \exp \left[-\exp \left(\log(2)(\alpha_{ij} + \delta_i)\right)\right].$$

Following our definition of accounts’ influences in the previous chapter, the influence of $j$ on topic set $\Gamma_1$ is given by

$$\Xi_j^{(1)} = \log \left[ \sum_{1 \leq k \leq n, k \neq j} L_{jk} \cdot \exp \left(\log(M_j^{(1)} + 1)\alpha_{jk}\right) \right], \quad (3.4)$$

and its influence on topic set $\Gamma_2$ can be written as

$$\Xi_j^{(2)} = \log \left[ \sum_{1 \leq k \leq n, k \neq j} L_{jk} \cdot \exp \left(\log(M_j^{(2)} + 1)(\alpha_{jk} + \delta_j)\right) \right], \quad (3.5)$$

where $M_j^{(1)}$ and $M_j^{(2)}$ are node $j$’s average number of actions on topic set $\Gamma_1$ and $\Gamma_2$ respectively. Therefore, on topic set $\Gamma_1$, on edge $(i, j)$, the influence $i$ can borrow from $j$ with a single action, can be expressed as

$$\varsigma_{ij}^{(1)} = P_{ij}^{(1)} \cdot \Xi_j^{(1)}
= (1 - \exp \left[-t_0 \exp \left(\log(2)\alpha_{ij}\right)\right]) \log \left[ \sum_{1 \leq k \leq n, k \neq j} L_{jk} \cdot \exp \left(\log(M_j^{(1)} + 1)\alpha_{jk}\right) \right], \quad (3.6)$$
and on topic set $\Gamma_2$, the influence $i$ that can be borrowed can be given by

$$\varsigma_{ij}^{(2)} = P_{ij}^{(2)} \cdot \Xi_j$$

$$= (1 - \exp[-t_0 \exp(\log(2)(\alpha_{ij} + \delta_i))]) \log \left[ \sum_{1 \leq k \leq n, k \neq j} L_{jk} \cdot \exp \left( \log(\bar{M}_j^{(2)} + 1)(\alpha_{jk} + \delta_j) \right) \right].$$

We define $\varsigma_{ij}^{(1)}$ and $\varsigma_{ij}^{(2)}$ to be edge $(i, j)$’s importance on topic set $\Gamma_1$ and $\Gamma_2$, respectively. When $L_{ij} = 0$, let $\varsigma_{ij}^{(1)} = \varsigma_{ij}^{(2)} = 0$.

### 3.2.2 The set of influential edges

In this subsection, we use our definition of edge importance to capture the edges that are essential for propagating or impairing the information flow process. Let $S_L^{(c)} = \{\varsigma_{ij}^{(c)} : L_{ij} = 1\}, c = 1, 2$, denote all the edge importance values on existing edges, on the two topic sets. For any given probability $p$, we can build the following two edge collections, which correspond to the $p$ proportion of the most spawning and jamming edges as follows.

$$S_1(p)^{(c)} = \{(i, j) : \varsigma_{ij}^{(c)} \geq q_p^{(c)} \}$$

and

$$S_2(p)^{(c)} = \{(i, j) : \varsigma_{ij} \leq q_p^{(c)} \}$$

where $q_p^{(c)}$ are the quantile of set $S_L^{(c)}$ of probability $p$, $c = 1, 2$. 
3.3 Computation and Inference

Next, we present a Newton-type algorithm for computing the parameter estimates $\Omega$. The logarithm of the partial likelihood function (3.3) is given by

$$LL_0(t) = \log(PL(t))$$

$$= \sum_{l \in \Gamma_1} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l, T_{j,k}^l \leq t} \sum_{i \neq j} L_{ij} \alpha_{ij} \log(M_i(T_{j,k}^l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l, T_{j,k}^l \leq t} \log \left[ \sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} \alpha_{ui} \log(M_u(T_{j,k}^l) + 1) \right) \right] \right\}$$

$$+ \sum_{l \in \Gamma_2} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l, T_{j,k}^l \leq t} \sum_{i \neq j} L_{ij} (\alpha_{ij} + \delta_i) \log(M_i(T_{j,k}^l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l, T_{j,k}^l \leq t} \log \left[ \sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} (\alpha_{ui} + \delta_u) \log(M_u(T_{j,k}^l) + 1) \right) \right] \right\}$$

(3.10)
To introduce sparsity on the differences of influence levels among topics, we add an $L_1$ penalty on $\Delta$ to obtain the following penalized partial likelihood function.

$$LL(t) = \log(PL(t)) - \sum_{i=1}^{n} \gamma |\delta_i|$$

$$= \sum_{t \in \Gamma_1} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \sum_{T_{j,k}^l \leq t} \sum_{i \neq j} L_{ij} \alpha_{ij} \log(M_i(T_{j,k}^l, l) + 1) \right. $$

$$- \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \sum_{T_{j,k}^l \leq t} \log \left[ \sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} \alpha_{ui} \log(M_u(T_{j,k}^l, l) + 1) \right) \right]$$

$$+ \sum_{t \in \Gamma_2} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \sum_{T_{j,k}^l \leq t} \sum_{i \neq j} L_{ij} (\alpha_{ij} + \delta_i) \log(M_i(T_{j,k}^l, l) + 1) \right. $$

$$- \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j^l} \sum_{T_{j,k}^l \leq t} \log \left[ \sum_{1 \leq i \leq n} \exp \left( \sum_{u \neq i} L_{ui} (\alpha_{ui} + \delta_u) \log(M_u(T_{j,k}^l, l) + 1) \right) \right]$$

$$- \sum_{i=1}^{n} \gamma |\delta_i|$$

(3.11)

The objective function corresponds to $LL(t_0)$, which considers all events $k$ in its equation (3.11). For the sake of notational simplicity, we will use $LL$ to represent $LL(t_0)$ in the rest of the paper. To maximize $LL(t_0)$, due to the smoothness of $LL_0(t_0)$, we first compute the gradient $G_0 \equiv \nabla_\Omega LL_0(t_0)$ and the Hessian $H \equiv \nabla_\Omega \nabla_\Omega (LL_0(t_0))$ of $LL_0(t_0)$. Then on the last $n$ entries of $G_0$, which is represented as $G_n$, update it to $G_n - \gamma \text{sign}(\Delta)$. Denote the updated $G_0$ as $G$. $G$ and $H$ are used as the approximate gradient and Hessian of $LL(t_0)$ in Newton’s algorithm. The detailed expressions for $G_0$ and the Hessian $H$ are given in the Appendix.

