Richardson Varieties in a Toric Degeneration of the Flag Variety

by

Giwan Kim

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Doctoral Committee:

Associate Professor David E Speyer, Chair
Professor Sergey Fomin
Professor Thomas Lam
Associate Professor Aaron T. Pierce
Professor Karen E. Smith

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CHAPTER I

Introduction

The purpose of this thesis is to complete the work of Kogan and Miller [KM05b] in degenerating a Schubert variety into a reduced union of toric subvarieties of the Gelfand-Tsetlin toric variety. This semi-toric degeneration[1] of a Schubert variety is stated in [KM05b, Theorem 8], however, the proof contained therein is incomplete. We also obtain as a corollary, semi-toric degenerations of Richardson varieties.

Let M_n denote the space of $n \times n$ matrices over the complex numbers, GL_n denote the general linear group of $n \times n$ invertible matrices, and B^- denote the Borel subgroup of lower triangular matrices of GL_n. Let $F\ell_n$ be the flag variety of complete flags in $\mathbb{C}^n$. For the permutation $w \in S_n$, the flag variety $F\ell_n$ has subvarieties X_w and X^w called a Schubert variety and an opposite Schubert variety, respectively. For permutations $u \leqslant w$, let X_u^w denote the Richardson variety defined as the intersection $X_u^w := X_u \cap X^w$.

Following [KM05b], we construct the degeneration of $F\ell_n$ as the GIT quotient $B^- \backslash\backslash (M_n \times \mathbb{A}^1)$. In particular, the space $M_n \times \mathbb{A}^1$ is the total space of Gröbner degenerations of matrix Schubert varieties studied in [KM05a]. Matrix Schubert varieties are subvarieties of M_n indexed by permutations and are closely related to Schubert varieties. For $w \in S_n$, we denote the matrix Schubert variety and opposite

[1] In a semi-toric degeneration, the irreducible components are toric varieties.

matrix Schubert variety by $\widetilde{X}_w$ and $\widetilde{X}^w$, respectively.

To describe the relation between the degenerations of a Schubert variety and a matrix Schubert variety, let

$$\rho : M_n \times \mathbb{A}^1 \longrightarrow N^- \backslash\backslash (M_n \times \mathbb{A}^1)$$

be the quotient map of affine varieties over $\mathbb{A}^1$. We denote by ρ_0 the specialization of ρ to the $t = 0$ fiber. Let $\widetilde{\mathcal{X}}_w$ be the flat family degenerating $\widetilde{X}_w$ and $\mathcal{X}_w :=$ $\overline{\rho}(\widetilde{\mathcal{X}}_w)$ be its scheme-theoretic image that is flat over $\mathbb{A}^1$ as well. The affine variety $N^- \backslash\backslash (M_n \times \mathbb{A}^1)$ is the multi-cone over $B^- \backslash\backslash (M_n \times \mathbb{A}^1)$ so that $\mathcal{X}_w$ is the multi-cone over the degeneration of X_w inside $B^- \backslash\backslash (M_n \times \mathbb{A}^1)$. Then,

- the irreducible components of the initial scheme $\lim_{t\to 0} \widetilde{X}_w$ are parametrized by reduced pipe dreams [KM05a, Theorem B]; so,
- the image under ρ_0 of the components of $\lim_{t\to 0} \widetilde{X}_w$ correspond to a union of faces of the GT-cone parametrized by reduced pipe dreams.

It does not follow from the above two facts that the initial scheme $\lim_{t\to 0} \mathcal{X}_w$ is equal to the reduced union of affine toric subvarieties corresponding to faces of the GT-cone. In general, the fiber of the image may properly contain the image of the fiber [EH00, pg. 216]. A specific example of this phenomenon in our context is included in Chapter IV.

The additional ingredient in our proof is the application of Standard Monomial Theory (SMT) [LS86] to parametrize the lattice points of the GT-cone. In particular, we show that the SMT basis of a Schubert variety are in bijection with lattice points of faces of the GT-cone corresponding to reduced pipe dreams.

Similar results have been obtained by [GL96, DY01] for a subset of Schubert varieties and by [Chi00] for all Schubert varieties. The degeneration of [Cal02] applied deep results from Kashiwara-Lusztig's parametrization of dual canonical basis and [MG08] degenerated generalized Richardson varieties after [Cal02].

In a recent work of [KST12], the Gelfand-Tsetlin polytope appears in the description of the cohomology ring of the flag variety and a certain union of its faces is associated with global sections of line bundles $\mathcal{L}_\lambda$ restricted to the Schubert variety X_w. Our works differ in that for us these results follow as a corollary of flat degenerations of Schubert and Richardson varieties. We also show that standard monomials for Richardson varieties correspond naturally to faces of Gelfand-Tsetlin polytope indexed by reduced pipe dreams.

This thesis is organized as follows. In Chapter II, we introduce the combinatorial objects that encode much of the structure of algebraic varieties that we study. In Chapter III, we define those algebraic varieties and present the background on Gröbner degeneration. In Chapter IV, we present the toric degeneration of the flag variety adapted from [KM05a] and construct an involution that maps the degeneration of a Schubert variety to that of an opposite Schubert variety. We also present our results on semi-toric degenerations of Schubert varieties localized to the degeneration of the opposite big cell. In Chapter V, we apply SMT to conclude that Richardson varieties degenerate to a reduced union of toric subvarieties indexed by pairs of reduced pipe dreams.

The following table summarizes the notation of this thesis.

$\mathbb{N}$	nonnegative integers
$[a,b]$	set of integers $\{a, a+1, \ldots, b\}$
$[n]$	set of integers $\{1, 2, \ldots, n\}$
$2^{[n]}$	power set on $[n]$
$\binom{[n]}{k}$	set of k-element subsets of n
λ	partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$
Λ_n^+	the set of partitions λ with at most n parts
Λ_n^{++}	the set of partitions λ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$
S_n	the symmetric group of permutations
$\mathsf{SSYT}(n)$	set of semistandard tableaux with entries in $[n]$
$\mathsf{SSYT}(n;\lambda)$	subset of $\mathsf{SSYT}(n)$ consisting of tableaux of shape λ
$K_\pm(T)$	the left/right key tableau for T
$w_\pm(T)$	the canonical lift of $K_\pm(T)$
$\mathcal{PD}_n$	the set of pipe dreams of rank n
$\mathcal{RP}_w$	the set of reduced pipe dreams associated with $w \in S_n$
$\mathsf{GT}(n)$	the semigroup of integer Gelfand-Tsetlin patterns of rank n
$\mathsf{GT}(n;\lambda)$	subset of $\mathsf{GT}(n)$ with shape λ
$\mathcal{P}_\lambda$	Gelfand-Tsetlin polytope of shape λ
M_n	the set of $n \times n$ matrices over $\mathbb{C}$
GL_n	the set of invertible matrices of M_n
B^-	the Borel subgroup of GL_n consisting of lower triangular matrices
N^-	the subgroup of B^- consisting of matrices with 1's on the diagonal
$F\ell_n$	the flag variety of complete flags in $\mathbb{C}^n$
$\widetilde{X}_w$, $\widetilde{I}_w$	matrix Schubert variety and Schubert determinantal ideal for w
$\widetilde{X}^w$, $\widetilde{I}^w$	opposite matrix Schubert variety and its ideal
X_w, I_w	Schubert variety and Schubert ideal for $w \in S_n$
X^w, I^w	opposite Schubert variety and opposite Schubert ideal for $w \in S_n$
X_u^w, I_u^w	Richardson variety $X_u \cap X^w$ and its ideal
$\mathrm{in}_\omega(\cdot)$	the initial term with respect to weight ω
Z	the generic $n \times n$ matrix of indeterminates (z_{ij})
P	the set of Plücker variables $\{p_I : I \in 2^{[n]}\}$
$P^{(k)}$	the subset of P consisting of $\{p_I : I \in \binom{[n]}{k}\}$
Q	the set of degenerated Plücker variables $\{q_I : I \in 2^{[n]}\}$
$Q^{(k)}$	the subset of Q consisting of $\{q_I : I \in \binom{[n]}{k}\}$
X	the set of indeterminates $\{x_I : I \in 2^{[n]}\}$
$X^{(k)}$	the set of indeterminates $\{x_I : I \in \binom{[n]}{k}\}$

CHAPTER II

Combinatorial Background

2.1 Conventions and notation

For integers a and b, let $[a, b]$ denote the interval $\{a, a + 1, \ldots, b\}$ and $[n]$ denote the initial interval $\{1, 2, \ldots, n\}$. Let $2^{[n]}$ denote the power set on $[n]$ and $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. We identify partitions that differ by trailing zeros, so for example, $(5, 3, 2, 1, 0, 0)$ is identified with $(5, 3, 2, 1)$. A partition is represented by its (Young or Ferrers) diagram, which is a left-justified array of boxes (or cells) with λ_i boxes in row i. Let Λ_n^+ denote the set of partition with at most n parts and Λ_n^{++} denote the subset of Λ_n^+ consisting of partitions that are strictly decreasing: $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

Let S_n denote the permutation group on $[n] = \{1, 2, \ldots, n\}$. We write permutations in one-line notation representing $w \in S_n$ as the list $w(1)\, w(2) \ldots w(n)$. Our convention for multiplying permutations is to read the product as composition of maps so that $(uw)(i) = u(w(i))$ for $u, w \in S_n$. As a consequence of this convention, the product $w \cdot (i, j)$ transposes values in positions i and j so $w(i)$ and $w(j)$, whereas multiplying $(i,j) \cdot w$ transposes values i and j. For example, in S_5

$$24531 \cdot (2,5) = 21534 \quad \text{and} \quad (2,5) \cdot 24531 = 54231.$$

2.2 Permutations

2.2.1 Bruhat order

Let $s_i = (i, i+1)$ be the adjacent transposition that interchanges i and $i+1$. As a Coxeter group, S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations, $s_i^2 = \mathrm{id}$, $s_i s_j = s_j s_i$ if $|i-j| > 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

A word of size q is an ordered sequence $Q = (s_{a_1}, s_{a_2}, \ldots, s_{a_q})$ of adjacent transpositions of S_n. An ordered subsequence P of Q is called a subword of Q. An inversion of w is a pair $(i,j) \in [n] \times [n]$ such that $i < j$ and $w(i) > w(j)$. The length $\ell(w)$ of a permutation $w \in S_n$ is the number of its inversions. Each permutation $w \in S_n$ can be written as a product of simple transpositions as in $w = s_{i_1} s_{i_2} \ldots s_{i_q}$. If q is minimal among all such expressions for w, then the word $s_{i_1} s_{i_2} \ldots s_{i_q}$ or $(s_{i_1}, s_{i_2}, \ldots, s_{i_q})$ is called a reduced word for w. The minimal number of generators appearing in a reduced word for w is equal to the number of inversions of w so $q = \ell(w)$. The word Q represents $w \in S_n$ if the ordered product of the simple reflection comprising Q is a reduced word for w and Q contains w if some subsequence of P represents w.

The permutation matrix for $w \in S_n$ is the $n \times n$ matrix with 1's in coordinates $(i, w(i))$ for $i = 1, 2, \ldots, n$ and 0's elsewhere. For instance, the permutation matrix for $w = 2413$ is $\begin{bmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&0&1&0 \end{bmatrix}$.

For $k = 1, 2, \ldots, n$, let $S_k \times S_{n-k} \subseteq S_n$ denote the subgroup of permutations that preserve the subsets $[k]$ and $[k+1, n]$. A permutation $w \in S_n$ is called anti-Grassmannian if it is equal to the maximal length permutation in its coset $(S_k \times$

$S_{n-k})w$. More explicitly, the set of anti-Grassmannian for a fixed k is equal to

$$S_n^{(k)} := \{w \in S_n : w(1) > w(2) > \cdots > w(k) \text{ and } w(k+1) > w(k+2) > \cdots > w(n)\}.$$

These permutations form a system of representatives of the coset space $(S_k \times S_{n-k})\backslash S_n$.

Let $\pi_k : S_n \to \binom{[n]}{k}$ be the map defined by $\pi_k(w) = \{w(1), w(2), \ldots, w(k)\}$ for $w \in S_n$. For convenience of notation, we write π for π_k when the value of k is clear from the context. Notice that π_k restricts to a bijection of $S_n^{(k)}$ with $\binom{[n]}{k}$ which is the way in which we identify anti-Grassmannian permutations with k-element subsets.

Definition II.1. For $w \in S_n$, we define the rank function $r_w : [n] \times [n] \longrightarrow \mathbb{Z}$ by

$$r_w(i,j) := \#(w[i] \cap [j]).$$

Notice that $r_w(i,j)$ is equal to the rank of the $i \times j$ submatrix on the upper left corner of the permutation matrix for w or, equivalently, $r_w(i,j)$ counts the number of nonzero entries in the upper left $i \times j$ corner of the permutation matrix.

Definition II.2. (Strong) Bruhat order is a partial order on S_n defined by $u \leqslant w$ for $u, w \in S_n$ if $r_u(i,j) \geqslant r_w(i,j)$ for all i,j. A partial order on $\binom{[n]}{k}$ closely related to the Bruhat order is given by $I \leqslant J$ if $i_m \leqslant j_m$ where $I = \{i_1 > i_2 > \cdots > i_k\}$ and $J = \{j_1 > j_2 > \cdots > j_k\}$.

A well-known criterion for comparison in Bruhat order says that $u \leqslant w$ if and only if $\pi_k(w) \leqslant \pi_k(w)$ for all $k \in [n]$. The subword property and chain property are two fundamental properties of Bruhat order.

Theorem II.3. *[BB05, Theorem 2.2.2 (Subword Property)] Let $w = s_{a_1} s_{a_2} \ldots s_{a_q}$ be a reduced expression. Then, $u \leqslant w$ if and only if there exists a reduced expression $u = s_{a_{i_1}} s_{a_{i_2}} \ldots s_{a_{i_p}}$ where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_p \leqslant q$.*

Theorem II.4. *[BB05, Theorem 2.2.6 (Chain Property)] If $u < w$, there exists a chain $u = v_0 < v_1 < \cdots < v_k = w$ such that $\ell(v_i) = \ell(u) + i$ for $i = 1, 2, \ldots, k$.*

Let $w_0 \in S_n$ be the permutation that sends $i \longmapsto n - i + 1$ for each $i \in [n]$ so that in one-line notation $w_0 = n(n-1)\ldots 1$. We call w_0 the longest permutation, or long word of S_n. It is characterized by the fact that it is the unique maximal element in Bruhat order so that $w < w_0$ for all $w \in S_n \setminus \{w_0\}$.

2.2.2 Demazure product

For the permutation w and adjacent transposition $s \in S_n$, we define the product $w * s \in S_n$ by

$$w * s = \begin{cases} ws & \text{if } ws > w, \\ w & \text{if } ws < w. \end{cases} \tag{2.1}$$

Then, define $w * v$ by choosing a reduced expression $s_{a_1} s_{a_2} \ldots s_{a_p}$ for v and setting $w * v := (((w * s_{a_1}) * s_{a_2}) * \ldots) * s_{a_p}$. In particular, if wv is length-additive, then $w * v = wv$. It turns out that the product $w * v$ is independent of choice of reduced word for v. Further background on the Demazure product can be found in [KM05b].

Definition II.5. Let $Q = (s_{a_1}, s_{a_2}, \ldots, s_{a_q})$ be a word. Then, let $\mathsf{Dem}(Q)$ to be the permutation of S_n defined by $\mathsf{Dem}(Q) := ((s_{a_1} * s_{a_2}) * \ldots) * s_{a_q}$.

The main property of Demazure products used in this thesis is that the Bruhat order on Demazure products detects reduced subwords of arbitrary words just as Bruhat order detects reduced subwords of reduced words.

Lemma II.6. *[KM05b, Lemma 3.4] Let Q be a word in S_n and let $w \in S_n$. Then, $\mathsf{Dem}(Q) \geqslant w$ if and only if Q contains w as a subword.*

2.3 Tableaux

2.3.1 Jeu de taquin

Definition II.7. A semistandard Young tableau [1] is a filling of the boxes of a diagram by integers so that the rows are weakly decreasing and columns are strictly decreasing. We call these integers the entries of the tableau. Formally, a semistandard Young tableau of shape λ is an array of positive integers $T = (t_{ij})$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \lambda_i$ such that

- rows weakly decrease: $t_{i1} \geqslant t_{i2} \geqslant \cdots \geqslant t_{i,\lambda_i}$;
- columns strictly decrease: $t_{1j} > t_{2j} > \cdots > t_{\lambda'_j,j}$.

Given partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we write $\mu \subseteq \lambda$, and say that λ contains μ, to mean that $m \leqslant n$ and $\mu_i \leqslant \lambda_i$ for $i = 1, 2, \ldots, m$. A skew diagram or skew partition is the diagram obtained by removing a smaller Young diagram contained in a larger Young diagram after aligning the upper left corners of the two diagrams. The skew diagram resulting from removing μ from λ is denoted λ/μ.

Analogously, a skew tableau is a filling of skew diagram with positive integers such that the filling is strictly decreasing in the columns and weakly decreasing in the rows. The (possibly skew) partition associated to the diagram of a tableau is called its shape. When it is necessary to make the distinction, we say that a tableau T has straight shape if the shape of T is a partition or that T has a skew shape if the shape of T is a skew partition.

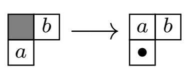
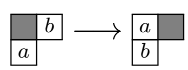
Notation II.8. Let $\mathsf{SSYT}(n)$ denote the set of semistandard tableaux with entries in $[n]$ and $\mathsf{SSYT}(n; \lambda)$ denote the subset of $\mathsf{SSYT}(n)$ consisting of tableaux of shape

[1] Our decreasing convention on the entries of a tableau is the opposite of the usual increasing convention.

λ, so $\mathsf{SSYT}(n) = \bigsqcup_\lambda \mathsf{SSYT}(n;\lambda)$.

Let λ/μ be a skew shape. An inside corner is a box in the removed diagram μ such that neither the box directly below nor directly to the right are in μ. Notice that a skew shape that is not a partition has one or more inside corners. An outside corner is a box in λ such that neither box below or to the right is in λ. Each skew tableau determines a unique tableau of straight shape called its rectification (*redressement*) which can be obtained by applying jeu de taquin or sliding algorithm. The sliding algorithm takes a skew tableau S and an inside corner x, which we regard as an empty box, and successively slides the empty box through the skew tableau by interchanging the empty box with larger of its neighbors either directly to the right or directly below. If the entries in the two neighbors are equal, then the empty box is interchanged with the box below; if only one of the two neighbors is in the skew tableau, then the empty box is interchanged with that neighboring box.

Locally, a typical step in the sliding algorithm looks like

 $(a \geqslant b)$,  $(a > b)$.

This process of interchanging neighbors is repeated until the empty box arrives at an outside corner, i.e., there are no neighbors to the right nor below. Sliding an inside corner through a skew tableau as described above results in another skew tableau.

This sliding algorithm is reversible by running the sliding algorithm backwards. Reverse jeu de taquin or reverse slide takes as input a skew tableau S' together with an outer corner y, and outputs a skew tableau. Reverse sliding a skew tableau results in another skew tableau and reverse slide and forward slides are inverse operations on tableaux.

Notation II.9. Let S be a skew tableau with inside corner x and outside corner y.

We write $\mathrm{jdt}_x(S)$ to denote the skew tableau obtained by sliding x through S and $\mathrm{jdt}^y(S)$ to denote the skew tableau resulting from reverse sliding y through S.

Given a skew tableau S, the sliding algorithm can successively be applied until there are no inside corners; the result is a tableau of straight shape. It is a fundamental result from tableaux theory that this resulting tableau is independent of all intermediate choices of inside corners. It follows that there exists a unique straight-shaped tableau that can be computed from S by applying any sequence of jeu de taquin slides. We call this resulting tableau of straight shape, the rectification of S and write $\mathrm{rect}(S)$.

The following example illustrates how a sequence of slides can be applied to the skew tableau $S = \begin{ytableau} \none & \none & 4 & 2 & 2 \\ \none & \none & 2 \\ \none & \none & 1 \\ 3 & 2 \\ 2 \end{ytableau}$ to obtain its rectification $\mathrm{rect}(S)$:

(2.2)

$$\begin{ytableau} \none & \none & 4 & 2 & 2 \\ \none & \none & 2 \\ \none & *(gray) & 1 \\ 3 & 2 \\ 2 \end{ytableau} \to \begin{ytableau} \none & \none & 4 & 2 & 2 \\ \none & \none & 2 \\ *(gray) & 2 & 1 \\ 3 \\ 2 \end{ytableau} \to \begin{ytableau} \none & \none & 4 & 2 & 2 \\ \none & *(gray) & 2 \\ 3 & 2 & 1 \\ 2 \end{ytableau} \to \begin{ytableau} \none & \none & 4 & 2 & 2 \\ *(gray) & 2 & 2 \\ 3 & 1 \\ 2 \end{ytableau} \to \begin{ytableau} \none & *(gray) & 4 & 2 & 2 \\ 3 & 2 & 2 \\ 2 & 1 \end{ytableau} \to \begin{ytableau} *(gray) & 4 & 2 & 2 & 2 \\ 3 & 2 \\ 2 & 1 \end{ytableau} \to \begin{ytableau} 4 & 2 & 2 & 2 & 2 \\ 3 & 1 \\ 2 \end{ytableau}.$$

2.3.2 Key tableaux

Key tableaux were introduced in [LS90] as a combinatorial tool for understanding certain bases of global sections of line bundles called standard monomials (Chapter V). Our account follow that of [FL94, Ful97] in applying jeu de taquin to compute keys. See [Ful97] for the proofs of facts cited here, and [RS95] for a parallel account from the persepective of the Plactic monoid.

A skew tableau is called frank if its column heights are a permutation of the column heights of its rectification. For example, the first, second, third, sixth, and seventh skew tableaux in (2.2) are frank.

Proposition II.10. *[Ful97, Appendix A] Let T be a tableau of shape λ and ν/μ be a skew diagram whose column heights are a permutation of the column heights of λ. Then, there exists a unique skew tableau S on ν/μ that rectifies to T. In other words, there exists a unique frank skew tableau S of given skew shape that rectifies to T.*

In fact, the proof in [Ful97] implies that the entries of S depend only on the ordered heights of its columns. For a given permutation of column heights, the most compact frank skew tableau is obtained by aligning each successive pair of columns at the top if the left column is longer or at the bottom if the right column is longer. The S of Proposition II.10 for any other skew shape with these ordered column heights is obtained by shifting the columns of the compact form further apart. For example, for $T = \begin{array}{cccc} 4&4&3&3 \\ 3&2&2& \\ 1&&& \end{array}$ and column heights $(2,3,2,1)$, $S = \begin{array}{cccc} &4&3&3 \\ 4&2&2& \\ 3&1&& \end{array}$ is in compact form and $S' = \begin{array}{cccc} &&&3 \\ &4&3& \\ &2&2& \\ 4&1&& \\ 3&&& \end{array}$ is another franks skew tableau rectifying to T. We will usually write frank skew tableaux in their compact form.

When T has two columns finding S is relatively easy: reverse slide the empty boxes at the bottom of the second column. For example,

$$T = \begin{array}{cc} 5&4 \\ 3&2 \\ 2&\blacksquare \\ 1& \end{array} \longrightarrow \begin{array}{cc} 5&4 \\ 3&\blacksquare \\ 2&2 \\ 1& \end{array} \longrightarrow \begin{array}{cc} 5&4 \\ \blacksquare&3 \\ 2&2 \\ 1& \end{array} \longrightarrow \begin{array}{cc} \blacksquare&4 \\ 5&3 \\ 2&2 \\ 1& \end{array} \longrightarrow \begin{array}{cc} &4 \\ 5&3 \\ 2&2 \\ 1&\blacksquare \end{array} \longrightarrow \begin{array}{cc} &4 \\ 5&3 \\ 2&2 \\ \blacksquare&1 \end{array} \longrightarrow \begin{array}{cc} &4 \\ 5&3 \\ 2&2 \\ \blacksquare&1 \end{array} \longrightarrow \begin{array}{cc} &4 \\ \blacksquare&3 \\ 5&2 \\ 2&1 \end{array} = S.$$

We call this process or its inverse (forward sliding the empty boxes at the top of the first column) on adjacent columns, an elementary move. It follows that we can find all frank skew tableau S rectifying to a given tableau T by successively applying elementary moves to adjacent columns. Independence of the result from intermediate choices is a consequence of the the fact that entries of S are already determined by ordered column heights. In fact, this method of computing frank skew tableaux implies additional properties for the left-most and right-most columns.

