Electrical Networks and Electrical Lie Theory of Classical Types

by

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For my son Ian Yang Su

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ABSTRACT

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In this thesis we investigate the structure of **electrical Lie algebras of finite Dynkin type**. These Lie algebras were introduced by Lam-Pylyavskyy in the study of **circular planar electrical networks**. Among these electrical Lie algebras, the Lie group corresponding to type A electrical Lie algebra acts on such networks via some combinatorial operations studied by Curtis-Ingerman-Morrow and Colin de Verdière-Gitler-Vertigan. Lam-Pylyavskyy studied the type A electrical Lie algebra of even rank in detail, and gave a conjecture for the dimension of electrical Lie algebras of finite Dynkin types. We prove this conjecture for all classical Dynkin types, that is, A, B, C, and D. Furthermore, we are able to explicitly describe the structure of some electrical Lie algebras of classical types as the semisimple product of the symplectic Lie algebra with its finite dimensional irreducible representations.

We then introduce **mirror symmetric circular planar electrical networks** as the mirror symmetric subset of circular planar electrical networks studied by Curtis-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV]. These mirror symmetric networks can be viewed as the type B generalization of circular planar electrical networks. We show that most of the properties of circular planar electrical

networks are well inherited by these mirror symmetric electrical networks. In particular, the type B electrical Lie algebra has an infinitesimal action on such networks. Inspired by Lam [Lam], the space of mirror symmetric circular planar electrical networks can be compactified using **mirror symmetric cactus networks**, which admit a stratification indexed by mirror symmetric matchings on $[4n]$. The partial order on the mirror symmetric matchings emerging from mirror symmetric electrical networks is dual to a subposet of affine Bruhat order of type C. We conjecture that this partial order is the closure partial order of the stratification of mirror symmetric cactus networks.

CHAPTER 1

Introduction

1 Introduction

The study of electrical networks dates back to Georg Ohm and Gustav Kirchhoff more than a century ago, and it is still a classical object in the study of many branches of mathematics including graph theory (see for example [KW]). It also has many applications in other fields including material science and medical imaging (see for example [BVM]). In this thesis, we will focus on the class of **circular planar electrical networks**.

A **circular planar electrical network** Γ is an undirected weighted planar graph which is bounded inside a disk (See Figure 1.1). The weights can be thought as the conductances of electrical networks. The vertices on the boundary are called **boundary vertices**, say there are n of them. When voltages are put on the boundary vertices, there will be current flowing in the edges. This transformation

$$\Lambda(\Gamma) : \mathbb{R}^{|n|} \longrightarrow \mathbb{R}^{|n|}$$

from voltages on the boundary vertices to current flowing in or out of the boundary vertices is linear, and called the **response matrix** of Γ. If the response matrices of two circular planar electrical networks are the same, then they are called **electrically-**

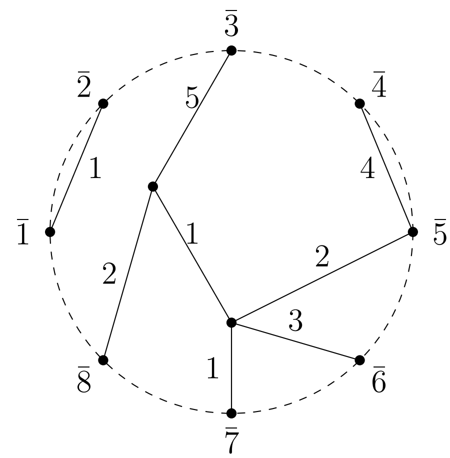
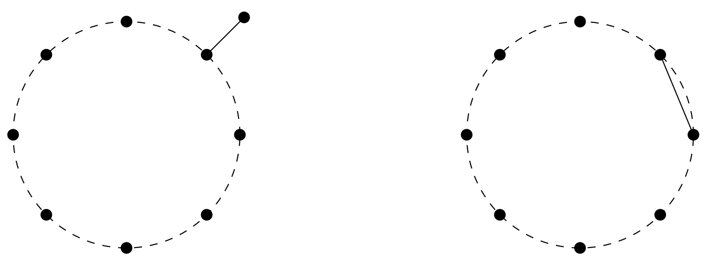


Figure 1.1: An Example of a Circular Planar Electrical Network



Adjoining a boundary spike   Adjoining a boundary edge

Figure 1.2: Boundary Operations on Circular Planar Electrical Networks

equivalent. Curtis-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV] gave a robust theory of circular planar electrical networks. They classified the response matrices of circular planar electrical networks, which form a space that can be decomposed into a disjoint union of $\mathbb{R}^{d_i}_{>0}$. They also studied two operations of **adjoining a boundary spike** and **adjoining a boundary edge** to a circular planar electrical network (See Figure 1.2).

These two operations generate the set of circular planar electrical networks modulo the electrical equivalences. The study of circular planar electrical networks can be seen as of type A, since these two operations are viewed by Lam-Pylyavskyy as one-parameter subgroups of the **electrical Lie group of type** A, namely $E_{A_{2n}}$, whose

Lie algebra will be defined shortly [LP].

In [KW], Kenyon and Wilson studied **grove measurements** $L_\sigma(\Gamma)$ of Γ, which is a generating function of all spanning subforests given that the roots on the boundary disk of each subtree form a fixed **non-crossing partition** σ. They also drew connection between grove measurements and response matrix of circular planar electrical networks.

In [Lam], Lam viewed the map $\Gamma \longrightarrow \mathcal{L}(\Gamma)$ as projective coordinates of circular planar electrical networks, where $\mathcal{L}(\Gamma) := (L_\sigma(\Gamma))_\sigma$ is in the projective space $\mathbb{P}^{\mathcal{NC}_n}$ indexed by non-crossing partitions. The Hausdorff closure E_n of the image of this map can be seen as the compactification of the space of circular planar electrical networks. The preimage of E_n is the space of **cactus networks**, which are obtained by contracting some of edges and identifying the corresponding boundary vertices. A cactus network can also be seen as a union of circular networks, whose shape looks like a cactus. He showed that E_n admits a cell decomposition

$$E_n = \bigsqcup_{\tau \subset \mathcal{P}_n} E_\tau,$$

where $\mathcal{P}_n$ is the set of matchings of $\{1, 2, \ldots, 2n\}$. There is a graded poset structure on $\mathcal{P}_n$ which is dual to an induced subposet of the affine Bruhat order of type A. Lam [Lam] showed that this is exactly the closure partial order of the above cell decomposition, that is,

$$\overline{E}_\tau = \bigsqcup_{\tau' \leq \tau} E_{\tau'}.$$

At the end of [LP], Lam and Pylyavskyy generalize the electrical Lie theory to other finite Dynkin type. They first begin with the definition of general electrical Lie algebra:

Definition 1.1. Let X be a Dynkin diagram of finite type, $I = I(X)$ be the set of nodes in X, and $A = (a_{ij})$ be the associated Cartan matrix . Define the electrical

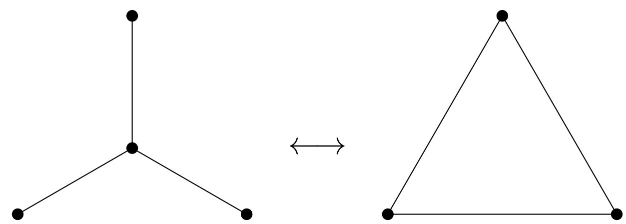


Figure 1.3: Star-Triangle Transformation

Lie algebra $\mathfrak{e}_X$ associated to X to be the Lie algebra generated by $\{e_i\}_{i\in I}$ modulo the relations

$$\mathfrak{ad}(e_i)^{1-a_{ij}}(e_j) = \begin{cases} 0 & \text{if } a_{ij} \neq -1, \\ -2e_i & \text{if } a_{ij} = -1. \end{cases}$$

where $\mathfrak{ad}$ is the adjoint representation.

Note that our convention for a_{ij} is that if the simple root corresponding to i is shorter than the one corresponding to an adjacent node j, then $|a_{ij}| > 1$. Equivalently, the arrows in the Dynkin diagram point towards the nodes which correspond to the shorter roots.

These relations can be seen as a deformation of the upper half of semisimple Lie algebras. For ordinary semisimple Lie algebras, the corresponding relations are

$$\mathfrak{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \text{ for all } i, j.$$

In the case of the electrical Lie algebra of type A, the famous star-triangle (See Figure 1.3) (or Yang-Baxter) transformation of electrical networks translates into **the electrical Serre relation**:

$$[e_i, [e_i, e_{i\pm1}]] = -2e_i,$$

whereas the usual Serre relation for the semisimple Lie algebra of type A is

$$[e_i, [e_i, e_{i\pm 1}]] = 0.$$

where $i \in [n]$ are labels of the nodes of the Dynkin diagram A_n. Lam-Pylyavskyy looked at the algebraic structure of electrical Lie groups and Lie algebras of finite Dynkin types. They showed that $\mathfrak{e}_{A_{2n}}$ is semisimple and isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}$. Moreover, they conjectured that the dimension of the electrical Lie algebra $\mathfrak{e}_X$ equals the number of positive roots $|\Phi(X)^+|$, where $\Phi(X)^+$ is the set of positive roots of root system $\Phi(X)$ with Dynkin diagram X.

In first part of this thesis, we will not only prove Lam-Pylyavskyy's conjecture regarding the dimension for all classical types, but also explore the structure of certain electrical Lie algebras of classical types.

In fact, the semisimplicity of $\mathfrak{e}_{A_{2n}}$ is not a general property of electrical Lie algebras. For example, in Chapter 2 we will see that $\mathfrak{e}_{C_{2n}}$ has a nontrivial solvable ideal. This also makes the structure of such Lie algebras difficult to describe. There is no uniform theory for electrical Lie algebras of classical type, so we will explore the structure of such electrical Lie algebras in a case by case basis:

Consider three irreducible representations of $\mathfrak{sp}_{2n}$: let $V_{\mathbf{0}}$ be the trivial representation, V_ν be the standard representation, that is, with the highest weight vector $\nu = \omega_1$, and V_λ be the irreducible representation with the highest weight vector $\lambda = \omega_1 + \omega_2$, where ω_1, and ω_2 are the first and second fundamental weights.

For type A, the electrical Lie algebra of even rank $\mathfrak{e}_{A_{2n}}$ is isomorphic to $\mathfrak{sp}_{2n}$ [LP]. We show that $\mathfrak{e}_{A_{2n+1}}$ is isomorphic to an extension of $\mathfrak{sp}_{2n} \ltimes V_\nu$ by the $\mathfrak{sp}_{2n}$-representation $V_{\mathbf{0}}$. Note that $\mathfrak{e}_{A_{2n+1}}$ is also isomorphic to the odd symplectic Lie algebra $\mathfrak{sp}_{2n+1}$ studied by Gelfand-Zelevinsky [GZ] and Proctor [RP].

For type B, we show that $\mathfrak{e}_{B_n} \cong \mathfrak{sp}_n \oplus \mathfrak{sp}_{n-1}$ by constructing an isomorphism between these two Lie algebras, where the odd symplectic Lie algebra is the same as the one appearing in the case of type A.

For type C, we first consider the case of even rank. We find an abelian ideal $I \subset \mathfrak{e}_{C_{2n}}$ and prove that this quotient $\mathfrak{e}_{C_{2n}}/I$ is isomorphic to $\mathfrak{e}_{A_{2n}}$. So we can define a Lie algebra action of $\mathfrak{e}_{A_{2n}}$ (or $\mathfrak{sp}_{2n}$) on I. Consequently, we show that $\mathfrak{e}_{C_{2n}}$ is isomorphic to $\mathfrak{sp}_{2n} \ltimes (V_\lambda \oplus V_{\mathbf{0}})$. As for the odd case, it is a Lie subalgebra of $\mathfrak{e}_{C_{2n+2}}$, so we are able to conclude that its dimension is $(2n+1)^2$, the number of positive roots $|\Phi^+(C_{2n+1})|$.

For type D, we find that $\mathfrak{e}_{D_{n+1}}$ contains a Lie subalgebra isomorphic to $\mathfrak{e}_{C_n}$, and use the structure theorem of $\mathfrak{e}_{C_{2n}}$ to find that the dimension of $\mathfrak{e}_{D_{2n+1}}$ is equal to the number of positive roots $|\Phi^+(D_{2n+1})|$. Similarly to type C, we are also able to conclude that the dimension of $\mathfrak{e}_{D_{2n}}$ is the one expected as in the conjecture.

In the second part of the thesis, we first introduce the previous work on circular planar electrical networks and type A electrical Lie theory by Curtis-Ingerman-Morrow, and Lam-Pylyavskyy. Then we would like to develop a similar theory of electrical networks versus electrical Lie algebra in the case of Type B. In this case the combinatorial object is the space of **mirror symmetric circular planar electrical networks**, which are circular planar electrical networks on $2n$ boundary vertices, such that each network is mirror symmetric with respect to some fixed mirror line.

We introduce a new electrically-equivalent transformation, the **square transformation** or **square move** (Figure 1.4) for mirror symmetric circular planar electrical networks. The square move cannot be decomposed into symmetric star-triangle transformations. Thus it can be seen as one of fundamental transformations in electrical equivalence for mirror symmetric circular planar electrical networks. In fact, we show that any two **critical** or **reduced** electrically-equivalent mirror symmetric networks can be transformed from each other only by symmetric star-triangle or (Y-Δ) transformations and square moves.

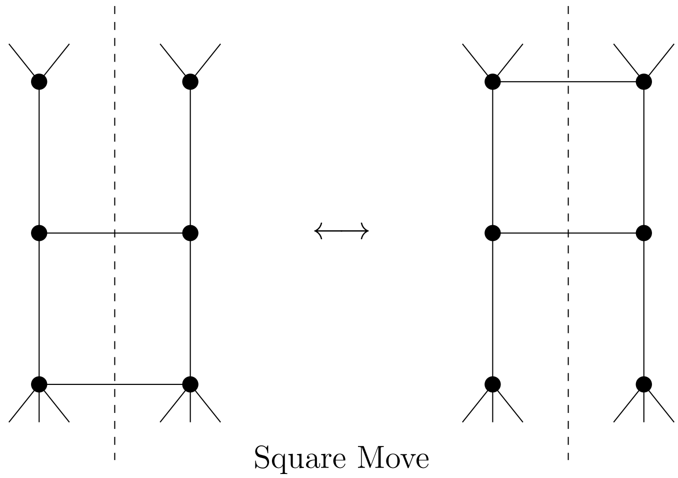


Figure 1.4: An Example of Mirror Symmetric Circular Planar Electrical Networks

On the level of **medial graphs**, the square move translates into the **crossing interchanging transformation** (See Figure 1.5). Later we will see that crossing interchanging transformation can be seen as the type B version of Yang–Baxter transformation. By trying to extend the **medial pairings** to all possible mirror symmetric matchings on $4n$ vertices, we compactify the space of mirror symmetric circular planar electrical networks to mirror symmetric cactus networks. These cactus networks were first introduced by [Lam] in the study of type A electrical Lie theory and ordinary circular planar electrical networks. We show that these mirror symmetric medial pairings has a natural partial order by uncrossing the intersection points, and this partial order is dual to a subposet of Bruhat order of affine permutations of type C. Furthermore, we conjecture that this partial order is the closure partial order of the natural cell decomposition of the space of mirror symmetric cactus networks indexed by mirror symmetric medial pairings.

The structure of this paper goes as follows: Section 1.1 to 1.4 of Chapter 2 give the proofs of the structure theorems of the electrical Lie algebras of type A, B, C, and D, whereas the proofs of some technical lemmas in Section 1.3 and 1.4 are left in Section A.1 and A.2 respectively. Chapter 3 contains the definition and some known results

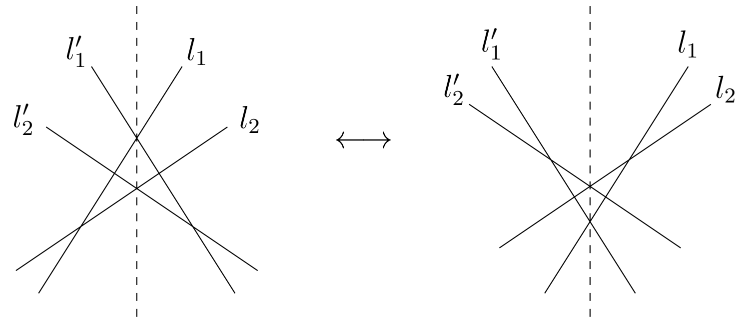


Figure 1.5: Crossing Interchanging Transformation

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about the type A electrical networks, that is, circular planar electrical networks. Chapter 4 contains the author's results about the mirror symmetric electrical networks and the type B electrical Lie algebra. Chapter 5 is a summary of this thesis and also gives possible research directions.

CHAPTER 2

Electrical Lie Algebra of Classical Types

1 Dimension of Electrical Lie Algebra of Classical Types

Lam-Pylyavskyy [LP] proved the following proposition that gives an upper bound for the dimension of electrical Lie algebras of finite Dynkin type.

Proposition 2.1 ([LP])**.** *Let X be a Dynkin diagram of finite type, $\Phi^+(X)$ be the set of positive roots of X. The dimension of $\mathfrak{e}_X$ is less than or equal to $|\Phi^+(X)|$. Moreover, $\mathfrak{e}_X$ has a spanning set indexed by positive roots in $\Phi^+(X)$.*

Remark 2.2. In the proof of Proposition 2.1, Lam-Pylyavskyy gave an explicit spanning set of this Lie algebra: Let α be any element in $\Phi^+(X)$. Write $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_t}$ such that each α_{i_j} is a simple root, and $\sum_{j=1}^{s} \alpha_{i_j}$ is a positive root for all $s \in [t]$. Set $e_\alpha \coloneqq [e_{\alpha_{i_1}}[e_{\alpha_{i_2}}[\ldots[e_{\alpha_{i_{t-1}}}e_{\alpha_{i_t}}]\ldots]$. Then the set $\{e_\alpha\}_{\alpha\in\Phi^+(X)}$ is a spanning set of $\mathfrak{e}_X$.

We will also need the following lemma in exploring the structure of electrical Lie algebras:

Lemma 2.3. *Let L be a Lie algebra, and I be an ideal of L with $[I, I] = 0$. Then the quotient Lie algebra L/I has a Lie algebra action on I.*

Proof. Let $\bar{a} \in L/I$, where $a \in L$. Let $x \in I$. Define $\bar{a} \cdot x = [a, x]$. Let $b \in L$ such that $\bar{b} = \bar{a} \in L/I$. Hence, $a - b = y \in I$. $(\bar{a} - \bar{b}) \cdot x = [a - b, x] = [y, x] = 0$ because

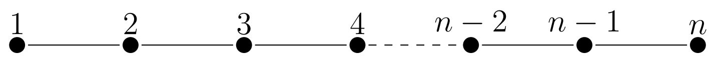


Figure 2.1: Dynkin Diagram of A_n

of $[I, I] = 0$, which shows the well-definedness. Since this action is induced by the adjoint representation of L on I, it defines a valid representation of L/I. $\square$

We will show that the upper bound in Proposition 2.1 is indeed the dimension for each Dynkin diagram of classical type. Furthermore, we will also give the explicit structure of electrical Lie algebras $\mathfrak{e}_{A_n}$, $\mathfrak{e}_{B_n}$, and $\mathfrak{e}_{C_{2n}}$. The notations will only be used within each of following Sections 1.1 to 1.4.

1.1 Type A

By definition, according to Figure 2.1 Lie algebra $\mathfrak{e}_{A_n}$ is generated by generators $\{e_i\}_{i=1}^n$ under the relations:

$$[e_i, e_j] = 0 \qquad \text{if } |i-j| \geq 2,$$
$$[e_i, [e_i, e_j]] = -2e_i \text{ if } |i-j| = 1.$$

Lam-Pylyavskyy studied the structure of $\mathfrak{e}_{A_{2n}}$:

Theorem 2.4 ([LP])**.** *We have*

$$\mathfrak{e}_{A_{2n}} \cong \mathfrak{sp}_{2n}.$$

We will explore the structure of $\mathfrak{e}_{A_{2n+1}}$:

Proposition 2.5. *The dimension of* $\mathfrak{e}_{A_{2n+1}}$ *is* $(n+1)(2n+1)$.

Proof. Let $\{e_i\}_{i=1}^{2n+1}$ and $\{\tilde{e}_i\}_{i=1}^{2n+2}$ be generators of $\mathfrak{e}_{A_{2n+1}}$ and $\mathfrak{e}_{A_{2n+2}}$ respectively. Then there is a Lie algebra homomorphism:

$$\psi : \mathfrak{e}_{A_{2n+1}} \longrightarrow \mathfrak{e}_{A_{2n+2}}, \quad e_i \longmapsto \tilde{e}_i.$$

We claim this is an injection. By Remark 2.2, $\{e_\alpha\}_{\alpha\in\Phi^+(A_{2n+1})}$ and $\{\tilde{e}_\alpha\}_{\alpha\in\Phi^+(A_{2n+2})}$ are spanning sets of $\mathfrak{e}_{A_{2n+1}}$ and $\mathfrak{e}_{A_{2n+2}}$ respectively. By Theorem 2.4, we know that $\{\tilde{e}_\alpha\}_{\alpha\in\Phi^+(A_{2n+2})}$ is a basis of $\mathfrak{e}_{A_{2n+2}}$. On the other hand $\psi(e_\alpha) = \tilde{e}_\alpha$, so $\{e_\alpha\}_{\alpha\in\Phi^+(A_{2n+1})}$ is a basis of $\mathfrak{e}_{A_{2n+1}}$. Therefore, ψ is an injective Lie algebra homomorphism, and the dimension of $\mathfrak{e}_{A_{2n+1}}$ is $(n+1)(2n+1)$. $\square$

We consider a Lie subalgebra of $\mathfrak{sp}_{2n+2}$ introduced by Gelfand-Zelevinsky [GZ]. The definition of this Lie algebra that we will use comes from Proctor [RP]. Let $V = \mathbb{C}^{2n+2}$ be a complex vector space with standard basis $\{x_i\}_{i=1}^{2n+2}$. Let $\{y_i\}_{i=1}^{2n+2}$ be the corresponding dual basis of V^*. Also let B be a nondegenerate skew-symmetric bilinear form of V, and β be one nonzero element in V^*. Let $GZ(V, \beta, B)$ be the Lie subgroup of $GL(V)$ which preserves both β and B. Its Lie algebra is denoted as $gz(V, \beta, B)$. Since $Sp_{2n+2} = \{M \in GL(V) | M \text{ preserves } B\}$, we have $gz(V, \beta, B) \subset \mathfrak{sp}_{2n+2}$. Now we fix $\beta = y_{n+1}$ and the matrix representing B to be $\left(\begin{smallmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{smallmatrix}\right)$. Then we define the **odd symplectic Lie algebra** $\mathfrak{sp}_{2n+1}$ to be $gz(\mathbb{C}^{2n+2}, y_{n+1}, B)$.

Theorem 2.6. *We have $\mathfrak{e}_{A_{2n+1}} \cong \mathfrak{sp}_{2n+1}$ as Lie algebras.*

Proof. We use a specific isomorphism of $\mathfrak{e}_{A_{2n+2}}$ and $\mathfrak{sp}_{2n+2}$ in Theorem 3.1 of [LP]. Let $\epsilon_i \in \mathbb{C}^{2n+2}$ denote the column vector with 1 in the ith position and 0 elsewhere. Let $a_1 = \epsilon_1, a_2 = \epsilon_1 + \epsilon_2, \ldots, a_{n+1} = \epsilon_n + \epsilon_{n+1}$, and $b_1 = \epsilon_1, b_2 = \epsilon_2, \ldots, b_{n+1} = \epsilon_{n+1}$. Now define $\phi : \mathfrak{e}_{A_{2n+2}} \longrightarrow \mathfrak{sp}_{2n+2}$ as follows:

$$e_{2i-1} \mapsto \begin{pmatrix} 0 & a_i \cdot a_i^T \\ 0 & 0 \end{pmatrix}, \qquad e_{2i} \mapsto \begin{pmatrix} 0 & 0 \\ b_i \cdot b_i^T & 0 \end{pmatrix}.$$

and extend this to a Lie algebra homomorphism.

It can be found that $\mathfrak{sp}_{2n+1}$ consists of the matrices of $\mathfrak{sp}_{2n+2}$ whose entries in $(n+1)$-th column and $(2n)$-th row are all zero. The dimension of this Lie algebra is $(n+1)(2n+1)$.

Due to Proposition 2.5, both $\mathfrak{e}_{A_{2n+1}}$ and $\mathfrak{sp}_{2n+1}$ have dimension $(n+1)(2n+1)$. Since $\phi \circ \psi$ is injective, it suffices to show that $\phi \circ \psi(\mathfrak{e}_{A_{2n+1}}) \subset \mathfrak{sp}_{2n+1}$. Because $\phi \circ \psi(\mathfrak{e}_{A_{2n+1}}) \subset \mathfrak{sp}_{2n+2}$ and

$$\mathfrak{sp}_{2n+2} = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \middle| m, n, p, q \text{ are } (n+1) \times (n+1) \text{ matrices}, m = -q^T, p = p^T, n = n^T \right\},$$

we only need to show the entries of the $(n+1)$-th column of $\phi \circ \psi(\mathfrak{e}_{A_{2n+1}})$ are all zeros. Clearly the entries of the $(n+1)$-th column of $\phi \circ \psi(e_i)$ for $i = 1, 2, \ldots, 2n+1$ are all zeros. Now assume $R_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right)$ for $i = 1, 2$ be block matrices, where all entries are matrices of size $(n+1) \times (n+1)$, and the entries of $(n+1)$-th column of A_i and C_i are all zeros. Notice that if M, N are two square matrices, and the entries of last column of N are all zeros, then the last column of MN is a zero column vector. Now

$$R_1 R_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ A_2 C_1 + D_1 C_2 & B_2 C_1 + D_1 D_2 \end{pmatrix}.$$

Therefore, the last columns of $A_1 A_2 + B_1 C_2$ and $A_2 C_1 + D_1 C_2$ are both zero. So the $(n+1)$-th column of the product $R_1 R_2$ is also zero. Since this property of having $(n+1)$-th column being zero is closed among set of $(2n+2) \times (2n+2)$ matrices under matrix multiplication, it is also closed under Lie bracket. Thus, $\phi \circ \psi(\mathfrak{e}_{A_{2n+1}}) \subset \mathfrak{sp}_{2n+1}$. $\square$

Theorem 2.7. *We have that $\mathfrak{e}_{A_{2n+1}}$ is isomorphic to an extension of $\mathfrak{sp}_{2n} \ltimes V_\nu$ by the representation $V_{\mathbf{0}}$, where V_ν and $V_{\mathbf{0}}$ are standard and trivial representation of $\mathfrak{sp}_{2n}$*

respectively. In another word, there is a short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{e}_{A_{2n+1}} \longrightarrow \mathfrak{sp}_{2n} \ltimes \mathbb{C}^{2n} \longrightarrow 0.$$

Proof. We will use the matrix presentation of $\mathfrak{e}_{A_{2n+1}}$ in Theorem 2.6 to prove this theorem.

Let

$$A = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ a_1 & \cdots & a_n & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \cdots & 0 & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n \\ b_1 & \cdots & b_n & b_{n+1} \end{pmatrix}, \quad C = 0.$$

Let I be the set of matrices of the form $\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)$, where a_i, b_i's are arbitrary. It is clear that I is an ideal of $\mathfrak{e}_{A_{2n+1}}$, and $\mathfrak{e}_{A_{2n+1}}/I \cong \mathfrak{e}_{A_{2n}}$. On the other hand, let I' be the one dimensional subspace of $\mathfrak{e}_{A_{2n+1}}$ generated by $\left(\begin{smallmatrix} 0 & E_{n+1,n+1} \\ 0 & 0 \end{smallmatrix}\right)$. Then I' is in the center of $\mathfrak{e}_{A_{2n+1}}$. With calculation we know that $[I, I] \subset I'$. Let $\bar{I}$ be the image of I in $\mathfrak{e}_{A_{2n+1}}/I'$. Thus in the quotient algebra $\mathfrak{e}_{A_{2n+1}}/I'$, we have $[\bar{I}, \bar{I}] = 0$. By Lemma 2.3, this shows that $\mathfrak{e}_{A_{2n+1}}/I'/\bar{I} \cong \mathfrak{e}_{A_{2n}} \cong \mathfrak{sp}_{2n}$ has an action on $\bar{I}$. Since $\bar{I}$ is cyclic and $\dim \bar{I} = 2n$, it has to be isomorphic to the standard representation V_ν. Thus,

$$\mathfrak{e}_{A_{2n+1}}/I' \cong \mathfrak{sp}_{2n} \ltimes V_\nu.$$

Because I' is in the center of $\mathfrak{e}_{A_{2n+1}}$, it is isomorphic to the trivial representation $V_{\mathbf{0}}$ of $\mathfrak{sp}_{2n}$.