The steps of the optimization, with a given penalty $\gamma$ in computing $G$, are given in Algorithm 1. To speed up calculations, we take advantage of the structure of the problem in actual applications, as explained in detail in the Appendix. Then $\gamma$ is selected with the one that minimize the estimation of the parameter. As stated in the
Algorithm 3 Estimating the parameters by Newton’s algorithm

1: Initialize Ω value by \( \alpha_{ij} = 0, \delta_i = 0, 1 \leq i, j \leq n \)
2: Define \( s \) as a positive thresholding constant for the minimum step size
3: while \( \tau > s \) do
4: Calculate \( G \) by using (3.16), (3.20) and (3.22)
5: Calculate \( H \) by using (3.18) to (3.21)
6: Find the optimum positive \( \tau \) value such that \( \Omega - \tau \cdot H^{-1}G \) will maximize the log-partial-likelihood (3.11), when \( H \) is non-singular and find the optimum positive \( \tau \) value such that \( \Omega - \tau \cdot (H - \theta I)^{-1}G \) will maximize the log-partial-likelihood (3.11), otherwise.
7: Update \( \Omega \leftarrow \Omega - \tau \cdot H^{-1}G \) or \( \Omega - \tau \cdot (H - \theta I)^{-1}G \), depending on \( H \)’s singularity
8: end while
9: return \( \Omega \)

algorithm, \( s, \theta \) are positive constants to judge the convergence of the the Newton’s algorithm and to solve singularity problems when computing the inverse matrices.

The computational complexity of this algorithm is dominated by the computation of \( H \). Denote by \( m_n = \max_{1 \leq j \leq n} \{ n_j \} \). Based on (3.16), (3.20) and (3.22), it costs \( O(\Gamma n^2 m_n) \) operations to calculate an entry of \( G \). Further, since \( G \) is of dimension \( 2n \), it takes \( O(\Gamma n^3 m_n) \) to obtain the entire \( G \) vector. Analogously, based on (3.18) to (3.21), it costs \( O(\Gamma n m_n) \) operations to calculate an entry of \( H \), if proper bookkeeping is used on the results obtained for the gradient \( G \). Further, since \( H \) is of dimension \( n^4 \), it takes \( O(\Gamma n^5 m_n) \) to obtain the entire \( H \) matrix. Hence, the overall time complexity for each iteration of the algorithm is of the order \( O(\max\{\Gamma n^5 m_n\}) \).

### 3.4 Properties of the \( \hat{\Omega} \) estimates

Next, we establish that the estimator \( \hat{\Omega} \) which maximizes (3.11) will converge to the true parameter \( \Omega = \{ \Psi \cdot L, \Delta \} \) in probability under certain regularity conditions.

**Theorem 2.** Conditions:

A. (Bounded hazard rate) \( C_0 \leq \lambda_{0,l}(t) \leq C_1 \) for \( 0 \leq t \leq t_0 \) \( 1 \leq l \leq \Gamma \),

B. (Bounded parameters) \( \max_{1 \leq i,j \leq n} \{ |\alpha_{ij}|, |\delta_i| \} \leq C_2 \),
C. (Limited posting frequencies)

\[ P(A_j(t+h,l) - A_j(t,l) \geq 1) \leq C_3 \cdot h, \text{ when } h > 0, t + h \leq t_0, \quad (3.12) \]

D. (Balanced sets sizes) \( \max \left\{ \frac{\|\Gamma_1\|_0}{\|\Gamma_2\|_0}, \frac{\|\Gamma_2\|_0}{\|\Gamma_1\|_0} \right\} < C_4 \)

E. (Positive definite limit of Hessian) Let \( \Omega' \) be any choosable parameter vector satisfying (B). For large enough \( \Gamma \) and some \( C_5 \), we have the holding condition to hold on the smallest eigenvalue of \(-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)\), at \( \Omega' = \Omega, t = t_0 \),

\[ P(\lambda_{\min}(-\nabla_{\Omega'} \nabla_{\Omega'} LT(\Omega', t)) |_{\Omega'=\Omega,t=t_0} > C_5) \rightarrow 1, \text{ as } \|\Gamma_1\|_0 \rightarrow \infty, \]

where

\[
LT(\Omega', t) \equiv (\|\Gamma_1\|_0)^{-1} \left\{ \sum_{l \in \Gamma_1} -\lambda_{0,l}(u) \log \left( \sum_j \exp \left( \sum_{i \neq j} L_{ij} \alpha'_{ij} \log(M_i(t,l) + 1) \right) \right) \right.
\]
\[
\cdot \left( \sum_{j=1}^n \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij} \alpha_{ij} \log(M_i(t,l) + 1) \right) \right) \}
\]
\[
+ (\|\Gamma_1\|_0)^{-1} \left\{ \sum_{l \in \Gamma_1} -\lambda_{0,l}(u) \log \left( \sum_j \exp \left( \sum_{i \neq j} L_{ij} (\alpha'_{ij} + \delta'_{ij}) \log(M_i(t,l) + 1) \right) \right) \right.
\]
\[
\cdot \left( \sum_{j=1}^n \exp \left( \sum_{1 \leq i \leq n, i \neq j} L_{ij} (\alpha_{ij} + \delta_i) \log(M_i(t,l) + 1) \right) \right) \}
\]

(3.13)

where \( \| \cdot \|_0 \) computes the \( L_0 \) (size) of a set.

In the five Conditions A through E above, \( C_0, C_1, C_2, C_3, C_4 \) and \( C_5 \) are all positive constants. Under these conditions, we will have:

\[ \hat{\Omega} \rightarrow_P \Omega \text{ as } \|\Gamma_1\|_0, \|\Gamma_2\|_0 \rightarrow \infty. \]

The detailed proof is given in the Appendix.
Since Condition C in Theorem 2 is the same as the one in Theorem 1, Lemma 1 in Section 2.4 already gives one example in which this condition will hold. Based on Theorem 2, by leveraging the properties of continuous functions, we can establish the consistency of the proposed edge importance measure.