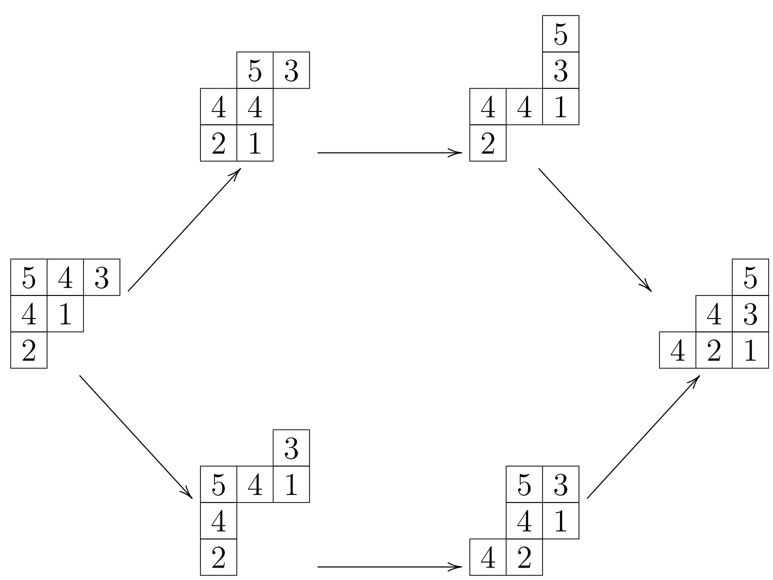
Corollary II.11. *For a given fixed rectification T, the entries of the left-most column of S are determined by the height of that column and an analogous claim holds true for the right-most column.*

So for a given column length c of T, it makes sense to talk about the left-most and right-most columns of S of height c. Let $\mathcal{L}_c$ and $\mathcal{R}_c$ denote the sets of elements in the left-most and respectively, right-most columns of S.

Corollary II.12. *If $c < d$, then $\mathcal{L}_c \subset \mathcal{L}_d$ and $\mathcal{R}_c \subset \mathcal{R}_d$.*

Definition II.13. A tableau is called a key, or a key tableau if the j^{th} column contains that of the $(j+1)^{\text{st}}$ column for all j. For a tableau T, let left and right key of T be the tableaux of identical shape as T whose columns of height c consists of the elements of $\mathcal{L}_c$ and $\mathcal{R}_c$, respectively. We write $K_-(T)$ and $K_+(T)$ to denote the left and right key, respectively.

Example II.14. We apply a sequence of elementary moves to $T = \begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ to see that



so $K_-(T) = \begin{array}{|c|c|c|} \hline 5 & 4 & 4 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ and $K_+(T) = \begin{array}{|c|c|c|} \hline 5 & 3 & 3 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$.

A decreasing chain $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots \supseteq \mathcal{C}_\ell$ of subsets of $[n]$ determines two permutation in S_n with the one being "minimal" and the other "maximal." The minimal lift in one-line notation is obtained by listing the elements of $\mathcal{C}_\ell$ in increasing order followed by the elements of $\mathcal{C}_\ell \setminus \mathcal{C}_{\ell-1}$ in increasing order and so forth, until finally one lists the elements of $[n] \setminus \mathcal{C}_\ell$ in increasing order. Similarly, the maximal lift is obtained by listing the elements in decreasing order.

Definition II.15. For $T \in \mathsf{SSYT}(n)$, we define the canonical lift $w_-(T)$ of $K_-(T)$ to be the permutation obtained as the minimal lift of $\{\mathcal{L}_c\}$ and similarly, we define the canonical lift $w_+(T)$ of $K_+(T)$ to be the permutation obtained as the maximal lift of $\{\mathcal{R}_c\}$. For $T = \begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}$ as in the above example, $w_-(T) = 42513$ and $w_+(T) = 31542$.

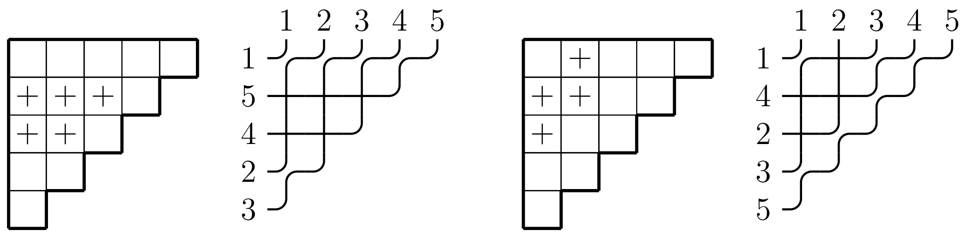
2.4 Pipe dreams

Reduced pipe dreams index the monomials of a Schubert polynomial generalization the role of semistandard Young tableaux for a Schur polynomial. For further background, see [MS05, Chapter 16] and [KM05a, BB93].

Definition II.16. A pipe dream of rank n is a tiling of a $n \times n$ square diagram by crosses "$+$" and elbows "╯╭." Let $\mathcal{PD}(n)$ denote the set of pipe dreams of size n. We only consider pipe dreams that are subsets of the pipe dream $D_0 \subseteq [n] \times [n]$ that has crosses in the triangular region strictly above the main antidiagonal ($(i, j) \in D_0$ if $i + j \leqslant n$) and elbow joints elsewhere. Consequently, our pipe dreams always fit inside the staircase shape $(n, n-1, \ldots, 1)$.

We often identify a pipe dream with its crossing tiles and consider a pipe dream as a subset of $[n] \times [n]$ consisting of the coordinates of its crossing tiles. Similarly, when we draw pipe dreams, we often do not draw elbows for ease of notation.

Example II.17. The pipe dreams with $n = 5$ corresponding to $\{(2,1),(2,2),(2,3),(3,1),(3,2)\}$ and $\{(1,2),(2,1),(2,2),(3,1)\}$ are



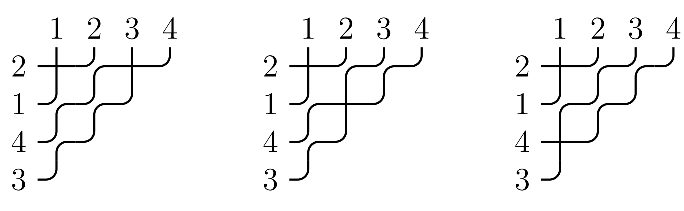
(2.3)

We label the pipe entering the diagram horizontally by its exit column. Reading the labels on pipes from top to bottom yields a permutation in one-line notation. Let $\mathsf{perm}(D)$ denote the resulting permutation for pipe dream D; the pipe entering through row i exits through column $\mathsf{perm}(D)(i)$. The permutations for pipe dreams in (2.3) are 15423 and 14235.

We call a pipe dream reduced if each pair of pipes crosses at most once. For instance, the first pipe dream in (2.3) is reduced but the second is not.

Definition II.18. For $w \in S_n$, let $\mathcal{RP}_w$ denote the set of reduced pipe dreams such that $\mathsf{perm}(D) = w$ for all $D \in \mathcal{RP}_w$. So for every pipe dream in $\mathcal{RP}_w$, the pipe entering row i exits the diagram through column $w(i)$.

For example, $\mathcal{RP}_{2143}$ consists of three reduced pipe dreams:



For a pipe dream $D \in \mathcal{PD}(n)$, let $Q(D)$ be the word obtained from D by reading a crossing tile in position (i,j) as the adjacent transposition s_{i+j-1}, where the reading order is from right to left in each row starting from top row and ending with the bottom row. For example, the words associated with the two pipe dreams in (2.3) are $s_4s_3s_2s_4s_3 = 15423$ and $s_2s_3s_2s_3 = 14235$. In particular, the pipe dream D_0 corresponds to the word

$$Q(D_0) = (s_{n-1}s_{n-2}\ldots s_1)(s_{n-1}s_{n-2}\ldots s_2)\ldots(s_{n-1}s_{n-2})(s_{n-1}),$$

which is the triangular form of the long word $w_0 = n(n-1)\ldots 1$. Moreover, since we only consider pipe dreams that are subsets of D_0, we may think of pipe dreams as subwords of $Q(D_0)$.

Lemma II.19. *[KM05a, Lemma 1.4.5] Let $D \in \mathcal{PD}(n)$ be a pipe dream. Then, the product of $Q(D)$ equals the permutation $\mathsf{perm}(D)$. Furthermore, the number of crossing tiles in D is at least $\ell(\mathsf{perm}(D))$ with equality if and only if D is a reduced pipe dream in $\mathcal{RP}_{\mathsf{perm}(D)}$.*

So a reduced pipe dream $D \in \mathcal{RP}_w$ corresponds to a reduced subword of w_0 and $\mathsf{perm}(D)$ is equal to the product of $Q(D)$.

Definition II.20. Let $D \in \mathcal{PD}(n)$ be a pipe dream. We define $\mathsf{Dem}(D)$ to be the Demazure product of $Q(D)$.

As a consequence of Lemma II.19, if $D \in \mathcal{RP}_w$ is reduced, then $\mathsf{Dem}(D) = \mathsf{perm}(D) = w$, whereas if $D \in \mathcal{PD}(n)$ is not reduced, then $\mathsf{Dem}(D) \geqslant \mathsf{perm}(D)$ by Lemma II.6.

Figure 2.1: D_0 for $w_0 = 654321$

2.5 Gelfand-Tsetlin polytope

Definition II.21. A Gelfand-Tsetlin pattern (GT-pattern) of rank n is a triangular array $\Gamma = (\gamma_{i,j})_{i+j \leqslant n+1}$ such that $\gamma_{i,j} \geqslant \gamma_{i,j+1} \geqslant \gamma_{i+1,j}$. We typically represent a GT-pattern Γ as

$$\begin{array}{ccccccc}
\gamma_{1,1} & \geqslant & \gamma_{1,2} & \geqslant & \cdots & \geqslant & \gamma_{1,n} \\
\rotatebox{-90}{$\geqslant$} & \rotatebox{-45}{$\geqslant$} & \rotatebox{-90}{$\geqslant$} & \rotatebox{-45}{$\geqslant$} & \cdots & \rotatebox{-45}{$\geqslant$} & \\
\gamma_{2,1} & \geqslant & \gamma_{2,2} & \geqslant & \cdots & & \\
\rotatebox{-90}{$\geqslant$} & \rotatebox{-45}{$\geqslant$} & \cdots & \rotatebox{-45}{$\geqslant$} & & & \\
\vdots & \geqslant & \cdots & & & & \\
\rotatebox{-90}{$\geqslant$} & \rotatebox{-45}{$\geqslant$} & & & & & \\
\gamma_{n,1} & & & & & &
\end{array}$$

We denote the semigroup of integer GT-patterns by $\mathsf{GT}(n)$. We define the shape of an integer GT-pattern $\Gamma \in \mathsf{GT}(n)$ to be the partition $\lambda = (\gamma_{1,1}, \gamma_{2,1}, \ldots, \gamma_{n,1})$, which is the first column of Γ. For a given partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n)$, let $\mathsf{GT}(n;\lambda)$ denote the subset of GT-patterns in $\mathsf{GT}(n)$ with shape λ.

Definition II.22. Let $\mathcal{P}_\lambda \subseteq \mathbb{R}^{\binom{n}{2}}$ be the lattice polytope defined as the convex hull of integer Gelfand-Tsetlin patterns of shape λ called the Gelfand-Tsetlin polytope (GT-polytope). The lattice points of $\mathcal{P}_\lambda$ are integer GT-patters of $\mathsf{GT}(n;\lambda)$.

GT-polytope is normal meaning that if $d\mathbf{m} \in d\mathcal{P}_\lambda \cap \mathbb{Z}^{\binom{n}{2}}$ then $\mathbf{m} \in \mathcal{P}_\lambda \cap \mathbb{Z}^{\binom{n}{2}}$ for all $d \geqslant 1$. So normality means that $\mathcal{P}_\lambda$ has enough lattice points to generate the lattice points in all integer multiples of $\mathcal{P}_\lambda$. Faces of normal polytopes are normal as well.

GT-patterns were introduced in [GC50] to index a basis of irreducible representations of GL_n compatible with decomposition into irreducible representations for subgroups $\mathrm{GL}_k \leqslant \mathrm{GL}_n$ for $k = 1, 2, \ldots, n$.

There exists a well-known bijection between integer GT-patterns of rank n and tableaux with entries in $[n]$ preserving shapes. For partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell)$ and $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m)$, we write $\lambda \trianglerighteq \mu$ if $\ell \geqslant m$ and $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant \mu_m \geqslant \lambda_{m+1} \geqslant \cdots \geqslant \lambda_\ell)$. To describe the bijection $\mathsf{GT}(n;\lambda) \longrightarrow \mathsf{SSYT}(n;\lambda)$, we consider $\Gamma \in \mathsf{GT}(n;\lambda)$ as a interlaced sequence of partitions $\lambda = \lambda^{(1)} \trianglerighteq \lambda^{(2)} \trianglerighteq \cdots \trianglerighteq \lambda^{(n)}$ where $\lambda^{(j)}$ is equal to the j^{th} column of Γ. Then, map Γ to the tableau of shape λ such that the boxes of the skew shape $\lambda^{(j)}/\lambda^{(j+1)}$ are labeled j. The defining conditions of a Gelfand-Tsetlin pattern imply that $\lambda^{(j)}/\lambda^{(j+1)}$ is a horizontal strip so the above map results in a semistandard tableau.

For the inverse map $\mathsf{SSYT}(n;\lambda) \longrightarrow \mathsf{GT}(n;\lambda)$, send $T \in \mathsf{SSYT}(n;\lambda)$ to the sequence of partitions

$$\lambda = \lambda^{(1)} \supseteq \lambda^{(2)} \supseteq \cdots \supseteq \lambda^{(n)} \tag{2.4}$$

where each $\lambda^{(j)}$ for $j = 1, 2, \ldots, n$ is the shape of the sub-tableau of T consisting of those boxes containing entries from the set $[j, n]$. Semistandardness of T is equivalent to $\lambda^{(j)}/\lambda^{(j+1)}$ being a horizontal strip for $j = 1, 2, \ldots, n-1$ so the partitions in (2.4) are interlaced. We denote the GT-pattern equivalent to this chain of partitions, $\Gamma(T)$. For example,

$$T = \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad \longleftrightarrow \quad \begin{array}{ccccc} 3 & 3 & 3 & 2 & 1 \\ 2 & 1 & 1 & 1 & \\ 1 & 1 & 0 & & \\ 0 & 0 & & & \\ 0 & & & & \end{array} = \Gamma$$

where $(3,2,1,0,0) \trianglerighteq (3,1,1,0) \trianglerighteq (3,1,0) \trianglerighteq (2,1) \trianglerighteq (1)$ is the nested sequence of partitions.

CHAPTER III

Geometric Background

3.1 Richardson varieties

Let GL_n be the general linear group of invertible matrices in M_n and B^- be the Borel subgroup of GL_n of lower-left triangular matrices. Let $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$.

Definition III.1. A complete flag, or flag $F_\bullet = (F_1 \subset F_2 \subset \cdots \subset F_n)$ is an increasing sequence of subspaces of $\mathbb{C}^n$ such that F_i has dimension i for $i = 1, 2, \ldots, n$. For $w \in S_n$, the coordinate flag $wE_\bullet$ is defined as

$$wE_\bullet := \mathrm{span}\{e_{w(1)}\} \subset \mathrm{span}\{e_{w(1)}, e_{w(2)}\} \subset \cdots \subset \mathrm{span}\{e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\}.$$

In particular, we call $E_\bullet := 1E_\bullet$, the forward flag and $\widetilde{E}_\bullet := w_0 E_\bullet$, the opposite, or backward flag.

Definition III.2. Flag variety $F\ell_n$ is the set of complete flags in $\mathbb{C}^n$. We also identify $F\ell_n$ with the homogeneous space $B^- \setminus \mathrm{GL}_n$ by the right GL_n-equivariant map that sends $B^- g$ to $F_\bullet$ where F_i of $F_\bullet$ is given by the span of the first i-rows of the matrix $g \in \mathrm{GL}_n$.

Definition III.3. We define the Schubert cell $X_w^\circ \subseteq F\ell_n$ as the set

$$X_w^\circ := \{F_\bullet \in F\ell_n : \mathrm{rank}(F_i \to E_j) = r_w(i,j) \text{ for } 1 \leqslant i, j \leqslant n\}$$

where r_w is the rank function defined in Section 2.2.1 and the linear map $F_i \to E_j$ is the restriction of the linear projection $\mathbb{C}^n \twoheadrightarrow E_j$.

We define the Schubert variety $X_w := \overline{X_w^\circ}$ as the closure of the Schubert cell X_w° in $F\ell_n$. As a set,

$$X_w = \{F_\bullet \in F\ell_n : \mathrm{rank}(F_i \to E_j) \leqslant r_w(i,j) \text{ for } 1 \leqslant i,j \leqslant n\}.$$

The coordinate flag $wE_\bullet$ is a point in X_w° and $vE_\bullet$ is a point in X_w if $v \geqslant w$. Both sets, X_w° and X_w have codimension $\ell(w)$ in $F\ell_n$. There are well-known cell decompositions,

$$F\ell_n = \bigsqcup_{w \in S_n} X_w^\circ, \qquad X_w = \bigsqcup_{v \geqslant w} X_v^\circ.$$

Similarly, we define the opposite Schubert cell by

$$(X^w)^\circ := \{F_\bullet \in F\ell_n : \mathrm{rank}(F_i \to \widetilde{E}_j) = r_{w_0 w}(i,j) \text{ for } 1 \leqslant i,j \leqslant n\}$$

and the opposite Schubert variety by

$$X^w := \{F_\bullet \in F\ell_n : \mathrm{rank}(F_i \to \widetilde{E}_j) \leqslant r_{w_0 w}(i,j) \text{ for } 1 \leqslant i,j \leqslant n\}.$$

We can reinterpret the above definitions from the perspective of the homogeneous space $B^- \backslash \mathrm{GL}_n$. Let g be an invertible matrix and $F_\bullet$ be the flag determined by the row spans of g. Then, the flag $F_\bullet$ is in X_w° (respectively, X_w) if and only if, for all $1 \leqslant i,j \leqslant n$, the rank of the upper-left $i \times j$-submatrix of g is the same as (respectively, less than or equal to) the rank of the corresponding submatrix of the permutation matrix for w. Similarly, $F_\bullet$ is in $(X^w)^\circ$ (respectively, X^w) if the ranks of the upper-right submatrices of g are equal to (respectively, less than or equal to) those of w.

Definition III.4. For $u, w \in S_n$, Richardson variety is defined as the intersection,

$$X_u^w = X_u \cap X^w \quad \text{and} \quad (X_u^w)^\circ = X_u^\circ \cap (X^w)^\circ.$$

The varieties X_u^w and $(X_u^w)^\circ$ are nonempty if and only if $u \leqslant w$, in which case both varieties have dimension $\ell(w) - \ell(u)$ and X_u^w is reduced and irreducible. The coordinate flag $vE_\bullet$ is a point in X_u^w if and only if $u \leqslant v \leqslant w$.

We refer to [Ful97, RS97, Man01] and [Bri05] for further background on the geometry and combinatorics of flag and Schubert varieties.

3.2 Matrix Schubert varieties

Let M_n be the variety of $n \times n$ matrices over $\mathbb{C}$ and $Z = (z_{ij})$ be a generic matrix of indeterminates though occasionally Z will denote an element of M_n. We write $\mathbb{C}[Z]$ for the polynomial ring over $\mathbb{C}$ with indeterminates z_{ij}, $1 \leqslant i, j \leqslant n$ so that $M_n = \mathrm{Spec}(\mathbb{C}[Z])$.

Definition III.5. Matrix Schubert variety $\widetilde{X}_w$ is the subvariety of M_n defined by

$$\widetilde{X}_w = \{Z \in M_n : \mathrm{rank}(Z_{i\times j}) \leqslant r_w(i,j) \text{ for } 1 \leqslant i, j \leqslant n\}$$

where $Z_{i\times j}$ denotes the upper-left $i \times j$ submatrix of Z and r_w is the rank function for w.

Matrix Schubert varieties and their defining ideals were introduced in [Ful92] though in a slightly different language from ours.

Definition III.6. Let the Schubert determinantal ideal $\widetilde{I}_w$ be the ideal generated by the minors of $Z_{i\times j}$ of size $1 + r_w(i,j)$ for $1 \leqslant i, j \leqslant n$. Notice that the polynomials of $\widetilde{I}_w$ carve out $\widetilde{X}_w$ from M_n.

Schubert determinantal ideals are known to be prime so the ideals $\widetilde{I}_w$ and $I(\widetilde{X}_w)$ coincide. See [Ful92, KM05a, MS05] for further details on various algebraic and geometric properties of matrix Schubert varieties and Schubert determinantal ideals.

For example, five of the six matrix Schubert varieties for $n = 3$ are linear subspaces:

$$\widetilde{I}_{123} = 0, \quad \widetilde{X}_{132} = M_3$$

$$\widetilde{I}_{213} = \langle z_{11} \rangle, \quad \widetilde{X}_{213} = \{Z \in M_3 : z_{11} = 0\}$$

$$\widetilde{I}_{231} = \langle z_{11}, z_{12} \rangle, \quad \widetilde{X}_{231} = \{Z \in M_3 : z_{11} = z_{12} = 0\}$$

$$\widetilde{I}_{312} = \langle z_{11}, z_{21} \rangle, \quad \widetilde{X}_{312} = \{Z \in M_3 : z_{11} = z_{21} = 0\}$$

$$\widetilde{I}_{321} = \langle z_{11}, z_{12}, z_{21} \rangle, \quad \widetilde{X}_{321} = \{Z \in M_3 : z_{11} = z_{12} = z_{21} = 0\}.$$

For the remaining permutation $w = 132$,

$$\widetilde{I}_{132} = \langle z_{11}z_{22} - z_{12}z_{21} \rangle, \quad \widetilde{X}_{132} = \{Z \in M_3 : \mathrm{rank}(Z_{2\times 2}) \leqslant 1\},$$

that defines the set of matrices whose upper-left 2×2 block is singular.

3.3 Gröbner degeneration

Gröbner bases and their analogues for subalgebras allow us to degenerate interesting but "complicated" rings to simpler objects defined by monomials, hence accessible through combinatorial methods. Geometrically, Gröbner bases degenerate varieties into schemes defined by monomial ideals. Their subalgebra analogues degenerate parametrically presented varieties into toric varieties. Our references for the material presented here are [BC03, Stu96] and [Eis95, Chapter 15].

3.3.1 Gröbner/SAGBI bases

Let $S := \mathbb{C}[z_1, z_2, \ldots, z_m]$ be the polynomial ring in m indeterminates. The monomials in S are denoted $\mathbf{z}^{\mathbf{a}} := z_1^{a_1} z_2^{a_2} \ldots z_m^{a_m}$; we at times identify monomials with lattice points in $\mathbb{N}^m$ by identifying $\mathbf{z}^{\mathbf{a}}$ with $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ in $\mathbb{N}^m$, where $\mathbb{N}$ denotes the set of non-negative integers.

A total order $<$ on the monomials in S is a term order if $\mathbf{z}^{\mathbf{0}} = 1$ is the unique minimal element and $\mathbf{z}^{\mathbf{a}} < \mathbf{z}^{\mathbf{b}}$ implies that $\mathbf{z}^{\mathbf{a}} \cdot \mathbf{z}^{\mathbf{c}} < \mathbf{z}^{\mathbf{b}} \cdot \mathbf{z}^{\mathbf{c}}$ for all $\mathbf{c} \in \mathbb{N}^m$. Most widely used examples of term orders include the lexicographic order, graded lexicographic order, and graded reverse lexicographic order.