Therefore, $\mathfrak{e}_{A_{2n+1}}$ is isomorphic to an extension of $\mathfrak{sp}_{2n} \ltimes V_\nu$ by the trivial representation $V_{\mathbf{0}}$. According to the above matrix presentation of $\mathfrak{e}_{A_{2n+1}}$, we can find a short exact sequence

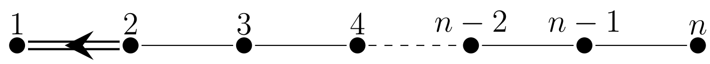


Figure 2.2: Dynkin Diagram of $\mathcal{B}_n$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{e}_{A_{2n+1}} \longrightarrow \mathfrak{sp}_{2n} \ltimes \mathbb{C}^{2n} \longrightarrow 0.$$

□

This concludes the study of the electrical Lie algebra of type A.

1.2 Type B

According to Figure 2.2 the electrical Lie algebra $\mathfrak{e}_{B_n}$ is generated by $\{e_1, e_2, \ldots, e_n\}$ under the relations:

$$\begin{aligned}
&[e_i, e_j] = 0 && \text{if } |i-j| \geq 2 \\
&[e_i, [e_i, e_j]] = -2e_i && \text{if } |i-j| = 1, i \neq 2 \text{ and } j \neq 1 \\
&[e_2, [e_2, [e_2, e_1]]] = 0 &&
\end{aligned}$$

Let $\{f_1, f_2, \ldots, f_n\}$ be a generating set of $\mathfrak{e}_{A_n}$. Then $\{f_2, \ldots, f_n\}$ is a generating set of $\mathfrak{e}_{A_{n-1}}$. Now consider a new Lie algebra $\mathfrak{e}_{A_n} \oplus \mathfrak{e}_{A_{n-1}}$: the underlying set is $(a, b) \in \mathfrak{e}_{A_n} \times \mathfrak{e}_{A_{n-1}}$, and the Lie bracket operation is $[(a, b), (c, d)] = ([a, c], [b, d])$.

Define a map $\phi : \mathfrak{e}_{B_n} \longrightarrow \mathfrak{e}_{A_n} \oplus \mathfrak{e}_{A_{n-1}}$ as follows:

$$\begin{aligned}
&\phi(e_1) = (f_1, 0), \qquad \phi(e_k) = (f_k, f_k) \quad \forall k \geq 2 \\
&\phi([e_{i_1}[e_{i_2}[\ldots[e_{i_{s-1}} e_{i_s}]\ldots]) = [\phi(e_{i_1})[\phi(e_{i_2})[\ldots[\phi(e_{i_{s-1}})\phi(e_{i_s})]\ldots]
\end{aligned}$$

Theorem 2.8. *ϕ is a Lie algebra isomorphism. Therefore, we have*

$$\mathfrak{e}_{B_n} \cong \mathfrak{sp}_n \oplus \mathfrak{sp}_{n-1},$$

where the odd symplectic Lie algebra is defined in Section 1.1.

Proof. First of all we would like to prove ϕ is a Lie algebra homomorphism. It suffices to show that $\phi(e_k)$'s also satisfy the defining relation of $\mathfrak{e}_{B_n}$.

For $k \geq 2$, we have

$$\begin{aligned}
[\phi(e_k)[\phi(e_k)\phi(e_{k+1})]] =& [(f_k, f_k), [(f_k, f_k), (f_{k+1}, f_{k+1})]] \\
=& ([f_k[f_k f_{k+1}]], [f_k[f_k f_{k+1}]] \\
=& -2(f_k, f_k) = -2\phi(e_k).
\end{aligned}$$

Similarly, for $k \geq 3$

$$[\phi(e_k)[\phi(e_k)\phi(e_{k-1})] = -2\phi(e_{k-1}).$$

And

$$\begin{aligned}
[\phi(e_1)[\phi(e_1)\phi(e_2)]] =& [(f_1, 0), [(f_1, 0), (f_2, f_2)]] \\
=& ([f_1[f_1 f_2]], 0) = -2(f_1, 0) \\
=& -2\phi(e_1), \\
[\phi(e_2)[\phi(e_2)[\phi(e_2)\phi(e_1)]]] =& [(f_2, f_2), [(f_2, f_2), [(f_2, f_2), (f_1, 0)]]] \\
=& [(f_2, f_2), (-2f_2, 0)] = 0.
\end{aligned}$$

It is also clear that if $|i - j| \geq 2$, $[\phi(e_i)\phi(e_j)] = 0$. Therefore, this is a Lie algebra homomorphism.

Next we claim that ϕ is surjective. We already know that $(f_1, 0) \in \Im(\phi)$, where $\Im(\phi)$ is the image of ϕ, so it suffices to show that $(f_k, 0)$, $(0, f_k) \in \Im(\phi)$ for all $k \geq 2$.

We go by induction. Base case:

$$\phi(-\frac{1}{2}[e_2[e_2e_1]]) = -\frac{1}{2}[(f_2, f_2), [(f_2, f_2), (f_1, 0)]] = (f_2, 0),$$
$$\phi(e_2 + \frac{1}{2}[e_2[e_2e_1]]) = (f_2, f_2) - (f_2, 0) = (0, f_2).$$

So $(f_2, 0),\ (0, f_2) \in \Im(\phi)$.

Now assume this is true for k. Without loss of generality, say $\phi(y) = (f_k, 0)$. Then

$$\phi(-\frac{1}{2}[e_{k+1}[e_{k+1}y]]) = -\frac{1}{2}[(f_{k+1}, f_{k+1}), [(f_{k+1}, f_{k+1}), (f_k, 0)]] = (f_{k+1}, 0),$$
$$\phi(e_{k+1} + \frac{1}{2}[e_{k+1}[e_{k+1}y]]) = (f_{k+1}, f_{k+1}) - (f_{k+1}, 0) = (0, f_{k+1}).$$

Thus, the claim is true. Note that $\dim \mathfrak{e}_{A_n} \oplus \mathfrak{e}_{A_{n-1}} = \binom{n+1}{2} + \binom{n}{2} = n^2$. By Proposition 2.1, $\dim \mathfrak{e}_{B_n} \leq n^2$. Then we get

$$n^2 = \dim \mathfrak{e}_{A_n} \oplus \mathfrak{e}_{A_{n-1}} \leq \dim \mathfrak{e}_{B_n} \leq n^2.$$

So we achieve equality, and ϕ is an isomorphism. By Section 1.1, we know that $\mathfrak{e}_{A_n} \cong \mathfrak{sp}_n$ for all n. Hence,

$$\mathfrak{e}_{B_n} \cong \mathfrak{sp}_n \oplus \mathfrak{sp}_{n-1}.$$

$\square$

This concludes the study of e_{B_n}.

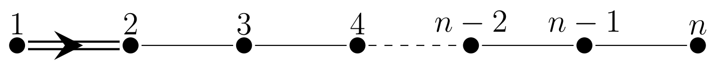


Figure 2.3: Dynkin Diagram of C_n

1.3 Type C

According to Figure 2.3, the electrical Lie algebra of type C_n is generated by generators $\{e_i\}_{i=1}^n$ with relations:

$$\begin{aligned}
&[e_i, e_j] = 0, && \text{if } |i-j| \geq 2,\\
&[e_i, [e_i, e_j]] = -2e_i, && \text{if } |i-j| = 1,\ i \neq 1,\\
&[e_1[e_1[e_1e_2]]] = 0.
\end{aligned}$$

By Remark 2.2, there is a spanning set of $\mathfrak{e}_{C_n}$ indexed by the positive roots $\Phi^+(C_n)$. More precisely, the spanning set is

$$\big\{[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots] : 1 \leq i < j \leq n\big\} \bigcup \big\{[e_i[\ldots[e_{j-1}e_j]\ldots] : 1 \leq i \leq j \leq n\big\}.$$

The way in which the generators of $\mathfrak{e}_{C_n}$ act on the elements in the spanning set is given in Lemma A.1.

Let S be the set $\big\{[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]|i < j\big\}\backslash\big\{[e_1[e_1e_2]]\big\}$. We have the following lemma:

Lemma 2.9. *Let I' be the vector space spanned by S. Then I' is an ideal of $\mathfrak{e}_{C_n}$.*

Proof. Based on Lemma A.1, we can see that $[e_i, s]$ is a linear combination of elements in S for all $s \in S$ and $i \in [n]$. $\square$

Furthermore, I' has a special property:

Lemma 2.10. *The ideal I' is abelian, that is, $[I', I'] = 0$.*

Proof. See Proof $A.2$. $\square$

We also need the following lemma for later.

Lemma 2.11.

$$
\begin{aligned}
&[[e_1e_2],[e_1[e_1[e_2e_3]]]] = [e_1[e_1[e_2e_3]]],\\
&[[e_3e_4],[e_1[e_1[e_2e_3]]]] = [e_1[e_1[e_2e_3]]],\\
&[[e_{2i+1}[\ldots[e_{j-1}e_j]\ldots],[e_1[e_1[e_2e_3]]]] = 0 \textit{ for all } j > 2i+1 \geq 3, j \neq 4,\\
&[[e_1[\ldots[e_{j-1}e_j]\ldots],[e_1[e_1[e_2e_3]]]] = 0 \qquad \textit{for all } j \geq 3,
\end{aligned}
$$

Proof. See Proof A.3. □

Now consider the case when n is even, that is, $\mathfrak{e}_{C_{2n}}$.

Lemma 2.12. *Let*

$$
\begin{aligned}
c = {} & 2n\cdot e_1 + n\cdot[e_1[e_1e_2]]\\
&+\sum_{i=1}^{n-1}(n-i)([e_{2i}[\ldots[e_1[e_1\ldots[e_{2i}e_{2i+1}]\ldots]+[e_{2i+1}[\ldots[e_1[e_1\ldots[e_{2i+1}e_{2i+2}]\ldots])\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{i+j-1}[e_{2j-1}[\ldots[e_1[e_1\ldots[e_{2i+1}e_{2i+2}]\ldots].
\end{aligned}
$$

Then c is in the center of $\mathfrak{e}_{C_{2n}}$.

Proof. See Proof A.4. □

Define I to be the vector space spanned by S together with c. Lemma 2.9, 2.10, and 2.12 show that (1) I is an ideal, and (2) $[I,I]=0$. Also, $\mathfrak{e}_{C_{2n}}/I$ is generated by $\bar{e}_i$'s via the relations $[\bar{e}_i[\bar{e}_i\bar{e}_{i\pm1}]]=-2\bar{e}_i$, for all i except $i=1$, and $[\bar{e}_i,\bar{e}_j]=0$ for $|i-j|\geq 2$. However, the element c in I gives us the relation $[\bar{e}_1[\bar{e}_1\bar{e}_2]=-2\bar{e}_1$. This shows that $\mathfrak{e}_{c_{2n}}/I \cong \mathfrak{e}_{A_{2n}} \cong \mathfrak{sp}_{2n}$.

Applying Lemma 2.3 to our case, we see that $\mathfrak{e}_{C_{2n}}/I$ has an action on I. Our goal is to find how I is decomposed into irreducible representations of $\mathfrak{sp}_{2n}$, and show that

$\mathfrak{e}_{C_{2n}} \cong \mathfrak{e}_{C_{2n}}/I \ltimes I \cong \mathfrak{sp}_{2n} \ltimes I.$

To understand the structure of $\mathfrak{e}_{C_{2n}}$ we first have to understand the structure of the $\mathfrak{sp}_{2n}$-representation I. The plan is to find the highest weight vectors in I.

First of all, let E_{ij} be the $n \times n$ matrix whose (i, j) entry is 1 and 0 otherwise. Thanks to [LP], we have an isomorphism ϕ from $\mathfrak{e}_{C_{2n}}/I \cong \mathfrak{e}_{A_{2n}}$ to $\mathfrak{sp}_{2n}$:

$$\bar{e}_1 \mapsto \begin{pmatrix} 0 & E_{11} \\ 0 & 0 \end{pmatrix}, \quad \bar{e}_{2i-1} \mapsto \begin{pmatrix} 0 & E_{(i-1)(i-1)} + E_{(i-1)i} + E_{i(i-1)} + E_{ii} \\ 0 & 0 \end{pmatrix} \text{ for } i \geq 2,$$
$$\bar{e}_{2i} \mapsto \begin{pmatrix} 0 & 0 \\ E_{ii} & 0 \end{pmatrix} \text{ for } i \geq 1.$$

We would like to find all generators of $\mathfrak{e}_{C_{2n}}/I \cong \mathfrak{sp}_{2n}$ which correspond to simple roots, i.e. the preimage of $\left(\begin{smallmatrix} E_{(i-1)i} & 0 \\ 0 & -E_{i(i-1)} \end{smallmatrix}\right)$ for $i \geq 2$ and $\left(\begin{smallmatrix} 0 & E_{nn} \\ 0 & 0 \end{smallmatrix}\right)$. Also we need to find the maximal toral subalgebra in $\mathfrak{e}_{C_{2n}}/I$, i.e. the preimage of $\left(\begin{smallmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{smallmatrix}\right)$ for $i \geq 1$.

Lemma 2.13. *We have the following identities:*

1.

$$\phi^{-1}\begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix} = [\bar{e}_{2k-1}, \bar{e}_{2k}] - [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]$$
$$+ [\bar{e}_{2k-5}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots] + \ldots + (-1)^{k-1}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots],$$

2.

$$\phi^{-1}\begin{pmatrix} E_{(k-1)k} & 0 \\ 0 & -E_{k(k-1)} \end{pmatrix} = [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots] - [\bar{e}_{2k-5}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]$$
$$+ (-1)^k[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots],$$

3.

$$\phi^{-1}\begin{pmatrix} E_{(k+1)k} & 0 \\ 0 & -E_{k(k+1)} \end{pmatrix} = [\bar{e}_{2k+1}, \bar{e}_{2k}] - [\bar{e}_{2k-1}, \bar{e}_{2k}] + [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]$$

$$- [\bar{e}_{2k-5}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots] + \ldots + (-1)^k[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots],$$

4.

$$\begin{aligned} \phi^{-1}\begin{pmatrix} 0 & E_{nn} \\ 0 & 0 \end{pmatrix} &= \bar{e}_1 + \bar{e}_3 + \ldots \bar{e}_{2n-1} \\ &- [\bar{e}_1[\bar{e}_2\bar{e}_3]] - [\bar{e}_3[\bar{e}_4\bar{e}_5]] - \ldots - [\bar{e}_{2n-3}[\bar{e}_{2n-2}\bar{e}_{2n-1}]] \\ &+ [\bar{e}_1[\ldots, [\bar{e}_4\bar{e}_5]\ldots] + [\bar{e}_3[\ldots, [\bar{e}_6\bar{e}_7]\ldots] + \ldots [\bar{e}_{2n-5}[\ldots, [\bar{e}_{2n-2}\bar{e}_{2n-1}]\ldots] \\ &+ \ldots \\ &+ (-1)^{n-1}[\bar{e}_1[\ldots, [\bar{e}_{2n-2}\bar{e}_{2n-1}]\ldots] \\ &= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1-l}(-1)^l[\bar{e}_{2k+1}[\ldots, [\bar{e}_{2k+2l}\bar{e}_{2k+2l+1}]\ldots]. \end{aligned}$$

Proof. See Proof A.5. □

This computation leads to the following lemma:

Lemma 2.14. *The elements c and $[e_1[e_1[e_2e_3]]]$ in I are annihilated by $\phi^{-1}\left(\begin{smallmatrix} E_{(k-1)k} & 0 \\ 0 & -E_{k(k-1)} \end{smallmatrix}\right)$ for all $k \geq 2$ and $\phi^{-1}\left(\begin{smallmatrix} 0 & E_{nn} \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. Since $c \in Z(\mathfrak{e}_{C_{2n}})$, the center of $\mathfrak{e}_{C_{2n}}$, clearly it is annihilated by the elements of $\mathfrak{e}_{C_{2n}}$. By Lemma 2.11, we notice that when $k \geq 2$, the commutator $[[e_{2i-1}[\ldots[e_{2k-1}e_{2k}]\ldots], [e_1[e_1[e_2e_3]]]] = 0$ for all $i \leq k-1$. From Lemma 2.13,we know that

$$\phi^{-1}\begin{pmatrix} E_{(k-1)k} & 0 \\ 0 & -E_{k(k-1)} \end{pmatrix} = \sum_{i=1}^{k-1}(-1)^{i+1}[\bar{e}_{2(k-i)-1}[\ldots,[\bar{e}_{2k-1}\bar{e}_{2k}]\ldots],$$

so the action of $\phi^{-1}\left(\begin{smallmatrix} E_{(k-1)k} & 0 \\ 0 & -E_{k(k-1)} \end{smallmatrix}\right)$ on $[e_1[e_1[e_2e_3]]]$ gives 0 for all $k \geq 2$.

Similarly, by Lemma 2.11, one knows $[[e_{2i-1}[\ldots[e_{2j-2}e_{2j-1}]\ldots],[e_1[e_1[e_2e_3]]]] = 0$ for all $i \leq j$. Again by Lemma 2.13,

$$\phi^{-1}\begin{pmatrix} 0 & E_{nn} \\ 0 & 0 \end{pmatrix} = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1-l}(-1)^{l}[\bar{e}_{2k+1}[\ldots,[\bar{e}_{2k+2l}\bar{e}_{2k+2l+1}]\ldots],$$

so the action of $\phi^{-1}\left(\begin{smallmatrix} 0 & E_{nn} \\ 0 & 0 \end{smallmatrix}\right)$ on $[e_1[e_1[e_2e_3]]]$ is 0. $\square$

By Lemma 2.14, it turns out that c and $[e_1[e_1[e_2e_3]]]$ are both highest weight vectors. We will find their weights. Because $c \in Z(\mathfrak{e}_{C_{2n}})$, its weight vector has to be a zero vector. Hence, the element c spans a trivial representation of $\mathfrak{e}_{C_{2n}}/I \cong \mathfrak{sp}_{2n}$. As for $[e_1[e_1[e_2e_3]]]$, we have the following lemma:

Lemma 2.15. *The weight of* $[e_1[e_1[e_2e_3]]]$ *is* $\omega_1+\omega_2$*, where* ω_1 *and* ω_2 *are first and second fundamental weights of* $\mathfrak{sp}_{2n}$*.*

Proof. Apply Lemma 2.11:

$$[[e_1e_2],[e_1[e_1[e_2e_3]]]] = [e_1[e_1[e_2e_3]]],$$

so $\phi^{-1}\left(\begin{smallmatrix} E_{11} & 0 \\ 0 & -E_{11} \end{smallmatrix}\right)$ acts on $[e_1[e_1[e_2e_3]]]$ by 1.

$$
\begin{aligned}
&[[e_3e_4] - [e_1[e_2[e_3e_4]]], [e_1[e_1[e_2e_3]]]] \\
=&[[e_3e_4], [e_1[e_1[e_2e_3]]]] - [[e_1[e_2[e_3e_4]]], [e_1[e_1[e_2e_3]]]] \\
=&[e_1[e_1[e_2e_3]]] - 0 = [e_1[e_1[e_2e_3]]],
\end{aligned}
$$

so $\phi^{-1}\left(\begin{smallmatrix} E_{22} & 0 \\ 0 & -E_{22} \end{smallmatrix}\right)$ acts on $[e_1[e_1[e_2e_3]]]$ by 1.

For $k \geq 3$, $[[e_{2i-1}[\ldots[e_{2k-1}e_{2k}]\ldots], [e_1[e_1[e_2e_3]]]] = 0$ when $i \leq k$. Therefore, $\phi^{-1}\left(\begin{smallmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{smallmatrix}\right) = \sum_{i=0}^{k}(-1)^i[\bar{e}_{2(k-i)-1}[\ldots,[\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]$ annihilates $[e_1[e_1[e_2e_3]]]$. This completes the lemma. $\square$

Let $V_{\mathbf{0}}$ be the trivial $\mathfrak{sp}_{2n}$-representation, and V_λ be the irreducible $\mathfrak{sp}_{2n}$-representation with highest weight $\lambda = \omega_1 + \omega_2$. Lemma 2.14 and 2.15 imply that $V_\lambda \oplus V_{\mathbf{0}}$ is isomorphic to an $\mathfrak{sp}_{2n}$-subrepresentation of I. By the Weyl character formula [HJ], $\dim V_\lambda = (2n+1)(n-1) = 2n^2 - n - 1$, and $\dim V_{\mathbf{0}} = 1$, so $\dim V_\lambda + \dim V_{\mathbf{0}} = 2n^2 - n \leq \dim I$. On the other hand, Since $S \cup \{c\}$ form a spanning set of I, $\dim I \leq |S \cup \{c\}| = 2n^2 - n$. So $\dim I = \dim V_\lambda + \dim V_{\mathbf{0}}$. Thus $I \cong V_\lambda \oplus V_{\mathbf{0}}$.

The above argument is based on the assumption that c and $[e_1[e_1[e_2e_3]]]$ are not equal to 0. We still need to show c and $[e_1[e_1[e_2e_3]]]$ are not zero.

Let $F[i,j]$ be the $(2n)^2 \times (2n)^2$ matrix with 1 in i,j entry and 0 elsewhere. Define the a Lie algebra homomorphism (this is actually the adjoint representation of $\mathfrak{e}_{C_{2n}}$) from $\mathfrak{e}_{C_{2n}}$ to $\mathfrak{gl}_{(2n)^2}$ by

$$
\begin{aligned}
e_1 \mapsto & F[3,2] + F[4,3] - F[8,16] + \sum_{j=3}^{2n}(F[(j-1)^2 + j, (j-1)^2 + j - 1] \\
& F[(j-1)^2 + j + 1, (j-1)^2 + j] + F[(j-1)^2 + j + 1, (j-1)^2 + j + 2]),
\end{aligned}
$$

$$
\begin{aligned}
e_k \mapsto & -\sum_{i=2}^{2k-2} F[(k-1)^2+i,(k-2)^2+i-1] + 2F[(k-1)^2+1,(k-1)^2+2] \\
& + \begin{cases} F[(k-1)^2+2k-2,(k-1)^2+2k-1] & \text{if } k \geq 3 \\ 2F[3,4] & \text{if } k=2 \end{cases} \\
& + F[k^2+2,k^2+1] - 2F[k^2+2,k^2+3] - \sum_{i=3}^{2k-1} F[(k-1)^2+i-1,k^2+i] \\
& + F[k^2+2k+1,k^2+2k] - F[(k+1)^2+2k+2,(k+2)^2+2k+5] \\
& + \sum_{j=k+2}^{2n} (F[(j-1)^2+j-k+1,(j-1)^2+j-k] \\
& \qquad + F[(j-1)^2+j-k+1,(j-1)^2+j-k+2] \\
& \qquad + F[(j-1)^2+j+k,(j-1)^2+j+k-1] \\
& \qquad + F[(j-1)^2+j+k,(j-1)^2+j+k+1]).
\end{aligned}
$$

It is straightforward to verify this map satisfies all of the relation among e_k, thus a Lie algebra homomorphism of $\mathfrak{e}_{C_{2n}}$. And the $(14,10)$ entry of the image of $[e_1[e_1[e_2e_3]]]$ is 1. Hence, $[e_1[e_1[e_2e_3]]]$ is not zero.

On the other hand, if we consider the map from $\mathfrak{e}_{C_{2n}}$ to $\mathfrak{gl}_1$:

$$e_1 \mapsto 1, \qquad e_k \mapsto 0.$$

It will also satisfy the relation among $e_k's$, and the image of c is $2n$, so c is not zero.

Theorem 2.16. *We have*

$$\mathfrak{e}_{C_{2n}} \cong \mathfrak{sp}_{2n} \ltimes (V_\lambda \oplus V_{\mathbf{0}}),$$

where λ is the sum of the first and second fundamental weights, and $\mathbf{0}$ is the trivial representation.

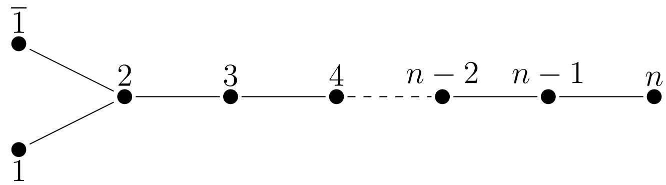


Figure 2.4: Dynkin Diagram of $\mathcal{D}_{n+1}$

Proof. We will use the second Lie algebra cohomology group $H^2(L,V)$, where L is a Lie algebra, and V is a representation of L. It is known that $H^2(L,V)$ is in bijection with extensions L^* with abelian kernel V [CE]. In our case, $\mathfrak{e}_{C_{2n}}$ is an extension of $\mathfrak{e}_{C_{2n}}/I \cong \mathfrak{e}_{A_{2n}}$ with abelian kernel $V \cong V_\lambda \oplus V_{\mathbf{0}}$. By Theorem 26.3 and 26.4 of [CE], since $\mathfrak{e}_{A_{2n}} \cong \mathfrak{sp}_{2n}$ is semisimple, and V is a finite dimensional representation of $\mathfrak{sp}_{2n}$, we know that $H^2(\mathfrak{sp}_{2n}, V) = 0$. So there is only one extension up to isomorphism, that is, $\mathfrak{sp}_{2n} \ltimes V$. Hence,

$$\mathfrak{e}_{C_{2n}} \cong \mathfrak{sp}_{2n} \ltimes (V_\lambda \oplus V_{\mathbf{0}}),$$

where V_λ is the irreducible representation of $\mathfrak{sp}_{2n}$ with the highest weight vector $\lambda = \omega_1 + \omega_2$, and $V_{\mathbf{0}}$ is the trivial representation.

One immediate corollary is:

Corollary 2.17. *The dimension of $\mathfrak{e}_{C_n}$ is n^2.*

Proof. $\dim \mathfrak{e}_{C_{2n}} = \dim \mathfrak{e}_{A_{2n}} + \dim V_\lambda + \dim V_{\mathbf{0}} = 2n^2+n+2n^2-n = 4n^2$. The spanning set of $\mathfrak{e}_{C_{2n+1}}$ of our choice is a subset of a basis of $\mathfrak{e}_{C_{2n+2}}$, so they have to be linearly independent, thus a basis of $\mathfrak{e}_{C_{2n+1}}$, so $\dim \mathfrak{e}_{C_{2n+1}} = (2n+1)^2$. □

1.4 Type D

We will study the case of odd rank first. According to Figure 2.4, electrical Lie algebra $\mathfrak{e}_{D_{2n+1}}$ is generated by generators $\{e_{\bar{1}}, e_1, e_2, e_3, \ldots, e_{2n}\}$ with the relations:

$$
\begin{aligned}
&[e_{\bar{1}}, [e_{\bar{1}}, e_2]] = -2e_{\bar{1}}, \quad [e_1, [e_1, e_2]] = -2e_1, \\
&[e_2, [e_2, e_{\bar{1}}]] = -2e_2, \quad [e_2, [e_2, e_1]] = -2e_2, \\
&[e_{\bar{1}}, e_i] = 0 \qquad\qquad \text{if } i \geq 3, \\
&[e_1, e_i] = 0 \qquad\qquad \text{if } i \geq 3, \\
&[e_i, [e_i, e_j]] = -2e_i \quad \text{if } |i-j| = 1, \\
&[e_j, e_j] = 0 \qquad\qquad \text{if } |i-j| \geq 2, \quad i, j \geq 2.
\end{aligned}
$$

By Proposition 2.1, $\mathfrak{e}_{D_{2n+1}}$ has a spanning set:

$$
\begin{aligned}
&\{e_{\bar{1}}, e_1\} \bigcup \{[e_i[\ldots[e_{j-1}e_j]\ldots]\}_{2\leq i\leq j\leq 2n} \bigcup \{[e_1[e_2[\ldots[e_{j-1}e_j]\ldots]\}_{2\leq j\leq 2n} \\
&\bigcup \{[e_{\bar{1}}[e_2[\ldots[e_{j-1}e_j]\ldots]\}_{2\leq j\leq 2n} \bigcup \{[e_{\bar{1}}[e_1[\ldots[e_{j-1}e_j]\ldots]\}_{2\leq j\leq 2n} \\
&\bigcup \{[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots]\}_{2\leq i<j\leq 2n}.
\end{aligned}
$$

The brackets of generators with the elements in the spanning set are entirely similar to type C, which we will omit here.

Let

$$
\begin{aligned}
c = {} & n\cdot e_1 + n\cdot e_{\bar{1}} + n\cdot [e_{\bar{1}}[e_1 e_2]] \\
& + \sum_{i=1}^{n-1}(n-i)([e_{2i}[\ldots[e_{\bar{1}}[e_1\ldots[e_{2i}e_{2i+1}]\ldots] + [e_{2i+1}[\ldots[e_{\bar{1}}[e_1\ldots[e_{2i+1}e_{2i+2}]\ldots]) \\
& + \sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{i+j-1}[e_{2j-1}[\ldots[e_{\bar{1}}[e_1\ldots[e_{2i+1}e_{2i+2}]\ldots]
\end{aligned}
$$

Lemma 2.18. *Let $\{f_i\}_{i\geq 1}$ be the generating set of $\mathfrak{e}_{C_{2n}}$. Consider the map $\phi : \mathfrak{e}_{C_{2n}} \to$*

$\mathfrak{e}_{D_{2n+1}}$:

$$\phi(f_1) = \frac{e_1 + e_{\bar{1}}}{2}, \qquad \phi(f_k) = e_k \forall k \geq 2$$
$$\phi([f_1[f_2, \ldots [f_{k-1}f_k] \ldots]) = [\phi(f_1)[\phi(f_2), \ldots [\phi(f_{k-1})\phi(f_k)] \ldots]$$

and extend it by linearity. Then ϕ is a Lie algebra homomorphism.