**Proposition 2.** Let \( \varsigma^{(1)} = (\varsigma^{(1)}_{ij})_{1 \leq i,j \leq n} \) and \( \varsigma^{(2)} = (\varsigma^{(2)}_{ij})_{1 \leq i,j \leq n} \) denote the \( n \times n \) dimensional matrix of edge importance. Further, denote by \( \hat{\varsigma}^{(1)} = (\hat{\varsigma}^{(1)}_{ij})_{1 \leq i,j \leq n} \) and \( \hat{\varsigma}^{(2)} = (\hat{\varsigma}^{(2)}_{ij})_{1 \leq i,j \leq n} \) their empirical estimates. Under the conditions of Theorem 1, we have that when \( \| \Gamma_1 \|_0, \| \Gamma_2 \|_0 \to \infty \)

\[
\| \hat{\varsigma}^{(1)} - \varsigma^{(1)} \|_2 + \| \hat{\varsigma}^{(2)} - \varsigma^{(2)} \|_2 \to_P 0, \tag{3.14}
\]

where \( \| \cdot \|_2 \) computes the \( L_2 \) norm. As a result, with any probability \( p \in (0, 1) \), when \( (\varsigma^{(1)}_{ij})_{1 \leq i,j \leq n} \) and \( (\varsigma^{(2)}_{ij})_{1 \leq i,j \leq n} \) values are all distinct, we have

\[
\| \hat{S}_1(p)^{(c)} \setminus S_1(p)^{(c)} \|_0 \to_P 0,
\]

where \( \| \cdot \|_0 \) is the \( L_0 \) norm of a set.

From Theorem 2, the proof of the proposition is straightforward, since each element of the matrix \( \hat{\varsigma}^{(1)} \) and \( \hat{\varsigma}^{(1)} \) is a continuous function of \( \hat{\Omega} \).

### 3.5 Performance evaluation

The key steps for obtaining the synthetic data are identical to those in the previous chapter; namely, generating the followers’ network and generating actions for the two sets of topics \( \Gamma_1 \) and \( \Gamma_2 \), respectively.

As before, we first illustrate the performance of the Newton estimation algorithm, on a random network of varying size. We set the parameter \( a = 0.5 \) for the baseline hazard rate and choose a time horizon of \( t_0 = 10 \), to emulate ten days worth of data.
We also select the parameters $\Omega$ uniformly at random in the interval $[-0.4, 0.4]$. Due to the bounded baseline hazard rate and simulated parameters, and since the retweets and mentions are generated as Poisson, Condition A, B, C of Theorem 1 have been satisfied. Then we empirically "check" Condition D. With a large $\|\Gamma_1\|_0 = \|\Gamma_2\|_0 = 500$, network size $n = 10, 30$, we repeated Step 1 and 2 for 20 times to simulate the network and actions. In each repetition, the square root of the smallest eigenvalue of $\lambda_{\min} \left( -\Gamma \nabla \Omega', \nabla \Omega' L(T, t) \right)|_{\Omega'=\Omega, t=t_0}$ is computed. The results are plotted in Figure 3.1. In the plot, it can be seen that smallest eigenvalues of $\lambda_{\min} \left( -\Gamma \nabla \Omega', \nabla \Omega' L(T, t) \right)|_{\Omega'=\Omega, t=t_0}$ are generally larger or not smaller than 0.5. Due to their large variations, we took square root to make the values of the smallest eigenvalues easier to be reflected in plots.

Then as we have "checked", with network sizes $n = 10, 30$ and number of topics generated $\|\Gamma_1\|_0 = \|\Gamma_2\|_0 = 500$ and another set of smaller topic sizes, and penalty $\gamma$, we obtain three sets of values to estimate the relative error of the parameter and importance estimates, $\frac{\|\hat{\Omega} - \Omega\|_2}{\|\Omega\|_2}$, $\frac{\|\hat{\varsigma}^{(1)} - \varsigma^{(1)}\|_2}{\|\varsigma^{(1)}\|_2}$ and $\frac{\|\hat{\varsigma}^{(2)} - \varsigma^{(2)}\|_2}{\|\varsigma^{(2)}\|_2}$. At $n = 10$, with $\gamma \in \{1, 2, \ldots, 50\}$, and $\|\Gamma_1\|_0 = \|\Gamma_2\|_0 = 500$, we first observed that $\frac{\|\hat{\Omega} - \Omega\|_2}{\|\Omega\|_2}$ was minimized around $\gamma = 10$ and actually the errors only had small differences, based on the
average of five replicates for each chosen \( \gamma \). Due to the high time complexity of the computation algorithm, we then only applied it to the network with \( n = 30 \) under \( \gamma = 0, 10 \) and 50. We obtain the following Figure 3.2 to show the relative error of the parameter and importance estimates.

The results are based on 20 replicates of the underlying followers networks, as well as the actions (postings, retweets and mentions) data. It can be seen in Figure 3.2 that with large enough topic sets, \( \| \Gamma_1 \| = \| \Gamma_2 \| = 500 \), the parameters and importance measures, exhibit a small (less than 10\%) relative error rate, where \( \| \cdot \|_0 \) corresponds to \( \ell_0 \) norm.

Based on the estimated parameters \( \hat{\alpha}_{ij} \) of the simulations, we examine more closely the estimation results in the setting with \( \| \Gamma_1 \|_0 = \| \Gamma_2 \|_0 = 500 \). We estimate the important edge collections \( \hat{S}_k(p)^{(c)} \) as defined in (3.8) and (3.9), \( c, k = 1, 2 \). Then, we check the performance of \( \hat{S}_k(p)^{(c)} \) by looking at the proportions of edges in the estimated sets, that are also coherent with the original ones:

\[
\frac{\| \hat{S}_k(p)^{(c)} \cap S_k(p)^{(c)} \|_0}{\| \hat{S}_k(p)^{(c)} \|_0},
\]

with \( c, k = 1, 2 \). The estimation results are shown in Figure 3.3. Since our interest mainly focus on capturing the most spawning and jamming edges, we may only plot with small \( p \) values.