Given a term order $<$ and a nonzero polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}}$ in S, we define the initial term of f with respect to $<$ to be the term $c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}}$ of f such that $\mathbf{z}^{\mathbf{a}'} < \mathbf{z}^{\mathbf{a}}$ for all $\mathbf{z}^{\mathbf{a}'}$'s in the support of f that are distinct from $\mathbf{z}^{\mathbf{a}}$. Let $\mathrm{in}_<(f)$ denote the initial term of f.

Definition III.7. Let I be an ideal of S. Define the initial ideal of I with respect to $<$ to be the monomial ideal,

$$\mathrm{in}_<(I) := \langle \mathrm{in}_<(f) : f \in I \rangle.$$

We note with emphasis that $\mathrm{in}_<(I)$ is not usually generated by the initial terms of a minimal generating set for I. Monomials that do not lie in $\mathrm{in}_<(I)$ are called standard monomials.

Definition III.8. A finite subset $\mathcal{G}_< = \{g_1, g_2, \ldots, g_s\}$ of I is called a Gröbner basis for I with respect to $<$, if $\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \ldots, \mathrm{in}_<(g_s) \rangle$.

Let R be a finitely generated subalgebra of S. Fix a term order $<$ on S. Define the initial algebra $\mathrm{in}_<(R)$ as the $\mathbb{C}$-vector space spanned by $\mathrm{in}_<(f)$ for $f \in R$.

Definition III.9. A finite subset $\mathcal{S}_<$ of R is called a SAGBI basis of R with respect to $<$, if $\mathrm{in}_<(R)$ is generated as a $\mathbb{C}$-algebra by $\mathrm{in}_<(f)$ for $f \in \mathcal{S}$. The term SAGBI is the acronym for "**S**ubalgebra **A**nalog to **G**röbner **B**ases for **I**deals."

The initial algebra $\mathrm{in}_<(R)$ need not be finitely generated so that there is no finite SAGBI basis for R with respect to $<$. On the other hand, if $\mathrm{in}_<(R)$ is finitely generated, then so is R, in which case R is generated as a $\mathbb{C}$-algebra over a SAGBI basis.

There are number of algorithms for computation of Gröbner and SAGBI bases with the most well-known being Buchberger's algorithm which takes as input a set of generators for I and outputs a Gröbner basis.

3.3.2 Flat families

To present the deformation of initial ideals and algebras we generalize the notions of initial objects with respect to term orders to initial objects with respect to weights. Then, we display a construction of a flat family connecting the original objects to their initial counterparts.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ be a non-negative vector in $\mathbb{R}^m_{\geqslant 0}$ called a weight vector. For a nonzero $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} \in S$, we define the initial term $\mathrm{in}_\omega(f)$ to be the sum of terms $c_{\mathbf{a}}\mathbf{z}^{\mathbf{a}}$ supported on f for which $\omega \cdot \mathbf{a} := \omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_m a_m$ is minimized. Also, define $\omega(f)$ to be the minimum value of $\omega \cdot \mathbf{a}$ as $\mathbf{a}$ varies over the support of f.

Let $S = \mathbb{C}[z_1, z_2, \ldots, z_m]$ as in the previous section with term order $<$. Let I be an ideal of S and R be a finitely generated subalgebra of S.

Definition III.10. Define the initial ideal $\mathrm{in}_\omega(I)$ with respect to ω to be the ideal generated by all initial terms so that

$$\mathrm{in}_\omega(I) := \langle \mathrm{in}_\omega(f) : f \in I \rangle.$$

A finite subset $\mathcal{G}_\omega = \{g_1, g_2, \ldots, g_s\}$ of I is called a Gröbner basis for I with respect to ω if $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(g_1), \mathrm{in}_\omega(g_2), \ldots, \mathrm{in}_\omega(g_s) \rangle$.

Definition III.11. Define the initial algebra $\mathrm{in}_\omega(R)$ with respect to ω as the $\mathbb{C}$-vector space spanned by $\mathrm{in}_\omega(f)$ for $f \in R$. If $\mathrm{in}_\omega(R)$ is finitely generated, a finite set

$\mathcal{S}_\omega = \{f_1, f_2, \ldots, f_s\}$ is called a SAGBI basis for R with respect to ω, if $\mathrm{in}_\omega(R)$ is generated by $\{\mathrm{in}_\omega(f_1), \mathrm{in}_\omega(f_2), \ldots, \mathrm{in}_\omega(f_s)\}$ as a $\mathbb{C}$-algebra.

Note that $\mathrm{in}_\omega(I)$ as defined above may not be a monomial ideal; however, it is when ω is chosen sufficiently generically, eliminating ties among monomials. Moreover, given $\omega \in \mathbb{R}^m_{\geqslant}$ such that $\mathrm{in}_\omega(I)$ is a monomial ideal, we can define a new term order $<_\omega$ such that $\mathrm{in}_\omega(I) = \mathrm{in}_{<_\omega}(I)$ [Stu96, Corollary 1.10]. Conversely, given an ideal, or subalgebra and a term order, there exists a weight vector such that the initial ideal, or initial algebra with respect to the term order can be realized in terms of the weight vector.

Proposition III.12. *[BC03, Proposition 3.8] If* $\mathrm{in}_<(R)$ *is finitely generated as a* $\mathbb{C}$-*algebra, then there exists an integral weight* $\omega \in \mathbb{N}^m$ *such that* $\mathrm{in}_<(R) = \mathrm{in}_\omega(R)$ *and* $\mathrm{in}_<(I) = \mathrm{in}_\omega(I)$.

Fix an ideal $I \subseteq S$ and weight vector $\omega \in \mathbb{R}^n_{\geqslant 0}$. We define the set $C[\omega] \subseteq \mathbb{R}^n_{\geqslant 0}$ by

$$C[\omega] := \{\omega' \in \mathbb{R}^n_{\geqslant 0} : \mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)\},$$

which is known to be a relative interior of a polyhedral cone inside $\mathbb{R}^m_{\geqslant 0}$ [Stu96, Proposition 2.3].

Lemma III.13. *Let* I *be homogeneous and* $\mathcal{G}_\omega = \{g_1, g_2, \ldots, g_s\}$ *be a Gröbner basis of* I *with respect to* ω. *Then,* $\{\omega' \in \mathbb{R}^n_{\geqslant 0} : \mathrm{in}_{\omega'}(g) = \mathrm{in}_\omega(g) \quad \text{for } g \in \mathcal{G}_\omega\} \subseteq C[\omega]$.

Proof. Suppose $\omega' \in \mathbb{R}^n_{\geqslant 0}$ such that $\mathrm{in}_{\omega'}(g) = \mathrm{in}_\omega(g)$ for $g \in \mathcal{G}_\omega$. Then, $\mathrm{in}_\omega(I) \subseteq \mathrm{in}_{\omega'}(I)$ since $\mathrm{in}_\omega(I)$ is generated by $\mathrm{in}_\omega(g_1), \mathrm{in}_\omega(g_2), \ldots, \mathrm{in}_\omega(g_s)$. But by flatness of passage from an ideal to its initial ideal, the Hilbert series of $S/\mathrm{in}_\omega(I)$ and $S/\mathrm{in}_{\omega'}(I)$ are the same. So the inclusion, $\mathrm{in}_\omega(I) \subseteq \mathrm{in}_{\omega'}(I)$ implies equality as in $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)$. $\square$

In the ensuing discussion, we assume that the weight vectors are integral.

Informally, the flat family of algebras degenerating an ideal to its initial ideal can be described as follows. For each $t \in \mathbb{C}^*$, there is an automorphism of S that sends x_i to $t^{\omega_i} x_i$. Let I_t be the image of I under this automorphism. Notice that for $t \in \mathbb{C}^*$, all of the rings S/I_t are isomorphic to S/I, but as t approaches 0, the initial terms of polynomials in I_t become dominant. Therefore, in the limit, the fiber over $t = 0$ is equal to $S/\operatorname{in}_\omega(I)$.

To make the above description more precise, for nonzero polynomial $f = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in S$, we define

$$\tilde{f} := t^{-\omega(f)} \sum_{\mathbf{a}} c_{\mathbf{a}}\, t^{\omega \cdot \mathbf{a}}\, \mathbf{x}^{\mathbf{a}} \in S[t].$$

By definition of $\omega(f)$, $\tilde{f}$ is equal to $\operatorname{in}_\omega(f)$ plus t times a polynomial in $S[t]$.

Definition III.14. Let I be an ideal of S. We define the ideal $\mathcal{I}$ of $S[t]$ by $\mathcal{I} := \langle \tilde{f} \in S[t] : f \in I \rangle$. Let $\mathcal{G}_\omega = \{g_1, g_2, \ldots, g_s\}$ be a Gröbner basis of I. Then, it is not difficult to show that $\mathcal{I} = \langle \tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_s \rangle$.

The following theorem is fundamental to the degeneration of I to $\operatorname{in}_\omega(I)$.

Theorem III.15. *[Eis95, Theorem 15.17] Let I be an ideal of S. The $\mathbb{C}[t]$-algebra $S[t]/\mathcal{I}$ is free and, thus flat as a $\mathbb{C}[t]$-module. Furthermore,*

$$S[t]/\mathcal{I} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong (S/I)[t, t^{-1}],$$

$$S[t]/\mathcal{I} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/\langle t \rangle \cong S/\operatorname{in}_\omega(I).$$

It follows that $S[t]/\mathcal{I}$ is a flat family over $\mathbb{C}[t]$ whose fiber over $t = 0$ is $S/\operatorname{in}_\omega(I)$, and fibers over $t \in \mathbb{C}^*$ is S/I.

Geometrically, Theorem III.15 says that $\operatorname{Spec}(S[t]/\mathcal{I}) \subseteq \mathbb{A}^m \times \mathbb{A}^1$ is a flat family over $\mathbb{A}^1$. Moreover, if I is homogeneous, then so is $\mathcal{I}$ with respect to the usual

$\mathbb{N}$-grading of S, hence

$$\mathrm{Proj}(S[t]/\mathcal{I}) \subseteq \mathbb{P}^{m-1} \times \mathbb{A}^1$$

is a flat family over $\mathbb{A}^1$. By construction the fiber over each $t \in \mathbb{C}^*$ is $\mathrm{Spec}(S/I)$, or $\mathrm{Proj}(S/I)$ while the fiber over $t = 0$ is $\mathrm{Spec}(S/\operatorname{in}_\omega(I))$ or $\mathrm{Proj}(S/\operatorname{in}_\omega(I))$, respectively. We call this flat degeneration from an affine or projective scheme to the scheme determined by the initial ideal a Gröbner degeneration.

Given a subalgebra with a finite SAGBI basis, we can similarly construct a flat family degenerating the subalgebra to its initial algebra, which we call a SAGBI degeneration. Let $\mathcal{S}_\omega$ be a SAGBI basis for $R \subseteq S$ with respect to $\omega \in \mathbb{N}^m$. Let $A := \mathbb{C}[x_1, x_2, \ldots, x_s]$ be the polynomial ring in s indeterminates and $\omega' := (\omega(f_1), \ldots, \omega(f_s)) \in \mathbb{N}^s$ be the weight vector on A.

Let $\mathcal{R}$ be the subalgebra of $S[t]$ generated by deformations of elements of the SAGBI basis as elements of S so that

$$\mathcal{R} := \mathbb{C}[t][\tilde{f} : f \in R].$$

Let I be an ideal of R and $\mathcal{I}$ be the ideal of $\mathcal{R}$ defined by

$$\mathcal{I} := \langle \tilde{f} \in \mathcal{R} : f \in I \rangle.$$

Lemma III.16. *[BC03, Lemma 2.2] Let $\varphi : A[t] \longrightarrow \mathcal{R}/\mathcal{I}$ be the $\mathbb{C}[t]$-algebra map defined by $\varphi(x_i) = \tilde{f}_i$. By restricting φ to the fibers over $t = 1$ and $t = 0$, there are maps $\varphi_1 : A \longrightarrow R/I$ and $\varphi_0 : A \longrightarrow \operatorname{in}_\omega(R)/\operatorname{in}_\omega(I)$ defined by $\varphi_1(x_i) = f_i$ and $\varphi_0(x_i) = \operatorname{in}_\omega(f_i)$. Then,*

$$\operatorname{in}_{\omega'}(\ker(\varphi_1)) = \ker(\varphi_0).$$

We write J for the ideal $\ker(\varphi_1) \subseteq A$ and $\mathcal{J}$ for the ideal of $A[t]$ obtained through deformation of J with respect to $\omega' \in \mathbb{N}^s$. Notice that $A[t]/\mathcal{J} \cong \mathcal{R}/\mathcal{I}$

and $A/\operatorname{in}_{\omega'}(J) \cong \operatorname{in}_\omega(R)/\operatorname{in}_\omega(I)$ since $\{f_1, f_2, \ldots, f_s\}$ form a SAGBI basis for R and $A/J \cong R/I$ by definition of J. We may then apply Theorem III.15 to obtain the following.

Corollary III.17. *The algebra $\mathcal{R}/\mathcal{I}$ is flat as a $\mathbb{C}[t]$-module and*

$$\mathcal{R}/\mathcal{I} \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}] \cong (R/I)[t, t^{-1}]$$

$$\mathcal{R}/\mathcal{I} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/\langle t\rangle \cong \operatorname{in}_\omega(R)/\operatorname{in}_\omega(I).$$

It follows that $\mathcal{R}/\mathcal{I}$ is a flat family over $\mathbb{C}[t]$ whose fiber over $t = 0$ is $\operatorname{in}_\omega(R)/\operatorname{in}_\omega(I)$ and a fiber over $t \in \mathbb{C}^*$ is R/I.

Notation III.18. Our notations for Gröbner and SAGBI degenerations are as follows. Let $\mathcal{X}$ be a flat family over $\mathbb{A}^1$ degenerating either an ideal $I \subseteq S$, in which case $\mathcal{X}$ corresponds to $\mathcal{I} \subseteq S[t]$, or a subalgebra $R \subseteq S$, in which case $\mathcal{X}$ corresponds to $\mathcal{R} \subseteq S[t]$. We write $\lim_{t\to 0} \mathcal{X}$, $\operatorname{in}_\omega(X)$, or $\mathcal{X}_0$ to denote the fiber of $\mathcal{X}$ over $t = 0$. In the Gröbner case, let $\lim_{t\to 0} \mathcal{I}$ denote $\operatorname{in}_\omega(I)$ and in the SAGBI case, $\lim_{t\to 0} \mathcal{R}$ denotes $\operatorname{in}_\omega(R)$.

3.3.3 Gröbner degeneration of matrix Schubert varieties

A term order $\leqslant$ on $\mathbb{C}[Z]$ is called antidiagonal if the initial term of every minor of the generic matrix Z is its antidiagonal term. Let $\Delta_{I,J}(Z)$ be the determinant of the square submatrix Z whose rows are indexed by $I = \{i_1 > i_2 > \cdots > i_k\}$ and columns by $J = \{j_1 > j_2 > \cdots > j_k\}$ so that

$$\Delta_{I,J}(Z) = \begin{vmatrix} z_{i_k,j_k} & z_{i_k,j_{k-1}} & \cdots & z_{i_k,j_1} \\ z_{i_{k-1},j_k} & z_{i_{k-1},j_{k-1}} & \cdots & z_{i_{k-1},j_1} \\ \vdots & & \ddots & \vdots \\ z_{i_1,j_k} & z_{i_1,j_{k-1}} & \cdots & z_{i_1,j_1} \end{vmatrix}$$

and

$$\mathrm{in}_{\leqslant}(\Delta_{I,J}(Z)) = (-1)^{\binom{k}{2}} z_{i_k,j_1} z_{i_{k-1},j_2} \dots z_{i_1,j_k}.$$

Examples of antidiagonal term orders include:

- the reverse lexicographic term order that winds its way from the northwest corner to the southeast corner so that $z_{11} > z_{12} > \cdots > z_{1n} > z_{21} > \cdots > z_{nn}$; and
- the lexicographic term order that winds its way from northeast corner to the southwest corner so that $z_{1n} > \cdots > z_{nn} > \cdots > z_{2n} > z_{11} > \cdots > z_{n1}$.

Definition III.19. Fix a vertex set $Q = \{1, 2, \dots, m\}$. Simplicial complexes on Q are in bijection with squarefree monomial ideals of $S = \mathbb{C}[z_1, z_2, \dots, z_m]$ through the correspondence that associates a simplicial complex Δ to its Stanley-Reisner ideal,

$$I_\Delta := \langle \prod_{i \in F} z_i : F \notin \Delta \rangle$$

so that $S/I_\Delta = \bigoplus_{\mathrm{supp}(\mathbf{a}) \in \Delta} \mathbb{C} \cdot \mathbf{z}^{\mathbf{a}}$. Geometrically, the Stanley-Reisner scheme, $\mathrm{Spec}(S/I_\Delta)$ is the reduced union of coordinate planes corresponding to the faces of Δ:

$$\mathrm{Spec}(S/I_\Delta) = \bigcup_{F \in \Delta} \mathbb{A}^F.$$

Theorem III.20. *[KM05a, Theorem B] The minors of size $1 + r_w(i,j)$ in $Z_{i \times j}$, for $1 \leqslant i, j \leqslant n$, form a Gröbner basis for $\widetilde{I}_w$ for any antidiagonal term order. Moreover, $\mathrm{in}_{\leqslant}(\widetilde{I}_w)$ is the Stanley-Reisner ideal of a simplicial complex whose facets correspond to reduced pipe dreams $D \in \mathcal{RP}_w$ so that*

$$\mathrm{in}_{\leqslant}(\widetilde{I}_w) = \bigcap_{D \in \mathcal{RP}_w} \langle z_{ij} : (i,j) \in D \rangle. \tag{3.1}$$

For $D \in \mathcal{RP}_w$, let L_D denote the coordinate subspace of M_n spanned by the coordinates z_{ij} such that $(i,j) \notin D$, hence $I(L_D) = \langle z_{ij} : (i,j) \in D \rangle$. As a consequence of Theorem III.20, the irreducible components of $\mathrm{in}_{\leqslant}(\widetilde{X}_w)$ are given by

$$\mathrm{in}_{\leqslant}(\widetilde{X}_w) = \bigcup_{D \in \mathcal{RP}_w} L_D.$$

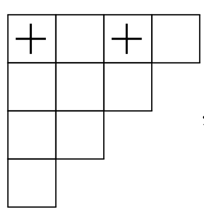
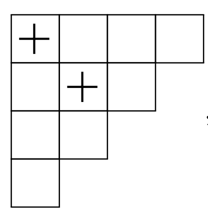
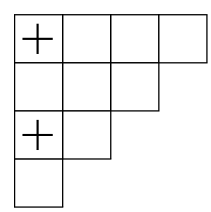
Example III.21. The matrix Schubert variety $\widetilde{X}_{2143}$ is the set of 4×4 matrices $Z = (z_{ij})$ whose upper-left entry is zero, and whose upper-left 3×3 block has rank at most two. The ideal of $\widetilde{X}_{2143}$ consists of the determinants

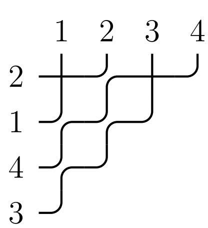
$$\widetilde{I}_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle = \langle z_{11}, -z_{13}z_{22}z_{31} + \dots \rangle$$

which has the initial ideal

$$\begin{aligned} \mathrm{in}_{\leqslant}(\widetilde{I}_{2143}) &= \langle z_{11}, -z_{13}z_{22}z_{31} \rangle = \langle z_{11}, z_{13} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{31} \rangle \\ &= I(L_{11,13}) \cap I(L_{11,22}) \cap I(L_{11,31}). \end{aligned}$$

On the geometry side, $\widetilde{X}_{2143}$ Gröbner degenerates to a union of three coordinate subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$ with ideals $\langle z_{11}, z_{13} \rangle$, $\langle z_{11}, z_{22} \rangle$, and $\langle z_{11}, z_{31} \rangle$, respectively. Pictorially, we represent the subspaces $L_{11,13}$, $L_{11,22}$, and $L_{11,31}$ as subsets

$\langle z_{11}, z_{13} \rangle =$  , $\langle z_{11}, z_{22} \rangle =$  , $\langle z_{11}, z_{31} \rangle =$ 



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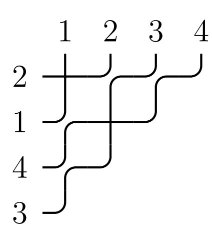

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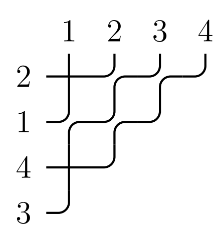


CHAPTER IV

Toric degeneration

In Section 4.1, we present the toric degeneration of the flag variety, then in Section 4.2 we construct an involution on the degeneration that maps Schubert varieties to opposite Schubert varieties. In Section 4.3, we examine the relation between the degeneration of a matrix Schubert variety and that of a Schubert variety.

4.1 Toric degeneration of the flag variety

In this section, we present a toric degeneration of $F\ell_n$ that is a slight modification of [KM05b]. We define a deformation of the action of B^- on M_n to a fiberwise action of B^- on the family $M_n \times \mathbb{A}^1$. We, then, identify the GIT quotient $\mathcal{X} = B^- \backslash\backslash (M_n \times \mathbb{A}^1)$ as the flat family degenerating $F\ell_n$. Lastly, in Section 4.1.4 we identify the Gröbner limit $\mathcal{X}_0$ as the projective toric variety of the Gelfand-Tsetlin polytope.

4.1.1 Degeneration of Borel group action

Definition IV.1. Let $\omega = (\omega_{ij})$ be the $n \times n$ matrix whose entries are

$$\omega_{ij} = \begin{cases} \binom{n+2-i-j}{2} & \text{if } i+j < n+1, \\ 0 & \text{if } i+j \geqslant n+1. \end{cases}$$

Notice that the entries of ω strictly above the main antidiagonal are triangular numbers and all other entries are zero.

For example, for $n = 5$,

$$\omega = \begin{bmatrix} 10 & 6 & 3 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We fix this definition of ω for the remainder of this thesis.

Given $t \in \mathbb{C}^*$, we define $\tilde{t} := (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ to be the element of $(\mathrm{GL}_n)^n$ where each $\tilde{t}_j := \mathrm{diag}(t^{\omega_{1j}}, t^{\omega_{2j}}, \ldots, t^{\omega_{nj}})$ for $j = 1, 2, \ldots, n$. Let $B^- \times \mathbb{C}^* \longrightarrow (\mathrm{GL}_n)^n \times \mathbb{A}^1$ be the embedding given by $(b, t) \longmapsto (\tilde{t}_1^{-1} b \tilde{t}_1, \tilde{t}_2^{-1} b \tilde{t}_2, \ldots, \tilde{t}_n^{-1} b \tilde{t}_n, t)$ for $b = (b_{ij}) \in B^-$ and $t \in \mathbb{C}^*$. For example, for $n = 5$, $(b, t) \in B^- \times \mathbb{C}^*$ is mapped to

$$\left(\begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ t^4 b_{21} & b_{22} & 0 & 0 & 0 \\ t^7 b_{31} & t^3 b_{32} & b_{33} & 0 & 0 \\ t^9 b_{41} & t^5 b_{42} & t^2 b_{43} & b_{44} & 0 \\ t^{10} b_{51} & t^6 b_{52} & t^3 b_{53} & t b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ t^3 b_{21} & b_{22} & 0 & 0 & 0 \\ t^5 b_{31} & t^2 b_{32} & b_{33} & 0 & 0 \\ t^6 b_{41} & t^3 b_{42} & t b_{43} & b_{44} & 0 \\ t^6 b_{51} & t^3 b_{52} & t b_{53} & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ t^2 b_{21} & b_{22} & 0 & 0 & 0 \\ t^3 b_{31} & t b_{32} & b_{33} & 0 & 0 \\ t^3 b_{41} & t b_{42} & b_{43} & b_{44} & 0 \\ t^3 b_{51} & t b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ t b_{21} & b_{22} & 0 & 0 & 0 \\ t b_{31} & b_{32} & b_{33} & 0 & 0 \\ t b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ t b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, t \right).$$

Definition IV.2. Let $(\mathcal{B}^-)^*$ be the family over $\mathbb{C}^*$ defined as the image of $B^- \times \mathbb{C}^*$ inside $(\mathrm{GL}_n)^n \times \mathbb{A}^1$ and $\mathcal{B}^-$ be the family over $\mathbb{A}^1$ defined as the closure, $\mathcal{B}^- := \overline{(\mathcal{B}^-)^*}$ in $(\mathrm{GL}_n)^n \times \mathbb{A}^1$.