Proof. We only need to show the relations $[\frac{e_1+e_{\bar{1}}}{2}, [\frac{e_1+e_{\bar{1}}}{2}, [\frac{e_1+e_{\bar{1}}}{2}, e_2]]] = 0$, $[e_2, [e_2, \frac{e_1+e_{\bar{1}}}{2}]] = -2e_2$, and $[\frac{e_1+e_{\bar{1}}}{2}, e_k] = 0$ for $k \geq 3$ in $\mathfrak{e}_{D_{2n+1}}$. These can all be verified by straightforward computation. ☐

Recall that the radical ideal I' of $\mathfrak{e}_{C_{2n}}$ is an abelian ideal which is a direct sum of its center

$$\begin{aligned} c' = {} & 2n \cdot f_1 + n \cdot [f_1[f_1 f_2]] \\ & + \sum_{i=1}^{n-1}(n-i)([f_{2i}[\ldots[f_1[f_1 \ldots [f_{2i}f_{2i+1}] \ldots] + [f_{2i+1}[\ldots[f_1[f_1 \ldots [f_{2i+1}f_{2i+2}] \ldots]) \\ & + \sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{i+j-1}[f_{2j-1}[\ldots[f_1[f_1 \ldots [f_{2i+1}f_{2i+2}] \ldots]. \end{aligned}$$

and an abelian ideal I_1' with basis vector of the form $[f_i[\ldots[f_1[f_1 \ldots [f_{j-1}f_j] \ldots]$, where $i < j$, $j \geq 3$. Now we calculate the image of I' under the map ϕ. We will use an identity $[e_i[\ldots[\frac{e_1+e_{\bar{1}}}{2}[\frac{e_1+e_{\bar{1}}}{2} \ldots [e_{j-1}e_j] \ldots] = \frac{1}{2}[e_i[\ldots[e_{\bar{1}}[e_1 \ldots [e_{j-1}e_j] \ldots]$ for $j \geq 3$, and $[\frac{e_1+e_{\bar{1}}}{2}, [\frac{e_1+e_{\bar{1}}}{2}, e_2]] = -\frac{1}{2}(e_1 + e_{\bar{1}}) + \frac{1}{2}[e_{\bar{1}}[e_1 e_2]]$.

Thus,

$$\phi([f_i[\dots[f_{\bar{1}}[f_1\dots[f_{j-1}f_j]\dots]) = \frac{1}{2}[e_i[\dots[e_{\bar{1}}[e_1\dots[e_{j-1}e_j]\dots] \quad \forall j \geq 3,$$

$$\begin{aligned}
\phi(c') &= n(e_1+e_{\bar{1}}) - \frac{n}{2}(e_1+e_{\bar{1}}) + \frac{n[e_{\bar{1}}[e_1e_2]]}{2} \\
&\quad + \frac{1}{2}(\sum_{i=1}^{n-1}(n-i)([e_{2i}[\dots[e_{\bar{1}}[e_1\dots[e_{2i}e_{2i+1}]\dots] + [e_{2i+1}[\dots[e_{\bar{1}}[e_1\dots[e_{2i+1}e_{2i+2}]\dots]) \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{i}(-1)^{i+j-1}[e_{2j-1}[\dots[e_{\bar{1}}[e_1\dots[e_{2i+1}e_{2i+2}]\dots]) \\
&= \frac{1}{2}c.
\end{aligned}$$

Lemma 2.19. *Let I be the image $\phi(I')$. Then I is an ideal of $\mathfrak{e}_{D_{2n+1}}$. Moreover, ϕ is injective.*

Proof. Let $y = \phi(x) \in I$. Clearly, for $k \geq 2$, $[e_k, y] = \phi([f_k, x]) \in I$ and $[e_k, c] = \phi([f_k, c']) = 0 \in I$. It suffices to show that $[e_{\bar{1}}, y] \in I$ and $[e_1, y] \in I$.

If $y = [e_i[\dots[e_{\bar{1}}[e_1\dots[e_{j-1}e_j]\dots]$ where $j \geq 3,\ i < j$. By straightforward computation, we have

$$[e_1, y] = \begin{cases} [e_{\bar{1}}[e_1[\dots[e_{j-1}e_j]\dots] & \text{if } i = 2, \\ [e_{\bar{1}}[e_1[e_2e_3]]] & \text{if } i = 3, j = 4, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

By symmetry, we will get the same results as above for $e_{\bar{1}}$. So $[e_1, y] = [e_{\bar{1}}, y] \in I$.

If $y = c$, we know that $[(e_1 + e_{\bar{1}})/2, c] = \phi([f_1, c']) = 0$. From Equation (1.1), $[e_1 - e_{\bar{1}}, c] = 0$. Thus $[e_1, c] = [e_{\bar{1}}, c] = 0 \in I$.

Therefore, I is an ideal of $\mathfrak{e}_{D_{2n+1}}$. Furthermore, the above shows that c is in the center of $\mathfrak{e}_{D_{2n+1}}$.

Consider the image $\phi(\mathfrak{e}_{C_{2n}})$. It naturally has a spanning set which is the image of basis elements of $\mathfrak{e}_{C_{2n}}$ under ϕ. If we compute the Lie bracket of this spanning set with itself, it is the same as the adjoint representation of $\mathfrak{e}_{C_{2n}}$, thus has dimension $4n^2-1$. It suffices to show that c is not zero. Let $I_1 = \phi(I_1')$, which is spanned by the elements of the form $[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots]$, where $j \geq 3$. By above computation, it is an ideal of $\mathfrak{e}_{D_{2n+1}}$. In $\mathfrak{e}_{D_{2n+1}}/I_1$, we have $\bar{c} = n(\bar{e}_1 + \bar{e}_{\bar{1}} + [\bar{e}_{\bar{1}}[\bar{e}_1\bar{e}_2])$. Note that $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_{\bar{1}}$ form a Lie subalgebra of $\mathfrak{e}_{D_{2n+1}}/I$, which is isomorphic to $\mathfrak{e}_{A_3}$. By the isomorphism in the proof of Theorem 2.6, we have $\bar{e}_1 + \bar{e}_{\bar{1}} + [\bar{e}_{\bar{1}}[\bar{e}_1\bar{e}_2]$ is not zero, and in the center of $\mathfrak{e}_{A_3}$. Thus $\bar{c}$ is not zero, neither is $c \in \mathfrak{e}_{D_{2n+1}}$. So we have $\dim\phi(\mathfrak{e}_{C_{2n}}) = 4n^2 = \dim\mathfrak{e}_{C_{2n}}$. We conclude that ϕ is injective. $\square$

Let J be the ideal generated by $e_{\bar{1}} - e_1$. It is clear that $\mathfrak{e}_{D_{2n+1}}/J$ is isomorphic to $\mathfrak{e}_{A_{2n}}$. We study the structure of the ideal J. Again by Equation (1.1) and computation of type A electrical Lie algebra, a spanning set for J is

$$\{e_1-e_{\bar{1}}, c\}\cup\{[e_1-e_{\bar{1}}[e_2[\ldots[e_{j-1}e_j]\ldots]\}_{j=2}^{2n}\cup\{[e_i[\ldots[e_1-e_{\bar{1}}[e_1-e_{\bar{1}}[\ldots[e_{j-1}e_j]\ldots]\}_{j\geq 2, i<j}.$$

Note that $[e_i[\ldots[e_1 - e_{\bar{1}}[e_1 - e_{\bar{1}}[\ldots[e_{j-1}e_j]\ldots] = -2[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots]$ for $j \geq 3$. The reason why $c \in J$ is because $[e_1-e_{\bar{1}}, [e_1-e_{\bar{1}}, e_2]] = -2e_1-2e_{\bar{1}}-2[e_{\bar{1}}[e_1e_2]] = \frac{2}{n}(c -$ linear combination of $[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots])$, where $j \geq 3$. By this observation, we have $I \subsetneq J$. Let $K = \{e_1-e_{\bar{1}}, [e_1-e_{\bar{1}}, e_2], \ldots, [e_1-e_{\bar{1}}, [e_2[\ldots[e_{2n-1}e_{2n}]\ldots]\}$. Then J is spanned by I and K as a vector space.

We claim that $[J, J] \in I$. If this is true, then because $[I, I] = 0$ from Lemma 2.10 and 2.19, J is the radical of $\mathfrak{e}_{D_{2n+1}}$.

To show the above claim, it suffices to find $[K, I]$ and $[K, K]$. Due to Equation (1.1), $[e_1-e_{\bar{1}}, I] = 0$. Hence by induction on k, we have $[[e_1-e_{\bar{1}}, [e_2[\ldots[e_{k-1}e_k]\ldots], I] = 0$. Hence $[K, I] = 0$.

Lemma 2.20. *Assume $i \geq j$. Then*

$$[[e_1[e_2[\ldots[e_{i-1}e_i]\ldots],[e_1[e_2[\ldots[e_{j-1}e_j]\ldots]] = \begin{cases} 2e_1 + [e_1[e_2e_3]] & \textit{if } i=2, j=3, \\ (-1)^i 2e_1 & \textit{if } j = i+1, i \neq 2, \\ 0 & \textit{if } |j-i| \geq 2 \textit{ or } j = i, \end{cases}$$

$$[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots],[e_{\bar{1}}[e_2[\ldots[e_{j-1}e_j]\ldots]] = \begin{cases} 2e_{\bar{1}} + [e_{\bar{1}}[e_2e_3]] & \textit{if } i=2, j=3, \\ (-1)^i 2e_{\bar{1}} & \textit{if } j = i+1, i \neq 2, \\ 0 & \textit{if } |j-i| \geq 2 \textit{ or } j = i, \end{cases}$$

$$[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots],[e_1[e_2[\ldots[e_{j-1}e_j]\ldots]]$$

$$= \begin{cases} -[e_2[e_{\bar{1}}[e_1[e_2e_3]]]] - [e_{\bar{1}}[e_1e_2]] + [e_{\bar{1}}[e_2e_3]] & \textit{if } i=2, j=3, \\ (-1)^{i-1}([e_{\bar{1}}[e_1e_2]] + \sum_{s=2}^{i}[e_s[\ldots[e_{\bar{1}}[e_1\ldots[e_se_{s+1}]\ldots]) & \textit{if } j=i+1, j \geq 4, \\ (-1)^{i-1}[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots] & \textit{if } |j-i| \geq 2, \\ [e_{\bar{1}}e_2] - [e_1e_2] & \textit{if } i=j=2, \\ 0 & \textit{if } i=j \geq 3. \end{cases}$$

Proof. See Proof A.6. □

By Lemma 2.20, we obtain that

$$[[e_1 - e_{\bar{1}}, [e_2[\ldots[e_{i-1}e_i]\ldots],[e_1 - e_{\bar{1}}, [e_2[\ldots[e_{j-1}e_j]\ldots]]$$

$$= \begin{cases} (-1)^i(2(e_1+e_{\bar{1}}) + 2[e_{\bar{1}}[e_1e_2]] + 2\sum_{s=2}^{i}[e_s[\ldots[e_{\bar{1}}[e_1\ldots[e_se_{s+1}]\ldots]) & \text{if } j=i+1, \\ (-1)^i 2[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots] & \text{if } |j-i| \geq 2. \end{cases}$$

Since $2(e_1+e_{\bar{1}})+2[e_{\bar{1}}[e_1e_2]]$ is a linear combination of c and $[e_i[\ldots[e_{\bar{1}}[e_1\ldots[e_{j-1}e_j]\ldots]$ where $j \geq 3,\ i < j$, we have $[K,K] \in I$. Combining the above calculation, we prove $[J,J] \in I$.

Then we have the following theorem:

Theorem 2.21. *J is the radical ideal of $\mathfrak{e}_{D_{2n+1}}$. Furthermore, $\mathfrak{e}_{D_{2n+1}}/J \cong \mathfrak{sp}_{2n}$.*

As a consequence of the above calculation, we have:

Theorem 2.22. *We have*

$$\dim \mathfrak{e}_{D_n} = n^2 - n.$$

Proof. First consider the case when n is odd. Using the above notations, we know $[K, K] \in I$, so for $\bar{K}$ as an ideal in $\mathfrak{e}_{D_{2n+1}}/I$, we have $[\bar{K}, \bar{K}] = \bar{0}$. Thus, by Lemma 2.3, $\bar{K}$ is a representation of the quotient Lie algebra $(\mathfrak{e}_{D_{2n+1}}/I)/\bar{K}$. Note that $(\mathfrak{e}_{D_{2n+1}}/I)/\bar{K} \cong \mathfrak{e}_{D_{2n+1}}/J \cong \mathfrak{sp}_{2n}$.

We claim that $\bar{K} \neq \bar{0}$. Otherwise, the injective homomorphism ϕ defined in Lemma 2.18 is an isomorphism. Then the element e_1 is not in $\mathfrak{e}_{D_{2n+1}}$, a contradiction.

Furthermore, $\bar{K}$ is cyclic generated by $\bar{e}_1 - \bar{e}_{\bar{1}}$. Thus $\bar{K}$ is irreducible. Because $\dim \bar{K} \leq 2n$ and the nontrivial irreducible representation of $\mathfrak{sp}_{2n}$ with the smallest dimension is the standard representation V_ν, where $\nu = \omega_1$, the first fundamental weight, we have that $\bar{K} \cong V_\nu$. On the other hand, by Lemma 2.19 the homomorphism $\phi : \mathfrak{e}_{C_{2n}} \longrightarrow \mathfrak{e}_{D_{2n+1}}$ is injective, so the spanning set of I is indeed a basis. Hence

$$\dim \mathfrak{e}_{D_{2n+1}} = \dim \mathfrak{sp}_{2n} + \dim V_\nu + \dim I = 2n^2 + n + 2n + 2n^2 - n = (2n+1)^2 - (2n+1)$$

Since $\mathfrak{e}_{D_{2n}}$ is a Lie subalgebra of $\mathfrak{e}_{D_{2n+1}}$, by an argument similar to the one for type C, we can prove that $\dim \mathfrak{e}_{D_{2n}} = (2n)^2 - (2n)$. $\square$

We also have the following conjecture:

Conjecture 2.23. *We conjecture that $\mathfrak{e}_{D_{2n+1}}$ is isomorphic to an extension of $\mathfrak{sp}_{2n} \ltimes V_\nu$ by $V_\lambda \oplus V_{\mathbf{0}}$, where V_ν is the standard representation, $V_{\mathbf{0}}$ is the trivial representation, and V_λ is the irreducible representation of $\mathfrak{sp}_{2n}$ with highest weight vector λ being the sum of the first and second fundamental weights. In other words, there is a short* *exact sequence*

$$0 \longrightarrow V_\lambda \oplus V_{\mathbf{0}} \longrightarrow \mathfrak{e}_{D_{2n+1}} \longrightarrow \mathfrak{sp}_{2n} \ltimes V_\nu \longrightarrow 0.$$

CHAPTER 3

Circular Planar Electrical Networks and Cactus Networks (Type A)

1 Circular Planar Electrical Networks and Type A Lie Theory

1.1 Circular Planar Electrical Networks and Response Matrices

This subsection is mainly attributed to [CIM] and [dVGV].

Definition 3.1. Let $G = (V, V_B, E)$ be a planar undirected graph with the vertex set V, the boundary vertex set $V_B \subseteq V$ and the edge set E. Assume that V_B is labeled by $[n]$ and nonempty. A **circular planar electrical network** $\Gamma = (G, \gamma)$ is a graph G together with a map $\gamma : E \to \mathbb{R}_{>0}^{|E|}$, called the **conductances** of G.

Example 3.2. The following is a circular planar electrical network with boundary vertex set $V_B = \{\bar{1}, \bar{2}, \ldots, \bar{8}\}$.

If we put voltages on each of the boundary vertices of Γ, by Ohm's Law and Kirchhoff's Law, electrical current will flow along edges. This electrical property is captured by the **response matrix** $\Lambda(\Gamma)$. We can interpret the response matrix $\Lambda(\Gamma)$ as a linear transformation in the following way: if one puts voltages $p = \{p(v_i)\}$ on each of the boundary vertices (think of p as column vector), then $\Lambda(\Gamma).p$ will be the

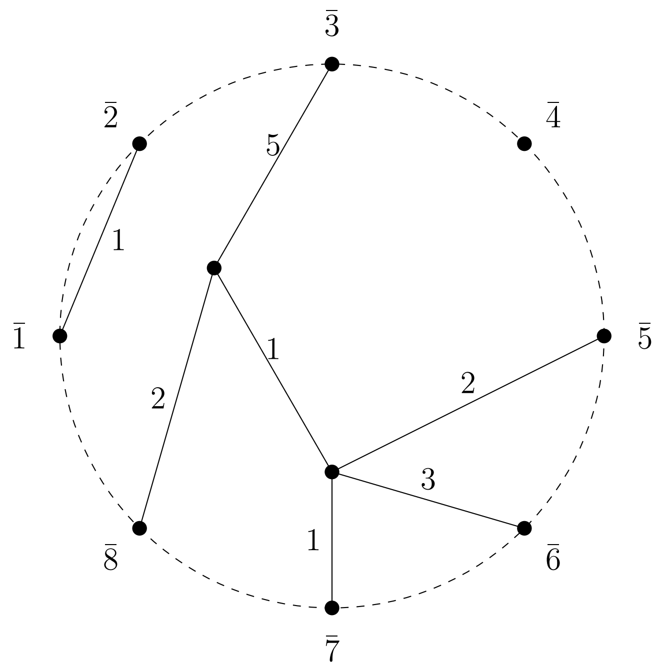


Figure 3.1: Circular Planar Electrical Network

current flowing into or out of each boundary vertex resulting from p. We say that two networks Γ and Γ' are **electrically-equivalent** if $\Lambda(\Gamma) = \Lambda(\Gamma')$.

1.2 Groves of Circular Planar Electrical Networks

The materials of this subsection are mainly from [KW] and [LP].

A **grove** of a circular planar electrical network Γ is a spanning subforest F that uses all vertices of Γ, and every connected component F_i of F has to contain some boundary vertices. The **boundary partition** $\sigma(F)$ encodes boundary vertices that are in the same connected component. Note that $\sigma(F)$ is a **non-crossing partition**. Let $\mathcal{NC}_n$ denote the set of non-crossing partitions of $[\bar{n}]$. Each non-crossing partition has a dual non-crossing partition on $[\tilde{n}]$, where $\tilde{i}$ is placed between $\bar{i}$ and $\overline{i+1}$, and the numbers are modulo n. For example in Figure 3.2 the partition $\{\{\bar{1}\}, \{\bar{2}, \bar{6}, \bar{8}\}, \{\bar{3}, \bar{4}, \bar{5}\}, \{\bar{7}\}\}$ is dual to $\{\{\tilde{1}, \tilde{8}\}, \{\tilde{2}, \tilde{5}\}, \{\tilde{3}\}, \{\tilde{4}\}, \{\tilde{6}, \tilde{7}\}\}$.

The set of non-crossing partition σ is in bijection to the set of non-crossing match-

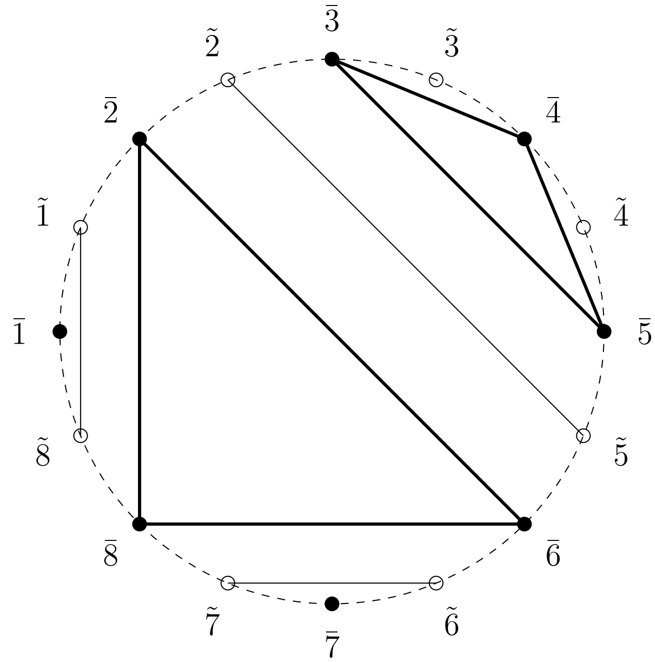


Figure 3.2: Non-crossing Partition

ing $\tau(\sigma)$ of $[2n]$. We put $2i-1$ between the labeling $\widetilde{i-1}$ and $\bar{i}$, and $2i$ between $\tilde{i}$ and $\bar{i}$. Then the non-crossing partition $\sigma = \{\{\bar{1}\}, \{\bar{2}, \bar{6}, \bar{8}\}, \{\bar{3}, \bar{4}, \bar{5}\}, \{\bar{7}\}\}$ is corresponding to a non-crossing matching $\tau(\sigma) = \{(1,2), (3,16), (4,11), (5,10), (6,7), (8,9), (12,15), (13,14)\}$ as in Figure 3.3.

Theorem 3.3. *The set of non-crossing partitions $\mathcal{NC}_n$ is in bijection with the set of non-crossing matchings on $[2n]$. Thus the number of non-crossing partitions is equal to the Catalan number $\dfrac{1}{n+1}\dbinom{2n}{n}$.*

For a non-crossing partition σ, we can define

$$L_\sigma(\Gamma) = \sum_{\{F|\sigma(F)=\sigma\}} \mathrm{wt}(F),$$

where the summation is over all groves F such that the boundary partition $\sigma(F)$ is equal to the non-crossing partition σ, and $\mathrm{wt}(F)$ is the product of all weights of edges in F.

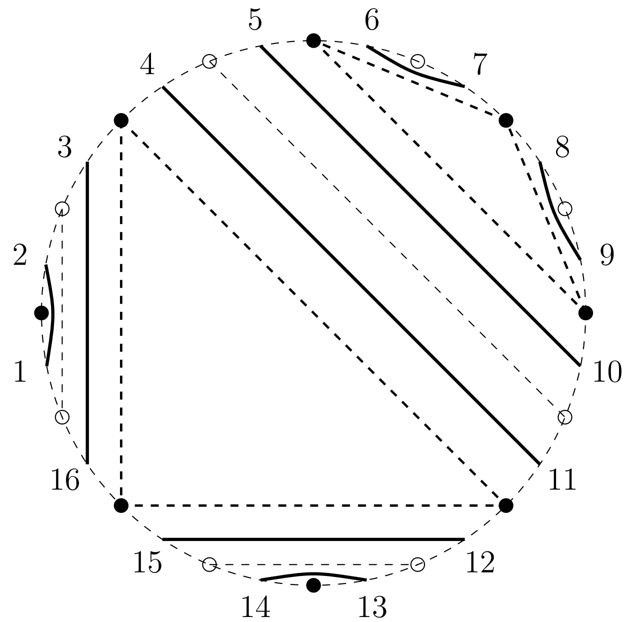


Figure 3.3: Noncrossing Matching from Noncrossing Partition

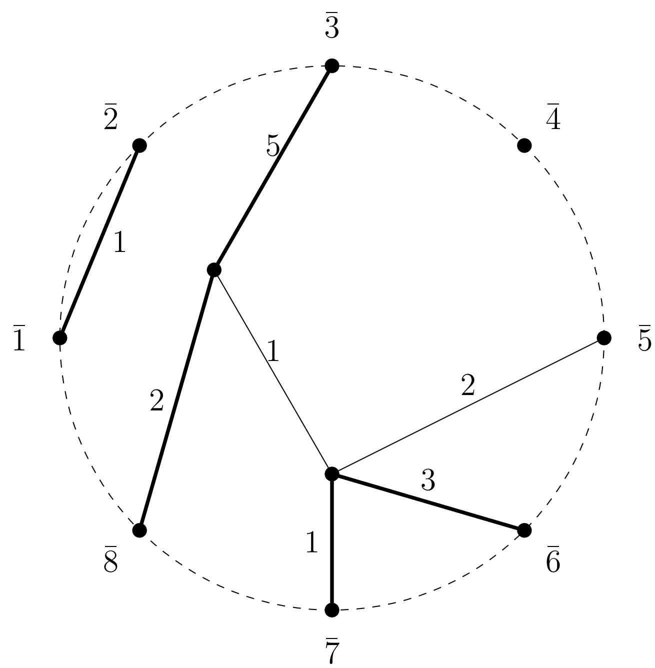


Figure 3.4: Grove F with boundary partition $\sigma(F) = \{\{\bar{1}, \bar{2}\}, \{\bar{3}, \bar{8}\}, \{\bar{4}\}, \{\bar{5}\}, \{\bar{6}, \bar{7}\}\}$, wt($F$)=30

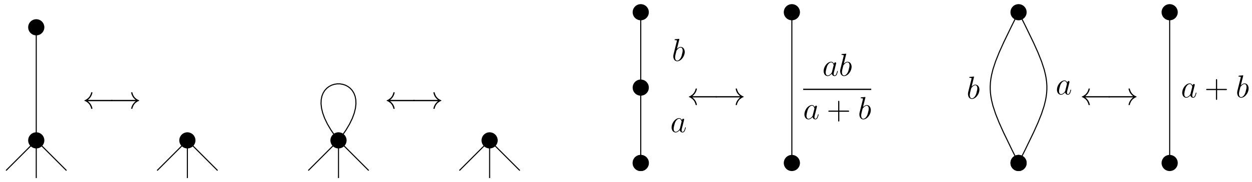


Figure 3.5: Reductions of Circular Planar Electrical Networks

Let $\sigma_{i,j}$ be the partition in which each part contains a single number except the part $\{i, j\}$. Let $\sigma_{\text{singleton}}$ be the partition with each part being a singleton. The following theorem can be found in [KW].

Theorem 3.4. *We have the following identity:*

$$\Lambda_{ij}(\Gamma) = -\frac{L_{\sigma_{i,j}}(\Gamma)}{L_{\sigma_{singleton}}(\Gamma)}.$$

1.3 Electrically-Equivalent Reductions and Transformations Networks

The following propositions can be found in [CIM] and [dVGV]. Recall that two circular planar electrical networks are electrically-equivalent if they have the same response matrix. In this subsection, we are exploring reductions and transformations which do not change response matrices.

Proposition 3.5. *Removing interior vertices of degree 1, removing loops, and series and parallel transformation as in Figure 3.5 do not change the response matrix of an electrical network.*

Note that the above operations reduce the number of resistors. We call these operations **reductions** of networks. We also have the following theorem due to [Ken].

Theorem 3.6 (Star-Triangle or Y-Δ Transformation)**.** *In an electrical network, changing between two configurations in Figure 3.6 locally does not change the response matrix of the network.*

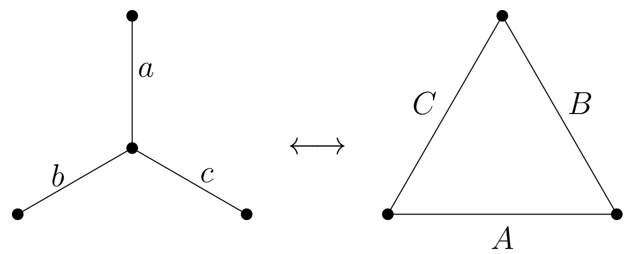


Figure 3.6: Star-Triangle (or Y-Δ) Transformation

where a, b, c, and A, B, C are related by the following relations:

$$A = \frac{bc}{a+b+c}, \qquad B = \frac{ac}{a+b+c}, \qquad C = \frac{ab}{a+b+c},$$

$$a = \frac{AB+AC+BC}{A}, \quad b = \frac{AB+AC+BC}{B}, \quad c = \frac{AB+AC+BC}{C}.$$

1.4 Generators of Circular Planar Electrical Networks and Electrical Lie Theory of Type A

Curits-Ingerman-Morrow [CIM] and Colin de Verdière-Gitler-Vertigan [dVGV] studied the generators of the circular planar electrical networks with n boundary vertices (See Figure 3.7):

Adjoining a boundary spike Define $v_{2i-1}(t) \cdot N$ to be the action on N by adding a vertex u into N, joining an edge of weight $1/t$ between this vertex and boundary vertex $\bar{i}$, and then treating u as the new boundary vertex $\bar{i}$, and old boundary vertex $\bar{i}$ as an interior vertex.

Adjoining a boundary edge Define $v_{2i}(t) \cdot N$ to be the action on N by adding an edge of weight t between boundary vertices $\bar{i}$ and $\overline{i+1}$.