From Figure 3.3, we can see that, generally \( \hat{S}_k(p)^{(c)} \) estimates \( S_k(p)^{(c)} \) well with its edges coinciding with those of \( S_k(p)^{(c)} \) with an accuracy of more than 80\%, \( c, k = 1, 2 \). Since the two sets (at the same \( p \)) contain almost the same number of edges, the rate in Figure 3.3 can also be looked at as the proportion of edges in \( S_k(p)^{(c)} \) that are captured by \( \hat{S}_k(p)^{(c)} \). To include most of the extreme edges \( S_k(p)^{(c)} \), we propose to use \( \hat{S}_k(p)^{(c)} \). And at the same time, we hope the other edges (not in \( S_k(p)^{(c)} \)) are still worth considering in the sense that they are included in \( S_k(4p)^{(c)} \), \( c, k = 1, 2 \). The
Figure 3.2: Mean relative error of the model parameter estimates $\Omega$ (up), $\varsigma^{(1)}$ (middle) and $\varsigma^{(2)}$ (down)
Figure 3.3: Proportion of correct edges, from up to down: \( n=10, S_1 \) (first); \( n=10, S_2 \) (second); \( n=30, S_1 \) (third); \( n=30, S_2 \) (fourth).
results of this idea are depicted in Figure 3.4 below. In the figure, we computed

\[
\frac{\|\hat{S}_k(2p)^{(c)} \cap S_k(p)^{(c)}\|_0}{\|S_k(p)^{(c)}\|_0},
\]

the coverage for real \(p\) proportional most important edges and

\[
\frac{\|\hat{S}_k(2p)^{(c)} \cap S_k(4p)^{(c)}\|_0}{\|\hat{S}_k(2p)^{(c)}\|_0},
\]

the accuracy rate, with \(c, k = 1, 2\) for the network \(n = 30\).

As can be seen from Figure 3.4, at small probability \(p\) values, generally more than 90% of edges in \(\hat{S}_k(2p)^{(c)}\) can be captured in \(S_k(p)^{(c)}\). With the exception of \(p = 0.01\), around (or more than) 90% of edges in \(S_k^{(c)}(p)\) have been included in \(\hat{S}_k(2p)^{(c)}\).

### 3.6 Identifying Important Connections between Senators

Next, we re-examine our Tweeter data. Recall that there are about 200,000 tweets and 4671 follower links within the set of 120 accounts from April 2009 to July 2014. The recorded accounts are registered to 55 Democratic politicians (U.S. Senators and the President of the U.S.), 46 Republican Senators, 2 government organizations (U.S. Army and the Federal Reserve Board), and 16 media outlets, including newspapers (Financial Times, Washington Post, New York Times, Huffington Post), television networks (MSNBC, Fox News, CNN, CSPAN), reporters (Nate Silver (538), Ezra Klein) and television hosts (Bill O’Reilly, Sean Hannity).

In the previous chapter, the focus was on identifying the most influential senators. This time, we are interested in exploring the importance of edges (connections) in the followers network. Also, as both in Table 1 in the previous chapter and in Golbeck et al. (2010), it is understood that senators tend to retweet and mention as a means of self or legislative promotion. We have increased frequencies of data during
Figure 3.4: Coverage and accuracy, from up to down: coverage of $S_1$ (first); coverage of $S_2$ (second); accuracy of $S_1$ (third); accuracy of $S_2$ (fourth).
some periods of hot topics, such as in the months surrounding the inauguration of
President Obama (January 2013), the debate on raising the debt ceiling of the US
government and its temporary suspension around April 2013 and the summer of 2014
(soccer World Cup). We would also like to understand whether the edges’ impor-
tances are different or not within and outside of these time periods. Given the high
computational complexity of the algorithm and the overall high volume of data, we
focus on the summer of 2014 (soccer World Cup).

Observations collected from April to June, 2014 form Group 2 of the data, and
others form Group 1. Topics sets $\Gamma_1$ and $\Gamma_2$ then correspond to these two time periods.
Although the 2014 summer period is relatively short in duration, it nevertheless
contains 55455 actions, which accounts for 40.88% of the tweets count. Given the
fast moving landscape of social media, new topics are assigned each week. Combined
with pre-assigned topic grouping based on key words, we get 2770 topics in total for
the entire data set.

Although not the main focus of this chapter, recall from (3.4) and (3.5), we can
still compute the nodes’ influences under our proposed model, for sets $\Gamma_1$ and $\Gamma_2$. To
rigorously justify our modeling of the data, we perform a regression analysis to assess
how well our influence measure can explain legislative leadership in Congress, by com-
paring to regression results applied with PageRank, on the followers networks (which
constitutes the backbone of many ranking algorithms of Twitter accounts). Our re-
response variable is the leadership score, published by www.govtrack.us (GovTrack.us,
2014). GovTrack creates the leadership score by applying the PageRank algorithm to
the adjacency matrix of bill cosponsorship data. Thus, the leadership score for each
senator is a number between 0 and 1, where higher values denote greater legislative
leadership. The regression model we are interested in is

$$\text{Leadership} = \beta \text{Influence} + \Theta \text{Controls},$$  \hspace{1cm} (3.15)
where Influence contains the two influences computed on $\Gamma_1$ and $\Gamma_2$, from our proposed model and/or PageRank, and Controls includes party affiliation, gender, age, and number of years in the senate. Seniority endows a number of benefits including preferential assignment to committees. Thus, these control variables likely associate strongly with legislative leadership.

To estimate the parameters in our proposed model and the two influence measures, after preprocessing to get the topics, we apply Algorithm 3 to estimate the $\alpha$ and $\delta$ parameters using all the data. The final influence measures, $\hat{\Xi}^{(1)}$ and $\hat{\Xi}^{(2)}$, are constructed by using the average $M_i(T_m,l)$, at all time points of a retweet or mention happens, in Group 1 and Group 2, respectively.

Since the leadership score provided by GovTrack takes values between 0 and 1, we estimate two models. One model uses the raw leadership scores, and another uses $\log(\frac{\text{leadership}}{1-\text{leadership}})$ for the response variable. In both cases, as shown in Table 3.1, we consistently find that the 2 newly proposed influence measures explains more variation in leadership than our original proposed measure, and PageRank. This observation may serve to suggest that the new more flexible model is more suitable, at the cost of higher computational complexity due to proliferation of parameters from $2n$ for the original models to $n^2$ for the new model.