Lemma IV.3. *[KM05b, Lemma 2] There is an isomorphism $B^- \times \mathbb{A}^1 \longrightarrow \mathcal{B}^-$ over $\mathbb{A}^1$ that extends $B^- \times \mathbb{C}^* \xrightarrow{\cong} (\mathcal{B}^-)^*$ over $t = 0$.*

The fiber of $\mathcal{B}^-$ over $t = 0$, denoted $\mathcal{B}_0^-$, consists of sequences $(b_1, b_2, \ldots, b_n) \in (B^-)^n$ where $b_n \in B^-$ and b_j, for $j = 1, 2, \ldots, n-1$, is obtained from b_n by setting to 0 all entries in columns $1, 2, \ldots, n-j$ that are strictly below the main diagonal. For example, for $n = 5$, the elements of $\mathcal{B}_0^-$ look like

$$\left(\begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 \\ 0 & 0 & b_{53} & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 & 0 \\ 0 & b_{42} & b_{43} & b_{44} & 0 \\ 0 & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix} \right).$$

There is a $(\mathrm{GL}_n)^n$-action on M_n by column-wise matrix multiplication: if $Z_1, Z_2, \ldots, Z_n$ are the columns of $Z \in M_n$, then $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in (\mathrm{GL}_n)^n$ acts on Z by

$$\mathbf{g} \cdot Z = (g_1, g_2, \ldots, g_n) \cdot \begin{bmatrix} Z_1 & Z_2 & \ldots & Z_n \end{bmatrix} = \begin{bmatrix} g_1 Z_1 & g_2 Z_2 & \ldots & g_n Z_n \end{bmatrix}. \tag{4.1}$$

The family $\mathcal{B}^-$ considered as a subset of $(\mathrm{GL}_n)^n \times \mathbb{A}^1$ acts fiberwise on $M_n \times \mathbb{A}^1$ through (4.1). Furthermore, Lemma IV.3 allows us to view the fiberwise action of $\mathcal{B}^-$ as a single action of B^- on the total space $M_n \times \mathbb{A}^1$.

The actions of B^- on all fibers $M_n \times \{t\}$ for $t \in \mathbb{C}^*$ are isomorphic in the sense that the map $(Z, 1) \longmapsto (\tilde{t}^{-1} \cdot Z, t)$ is a B^--equivariant isomorphism between $M_n \times \{1\}$ with $M_n \times \{t\}$.

4.1.2 SAGBI basis of the Plücker algebra

Recall that $Z = (z_{ij})$ denotes the generic $n \times n$ matrix of indeterminates.

Definition IV.4. For a subset $I \subseteq [n]$ of size k, let $\Delta_I(Z) \in \mathbb{C}[Z]$ be the minor $\Delta_I(Z)$ whose columns are indexed by the set I and rows $1, 2, \ldots, k$. We define the Plücker variable p_I by

$$p_I := \Delta_I(Z) = \begin{vmatrix} z_{1,i_k} & z_{1,i_{k-1}} & \cdots & z_{1,i_1} \\ z_{2,i_k} & z_{2,i_{k-1}} & \cdots & z_{2,i_1} \\ \vdots & & \ddots & \vdots \\ z_{k,i_k} & z_{k,i_{k-1}} & \cdots & z_{k,i_1} \end{vmatrix}.$$

We call the subalgebra $\mathbb{C}[p_I : I \in 2^{[n]}]$ of $\mathbb{C}[Z]$ generated by the $2^n - 1$ Plücker variables, the Plücker algebra.

For notational convenience, we write $P := \{p_I : I \in 2^{[n]}\}$ and $P^{(k)} := \{p_I : I \in \binom{[n]}{k}\}$ for $k = 1, 2, \ldots, n$. Then, the Plücker algebra can be written as

$$\mathbb{C}[P] = \mathbb{C}[P^{(1)}] \otimes \mathbb{C}[P^{(2)}] \otimes \cdots \otimes \mathbb{C}[P^{(n)}].$$

We degenerate the Plücker algebra $\mathbb{C}[P] \subseteq \mathbb{C}[Z]$ by considering the matrix ω as a weight vector on the coordinate ring $\mathbb{C}[Z]$ of M_n, weighing each variable z_{ij} by ω_{ij}.

While ω may not induce an antidiagonal term order on $\mathbb{C}[Z]$, the following lemma implies that for the purpose of degenerating matrix Schubert varieties ω is sufficient.

Lemma IV.5. *If all of the variables dividing the antidiagonal term of $\Delta_{I,J}(Z) \in \mathbb{C}[Z]$ are on or above the main antidiagonal of Z, then the unique monomial in $\Delta_{I,J}(Z)$ with the lowest weight is its antidiagonal term.*

Proof. Let $I = \{i_1 > i_2 > \cdots > i_k\}$ and $J = \{j_1 > j_2 > \cdots > j_k\}$ be subsets of $[n]$ such that $i_m + j_{k-m+1} \leqslant n+1$ for $m = 1, 2, \ldots, k$. Then, $\Delta_{I,J}(Z)$ is a signed sum of monomials $\prod_{m=1}^{k} z_{i_m, j_{w(m)}}$ for $w \in S_k$. Let $\underline{w} \in S_k$ be such that the weight of $\prod_{m=1}^{k} z_{i_m, j_{\underline{w}(m)}}$ is minimized; so $\sum_{m=1}^{k} \omega_{i_m, j_{\underline{w}(m)}} < \sum_{m=1}^{k} \omega_{i_m, j_{w(m)}}$ for all $w \in S_k$.

To prove that $\underline{w} = w_0$, suppose s is the largest integer such that $\underline{w}(s) \neq k+1-s$. Let t be the integer in $[k]$ such that $w(t) = k+1-s$. Let $w' = \underline{w} \cdot (s,t)$ so that $w'(s) = \underline{w}(t) = k+1-s$ and $w'(t) = \underline{w}(s)$. To compare the weight of $\prod_{m=1}^{k} z_{i_m, j_{w'(m)}}$ against that of $\prod_{m=1}^{k} z_{i_m, j_{\underline{w}(m)}}$ it suffices to compare $\omega_{i_s, j_{w'(s)}} + \omega_{i_t, j_{w'(t)}}$ against $\omega_{i_s, j_{\underline{w}(s)}} + \omega_{i_t, j_{\underline{w}(t)}}$ since the two monomials only differ by factors of $z_{i_s, j_{w'(s)}} z_{i_t, j_{w'(t)}}$ and $z_{i_s, j_{\underline{w}(s)}} z_{i_t, j_{\underline{w}(t)}}$. Indeed,

$$\omega_{i_s, j_{w'(s)}} + \omega_{i_t, j_{w'(t)}} < \omega_{i_s, j_{\underline{w}(s)}} + \omega_{i_t, j_{\underline{w}(t)}}$$

since the coordinates $(i_s, j_{w'(s)})$, $(i_t, j_{w'(t)})$, $(i_s, j_{\underline{w}(s)})$, and $(i_t, j_{\underline{w}(t)})$ form a square in ω thought of as a matrix while $\omega_{i_s, j_{\underline{w}(s)}} > 0$ and $\omega_{i_t, \underline{w}(t)} = 0$. Therefore, $\underline{w} = w_0$ and the antidiagonal term in $\Delta_{I,J}(Z)$ is the unique monomial with the lowest weight. $\square$

Definition IV.6. Let $\omega_I := \sum_{s=1}^{k} \omega_{s,i_s}$ so that ω_I is equal to the weight of the antidiagonal term in p_I. We define the degenerated Plücker variable $q_I \in \mathbb{C}[t][Z]$ by

$$q_I := \widetilde{p_I} = t^{-\omega_I} \Delta_I(\tilde{t} \cdot Z)$$

where $\tilde{t} \cdot Z = (t^{\omega_{ij}} z_{ij})$. We call the $\mathbb{C}[t]$-subalgebra generated by the $2^n - 1$ degenerated Plücker variables $\mathbb{C}[t]\left[q_I : I \in 2^{[n]}\right]$, the degenerated Plücker algebra.

Again for notational convenience, let $Q := \{q_I : I \in 2^{[n]}\}$ and $Q^{(k)} := \{q_I : I \in \binom{[n]}{k}\}$ for $k = 1, 2, \dots, n$, so that the degenerated Plücker algebra can be written as

$$\mathbb{C}[t][Q] = \mathbb{C}[Q^{(1)}] \otimes \mathbb{C}[Q^{(2)}] \otimes \cdots \otimes \mathbb{C}[Q^{(n)}] \otimes \mathbb{C}[t].$$

Definition IV.7. Let N^- be the normal subgroup of B^- consisting of lower triangular matrices with 1's on the diagonal. We call N^-, the maximal unipotent subgroup of GL_n:

$$N^- = \left\{ \begin{bmatrix} 1 & 0 & \dots & 0 \\ b_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & 1 \end{bmatrix} \right\}.$$

The Borel subgroup B^- is equal to the product[1] TN^- where $T \cong (\mathbb{C}^*)^n$ is the n-dimensional torus of GL_n consisting of diagonal matrices.

Theorem IV.8. *[KM05b, Theorem 5] The ring of N^--invariant functions on M_n is the Plücker algebra and the Plücker variables form a SAGBI basis for any diagonal or antidiagonal term order.*

Notice that the N^--action on M_n through $(\mathcal{B}^-)_1$ coincides with matrix multiplication. The generators of $\mathbb{C}[Z]^{N^-}$ were known to classical invariant theory wherein the Plücker algebra was called the algebra of primary covariants.

Recall from the previous section our degeneration of the action of B^- on M_n to an action on $M_n \times \mathbb{A}^1$. Let $\widetilde{Z} = (\widetilde{z}_{ij})$ be a $n \times n$ matrix of indeterminates defined by $\widetilde{Z} := \tilde{t} \cdot Z$. To see that the variable q_I is N^--invariant, notice that for $u \in N^-$,

$$u \cdot q_I = t^{-\omega_I} \Delta_I(u_\Delta^{-1} \widetilde{Z}) = t^{-\omega_I} \Delta_I(\widetilde{Z}) = q_I$$

[1]More precisely, B^- is the semi-direct product, $T \rtimes N^-$.

where u_Δ^{-1} denotes the element $(u^{-1}, u^{-1}, \ldots, u^{-1}) \in (\mathrm{GL}_n)^n$. Therefore, the degenerated Plücker algebra is a subalgebra of the N-invariant ring of $\mathbb{C}[t][Z]$. As a matter of fact, the q_I's generate the invariant ring, $\mathbb{C}[t][Z]^{N^-}$.

Theorem IV.9. *[KM05b, Theorem 5] The $\mathbb{C}[t]$-algebra of N^--invariant functions on $M_n \times \mathbb{A}^1$ is the degenerated Plücker algebra.*

4.1.3 Toric degeneration

Let $\mathbb{C}[t][X] := \mathbb{C}[t][x_I : I \in 2^{[n]}]$ be the polynomial ring in $2^n - 1$ variables over $\mathbb{C}[t]$ such that

$$\mathbb{C}[t][X] = \mathbb{C}[X^{(1)}] \otimes \mathbb{C}[X^{(2)}] \otimes \cdots \otimes \mathbb{C}[X^{(n)}] \otimes \mathbb{C}[t]$$

where $X^{(k)} := \left\{ x_I : I \in \binom{[n]}{k} \right\}$ for $k = 1, 2, \ldots, n$. Let $\varphi : \mathbb{C}[t][X] \longrightarrow \mathbb{C}[t][Q]$ be the map of $\mathbb{C}[t]$-algebras defined by $\varphi(x_I) = q_I$. By restricting φ to the fiber over $t = 1$, we obtain the map $\varphi_1 : \mathbb{C}[X] \longrightarrow \mathbb{C}[P]$ that presents the Plücker algebra $\mathbb{C}[P]$ as a quotient of the polynomial ring $\mathbb{C}[X]$.

To define a multigrading on $\mathbb{C}[X]$, recall that Λ_n^+ denotes the set of partitions with at most n parts and Λ_n^{++} denotes the subset of Λ_n^+ consisting of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\varpi_k \in \Lambda^+$ be the integer vector defined by $\varpi_k := (\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{n-k \text{ times}})$ for $k = 1, 2, \ldots, n$.

We define a multigrading on $\mathbb{C}[X]$ by Λ_n^+ by setting $\deg(x_I) = \varpi_k$ for $I \in \binom{[n]}{k}$ so that

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{C}[X]_\lambda.$$

Let $I := \ker(\varphi_1)$ and notice that I is homogeneous with respect to this multigrading since I is generated by homogeneous elements (called Garnir elements; see, Section 4.2) so that

$$\mathbb{C}[P] = \bigoplus_{\lambda\in\Lambda_n^+} \mathbb{C}[P]_\lambda.$$

Definition IV.10. Let $\rho := (n-1, n-2, \ldots, 1, 0) \in \Lambda_n^{++}$ be the sum $\rho = \varpi_1 + \varpi_2 + \cdots + \varpi_{n-1}$. We define the Geometric Invariant Theory (GIT) quotient of M_n by B^- as the Proj of the subring of N^--invariant functions on M_n in degrees that are multiples of ρ. More precisely,

$$B^-\backslash\!\backslash M_n := \operatorname{Proj}\left(\bigoplus_{d\geqslant 0} \mathbb{C}[P]_{d\rho}\right).$$

Notice that the decomposition

$$\mathbb{C}[X]_{d\rho} = \mathbb{C}[X^{(1)}]_{d\varpi_1} \otimes \mathbb{C}[X^{(2)}]_{d\varpi_2} \otimes \cdots \otimes \mathbb{C}[X^{(n)}]_{d\varpi_n}$$

implies that $\operatorname{Proj}\left(\bigoplus_{d\geqslant 0}\mathbb{C}[X]_{d\rho}\right)$ is the Segre product $\prod_{k=1}^n \operatorname{Proj}(\mathbb{C}[X^{(k)}]) = \prod_{k=1}^n \mathbb{P}^{\binom{n}{k}-1}$. So Definition IV.10 defines $B^-\backslash\!\backslash M_n$ as a subscheme of a product of projective spaces. More importantly, $B^-\backslash\!\backslash M_n$ is equal to the Plücker embedding of $F\ell_n$ into $\prod_{k=1}^n \mathbb{P}^{\binom{n}{k}-1}$.

Remark IV.11. Notice that $\varpi_1, \varpi_2, \ldots, \varpi_n$ generate Λ_n^+ as a semigroup such that a partition $\lambda \in \Lambda_n^+$ can be written as a unique linear combination of them, hence $\Lambda_n^+ \cong \mathbb{N}^n$. To a partition $\lambda \in \Lambda_n^+$, we can associate the integer vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ defined by $\mathbf{a} := (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n)$ so that $\lambda = a_1\varpi_1 + a_2\varpi_2 + \cdots + a_n\varpi_n$.

Definition IV.10 works equally well with any $\lambda \in \Lambda_n^{++}$ in place of ρ. The only difference is that the embedding with respect to λ composes the above Segre product with a_k-uple Veronese embedding of $\mathbb{P}^{\binom{[n]}{k}-1}$.

The rings $\mathbb{C}[t][X]$ and $\mathbb{C}[t][Q]$ are similarly multigraded by Λ_n^+ by setting

$$\deg(x_I) = \varpi_k, \quad \deg(q_I) = \varpi_k$$

for $I \in \binom{[n]}{k}$, and $\deg(t) = \mathbf{0}$. Let $\mathcal{I} := \ker(\varphi)$ and notice that $\mathcal{I}$ is homogeneous since it is a deformation of I.

Definition IV.12. The GIT quotient $B^- \backslash\backslash (M_n \times \mathbb{A}^1)$ is the Proj of the $\mathbb{C}[t]$-algebra of N^--invariant functions on $M_n \times \mathbb{A}^1$ generated in degree ρ where $\rho := (n-1, n-2, \ldots, 1, 0) \in \Lambda_n^{++}$. Let $\mathcal{X}$ be the family over $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[t])$ defined by

$$\mathcal{X} := B^- \backslash\backslash (M_n \times \mathbb{A}^1) = \mathrm{Proj}\left(\bigoplus_{d \geqslant 0} \mathbb{C}[t][Q]_{d\rho}\right).$$

Notice that $\mathcal{X}$ is a subscheme of $\mathrm{Proj}\left(\bigoplus_{d\geqslant 0} \mathbb{C}[t][X]_{d\rho}\right) = \prod_{k=1}^{n} \mathbb{P}^{\binom{n}{k}-1} \times \mathbb{A}^1$ and that Corollary III.17 implies that $\mathcal{X} \longrightarrow \mathbb{A}^1$ is a flat family. Remark IV.11 also applies in this case.

4.1.4 Gelfand-Tsetlin toric variety

We show that the zero fiber

$$\mathcal{X}_0 = \mathrm{Proj}\left(\bigoplus_{d \geqslant 0} \mathrm{in}_\omega(\mathbb{C}[P])_{d\lambda}\right)$$

for $\lambda \in \Lambda_n^{++}$ is the toric variety of the GT-polytope $\mathcal{P}_\lambda$. Our reference for toric varieties is [CLS11].

Definition IV.13. Let $\mathcal{P} \subseteq \mathbb{R}^N$ be a full dimensional lattice polytope and let the cone of $\mathcal{P}$ be defined by

$$\mathcal{C}(\mathcal{P}) := \mathrm{Cone}(\mathcal{P} \times \{1\}) \subseteq \mathbb{R}^N \times \mathbb{R}.$$

The key feature of this cone is that $d\mathcal{P}$ is the "slice" of $\mathcal{C}(\mathcal{P})$ at height d, from which it follows that the lattice points $m \in d\mathcal{P} \cap \mathbb{Z}^N$ corresponds to points $(m, d) \in \mathcal{C}(\mathcal{P}) \cap (\mathbb{Z}^N \times \mathbb{Z})$.

Definition IV.14. Let $S_\mathcal{P}$ be the subring of $\mathbb{C}[\mathbb{Z}^N \times \mathbb{Z}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_N^{\pm 1}, t^{\pm 1}]$ defined by

$$S_\mathcal{P} := \mathbb{C}[\mathcal{C}(\mathcal{P}) \cap (\mathbb{Z}^N \times \mathbb{Z})] = \bigoplus_{d \geqslant 0,\, m \in d\mathcal{P} \cap \mathbb{Z}^N} \mathbb{C} \cdot \mathbf{x}^m t^d.$$

Notice that $S_{\mathcal{P}}$ is the semigroup algebra of the cone $\mathcal{C}(\mathcal{P}) = \mathrm{Cone}(P \times \{1\}) \subseteq \mathbb{R}^N \times \mathbb{R}$. There is a $\mathbb{N}$-grading on $S_{\mathcal{P}}$ defined by $\deg(\mathbf{x}^m t^d) = d$. We define $X_{\mathcal{P}} := \mathrm{Proj}(S_{\mathcal{P}})$ to be the toric variety of $\mathcal{P}$.

Notice that if $\mathcal{P}$ is normal, as is the case with GT-polytopes, then $S_{\mathcal{P}}$ is generated in degree one, so that $X_{\mathcal{P}}$ is a subscheme of a projective space.

Lemma IV.15. *[KM05b, Proposition 7] The inital algebra* $\mathrm{in}_\omega(\mathbb{C}[P])$ *is isomorphic to* $\mathbb{C}[\mathsf{GT}(n)]$ *as multigraded semigroup rings so that* $\mathrm{in}_\omega(\mathbb{C}[P])_\lambda \cong \mathbb{C}[\mathsf{GT}(n;\lambda)] = \mathbb{C}[\mathsf{GT}(n)]_\lambda$ *for* $\lambda \in \Lambda_n^+$.

Proof. Let $\phi : \mathbb{Z}^{n^2} \longrightarrow \mathbb{Z}^{\binom{n}{2}}$ be the linear map defined by $\phi(\mathbf{a})_{ij} = \gamma_{ij} = a_{i,j} + a_{i,j+1} + \cdots + a_{i,n-i+1}$ and $\psi : \mathbb{Z}^{\binom{n}{2}} \longrightarrow \mathbb{Z}^{n^2}$ be the map defined by $\psi(\Gamma)_{ij} = a_{ij} = \gamma_{i,j+1} - \gamma_{i,j}$. Then, the isomorphism is given by considering the exponent vectors of monomials in $\mathrm{in}_\omega(\mathbb{C}[P])$ as a subset of $\mathbb{Z}^{n^2}$ and checking that ϕ and ψ are inverses identifying exponent vectors with GT-patterns. See [KM05b] for further details. $\square$

It now follows from Lemma IV.15 that $\mathcal{X}_0$ is the toric variety $X_{\mathcal{P}_\lambda}$.

Subvarieties of $X_{\mathcal{P}_\lambda}$ are torus orbit closures that correspond to faces of $\mathcal{P}_\lambda$ by the toric orbit-cone correspondence; each face $\mathcal{Q}$ of $\mathcal{P}_\lambda$ corresponds to a toric subvariety of $X_{\mathcal{P}_\lambda}$ isomorphic to $X_{\mathcal{Q}}$.

4.2 Involution on the degeneration

While so far we have indexed the variables x_I, p_I, and q_I by subsets of $[n]$, in this section we index variables by finite strings in the alphabet $[n]^*$. Given integers $n \geqslant i_1 > i_2 > \cdots > i_k \geqslant 1$, we define $x_{i_1 i_2 \cdots i_k} := x_{\{i_1, i_2, \ldots, i_k\}}$, $p_{i_1 i_2 \cdots i_k} := p_{\{i_1, i_2, \ldots, i_k\}}$, and $q_{i_1 \cdots i_k} := q_{\{i_1, i_2, \ldots, i_k\}}$. For an arbitrary string $i_1 i_2 \ldots i_k$, we define $x_{i_1 i_2 \cdots i_k}$, $p_{i_1 i_2 \cdots i_k}$, and $q_{i_1 \cdots i_k}$ to be alternating in the k-tuple $(i_1, i_2, \ldots, i_k)$, so for example, $x_{21} = -x_{12}$ and $x_{11} = 0$.

Recall the surjection $\varphi : \mathbb{C}[t][X] \longrightarrow \mathbb{C}[t][Q]$ defined by $\varphi(x_I) = q_I$ for $I \subseteq [n]$. In the ensuing discussion, we index the sets X, P, and Q by finite substrings of $[n]^*$. Let $\varphi_1 : \mathbb{C}[X] \longrightarrow \mathbb{C}[P]$ and $\varphi_0 : \mathbb{C}[X] \longrightarrow \text{in}_\omega(\mathbb{C}[P])$ be the restriction of φ to fibers over $t = 1$ and $t = 0$ so that $\varphi_1(x_i) = p_i$ and $\varphi_0(x_i) = \text{in}_\omega(p_i)$. In particular, $\ker(\varphi_1)$ is called the ideal of Plücker relations whose generators are described as follows.

Notation IV.16. Let s and t be positive integers satisfying $n \geqslant s \geqslant t$ and A, B, C, D, and E be subsets that partition $[n]$. If we denote the cardinalities of the sets A, B, C, D, and E by a, b, c, d, and e, respectively, then they satisfy

$$a + b + 2c + e = s + t,$$
$$\text{and } c + e \geqslant s + 1.$$

Let $E = \{k_1 > k_2 > \cdots > k_e\}$ and $E_1 = \{k_{t-b-c+1} > \cdots > k_e\}$ and $E_2 = \{k_1 > \cdots > k_{t-b-c}\}$ be sets that further partition E into $E_1 \sqcup E_2$. For each $w \in S_e$, we define the ordered strings $w(E_1)$ and $w(E_2)$ by

$$w(E_1) := k_{w(t-b-c+1)} k_{w(t-b-c+2)} \dots k_{w(e)},$$
$$\text{and } w(E_2) := k_{w(1)} k_{w(2)} \dots k_{w(t-b-c)}.$$

Definition IV.17. For subsets A, B, C, D, E_1, E_2 satisfying the conditions above, let $R(A, B, C, D, E_1, E_2)$ be the element of $\ker(\varphi_1)$ defined by

$$R(A, B, C, D, E_1, E_2) := \sum_{w \in S_e} (-1)^w x_{ACw(E_1)}\, x_{BCw(E_2)}.$$

We call such elements of $\ker(\varphi_1)$, Garnir elements. The fact that the ideal of Plücker relations is generated by Garnir elements is well-known; see, for example, [MS05, Theorem 14.6] for a Gröbner basis consisting of Garnir elements.