These two operations can be seen as the generators of circular planar electrical networks. Recall from Chapter 2 the electrical Lie algebra of type A of even rank,

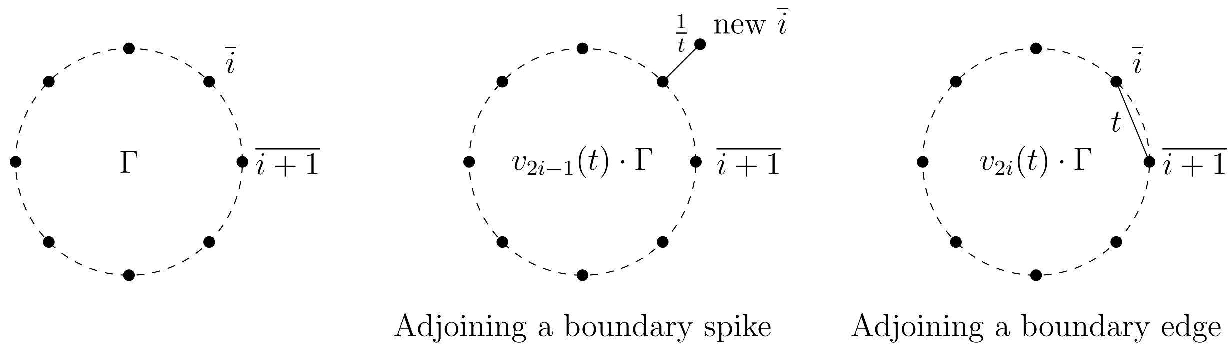


Figure 3.7: Generators of Circular Planar Electrical Networks

$\mathfrak{e}_{A_{2n}}$, generated by $\{e_1, e_2, \ldots, e_{2n}\}$ under the relations:

$$[e_i, e_j] = 0 \qquad \text{if } |i-j| \geq 2,$$
$$[e_i, [e_i, e_j]] = -2e_i \text{ if } |i-j| = 1.$$

Let Sp_{2n} be the **electrical Lie group** $E_{A_{2n}}$. Let $u_i(t) = \exp(te_i)$ for all i. Define the **nonnegative part** $(E_{A_{2n}})_{\geq 0}$ of $E_{A_{2n}}$ to be the Lie subsemigroup generated by all $u_i(t)$ for $t \geq 0$. The following theorem is due to [LP].

Theorem 3.7. *If $a, b, c > 0$, then the elements $u_i(a)$, $u_i(b)$, and $u_i(c)$ satisfy the relations:*

(1) $u_i(a)u_i(b) = u_i(a+b)$

(2) $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| \geq 2$

(3) $u_i(a)u_j(b)u_i(c) = u_j\left(\dfrac{bc}{a+c+abc}\right) u_i(a+c+abc) u_j\left(\dfrac{ab}{a+c+abc}\right)$ *if* $|i-j| = 1$

Furthermore, these three relations are the only ones satisfied by $(E_{A_{2n}})_{\geq 0}$. Thus $(E_{A_{2n}})_{\geq 0}$ has a semigroup action on the set of response matrices of the circular planar electrical network Γ via $u_i(t) \cdot \Lambda(\Gamma) := \Lambda(v_i(t) \cdot \Gamma)$.

Remark 3.8. Relation (3) in Theorem 3.7 translates into to the star-triangle transformations on the boundary of electrical networks.

1.5 Medial Graphs

A circular planar electrical network Γ can be associated with a **medial graph** $\mathcal{G}(\Gamma)$. The construction of medial graphs go as follows: say Γ has n boundary vertices $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. We put vertices $\{t_1, \ldots, t_{2n}\}$ on the boundary circle in the order $t_1 < \bar{1} < t_2 < t_3 < \bar{2} < t_4 < \ldots < t_{2n-1} < \bar{n} < t_{2n}$, and for each edge e of Γ, a vertex t_e of $\mathcal{G}(\Gamma)$ is placed in the middle of the edge e. Join t_e and $t_{e'}$ with an edge in $\mathcal{G}(\Gamma)$ if e and e' share a vertex in Γ and are incident to the same face. As for the boundary vertex t_{2i-1} or t_{2i}, join it with the "closest" vertex t_e where e is among the edges incident to $\bar{i}$. If $\bar{i}$ is an isolated vertex, then join t_{2i-1} with t_{2i}. Note that each interior vertex t_e of $\mathcal{G}(\Gamma)$ has degree 4, each boundary vertex has degree 1, and $\mathcal{G}(\Gamma)$ only depends on the underlying graph of Γ.

A **strand** T of $\mathcal{G}(\Gamma)$ is a maximum sequence of connected edges such that it goes straight through any encountered 4-valent vertex. By definition, a strand either joins two boundary vertices t_i and t_j, or forms a loop. Hence, a medial graph contains a pairing on $[2n]$, which we call the **medial pairing** $\tau(\Gamma)$ (or $\tau(\mathcal{G}(\Gamma))$) of Γ. Medial pairings can also be regarded as matchings of $[2n]$.

The underlying graph of Γ can also be recovered from $\mathcal{G}(\Gamma)$ as follows: the edges of $\mathcal{G}(\Gamma)$ divides the interior of the boundary circle into regions. Color the regions into black and white so that the regions sharing an edge have different colors. Put a vertex in each of the white regions. By convention, the regions containing boundary vertices of $\mathcal{G}$ are colored white. When two white region share a vertex in $\mathcal{G}(\Gamma)$, join the corresponding vertices in Γ by an edge. The resulting graph is the underlying graph of Γ.

Example 3.9. Figure 3.8 is a network from Example 3.2 together with its medial graph in dashed lines. The medial pairing is {(1,3),(2,4),(5,13),(6,15),(7,8),(9,12),(10,16),(11,14)}

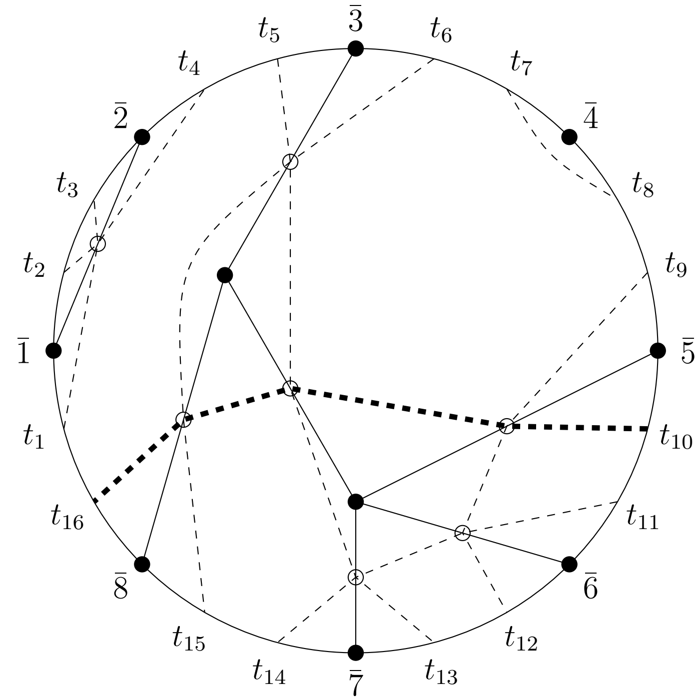


Figure 3.8: Medial Graph of a Circular Planar Electrical Network

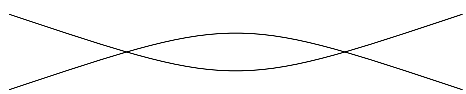


Figure 3.9: Lens

A **lens** consists of two medial strands intersecting with each other at two different vertices of the medial graph as in Figure 3.9.

A medial graph $\mathcal{G}(\Gamma)$ is **lensless** if it does not contain lenses or loops, and every strand starts and ends on the boundary circle. Say Γ is **critical** or **reduced** if $\mathcal{G}(\Gamma)$ is lensless. Usually we talk about medial pairing only when Γ is critical. Let $c(\tau)$ be the number of crossings of the medial pairing τ. This number is independent of the choice of medial graph, as long as this medial graph is lensless.

Proposition 3.10. *[CIM] If two networks are related by relations in Proposition 3.5 then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by lens removal and loop removal in Figure 3.10. If Γ and Γ' are critical and related by star-triangle transformation in Theorem 3.6, then*

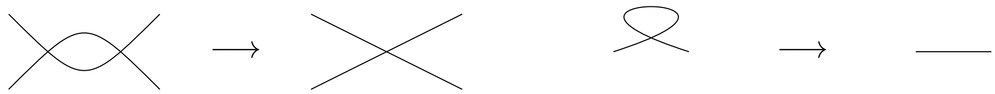


Figure 3.10: Lens and Loop Removal

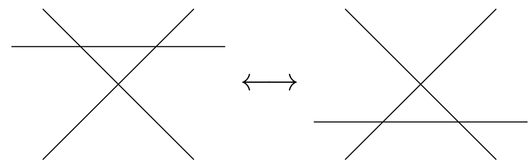


Figure 3.11: Yang-Baxter Move

$\mathcal{G}(\Gamma)$ *and* $\mathcal{G}(\Gamma')$ *are lensless and related by the Yang-Baxter move in Figure 3.11.*

Let $P = \{p_1, p_2, \ldots, p_k\}$, $Q = \{q_1, q_2, \ldots, q_k\}$ be two disjoint ordered subsets of $[2n]$. We say (P, Q) is a **circular pair** if $p_1, p_2 \ldots, p_k, q_k, \ldots, q_2, q_1$ is in circular order. The minor $\Lambda(P, Q)$ is said to be a circular minor of $\Lambda = \Lambda(\Gamma)$ if (P, Q) is a circular pair. The following theorem of circular planar electrical networks can be found in [CIM] and [dVGV].

Theorem 3.11.

1. *Every circular planar electrical network is electrically-equivalent to some critical network.*

2. *The set of response matrices of all circular planar electrical networks consists of matrices* M *such that* $(-1)^k M(P, Q) \geq 0$ *for all* k *and all circular pairs* (P, Q) *such that* $|P| = |Q| = k$.

3. *If two circular planar electrical networks* Γ *and* Γ' *have the same response matrix, then they can be connected by leaf removal, loop removal, series-parallel transformations (in Proposition 3.5), and star-triangle transformations (in Theorem 3.6). Furthermore, if both* Γ *and* Γ' *are critical, only the star-triangle transformations are required.*

4. The conductances of a critical circular planar electrical network can be recovered uniquely from its response matrix.

5. The spaces E'_n of the response matrices of circular planar electrical networks with n boundary vertices has a stratification $E'_n = \sqcup_i D_i$, where $D_i \cong \mathbb{R}_{>0}^{d_i}$ can be obtained as the response matrices of some fixed critical circular planar electrical networks with its conductances varying.

2 Compactification of the Space of Circular Planar Electrical Networks

This section is attributed to [Lam]. From Section 1.5, we note that not every partition of $[2n]$ into pairs can be obtained as a medial pairing of some circular planar electrical network. We would like to generalize the notion of circular planar electrical networks to resolve this issue.

2.1 Cactus Networks

Let σ be a non-crossing partition on $[\bar{n}]$. Let S be a circle with vertices $\{\bar{1}, \dots, \bar{n}\}$. A **hollow cactus** S_σ is obtained from S by gluing boundary vertices according to parts of σ. S_σ can be seen as a union of circles glued together by the identified point according to σ. The interior of a hollow cactus is the union of the open disk bounded by the circles. A **cactus** is a hollow cactus together with its interior. A **cactus network** is a planar electrical network embedded in a cactus, which can also be seen as a union of circular planar networks. σ is called the **shape** of a cactus network. One can think of a cactus network as a circular planar electrical network where the conductance between any two identified boundary points goes to infinity.

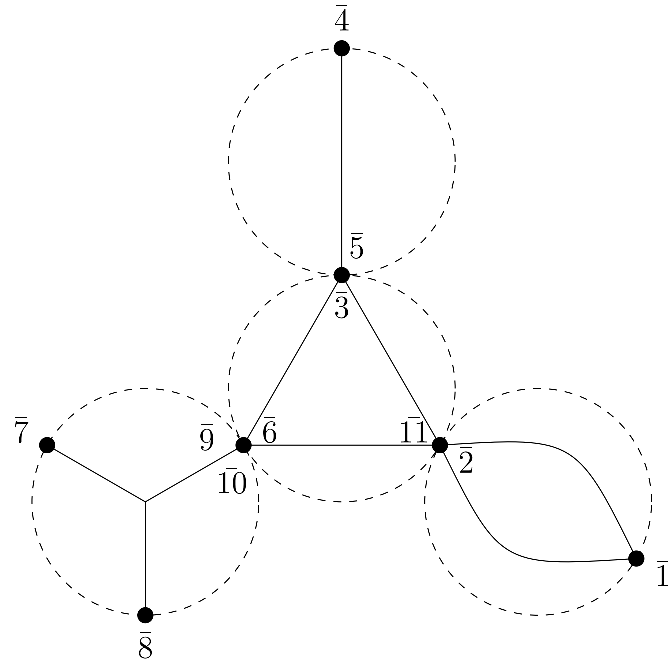


Figure 3.12: Cactus Network

Example 3.12. Figure 3.12 is a cactus network with all conductances equal to 1 and shape $\sigma = \{\{\bar{1}\}, \{\bar{2}, \bar{1}\bar{1}\}, \{\bar{3}, \bar{5}\}, \{\bar{4}\}, \{\bar{6}, \bar{9}, \bar{1}\bar{0}\}, \{\bar{7}\}, \{\bar{8}\}\}$.

A medial graph can also be defined for a given cactus network under the assumption that every edge of the medial graph is contained in one circle of the hollow cactus. Sometimes it is more convenient to draw the medial graph of a cactus network in a disk instead of in a cactus (See Figure 3.13). Similarly, we say a cactus network is **critical** if its medial graph is lensless.

Note that for a cactus network Γ if we put the same voltage on the identified vertices, we can still measure the electrical current flowing into or out of the boundary vertices. Thus, the response matrix $\Lambda(\Gamma)$ can also be defined. We have the following propositions.

Proposition 3.13 ([Lam])**.**

1. *Every cactus network is electrically-equivalent to a critical cactus network.*

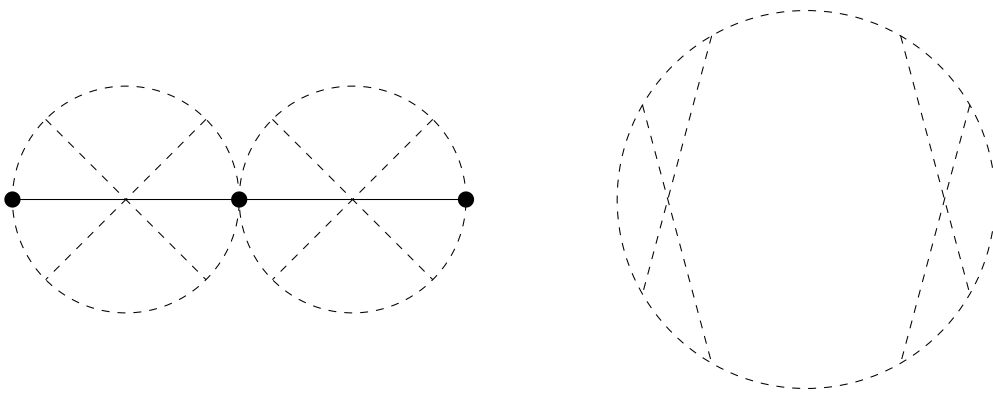


Figure 3.13: Medial Graph in a Cactus Network vs in a Disk

2. *If two critical cactus networks have the same response matrix, then they are related by doing a sequence star-triangle transformations.*

3. *Any medial pairing can be obtained as the medial graph of some cactus network.*

An easy enumeration shows that there are $\frac{(2n)!}{2^n n!}$ medial pairings for cactus networks.

2.2 Grove Measurements as Projective Coordinates

The definition of grove $L_\sigma(\Gamma)$ for a circular planar electrical network Γ and a non-crossing planar partition σ in Subsection 1.2 can be naturally extended to cactus networks. Let $\mathbb{P}^{\mathcal{NC}_n}$ be the projective space with homogeneous coordinates indexed by non-crossing partitions. The map

$$\Gamma \longmapsto (L_\sigma(\Gamma))_\sigma$$

sends a cactus network Γ to a point $\mathcal{L}(\Gamma) \in \mathbb{P}^{\mathcal{NC}_n}$.

Remark 3.14. If the shape of a cactus network Γ is a union of more than one disks, then $L_{\sigma_{singleton}}(\Gamma) = 0$.

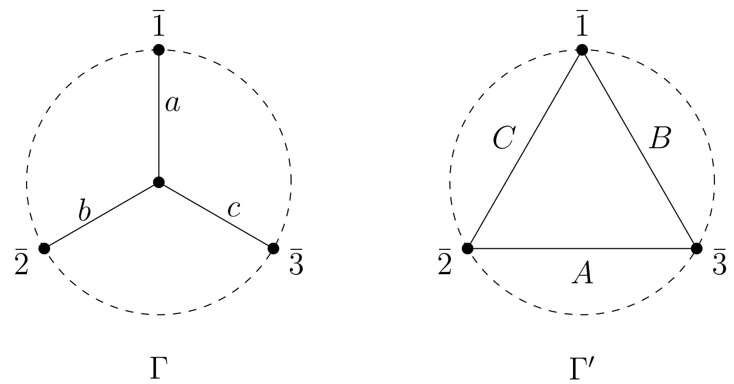


Figure 3.14: Electrically Equivalent Cactus Networks

Proposition 3.15 ([Lam]). *If Γ and Γ' are electrically-equivalent cactus networks, then $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.*

Remark 3.16. From Proposition 3.15, the map $\Gamma \longmapsto \mathcal{L}(\Gamma)$ can be lifted to a map from the electrically-equivalent classes of cactus networks to $\mathbb{P}^{\mathcal{NC}_n}$.

Example 3.17. Figure 3.14 are two electrically-equivalent circular planar electrical networks (a special case of cactus networks) Γ and Γ'. We would like to see that their images $\mathcal{L}(\Gamma)$ and $\mathcal{L}(\Gamma')$ are equal.

We see that

$$L_{\{\bar{1}\},\{\bar{2}\},\{\bar{3}\}}(\Gamma) = a+b+c, \quad L_{\{\bar{1},\bar{2}\},\{\bar{3}\}}(\Gamma) = ab, \quad L_{\{\bar{1}\},\{\bar{2},\bar{3}\}}(\Gamma) = bc, \quad L_{\{\bar{1},\bar{3}\},\{\bar{2}\}}(\Gamma) = ac,$$
$$L_{\{\bar{1},\bar{2},\bar{3}\}}(\Gamma) = abc.$$
$$L_{\{\bar{1}\},\{\bar{2}\},\{\bar{3}\}}(\Gamma') = 1, \quad L_{\{\bar{1},\bar{2}\},\{\bar{3}\}}(\Gamma') = C, \quad L_{\{\bar{1}\},\{\bar{2},\bar{3}\}}(\Gamma') = A, \quad L_{\{\bar{1},\bar{3}\},\{\bar{2}\}}(\Gamma') = B,$$
$$L_{\{\bar{1},\bar{2},\bar{3}\}}(\Gamma') = AC + AB + BC$$

By Theorem 3.6, we have $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$ in $\mathbb{P}^{\mathcal{NC}_n}$.

2.3 Compactificaiton and Main Results for Cactus Networks

By Theorem 3.4, and the fact that $L_{\sigma_{singleton}}(\Gamma) \neq 0$ for every circular planar electrical network Γ, we know that there is a one-to-one correspondence between the grove measurements $\mathcal{L}(\Gamma)$ of Γ and the response matrix of $\Lambda(\Gamma)$ of Γ. Thus, we can also define the space of circular planar electrical networks E_n' with n boundary vertices to be the space of grove measurements of such networks.

Define the closure in the Hausdorff topology $E_n = \overline{E_n'} \subset \mathbb{P}^{\mathcal{NC}_n}$ to be the compactification of the space of circular planar electrical networks. Let P_n be the set of medial pairings (or matchings) of $[2n]$. Note that two electrically-equivalent cactus networks will have the same medial pairing.

Theorem 3.18 ([Lam])**.**

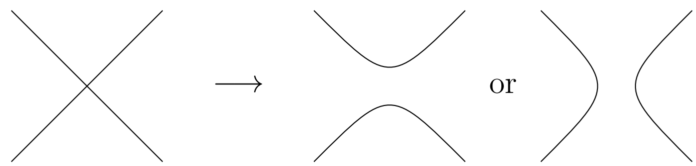
1. *The space E_n is exactly the set of grove measurements of cactus networks. A cactus network is determined by its grove measurement uniquely up to electrical equivalences.*

2. *Let $E_\tau = \{\mathcal{L}(\Gamma) | \tau(\Gamma) = \tau\} \subset E_n$. Each stratum E_τ is parameterized by choosing a cactus network Γ such that $\tau(\Gamma) = \tau$ with edge weights being the parameters. So we have $E_\tau = \mathbb{R}_{>0}^{c(\tau)}$. Moreover,*

$$E_n = \bigsqcup_{\tau \in P_n} E_\tau.$$

2.4 Matching Partial Order on P_n and Bruhat Order

A partial order on P_n can be defined as follows: Let τ be a medial pairing and $\mathcal{G}$ be any lensless medial graph representing τ, denoted as $\tau(\mathcal{G}) = \tau$. Next uncrossing one crossing of $\mathcal{G}$ in either of the following two ways:



Suppose the resulting graph $\mathcal{G}'$ is also lensless. Let $\tau' = \tau(\mathcal{G}')$. Then we say $\tau' \lessdot \tau$ is a covering relation on P_n. The transitive closure of $\lessdot$ defines a partial order on P_n.

Remark 3.19. We can define another partial order on P_n, by uncross the crossings of $\mathcal{G}$, and use lens and loop removal in Figure 3.10 to reduce the resulting graph to a lensless graph $\mathcal{G}'$. Let $\tau' = \tau(\mathcal{G}')$. Then define $\tau' < \tau$ to be the partial order on P_n. Lam [Lam] and Alman-Lian-Tran [ALT] independently showed that these two partial orders on P_n are the same.

Recall that $c(\tau)$ be the number of crossings in a lensless representative of τ. We have the following two theorems.

Theorem 3.20 ([Lam])**.**

1. *The poset $(P_n, \leq)$ is graded by the crossing number $c(\tau)$.*
2. *This partial order on P_n is exactly the closure partial order for the stratification $E_n = \bigsqcup_{\tau \in P_n} E_\tau$. In another word, $\overline{E_\tau} = \bigsqcup_{\tau' \leq \tau} E_{\tau'}$.*

A **bounded affine permutation** of type (k, n) is a bijection $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying

1. $i \leq f(i) \leq i + n$,
2. $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$,
3. $\sum_{i=1}^{n} (f(i) - i) = kn$.

We can associate $\tau \in P_n$ with an affine permutation g_τ as the following,

$$g_\tau(i) = \begin{cases} \tau(i) & \text{if } i < \tau(i), \\ \tau(i) + 2n & \text{if } i > \tau(i). \end{cases}$$

It is straightforward to check that g_τ is a bounded affine permutation of type $(n, 2n)$.

Theorem 3.21 ([Lam])**.** *We have $l(g_\tau) = 2(\binom{n}{2} - c(\tau))$. Then $\tau \longmapsto g_\tau$ gives an isomorphism between $(P_n, \leq)$ and an induced subposet of the dual Bruhat order of bounded affine permutations of type A. In other words, $g_\tau \leq g_{\tau'}$ in the affine Bruhat order if and only if $\tau' \leq \tau$ in $(P_n, \leq)$.*

Remark 3.22. By Theorem 3.21, we can view the decomposition of the space of circular planar electrical networks as an analogy of type A Bruhat decomposition.

CHAPTER 4

Mirror Symmetric Circular Planar Electrical Networks and Mirror Symmetric Cactus Networks (Type B)

In this chapter, we will develop the analogous theory for mirror symmetric circular planar electrical networks and type B electrical Lie group and Lie algebra.

1 Mirror Symmetric Circular Planar Electrical Networks and Type B Electrical Lie Theory

1.1 Mirror Symmetric Circular Planar Electrical Networks

Definition 4.1. A **mirror symmetric circular planar electrical network** of rank n is a circular planar electrical network with $2n$ boundary vertices, which is also mirror symmetric to itself with respect to a mirror line that does not contain any boundary vertex.

The boundary vertices are labeled as $\{\bar{1}, \bar{2}, \ldots, \bar{n}, \bar{1}', \bar{2}', \ldots, \bar{n}'\}$.

Example 4.2. Figure 4.1 is a mirror symmetric circular planar electrical network with boundary vertices $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{1}', \bar{2}', \bar{3}', \bar{4}'\}$.

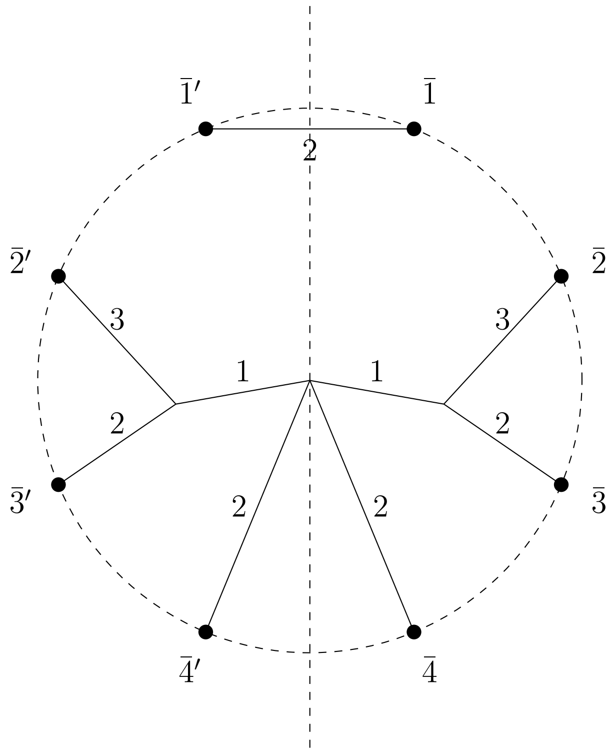


Figure 4.1: Mirror Symmetric Circular Planar Electrical Network

The definitions of response matrices and grove measurements of mirror symmetric electrical networks are the same as the ones for circular planar electrical networks in Subsection 1.1 and 1.2. Let ME_n' be the space of response matrices of mirror symmetric circular planar electrical networks.

1.2 Electrically-Equivalent Transformations of Mirror Symmetric Networks

In this section, we discuss under what reductions and transformations the response matrix or the grove measurements will be unchanged. Note that after each transformation, the resulting electrical network should still be mirror symmetric.

The following proposition is an easy consequence of Theorem 3.5.

Proposition 4.3. *Mirror symmetrically performing the actions in Proposition 3.5, that is, removing interior vertices of degree 1, removing loops, and series and parallel*

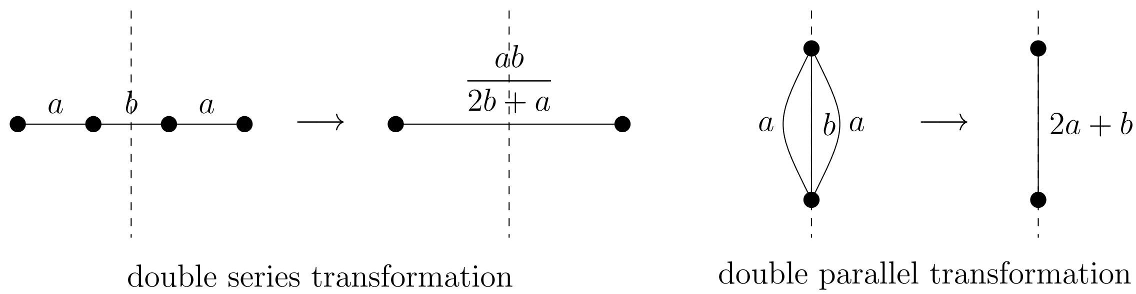


Figure 4.2: Double Series and Parallel Transformation

transformations, plus double series and parallel transformation (see Figure 4.2) will not change the response matrix or the grove measurement of a mirror symmetric electrical network.

We call these operations **symmetric reductions** of a mirror symmetric network. As for non-reduction transformations, we have:

Theorem 4.4.

1. *Mirror symmetrically performing star-triangle transformations will not change the response matrix or the grove measurement of the mirror symmetric network.*

2. *(**Square transformation** or **square move**) locally changing between the two configurations in Figure 4.3 will not change the response matrix or the grove measurement of the mirror symmetric network, where the weights are given by the rational transformation $\phi : (a, b, c, d) \longrightarrow (A, B, C, D)$.*

$$A = \frac{ad + bc + cd + 2bd}{b}$$
$$B = \frac{(ad + bc + cd + 2bd)^2}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$
$$C = \frac{ac(ad + bc + cd + 2bd)}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$
$$D = \frac{a^2 bd}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}$$

We also have $\phi(A, B, C, D) = (a, b, c, d)$, so ϕ is an involution.

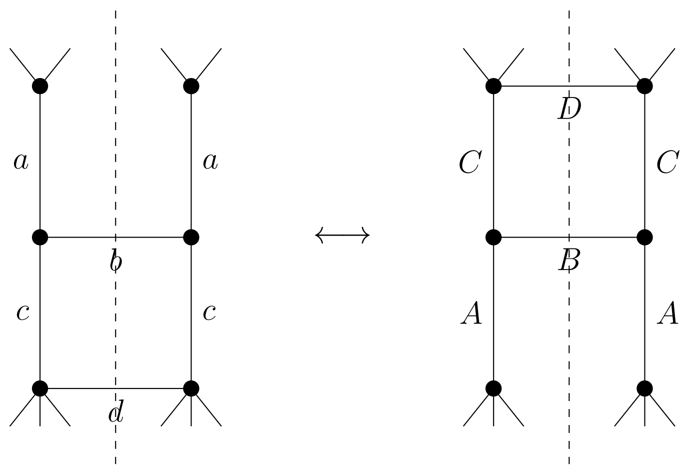


Figure 4.3: Square Move

Proof. By Theorem 3.6, (1) is clear. As for (2), we would like to decompose this transformation into star-triangle transformations as in Figure 4.4, where the numbers on the edge keep track of the corresponding edges changes in the star triangle transformations.