In the Senators’ social network, we are interested to find the most spawning and jamming edges. As suggested by Figure 3.4, if we use 40 edges with largest (smallest) importances to capture the true 20 most extreme (largest or smallest) edges, we should have enough accuracy. Our estimated 40 edges with the largest importances, on topic $\Gamma_1$, are listed in Table 3.2. The list of the edges with the smallest edge importances on the same topic set is given in Table 3.3. Similarly, the 40 estimated edges with the largest importances, on topic $\Gamma_2$, are listed in Table 3.4, while those with the smallest edge importances on the second topic set are listed in Table 3.5.

A summary of the main findings is given next:
Table 3.1: Estimated R-squared values for different regression models, where the two new proposed influence measures, the original proposed measure or PageRank is included in the set of independent variables and the influence is computed for the entire data sample. We consistently find that the 2 newly proposed measure is a better indicator of legislative importance.

<table>
<thead>
<tr>
<th>Response</th>
<th>New Influence on $\Gamma_1$</th>
<th>New Influence on $\Gamma_2$</th>
<th>Original Proposed</th>
<th>PageRank</th>
<th>$R^2$</th>
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<td>0.276</td>
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<tr>
<td>$\log(\frac{\text{leadership}}{1-\text{leadership}})$</td>
<td>$V$</td>
<td>$V$</td>
<td>$V$</td>
<td>$V$</td>
<td>0.361</td>
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<td></td>
<td></td>
<td></td>
<td>0.098</td>
</tr>
</tbody>
</table>

- There is a great deal of agreement between the most important edges for both topics sets. This should be expected due to the fact that the two sets are not separated by a careful topics selection. Obviously, $\Gamma_2$ contains many more discussions related to the World Cup, but on the other hand this are not contentious issues that may produce disparate results.

- The results indicate that Jon Tester (Senator from Montana) is least influential, which is consistent with his joining the Twitter platform in March 2012 and overall being a low content producer (740 total tweets since April 15, 2015). His Twitter activity should be juxtaposed with a prolific user like Cory Booker (Senator from New Jersey) who has sent over 47.5K in less than 7 years and a moderate user like John McCain (Senator from Arizona) who has tweeted around 8.5K times since the beginning of 2009.

- It is worth noting that John McCain features prominently in both topics sets, given his foreign policy expertise and immigration views. These discussion topics feature prominently in both sets.

- In general, the strongest influences are between more “senior” Senators (e.g.
Reid, Senate Majority Leader at the time, Levin -36 years in office and Chairman of the powerful Armed Services Committee-, Collins -18 years in office and ranking member of the powerful Committee on Appropriations, Sessions -18 years in office and ranking member of the influential Committee on the Judiciary, and so forth).

### 3.7 Summary

In this chapter, we have proposed a novel measure of the edge importances in large social platform, by considering the amount of influence an account can "borrow" from the follower the edge connects to, with a single action. The method is based on using counting processes with exponential hazard rates, to model the time sequences.
Table 3.3: Top forty edges with smallest proposed edge importance values from April 16, 2009 - April 30, 2014 (topic set $\Gamma_1$).

<table>
<thead>
<tr>
<th>Rank</th>
<th>Followed</th>
<th>Follower</th>
<th>Rank</th>
<th>Followed</th>
<th>Follower</th>
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<tr>
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<td>Debbie Stabenow</td>
<td>Jon Tester</td>
<td>21</td>
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of the actions users take on the platform. In the hazard rates, we use independent parameters to model the long term capability for an account to bring on an action from each of its followers and the parameters can be different on separate topic sets. The parameters are then estimated by maximizing the log of a partial likelihood function, with lasso penalty included to introduce sparsity on the differences between topic sets. With the estimated parameters, for each edge, we then compute the probability for a follower on the other end of the edge, to take an action due to the action the followed account takes, within our observation time. The importance of the edge is then estimated by the probability multiplied by the influence of the follower, which is computed following our influence definition in our previous project. Applications of our new model to the US senators data shows the larger flexibility in the hazard rate model illustrate superior performance on explaining Senators’ leadership scores.
Table 3.4: Top forty edges with largest proposed edge importance values from May 1, 2014 - July 31, 2014 (topic set $\Gamma_2$).

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<td>Barbara Boxer</td>
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</table>

in real life. And the estimated edge importances are consistent with our observation.

Recalling the content of Chapter 2, from the computation complexity and application examples given in this thesis, it can be seen that our proposed models in the two chapters are most useful when looking at an small scale ecosystem of related users like the US Senators. They may also be useful for getting better insights into the influence of subsets of users in a bigger network. But Due to scalability issues, they are not yet appropriate to analyze huge network like the entire Twitter space.
<table>
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<th>Rank</th>
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<th>Rank</th>
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</tbody>
</table>

### 3.8 Estimation Algorithm and Proofs

#### 3.8.1 Computation equations for Newton’s update to maximize $LL$

Note that $LL = LL_0(t_0) - \gamma \sum_{i=1} |\delta_i|$. To simplify notations, we will use $LL_0$ to represent $LL_0(t_0)$ in the rest of the paper. We will first give expressions for the gradient vector and Hessian matrix of the $LL_0$ function and modify them to maximize $LL$. Some rather straightforward algebra yields the following expressions for the
elements of the gradient vector $G_0 \equiv \nabla_{\Omega} LL_0$:

$$\frac{\partial LL_0}{\partial \alpha_{ij_1}} = \sum_{l \in \Gamma_1} \left\{ \sum_{1 \leq k \leq n'_{j_1}} \sum_{i} L_{ij_1} \log(M_i(T_{j_1,k,l}^l + 1) \right.$$ 

$$- \sum_{j=1}^{n} \sum_{1 \leq k \leq n'_{j_1}} L_{ij_1} \log(M_i(T_{j_1,k,l}^l + 1) \exp \left( \sum_{u \neq j_1} L_{uj_1} \alpha_{ij_1} \log(M_u(T_{j_1,k,l}^l + 1) \right) \right\}$$ 

$$- \sum_{j=1}^{n} \sum_{1 \leq k \leq n'_{j_1}} \sum_{v} \L_{iv} \log(M_i(T_{j_1,k,l}^l + 1) \exp \left( \sum_{u \neq v} \L_{uv} (\alpha_{uv} + \delta_u) \log(M_u(T_{j_1,k,l}^l + 1) \right) \right\}$$ 

(3.16)

for $1 \leq i, j_1 \leq n, i \neq j_1$, and

$$\frac{\partial LL_0}{\partial \delta_i} = \sum_{l \in \Gamma_2} \left\{ \sum_{j=1}^{n} \sum_{1 \leq k \leq n'_{j_1}} \sum_{i} L_{ij} \log(M_i(T_{j,k,l}^l + 1) \right.$$ 

$$- \sum_{j=1}^{n} \sum_{1 \leq k \leq n'_{j_1}} \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} \L_{uv} (\alpha_{uv} + \delta_u) \log(M_u(T_{j,k,l}^l + 1) \right)$$ 

$$\cdot \exp \left( \sum_{u \neq v} \L_{uv} (\alpha_{uv} + \delta_u) \log(M_u(T_{j,k,l}^l + 1) \right) \right\}$$ 

(3.17)

for $1 \leq i \leq n$.