Example IV.18. For $n = 7$,

$$R(6, \emptyset, 3, 7, 21, 54) = 4(x_{63\mathbf{21}}x_{3\mathbf{54}} - x_{63\mathbf{5}1}x_{3\mathbf{2}4} - x_{63\mathbf{4}1}x_{3\mathbf{52}} + x_{63\mathbf{42}}x_{3\mathbf{5}1} - x_{63\mathbf{2}5}x_{3\mathbf{1}4} + x_{63\mathbf{54}}x_{3\mathbf{21}})$$

$$= 4(x_{6321}x_{543} + x_{6531}x_{432} - x_{6431}x_{532} + x_{6432}x_{531} - x_{6532}x_{431} + x_{6543}x_{321})$$

where elements of E_1 and E_2 in the first line are boldfaced.

Let $\underline{w}$ be the permutation of S_e that realizes the minimum of $\omega_{ACw(E_1)} + \omega_{BCw(E_2)}$ as w varies over S_e. For each $w \in S_e$, let $\mu(A, B, C, D, E_1, E_2; w)$ be the integer defined by

$$\mu(A, B, C, D, E_1, E_2; w) := \omega_{ACw(E_1)} + \omega_{BCw(E_2)} - \omega_{AC\underline{w}(E_1)} - \omega_{BC\underline{w}(E_2)}.$$

We define a degenerated Garnir element $\widetilde{R}(A, B, C, D, E_1, E_2) \in \mathbb{C}[t][X]$ by

$$\widetilde{R}(A, B, C, D, E_1, E_2) := \sum_{w \in S_e} (-1)^w t^{\mu(A,B,C,D,E_1,E_2;w)} x_{ACw(E_1)}\, x_{BCw(E_2)},$$

which is the deformation of $R(A, B, C, D, E_1, E_2)$ with respect to weight vector ω' obtained from ω as in Lemma III.16.

Definition IV.19. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[t][X]$ corresponding to a flat family deforming $\ker(\varphi_1) \subseteq \mathbb{C}[X]$ with respect to ω' so that $\ker(\varphi) = \mathcal{I}$. Let $\mathcal{J}$ be the ideal of $\mathbb{C}[t][X]$ generated by degenerated Garnir relations so that $\mathcal{J} = \langle \widetilde{R}(A, B, C, D, E_1, E_2) \rangle$. We observe that $\mathcal{J} \subseteq \mathcal{I}$.

Notice that for $\tau \in \mathbb{C}^*$, the fibers of $\mathbb{C}[t][X]/\mathcal{I}$ and $\mathbb{C}[t][X]/\mathcal{J}$ over $(t - \tau)$ are equal:

$$\frac{\mathbb{C}[t][X]}{\mathcal{I}} \otimes_{\mathbb{C}[t]} \frac{\mathbb{C}[t]}{(t-\tau)} \cong \frac{\mathbb{C}[t][X]}{\mathcal{J}} \otimes_{\mathbb{C}[t]} \frac{\mathbb{C}[t]}{(t-\tau)}.$$

The flat family $\mathcal{I}$ is equal to the saturation of $\mathcal{J}$ with respect to t so that $\mathcal{I} = \langle \mathcal{J} : t^\infty \rangle := \bigcup_{k \geqslant 0} \langle \mathcal{J} : t^k \rangle$.

Let K and K^c be pairwise distinct finite strings in the alphabet $[n]^*$ such that K^c considered as a subset of $[n]$ is equal to the complement of K in $[n]$ though not necessarily in strictly decreasing order. Let $\epsilon_{K,K^c} \in \{\pm 1\}$ denote the sign of the permutation in S_n that rearranges the concatenation of K and K^c in decreasing order from n to 1.

Definition IV.20. Let $\widetilde{\tau} : \mathbb{C}[t][X] \longrightarrow \mathbb{C}[t][X]$ be the $\mathbb{C}[t]$-algebra map [2] defined by $\widetilde{\tau}(x_I) = \epsilon_{I,I^c}\, x_{I^c}$ for finite string I in the alphabet $[n]^*$.

Notice that $\epsilon_{K,K^c}\, \epsilon_{K^c,K} = (-1)^{k(n-k)}$, so $\widetilde{\tau}$ is a "signed" involution of $\mathbb{C}[t][X]$. In the remainder of this section, we prove the following proposition.

Proposition IV.21. *The involution $\widetilde{\tau}$ preserves $\mathcal{I}$ so that $\widetilde{\tau}$ induces the involution $\tau : \mathbb{C}[t][Q] \longrightarrow \mathbb{C}[t][Q]$ as in the following diagram:*

$$\begin{tikzcd} \mathbb{C}[t][X] \arrow[r, "\widetilde{\tau}"] \arrow[d, "\varphi"'] & \mathbb{C}[t][X] \arrow[d, "\varphi"] \\ \mathbb{C}[t][Q] \arrow[r, "\tau"'] & \mathbb{C}[t][Q]. \end{tikzcd}$$

Notice that to prove that $\widetilde{\tau}(\mathcal{I}) = \mathcal{I}$, it suffices to show that $\widetilde{\tau}(\mathcal{J}) = \mathcal{J}$. Indeed, if $f \in \mathcal{I}$ then $t^N f \in \mathcal{J}$ for some $N \gg 0$. Then, observe that $t^N \widetilde{\tau}(f) = \widetilde{\tau}(t^N f) \in \mathcal{J}$ so that $\widetilde{\tau}(f) \in \mathcal{I}$. We will return to discussing Proposition IV.21 after the following lemma.

Lemma IV.22. *Let A, B, C, D, E_1, and E_2 be as in Notation IV.16. Then,*

$$\mu(A, B, C, D, E_1, E_2; w) = \mu(A, B, D, C, E_1, E_2; w)$$

for all $w \in S_e$.

Proof. For notational convenience, we write $I := A \cup C \cup \underline{w}(E_1)$, $J := B \cup C \cup \underline{w}(E_2)$, $I' := A \cup C \cup w(E_1)$, and $J' := B \cup C \cup w(E_2)$. Notice that in terms of these notations, $\mu(A, B, C, D, E_1, E_2; w)$ can be rewritten as $\omega_{I'} + \omega_{J'} - \omega_I - \omega_J$.

[2] The map $\widetilde{\tau}$ is motivated by the Hodge star operator, $* : \bigwedge^k \mathbb{C}^n \longrightarrow \bigwedge^{n-k} \mathbb{C}^n$.

We claim that

$$\omega_{I'} + \omega_{J'} - \omega_I - \omega_J = \omega_{I'^c} + \omega_{J'^c} - \omega_{I^c} - \omega_{J^c}. \tag{4.2}$$

Indeed, we first observe that we may replace $\omega_{ij} = \binom{n+2-i-j}{2}$ with $\omega_{ij} = ij$, since in evaluating $\omega_{I'} + \omega_{J'} - \omega_I - \omega_J$, the contribution from terms other than ij in the definition,

$$\omega_{ij} = \binom{n+2-i-j}{2} = \frac{1}{2}(n+2)(n+1) - \frac{1}{2}(2n+3)(i+j) + \frac{1}{2}(i^2+j^2) + ij,$$

cancel.

For a subset K of $[n]$, we define $\mathrm{pos}_K : [n] \longrightarrow [n]$ by

$$\mathrm{pos}_K(i) := \begin{cases} \#\{k \in K : k \geqslant i\} & \text{if } i \in K \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $i \in K$, then $\mathrm{pos}_K(i)$ records the position of i in K listed in decreasing order. Then,

$$\omega_I = \sum_{k=1}^{s} k \cdot i_k = \sum_{i=1}^{n} \mathrm{pos}_I(i) \cdot i.$$

Let $\mathrm{pos}(I, J, I', J'; i)$ denote the difference,

$$\mathrm{pos}(I, J, I', J'; i) := \mathrm{pos}_{I'}(i) + \mathrm{pos}_{J'}(i) - \mathrm{pos}_I(i) - \mathrm{pos}_J(i).$$

Observe that (4.2) is implied by the stronger assertion that

$$\mathrm{pos}(I, J, I'J'; i) = \mathrm{pos}(I^c, J^c, I'^c, J'^c; i) \tag{4.3}$$

for $i = 1, 2, \ldots, n$. We proceed to prove (4.3) by induction on $\#(I \cap J) + \#(I^c \cap J^c)$.

If $\#(I \cap J) = \#(I^c \cap J^c) = 0$, then the pairs (I, J) and (I', J') partition $[n]$ and $I = J^c$, $J = I^c$, $I' = J'^c$, and $J' = I'^c$, which in turn implies (4.3). Next, suppose that $\#(I \cap J) + \#(I^c \cap J^c) > 0$, and notice that we may assume that $\#(I^c \cap J^c) > 0$

by interchanging I with J^c and J with I^c, if necessary. Let k be an element of $I \cap J$. Then, $\text{pos}(I, J, I', J'; k) = 0$ since the pair (I', J') reshuffles the elements of (I, J). On the other hand, $\text{pos}(I^c, J^c, I'^c, J'^c; k) = 0$ since k is not an element of I^c, J^c, I'^c, or J'^c.

To evaluate (4.3) for $i \neq k$, let $K_\downarrow$ be the subset obtained from the set $K \subseteq [n]$ containing k by omitting k then decreasing by 1 those entries in K greater than k, while keeping constant those entries less than k. We consider $K_\downarrow$ as a subset of $[n-1]$ so that $K_\downarrow^c = [n-1] \setminus K_\downarrow$. For example, for $n = 6$, $k = 4$, $K = 6431$, so that $K^c = 52$, $K_\downarrow = 531$ and $K_\downarrow^c = 642$.

Since k in $I \cap J$ is omitted in passing from I and J to $I_\downarrow$ and $J_\downarrow$, $\#(I_\downarrow \cap J_\downarrow) = \#(I \cap J) - 1$. Also $\#(I_\downarrow^c \cap J_\downarrow^c) = \#(I^c \cap J^c)$ since there is a bijection from $I^c \cap J^c \longrightarrow I_\downarrow^c \cap J_\downarrow^c$ that maps $i \longmapsto i - 1$ for $i > k$ and $i \longmapsto i$ for $i < k$. So we may apply the induction hypothesis to see that $\text{pos}(I_\downarrow, J_\downarrow, I'_\downarrow, J'_\downarrow; i) = \text{pos}(I_\downarrow^c, J_\downarrow^c, I_\downarrow'^c, J_\downarrow'^c; i)$ for all $i = 1, 2, \ldots, n$.

We consider (4.3) when $i < k$. Indeed, notice that $\text{pos}_K(i) = \text{pos}_{K_\downarrow}(i) + 1$ if $i \in K$, hence $i \in K_\downarrow$ as well, and $\text{pos}_K(i) = \text{pos}_{K_\downarrow}(i) = 0$ if $i \notin K$. It follows that $\text{pos}(I, J, I', J'; i) = \text{pos}(I_\downarrow, J_\downarrow, I'_\downarrow, J'_\downarrow; i)$. Also, the relative positions of i in K^c and $K_\downarrow^c$ are the same, hence $\text{pos}_{K^c}(i) = \text{pos}_{K_\downarrow^c}(i)$. These observations imply that $\text{pos}(I^c, J^c, I'^c, J'^c; i) = \text{pos}(I_\downarrow^c, J_\downarrow^c, I_\downarrow'^c, J_\downarrow'^c; i)$, and combining the above with the induction hypothesis implies (4.3).

We next consider the case $i > k$. Observe that $\text{pos}_K(i) = \text{pos}_{K_\downarrow}(i-1)$ implies that $\text{pos}(I, J, I', J'; i) = \text{pos}(I_\downarrow, J_\downarrow, I'_\downarrow, J'_\downarrow; i-1)$ and $\text{pos}_{K^c}(i) = \text{pos}_{K_\downarrow^c}(i-1)$ implies that $\text{pos}(I^c, J^c, I'^c, J'^c; i) = \text{pos}(I_\downarrow^c, J_\downarrow^c, I_\downarrow'^c, J_\downarrow'^c; i-1)$. The induction hypothesis combined with these observations imply (4.3).

Let $\underline{w}' := \arg\min_{u \in S_e} \omega_{ADu(E_1)} + \omega_{BDu(E_2)}$ and $C := \omega_{AD\underline{w}(E_1)} + \omega_{BD\underline{w}(E_2)} -$

$\omega_{AD\underline{w}'(E_1)} - \omega_{BD\underline{w}'(E_2)}$. Notice that C is a nonnegative integer independent of w. To see that (4.2) implies the lemma, observe that

$$\begin{aligned}
\mu(A,B,C,D,E_1,E_2;w) &= \omega_{I'} + \omega_{J'} - \omega_I - \omega_J \\
&= \omega_{I'^c} + \omega_{J'^c} - \omega_{I^c} - \omega_{J^c} \\
&= \mu(A,B,D,C,E_1,E_2;w) - C
\end{aligned}$$

Specialize the above equation to $w = \underline{w}'$ to see that $C = 0$ and the lemma follows. □

Proof. (Proposition IV.21) Recall that it suffices to show that $\widetilde{\tau}(\mathcal{J}) = \mathcal{J}$. Indeed, notice that

$$\begin{aligned}
\widetilde{\tau}(\widetilde{R}(A,B,C,D,E_1,E_2;w)) &= \sum_{w\in S_e} (-1)^w t^{\mu(A,B,C,D,E_1,E_2;w)} \widetilde{\tau}(x_{ACw(E_1)})\, \widetilde{\tau}(x_{BCw(E_2)}) \\
&= \sum_{w\in S_e} \epsilon_{ACw(E_1),BDw(E_2)}\, \epsilon_{BCw(E_2),ADw(E_1)} \\
&\qquad (-1)^w t^{\mu(A,B,C,D,E_1,E_2;w)} x_{BDw(E_2)} x_{ADw(E_1)}.
\end{aligned}$$

To see that

$$\epsilon_{ACE_1,BDE_2}\, \epsilon_{BCE_2,ADE_1} = \epsilon_{ACw(E_1),BDw(E_2)}\, \epsilon_{BCw(E_2),ADw(E_1)}$$

for all $w \in S_e$, observe that $\epsilon_{ACw(E_1),BDw(E_2)} = (-1)^w \epsilon_{ACE_1,BDE_2}$ and $\epsilon_{BCw(E_2),ADw(E_1)} = (-1)^w \epsilon_{BCE_2,ADE_1}$. Then, apply Lemma IV.22 to see that

$$\begin{aligned}
\widetilde{\tau}(\widetilde{R}(A,B,C,D,E_1,E_2;w)) &= \pm \sum_{w\in S_e} (-1)^w t^{\mu(A,B,C,D,E_1,E_2;w)} x_{ADw(E_1)}\, x_{BDw(E_2)} \\
&= \pm \sum_{w\in S_e} (-1)^w t^{\mu(A,B,D,C,E_1,E_2;w)}\, x_{ADw(E_1)} x_{BDw(E_2)} \\
&= \pm R(A,B,D,C,E_1,E_2) \in \mathcal{J}
\end{aligned}$$

where the sign is equal to $\epsilon_{ACE_1,BDE_2}\, \epsilon_{BCE_2,ADE_1}$. □

4.3 Matrix Schubert variety and Schubert variety together

In this section, we state the main theorem of this thesis as Theorem IV.26 and following [KM05b] examine the relation between the degeneration of a matrix Schubert variety and that of a Schubert variety. Example IV.27 highlights the gap in the proof of [KM05b] and Section 4.3.2 motivates our application of Standard Monomial Theory in Chapter V. In Section 4.3.3, we present the semi-toric degeneration of a Schubert variety localized to affine open subsets.

4.3.1 Relating the two degenerations

Let $\rho : M_n \times \mathbb{A}^1 \longrightarrow N^- \backslash\backslash (M_n \times \mathbb{A}^1)$ be the quotient map dual to the inclusion

$$\rho^{\#} : \mathrm{Spec}(\mathbb{C}[t][Z]^{N^-}) = \mathrm{Spec}(\mathbb{C}[t][Q]) \hookrightarrow \mathrm{Spec}(\mathbb{C}[t][Z]).$$

We think of $N^- \backslash\backslash (M_n \times \mathbb{A}^1)$ as the multi-cone over $B^- \backslash\backslash (M_n \times \mathbb{A}^1)$. Recall from Section 3.3.3, the Schubert determinantal ideal $\widetilde{I}_w$ in $\mathbb{C}[Z]$ and let $\widetilde{\mathcal{I}}_w$ be the ideal of $\mathbb{C}[t][Z]$ defined as the deformation of $\widetilde{I}_w$ by ω. The combination of Lemma III.13 and Lemma IV.5 implies that $\widetilde{\mathcal{I}}_w$ degenerates $\widetilde{I}_w$ as described in Theorem III.20.

Definition IV.23. Let I_w be the ideal of $\mathbb{C}[P]$ defined by $I_w := \widetilde{I}_w \cap \mathbb{C}[P]$ called the Schubert ideal. It is the ideal of the Schubert variety X_w inside the Plücker algebra. The Schubert ideal I_w is generated by Plücker variables $p_I \in \mathbb{C}[P^{(k)}]$ such that $\pi_k(w) \nleq I$ for $k = 1, 2, \ldots, n$ [RS97, Theorem 4].

Definition IV.24. Let $\mathcal{X}_w$ be the scheme-theoretic image $\mathcal{X}_w := \overline{\rho}(\widetilde{\mathcal{X}}_w)$ in $N^- \backslash\backslash (M_n \times \mathbb{A}^1)$. Let $\mathcal{I}_w := \widetilde{\mathcal{I}}_w \cap \mathbb{C}[t][Q]$ be the ideal corresponding to $\mathcal{X}_w$. Notice that $\mathcal{I}_w$ is the deformation of I_w as in Lemma III.16.

Notation IV.25. Let $\widetilde{\mathcal{X}}_{w,t}$ denote the fiber of $\widetilde{\mathcal{X}}_w$ over $t \in \mathbb{C}$ and similarly, $\mathcal{X}_{w,t}$ for a fiber of $\mathcal{X}_w$.

For a reduced pipe dream $D \in \mathcal{RP}_w$, let

- L_D be the coordinate subspace of M_n consisting of matrices whose coordinates z_{ij} are zero for $(i,j) \in D$. Recall that L_D is an irreducible component of $\mathrm{in}_\omega(\widetilde{X}_w)$;
- $\mathcal{F}_D$ be the face of the GT-polytope defined by setting $\gamma_{i,j} = \gamma_{i+1,j}$ for each $(i,j) \in D$; and
- $X_{\mathcal{F}_D}$ be the toric subvariety of the Gelfand-Tsetlin toric variety associated with face $\mathcal{F}_D$.

Our main theorem is as follows.

Theorem IV.26. *The family $\mathcal{X} = B^- \backslash\backslash (M_n \times \mathbb{A}^1)$ induces a flat degeneration of Schubert variety X_w to a reduced union $\bigcup_{D \in \mathcal{RP}_w} X_{\mathcal{F}_D}$ of toric subvarieties of the Gelfand-Tsetlin toric variety $X_{\mathcal{P}_\lambda}$.*

The two objects central to the argument of [KM05b] are $\mathcal{X}_{w,0} = \overline{\rho}(\widetilde{\mathcal{X}}_w)_0$ corresponding to $\mathrm{in}_\omega(I_w)$ and $\overline{\rho_0}(\widetilde{\mathcal{X}}_{w,0})$ corresponding to $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$. Kogan and Miller assume that these two objects are equal; however, in general, given a family of morphisms of schemes over a parameter space, the fiber of the image may properly contain the image of the fiber (see, [EH00, pg. 216]). Indeed, it is not difficult to see that $\mathrm{in}_\omega(I)$ is a subset of $\mathrm{in}(\widetilde{I}) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ for ideals, $\widetilde{I} \subseteq \mathbb{C}[Z]$ and $I = \widetilde{I} \cap \mathbb{C}[P]$. The following example, however, shows that $\mathrm{in}_\omega(I)$ can be a proper subset of $\mathrm{in}_\omega(\widetilde{I}) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ where $\widetilde{I}$ is an opposite Schubert determinantal ideal and I is an opposite Schubert ideal.

Example IV.27. For $n = 3$, the degenerated Plücker algebra is

$$\mathbb{C}[t][q_1, q_2, q_3, q_{12}, q_{13}, q_{23}, q_{123}] \subseteq \mathbb{C}[t][z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}, z_{31}, z_{32}, z_{33}]$$

where $q_1 = z_{11}$, $q_2 = z_{12}$, $q_3 = z_{13}$, $q_{12} = tz_{11}z_{22} - z_{12}z_{21}$, $q_{13} = t^2 z_{11}z_{23} - z_{13}z_{21}$, $q_{23} = tz_{12}z_{23} - z_{13}z_{22}$, $q_{123} = t^3 z_{11}z_{22}z_{33} - t^3 z_{11}z_{23}z_{32} - t^2 z_{12}z_{21}z_{33} + tz_{12}z_{23}z_{31} + tz_{13}z_{21}z_{32} - z_{13}z_{22}z_{31}$. The equation $tq_1q_{23} - q_2q_{13} + q_3q_{12} = 0$ generates the relations of $\mathbb{C}[t][Q]$, hence

$$\frac{\mathbb{C}[t][x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]}{\langle tx_1x_{23} - x_2x_{13} + x_3x_{12}\rangle} \cong \mathbb{C}[t][q_1, q_2, q_3, q_{12}, q_{13}, q_{23}, q_{123}].$$

Consider the opposite Schubert variety $X^{231} \subseteq F\ell_3$ along with its ideal $I^{231} = \langle p_3\rangle$ and the opposite matrix Schubert variety $\widetilde{X}^{231}$ along with its ideal $\widetilde{I}^{231} = \langle z_{13}\rangle \cdot \mathbb{C}[Z]$. Notice that $\widetilde{\mathcal{I}}^{231} = \langle z_{13}\rangle \cdot \mathbb{C}[t][Z]$ while $\mathcal{I}^{231} = \widetilde{\mathcal{I}}^{231} \cap \mathbb{C}[t][Q] = \langle q_3\rangle$. We show that $\mathrm{in}_\omega(I^{231})$ is a proper subset of $\mathrm{in}_\omega(\widetilde{I}^{231}) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ by displaying an element of $\mathrm{in}_\omega(\widetilde{I}^{231}) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ that is not contained in $\mathrm{in}_\omega(I^{231})$.

Clearly, $\mathrm{in}_\omega(p_{23}) = -z_{13}z_{22}$ is an element of $\mathrm{in}_\omega(\widetilde{I}^{231}) \cap \mathrm{in}_\omega(\mathbb{C}[P]) = \langle \mathrm{in}_\omega(p_3)\rangle \cdot \mathrm{in}_\omega(\mathbb{C}[P])$. Suppose $\mathrm{in}_\omega(p_{32})$ is also an element of $\mathrm{in}_\omega(I^{231})$ implying that $\mathrm{in}_\omega(p_{32})$ is divisible by $\mathrm{in}_\omega(p_3)$ as elements of $\mathrm{in}_\omega(\mathbb{C}[P])$. Then, $\mathrm{in}_\omega(p_{32})/\mathrm{in}_\omega(p_3) = -z_{22}$ implying that z_{22} is an element of $\mathrm{in}_\omega(\mathbb{C}[P])$, which is absurd since monic monomials of $\mathrm{in}_\omega(\mathbb{C}[P])$ corresponds to GT-patterns.