The transformation ϕ is obtained by the composition of star-triangle transformations. The claim that ϕ is an involution can be directly verified. □

1.3 Generators of Mirror Symmetric Networks and Electrical Lie Theory of Type B

We introduce the following operations on mirror symmetric circular planar electrical networks with $2n$ boundary vertices (See Figure 4.5).

Adjoining two boundary spikes mirror symmetrically For all $i \in [n]$, define $v_{2i}(t) \cdot \Gamma$ to be the action of adding boundary spikes with weights $1/t$ on both vertices $\bar{i}$ and $\bar{i}'$, and treating the newly added vertices as new boundary vertices $\bar{i}$ and $\bar{i}'$, and old boundary vertices $\bar{i}$ and $\bar{i}'$ as interior vertices.

Adjoining two boundary edges mirror symmetrically For $i \in [n] \setminus \{1\}$, define $v_{2i-1}(t) \cdot \Gamma$ to be the action of adding boundary edges between vertices $\overline{i-1}$ and

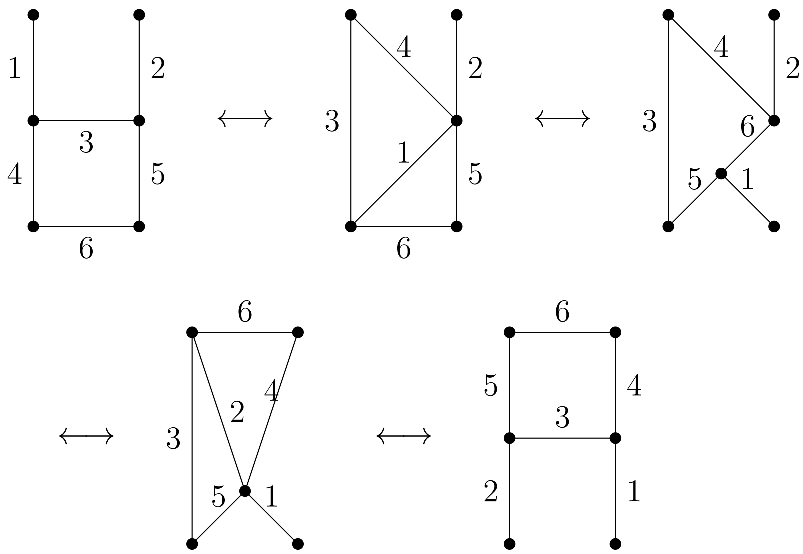


Figure 4.4: Pictorial Proof of Square Move

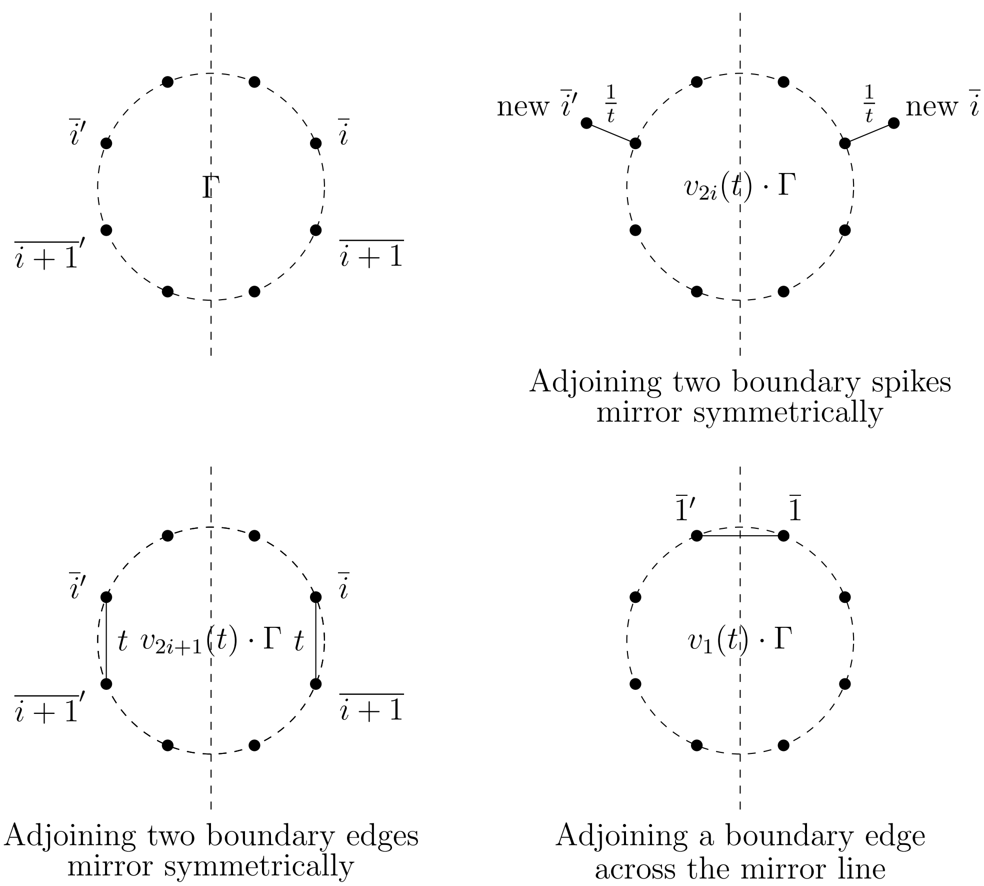


Figure 4.5: Generators of Mirror Symmetric Circular Planar Electrical Networks

$\bar{i}$, and between $\overline{i-1}'$ and $\bar{i}'$, both with weight t.

Adjoining a boundary edge across the mirror line Define $v_1(t)$ be the action of adding an boundary edge between vertices $\bar{1}$ and $\bar{1}'$ with weight $t/2$.

These operations can be seen as the generators of mirror symmetric circular planar electrical networks. Now recall the electrical Lie algebra of type B of even rank, namely $e_{B_{2n}}$, defined in Chapter 2 Subsection 1.2. It is generated by $\{e_1, e_2, \ldots, e_{2n}\}$ under the relations:

$$\begin{aligned}
&[e_i, e_j] = 0 && \text{if } |i-j| \geq 2 \\
&[e_i, [e_i, e_j]] = -2e_i && \text{if } |i-j| = 1, i \neq 2 \text{ and } j \neq 1 \\
&[e_2, [e_2, [e_2, e_1]]] = 0
\end{aligned}$$

By Theorem 2.8, we know that $\mathfrak{e}_{B_{2n}} \cong \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2n-1}$. So we can see $E_{B_{2n}}$ as the direct product of Sp_{2n} and Sp_{2n-1}. Let $u_i(t) = \exp(te_i)$ for all i. Define the nonnegative part $(E_{B_{2n}})_{\geq 0}$ to be the Lie subsemigroup generated by all $u_i(t)$ for all $t \geq 0$.

Theorem 4.5 ([LP])**.** *If $t > 0$, then $u_i(t)$'s satisfy the following relations:*

1. $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| \geq 2$,

2. $u_i(a)u_i(b) = u_i(a+b)$,

3. $u_i(a)u_j(b)u_i(c) = u_j\left(\frac{bc}{a+c+abc}\right)u_i(a+c+abc)u_j\left(\frac{ab}{a+c+abc}\right)$ *if* $|i-j| = 1, i, j \neq 1$,

4. $u_2(t_1)u_1(t_2)u_2(t_3)u_1(t_4) = u_1(p_1)u_2(p_2)u_1(p_3)u_2(p_4)$, *with*

$$p_1 = \frac{t_2t_3^2t_4}{\pi_2}, p_2 = \frac{\pi_2}{\pi_1}, p_3 = \frac{\pi_1^2}{\pi_2}, p_4 = \frac{t_1t_2t_3}{\pi_1},$$

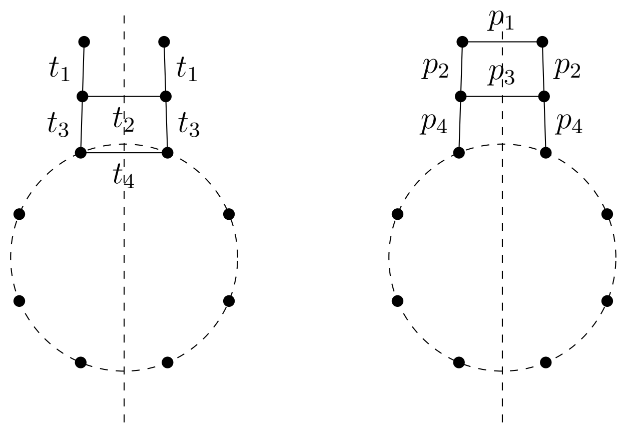


Figure 4.6: Pictorial Proof of Theorem 4.6

where

$$\pi_1 = t_1t_2 + (t_1 + t_3)t_4 + t_1t_2t_3t_4, \quad \pi_2 = t_1^2t_2 + (t_1 + t_3)^2t_4 + t_1t_2t_3t_4(t_1 + t_3).$$

This Lie subsemigroup $(E_{B_{2n}})_{\geq 0}$ is related to the operation $v_i(t)$'s in the following way.

Theorem 4.6. *The generators $v_i(t)$'s also satisfy the relation of $u_i(t)$'s in Theorem 4.5 for $t > 0$. Therefore, $e_{B_{2n}}$ has an infinitesimal action on the space of mirror symmetric circular planar electrical networks.*

Proof. $v_i(t)$'s satisfying the first three relations is a consequence of Theorem 3.7. It suffices to show relation (4) in Theorem 4.5. Note that the action $v_2(t_1)v_1(t_2)v_2(t_3)v_1(t_4)$ and $u_1(p_1)u_2(p_2)u_1(p_3)u_2(p_4)$ will give two configurations in Figure 4.6.

We note that this is exactly the square transformation in Theorem 4.4 with $\phi(t_1, t_2, t_3, t_4) = (p_4, p_3, p_2, p_1)$.

As for the second claim, since the derivatives of the relations in this theorem are exactly the defining relation of $\mathfrak{e}_{B_{2n}}$, we know that $\mathfrak{e}_{B_{2n}}$ has an infinitesimal action on ME'_n. $\square$

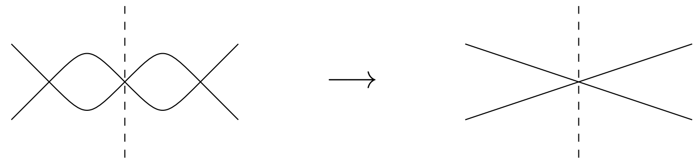


Figure 4.7: Double Lenses Removal

1.4 Medial Graph and Some Results for Mirror Symmetric Electrical Networks

Each mirror symmetric circular planar electrical network Γ is associated with a **(Symmetric) medial graph** $\mathcal{G}(\Gamma)$. The notion of **lensless** medial graph and its medial pairing are the same as the ones for ordinary electrical networks. A **mirror symmetric medial pairing** is a matching of $\{\bar{1}, \bar{2}, \ldots, \overline{2n}, \bar{1}', \bar{2}', \ldots, \overline{2n}'\}$ such that if $\{\bar{i}, \bar{j}\}$, $\{\bar{i}', \bar{j}'\}$ or $\{\bar{i}, \bar{j}'\}$ are in the matching, so are $\{\bar{i}', \bar{j}'\}$, $\{\bar{i}, \bar{j}\}$ or $\{\bar{i}', \bar{j}\}$, respectively. Similar to ordinary medial pairings, the number of pairs of mirror symmetric crossings (if the crossing is on the mirror line, it counts as one pair of symmetric crossing) of a mirror symmetric medial pairing τ is independent of the choice of medial graph. Define $mc(\tau)$ to be the number of pairs of symmetric crossings.

Proposition 4.7. *Let Γ and Γ' be two mirror symmetric networks. Then we have the following:*

1. *If Γ and Γ' are related by symmetric leaf and loop removals, series and parallel transformations, and double series and parallel transformations, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by symmetric lens and loop removals and double lenses removals (See Figure 4.7).*

2. *If Γ and Γ' are related by symmetric star-triangle transformations, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by two mirror symmetric Yang-Baxter transformations.*

3. *If Γ and Γ' are related by the square move in Theorem 4.4, then $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma')$ are related by the* ***crossing interchanging transformation*** *(See Figure 4.8).*

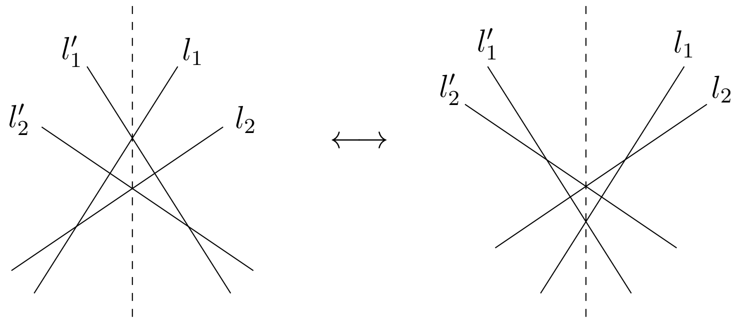


Figure 4.8: Crossing Interchanging Transformation

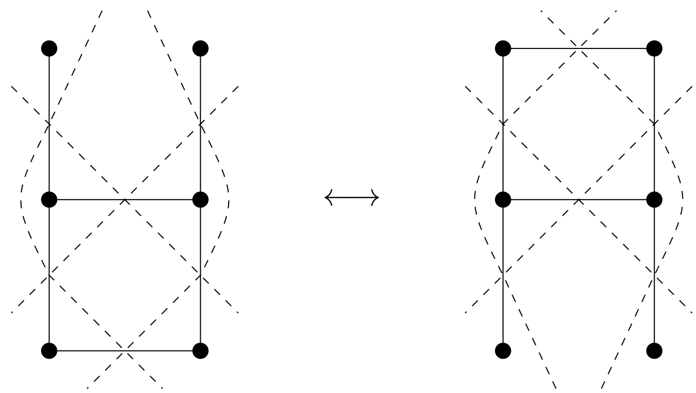


Figure 4.9: Pictorial Proof of Theorem 4.7

Proof. The first two claims are consequences of Theorem 3.10. The third claim is proved via Figure 4.9. □

Theorem 4.8. *Every mirror symmetric circular planar electrical network can be transformed into a critical mirror symmetric network through the symmetric operations in Proposition 4.3 and Theorem 4.4.*

Proof. Let Γ be a mirror symmetric electrical network. Consider its symmetric medial graph $\mathcal{G}$. Claim that we can use symmetric lens and loop removal, double lenses removal, symmetric Yang-Baxter transformations, and crossing interchanging transformations to remove all lenses in $\mathcal{G}$. Thus on the level of mirror symmetric electrical network, the resulting network Γ' will be a critical network, and it is obtained from Γ by doing symmetric leaf and loop removals, series-parallel transformations, double series-parallel transformations, symmetric star-triangle transformations, and square transformations. Thus, we finish the theorem.

It suffices to show the above claim. First pick a lens in $\mathcal{G}$. Two medial strands of this lens are denoted as l_1 and l_2, which intersect at a and b. Let al_1b be the arc of this lens which lies on l_1. Define al_2b likewise. Let the interior of the lens be the region enclosed by al_1b and al_2b. We can assume this lens does not contain any other lens in its interior. Otherwise, pick a smaller lens in its interior. We have two cases: (1) the mirror line does not pass through both of a and b; (2) the mirror line pass through both a and b, which means l_1 is symmetric to l_2. Our goal is to remove this lens via the operations in Proposition 4.7.

Case (1): There is another lens enclosed by l_1' and l_2' which are the mirror symmetric counterparts of l_1 and l_2 respectively. In the following discussion, for every transformation we perform mirror symmetric operations simultaneously on both lenses.

Let $H = \{h_1, h_2, \ldots, h_k\}$ be the set of medial strands that intersect both of al_1b and al_2b. Then h_i can possibly intersect with other h_j's in the interior of the lens. We claim that we can use symmetric Yang-Baxter transformations to make h_i's have no intersections among themselves in the interior of the lens, and the same is true for the mirror images of h_i's.

We proceed by induction. The base case is trivial. Now assume there is at least one intersection. Among those intersections, let r_i be the intersection point on h_i that is closest to the arc al_1b. Two medial strands which intersect at r_i and the arc al_1b form a closed region D_i. We pick r_k such that the number of regions in D_k is minimized. We do the same construction for the mirror symmetric lens (these two lenses could possibly be the same). Note the number of regions in D_k has to be one. Otherwise, there must be another strand h_s intersecting D_k at r_s, and the region enclosed by two strands which intersect at r_s and the arc al_1b will have smaller number of regions in it, contradiction. Hence, we can use symmetric Yang-Baxter transformation to remove the region D_k and its mirror image in order to reduce the number of intersections by one in this lens and in its mirror image. Thus the claim is true.

Note that at least one of a and b is not on the mirror line. Without loss of generality, say the point a does not lie on the mirror line. Among the strands in H, pick the strand h_t that is the "closest" to a. Then l_1, l_2, and h_t enclose a region which does not have subregions inside. So we can use symmetric Yang-Baxter transformation to move h_t such that it does not intersect the arcs al_1b and al_2b, and likewise for the mirror image. Repeat this until no strand intersects this lens and its mirror image.

Lastly, use the symmetric lens removal or double lenses removal to remove these two lenses symmetrically.

Case (2): Again let $H = \{h_1, h_2, \ldots, h_k\}$ be the set of medial strands that intersect both of al_1b and al_2b. We can use the same argument as in Case (1) to reduce the number of intersections among h_i's which are in the lens, but not on the mirror line. Thus, we can assume all of the intersections stay on the mirror line. Pick an intersection point r_k that is closest to the point a. Say two mirror symmetric strands h_1 and h_1' intersect at r_k. Then we can use crossing interchanging transformation to move h_1 and h_1' so that they intersect neither al_1b nor al_2b. Repeat this until no strand intersects with this lens. Then we use symmetric lens removal to decrease the number of lenses.

By iterating the above lens removal procedures, we can change $\mathcal{G}$ into a lensless medial graph. □

Next we prove a lemma.

Lemma 4.9. *Let $\mathcal{G}$ and $\mathcal{H}$ be two lensless mirror symmetric medial graphs. If $\mathcal{G}$ and $\mathcal{H}$ have the same medial pairing, then they can be obtained from each other by symmetric Yang-Baxter transformations and crossing interchanging transformations* *of Proposition 4.7 (2) and (3).*

Proof. We will proceed by induction on the number of medial pairings. The base case is the empty network, which is trivially true. Pick a medial strand L such that L divides the circle into two parts, say A and B, and there is no other chord completely contained within A. Consequently, the mirror image L' of L divides the circle into two parts A' and B', mirror image of A and B. There are three cases:

1 L does not intersect L'.

2 L coincides with L'.

3 L intersects L' at one point.

Let $P_1, P_2, \ldots, P_k$ be the medial strands intersecting L. We claim that one can use symmetric Yang-Baxter transformations and crossing interchanging transformations to make $P_1, P_2, \ldots, P_k$ have no intersection point among themselves in the region A. Suppose otherwise. Let r_j be the intersection point on the medial strand P_j that is closest to L. For each r_j, we know that L and two medial strand where r_j lies on form a closed region S_j. Pick r_i such that the number of subregions in S_i is minimized. Similarly we can define the mirror image $L', P'_1, P'_2, \ldots, P'_k, r'_j$ and S'_i. Next consider three cases separately.

Case 1 (Figure 4.10): First note that the number of regions in S_i must be one, otherwise, one can find $r_{i'}$ on one of the medial strand where r_i lies on, such that $S_{i'}$ has fewer regions, contradiction. Then by mirror symmetry, the number of regions in S'_i is also 1. By symmetric Yang-Baxter transformations, we can remove the closed regions S_i and S'_i so that the number of intersections in A and A' decreases respectively, while the medial parings remain the same. So without loss of generality, we can assume the number of intersections in A and A' is 0. Hence we can use symmetric Yang-Baxter transformations to change both $\mathcal{G}$ and $\mathcal{H}$ as in the following picture. The regions C

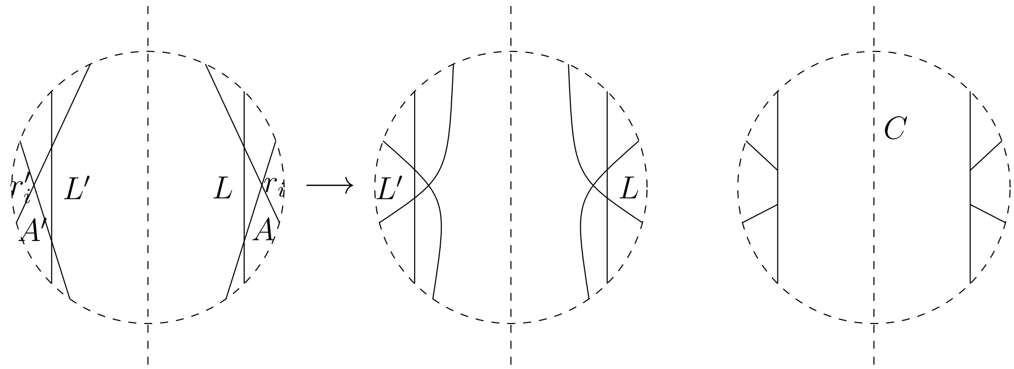


Figure 4.10: Case 1 of Proof of Lemma 4.9

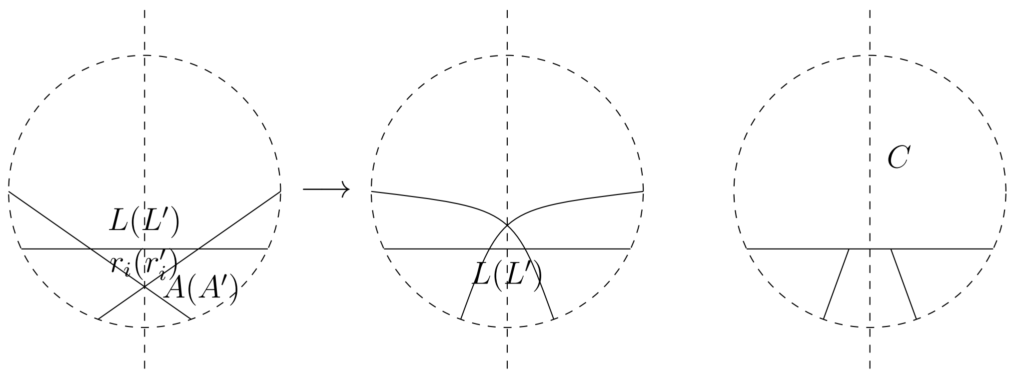


Figure 4.11: Case 2 of Proof of Lemma 4.9

enclosed by L, L' and the rest of the boundary circle has fewer number of medial pairings. Therefore, by induction, we are done.

Case 2 (Figure 4.11): A and A' coincide with each other. The argument is almost the same as above except that the region S_i and S_i' can coincide with each other, which means the region S_i is symmetric to itself and r_i is on the mirror line. In this situation, we can still apply symmetric Yang-Baxter move to decrease the number of intersections in region A, and apply the induction hypothesis.

Case 3 (Figure 4.12): L and L' intersect at point P. One can assume that all the medial strands cross the mirror line, otherwise, we can perform Case 1 first. Then by similar argument to Case 1, regions S_i and $S_{i'}$ both have one region in it. Moreover, r_i coincides with r_i'. And S_i is mirror symmetric to S_i' and they only intersect at r_i.

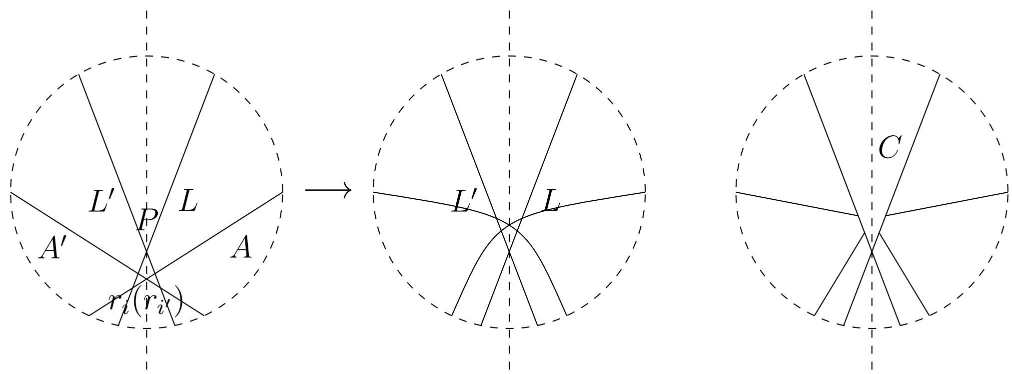


Figure 4.12: Case 1 of Proof of Lemma 4.9

The region enclosed by L, L' and the two medial strands whose intersection is r_i only has one region in it. In this case we can perform crossing exchanging transformations to swap P and r_i, so that the number of intersections in A and A' decrease. By repeating these operations, we can also assume the number of intersections in A and A' is 0. Hence we can use symmetric Yang-Baxter transformations to change both $\mathcal{G}$ and $\mathcal{H}$ as in the following picture, and apply induction hypothesis. □

The following theorem is analogus to Theorem 3.11 (3).

Theorem 4.10. *If two mirror symmetric planar electrical networks* Γ *and* Γ' *have the same response matrix, then they can be connected by symmetric leaf and loop removal, symmetric series-parallel transformations (in Proposition 4.3), symmetric star-triangle transformations, and square transformations (in Theorem 4.4). Furthermore, if both* Γ *and* Γ' *are critical, only the symmetric star-triangle transformations and square transformations are required.*

Proof. By Theorem 4.8, we can assume that Γ and Γ' are both critical.

As Γ and Γ' have the same response matrix, by Theorem 3.4 and 3.18, we know that the medial pairing of these two networks $\tau = \tau(\mathcal{G}(\Gamma))$ and $\tau' = \tau(\mathcal{G}(\Gamma'))$ are the same. Then by Lemma 4.9, τ and τ' can be obtained from each other by symmetric Yang-Baxter transformations and crossing interchanging transformations. Therefore, the underlying graph of Γ and Γ' are related by mirror symmetric star-triangle transformations and square transformations. Now let $T(\Gamma)$ be the network by doing such transformations on the underlying graph of Γ as well as its weights on the edges. It can be seen that the underlying graph of $T(\Gamma)$ is the same as underlying graph of Γ'. On the other hand, $T(\Gamma)$ and Γ' have the same response matrix. Thus by Theorem 3.11 (4), they must have the same weight on each edge, which means $T(\Gamma) = \Gamma'$. This proves the theorem. $\square$

We also have the following theorem:

Theorem 4.11. *The space ME'_n of response matrices of mirror symmetric circular planar networks has a stratification by cells $ME'_n = \bigsqcup D_i$ where each $D_i \cong \mathbb{R}^{d'_i}_{>0}$ can be obtained as the set of response matrices for a fixed critical mirror symmetric circular planar network with varying weights on the pairs of symmetric edges.*

Proof. Recall that E'_n is the space of response matrices of circular planar electrical networks. By Theorem 3.11 (5), $E'_n = \bigsqcup C_i$ where $C_i \cong \mathbb{R}^{d_i}_{>0}$ can be obtained as a set of response matrices for a fixed critical network with varying edge weights. Now for each C_i, if possible, we pick the representative critical network such that the underlying graph is mirror symmetric. Let D_i be the subspace of each such C_j with mirror symmetric edge weights to be the same. It is clear that every mirror symmetric circular planar electrical network is obtained in this way up to electrical equivalences. Thus, $ME'_n = \bigsqcup D_i$. $\square$

2 Compactification of the Space of Mirror Symmetric Circular Planar Electrical Networks

Again, not every mirror symmetric medial pairing can be obtained as a medial pairing of some mirror symmetric circular planar electrical network. We will use a definition similar to cactus networks in Section 3.12 to resolve this.

2.1 Mirror Symmetric Cactus Network

A **mirror symmetric cactus network** is a cactus network which is symmetric with respect to the mirror line. The medial graph of mirror symmetric medial graph again is typically drawn in a circle. Similar to the usual cactus networks, we have the following proposition.

Proposition 4.12.