Next, we obtain the necessary expressions for the Hessian matrix $H(LL_0)$. We start by computing the sub-matrix of $H$ that includes the second partial derivatives.
of $LL_0$ with respect to the $\alpha$ parameters and obtain

$$
\frac{\partial^2 LL_0}{\partial \alpha_{i_1j_1} \partial \alpha_{i_2j_1}} = \sum_{l \in \Gamma_1} \left\{ -\sum_{j=1}^{n} \sum_{1 \leq k \leq n^l_j} \frac{L_{ij_1} L_{i_2j_1} \log(M_{ij_1}(T_{j,k}^l, l) + 1) \log(M_{i_2j_1}(T_{j,k}^l, l) + 1)}{\sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} \alpha_{uv} \log(M_u(T_{j,k}^l, l) + 1) \right)} \cdot \exp \left( \sum_{u \neq v} L_{uj_1}(T_{j,k}^l) \alpha_{uj_1} \log(M_u(T_{j,k}^l, l) + 1) \right) \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n^l_j} \frac{L_{ij_1} L_{i_2j_1} \log(M_{ij_1}(T_{j,k}^l, l) + 1) \log(M_{i_2j_1}(T_{j,k}^l, l) + 1)}{\left[ \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} \alpha_{uv} \log(M_u(T_{j,k}^l, l) + 1) \right) \right]^2} \cdot \exp \left( 2 \sum_{u \neq j_1} L_{uj_1} \alpha_{uj_1} \log(M_u(T_{j,k}^l, l) + 1) \right) \right\} \\
+ \sum_{l \in \Gamma_2} \left\{ -\sum_{j=1}^{n} \sum_{1 \leq k \leq n^l_j} \frac{L_{ij_1} L_{i_2j_1} \log(M_{ij_1}(T_{j,k}^l, l) + 1) \log(M_{i_2j_1}(T_{j,k}^l, l) + 1)}{\sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} \alpha_{uv} \log(M_u(T_{j,k}^l, l) + 1) \right)} \cdot \exp \left( \sum_{u \neq v} L_{uj_1}(T_{j,k}^l) \alpha_{uj_1} + \delta_u \right) \log(M_u(T_{j,k}^l, l) + 1) \right\} \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n^l_j} \frac{L_{ij_1} L_{i_2j_1} \log(M_{ij_1}(T_{j,k}^l, l) + 1) \log(M_{i_2j_1}(T_{j,k}^l, l) + 1)}{\left[ \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_{uv} + \delta_u) \log(M_u(T_{j,k}^l, l) + 1) \right) \right]^2} \cdot \exp \left( 2 \sum_{u \neq j_1} L_{uj_1} (\alpha_{uj_1} + \delta_u) \log(M_u(T_{j,k}^l, l) + 1) \right) \right\}
$$

(3.18)
when \( 1 \leq i_1, j_1, i_2 \leq n, i_1 \neq j_1, i_2 \neq j_1 \), and also

\[
\frac{\partial^2 LL_0}{\partial \alpha_{i_1j_1} \partial \alpha_{i_2j_2}} = \sum_{l \in \Gamma_1} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n} L_{i_1j_1} L_{i_2j_2} \log(M_{i_1}(T_{j,k,l}^q) + 1) \log(M_{i_2}(T_{j,k,l}^q) + 1) \right. \\
\cdot \exp \left( \sum_{u \neq j_1} L_{u j_1} \alpha_{uj_1} \log(M_u(T_{j,k,l}^q) + 1) \right) \exp \left( \sum_{u \neq j_2} L_{u j_2} \alpha_{uj_2} \log(M_u(T_{j,k,l}^q) + 1) \right) \\
\left. \left[ \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} \alpha_{uv} \log(M_u(T_{j,k,l}^q) + 1) \right) \right]^2 \right\} \\
+ \sum_{l \in \Gamma_2} \left\{ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n} L_{i_1j_1} L_{i_2j_2} \log(M_{i_1}(T_{j,k,l}^q) + 1) \log(M_{i_2}(T_{j,k,l}^q) + 1) \right. \\
\cdot \exp \left( 2 \sum_{u \neq j_1} L_{u j_1} (\alpha_{uj_1} + \delta_u) \log(M_u(T_{j,k,l}^q) + 1) \right) \\
\left. \cdot \exp \left( 2 \sum_{u \neq j_2} L_{u j_2} (\alpha_{uj_2} + \delta_u) \log(M_u(T_{j,k,l}^q) + 1) \right) \right\} \\
\]

(3.19)

when \( 1 \leq i_1, j_1, i_2 \leq n, i_1 \neq j_1, i_2 \neq j_1 \) and especially \( j_1 \neq j_2 \).

Next, we obtain the sub-matrix of \( H \) that includes the second partial derivatives
of $LL$ with respect to the $\delta$ parameters and get

$$
\frac{\partial^2 LL_0}{\partial \delta_{i_1} \partial \delta_{i_2}} = \sum_{l \in T_2} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \right.
\cdot \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right)
\cdot \sum_{u \neq v} L_{uv} \left( \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \right)
\cdot \sum_{v \neq i_2} L_{iv} \log (M_i(T^l_{j,k}, l) + 1) \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \left\},
$$

(3.20)

when $1 \leq i_1, i_2 \leq n$.