Therefore, while $\mathrm{in}_\omega(I_w)$ is a subset of $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$, it requires further justification to conclude that two ideals are equal. To identify the image of $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ in $\mathbb{C}[\mathsf{GT}(n)]$, we introduce the following definitions.

Definition IV.28. Let RP_w be the subset of GT-patterns defined by

$$\mathsf{RP}_w := \bigcup_{D \in \mathcal{RP}_w} \{\Gamma \in \mathsf{GT}(n) : \gamma_{i,j} = \gamma_{i,j+1} \text{ for } (i,j) \in D\}.$$

For $\lambda \in \Lambda_n^+$, let $\mathsf{RP}_w(\lambda)$ be the subset of RP_w consisting of patterns of shape λ. Let $\mathbb{C}\{\mathsf{RP}_w\}$ denote the subspace of $\mathbb{C}[\mathsf{GT}(n)]$ spanned by GT-patterns in RP_w and similarly define $\mathbb{C}\{\mathsf{RP}_w(\lambda)\}$ as a subspace spanned by patterns of RP_w with shape λ.

Definition IV.29. Let A_w be the subset of GT-patterns defined by

$$\mathsf{A}_w := \bigcap_{D \in \mathcal{RP}_w} \{\Gamma \in \mathsf{GT}(n) : \gamma_{i,j} > \gamma_{i,j+1} \text{ for } (i,j) \in D\}.$$

The set A_w is an ideal of the semigroup $\mathsf{GT}(n)$, so $\langle \mathsf{A}_w \rangle := \mathbb{C}\{\mathsf{A}_w\}$ is an ideal of $\mathbb{C}[\mathsf{GT}(n)]$.

Since

$$\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P]) = \bigcap_{D \in \mathcal{RP}_w} \langle z_{i,j} : (i,j) \in D \rangle \cap \mathrm{in}_\omega(\mathbb{C}[P])$$

and the image of $\langle z_{i,j} : (i,j) \in D \rangle \cap \mathrm{in}_\omega(\mathbb{C}[P])$ for $D \in \mathcal{RP}_w$ in $\mathbb{C}[\mathsf{GT}(n)]$ is spanned by $\{\Gamma \in \mathsf{GT}(n) : \gamma_{i,j} > \gamma_{i,j+1} \text{ for } (i,j) \in D\}$, we have that

$$\frac{\mathbb{C}[\mathsf{GT}(n)]}{\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])} \cong \frac{\mathbb{C}[\mathsf{GT}(n)]}{\langle \mathsf{A}_w \rangle} = \mathbb{C}\{\mathsf{RP}_w\}.$$

So the equality of $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ and $\mathrm{in}_\omega(I_w)$ implies Theorem IV.26.

4.3.2 Why more machinery?

Recall from Definition IV.23 that the Schubert ideal I_w is generated by Plücker variables. The following examples show, however, that the ideal $\langle \mathsf{A}_w \rangle \cong \mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ can have generators that are the initial terms of products of Plücker variables. Such examples show that $\langle \mathsf{A}_w \rangle$ does not inherit the property of having simple generators from I_w and indicate the need for a more systematic way of parametrizing elements of $\langle \mathsf{A}_w \rangle$.

Example IV.30. The Schubert ideal I_{1342} is generated by $\langle p_{21}, p_{321}, p_{421} \rangle$ so that $\mathrm{in}_\omega(p_{21})$, $\mathrm{in}_\omega(p_{321})$, and $\mathrm{in}_\omega(p_{421})$ are elements of $\langle \mathsf{A}_{1342} \rangle$. Since $q_{321}q_4 - q_{421}q_3 + tq_{431}q_2 - t^3 q_{432}q_1 = 0$ as a Garnir element and q_{321} and $q_{421} \in \mathcal{I}_{1342}$, $q_{431}q_2 - t^2 q_{432}q_1 \in \mathcal{I}_{1342}$. Therefore, $\mathrm{in}_\omega(p_{431}) \cdot \mathrm{in}_\omega(p_2) \in \langle \mathsf{A}_{1342} \rangle$. As a matter of fact,

$$\langle \mathsf{A}_{1342} \rangle = \langle \mathrm{in}_\omega(p_{21}),\, \mathrm{in}_\omega(p_{321}),\, \mathrm{in}_\omega(p_{421}),\, \mathrm{in}_\omega(p_{431}) \cdot \mathrm{in}_\omega(p_2) \rangle.$$

In fact, Example IV.30 is a special case of the next example.

Example IV.31. Let $n = 2d+2$ and $w = 1\widehat{2}3\dots(2d+2)2 \in S_{2d+2}$. We claim that $\mathrm{in}_\omega(p_{I_1}p_{I_2}\dots p_{I_{d+1}})$ for sets $I_1, I_2, \dots, I_{d+1}$ described in Figure 4.1 is an element of $\langle \mathcal{A}_w \rangle$ indivisible by any $\mathrm{in}_\omega(p_I) \in \langle \mathcal{A}_w \rangle$.

	# of elements	elements
I_1	$2d+1$	$2d+2, 2d+1, \dots, 3, 1$
I_2	$2d-1$	$2d+2, 2d+1, \dots, 5, 2$
	$\vdots$	
I_m	$2d-2m+3$	$2d+2, 2d+1, \dots, 2m+1, m$
	$\vdots$	
I_{d+1}	1	$d+1$

Figure 4.1: A generator for $\langle \mathsf{A}_w \rangle$ for $w = 1\widehat{2}3\dots(2d+2)$

There exists a Garnir element,

$$q_{\dots321}q_{\dots4}\dots q_{d+1} - q_{\dots421}q_{\dots3}\dots q_{d+1} + tq_{\dots431}q_{\dots2}\dots q_{d+1} - t^3q_{\dots432}q_{\dots1}\dots q_{d+1} = 0.$$

Since $q_{\dots321}, q_{\dots421} \in \mathcal{I}_w$, the relation $q_{\dots431}q_{\dots2}\dots q_{d+1} - t^2q_{\dots432}q_{\dots1}\dots q_{d+1} \in \mathcal{I}_w$, hence $\mathrm{in}_\omega(p_{I_1}p_{I_2}\dots p_{I_{d+1}})$ is an element of $\langle \mathcal{A}_w \rangle$.

We think of sets $I_1, I_2, \dots, I_{d+1}$ as the column entries of the tableau,

$$T = \begin{array}{|c|c|ccc|c|c|c|}
\hline
2d{+}2 & 2d{+}2 & & \dots & & 2d{+}2 & 2d{+}2 & d{+}1 \\
\hline
2d{+}1 & 2d{+}1 & & \dots & & 2d{+}1 & 2d{+}1 & \\
\hline
2d & 2d & & \dots & & 2d & d & \\
\hline
\vdots & \vdots & & & & \iddots & & \\
\hline
4 & 2 & & & & & & \\
\hline
3 & & & & & & & \\
\hline
1 & & & & & & & \\
\hline
\end{array}$$

with the corresponding GT-pattern,

$$
\Gamma(T) \quad = \quad \begin{array}{cccccccccccc}
d{+}1 & d{+}1 & & \ldots & & d{+}1 & d & d & & \ldots & & d \\
d & d & & & d & d & & & & & d & \\
d & d & & & d & d{-}1 & & & & d{-}1 & & \\
\vdots & \vdots & & & & & & \iddots & & & & \\
2 & 2 & 2 & 2 & 2 & & & & & & & \\
2 & 2 & 1 & 1 & & & & & & & & \\
1 & 1 & 1 & & & & & & & & & \\
1 & 0 & & & & & & & & & & \\
0 & & & & & & & & & & &
\end{array}
$$

It is not difficult to see that $\Gamma(T)$ has a unique factorization into $\prod_{k=1}^{d+1} \mathrm{in}_\omega(p_{I_k})$, say by the method of [PPS10, Section 5], so our claim follows.

4.3.3 Inside the opposite big cell

In this section, we localize Theorem IV.26 to an open subset. Then, Theorem IV.34 says that a Schubert variety intersected with the open subset degenerates into a reduced union of toric subvarieties of the affine toric variety obtained by localizing the Gelfand-Tsetlin toric variety at a vertex.

Definition IV.32. Let $\mathcal{V}(q_n q_{n,n-1} \ldots q_{[n]})$ denote the closed subscheme of $M_n \times \mathbb{A}^1$ defined by $\langle q_n, q_{n,n-1}, \ldots, q_{[n]} \rangle$ of $\mathbb{C}[t][Q]$. Let U be the multiplicative subset of $\mathbb{C}[t][Q]$ generated by $\left\{q_n, q_{n,n-1}, \ldots, q_{[n]}\right\}$. Let $\mathcal{G}$ be the open subscheme of $M_n \times \mathbb{A}^1$ defined by

$$\mathcal{G} := (M_n \times \mathbb{A}^1) \setminus \mathcal{V}(q_n q_{n,n-1} \ldots q_{[n]}) = \mathrm{Spec}(\mathbb{C}[t][Q][U^{-1}]).$$

It is not difficult to see by Buchberger's algorithm that $\{p_n, p_{n,n-1}, \ldots, p_{[n]}\}$ is a Gröbner basis for $\langle p_n, p_{n,n-1}, \ldots, p_{[n]} \rangle$. Flatness is local so $\mathcal{G}$ is a flat family over $\mathbb{A}^1$. Notice that $\mathcal{G}_1$ is the subset of GL_n consisting of matrices that have a LU-decomposition (Gaussian decomposition). We may then think of $\mathcal{G} \subseteq M_n \times \mathbb{A}^1$ as a deformation of $\mathcal{G}_1$.

Definition IV.33. Let $N^-\backslash\backslash\mathcal{G}$ be the affine GIT quotient defined by

$$N^-\backslash\backslash\mathcal{G} := \mathrm{Spec}(\mathbb{C}[t][Z][U^{-1}]^{N^-}) = \mathrm{Spec}(\mathbb{C}[t][Q][U^{-1}])$$

where $\mathbb{C}[t][Z][U^{-1}]^{N^-} = \mathbb{C}[t][Q][U^{-1}]$ since U consists of N^--invariants.

For notational convenience, we denote the restriction $\rho|_{\mathcal{G}} : \mathcal{G} \longrightarrow N^-\backslash\backslash\mathcal{G}$ by ρ and similarly denote the restriction $\rho_0|_{\mathcal{G}_0} : \mathcal{G}_0 \longrightarrow (N^-\backslash\backslash\mathcal{G})_0$ by ρ_0. Let U_0 be the multiplicative subset of $\mathrm{in}_\omega(\mathbb{C}[P])$ generated by $\{\mathrm{in}_\omega(p_n), \mathrm{in}_\omega(p_{n,n-1}), \ldots, \mathrm{in}_\omega(p_{[n]})\}$.

Theorem IV.34. *The $t = 0$ fiber of the image of $\widetilde{\mathcal{X}}_w$ under $\rho : \mathcal{G} \longrightarrow N\backslash\backslash\mathcal{G}$ is equal to the image of $t = 0$ fiber of $(\widetilde{\mathcal{X}}_w \cap \mathcal{G})_0$ under $\rho_0 : \mathcal{G}_0 \longrightarrow (N^-\backslash\backslash\mathcal{G})_0$. Equivalently,*

$$\mathrm{in}_\omega(I_w \cdot \mathbb{C}[P])[U_0^{-1}] = \mathrm{in}_\omega(\widetilde{\mathcal{I}}_w \cdot \mathbb{C}[Z])[U_0^{-1}] \cap \mathrm{in}_\omega(\mathbb{C}[P])[U_0^{-1}].$$

To prove the above theorem, we will extend the LU-decomposition of $\mathcal{G}_1$ to all of $\mathcal{G}$.

Definition IV.35. Let $\mathcal{N}^- := N^- \times \mathbb{A}^1$ and $\mathcal{A} := w_0 B^- \times \mathbb{A}^1$ be trivial families over $\mathbb{A}^1$. Let $X = (x_{ij})_{i>j}$ be the matrix of indeterminates arranged in strictly lower-triangular form and $Y = (y_{kl})_{k+l\leqslant n+1}$ be the matrix of indeterminates y_{kl} arranged as an upper-left triangular matrix. Then, $\mathcal{N}^- = \mathrm{Spec}(\mathbb{C}[t][X])$ and $\mathcal{A} = \mathrm{Spec}(\mathbb{C}[t][Y][V^{-1}])$ where V is the multiplicative subset of $\mathbb{C}[t][Y]$ generated by $\{y_{1,n}, y_{2,n-1}, \ldots, y_{n,1}\}$.

Remark IV.36. We apologize for the abuse of notation for X in the above definition. Our current definition for X is local to the current section.

The following lemma says that the space $\mathcal{G}$ factors over $\mathbb{A}^1$ as the product of $\mathcal{N}^-$ and $\mathcal{A}$.

Lemma IV.37. *Let $\mu : \mathcal{N}^- \times_{\mathbb{A}^1} \mathcal{A} \longrightarrow \mathcal{G}$ be the map defined by the $\mathcal{B}^-$-action on $\mathcal{A}$ considered as a subset of $M_n \times \mathbb{A}^1$. Then, μ is an isomorphism.*

Proof. Let $X' := (x'_{ij})$ be the $n \times n$ matrices of indeterminates given by

$$x'_{ij} = \begin{cases} x_{ij} & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

Define the $\mathbb{C}[t]$-algebra map $\mu^\# : \mathbb{C}[t][Z][U^{-1}] \longrightarrow \mathbb{C}[t][X,Y][V^{-1}]$ by

$$\mu^\#(z_{ij}) = \sum_{k=1}^{n} t^{\omega_{kj}-\omega_{ij}} x'_{ik} y_{kj}. \tag{4.4}$$

We can encode $\mu^\#$ more succinctly using matrix multiplication as

$$\mu^\#(\widetilde{Z}) = X'\widetilde{Y}. \tag{4.5}$$

For $I = \{n, n-1, \ldots, n-k+1\}$,

$$\begin{aligned} \mu^\#(q_I) &= t^{-\omega_I}\Delta_I(\mu^\#(\widetilde{Z})) = t^{-\omega_I}\Delta_I(X'\widetilde{Y}) = t^{-\omega_I}\Delta_I(\widetilde{Y}) \\ &= (-1)^{\binom{k}{2}} y_{k,n-k+1} y_{k-1,n-k+2} \cdots y_{1,n} \end{aligned} \tag{4.6}$$

which is a unit in $\mathbb{C}[t][X,Y][V^{-1}]$ so (4.4) determine a map from $\mathbb{C}[t][Z][U^{-1}]$. Notice that in (4.6) we used the fact that $\Delta_I(X'\widetilde{Y}) = \Delta_I(\widetilde{Y})$ which follows from the fact that X' is lower triangular so $X'\widetilde{Y}$ has the same top-justified row span as $\widetilde{Y}$.

Let $\nu^\# : \mathbb{C}[t][X,Y][V^{-1}] \longrightarrow \mathbb{C}[t][Z][U^{-1}]$ be the $\mathbb{C}[t]$-algebra map defined by

$$\begin{aligned} \nu^\#(x_{ij}) &= (-1)^{\binom{j+1}{2}} \frac{\Delta_{[j-1]\cup\{i\},[n]\setminus[j]}(\widetilde{Z})}{y_{1,n}y_{2,n-1}\cdots y_{j,n-j+1}}, \\ \nu^\#(y_{kl}) &= \theta^\#(y_{kl}). \end{aligned}$$

Notice that $\mu^\# \circ \nu^\# = \mathrm{Id}$ is a sufficient condition for μ and ν to be inverses because it implies that $\mu^\#$ is a surjective map between integral domains of equal Krull dimension

$n^2 + 1$ so is an isomorphism. Indeed,

$$\mu^\# \circ \nu^\#(y_{1l}) = \mu^\#(q_l) = \mu^\#(z_{1l}) = y_{1l}$$

and for $k > 1$,

$$\begin{aligned}
\mu^\# \circ \nu^\#(y_{kl}) = \mu^\# \left((-1)^{1+k} \frac{q_{n,n-1,\dots,n-k+2,l}}{q_{n,n-1,\dots,n-k+2}} \right) &= (-1)^{1+k} t^{-\omega_{kl}} \frac{\Delta_{n,n-1,\dots,n-k+2,l}(\mu^\#(\widetilde{Z}))}{\Delta_{n,n-1,\dots,n-k+2}(\mu^\#(\widetilde{Z}))} \\
&= (-1)^{1+k} t^{-\omega_{kl}} \frac{\Delta_{n,n-1,\dots,n-k+2,l}(X'\widetilde{Y})}{\Delta_{n,n-1,\dots,n-k+2}(X'\widetilde{Y})} \\
&= (-1)^{1+k} t^{-\omega_{kl}} \frac{\Delta_{n,n-1,\dots,n-k+2,l}(\widetilde{Y})}{\Delta_{n,n-1,\dots,n-k+2}(\widetilde{Y})}. \qquad (4.7)
\end{aligned}$$

We expand $\Delta_{n,n-1,\dots,n-k+2,l}(\widetilde{Y})$ along the k^{th}-row of $\widetilde{Y}$ to find that

$$\Delta_{n,n-1,\dots,n-k+2,l}(\widetilde{Y}) = (-1)^{1+k} t^{\omega_{kl}} y_{kl} \Delta_{n,n-1,\dots,n-k+2}(\widetilde{Y}).$$

It follows from substituting the above equation into (4.7) that

$$\mu^\# \circ \nu^\#(y_{kl}) = y_{kl}.$$

Before we show by direct computation that $\mu^\# \circ \nu^\#(x_{ij}) = x_{ij}$, we make the following auxiliary computation. Multiply both sides of (4.5) on the right by w_0, which flips columns left-to-right, then observe that $\widetilde{Y} w_0$ is invertible as an element of $\mathrm{Mat}_{n\times n}(\mathbb{C}[t][X,Y][V^{-1}])$ since both $\widetilde{Y}$ and w_0 are invertible. Solving for X,

$$X = \mu^\#(\widetilde{Z}) w_0 (\widetilde{Y} w_0)^{-1}. \qquad (4.8)$$

Let $i > j$ and apply $\Delta_{[j-1]\cup\{i\},[j]}(\cdot)$ to both sides of (4.8) to see that

$$\begin{aligned}
x_{ij} &= \Delta_{[j-1]\cup\{i\},[j]}(X) = \Delta_{[j-1]\cup\{i\},[j]}(\mu^\#(\widetilde{Z}) w_0 (\widetilde{Y} w_0)^{-1}) \\
&= \frac{\Delta_{[j-1]\cup\{i\},[j]}(\mu^\#(\widetilde{Z}) w_0)}{y_{1,n} y_{2,n-1} \cdots y_{j,n-j+1}} = (-1)^{\binom{j+1}{2}} \frac{\Delta_{[j-1]\cup\{i\},[n]\setminus[j]}(\mu^\#(\widetilde{Z}))}{y_{1,n} y_{2,n-1} \cdots y_{j,n-j+1}} \qquad (4.9)
\end{aligned}$$

where in passing from the first line to the second we used the fact that $(\widetilde{Y} w_0)^{-1}$ is upper-triangular with $y_{1,n}^{-1}, y_{2,n-1}^{-1}, \cdots, y_{n,1}^{-1}$ ($\omega_{1,n} = \omega_{2,n-1} = \cdots = \omega_{n,1} = 0$) on the diagonal and that right multiplication by upper-triangular matrices "sweeps" through columns from left to right. It now follows from (4.9) and $\mu^{\#} \circ \nu^{\#}(y_{kl}) = y_{kl}$ that

$$\begin{aligned}
\mu^{\#} \circ \nu^{\#}(x_{ij}) &= (-1)^{\binom{j+1}{2}} \frac{\Delta_{[j-1]\cup\{i\},[n]\setminus[j]}(\mu^{\#}(\widetilde{Z}))}{\mu^{\#} \circ \nu^{\#}(y_{1,n}y_{2,n-1}\cdots y_{j,n-j+1})} \\
&= (-1)^{\binom{j+1}{2}} \frac{\Delta_{[j-1]\cup\{i\},[n]\setminus[j]}(\mu^{\#}(\widetilde{Z}))}{y_{1,n}y_{2,n-1}\cdots y_{j,n-j+1}} \\
&= x_{ij}.
\end{aligned}$$

□

Factoring as in the above lemma means that the quotient $N^{-1}\backslash\mathcal{G}$ is isomorphic to $\mathcal{A}$. In fact, we next show that $N^{-}\backslash\backslash\mathcal{G}$ is isomorphic to $\mathcal{A}$, hence $N^{-}\backslash\backslash\mathcal{G} = N^{-}\backslash\mathcal{G}$.

Let $\iota^{\#} : \mathbb{C}[t][Z][U^{-1}] \twoheadrightarrow \mathbb{C}[t][Y][V^{-1}]$ be the surjection defined by

$$\iota^{\#}(z_{ij}) = \begin{cases} y_{ij} & \text{if } i+j \leqslant n+1, \\ 0 & \text{otherwise} \end{cases}$$

corresponding to the inclusion $\iota : \mathcal{A} \hookrightarrow \mathcal{G}$.

Lemma IV.38. *Let $\theta : N^{-}\backslash\backslash\mathcal{G} \longrightarrow \mathcal{A}$ be the map corresponding to the $\mathbb{C}[t]$-algebra map $\theta^{\#} : \mathbb{C}[t][Y][V^{-1}] \longrightarrow \mathbb{C}[t][Q][U^{-1}]$ defined by*

$$\theta^{\#}(y_{kl}) = \begin{cases} q_l & \text{if } k = 1, \\ (-1)^{1+k} \frac{q_{n,n-1,\ldots,n-k+2,l}}{q_{n,n-1,\ldots,n-k+2}} & \text{if } k > 1. \end{cases}$$

Then, θ is an isomorphism and is inverse to $\rho \circ \iota : \mathcal{A} \longrightarrow N^{-}\backslash\backslash\mathcal{G}$.