1. *Every mirror symmetric cactus network is electrically-equivalent to a critical cactus network through symmetric reductions.*

2. *If two critical mirror symmetric cactus network have the same response matrix, then they are related by doing a sequence of symmetric star-triangle and square transformations.*

3. *Any symmetric medial pairing can be obtained as the medial pairing of some mirror symmetric cactus network.*

Proof. The proof (1) and (2) are similar to the proof for mirror symmetric circular planar networks. (3) is proved as following: Let τ be a symmetric medial pairing of $\{-2n+1, \ldots, 0, \ldots, 2n-1, 2n\}$ and $\mathcal{G}$ be any medial graph with $\tau(\mathcal{G}) = \tau$. Then the medial strands divide the disk into different regions. If some vertices in $\{\bar{1}, \bar{2}, \ldots, \bar{n}, \bar{1}', \bar{2}', \ldots, \bar{n}'\}$ are in one region, then identify them as one vertex in the mirror symmetric cactus network. This gives a mirror symmetric hollow cactus. Now within each pair of symmetric disks we have symmetric medial pairings. We can reconstruct a pair of circular planar networks which is mirror symmetric to each other within each pair of such disks. Thus we are done. $\square$

2.2 Grove Measurements as Projective Coordinates of Mirror Symmetric Networks

Similar to ordinary electrical networks, we can also define grove measurements for mirror symmetric cactus networks. Two partitions σ and σ' are called **mirror images of each other** or **mirrors** if interchanging $\bar{i}$ and $\bar{i}'$ for all $i \in [n]$ in σ and σ' will swap these two partitions. Clearly, if Γ is a mirror symmetric cactus network, $L_\sigma(\Gamma) = L_{\sigma'}(\Gamma)$, where σ and σ' are mirrors. Let $\mathbb{P}^{\mathcal{SNC}_n}$ be the subspace of $\mathbb{P}^{\mathcal{NC}_{2n}}$ in which the grove measurements indexed by a pair of symmetric non-crossing partitions are the same. Thus the map

$$\Gamma \longmapsto (L_\sigma(\Gamma))_\sigma$$

sends a mirror symmetric cactus network to a point $\mathcal{L}(\Gamma) \in \mathbb{P}^{\mathcal{SNC}^n}$.

Proposition 4.13. *If Γ and Γ' are electrically equivalent mirror symmetric cactus networks, then $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.*

Proof. Since each symmetric reduction, symmetric star-triangle transformation and square transformation can all be decomposed into a sequence of ordinary reductions and star-triangle transformations. Thus by Theorem 3.15, we have $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$.□

2.3 Compactification and Some Result for Cactus networks

Recall ME_n' is the space of response matrices of mirror symmetric circular planar electrical networks. Since the grove measurements of mirror symmetric circular planar electrical network is in one-to-one correspondence with the response matrix of $\Lambda(\Gamma)$ by a similar argument in Subsection 2.3, we can regard ME_n' as the space of grove measurements of electrical equivalence classes of mirror symmetric circular planar electrical networks. Theorem 4.10 implies that $ME_n' \longrightarrow \mathbb{P}^{\mathcal{SNC}_n}$ is an injection.

Define the closure in the Hausdorff topology $ME_n = \overline{ME_n'} \subset \mathbb{P}^{\mathcal{SNC}_n}$ to be the **compactification** of the space of mirror symmetric circular planar electrical networks. Let MP_n be the set of mirror symmetric medial pairings of $\{-2n+1,\ldots,0,\ldots,2n-1,2n\}$. Note that two electrically equivalent cactus networks have the same medial pairing.

Theorem 4.14.

1. *The space ME_n is exactly the set of grove measurements of mirror symmetric cactus networks. A mirror symmetric cactus network is determined uniquely by its grove measurement up to symmetric electrical equivalences.*

2. *Let $ME_\tau := \{\mathcal{L}(\Gamma) \in ME_n \mid \tau(\Gamma) = \tau\} \subset ME_n$. Then we have*

$$ME_n = \bigsqcup_{\tau \in MP_n} ME_\tau$$

Where each stratum ME_τ is parametrized by choosing a mirror symmetric cactus network Γ such that $\tau(\Gamma) = \tau$ with varying edge weights. So we have $ME_\tau \cong \mathbb{R}_{>0}^{mc(\tau)}$, where $mc(\tau)$ is the number of pair of symmetric crossings.

Proof. First we prove (1). By definition, any point $\mathcal{L}$ in ME_n is a limit point of points in ME'_n. On the other hand, the top cell of ME'_n is dense in ME'_n. Thus we can assume $\mathcal{L} = \lim_{i\to\infty} \mathcal{L}(\Gamma_i)$, where Γ_i's are mirror symmetric circular planar electrical networks whose underlying graphs are the same, say G. Note that the response matrices depend continuously on the edge weights. Thus $\mathcal{L}$ is obtained by sending some of mirror symmetric edge weights of G to ∞. By doing this we identify some of the boundary vertices, and obtain a mirror symmetric cactus network.

To see how $\mathcal{L}(\Gamma)$ determines Γ, first notice that Γ is a union of circular planar networks Γ_r. We observe that the shape σ of Γ is determined by $\mathcal{L}(\Gamma)$. To reconstruct Γ, it suffices to recover each circular planar networks Γ_r in a symmetric way. Let σ_{ij} be obtained from σ by combining the parts containing i and j. Then we can determine

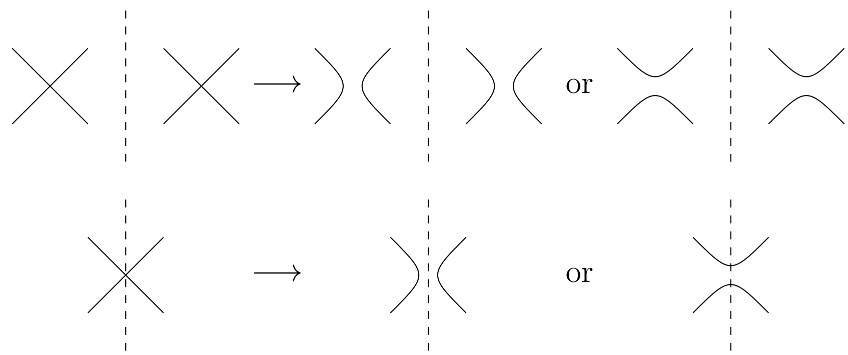


Figure 4.13: Definition of $\mathcal{MP}_n$

the response matrix $\Lambda(\Gamma_r)$ of Γ_r by the following identity:

$$\Lambda(\Gamma_r)_{ij} = \frac{L_{\sigma_{ij}}(\tau)}{L_\sigma(\tau)}.$$

Each grove of Γ is a union of groves in Γ_r, so in the above ratio the contribution of groves from Γ_s where $s \neq r$ gets cancelled. Thus by Theorem 3.4, the above identity is true. Hence we can recover response matrices $\Lambda(\Gamma_r)$ for each Γ_r. By Theorem 3.11 (3) we can reconstruct Γ_r and its mirror image Γ_r' simultaneously in a mirror symmetric way uniquely up to electrical equivalences.

As for (2), By definition we have $ME_n = \bigsqcup_{\tau \in MP_n} ME_\tau$. Note that each mirror symmetric cactus network is a union of pairs of mirror symmetric circular networks, and then we apply Theorem 3.11 and Lemma 4.9 to obtain the second statement.$\square$

2.4 Symmetric Matching Partial Order on $\mathcal{MP}_n$ and Bruhat Order

A partial order $(\mathcal{MP}_n, \leq)$ with underlying set $\mathcal{MP}_n$ can be defined as follows: Let τ be a medial pairing and $\mathcal{G}$ be a medial graph such that $\tau(\mathcal{G}) = \tau$. Uncross the two different kinds of crossings as in Figure 4.13.

Suppose the resulting medial graph $\mathcal{G}'$ is also lenseless. Let $\tau' = \tau(\mathcal{G}')$. Then we say $\tau' \lessdot \tau$ is a covering relation in $\mathcal{MP}_n$. The partial order on $\mathcal{MP}_n$ is the transitive closure of these relations. This poset is an induced subposet of the poset $\mathcal{P}_{2n}$ studied in [Lam].

Lemma 4.15. *Let $\mathcal{G}$ be a medial graph with $\tau(\mathcal{G}) = \tau$, where the labels of vertices of $\mathcal{G}$ on the boundary are $\{-2n+1, \ldots, -1, 0, 1, \ldots, 2n\}$, and i is the mirror image of $-i+1$. Suppose that (a, b, c, d) are in clockwise order (some of them possibly coincide), and τ has strands from a to c, and b to d. Correspondingly $(-a+1, -b+1, -c+1, -d+1)$ are in counterclockwise order, and τ has strands from $-a+1$ to $-c+1$, and $-b+1$ to $-d+1$. Let $\mathcal{G}'$ be obtained from uncrossing the intersections and joining a to d, b to c, $-a+1$ to $-d+1$, and $-b+1$ to $-c+1$. Then G' is lenseless if and only if no other medial strand goes from the arc (a, b) to (c, d) and from the arc $(-a+1, -b+1)$ to $(-c+1, c-d+1)$ and the positions of (a, b, c, d) and $(-a+1, -b+1, -c+1, -d+1)$ are one of the configurations in Figure 4.14.*

Proof. Straightforward case analysis. □

Recall that $mc(\tau)$ is the number of pairs of symmetric crossings.

Lemma 4.16. *$\mathcal{MP}_n$ is a graded poset with grading given by $mc(\tau)$.*

By straightforward enumeration the highest rank is n^2, and the number of elements on rank 0 is $\binom{2n}{n}$ which is the number of mirror symmetric medial pairings.

Example 4.17. Figure 4.15 is the poset $\mathcal{MP}_2$.

Now consider another partial order $(\mathcal{MP}_n, \preceq)$ on $\mathcal{MP}_n$. Let $\tau \in \mathcal{MP}_n$. Pick a medial graph $\mathcal{G}$ with $\tau(\mathcal{G}) = \tau$. We break a crossing of $\mathcal{G}$ to obtain a graph $\mathcal{H}$. Note that $\mathcal{H}$ may not be lensless any more. Let $\mathcal{H}'$ be the lensless graph obtained from removing all lenses in $\mathcal{H}$. Let $\tau' = \tau(\mathcal{H}') = \tau(\mathcal{H})$. In this case, we say $\tau' \prec \tau$. We claim these two partial orders are the same.

Theorem 4.18. *Let $\tau, \nu \in (\mathcal{MP}_n, \prec)$. Then τ covers ν if and only if there is a lensless medial graph $\mathcal{G}$ with $\tau(\mathcal{G}) = \tau$, such that a lensless medial graph $\mathcal{H}$ with*

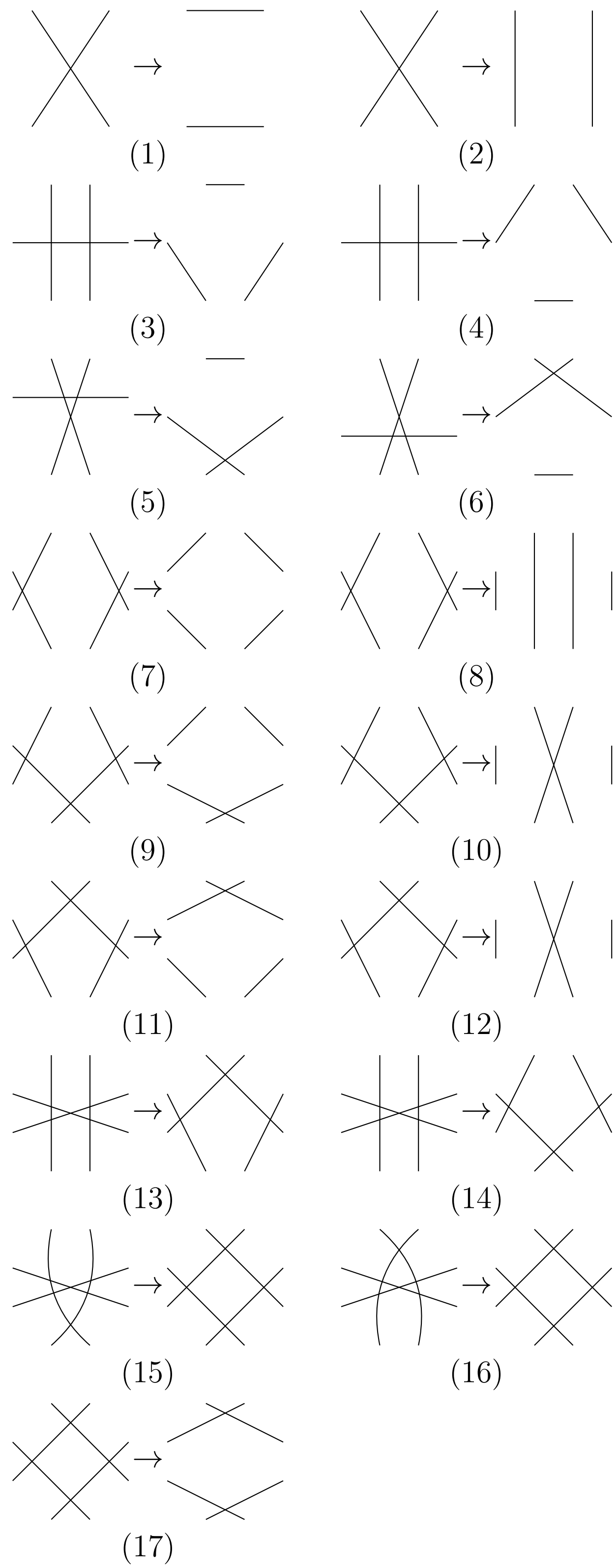


Figure 4.14: Covering Relation for $\mathcal{MP}_n$

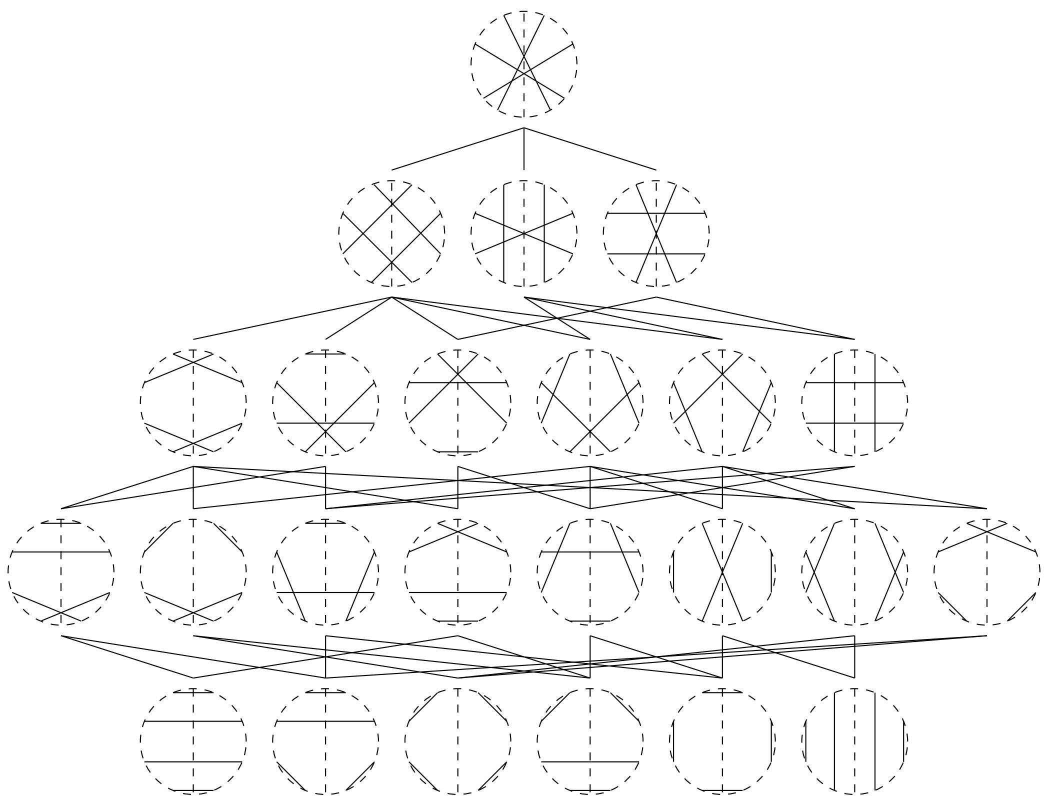


Figure 4.15: Poset on $\mathcal{MP}_2$

$\tau(\mathcal{H}) = \nu$ can be obtained by uncrossing one pair of symmetric intersection points (one intersection point if the point stays on the mirror line) from $\mathcal{G}$.

Proof. First assume that a lensless medial graph $\mathcal{H}$ is obtained from $\mathcal{G}$ by uncrossing one pair of intersection points, and we want to prove τ covers ν. Suppose otherwise. Then there must be a medial graph $\mathcal{G}'$ such that $\tau = \tau(\mathcal{G}) \succ \tau(\mathcal{G}') \succ \tau(\mathcal{H}) = \nu$. But the number of pairs of symmetric crossings in $\mathcal{G}$ is at least two more than that in $\mathcal{H}$, contradicting the hypothesis. Therefore, $\tau(\mathcal{G})$ covers $\tau(\mathcal{H})$.

Now we show the contrapositive of the other direction. Suppose $\mathcal{H}$ is obtained from $\mathcal{G}$ by uncrossing one pair of symmetric intersection points and $\tau(\mathcal{H}) = \nu$, but $\mathcal{H}$ is not lensless. Then we would like to show that there is a lensless medial graph $\mathcal{G}'$ such that $\tau(\mathcal{G}) \succ \tau(\mathcal{G}') \succ \tau(\mathcal{H})$.

Suppose points a, b, c and d lie on a circle in a clockwise direction, and strands a, c and b, d intersect at q, and their mirror image strands a', c' and b', d' intersect at q'. Suppose we uncross mirror-symmetrically the intersection q and q' so that the new strands are a, d and b, c, as well as a', d' and b', c'. Since H is not lensless, then there must be a strand connecting a boundary point between a and b to a boundary point between c and d, and likewise in the mirror-symmetric side. Without loss of generality assume that one such strand intersects the sector aqd. Let $L = \{l_1, l_2, \ldots, l_k\}$ be the set of strands intersecting both aq and dq. We want to prove the following claim:

Claim. We can use mirror symmetric Yang-Baxter transformations to change $\mathcal{G}$ into a medial graph such that the intersection point among strands in L are all outside of the sector aqd.

Let x_i be the intersection of l_i and aq. Let D_i be the closest intersection point to x_i on l_i among all l_j's that have intersection with l_i within the sector aqd. Let D be the set of D_i's. If D is empty, the statement is already true. Otherwise, we pick D_i such that the number r of regions enclosed by l_i, l_j and aq is smallest, where l_j is the strand that intersects with l_i at D_i. We claim $r = 1$. Otherwise, there is another medial strand l_k intersecting the region enclosed by l_i, l_j and aq. For example, if l_k intersects l_i and aq, then the region enclosed by l_i, l_k, aq has strictly fewer number of subregions, contradicting the minimality of r. Hence, $r = 1$.

The same is true for the mirror image. Thus we can use mirror symmetric Yang-Baxter transformation to move the intersection of l_i, l_j as well as its mirror image out of the sector aqd and its mirror image. $|D|$ gets decreased by 1. By induction, we can reduce D to be empty set. We are done with this claim.

Now pick $l \in L$ such that l intersects with aq at the point x closest to q. And say l intersects dq at y. Then given the above claim, we also want to prove the following claim:

Claim. We can use mirror symmetric Yang-Baxter transformations to change $\mathcal{G}$ into a medial graph such that no other medial strand intersects the region xqy.

Let X be the set of medial strands that intersect aq and l. By an argument similar to above, we can show that using mirror symmetric Yang-Baxter moves, any intersection between two medial strands in X can be moved outside of sector aqd. Then we can use mirror symmetric Yang-Baxter move at x and its mirror image to move all medial strands in X out of sector aqd. Now let Y be the set of medial strands that intersect dq and l. With similar argument, we can show that all medial strand in Y can be moved out of sector aqd and its mirror image by mirror symmetric Yang-Baxter move. We finish proving this claim.

With the two claims above, we can assume that $\mathcal{G}$ is lensless, and ac, bd, and l form a triangle where no other medial strand enters, and the same configuration on the mirror symmetric image. We are ready to prove the rest of the theorem.

Say $l = ef$, where e, f are two end points of l, e is between arc ab, and f is between arc cd. If a is not mirror symmetric to d or b is not mirror symmetric to c, then we uncross the intersection of strands ac, ef to get a mirror symmetric medial graph $\mathcal{G}'$ with new strands af, ec. After that uncross the intersection of af, bd to get new strands ad, bf. Lastly uncross the intersection of ec, bf to get new strands ef, bc. And perform the same operation for the mirror image simultaneously. If a is mirror symmetric to d and b is mirror symmetric to c, we uncross the intersections of bd with ef and ac with ef respectively to obtain strands ad, fb, and ce (See Figure 4.14 (5)). Again call this resulting mirror symmetric medial graph $\mathcal{G}'$. Then uncross the intersection between fb and ce to obtain strands bc and fe. At each step, we always get a lensless medial graph. In particular, $\mathcal{G}'$ is lensless. The whole procedure above is equivalent of uncrossing the intersection of strands ac and bd as well as its mirror image to get $\mathcal{H}$, and remove lenses from $\mathcal{H}$. Thus $\tau(\mathcal{G}) \succ \tau(\mathcal{G}') \succ \tau(\mathcal{H})$. $\square$

The above theorem implies that two posets are the same. In the following, we would like to explore the relations between mirror symmetric matching partial order and Bruhat order.

For each $\tau \in \mathcal{MP}_n$, we associate an affine bounded permutation g_τ to τ as follows:

$$g_\tau(i) = \begin{cases} \tau(i) & \text{if } \tau(i) > i \\ \tau(i) + 4n & \text{if } \tau(i) < i \end{cases}$$

where τ is thought of as a fixed point free involution on the set $\{-(2n-1), \ldots, -1, 0, 1, \ldots, 2n\}$. Note g_τ is a bounded affine permutation of type $(2n, 4n)$. And $g_0 := g_{\tau_{\text{top}}}$ is defined as $g_0(i) = i + 2n$ with length 0. Then g_0 plays the role of the identity permutation. Let $t_{a,b}$ denote the transposition swapping a and b. Note we have $t_{i,i+1}g_0 = g_0 t_{i+2n,i+1+2n}$.

Let

$$s_0 = \prod_{k\in\mathbb{Z}} t_{4nk,4nk+1}, \quad s_{2n} = \prod_{k\in\mathbb{Z}} t_{2n+4nk,2n+1+4nk},$$

$$s_i = \prod_{k\in\mathbb{Z}} t_{i+4nk,i+1+4nk} t_{-i+4nk,-i+1+4nk} \text{ for } 1 \le i \le 2n-1.$$

The group of affine permutations of type C, denoted as $\tilde{S}^C_{2n}$, is defined to be the group generated by $\{s_i\}_{i=0}^{2n}$. More precisely, $\tilde{S}^C_{2n}$ is the set of injective maps $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ with the conditions:

$$\sum_{i=-2n+1}^{2n} (f(i) - i) = 0,$$
$$f(i + 4n, j + 4n) = f(i, j),$$
$$f(i, j) = f(-i + 1, -j + 1) \text{ for all } i, j \in \mathbb{Z}.$$

Note that $\tilde{S}^C_{2n}$ is a subgroup of $\tilde{S}^0_{4n}$, and the affine Bruhat order on $\tilde{S}^C_{2n}$ is induced from a subposet of the affine Bruhat order on $\tilde{S}^0_{4n}$. Let l_C be the length function of $\tilde{S}^C_{2n}$. It is clear that each g_τ can be viewed as an element in affine permutations of type C with identity shifted to g_0.

Lemma 4.19. *Let $\tau \in \mathcal{MP}_n$. Then there is $w \in \tilde{S}^C_{2n}$ such that*

$$g_\tau = w g_0 w^{-1}$$

where $l_C(w) = n^2 - mc(\tau)$, and $l_C(g_\tau) = 2l_C(w)$

Proof. The claim is trivial if $\tau = \tau_{top}$, where $g_\tau = g_0$. Suppose that τ is not the top element. Then there exists some i such that $g_\tau(i) > g_\tau(i + 1)$. By symmetry, $g_\tau(-i) > g_\tau(-i + 1)$. Then we can swap $i, i + 1$ and $g_\tau(i), g_\tau(i + 1)$, $-i, -i + 1$ and $g_\tau(-i), g_\tau(-i + 1)$ (all taken modulo $4n$) to obtain τ'. Thus, we have $\tau \lessdot \tau'$, with $g_\tau = s_i g_{\tau'} s_i$, and $l_C(g_\tau) = l_C(g_{\tau'}) + 2$. By induction on $mc(\tau)$, the claim is true. □

The factorization in Lemma 4.19 is not unique. In fact, later we will see that the number of such w is equal to the number of different paths from τ to τ_{top} in $\mathcal{MP}_n$.

Define the (infinite by infinite) affine rank matrix of an affine permutation f by

$$r_f(i, j) = |\{a \in \mathbb{Z} | a \leq i, f(a) \geq j\}|.$$

Note that $r(i,j)$ satisfies $r(i,j) = r(i+4n, j+4n)$.

Theorem 4.20 ([BB], Theorem 8.4.8)**.** *Let $u, v \in \tilde{S}^C_{2n}$. Then $u \leq v$ if and only if $r_u(i,j) \leq r_v(i,j)$ for all $i, j \in \mathbb{Z}$.*

The following theorem identifies the partial order on $\mathcal{MP}_n$ with a subposet of the affine Bruhat order.

Theorem 4.21. *We know $l_C(g_\tau) = 2(n^2 - mc(\tau))$. Thus the map $\tau \to g_\tau$ identifies $\mathcal{MP}_n$ with an induced subposet of the dual Bruhat order of affine permutation of type C. In other words, $\tau' \leq \tau$ if and only if $g_\tau \leq g_{\tau'}$.*

Proof. First assume $\tau' \lessdot \tau$. According to Lemma 4.15, τ' is obtained from τ by uncrossing the intersection points of strands $a \leftrightarrow c$ and $b \leftrightarrow d$, $-a+1 \leftrightarrow -c+1$ and $-b+1 \leftrightarrow -d+1$, and join a to d, b to c, $-a+1$ to $-d+1$, $-b+1$ to $-c+1$ (if the intersection point is on the mirror line, then there will only be one pair of strands) as in Figure 4.14. Then $g_{\tau'} = t_{-a+1,-b+1} t_{a,b} g_\tau t_{a,b} t_{-a+1,1b+1}$ (if the intersection point of $a \leftrightarrow c$ and $b \leftrightarrow d$ not on the mirror line), or $g_{\tau'} = t_{a,b} g_\tau t_{a,b}$ (if the intersection point lies on the mirror line). In both cases, we have that $g_{\tau'} > g_\tau$.

Next assume $g_\tau < g_{\tau.}$. Denote the type A Bruhat order for $\tilde{S}^0_{4n}$ as $\leq_A$. Then there exist $a < b$ such that: (i) $g_{\tau'} >_A t_{-a+1,-b+1} t_{a,b} g_\tau \gtrdot_A t_{a,b} g_\tau \gtrdot_A g_\tau$ (if the strands $a \leftrightarrow c$ and $b \leftrightarrow d$ are not mirror symmetric), or (ii) $g_{\tau'} >_A t_{a,b} g_\tau \gtrdot_A g_\tau$ (if the strands $a \leftrightarrow c$ and $b \leftrightarrow d$ are mirror symmetric). The Case (ii) is corresponding to configuration (1) and (2) in Figure 4.14 and was proved in [Lam], Theorem 4.15. We only focus on Case (i).

For Case (i), let $c := g_\tau(a), d := g_\tau(b)$. Now claim $g_{\tau'} >_A g_\tau t_{a,b} t_{-a+1,-b+1}$. Define the group isomorphism $\phi : \tilde{S}^0_{4n} \to \tilde{S}^0_{4n}$ by $t_{i,i+1} \mapsto t_{i+2n,i+2n+1}$. By Lemma 4.19 , $g_\tau = u g_0 u^{-1}$, $g_{\tau'} = v g_0 v^{-1}$, where $u, v \in \tilde{S}^C_{2n} \subset \tilde{S}^0_{4n}$. Therefore $g_{\tau'} >_A t_{-a+1,-b+1} t_{a,b} g_\tau$ is equivalent to $u g_0 u^{-1} >_A t_{-a+1,-b+1} t_{a,b} v g_0 v^{-1}$, which implies $u\phi(u^{-1}) g_0 >_A t_{-a+1,-b+1} t_{a,b} v \phi(v^{-1}) g_0$. Hence $u\phi(u^{-1}) >_A t_{-a+1,-b+1} t_{a,b} v \phi(v^{-1})$. Taking the inverse of both sides and multiply g_0 on the left side of both terms, we get $g_0\phi(u)u^{-1} >_A g_0\phi(v)v^{-1}t_{a,b}t_{-a+1,-b+1} \iff ug_0u^{-1} >_A vg_0v^{-1}t_{a,b}t_{-a+1,-b+1}$, which is $g_{\tau'} >_A g_\tau t_{a,b}t_{-a+1,-b+1}$.