Finally, we provide expressions for the cross-partials

$$
\frac{\partial^2 LL_0}{\partial \alpha_{i_1,j_1} \partial \delta_{i_2}} = \sum_{l \in T_2} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j} \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv} (\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \right.
\cdot \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right)
\cdot \sum_{u \neq j_1} L_{uj_1} \left( \sum_{1 \leq v \leq n} \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \right)
\cdot \sum_{v \neq i_2} L_{iv} \log (M_i(T^l_{j,k}, l) + 1) \exp \left( \sum_{u \neq v} L_{uv}(\alpha_{uv} + \delta_u) \log (M_u(T^l_{j,k}, l) + 1) \right) \left\},
$$

(3.21)

when $1 \leq i_1, i_2, j_1 \leq n, i_1 \neq j_1$. 

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After getting $G_0$ and $H$, we let

$$G = G_0 - \gamma \text{sign}(\Delta) \tag{3.22}$$

and use $G$ and $H$ in Newton’s update. The function sign(·) gives a (vector) of the signs of the values.

## 3.8.2 Implementation Issues

As outlined above, the maximum likelihood estimator is also obtained by Newton’s algorithm and detailed expressions for the respective gradient and Hessian are given in the previous subsection. However, luckily enough, the structure of the problem again allows us to precompute and store several quantities for repeated use, thus saving on computational time in practice. Note that the data containing the actions are stored according to their time stamps. We again start by computing four groups of quantities introduced by an action, labeled respectively by indices $j$, $l$, $k$ and possibly some other parameters, where $j$ indicates the node that takes the activity, $l$ is the topic label and $k$ represents the relative sequence number of the action, in all the actions that node $j$ has taken under topic $l$.

First, we define

$$E^{(1)}_{j,v,k,l} = \exp \left( \sum_{u \neq v} L_{uv} \alpha_{uv} \log(M_u(T_{j,k}^l, l) + 1) \right).$$

and

$$E^{(2)}_{j,v,k,l} = \exp \left( \sum_{u \neq v} L_{uv} (\alpha_{uv} + \delta_u) \log(M_u(T_{j,k}^l, l) + 1) \right).$$

We also define

$$O^{(c)}_{j,i,k,l} = L_{ij} \log(M_i(T_{j,k}^l, l) + 1) E^{(c)}_{j,v,k,l}, c = 1, 2.$$
Then, we compute

\[ SE_{j,k,l}^{(c)} = \sum_{1 \leq v \leq n} E_{j,v,k,l}^{(c)} \]

and

\[ ME_{j,i,k,l}^{(c)} = \sum_{v \neq i} L_{iv} \log(M_i(T_{j,k}^l, l) + 1)E_{j,v,k,l}^{(c)} \]

\( c = 1, 2 \). Also, we have

\[ LM_{j,s,k,l} = \sum_{u \neq j} L_{uj} \log(M_u(T_{s,k}^l, l) + 1)E_{j,v,k,l}^{(1)} \]

Then, based on the precomputed components values, the elements of the gradient vector \( G \equiv \nabla_{\Omega} LL \) are obtained as follows:

\[
\frac{\partial LL_0}{\partial \alpha_{ij}} = \sum_{l \in \Gamma_1} \left\{ \sum_{1 \leq k \leq n_j^l} L_{ij} \log(M_i(T_{j,k}^l, l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l^j} O_{j,i,k,l}^{(1)} SE_{j,k,l}^{(1)} \right\} \\
+ \sum_{l \in \Gamma_2} \left\{ \sum_{1 \leq k \leq n_j^l} L_{ij} \log(M_i(T_{j,k}^l, l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l^j} O_{j,i,k,l}^{(2)} SE_{j,k,l}^{(2)} \right\}
\]

for \( 2 \leq i \leq n \), and

\[
\frac{\partial LL_0}{\partial \delta_i} = \sum_{l \in \Gamma_1} \left\{ \sum_{1 \leq j \leq n, j \neq i} \sum_{1 \leq k \leq n_l^j} L_{ij} \log(M_i(T_{j,k}^l, l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l^j} ME_{j,i,k,l}^{(1)} SE_{j,k,l}^{(1)} \right\} \\
+ \sum_{l \in \Gamma_2} \left\{ \sum_{1 \leq j \leq n, j \neq i} \sum_{1 \leq k \leq n_l^j} L_{ij} \log(M_i(T_{j,k}^l, l) + 1) - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_l^j} ME_{j,i,k,l}^{(2)} SE_{j,k,l}^{(2)} \right\}
\]

for \( 1 \leq i, j_1 \leq n, i \neq j_1 \).

Regarding the Hessian, based on the four precomputed groups of quantities, we start by computing the sub-matrix of \( H \) that includes the second partial derivatives of
\( LL \) with respect to the \( \alpha \) parameters. We get when \( 1 \leq i_1, j_1, i_2 \leq n, i_1 \neq j_1, i_2 \neq j_1, \)

\[
\frac{\partial^2 LL_0}{\partial \alpha_{i_1,j_1} \partial \alpha_{i_2,j_2}} = \sum_{l \in \Gamma_1} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(1)}_{j_1,i_1,k,l} O^{(1)}_{j_1,i_2,k,l}}{SE^{(1)}_{j,k,l}} + \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(1)}_{j_1,i_1,k,l} O^{(1)}_{j_1,i_2,k,l}}{(SE^{1}_{j,k,l})^2} \right\} \\
+ \sum_{l \in \Gamma_2} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(2)}_{j_1,i_1,k,l} O^{(2)}_{j_1,i_2,k,l}}{SE^{(1)}_{j,k,l}} + \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(2)}_{j_1,i_1,k,l} O^{(2)}_{j_1,i_2,k,l}}{(SE^{2}_{j,k,l})^2} \right\}
\]

When \( 1 \leq i_1, j_1, i_2 \leq n, i_1 \neq j_1, i_2 \neq j_1 \) and especially \( j_1 \neq j_2 \), we similarly have

\[
\frac{\partial^2 LL_0}{\partial \alpha_{i_1,j_1} \partial \alpha_{i_2,j_2}} = \sum_{l \in \Gamma_1} \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(1)}_{j_1,i_1,k,l} O^{(1)}_{j_2,i_2,k,l}}{(SE^{1}_{j,k,l})^2} + \sum_{l \in \Gamma_2} \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O^{(2)}_{j_1,i_1,k,l} O^{(2)}_{j_2,i_2,k,l}}{(SE^{2}_{j,k,l})^2}
\]

Next, we obtain the sub-matrix of \( H \) that includes the second partial derivatives of \( LL \) with respect to the \( \delta \) parameters and get

\[
\frac{\partial^2 LL_0}{\partial \delta_{i_1} \partial \delta_{i_2}} = \sum_{l \in \Gamma_2} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{\sum_{v \neq i_1, i_2} \log(M_{i_1}(T_{j,k,l}^l) + 1) \log(M_{i_2}(T_{j,k,l}^l) + 1) E_{j,v,k,l}^{(1)}}{SE_{j,k,l}^{(2)}} \right\} \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{ME_{j,1,k,l}^{(1)} ME_{j,2,k,l}^{(2)}}{[SE_{j,k,l}^{(2)}]^2}
\]

when \( 1 \leq i_1, i_2 \leq n \).