Proof. It suffices to show that $\theta^{\#}$ and $\iota^{\#} \circ \rho^{\#}$ are inverses. Let $\widetilde{Y} = (\widetilde{y}_{ij})$ be the $n \times n$ matrix of indeterminates defined by

$$\widetilde{y}_{ij} = \begin{cases} t^{\omega_{ij}} y_{ij} & \text{if } i+j \leqslant n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$
\begin{aligned}
(\iota^{\#} \circ \rho^{\#}) \circ \theta^{\#}(y_{kl}) &= (-1)^{1+k} \frac{\iota^{\#}(q_{n,n-1,\ldots,n-k+2,l})}{\iota^{\#}(q_{n,n-1,\ldots,n-k+2})} \\
&= (-1)^{1+k} \frac{t^{-\omega_{n,n-1,\ldots,n-k+2,l}} \Delta_{n,n-1,\ldots,n-k+2,l}(\widetilde{Y})}{t^{-\omega_{n,n-1,\ldots,n-k+2}} \Delta_{n,n-1,\ldots,n-k+2}(\widetilde{Y})} \\
&= (-1)^{1+k} t^{-\omega_{kl}} \frac{\Delta_{n,n-1,\ldots,n-k+2,l}(\widetilde{Y})}{\Delta_{n,n-1,\ldots,n-k+2}(\widetilde{Y})}. \qquad (4.10)
\end{aligned}
$$

Expand $\Delta_{n,n-1,\ldots,n-k+2,l}(\widetilde{Y})$ along the k^{th} row to see that

$$
\Delta_{n,n-1,\ldots,n-k+2,l}(\widetilde{Y}) = (-1)^{1+k} t^{\omega_{kl}} y_{kl}\, \Delta_{n,n-1,\ldots,n-k+2}(\widetilde{Y}). \qquad (4.11)
$$

We substitute (4.11) into (4.10) to see that $(\iota^{\#} \circ \rho^{\#}) \circ \theta^{\#}(y_{kl}) = y_{kl}$. It follows that $\iota^{\#} \circ \rho^{\#}$ is a surjective map of integral domains of equal Krull dimension $\binom{n+1}{2} + 1$, so $\iota^{\#} \circ \rho^{\#}$ is an isomorphism, which in turn implies that $\theta^{\#}$ is an isomorphism inverse to $\iota^{\#} \circ \rho^{\#}$. □

Lemma IV.39. *The ideal $\widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z][U^{-1}]$ is generated by elements of $\mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$.*

Proof. Let I and J be subsets of $[n]$ such that $q_{I,J} \in \widetilde{\mathcal{I}}_w$. Recall (4.5) to see that

$$
\begin{aligned}
\mu^{\#}(q_{I,J}) &= \mu^{\#}(t^{-\omega_{I,J}} \Delta_{I,J}(\widetilde{Z})) = t^{-\omega_{I,J}} \Delta_{I,J}(\mu^{\#}(\widetilde{Z})) \\
&= t^{-\omega_{I,J}} \Delta_{I,J}(X\widetilde{Y}) = \sum_{I' \leqslant I} t^{-\omega_{I,J}} f_{I'}(X)\, \Delta_{I',J}(\widetilde{Y}) \qquad (4.12)
\end{aligned}
$$

where $f_{I'}(X) \in \mathbb{C}[x_{ij}]_{i>j}$. Applying $\nu^{\#}$ to both sides of (4.12),

$$
q_{I,J} = (\nu^{\#} \circ \mu^{\#})(q_{I,J}) = \sum_{I' \leqslant I} t^{\omega_{I,J}} (\nu^{\#} \circ f_{I'})(X)\, \nu^{\#}(\Delta_{I',J}(\widetilde{Y})) \qquad (4.13)
$$

and notice that $\nu^{\#}(\Delta_{I',J}(\widetilde{Y})) \in \mathbb{C}[t][Q][U^{-1}]$. Furthermore, to see that $\nu^{\#}(\Delta_{I',J}(\widetilde{Y}))$ is generated by elements of $\widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z][U^{-1}]$, substitute in $\widetilde{Y} = X^{-1}\mu^{\#}(\widetilde{Z})$ from (4.5) so that

$$
\nu^{\#}(\Delta_{I',J}(\widetilde{Y})) = \nu^{\#}(\Delta_{I',J}(X^{-1}\mu^{\#}(\widetilde{Z}))) = \sum_{I'' \leqslant I'} \nu^{\#}(f_{I''}(X^{-1})\, \Delta_{I'',J}(\widetilde{Z})
$$

where $f_{I''}(X^{-1}) \in \mathbb{C}[X]$ as $X \longmapsto X^{-1}$ is algebraic $(\det(X) = 1)$. Now, notice that $\Delta_{I'',J}(\widetilde{Z}) \in \widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z][U^{-1}]$ since the rank condition making $q_{I,J}$ an element of $\widetilde{\mathcal{I}}_w$ implies that all degenerated minors $q_{K,L}$ such that $K \leqslant I$ and $L \leqslant J$ is an element of $\widetilde{\mathcal{I}}_w$. Therefore, $\nu^{\#}(\Delta_{I',J}(\widetilde{Y})) \in \mathbb{C}[t][Q][U^{-1}] \cap \widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z][U^{-1}] = \mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$ and (4.13) now implies the lemma. □

We now turn to the proof of Theorem IV.34.

Proof. To prove that

$$\mathrm{in}_\omega(\mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]) = \mathrm{in}_\omega(\widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z])[U_0^{-1}] \cap \mathrm{in}_\omega(\mathbb{C}[t][Q])[U_0^{-1}],$$

it suffices to show that the right-hand side is included in the left-hand side. Indeed, let $f_0 \in \mathrm{in}_\omega(\widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z])[U_0^{-1}] \cap \mathrm{in}_\omega(\mathbb{C}[t][Q])[U_0^{-1}]$ and $f \in \widetilde{\mathcal{I}}_w \cdot \mathbb{C}[t][Z][U^{-1}]$ be such that $\mathrm{in}_\omega(f) = f_0$. We show that $(\theta^{\#} \circ \iota^{\#})(f)$ is an element of $\mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$ such that $\mathrm{in}_\omega((\theta^{\#} \circ \iota^{\#})(f)) = f_0$ where the maps are

$$\begin{tikzcd}
 & \mathbb{C}[t][Z][U^{-1}] \arrow[dl, "\iota^{\#}"'] & \\
\mathbb{C}[t][Y][V^{-1}] \arrow[rr, "\theta^{\#}"'] & & \mathbb{C}[t][Q][U^{-1}]. \arrow[ul, hook, "\rho^{\#}"']
\end{tikzcd}$$

It follows from Lemma IV.39 that there exists $a_i \in \mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$ and $b_i \in \mathbb{C}[t][Z][U^{-1}]$ such that $f = \sum_i a_i b_i$. Then,

$$(\theta^{\#} \circ \iota^{\#})(f) = \sum_i (\theta^{\#} \circ \iota^{\#})(a_i) \cdot (\theta^{\#} \circ \iota^{\#})(b_i) = \sum_i a_i \, (\theta^{\#} \circ \iota^{\#})(b_i)$$

where $(\theta^{\#} \circ \iota^{\#})(a_i) = a_i$ by Lemma IV.38. It follows that $(\theta^{\#} \circ \iota^{\#})(f) \in \mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$.

To see that $\mathrm{in}_\omega((\theta^{\#} \circ \iota^{\#})(f)) = f_0$, let $f' \in \mathbb{C}[t][Q][U^{-1}]$ be another lift of f_0. Notice that $(\theta^{\#} \circ \iota^{\#})(f') = f'$ by Lemma IV.38, so that

$$\mathrm{in}_\omega((\theta^{\#} \circ \iota^{\#})(f')) = \mathrm{in}_\omega(f') = f_0$$

on the one hand, and

$$\mathrm{in}_\omega((\theta^\# \circ \iota^\#)(f')) = (\theta_0^\# \circ \iota_0^\#)(f_0) = \mathrm{in}_\omega((\theta^\# \circ \iota^\#)(f))$$

on the other. It follows that $\mathrm{in}_\omega((\theta^\# \circ \iota^\#)(f)) = f_0$ so $(\theta^\# \circ \iota^\#)(f)$ is an element of $\mathcal{I}_w \cdot \mathbb{C}[t][Q][U^{-1}]$ that realizes f_0 as an element of $\mathrm{in}(\mathcal{I}_w \cdot \mathbb{C}[t][Q])[U_0^{-1}]$. $\square$

CHAPTER V

Standard Monomial Theory

In this chapter, which is the core of this thesis, we present a complete proof of Theorem IV.26 by applying Standard Monomial Theory. In Section 5.1, we discuss the relevant background on Standard Monomial Theory. In Section 5.2, we show that standard monomials parametrize lattice points on RC-faces of the Gelfand-Tsetlin cone. In Section 5.2.3, we deduce a semi-toric degeneration of a Richardson variety as a further application of standard monomials and the involution constructed in Section 4.2.

5.1 Standard Monomial Theory

5.1.1 Standard monomials and defining chains

Consider the Grassmannian of k-planes in $\mathbb{C}^n$ embedded in $\mathbb{P}^{\binom{n}{k}-1}$ via its Plücker embedding. The Hodge-Young basis [Hod43] of the homogeneous coordinate ring of this embedding consists of products of Plücker variables called standard monomials. Standard monomials reflect the Schubert geometry of the Grassmannian in the sense that standard monomials not only restrict to a basis of the homogeneous coordinate ring of a Schubert variety, but do so by either vanishing or remaining linearly independent.

Standard Monomial Theory (SMT) [LS86] generalizes Hodge's basis to flag varieties and their Schubert varieties. Our reference for SMT are [RS97, Ses85] and [BL03, LL03] for its applications to Richardson varieties.

To a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ we can associate a line bundle $\mathcal{L}_\lambda$ over $F\ell_n$ as follows. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ be the integer vector defined by $\mathbf{a} := (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n)$. Recall the Plücker embedding of $F\ell_n = B^- \backslash\backslash M_n$ inside $\prod_{k=1}^n \mathbb{P}_k := \prod_{k=1}^n \mathbb{P}^{\binom{n}{k}-1}$ and let $\mathcal{L}_k$, for $k = 1, 2, \ldots, n$, be the line bundle on $F\ell_n$ defined as the pullback of $\mathcal{O}_{\mathbb{P}_k}(1)$ through the composition $F\ell_n \hookrightarrow \prod_k \mathbb{P}_k \twoheadrightarrow \mathbb{P}_k$.

Definition V.1. Let $\mathcal{L}_\lambda$ be the line bundle on $F\ell_n$ defined by

$$\mathcal{L}_\lambda := \mathcal{L}_n^{\otimes a_n} \otimes \mathcal{L}_{n-1}^{\otimes a_{n-1}} \otimes \cdots \otimes \mathcal{L}_1^{\otimes a_1}.$$

The n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the multidegree of sections in $H^0(F\ell_n, \mathcal{L}_\lambda)$.

Notice that we can consider Plücker variables p_I's as sections of $H^0(F\ell_n, \mathcal{L}_{|I|})$ as the pullback of homogeneous coordinate functions x_I's from the $|I|^{\text{th}}$-projective space in the Plücker embedding.

Definition V.2. Let $T \in \mathsf{SSYT}(n; \lambda)$ and $I_1, I_2, \ldots, I_{\lambda_1}$ be the columns of T indexed from left to right. We define the monomial $p_T \in H^0(F\ell_n, \mathcal{L}_\lambda)$ as the (tensor) product of sections corresponding to the columns of T:

$$p_T := p_{I_1} p_{I_2} \ldots p_{I_{\lambda_1}} = \prod_{k=1}^{\lambda_1} p_{I_k}.$$

We say that p_T is a standard monomial on $F\ell_n$.

Remark V.3. Notice that standard monomials of SMT correspond to semistandard tableaux rather than standard tableaux. This is because the notion of standard tableaux is already reserved for the set of tableaux associated with the representation theory of the symmetric group.

That standard monomials form a basis of $H^0(F\ell_n, \mathcal{L}_\lambda)$ was known to [You01]. Notice that it identifies the section ring $\bigoplus_{d\geqslant 0} H^0(F\ell_n, \mathcal{L}_{d\lambda})$ with the homogeneous coordinate ring $\bigoplus_{d\geqslant 0} \mathbb{C}[P]_{d\lambda}$. We define SMT basis for a Schubert variety X_w (Definition V.6) as a subset of standard monomials on $F\ell_n$ so that the standard monomial basis of $H^0(F\ell_n, \mathcal{L}_\lambda)$ restricts to the SMT basis of $H^0(X_w, \mathcal{L}_\lambda|_{X_w})$. The following example shows, however, that restricting standard monomials to a Schubert variety may create linear dependencies among non-vanishing standard monomials. Note this is in contrast to Hodge's standard monomials on the Grassmannian that either vanish or restrict to standard monomials of a Schubert variety.

Example V.4. Let $T_1 = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$. Consider the restriction of p_{T_1} and p_{T_2} to $X_{132} \subseteq F\ell_3$. Plücker variables satisfy $p_{21}p_3 - p_{31}p_2 + p_{32}p_1 = 0$ on $F\ell_3$ while $p_{21} = 0$ on X_{132}, therefore $p_{T_1}|_{X_{132}} = p_{T_2}|_{X_{132}}$ on X_{132}. So p_{T_1} and p_{T_2} are part of a basis for $H^0(F\ell_3, \mathcal{L}_{(1,1,0)})$, but become linearly dependent when restricted to X_{132}.

To define standard monomials, standard on a Schubert variety recall the map $\pi_k : S_n \longrightarrow \binom{[n]}{k}$ from Section 2.2.1 which sends $w \in S_n$ to $\pi_k(w) = \{w(1), w(2), \ldots, w(n)\} \in \binom{[n]}{k}$.

Definition V.5. A lift for the tableau $T \in \mathsf{SSYT}(n;\lambda)$ is a sequence $\mathbf{w} = (w_1, w_2, \ldots, w_{\lambda_1})$ of elements in S_n such that $\pi_{\lambda'_j}(w_j) = I_j$ for $j = 1, 2, \ldots, \lambda_1$. A lift $\mathbf{w} = (w_1, w_2, \ldots, w_{\lambda_1})$ for T is called a defining chain for T if $\mathbf{w}$ is linearly ordered with respect to Bruhat order, i.e., $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_{\lambda_1}$.

As a matter of fact, it can be shown that a tableau T is semistandard if and only if T admits a defining chain.

Definition V.6. The monomial p_T associated to a tableau $T \in \mathsf{SSYT}(n, \lambda)$ is called standard on X_u^w if there exists defining chains $\mathbf{w} = (w_1, w_2, \ldots, w_{\lambda_1})$ and

$\mathbf{w}' = (w'_1, w'_2, \ldots, w'_{\lambda_1})$ for T such that $w \geqslant w_1$ and $w'_{\lambda_1} \geqslant u$. We also say that T is standard on X_u^w to mean that p_T in standard.

Example V.7. The tableau $T_2 = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$ has a unique defining chain $(312, 213)$ and X_{213} is the only Schubert variety (excluding $X_{123} = F\ell_3$) on which p_{T_2} is standard. The tableau $T_1 = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ has four different defining chains $(321, 132)$, $(321, 123)$, $(231, 132)$, $(231, 123)$. Recall from Example V.4 that $p_{T_1}|_{X_{132}} = p_{T_2}|_{X_{132}}$, so the notion of standardness on X_{132} can be understood as making a choice between monomials p_{T_1} and p_{T_2} whose restrictions give the same function.

Notation V.8. For $u, w \in S_n$, let SM_u^w denote the set of tableaux standard on X_u^w and $\mathsf{SM}_u^w(\lambda)$ denote the subset of SM_u^w consisting of tableaux of shape λ. We write SM_u for $\mathsf{SM}_u^{w_0}$ and SM^w for $\mathsf{SM}_{\mathrm{id}}^w$, and similarly, $\mathsf{SM}_u(\lambda)$ for $\mathsf{SM}_u^{w_0}(\lambda)$ and $\mathsf{SM}^w(\lambda)$ for $\mathsf{SM}_{\mathrm{id}}^w(\lambda)$.

We have seen that lifts and defining chains for a tableau $T \in \mathsf{SSYT}(n; \lambda)$ are in general not unique. For a given tableau T, however, we can define a partial order on the set of defining chains such that there exists a unique minimal defining chain and a unique maximal defining chain.

Lemma V.9. *[Ses85] Let $T \in \mathsf{SSYT}(n; \lambda)$ be a tableau. There exists a unique minimal defining chain $\mathbf{w}^- = (w_1^-, w_2^-, \ldots, w_{\lambda_1}^-)$ and maximal defining chain $\mathbf{w}^+ = (w_1^+, w_2^+, \ldots, w_{\lambda_1}^+)$ for T, such that if $\mathbf{w} = (w_1, w_2, \ldots, w_{\lambda_1})$ is any defining chain for T then $w_j^+ \geqslant w_j \geqslant w_j^-$ for $j = 1, 2, \ldots, \lambda_1$.*

5.1.2 Defining chains and key tableaux

It follows from Lemma V.9 that T is standard on X_u^w if and only if $w \geqslant w_1^-$ and $w_{\lambda_1}^+ \geqslant u$. Consequently, it would be desirable to have a computational method of obtaining the maximal and minimal defining chains of a given tableau T. In fact, the notions of right and left key tableaux (Section 2.3.2) were introduced for this purpose [LS88, LS90].

Notation V.10. For a nonempty partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, let λ^* be the partition defined by $\lambda^* = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0)$, called the dual partition of λ.

Definition V.11. Let $T \in \mathsf{SSYT}(n;\lambda)$ be a tableau of shape λ such that I_1, I_2, $\ldots$, I_{λ_1} are the columns of T from left to right. Let $*T$ be a filling of shape λ^* such that the columns of $*T$ are $I^c_{\lambda_1} = [n] \setminus I_{\lambda_1}, I^c_{\lambda_1-1} = [n] \setminus I_{\lambda_1-1}, \ldots, I^c_1 = [n] \setminus I_1$ from left to right. If we arrange this filling of $*T$ to be decreasing in the columns, then $*T$ is a semistandard tableau as observed in [Ava08, Proposition 2]. We call $*T$ the complement of T.

Define the involution $* : \mathsf{SSYT}(n;\lambda) \longrightarrow \mathsf{SSYT}(n;\lambda^*)$ by sending a tableau T to its complement tableau $*T$.

Lemma V.12. *[LS90, RS97] Let $T \in \mathsf{SSYT}(n;\lambda)$ and let T^J denote the tableau consisting of columns of T labeled by J for subsets J of $[\lambda_1]$. Let $\mathbf{w}^+$ and $\mathbf{w}^-$ be defining chains for T as in Lemma V.9. Then, $w^+_j = w_+(T^{[j]})$ and $w^-_j = w_-(T^{[\lambda_1]\setminus[j-1]})$ for $j = 1, 2, \ldots, \lambda_1$.*

Proof. The maximal defining chain half of the lemma is [RS97, Lemma 8]. By [Ava08, Theorem 8], the complement of the right key of T is the left key of the complement of T. Therefore, $K_-(T) = *K_+(*T)$, which in turn implies that

$$w_-(T) = w_+(*T)w_0. \tag{5.1}$$

We deduce the minimal defining chain half of the lemma from (5.1). Assume without loss of generality that $\lambda_n = 0$ by replacing T by $*T$ if necessary and applying (5.1).

Let $\mathbf{u}^+ = (u_1^+, u_2^+, \ldots, u_{\lambda_1}^+)$ be the maximal defining chain for $*T$ so that $u_j^+ = w_+((*T)^{[j]})$ where $T^{[j]}$ denotes the tableau consisting of the first j columns of T. The sequence $(\mathbf{u}^+)^* := (u_{\lambda_1}^+ w_0, u_{\lambda_1-1}^+ w_0, \ldots, u_1^+ w_0)$ is the minimal defining chain for T because right multiplication by w_0 reverses strings representing permutations in one-line notation and minimal for T since $\mathbf{u}^+$ is maximal for $*T$. We may now apply (5.1) to see that

$$w_j^- = u_{\lambda_1-j+1}^+ w_0 = w_+((*T)^{[\lambda_1-j+1]})w_0 = w_-(T^{[\lambda_1]\setminus[j-1]})$$

as was to be shown. $\square$

In particular, $w_{\lambda_1}^+ = w_+(T)$ and $w_1^- = w_-(T)$ so that our criteria for determining whether T is standard on X_u^w can be rephrased as follows.

Proposition V.13. *(SMT for Richardson varieties) A tableau T is standard on X_u^w if and only if $w_+(T) \geqslant u$ and $w \geqslant w_-(T)$.*

Finally, the following theorem is fundamental to SMT.

Theorem V.14. *[LL03, Theorem 34] Let $\lambda \in \Lambda_n^+$ be a partition with at most n parts and X_u^w be a Richardson variety in $F\ell_n$. Then, the standard monomials, standard on X_u^w of multidegree $d\lambda$ form a basis for $H^0(X_u^w, \mathcal{L}_{d\lambda})$ for $d \geqslant 1$.*

5.2 Pipe dreams and SMT

In this section, which is the core of our proof of Theorem IV.26, and therefore this thesis, we show that Standard Monomials, standard on a Schubert variety correspond to lattice points on RC-faces of GT-polytope. The proof of this correspondence is in Section 5.2.2 where we show that canonical lifts of key tableaux and Demazure products of pipe dreams are equal. In Section 4.2, we deduce Corollary V.26 for Richardson varieties from Theorem IV.26.

5.2.1 Schubert varieties

Recall that $\mathrm{in}_\omega(\mathbb{C}[P]) \cong \mathbb{C}[\mathsf{GT}(n)]$ as semigroup rings and that $\langle \mathsf{A}_w \rangle$ is the image of $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ under this isomorphism. We also have the graded vector space $\mathbb{C}\{\mathsf{RP}_w\} = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{C}[\mathsf{RP}_w]_\lambda = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{C}\{\mathsf{RP}_w(\lambda)\}$ that is the subspace of $\mathbb{C}[\mathsf{GT}(n)]$ spanned by monomials corresponding to faces of the GT-cone associated with reduced pipe dreams in $\mathcal{RP}_w$. See, Section 4.3 for details.

The inclusion of $\mathrm{in}_\omega(I_w)$ inside $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])$ induces the surjection

$$\frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}_\omega(I_w)} \twoheadrightarrow \frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P])} = \frac{\mathbb{C}[\mathsf{GT}(n)]}{\langle \mathsf{A}_w \rangle}.$$

Recall that $\mathbb{C}[\mathsf{GT}(n)]/\langle \mathsf{A}_w \rangle$ and $\mathbb{C}\{\mathsf{RP}_w\}$ are identified as graded vector spaces, so in fact, the above surjection implies that

$$\frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}_\omega(I_w)} \twoheadrightarrow \mathbb{C}[\mathsf{RP}_w] \tag{5.2}$$

as graded vector spaces. In terms of dimension count, (5.2) implies that

$$\#\{p_T \in \mathbb{C}[P] : T \in \mathsf{SSYT}(n;\lambda) \text{ is standard on } X_w\} = \#\mathsf{SM}_w(\lambda) \geqslant \#\mathsf{RP}_w(\lambda) \tag{5.3}$$

where $\dim_{\mathbb{C}}(\mathrm{in}_\omega(\mathbb{C}[P])/\mathrm{in}_\omega(I_w)) = \#\mathsf{SM}_w(\lambda)$ by flatness of the degeneration from $\mathbb{C}[P]/I_w$ to $\mathrm{in}_\omega(\mathbb{C}[P])/\mathrm{in}_\omega(I_w)$.

In the next section, we prove that $\#\mathsf{SM}_w(\lambda) = \#\mathsf{RP}_w(\lambda)$ from which it follows that the surjection in (5.2) is, in fact, an isomorphism. This isomorphism proves that $\mathrm{in}_\omega(\widetilde{I}_w) \cap \mathrm{in}_\omega(\mathbb{C}[P]) = \mathrm{in}_\omega(I_w)$ which is sufficient for Theorem IV.26.

5.2.2 Combinatorial lemmas

In this section, we prove the following proposition.

Proposition V.15. *Let T be a tableau and $\Gamma(T)$ be the corresponding GT-pattern. Then, p_T is standard on X_w if and only if $\Gamma(T)$ is in RP_w.*

Before discussing the proof of Proposition V.15, we observe that the proposition implies that $\#\mathsf{SM}_w(\lambda) = \#\mathsf{RP}_w(\lambda)$ as follows. By (5.3), it suffices to show that $\#\mathsf{SM}_w(\lambda) \leqslant \#\mathsf{RP}_w(\lambda)$ and as a consequence of Proposition V.15, the map $\Gamma : \mathsf{SSYT}(n) \longrightarrow \mathsf{GT}(n)$ restricts to an injective map of SM_w into RP_w preserving shapes as in

$$\begin{tikzcd}
\mathsf{SM}_w \arrow[r, hook, dashed] \arrow[d, hook] & \mathsf{RP}_w \arrow[d, hook] \\
\mathsf{SSYT}(n) \arrow[r] & \mathsf{GT}(n)
\end{tikzcd}$$

Therefore, $\#\mathsf{SM}_w(\lambda) \leqslant \#\mathsf{RP}_w(\lambda)$ as desired.

Proposition V.15 follows as an immediate consequence of Lemma V.19 combined with Lemma V.20. The next set of definitions provide a way to interpret GT-patterns, or equivalently tableaux as non-reduced pipe dreams.

Definition V.16. Let $\mathcal{D}$ be the map from skew tableaux with entries in $[n]$ to pipe dreams of rank n defined by

$$S = (s_{ij}) \longmapsto \mathcal{D}(S) := D_0 \setminus \{(i, s_{ij}) \in [n] \times [n] : i + s_{ij} \leqslant n\}. \tag{5.4}$$

For skew tableaux S with entries in $[n]$, let $Q(S) := Q(\mathcal{D}(S))$ denote the word read from the pipe dream $\mathcal{D}(S)$ and $\mathsf{Dem}(S) := \mathsf{Dem}(Q(S))$, the Demazure product of $Q(S)$.

For example,

$$S = \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 5 \\ \hline 4 & 4 & 3 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad \longmapsto \quad \mathcal{D}(S) = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & + & & \multicolumn{1}{c}{} \\ \cline{1-4} + & + & & \multicolumn{2}{c}{} \\ \cline{1-3} & & \multicolumn{3}{c}{} \\ \cline{1-2} & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

so $Q(S) = (s_4, s_4, s_3)$ and $\mathsf{Dem}(S) = s_4 * s_4 * s_3 = s_4 * s_3 = 12534$.