Let $N = 4n$. We claim that $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1}$. Note that modulo symmetry, the order of (a, b, c, d) has to be one of the configurations in Figure 4.14. For (a, b, c, d) as in configurations (3), (4), (5), (6), (8), (10), (12), (13), (14), (15), and (16), the proofs will be similar. Let's first assume it is configuration (8) of Figure 4.14. Then we have

$$-d+1 < -c+1 < -b+1 < -a+1 < a < b < c < d < -d+N$$

Let R_1 be the rectangle with corners $(-d+1+N, -b+2+N), (-d+1+N, -a+1+N), (-c+N, -b+2+N), (-c+N, -a+1+N)$, R_2 be the one with corners $(-b+1, -d+2+N), (-b+1, -c+1+N), (-a, -d+2+N), (-a, -c+1+N)$, R_3 be the one with corners $(a, c+1), (a, d), (b-1, c+1), (b-1, d)$, and R_4 be the one with corners $(c, a+1+N), (c, b+N), (d-1, a+1+N), (d-1, b+N)$. Then the rank matrix of $t_{-a+1,-b+1}t_{a,b}g_\tau$ will only increase by 1 in the rectangles R_3 and R_4 as well as its periodic shifts. The rank matrix of $g_\tau t_{a,b}t_{-a+1,-b+1}$ will only increase by 1 in the rectangles R_1 and R_2 as well as its periodic shifts. Note that R_1, R_2, R_3, R_4 and their periodic shifts will never intersect, which implies:

$$\begin{aligned} r_{g_{\tau'}}(i,j) &\geq \max(r_{t_{-a+1,-b+1}t_{a,b}g_\tau}(i,j), r_{g_\tau t_{a,b}t_{-a+1,-b+1}}(i,j)) \\ &= r_{t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1}}(i,j) \text{ for all } i,j \in \mathbb{Z} \end{aligned}$$

Hence, $g_{\tau'} >_A t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1}$. On the other hand $t_{-a+1,-b+1}t_{a,b}g_\tau t_{a,b}t_{-a+1,-b+1} = g_{\tau''}$, where $\tau'' < \tau$ is in $\mathcal{MP}_n$ obtained by uncrossing the intersection of strands $a \leftrightarrow c$ and $b \leftrightarrow d$, as well as strands $(-a+1) \leftrightarrow (-c+1)$ and $(-b+1) \leftrightarrow (-d+1)$. Consequently $g_{\tau'} \geq g_{\tau''} > g_\tau$. By induction on $l_C(g_{\tau'}) - l_C(g_\tau)$, we have $\tau' \leq \tau''$. Thus, $\tau' < \tau$.

For (a,b,c,d) as in the rest of the configuration (7), (9), (11), (17), we will use a different argument.

We will use configuration (7) to illustrate the argument, in which case we have:

$$-d+1<-c+1<-b+1<-a+1<a<b<c<d<-d+1+N$$

Let R_1 be the rectangle with corners $(-c+1,-a+2),(-c+1,-d+1+N),(-b,-a+2),(-b,-d+1+N)$, R_2 be the one with corners $(d-N,b+1),(d-N,c),(a-1,b+1),(a-1,c)$, R_3 be the one with corners $(b,d+1),(b,a+N),(c-1,d+1),(c-1,a+N)$, and R_4 be the one with corners $(-a+1,-c+2+N),(-a+1,-b+1+N),(-d+N,-c+2+N),(-d+N,-b+1+N)$. Then the rank matrix of $t_{-a+1,-b+1}t_{a,b}g_\tau$ will only increase by 1 in the rectangles R_1 and R_4 as well as its periodic shifts. The rank matrix of $g_\tau t_{a,b}t_{-a+1,-b+1}$ will only increase by 1 in the rectangles R_2 and R_3 as well as its periodic shifts. However, R_1 and R_2 intersect in the rectangle R_5 with corners $(-c+1,b+1),(-c+1,c),(-b,b+1),(-b,c)$ as well as its periodic shifts, and R_3 and R_4 intersect in the rectangle R_6 with corners $(b,-c+2+N),(b,-b+1+N),(c-1,-c+2+N),(c-1,-b+1+N)$ as well as its periodic shifts. Consider $t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1}$, the entries of the rank matrix of which decrease at regions R_5 and R_6 by 1. Note that $g_{\tau'}>_A t_{-a+1,-b+1}t_{a,b}g_\tau >_A g_\tau$ and $g_{\tau'}>_A g_\tau t_{a,b}t_{-a+1,-b+1}>_A g_\tau$. Therefore

$$r_{t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1}}(i,j)\geq r_{g_\tau}(i,j) \text{ for all } i,j\in\mathbb{Z}.$$

On the other hand $t_{-b+1,-c+1}g_{\tau'}t_{-b+1,-c+1}=g_{\tau''}$ where τ' is obtained from τ'' by uncrossing the intersection of strands $b(-c+1)$ and $(-b+1)c$. Thus, $\tau''>\tau'\in\mathcal{MP}_n$. By induction on $l_C(g_{\tau'})-l_C(g_\tau)$, we have $\tau''\leq\tau$. Thus, $\tau'<\tau$. $\square$

Note that the poset $\mathcal{MP}_n$ has a unique maximum element, and $\binom{2n}{n}$ minimum elements, which is the Catalan number of type B (see [CA]). Let $\widehat{\mathcal{MP}}_n$ denote $\mathcal{MP}_n$ with a minimum $\hat{0}$ adjoined, where we let $mc(\hat{0})=-1$. Recall that a graded poset

P with a unique maximum and a unique minimum, is **Eulerian** if for every interval $[x, y] \in P$ where $x < y$, the number of elements with odd rank in $[x, y]$ is equal to the number of elements with even rank in $[x, y]$. We have the following theorem.

Theorem 4.22. *$\widehat{\mathcal{MP}}_n$ is an Eulerian poset.*

We need some terminology and a few lemmas before proving the above theorem. For a subset $S \subset \widehat{\mathcal{MP}}_n$, we write $\chi(S) = \sum_{\tau\in S}(-1)^{mc(\tau)}$. We need to show that $\chi([\tau, \eta]) = 0$ for all $\tau < \eta$. Recall that $\tilde{S}^C_{2n}$ is denoted as the poset of affine permutation of type C. By Theorem 4.21, we know that there is an injection $\rho : \mathcal{MP} \hookrightarrow \tilde{S}^C_{2n}, \tau \mapsto g_\tau$ such that $\mathcal{MP}_n$ is dual to an induced subposet of $\tilde{S}^C_{2n}$. For $f \subset \tilde{S}^C_{2n}$, let

$$D_L f := \{i \in \mathbb{Z}/2n\mathbb{Z} \mid s_i f < f\}, \qquad D_R f := \{i \in \mathbb{Z}/2n\mathbb{Z} \mid f s_i < f\},$$

be the left and right descent set of f. We have the following lemma.

Lemma 4.23 ([BB], Proposition 2.2.7)**.** *Suppose $f \leq g$ in $\tilde{S}^C_{2n}$.*

- *If $i \in D_L(g)\backslash D_L(f)$, then $f \leq s_i g$ and $s_i f \leq g$,*
- *If $i \in D_R(g)\backslash D_R(f)$, then $f \leq g s_i$ and $f s_i \leq g$.*

For $\tau \in \mathcal{MP}_n$, We label the medial strands of a representative of τ by $\{-2n + 1, \ldots, 0, \ldots, 2n - 1, 2n\}$ and $i \in \{0, 1, 2, \ldots, 2n\}$, let

$$i \in \begin{cases} A(\tau), & \text{if the strands } i \text{ and } i+1 \text{ do not cross, and } -i+1 \text{ and } -i \text{ do not cross} \\ B(\tau), & \text{if the strands } i \text{ and } i+1 \text{ cross, and } -i+1 \text{ and } -i \text{ cross} \\ C(\tau), & \text{if } i \text{ is adjoined with } i+1 \text{, and } -i+1 \text{ is joined with } -i \end{cases}$$

Note that when $i = 0$, and $i = 2n$, the two mirror symmetric pairs of medial strands $\{i, i + 1\}$ and $\{-i + 1, -i\}$ will be the same.

Thus, we have $\{0, 1, 2, \ldots, 2n\} = A(\tau) \cup B(\tau) \cup C(\tau)$, where $\{0, 1, 2, \ldots, 2n\} \in \mathbb{Z} \backslash 4n\mathbb{Z}$. For $s_i \in \tilde{S}^C_{2n}$, $\tau \in \mathcal{MP}_n$, define $s_i \cdot \tau$ such that $g_{s_i \cdot \tau} = s_i g_\tau s_i$. Then by Theorem 4.21 the above conditions translate into the following

$$i \in \begin{cases} A(\tau), & \text{if } s_i g_\tau s_i < g_\tau, \text{ or equivalently } s_i \cdot \tau \gtrdot \tau, \\ B(\tau), & \text{if } s_i g_\tau s_i > g_\tau, \text{ or equivalently } s_i \cdot \tau \lessdot \tau, \\ C(\tau), & \text{if } s_i g_\tau s_i = g_\tau, \text{ or equivalently } s_i \cdot \tau = \tau, \end{cases}$$

For instance, if $i \in B(\tau)$, then $s_i \cdot \tau$ is obtained from τ by uncrossing pairs of medial strands i and $i+1$, and $-i+1$ and $-i$.

Lemma 4.24. *If $\tau < \sigma$, and $i \in A(\tau) \cap B(\sigma)$, then we have $s_i \cdot \tau \leq \sigma$ and $\tau \leq s_i \cdot \sigma$.*

Proof. Since $i \in A(\tau)$, we have $i \in D_R(g_\tau) \cap D_L(g_\tau)$. Similarly, since $i \in B(\sigma)$, we have $i \notin D_R(g_\sigma) \cap D_L(g_\sigma)$. Then we know $i \in D_R(g_\tau) \backslash D_R(g_\sigma)$. Thus, by Lemma 4.23, we have $g_\sigma s_i < g_\tau$. We also know $s_i \notin D_L(g_\sigma s_i)$ because $s_i g_\sigma s_i > g_\sigma s_i$. Again, since $i \in D_L(g_\tau) \backslash D_L(g_\sigma s_i)$, by Lemma 4.23, we have $s_i g_\sigma s_i < g_\tau$, or equivalently $\tau < s_i \cdot \sigma$. Similarly, we can show $s_i \cdot \tau < \sigma$. □

The following lemma is trivially true by the definition of sets $A(\tau)$ and $C(\tau)$.

Lemma 4.25. *If $\tau \leq \sigma$, and $i \in A(\tau)$, then $i \notin C(\sigma)$.*

Proof of Theorem 4.22. We first prove this theorem for the interval $[\tau, \sigma]$ of $\mathcal{MP}_n$, that is, $\tau \neq \hat{0}$. We will prove by descending induction on $mc(\tau) + mc(\sigma)$.

The base case is that σ is the maximum element, and $mc(\tau) = n^2 - 1$. It is trivial. If $mc(\sigma) - mc(\tau) = 1$, it is also clear. Thus, we may assume $mc(\sigma) - mc(\tau) \geq 2$. Since τ is not maximal, $D_L(g_\tau)$ and $D_R(g_\tau)$ are not empty. So we can pick $i \in A(\tau)$. We have the following three cases:

Case 1: If $i \in A(\sigma)$, then we have

$$[\tau, \sigma] = [\tau, s_i \cdot \sigma] \backslash \{\delta \mid \tau < \delta < s_i \cdot \sigma, \delta \not\leq \sigma\}.$$

We claim that

$$\{\delta \mid \tau < \delta < s_i \cdot \sigma, \delta \not\leq \sigma\} = \{\delta \mid s_i \cdot \tau < \delta < s_i \cdot \sigma, \delta \not\leq \sigma\}.$$

Suppose that $\tau \leq \delta \leq s_i\sigma$, and $\delta \not\leq \sigma$. If $i \in A(\delta)$, then because $i \in B(s_i \cdot \sigma)$, we can apply Lemma 4.24 to $\delta < s_i \cdot \sigma$ and get $\sigma < s_i \cdot (s_i \cdot \sigma) = \sigma$, contradiction. On the other hand, $i \notin C(\delta)$ because of $\tau \leq \delta$, $i \in A(\tau)$ and Lemma 4.25. Therefore, we have $i \in B(\delta)$, or equivalently $s_i \cdot \delta \lessdot \delta$. Since $i \in A(\tau) \cap B(\delta)$, applying Lemma 4.24, we get $\tau \leq s_i \cdot \delta$. Thus, we proved the claim.

Furthermore, we have

$$\{\delta \mid s_i \cdot \tau < \delta < s_i \cdot \sigma, \delta \not\leq \sigma\} = [s_i \cdot \tau, s_i \cdot \sigma] \backslash [s_i \cdot \tau, \sigma].$$

By induction, we have $\chi([\tau, \sigma]) = \chi([\tau, s_i \cdot \sigma]) - (\chi([s_i \cdot \tau, s_i \cdot \sigma]) - \chi([s_i \cdot \tau, \cdot \sigma])) = 0$. Note that the assumption $mc(\sigma) - mc(\tau) \geq 2$ implies that none of these intervals will contain only one element.

Case (2): $i \in B(\sigma)$. Let $\delta \in [\tau, \sigma]$. Since $i \in A(\tau)$, apply Lemma 4.25 on $\tau \leq \delta$, we have $i \notin C(\delta)$. Now if we apply Lemma 4.24, we then get $s_i \cdot \delta \in [\tau, \sigma]$. Hence, we construct an involution of elements in $[\tau, \sigma]$, where the parity of the rank of the elements gets swapped. Thus,

$$\chi([\tau, \sigma]) = 0.$$

Case (3): $i \in C(\sigma)$. Since $i \in A(\tau)$, apply Lemma 4.25 on $\tau \leq \sigma$, we know $i \notin C(\sigma)$. So this case is vacuous.

Therefore, we have proved that $\chi([\tau, \sigma]) = 0$ for intervals $\tau < \sigma$ where $\tau \neq \hat{0}$.

Next assume that $\tau = \hat{0}$, and $\tau \leq \sigma$. We can further assume $mc(\sigma) \geq 1$. Thus, $B(\sigma)$ is nonempty. Pick $i \in B(\sigma)$. By applying Lemma 4.24 on $\tau \leq \sigma$, we construct an involution on $\{\delta \in [\hat{0}, \sigma] | \ i \notin C(\delta)\}$. This involution will swap the parity of $mc(\delta)$. Now let $S = \{\delta \in [\hat{0}, \sigma) | \ i \in C(\delta) \text{ or } \delta = \hat{0}\}$. We claim that S is an interval with a unique maximal element. Then we may delete the strands connecting i and $i+1$ and use induction.

We prove the claim by constructing explicitly the maximal element ν. Let G be a representative medial graph of $\sigma \in \mathcal{MP}_n$. The strands l_i and l_{i+1} starting at i and $i+1$ respectively cross each other at some point p since $i \in B(\sigma)$. Similarly, the mirror image strands l_{-i+1} and l_{-i} starting at $-i+1$ and $-i$ respectively will cross at point p'. Assume these strands cross before intersecting with other strands. Let G' be obtained from G by uncrossing p and p' such that i is matched with $i+1$ and $-i+1$ is matched with $-i$. Let ν be the medial pairing of G'.

We need to show that for $\delta \in S$, we have $\delta \leq \nu$. Let H be a lensless medial graph representing δ. Then by Theorem 4.18, H can be obtained from G by uncrossing some subset $\mathcal{A}$ of the mirror symmetric pairs of crossings of G. Because $i \in C(\delta)$, we must have $\{p, p'\} \in \mathcal{A}$. If $\{p, p'\}$ are resolved in H in the same way as in G', then H can be obtained from G' by uncrossing a number of mirror symmetric pairs of crossings, which implies $\delta \leq \nu$. If $\{p, p'\}$ are resolved in H in the different direction to the one in G', then let H' be obtained from H by uncrossing the pair $\{p, p'\}$ of mirror symmetric crossing in another direction. We observe that H' and H both represent δ. H' may contain a closed loop in the interior which can be removed. Thus, we complete the proof of $S = [\hat{0}, \sigma]$, and consequently the theorem. □

Similar to the case of cactus networks, we conjecture that the partial order on $\mathcal{MP}_n$ is also the closure partial order of the decomposition $ME_n = \bigsqcup_{\tau \in \mathcal{MP}_n} ME_\tau$:

Conjecture 4.26.

$$ME_{\tau} = \bigsqcup_{\tau' \leq \tau} ME_{\tau}.$$

Remark 4.27. Lam [Lam] found an injection from the space of cactus networks E_n to the nonnegative part of Grassmannian $Gr(n-1, 2n)$, where the stratification of E_n coincides with the stratification of positive part of $Gr(n-1, 2n)$ intersecting with a projective plane. A possible approach of proving the above conjecture is to use this injection and look more closely at the subspace of the Grassmannian where the Plücker coordinates possess some symmetry.

CHAPTER 5

Conclusion and Future Work

In this thesis, among classical types we have only found the explicit structure of electrical Lie algebras $\mathfrak{e}_{A_n}$, $\mathfrak{e}_{B_n}$, and $\mathfrak{e}_{C_{2n}}$. Among the exceptional Lie types, only $\mathfrak{e}_{G_2}$ has been studied by Lam-Pylyavskyy [LP]. In order to finish the classification theory of electrical Lie algebras of all finite types, we still need to study the structure of $\mathfrak{e}_{C_{2n+1}}$, $\mathfrak{e}_{D_n}$, $\mathfrak{e}_{E_6}$, $\mathfrak{e}_{E_7}$, $\mathfrak{e}_{E_8}$, and $\mathfrak{e}_{F_4}$.

On the other hand, in the definition of electrical Lie algebra of finite type, when $|i-j|=1$, if we replace the relation $[e_i,[e_i,e_j]]=-2e_i$ by $[e_i,[e_i,e_j]]=-\alpha e_i$, we can define a one parameter family of electrical Lie algebra $\mathfrak{e}_{X_\alpha}$, where $\alpha>0$. It is not hard to see that by rescaling the generators, we have $\mathfrak{e}_{X_\alpha}\cong\mathfrak{e}_X$. When α goes to 0, the upper half of the ordinary semisimple Lie algebra can be seen as a direct limit of electrical Lie algebras. More detailed connection between ordinary semisimple Lie algebra and electrical Lie algebra of finite type may be extracted from this observation.

As noted in the previous paragraph, at this point, we can only draw the analogy between electrical Lie algebras of finite type with upper half of ordinary semisimple Lie algebras. It would be very interesting to see the analogy of "lower half" and "diagonal" of electrical Lie algebras with the remaining part of the ordinary semisimple Lie algebras.

On the level of electrical networks, we have successfully found an appropriate class of electrical networks, that is, mirror symmetric circular planar electrical networks for type B electrical Lie theory. Furthermore, we have studied extensively the properties of the space of such networks (more generally the cactus network). In a recent research project of the author and Rachel Karpman, we manage to draw connections between positive Lagrangian Grassmannian and the positroids varieties with some symmetry, which may help proving Conjecture 4.26.

In the future, we would like to find the class of electrical networks of other finite types. So far, some effort has been made in the type C and type D.

APPENDIX

APPENDIX A

Facts and Proofs of Lemmas in Chapter 2

A.1 Facts and Proofs for Type C in 1.3

The following lemma describes how the generators e_i act on the spanning set of $\mathfrak{c}_{C_n}$.

Lemma A.1. *Bracket e_k with the elements in the spanning set: $[e_k, [e_i[\ldots[e_{j-1}e_j]\ldots]]$: If $j = i$ and $j = i+1$ it is given in the defining relation; if $k < i-1$ or $k > j+1$, $[e_k, [e_i[\ldots[e_{j-1}e_j]\ldots]] = 0$; if $j \geq i+2$ and $i-1 \leq k \leq j+1$:*

(1) $j = i+2$

$$[e_{i-1}, [e_i[e_{i+1}e_{i+2}]]] = [e_{i-1}[\ldots[e_{i+1}e_{i+2}]\ldots],$$

$$[e_i, [e_i[e_{i+1}e_{i+2}]]] = \begin{cases} 0 & \text{if } i \neq 1, \\ [e_1[e_1[e_2e_3]]] & \text{if } i = 1, \end{cases}$$

$$[e_{i+1}, [e_i[e_{i+1}e_{i+2}]]] = -[e_ie_{i+1}] + [e_{i+1}e_{i+2}],$$

$$[e_{i+2}, [e_i[e_{i+1}e_{i+2}]]] = 0,$$

$$[e_{i+3}, [e_i[e_{i+1}e_{i+2}]]] = -[e_i[e_{i+1}[e_{i+2}e_{i+3}]]].$$

(2) $j \geq i+3$

$$[e_{i-1},[e_i[\ldots[e_{j-1}e_j]\ldots]] = [e_{i-1}[\ldots[e_{j-1}e_j]\ldots],$$

$$[e_i,[e_i[\ldots[e_{j-1}e_j]\ldots]] = \begin{cases} 0 & \text{if } i \neq 1, \\ [e_1[e_1[\ldots[e_{j-1}e_j]\ldots] & \text{if } i = 1, \end{cases}$$

$$[e_{i+1},[e_i[\ldots[e_{j-1}e_j]\ldots]] = [e_{i+1}[\ldots[e_{j-1}e_j]\ldots],$$

$$[e_k,[e_i[\ldots[e_{j-1}e_j]\ldots]] = 0 \text{ if } i+2 \leq k \leq j-2 \ ,$$

$$[e_{j-1},[e_i[\ldots[e_{j-1}e_j]\ldots]] = -[e_i[\ldots[e_{j-2}e_{j-1}]\ldots],$$

$$[e_j,[e_i[\ldots[e_{j-1}e_j]\ldots]] = 0,$$

$$[e_{j+1},[e_i[\ldots[e_{j-1}e_j]\ldots] = -[e_i[\ldots[e_je_{j+1}]\ldots].$$

$[e_k,[e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]]$, *where* $j \geq i+1$*:*

If $k \geq j+2$, $[e_k,[e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots] = 0$. *If* $k \leq j+1$*:*

(3) $j \geq i+2$

$$[e_{j+1},[e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = -[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots],$$

$$[e_j,[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = 0,$$

$$[e_{j-1},[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = -[e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_{j-3}[e_{j-2}e_{j-1}]\ldots],$$

$$[e_k,[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = 0 \text{ if } i+2 \leq k \leq j-2,$$

$$[e_{i+1},[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = [e_{i+1}[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots],$$

$$[e_i,[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = 0,$$

$$[e_{i-1},[e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = [e_{i-1}[e_{i-2}[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots],$$

$$[e_k,[e_i[\ldots[e_1[e_1\ldots[e_{j-1}e_j]\ldots]] = 0 \text{ if } k \leq i-2.$$

(4) $j = i + 1$

$$[e_{i+2}, [e_i[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = -[e_i[\ldots[e_1[e_1\ldots[e_{i+1}e_{i+2}]\ldots],$$

$$[e_{i+1}, [e_i[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = \begin{cases} [e_{i-1}[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots] & \text{if } i \geq 2, \\ 2[e_1e_2] & \text{if } i = 1, \end{cases}$$

$$[e_i, [e_i[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = 0,$$

$$[e_{i-1}, [e_i[e_{i-1}[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = [e_{i-1}[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots],$$

$$[e_{i-2}, [e_i[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = -[e_{i-2}[\ldots[e_1[e_1\ldots[e_{i-1}e_i]\ldots],$$

$$[e_k, [e_i[\ldots[e_1[e_1\ldots[e_ie_{i+1}]\ldots]] = 0 \ \text{if } k \leq i-3.$$

Proof. The above identities can be achieved by the straightforward but lengthy computation. We will omit it here. □

Proof A.2 (of Lemma 2.10)**.**

We would like to show $[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots],[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]] = 0$, where $i_1 < j_1, i_2 < j_2$,and $j_1, j_2 \geq 3$. This will be done by induction on $i_1 + j_1 + i_2 + j_2$. Let $i_k + j_k$ be the *length* of $[e_{i_k}[\ldots[e_1[e_1\ldots[e_{j_k-1}e_{j_k}]\ldots]$ in S. First, we show a few base cases which will be used shortly.

Base cases:

1. $[[e_1[e_1[e_2[e_3e_4]]]], [e_1[e_1[e_2e_3]]]] = 0$.

   First of all

$$\begin{aligned} & [[e_2e_3], [e_1[e_1[e_2[e_3e_4]]]]] \\ = & [[e_2[e_1[e_1[e_2[e_3e_4]\ldots], e_3] + [e_2[e_3[e_1[e_1[e_2[e_3e_4]\ldots] \\ = & -[e_3[e_2[e_1[e_1[e_2[e_3e_4]\ldots] - [e_2[e_1[e_1[e_2e_3]]]], \end{aligned}$$

so

$$
\begin{aligned}
&[[e_1[e_1[e_2[e_3e_4]]]], [e_1[e_2e_3]]] \\
&= [[e_2e_3], [e_1[e_1[e_1[e_2[e_3e_4] \ldots]] - [e_1, [[e_2e_3][e_1[e_1[e_2[e_3e_4] \ldots]]] \\
&= [e_1[e_3[e_2[e_1[e_1[e_2[e_3e_4] \ldots] + [e_1[e_2[e_1[e_1[e_2e_3] \ldots] \\
&= - [e_1[e_1[e_2e_3]]] + [e_1[e_1[e_2e_3]]] \\
&= 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&[[e_1[e_1[e_2[e_3e_4]]]], [e_1[e_1[e_2e_3]]]] \\
&= [[e_1[e_2e_3]], [e_1[e_1[e_1[e_2[e_3e_4] \ldots]] - [e_1, [[e_1[e_2e_3]][e_1[e_1[e_2[e_3e_4] \ldots]]] \\
&= 0.
\end{aligned}
$$

2. $[[e_1[e_1[e_2e_3]]], [e_2[e_1[e_1[e_2e_3]]]]] = 0.$

   First we compute

$$
\begin{aligned}
&[[e_2e_3], [e_1[e_1[e_2e_3]]]] \\
&= - [e_3[e_2[e_1[e_1[e_2e_3] \ldots] + [e_2[e_3[e_1[e_1[e_2e_3] \ldots] \\
&= - [e_1[e_1[e_2e_3]]],
\end{aligned}
$$

and

$$
\begin{aligned}
&[[e_1[e_2e_3]],[e_1[e_1[e_2e_3]]]]\\
&= -\,[[e_3[e_1e_2]],[e_1[e_1[e_2e_3]]]]\\
&= [[e_1e_2],[e_3[e_1[e_1[e_2e_3]]]]] - [e_3,[[e_1e_2][e_1[e_1[e_2e_3]]]]]\\
&= 0 - [e_3[e_1[e_1[e_2e_3]]]]\\
&= 0.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&[[e_2[e_1[e_1[e_2e_3]]]],[e_1[e_2e_3]]]\\
&= [[e_2[e_1[e_2e_3]]],[e_1[e_1[e_2e_3]]]] + [[[e_1[e_2e_3]][e_1[e_1[e_2e_3]]]],e_2]\\
&= -\,[[e_1e_2][e_1[e_1[e_2e_3]]]] + [[e_2e_3],[e_1[e_1[e_2e_3]]]] + 0\\
&= -\,2[e_1[e_1[e_2e_3]]].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&[[e_1[e_1[e_2e_3]]],[e_2[e_1[e_1[e_2e_3]]]]]\\
&= [[e_1[e_2[e_1[e_1[e_2e_3]\ldots],[e_1[e_2e_3]]] + [[[e_2[e_1[e_1[e_2e_3]]]][e_1[e_2e_3]]],e_1]\\
&= [[e_1[e_1[e_2e_3]]],[e_1[e_2e_3]]] + 2[e_1[e_1[e_1[e_2e_3]]]]\\
&= 0.
\end{aligned}
$$

3. $[[e_1[e_1[e_2e_3]]],[e_2[e_1[e_1[e_2[e_3e_4]\ldots]] = 0.$

First we compute

$$
\begin{aligned}
&[[e_2e_3],[e_2[e_1[e_1[e_2[e_3e_4]\ldots]]\\
&=[[e_2[e_2[e_1[e_1[e_2[e_3e_4]\ldots],e_3]+[e_2[e_3[e_2[e_1[e_1[e_2[e_3e_4]\ldots]\\
&=0+[e_2[e_1[e_1[e_2[e_3e_4]\ldots]\\
&=[e_2[e_1[e_1[e_2[e_3e_4]\ldots].
\end{aligned}
$$

Then

$$
\begin{aligned}
&[[e_1[e_2e_3]],[e_2[e_1[e_1[e_2[e_3e_4]\ldots]]\\
&=[e_1,[[e_2e_3][e_2[e_1[e_1[e_2[e_3e_4]\ldots]]]-[[e_2e_3],[e_1[e_2[e_1[e_1[e_2[e_3e_4]\ldots]]\\
&=[e_1[e_2[e_1[e_1[e_2[e_3e_4]\ldots]-\underbrace{[[e_2e_3],[e_1[e_1[e_2[e_3e_4]]]]]}_{\text{by an equation in case (1)}}\\
&=[e_1[e_1[e_2[e_3e_4]]]]+[e_3[e_2[e_1[e_1[e_2[e_3e_4]\ldots]+[e_2[e_1[e_1[e_2e_3]\ldots].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&[[e_1[e_1[e_2e_3]]],[e_2[e_1[e_1[e_2[e_3e_4]\ldots]]\\
&=[[e_1[e_2[e_1[e_1[e_2[e_3e_4]\ldots],[e_1[e_2e_3]]]\\
&\quad+[e_1,[[e_1[e_2e_3]][e_2[e_1[e_1[e_2[e_3e_4]\ldots]]]\\
&=\underbrace{[[e_1[e_1[e_2[e_3e_4]]]],[e_1[e_2e_3]]]}_{\text{by an equation in case (1)}}+[e_1[e_1[e_1[e_2[e_3e_4]\ldots]\\
&\quad+[e_1[e_3[e_2[e_1[e_1[e_2[e_3e_4]\ldots]+[e_1[e_1[e_1[e_2e_3]\ldots]\\
&=0+0-[e_1[e_1[e_2e_3]]]+[e_1[e_1[e_2e_3]]]\\
&=0.
\end{aligned}
$$

This finishes the base case we will use. Next we proceed to induction step. Up to symmetry, there are three cases: (i) $i_1 \le i_2 < j_2 \le j_1$, (ii) $i_1 \le i_2 \le j_1 < j_2$, where two equalities cannot achieve at the same time, (iii) $i_1 < j_1 \le i_2 < j_2$. We will use underbrace to indicate where induction hypothesis is used.