Finally, we provide expressions for the cross-partial

\[
\frac{\partial^2 LL_0}{\partial \alpha_{i_1,j_1} \partial \delta_{i_2}} = \sum_{l \in \Gamma_2} \left\{ - \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{L_{i_1,j_1} L_{i_2,j_2} \log(M_{i_1}(T_{j,k,l}^l) + 1) \log(M_{i_2}(T_{j,k,l}^l) + 1) E_{j,j_1,k,l}^{(1)}}{SE_{j,k,l}^{(2)}} \right\} \\
+ \sum_{1 \leq j \leq n} \sum_{1 \leq k \leq n_j'} \frac{O_{j_1,i_1,k,l}^{(2)} ME_{j,j_2,k,l}^{(2)}}{[SE_{j,k,l}^{(2)}]^2}
\]

when \( 1 \leq i_1, i_2, j_1 \leq n, i_1 \neq j_1 \).
3.8.3 Proof of Theorem 2

To simplify the notations, similar to what we have done in the proof of Theorem 1 in Chapter 1, we first define the following notations:

$$E_{1,l}(t, \Omega') = \sum_{j=1}^{n} \lambda_{0,l}(t) \exp \left( \sum_{i, i \neq j} L_{ij}(t) \alpha'_{ij} \log(M_{i}(t, l) + 1) \right)$$

$$E_{2,l}(t, \Omega') = \sum_{j=1}^{n} \lambda_{0,l}(t) \exp \left( \sum_{i, i \neq j} L_{ij}(t) (\alpha'_{ij} + \delta'_i) \log(M_{i}(t, l) + 1) \right)$$

$$\Phi_{1,j} = (\phi'_{1,1j}, \ldots, \phi'_{1,nj}) \equiv (\alpha'_{1,1j}, \ldots, \alpha'_{1,nj})'$$

$$\Phi_{2,j} = (\phi'_{2,1j}, \ldots, \phi'_{2,nj}) \equiv (\alpha'_{1,1j} + \delta'_i, \ldots, \alpha'_{1,nj} + \delta'_i)'$$

$$E_{k,lj}^{(1)}(t, \Omega') = \left( \frac{\partial E_{k,l}(t, \Omega')}{\partial \phi'_{k,1j}}, \ldots, \frac{\partial E_{k,l}(t, \Omega')}{\partial \phi'_{k,nj}} \right), k = 1, 2$$

$$E_{k,lj}^{(2)}(t, \Omega') = \left( \frac{\partial^2 E_{k,l}(t, \Omega')}{\partial \phi'_{k,ij} \partial \phi'_{k,i'j'}} \right)_{1 \leq i, q \leq n}, k = 1, 2$$

Let we let

$$e_{k,l}(t, \Omega') \equiv E[E_{k,l}(t, \Omega')]$$

$$e_{k,lj}^{(1)}(t, \Omega') \equiv \left( \frac{\partial e_{k,l}(t, \Omega')}{\partial \phi'_{k,1j}}, \ldots, \frac{\partial e_{k,l}(t, \Omega')}{\partial \phi'_{k,nj}} \right)$$

$$e_{k,lj}^{(2)}(t, \Omega') \equiv \left( \frac{\partial^2 e_{k,l}(t, \Omega')}{\partial \phi'_{k,ij} \partial \phi'_{k,i'j'}} \right)_{1 \leq i, j, i', j' \leq n}$$

By Condition D of Theorem 2, we can let

$$X(\Omega', t) = \frac{1}{\|\Gamma_1\|_0} (LL(\Omega', t) - LL(\Omega, t))$$

Following the Step 1 to 3 in the proof of Theorem 1 in Chapter 1, and put the

$$\sum_{i=1}^{n} |\delta_i|$$

into $o_P(1)$, it can be shown that

$$|X(\Omega', t) - P(\Omega', t)| \to P 0, \text{ as } \|\Gamma_1\|_0, \|\Gamma_2\|_0 \to \infty,$$
where

$$P(\Omega', t_0) = \int_0^{t_0} \frac{1}{\|\Gamma_1\|_0} \sum_{l \in \Gamma_1} \left[ \sum_{j=1}^n (\Phi_j' - \Phi_j) e_{1,lj}^{(1)}(u, \Omega) - \log \left\{ \frac{e_{1,l}(u, \Omega')}{e_{1,l}(u, \Omega)} \right\} e_{1,l}(u, \Omega) \right] du$$

$$\int_0^{t_0} \frac{1}{\|\Gamma_1\|_0} \sum_{l \in \Gamma_2} \left[ \sum_{j=1}^n (\Phi_j' - \Phi_j) e_{2,lj}^{(1)}(u, \Omega) - \log \left\{ \frac{e_{2,l}(u, \Omega')}{e_{2,l}(u, \Omega)} \right\} e_{2,l}(u, \Omega) \right] du$$

Similar to our argument in Step 1 of the proof of Theorem 1, it can be shown that $P(\Omega', t_0)$ is strictly concave and uniquely maximized at $\Omega' = \Omega$. Further by Condition E, the smallest eigenvalue of $-\nabla_{\Omega'} \nabla_{\Omega'} P(\Omega', t_0)$ is not smaller than $C_4$, in probability. Following the argument of Step 3 of the proof of Theorem 1, it can be shown that $\hat{\Omega}$, the maximizer of $X(\Omega', t)$ converges in probability to $\Omega$. 

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