Definition V.17. Let $\mathcal{D}' : \mathsf{GT}(n) \longrightarrow \mathcal{PD}(n)$ be the map defined by

$$\Gamma = (\gamma_{ij}) \longmapsto \mathcal{D}'(\Gamma) := \{(i,j) \in [n] \times [n] : \gamma_{i,j} = \gamma_{i,j+1}\}.$$

In words, $\mathcal{D}'(\Gamma)$ is a pipe dream, possibly non-reduced, obtained by converting horizontal equalities in a GT-pattern into crossing tiles. For $\Gamma \in \mathsf{GT}(n)$, let $Q'(\Gamma)$ denote the word read from $\mathcal{D}'(\Gamma) \in \mathcal{PD}(n)$ and $\mathsf{Dem}'(\Gamma)$ denote the Demazure product of $Q'(\Gamma)$.

Remark V.18. The map $\mathcal{D}$ restricts to tableaux with straight shape as the composition $\mathsf{SSYT}(n) \xrightarrow{\Gamma} \mathsf{GT}(n) \xrightarrow{\mathcal{D}'} \mathcal{PD}_n$ so identifying $\mathsf{SSYT}(n)$ and $\mathsf{GT}(n)$, $\mathcal{D}'|_{\mathsf{SSYT}(n)} = \mathcal{D}$. Henceforth, we will only use the notation $\mathcal{D}(\bullet)$ and similarly write $Q(\bullet)$ and $\mathsf{Dem}(\bullet)$ instead of $Q'(\bullet)$ and $\mathsf{Dem}'(\bullet)$.

For example, for $T = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array}$ such that $\Gamma(T) = \begin{array}{ccc} 2 & 2 & 1 \\ 1 & 1 & \\ 0 & & \end{array}$,

$$\mathcal{D}(\Gamma(T)) = \begin{array}{|c|c|c|}\hline + & & \\ \hline + & & \\ \cline{1-2} & & \\ \cline{1-1} \end{array}$$

so that $Q(\Gamma(T)) = (s_1, s_2)$ and $\mathsf{Dem}(\Gamma(T)) = s_1 * s_2 = s_1 s_2 = 231$.

The following lemma provides a Bruhat-order criterion for determining whether a given GT-pattern is contained in an RC-face.

Lemma V.19. *Let $\Gamma \in \mathsf{GT}(n)$. Then, $\Gamma \in \mathsf{RP}_w$ if and only if $\mathsf{Dem}(\Gamma) \geqslant w$.*

Proof. The word $Q(\Gamma)$ converts equations defining RC-faces into adjacent transpositions. Consequently, $\Gamma \in \mathsf{RP}_w$ if and only if w is a subword of $Q(\Gamma)$. Applying Lemma II.6, w is a subword of $Q(\Gamma)$ if and only if $\mathsf{Dem}(\Gamma) \geqslant w$. $\square$

The next lemma states that the left-hand side and the the right-hand side of Figure 5.1 commute.

$$\begin{tikzcd}
& \mathsf{GT}(n) \arrow[rr, leftrightarrow, "\Gamma"] \arrow[dl, "\mathcal{D}"'] & & \mathsf{SSYT}(n) \arrow[dr, "K_+"] & \\
\mathcal{PD}(n) \arrow[drr, "\mathsf{Dem}"'] & & & & \mathrm{Key}(n) \arrow[dll, "w_+"] \\
& & S_n & &
\end{tikzcd}$$

Figure 5.1: Maps in Lemma V.20

Lemma V.20. *Let $T \in \mathsf{SSYT}(n;\lambda)$ where $\lambda \in \Lambda_n^{++}$. Then, $\mathsf{Dem}(T) = w_+(T)$.*

Proof. The tableau T has columns of all possible heights since $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. For each column height $k = 1, 2, \ldots, n$, let

$$S_0^{(k)} \longrightarrow S_1^{(k)} \longrightarrow \ldots \longrightarrow S_m^{(k)} \tag{5.5}$$

be the sequence of skew tableaux beginning with the tableau $S_0^{(k)} := T$ and ending with the skew tableau $S^{(k)} := S_m^{(k)}$ whose right-most column has height equal to k. Consecutive pairs of tableaux in (5.5) are related by jdt such that $S_j^{(k)} = \mathrm{jdt}^{\boxed{j}}(S_{j-1}^{(k)})$, for $j = 1, 2, \ldots, m$, where the empty boxes that are used in reverse-slides are schematically labelled in order in Figure 5.2.

t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}	t_{19}
t_{21}	t_{22}	t_{23}	t_{24}	t_{25}	3	5	$\ldots$	$m-1$
t_{31}	t_{32}	t_{33}	1	2	4	6	$\ldots$	m
t_{41}								

Figure 5.2: Reverse slide order for (5.5) with $k = 3$

For example, for $T = \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$,

$$S_0^{(2)} = \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 1 \\ \cline{1-2} 2 & \blacksquare \\ \cline{1-2}\end{array} \longrightarrow S_1^{(2)} = \begin{array}{ccc} & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} \\ \cline{1-2} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{1-2}\end{array}$$

and

$$S_0^{(3)} = \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 4 & 1 \\ \cline{1-2} 2 & \blacksquare \\ \cline{1-2}\end{array} \longrightarrow S_1^{(3)} = \begin{array}{ccc} & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{\blacksquare} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \cline{1-2}\end{array} \longrightarrow S_2^{(3)} = \begin{array}{ccc} & & \multicolumn{1}{|c|}{5} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\blacksquare} \\ \hline\end{array} \longrightarrow S_3^{(3)} = \begin{array}{ccc} & & \multicolumn{1}{|c|}{5} \\ \cline{2-3} & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} \\ \hline\end{array}.$$

The ordering of empty boxes in reverse jdt slides is made to resemble the sequence of elementary moves of Section 2.3.2. Such sequence of elementary moves for the above example is

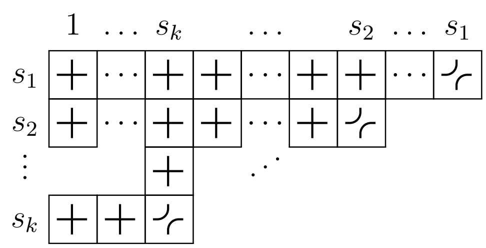
$$\begin{array}{|c|c|c|}\hline 5&4&3\\ \hline 4&1\\ \cline{1-2} 2\\ \cline{1-1}\end{array} \longrightarrow \begin{array}{ccc} &\multicolumn{1}{|c|}{5}&\multicolumn{1}{c|}{3}\\ \hline \multicolumn{1}{|c|}{4}&\multicolumn{1}{c|}{4}\\ \cline{1-2} \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{1}\\ \cline{1-2}\end{array} \longrightarrow \begin{array}{ccc} &&\multicolumn{1}{|c|}{5}\\ \cline{3-3} &&\multicolumn{1}{|c|}{3}\\ \hline \multicolumn{1}{|c|}{4}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{1}\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}.$$

Consequently, the right-most column of $S^{(k)}$ is equal to the column of $K_+(T)$ of height k.

We claim that the Demazure products of skew tableaux in (5.5) satisfy

$$\mathsf{Dem}(T)[k] = \mathsf{Dem}(S_1^{(k)})[k] = \cdots = \mathsf{Dem}(S^{(k)})[k], \tag{5.6}$$

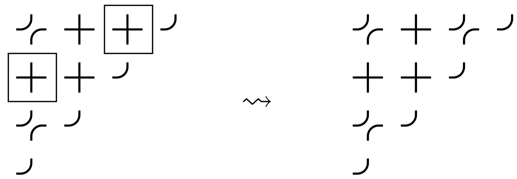
but isolate further discussion of (5.6) to Lemma V.21. We show that (5.6) implies that $\mathsf{Dem}(T) = w_+(T)$. Indeed, let $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k$ be the entries of the rightmost column of $S^{(k)}$. Then, for $1 \leqslant j \leqslant k$, the first s_j entries on the j^{th} row of $\mathcal{D}(S^{(k)})$ are a sequence of $s_j - 1$, $+$ tiles and a ╯╭ tile for the s_j^{th} entry. Therefore, the first k-rows of $\mathcal{D}(S^{(k)})$ look like

$\mathcal{D}(S^{(k)}) =$ 

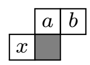
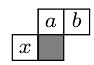
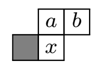
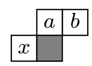
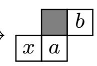
Notice that the sub-pipe dream formed by the first k-pipes is reduced so that $\mathsf{Dem}(S^{(k)})[k] = \{s_k, s_{k-1}, \ldots, s_1\}$. Since the height k column of $K_+(T)$ is equal to the right-most column of $S^{(k)}$, it then follows from (5.6) that $\mathsf{Dem}(T)[k] = w_+(T)[k]$, for $k = 1, 2, \ldots, n$. $\square$

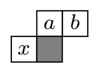
Lemma V.21. *Let $S_0^{(k)} \longrightarrow S_1^{(k)} \longrightarrow \ldots \longrightarrow S_m^{(k)}$ be the sequence of skew tableaux from* (5.5)*. Then, $\mathsf{Dem}(S_0^{(k)})[k] = \mathsf{Dem}(S_1^{(k)})[k] = \cdots = \mathsf{Dem}(S^{(k)})[k]$.*

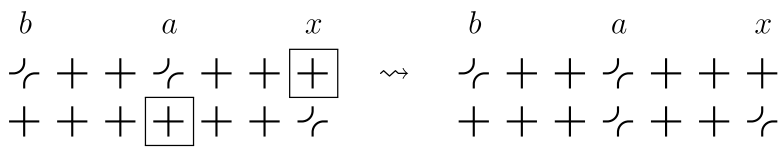
Proof. To compute the Demazure product of a non-reduced pipe dream, we iteratively reduce the number of crossing tiles until the pipe dream is reduced. Each reduction step in this process corresponds to the relation $s_i * s_i = s_i$ for some $i = 1, 2, \ldots, n-1$. For example,

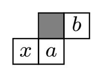


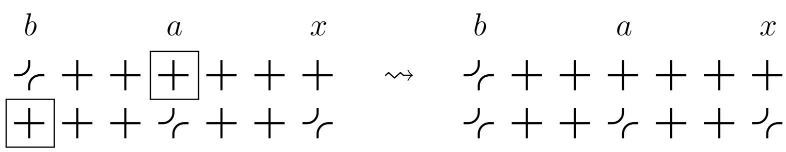
represents the computation $s_3 * s_2 * s_3 * s_2 = s_2 * s_3 * (s_2 * s_2) = s_2 * s_3 * s_2$. We show that reverse slides connecting T to $S^{(k)}$ preserves initial k-terms of $\mathsf{Dem}(S_0^{(k)})$.

Locally, a reverse slide is applied to  where $a \geqslant b$. We may assume without loss of generality that $x \geqslant a \geqslant b$, since the reverse slide  $\rightsquigarrow$  does not affect the image under $\mathcal{D}$. So the next reverse slide is  $\rightsquigarrow$ .

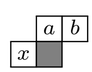
Case 1: $x > a > b$. The partial pipe dream mapped from  is

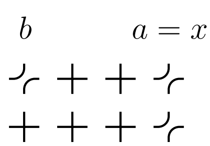
(5.7) 

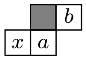
where in passing from the left to the right, we have reduced a double crossing of pipes into a single crossing. The partial pipe dream mapped from  is

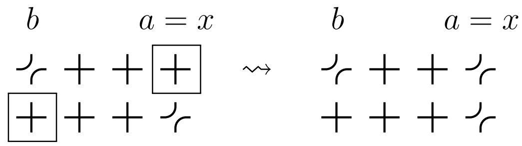
(5.8) 

The pipe dreams in (5.7) and (5.8) have the same pipe connectivity, so their Demazure products are equal.

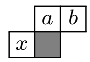
Case 2: $x = a > b$. The pipe dream mapped from  is

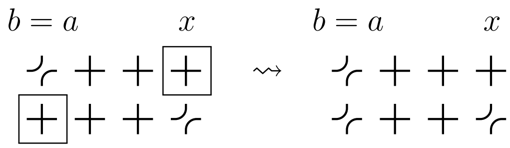


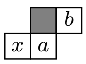
whereas  maps to

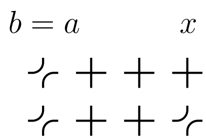


The two pipe dreams are the same.

Case 3: $x > a = b$. The map $\mathcal{D}$ sends  to

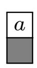
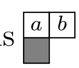
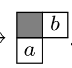


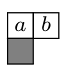
whereas  maps to

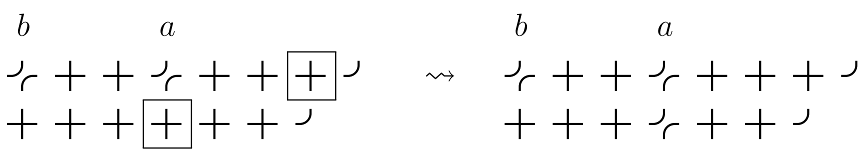


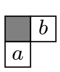
The two pipe dreams are the same.

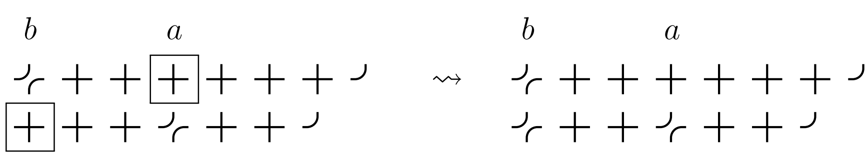
In the remaining cases, one or both entries x are b are not elements of the skew tableaux. We label the cases to indicate similarities, so Case 1 is similar to Case 1'.

In the next two cases,  is the left-most column so the reverse slide is  $\rightsquigarrow$ .

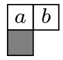
Case 1': $a > b$. The skew tableau  maps to

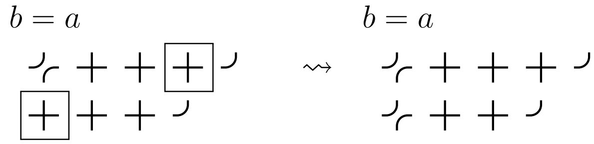


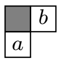
whereas  maps to

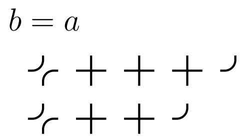


Demazure products of the two pipe dreams are the same.

Case 3': $a = b$. The skew tableau  maps to

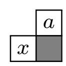
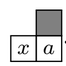


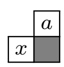
whereas  maps to

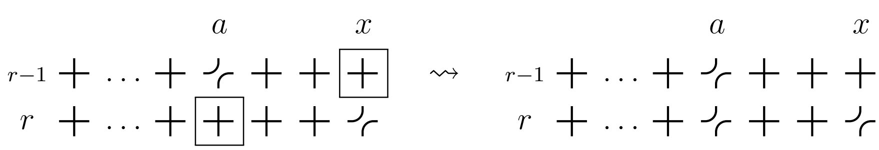


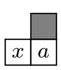
Demazure products of the two pipe dreams are the same.

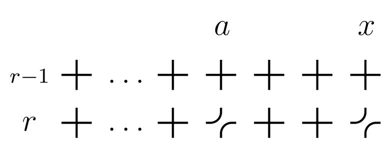
In the remaining three cases, pipes in rows $r-1$ and r are interchanged. Reverse slide, however, progresses monotonically towards the Northwest corner so that $r \leqslant k$ preserving the initial k-terms of the Demazure product.

In the next two cases, the skew tableau  is the right-most column of a skew tableau and the reverse slide we consider is  $\rightsquigarrow$ .

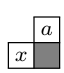
Case 4: $x > a$. The skew tableau  maps to

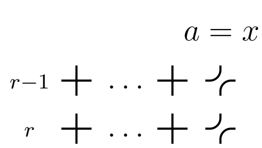


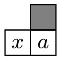
and  maps to

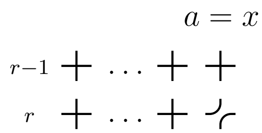


Demazure products of the two pipe dreams are different since pipes $r-1$ and r are switched. Nonetheless, the initial k-terms of the two Demazure products are the same.

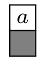
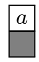
Case 5: $x = a$. The skew tableau  maps to

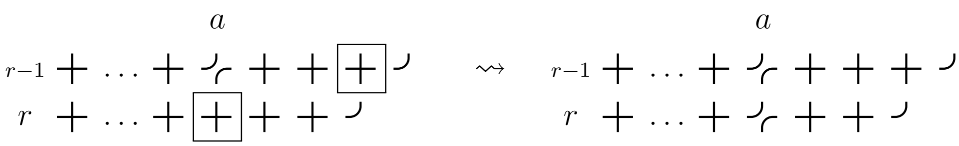


whereas  maps to

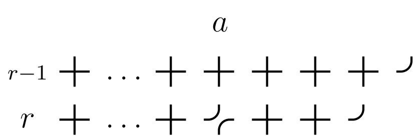


Initial k-terms of Demazure products are preserved by reverse slide as before.

Case 4': In this case, the reverse slide is  $\rightsquigarrow$  where in both skew tableaux, a is the only entry in its row. The skew tableau  maps to



whereas  maps to



Initial k-terms of Demazure products are preserved. $\square$

Now, the proposition follows from the two lemmas.

Proof. (Proposition V.15) Combine Lemma V.19 and Lemma V.20. $\square$

5.2.3 Richardson varieties

In this section, we deduce an analogue of Theorem IV.26 for Richardson varieties by applying the involution of Section 4.2 and SMT.

Definition V.22. Let RP^w be the subset of $\mathsf{GT}(n)$ defined by

$$\mathsf{RP}^w := \bigcup_{D \in \mathcal{RP}_{ww_0}} \{\Gamma \in \mathsf{GT}(n) : \gamma_{n-i-j+2,j} = \gamma_{n-i-j+1,j+1} \text{ for } (i,j) \in D\}.$$

We call the elements of RP^w the lattice points of opposite RC-faces for w of the GT-cone. Let $\mathsf{RP}_u^w := \mathsf{RP}_u \cap \mathsf{RP}^w$ and call its elements the lattice points of Richardson faces for X_u^w.

Definition V.23. Let $I_u^w := I_u + I^w$ where I_u is a Schubert ideal and I^w is an opposite Schubert ideal. By [LL03, Theorem 16], I_u^w is the ideal of the Richardson variety X_u^w in $\mathbb{C}[F\ell_n] = \mathbb{C}[P]$.

Definition V.24. For the tuple of reduced pipe dreams $(E, D) \in \mathcal{RP}_u \times \mathcal{RP}_{ww_0}$, let

- $\mathcal{F}^D$ be the face of the GT-polytope defined by setting $\gamma_{n-i-j+2,j} = \gamma_{n-i-j+1,j+1}$ for each $(i, j) \in D$.
- $\mathcal{F}_E^D$ be the face of the GT-polytope defined by $\mathcal{F}_E^D := \mathcal{F}_E \cap \mathcal{F}^D$.

We call $\mathcal{F}^D$ an opposite Schubert face and $\mathcal{F}_E^D$ a Richardson face of the GT-polytope.

Lemma V.25. *Let $\tau : \mathbb{C}[t][Q] \longrightarrow \mathbb{C}[t][Q]$ be the signed involution constructed in Section 4.2. Then, $\tau_1(\mathsf{SM}_{ww_0}) = \mathsf{SM}^w$ and $\tau_0(\mathsf{RP}_{ww_0}) = \mathsf{RP}^w$.*

Proof. For the first statement, notice that $\tau_1(p_T) = \pm p_{*T}$ for $T \in \mathsf{SSYT}(n)$ so it suffices to show that if T is standard on X_{ww_0} then $*T$ is standard on X^w. Indeed, $w_+(T) \geqslant ww_0$ is equivalent to $w_-(*T) \leqslant w$ by (5.1). Hence, $T \in \mathsf{SM}_{ww_0}$ if and only if $*T \in \mathsf{SM}^w$.

For the second statement, we claim that τ_0 maps $\Gamma \in \mathsf{GT}(n)$ to $\Gamma' = (\gamma'_{i,j})$ defined by

$$\gamma'_{i,j} = \gamma_{1,1} - \gamma_{n-i-j+2,j}. \tag{5.9}$$

To verify the claim, identify $\Gamma \in \mathsf{GT}(n)$ with the monomial $\mathrm{in}_\omega(p_T)$ for $T \in \mathsf{SSYT}(n)$ and observe that $\tau_0(\mathrm{in}_\omega(p_T)) = \mathrm{in}_\omega(p_{*T})$. It is not difficult to see that $\tau_0(\mathrm{in}_\omega(p_I)) = \mathrm{in}_\omega(p_{[n]\setminus I})$ corresponds to the GT-pattern obtained by (5.9). Hence, $\mathrm{in}_\omega(p_{*T})$ corresponds to GT-pattern obtained by (5.9) as well.

Then,

$$\begin{aligned}\tau_0(\mathsf{RP}_{ww_0}) &= \bigcup_{D\in\mathcal{RP}_{ww_0}} \{\tau_0(\Gamma)\in\mathsf{GT}(n) : \gamma_{i,j}=\gamma_{i,j+1} \text{ for } (i,j)\in D\}\\ &= \bigcup_{D\in\mathcal{RP}_{ww_0}} \{\Gamma'\in\mathsf{GT}(n) : \gamma'_{n-i-j+2,j}=\gamma'_{n-i-j+1,j+1} \text{ for } (i,j)\in D\}\\ &= \mathsf{RP}^w.\end{aligned}$$

□

Geometrically, Lemma V.25 says that $\mathcal{X}^w = \tau(\mathcal{X}_{ww_0})$ and the opposite Schubert variety X^w degenerates to the reduced union of toric subvarieties $\bigcup_{D\in\mathcal{RP}_{ww_0}} X_{\mathcal{F}^D}$. Next, as a further application of SMT, we show that the components of a degeneration of a Richardson variety correspond to Richardson faces.

Corollary V.26. *The family $\mathcal{X} = B^-\backslash\backslash(M_n\times\mathbb{A}^1)$ induces a flat degeneration of Richardson variety X_u^w to a reduced union $\bigcup_{(E,D)\in\mathcal{RP}_u\times\mathcal{RP}_{ww_0}} X_{\mathcal{F}_E^D}$ of toric subvarieties of the Gelfand-Tsetlin toric variety $X_{\mathcal{P}_\lambda}$.*

Proof. The inclusion of $\mathrm{in}_\omega(I_u)+\mathrm{in}_\omega(I^w)$ into $\mathrm{in}_\omega(I_u^w)$ imply that $\mathrm{in}_\omega(X_u^w)$ is a subscheme of the (scheme-theoretic) intersection $\mathrm{in}_\omega(X_u)\cap\mathrm{in}_\omega(X^w)$ and induces the surjection

$$\frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}_\omega(I_u)+\mathrm{in}_\omega(I^w)} \twoheadrightarrow \frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}_\omega(I_u^w)}. \tag{5.10}$$

Lemma V.25 implies that $\mathbb{C}\{\mathsf{RP}^w\}\cong\mathrm{in}_\omega(\mathbb{C}[P])/\mathrm{in}_\omega(I^w)$ as graded vector spaces so that

$$\mathbb{C}\{\mathsf{RP}_u^w\} = \mathbb{C}\{\mathsf{RP}_u\cap\mathsf{RP}^w\} \cong \frac{\mathrm{in}_\omega(\mathbb{C}[P])}{\mathrm{in}_\omega(I_u)+\mathrm{in}_\omega(I^w)}. \tag{5.11}$$

Notice that $\#\mathsf{RP}_u^w = \#\mathsf{SM}_u^w$ since $\#\mathsf{RP}_u = \#\mathsf{SM}_u$ and $\#\mathsf{RP}^w = \#\mathsf{SM}^w$. Now, counting dimensions in (5.10) and (5.11) implies that the map in (5.10) is an isomorphism. □

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