(i) If $i_1 \le i_2 < j_2 \le j_1$

- If $i_1 > 1$

$$
\begin{aligned}
&[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots], [e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]] \\
=\ &[\underbrace{[e_{i_1}[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]}_{\text{length decreases}}, [e_{i_1-1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]] \\
&+[\underbrace{[[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots], [e_{i_1-1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]]}, e_{i_1}] \\
=\ &0.
\end{aligned}
$$

- If $i_1 = 1$,and $j_1 - j_2 \ge 2$

$$
\begin{aligned}
&[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots], [e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]] \\
=&-[[e_{j_1}[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-2}e_{j_1-1}]\ldots]], [e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots] \\
=&-[\underbrace{[e_{j_1}[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]}_{\text{equals } 0}, [e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-2}e_{j_1-1}]\ldots]] \\
&-[\underbrace{[[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots][e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1-1}e_{j_1-1}]\ldots]]}_{\text{length decreases and } j_1-1 \ge j_2+1 \ge 3}, e_{j_1}] \\
=\ &0.
\end{aligned}
$$

- If $i_1 = 1$, $j_1 - j_2 = 1$, $j_2 - i_2 \geq 2$, and $j_2 \geq 4$

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&-[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{j_2}[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-2}e_{j_2-1}]\dots]]\\
=&-[[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-2}e_{j_2-1}]\dots],\underbrace{[e_{j_2}[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots]]}_{\text{length decreases}}\\
&-[e_{j_2},\underbrace{[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots][e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-2}e_{j_2-1}]\dots]]]}\\
=&\,0.
\end{aligned}
$$

- If $i_1 = 1$, $j_1 - j_2 = 1$, $j_2 - i_2 \geq 2$, and $j_2 = 3$, then we have to have $i_1 = 1, i_2 = 1, j_1 = 4, j_2 = 3$, this is base case (1).
- If $i_1 = 1$, $j_1 - j_2 = 1$, $j_2 - i_2 = 1$, and $i_2 \geq 3$, then

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&\,[\underbrace{[e_{i_2-1}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]}_{\text{length decreases}},\underbrace{[e_{i_2}[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots]]}_{\text{length decreases}}\\
&+[e_{i_2},\underbrace{[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots][e_{i_2-1}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]]}\\
=&\,0.
\end{aligned}
$$

- If $i_1 = 1$, $j_1 - j_2 = 1$, $j_2 - i_2 = 1$, and $i_2 = 2$, then $i_1 = 1, i_2 = 2, j_1 = 4, j_2 = 3$

$$
\begin{aligned}
&[e_1[e_1[e_2[e_3e_4]\dots],[e_2[e_1[e_1[e_2e_3]\dots]]\\
=&\,\underbrace{[[e_1[e_1[e_2e_3]]],[e_2[e_1[e_1[e_2[e_3e_4]\dots]]}_{\text{base case (3)}}+[e_2,\underbrace{[[e_1[e_1[e_2[e_3e_4]]]][e_1[e_1[e_2e_3]]]]]}_{\text{base case (1)}}\\
=&\,0.
\end{aligned}
$$

- If $i_1 = 1$, $j_1 - j_2 = 1$, $j_2 - i_2 = 1$, and $i_2 = 1$, then $j_2 = 2$. This case does not occur.

- If $i_1 = 1$, $j_1 = j_2 \geq 4$,

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-1}e_{j_1}]\dots], [e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-1}e_{j_2}]\dots]] \\
=& - [[e_{j_1}[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-2}e_{j_1-1}]\dots], [e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-1}e_{j_2}]\dots]] \\
=& - [\underbrace{[e_{j_1}[e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-1}e_{j_2}]\dots]}_{\text{length decrease}}, [e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-2}e_{j_1-1}]\dots]] \\
& - [\underbrace{[[e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-1}e_{j_2}]\dots][e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-2}e_{j_1-1}]\dots]]}, e_{j_1}] \\
=& \; 0.
\end{aligned}
$$

- If $i_1 = 1$, $j_1 = j_2 = 3$, then $i_2 = 1$ or 2. If $i_2 = 1$, it is trivially true. If $i_2 = 2$, this is the base case (2).

(ii) If $i_1 \leq i_2 \leq j_1 < j_2$

- If $j_2 - j_1 \geq 2$

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-1}e_{j_1}]\dots], [e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-1}e_{j_2}]\dots]] \\
=& - [[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-1}e_{j_1}]\dots], [e_{j_2}[e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-2}e_{j_2-1}]\dots]] \\
=& - [e_{j_2}, \underbrace{[[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-1}e_{j_1}]\dots][e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-2}e_{j_2-1}]\dots]]}] \\
& - [[e_{i_2}[\dots[e_1[e_1 \dots [e_{j_2-2}e_{j_2-1}]\dots], \underbrace{[e_{j_2}[e_{i_1}[\dots[e_1[e_1 \dots [e_{j_1-1}e_{j_1}]\dots]}_{\text{equals } 0}] \\
=& \; 0.
\end{aligned}
$$

- If $j_2 - j_1 = 1$, and $j_1 - i_2 \geq 2$

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&-[[e_{j_1}[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-2}e_{j_1-1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&-[\underbrace{[e_{j_1}[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]}_{\text{length decreases}},[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-2}e_{j_1-1}]\dots]\\
&-[\underbrace{[[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots][e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-2}e_{j_1-1}]\dots]]}, e_{j_1}]\\
=&\,0.
\end{aligned}
$$

- If $j_2 - j_1 = 1$, $j_1 - i_2 = 1$, and $i_2 - i_1 \geq 2$ or $j_2 - j_1 = 1$, and $j_1 = i_2$

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&[[e_{i_2-1}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots],\underbrace{[e_{i_2}[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots]]}_{\text{length decreases}}\\
&+[e_{i_2},\underbrace{[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots][e_{i_2-1}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]}]\\
=&\,0.
\end{aligned}
$$

- If $j_2 - j_1 = 1$, $j_1 - i_2 = 1$, $i_2 - i_1 = 1$, and $i_1 > 1$

$$
\begin{aligned}
&[[e_{i_1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots],[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]]\\
=&[\underbrace{[e_{i_1}[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots]}_{\text{length decreases}},[e_{i_1-1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots]\\
&+[\underbrace{[[e_{i_2}[\dots[e_1[e_1\dots[e_{j_2-1}e_{j_2}]\dots][e_{i_1-1}[\dots[e_1[e_1\dots[e_{j_1-1}e_{j_1}]\dots]]}, e_{i_1}]\\
=&\,0.
\end{aligned}
$$

- If $j_2 - j_1 = 1$, $j_1 - i_2 = 1$, $i_2 - i_1 = 1$, and $i_1 = 1$, then $j_1 = 3, i_2 = 2, j_2 = 4$. This is base case (3).

(iii) If $i_1 < j_1 \le i_2 < j_2$

- If $i_1 > 1$

$$
\begin{aligned}
&[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots][e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]\\
&= [\underbrace{[e_{i_1}[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]}_{\text{length decreases}}, [e_{i_1-1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]]\\
&\quad + [\underbrace{[[e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots][e_{i_1-1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]]}, e_{i_1}]\\
&= 0.
\end{aligned}
$$

- If $i_1 = 1$, $i_2 \ge j_1 + 2$ or $i_2 = j_1$

$$
\begin{aligned}
&[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots], [e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]\\
&= [[e_{i_2-1}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots], \underbrace{[e_{i_2}[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]]}_{\text{equals } 0}]\\
&\quad + [e_{i_2}, \underbrace{[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots][e_{i_2-1}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]}]\\
&= 0.
\end{aligned}
$$

- If $i_1 = 1$, $i_2 = j_1 + 1$

$$
\begin{aligned}
&[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots], [e_{i_2}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]\\
&= [[e_{i_2-1}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots], [e_{i_2}[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots]]\\
&\quad + [e_{i_2}, \underbrace{[[e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1-1}e_{j_1}]\ldots][e_{i_2-1}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots]]}]\\
&= -\underbrace{[[e_{i_2-1}[\ldots[e_1[e_1\ldots[e_{j_2-1}e_{j_2}]\ldots], [e_{i_1}[\ldots[e_1[e_1\ldots[e_{j_1}e_{j_1+1}]\ldots]]}_{\text{this is case (ii)}}\\
&= 0.
\end{aligned}
$$

This completes the proof. Therefore $[I', I'] = 0$. $\square$

Proof A.3 (Proof of Lemma 2.11)**.**

Based on identities in Lemma A.1

$$
\begin{aligned}
[[e_1e_2],[e_1[e_1[e_2e_3]]]] &= [[e_1[e_1[e_1[e_2e_3]]]],e_2] + [e_1[e_2[e_1[e_1[e_2e_3]\ldots] \\
&= [e_1[e_1[e_2e_3]]], \\
[[e_3e_4],[e_1[e_1[e_2e_3]]]] &= [[e_3[e_1[e_1[e_2e_3]]]],e_4] + [e_3[e_4[e_1[e_1[e_2e_3]\ldots] \\
&= [e_1[e_1[e_2e_3]]].
\end{aligned}
$$

If $k \geq 5$, then e_k is commute with e_1, e_2, e_3, so we have $[[e_{2i+1}[\ldots[e_{j-1}e_j]\ldots],[e_1[e_1[e_2e_3]]]] = 0$. Now if $i = 1$, then

$$
\begin{aligned}
[[e_1[e_1[e_2e_3]]],[e_3[\ldots[e_{j-1}e_j]\ldots]] &= [e_3,[[e_1[e_1[e_2e_3]]][e_4[\ldots[e_{j-1}e_j]\ldots]]] \\
&= [e_3,[e_1[e_1[\ldots[e_{j-1}e_j]\ldots]] \\
&= 0.
\end{aligned}
$$

By the base case (2) of Proof A.2, we know $[[e_1[e_1[e_2e_3]]],[e_1[e_2e_3]]] = 0$. As for $j = 4$,

$$
\begin{aligned}
&[[e_1[e_1[e_2e_3]]],[e_1[e_2[e_3e_4]]]] \\
&= -[[e_1[e_2e_3]],[e_1[e_1[e_2[e_3e_4]]]]]] + [e_1,[[e_1[e_2e_3]][e_1[e_2[e_3e_4]]]]] \\
&= ([e_1[e_1[e_2e_3]]] - [e_1[e_1[e_2e_3]]]) + ([e_1[e_1[e_2e_3]]] - [e_1[e_1[e_2e_3]]]) \\
&= 0.
\end{aligned}
$$

Now assume $j \geq 5$. We first consider the following two brackets.

$$\begin{aligned}
[[e_1[e_2e_3]],[e_1[e_1[\ldots[e_{j-1}e_j]\ldots]] &= [e_1,[[e_2e_3][e_1[e_1[\ldots[e_{j-1}e_j]\ldots]]] \\
&= -[e_1[e_3[e_2[e_1[e_1[\ldots[e_{j-1}e_j]\ldots] \\
&= 0, \\
[[e_1[e_2e_3]],[e_1[\ldots[e_{j-1}e_j]\ldots]] &= -[[e_2e_3],[e_1[e_1[\ldots[e_{j-1}e_j]\ldots]] \\
&\quad + [e_1,[[e_2e_3][e_1[\ldots[e_{j-1}e_j]\ldots]]] \\
&= [e_3[e_2[e_1[e_1[\ldots[e_{j-1}e_j]\ldots].
\end{aligned}$$

So,

$$\begin{aligned}
[[e_1[e_1[e_2e_3]]],[e_1[\ldots[e_{j-1}e_j]\ldots]] &= -[[e_1[e_2e_3]],[e_1[e_1[\ldots[e_{j-1}e_j]\ldots]] \\
&\quad + [e_1,[[e_1[e_2e_3]][e_1[\ldots[e_{2j-1}e_{2j}]\ldots]]] \\
&= 0.
\end{aligned}$$

$\square$

Proof A.4 (of Lemma 2.12)**.**

Clearly, $[e_1, c] = 0$. Now consider $[e_{2k+1}, c]$ for $k \geq 1$. There are only three summands in the right hand side of the equation which may contribute nontrivial commutators, that is, $i = k-1, k$ or $k+1$. Based on our case (3) and (4).

If $i = k-1$, we have the commutator equals

$$-(n-k+1)[e_{2k-1}[\ldots[e_1[e_1\ldots[e_{2k}e_{2k+1}]\ldots]+\sum_{j=1}^{k-1}(-1)^{j+k-1}[e_{2j-1}[\ldots[e_1[e_1\ldots[e_{2k}e_{2k+1}]\ldots].$$

If $i = k$, we have the commutator equals

$$(n-k)[e_{2k-1}[\ldots[e_1[e_1\ldots[e_{2k}e_{2k+1}]\ldots] + \sum_{j=1}^{k}(-1)^{j+k}[e_{2j-1}[\ldots[e_1[e_1\ldots[e_{2k}e_{2k+1}]\ldots].$$

If $i = k+1$, we have the commutator equals

$$(n-k-1)[e_{2k+1}[\ldots[e_1[e_1\ldots[e_{2k+2}e_{2k+3}]\ldots]-(n-k-1)[e_{2k+1}[\ldots[e_1[e_1\ldots[e_{2k+2}e_{2k+3}]\ldots].$$

The sum of these three terms is 0. Therefore, $[e_{2k+1}, c] = 0$ for all $k \geq 0$.

As for the even case: $[e_2, c] = 2n[e_2, e_1]+n[e_2, [e_1[e_1e_2]]] = -2n[e_1e_2]+2n[e_1e_2] = 0$. For $k > 1$, similarly we have the following nontrivial case:

If $i = k-1$, we have the commutator equals

$$-(n-k+1)[e_{2k-2}[\ldots[e_1[e_1\ldots[e_{2k-1}e_{2k}]\ldots]+(n-k+1)[e_{2k-2}[\ldots[e_1[e_1\ldots[e_{2k-1}e_{2k}]\ldots].$$

If $i = k$, we have the commutator equals

$$(n-k)[e_{2k}[\ldots[e_1[e_1\ldots[e_{2k+1}e_{2k+2}]\ldots] - [e_{2k}[\ldots[e_1[e_1\ldots[e_{2k+1}e_{2k+2}]\ldots].$$

If $i = k+1$, we have the commutator equals

$$-(n-k-1)[e_{2k}[\ldots[e_1[e_1\ldots[e_{2k+1}e_{2k+2}]\ldots].$$

If $i \geq k+2$, we have the commutator equals

$$(-1)^{i+k-1}[e_{2k}[\ldots[e_1[e_1\ldots[e_{2k+1}e_{2k+2}]\ldots] + (-1)^{i+k}[e_{2k}[\ldots[e_1[e_1\ldots[e_{2k+1}e_{2k+2}]\ldots].$$

The sum of above terms is 0. Hence, $[e_{2k}, c] = 0$ for all $k \geq 1$. This shows c is in the center of $\mathfrak{e}_{C_{2n}}$. $\square$

Proof A.5 (of Lemma 2.13)**.**

We prove by induction on (1) and (2). It is clear $\phi([\bar{e}_1, \bar{e}_2]) = \left(\begin{smallmatrix} E_{11} & 0 \\ 0 & -E_{11} \end{smallmatrix}\right)$, and $\phi([\bar{e}_3, \bar{e}_4]) = \left(\begin{smallmatrix} E_{12}+E_{22} & 0 \\ 0 & -E_{21}-E_{22} \end{smallmatrix}\right)$. So $\phi([\bar{e}_1[\bar{e}_2[\bar{e}_3\bar{e}_4]]]) = \phi([[\bar{e}_1, \bar{e}_2][\bar{e}_3, \bar{e}_4]]) = \left(\begin{smallmatrix} E_{12} & 0 \\ 0 & -E_{21} \end{smallmatrix}\right)$. This gives us the base case of (1) and (2). Now suppose (1) is true for $k-1$ and (2) is true for k. Because we know $\phi([\bar{e}_{2k-1}, \bar{e}_{2k}]) = \left(\begin{smallmatrix} E_{(k-1)k}+E_{kk} & 0 \\ 0 & -E_{k(k-1)}-E_{kk} \end{smallmatrix}\right)$, we have that $\phi([\bar{e}_{2k+1}, \bar{e}_{2k+2}]) = \left(\begin{smallmatrix} E_{k(k+1)}+E_{(k+1)(k+1)} & 0 \\ 0 & -E_{(k+1)k}-E_{(k+1)(k+1)} \end{smallmatrix}\right)$. So

$$
\begin{aligned}
&\begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix} \\
&= \begin{pmatrix} E_{(k-1)k} + E_{kk} & 0 \\ 0 & -E_{k(k-1)} - E_{kk} \end{pmatrix} - \begin{pmatrix} E_{(k-1)k} & 0 \\ 0 & -E_{k(k-1)} \end{pmatrix} \\
&= \phi([\bar{e}_{2k-1}, \bar{e}_{2k}]) - \phi([\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots] - \ldots + (-1)^k[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]) \\
&= \phi([\bar{e}_{2k-1}, \bar{e}_{2k}] - [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots] + \ldots + (-1)^{k-1}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\begin{pmatrix} E_{k(k+1)} & 0 \\ 0 & -E_{(k+1)k} \end{pmatrix} \\
&= \left[\begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix}, \begin{pmatrix} E_{k(k+1)} + E_{(k+1)(k+1)} & 0 \\ 0 & -E_{(k+1)k} - E_{(k+1)(k+1)} \end{pmatrix} \right] \\
&= [\phi([\bar{e}_{2k-1}, \bar{e}_{2k}] + \ldots + (-1)^{k-1}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots]), \phi([\bar{e}_{2k+1}, \bar{e}_{2k+2}])] \\
&= \phi([[\bar{e}_{2k-1}, \bar{e}_{2k}] + \ldots + (-1)^{k-1}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}]\ldots], [\bar{e}_{2k+1}, \bar{e}_{2k+2}]]) \\
&= \phi([\bar{e}_{2k-1}[\ldots, [\bar{e}_{2k+1}\bar{e}_{2k+2}]\ldots] + \ldots + (-1)^{k+1}[\bar{e}_1[\ldots, [\bar{e}_{2k}\bar{e}_{2k+2}]\ldots]).
\end{aligned}
$$

This proves (1) and (2).

For (3)

$$\begin{aligned}
&\begin{pmatrix} E_{(k+1)k} & 0 \\ 0 & -E_{k(k+1)} \end{pmatrix} \\
&= \begin{pmatrix} E_{(k+1)k} + E_{kk} & 0 \\ 0 & -E_{k(k+1)} - E_{kk} \end{pmatrix} - \begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix} \\
&= \phi([\bar{e}_{2k+1}, \bar{e}_{2k}]) \\
&\quad - \phi([\bar{e}_{2k-1}, \bar{e}_{2k}] - [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}] \ldots] + \ldots + (-1)^{k-1}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}] \ldots]) \\
&= \phi([\bar{e}_{2k+1}, \bar{e}_{2k}] \\
&\quad - [\bar{e}_{2k-1}, \bar{e}_{2k}] + [\bar{e}_{2k-3}[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}] \ldots] - \ldots + (-1)^{k}[\bar{e}_1[\ldots, [\bar{e}_{2k-1}\bar{e}_{2k}] \ldots]).
\end{aligned}$$

This finishes (3).

As for (4)

$$
\begin{aligned}
&\begin{pmatrix} 0 & E_{kk} \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} 0 & E_{(k-1)(k-1)} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & E_{(k-1)(k-1)} + E_{(k-1)k} + E_{k(k-1)} + E_{kk} \\ 0 & 0 \end{pmatrix} \\
&+ \Big[\begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix}, \begin{pmatrix} 0 & E_{(k-1)(k-1)} + E_{(k-1)k} + E_{k(k-1)} + E_{kk} \\ 0 & 0 \end{pmatrix} \Big], \\
= &\begin{pmatrix} 0 & E_{(k-1)(k-1)} \\ 0 & 0 \end{pmatrix} - \phi(\bar{e}_{2k-1}) + \Big[\begin{pmatrix} E_{kk} & 0 \\ 0 & -E_{kk} \end{pmatrix}, \phi(\bar{e}_{2k-1}) \Big] \\
= &\begin{pmatrix} 0 & E_{(k-1)(k-1)} \\ 0 & 0 \end{pmatrix} - \phi(\bar{e}_{2k-1}) \\
&+ \phi(2\bar{e}_{2k-1} + [\bar{e}_{2k-3}[\bar{e}_{2k-2}\bar{e}_{2k-1}]] + \ldots + (-1)^k [\bar{e}_1[\ldots, [\bar{e}_{2k-2}\bar{e}_{2k-1}] \ldots]) \\
= &\begin{pmatrix} 0 & E_{(k-1)(k-1)} \\ 0 & 0 \end{pmatrix} \\
&+ \phi(\bar{e}_{2k-1} + [\bar{e}_{2k-3}[\bar{e}_{2k-2}\bar{e}_{2k-1}]] + \ldots + (-1)^k [\bar{e}_1[\ldots, [\bar{e}_{2k-2}\bar{e}_{2k-1}] \ldots]).
\end{aligned}
$$

This gives a recursive relation of $\left(\begin{smallmatrix} 0 & E_{kk} \\ 0 & 0 \end{smallmatrix}\right)$. We can easily see that the formula in (4) satisfies this with initial condition $\left(\begin{smallmatrix} 0 & E_{11} \\ 0 & 0 \end{smallmatrix}\right) = \phi(\bar{e}_1)$. □

A.2 Facts and Proofs for Type D in 1.4

Proof A.6 (of Lemma 2.20)**.**

We obtain this by some recursive relations. First prove the first identity with

$j = i + 1$.

$$[[e_1e_2][e_1[e_2e_3]]] = [e_1[e_2[e_1[e_2e_3]]]] = -[e_1[e_1e_2]] + [e_1[e_2e_3]] = 2e_1 + [e_1[e_2e_3]],$$

$$\begin{aligned}
[[e_1[e_2e_3]][e_1[e_2[e_3e_4]]]] =& - [[e_3[e_1e_2]][e_1[e_2[e_3e_4]]]] \\
=& - [[e_3[e_1[e_2[e_3e_4]]]][e_1e_2]] - [[[e_1[e_2[e_3e_4]]][e_1e_2]]e_3] \\
=& - [[e_1e_2][e_1[e_2e_3]]] + [e_3[e_4[[e_1e_2][e_1[e_2e_3]]]]] \\
=& - 2e_1 - [e_1[e_2e_3]] + [e_1[e_2e_3]] = -2e_1.
\end{aligned}$$

Now the induction step:

$$\begin{aligned}
& [[e_1[e_2[\dots[e_{i-1}e_i]\dots], [e_1[e_2[\dots[e_ie_{i+1}]\dots]] \\
=& [[e_i[e_1[e_2[\dots[e_{i-2}e_{i-1}]\dots]][e_1[e_2[\dots[e_ie_{i+1}]\dots]] \\
=& - [[e_i[e_1[e_2[\dots[e_ie_{i+1}]\dots]][e_1[e_2[\dots[e_{i-2}e_{i-1}]\dots]] \\
& - [[[e_1[e_2[\dots[e_ie_{i+1}]\dots][e_1[e_2[\dots[e_{i-2}e_{i-1}]\dots]]e_i] \\
=& - [[e_1[e_2[\dots[e_{i-2}e_{i-1}]\dots][e_1[e_2[\dots[e_{i-1}e_i]\dots]] \\
& + [e_i[e_{i+1}[[e_1[e_2[\dots[e_{i-2}e_{i-1}]\dots][e_1[e_2[\dots[e_{i-1}e_i]\dots]]]] \\
=& - 2(-1)^{i-1}e_1 = 2(-1)^ie_1.
\end{aligned}$$

If $|j - i| \geq 2$, without loss of generality, assume $j - i \geq 2$, then

$$\begin{aligned}
& [[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_1[e_2[\dots[e_{i+1}e_{i+2}]\dots]] \\
=& - [[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_{i+2}[e_1[e_2[\dots[e_ie_{i+1}]\dots]]] \\
=& [e_{i+2}[[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_1[e_2[\dots[e_ie_{i+1}]\dots]]] \\
=& 2(-1)^{i+1}[e_{i+2}e_1] = 0.
\end{aligned}$$

So by induction

$$\begin{aligned}
&[[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_1[e_2[\dots[e_{j-1}e_j]\dots]]\\
=&-[[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_j[e_1[e_2[\dots[e_{j-2}e_{j-1}]\dots]]]\\
=&[e_j[[e_1[e_2[\dots[e_{i-1}e_i]\dots][e_1[e_2[\dots[e_{j-2}e_{j-1}]\dots]]]\\
=&[e_j,0]=0.
\end{aligned}$$

The second identity is the same as the first one when changing e_1 to $e_{\bar{1}}$. Next prove the third identity by induction. Some base cases:

$$\begin{aligned}
[[e_{\bar{1}}e_2],[e_1e_2]] &= [e_{\bar{1}}e_2]-[e_1e_2],\\
[[e_{\bar{1}}e_2][e_1[e_2e_3]]] &= -[e_2[e_{\bar{1}}[e_1[e_2e_3]]]]+[e_{\bar{1}}[e_2[e_1[e_2e_3]]]]\\
&= -[e_2[e_{\bar{1}}[e_1[e_2e_3]]]]-[e_{\bar{1}}[e_1e_2]]+[e_{\bar{1}}[e_2e_3]],\\
[[e_{\bar{1}}[e_2e_3]][e_1[e_2[e_3e_4]]]] &= -[[e_3[e_1e_2]][e_1[e_2[e_3e_4]]]]\\
&= -[[e_{\bar{1}}e_2][e_1[e_2e_3]]]+[e_3[e_4[[e_{\bar{1}}e_2][e_1[e_2e_3]]]]]\\
&=(-1)^2([e_{\bar{1}}[e_1e_2]]+\sum_{s=2}^{3}[e_s[\dots[e_{\bar{1}}[e_1\dots[e_se_{s+1}]\dots]).
\end{aligned}$$

By similar argument as the first identity, induction gives us

$$\begin{aligned}
&[[e_{\bar{1}}[e_2[\dots[e_{i-1}e_i]\dots][e_1[e_2[\dots[e_ie_{i+1}]\dots]]\\
=&-[[e_i[e_{\bar{1}}[e_2[\dots[e_{i-2}e_{i-1}]\dots]][e_1[e_2[\dots[e_ie_{i+1}]\dots]]\\
=&-[[e_{\bar{1}}[e_2[\dots[e_{i-2}e_{i-1}]\dots][e_1[e_2[\dots[e_{i-1}e_i]\dots]]\\
&+[e_i[e_{i+1}[[e_{\bar{1}}[e_2[\dots[e_{i-2}e_{i-1}]\dots][e_1[e_2[\dots[e_{i-1}e_i]\dots]]]]\\
=&\ (-1)^{i-1}([e_{\bar{1}}[e_1e_2]]+\sum_{s=2}^{i}[e_s[\dots[e_{\bar{1}}[e_1\dots[e_se_{s+1}]\dots]).
\end{aligned}$$

Consider

$$
\begin{aligned}
&[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_ie_{i+2}]\ldots]]\\
=&-[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_{i+2}[e_1[e_2[\ldots[e_ie_{i+1}]\ldots]]]\\
=&-[e_{i+2}[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_ie_{i+1}]\ldots]]]\\
=&[e_{i+2},(-1)^{i-1}([e_{\bar{1}}[e_1e_2]]+\sum_{s=2}^{i}[e_s[\ldots[e_{\bar{1}}[e_1\ldots[e_se_{s+1}]\ldots])]\\
=&\,0.
\end{aligned}
$$

Now if $j > i+2$, by induction

$$
\begin{aligned}
&[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_{j-1}e_j]\ldots]]\\
=&-[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_j[e_1[e_2[\ldots[e_{j-2}e_{j-1}]\ldots]]]\\
=&-[e_j[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_{j-2}e_{j-1}]\ldots]]]\\
=&\,0.
\end{aligned}
$$

If $j = i$, and $i \geq 3$

$$
\begin{aligned}
&[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_{i-1}e_i]\ldots]]\\
=&-[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_i[e_1[e_2[\ldots[e_{i-2}e_{i-1}]\ldots]]]\\
=&-[e_i[[e_{\bar{1}}[e_2[\ldots[e_{i-1}e_i]\ldots][e_1[e_2[\ldots[e_{i-2}e_{i-1}]\ldots]]]\\
=&[e_i,(-1)^{i-1}([e_{\bar{1}}[e_1e_2]]+\sum_{s=2}^{i-1}[e_s[\ldots[e_{\bar{1}}[e_1\ldots[e_se_{s+1}]\ldots])]\\
=&[e_{i-2}[\ldots[e_{\bar{1}}[e_1\ldots[e_{i-1}e_i]\ldots]-[e_{i-2}[\ldots[e_{\bar{1}}[e_1\ldots[e_{i-1}e_i]\ldots]\\
=&\,0.
\end{aligned}
$$

This completes the proof. $\square$

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