Essays in Financial Economics

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Economics)
in the University of Michigan
2015

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# Acknowledgments

I would like to express my deepest gratitudes to all my advisors for their guidance and support. I have been fortunate to receive financial support for University of Michigan Economics Department, Michigan Institute for Teaching and Research in Economics (MITRE) and Rackham Graduate School. I am grateful for the dissertation internship opportunity at the Office of Financial Stability Policy and Research at the Federal Reserve Board.

I would like to thank my partner Laura Colon-Melendez and my family for their love and patience, Kemal Ozhan and Ufuk Senveli for jokes and laughs, and all friends for amity.

# Contents

**Acknowledgments** **ii**

**List of Figures** **vi**

**List of Tables** **viii**

**Abstract** **ix**

**Chapter**

**1 Real Effects of Bank Capital Structure** **1**

1.1 Introduction . . . . . . . . . . 1

1.2 The Model . . . . . . . . . . 5

1.2.1 The Benchmark . . . . . . . . . . 7

1.2.2 The Governance Problem . . . . . . . . . . 8

1.2.3 Demandable Debt and Bank Runs . . . . . . . . . . 10

1.3 Equilibrium Liquidation Rule . . . . . . . . . . 12

1.3.1 Excessive Liquidation When $\kappa < 1 - \alpha$ . . . . . . . . . . 12

1.3.2 Excessive Continuation When $\kappa > 1 - \alpha$ . . . . . . . . . . 14

1.4 Investment in the Project . . . . . . . . . . 16

1.4.1 Under-capitalized Case $\kappa < 1 - \alpha$ . . . . . . . . . . 17

1.4.2 Over-capitalized Case $\kappa > 1 - \alpha$ . . . . . . . . . . 19

1.5 The Banker's Optimal Capital Structure . . . . . . . . . . 22

1.5.1 Perfect Competition among Outside Equity Investors . . . . . 25

1.5.2 No Private Benefits . . . . . . . . . . 27

1.6 Conclusion . . . 28

**Appendix 1** **30**

**2 Savings Portfolio Problem with Collateralized Debt**[1] **47**

2.1 Introduction . . . 47

2.2 The Model . . . 49

2.2.1 Preferences . . . 50

2.2.2 Borrowing and Lending . . . 50

2.2.3 Beliefs . . . 51

2.2.4 Margin-Interest Menu of Loans . . . 52

2.2.5 The Decision Problem(s) . . . 54

2.3 Levered Excess Returns . . . 55

2.4 Solution to the Decision Problem . . . 58

2.4.1 Demand for Leverage . . . 59

2.4.2 Demand for the Risky Asset . . . 62

2.4.3 The Optimal Portfolio and the Margin . . . 65

2.5 Comparative Statics . . . 66

2.5.1 Risk Aversion . . . 69

2.5.2 Initial Wealth . . . 70

2.5.3 The Lenders' Perception of Risk . . . 71

2.6 Conclusion . . . 72

**Appendix 2** **73**

**3 A General Equilibrium Model of Leverage**[2] **104**

3.1 Introduction . . . 104

3.2 Model . . . 106

3.2.1 The Supply of Loans . . . 109

[1]This chapter presents joint work with Lones A. Smith

[2]This chapter presents joint work with Lones A. Smith

3.2.2 The Borrower's Problem in Partial Equilibrium . . . . . . . . . 110
3.3 Deposit Interest Rate . . . . . . . . . . . . . . . . . . . . . . . . . . . 111
3.4 The Unique Equilibrium . . . . . . . . . . . . . . . . . . . . . . . . . 114
3.5 Comparative Statics . . . . . . . . . . . . . . . . . . . . . . . . . . 118
3.6 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 120

**Appendix 3** **121**

**Bibliography** **128**

# List of Figures


1.1 The Time Line of Events . . . . . . . . . . . . . . . . . . . . . . . . . 6
1.2 Payoffs for a given $I > 0$ . . . . . . . . . . . . . . . . . . . . . . . . 13
1.3 Liquidation Rule for a Given $I$ . . . . . . . . . . . . . . . . . . . . . . 15
1.4 Excessive Continuation and Investment $I' < I''$ . . . . . . . . . . . . 20
1.5 The Banker's Utility for $\eta < K$ . . . . . . . . . . . . . . . . . . . . 25

2.1 Margin-Interest Menu . . . . . . . . . . . . . . . . . . . . . . . . . . 53
2.2 Levered Excess Returns for Margins $m^H > m^L$ . . . . . . . . . . . . 56
2.3 Distribution of Levered Returns . . . . . . . . . . . . . . . . . . . . . 57
2.4 Expected Levered Excess Returns in $m$ . . . . . . . . . . . . . . . . . 58
2.5 Margins that Violate the Necessary Condition for Optimality (2.21) . 63
2.6 Points that Satisfy the First-Order Conditions given $\theta$ . . . . . . . . . 65
2.7 Comparative Statics in $\theta$ . . . . . . . . . . . . . . . . . . . . . . . . 68
2.8 Comparative Statics of Lenders' Belief and $R_f$ . . . . . . . . . . . . . 74
2.9 Proof of Lemma 2.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . 76
2.10 Levered Excess Returns $\lambda_k(S, m)$ for Margins $m^H > m^L$ . . . . . . . . 88
2.11 Unambiguous Comparative Statics of $x^*(\theta)$ and $q^*(\theta)$ . . . . . . . . . 91
2.12 Contradiction when $m^{*\prime}(\theta) > 0$ . . . . . . . . . . . . . . . . . . . . 96

3.1 Margin-Interest Menu . . . . . . . . . . . . . . . . . . . . . . . . . . 110
3.2 $x(R_d, \theta) \searrow R_d$ and $m(R_d, \theta) \searrow R_d$ . . . . . . . . . . . . . . . . . . 114
3.3 Equilibrium Conditions . . . . . . . . . . . . . . . . . . . . . . . . . 116
3.4 Unique Equilibrium . . . . . . . . . . . . . . . . . . . . . . . . . . . 117
3.5 Comparative Statics in $\theta$ . . . . . . . . . . . . . . . . . . . . . . . . 118

3.6 Equations (3.53) and (3.54) . . . . . . . . . . . . . . . . . . . . . . . . 125
3.7 Equations (3.58) and (3.59) . . . . . . . . . . . . . . . . . . . . . . . . 127

# List of Tables

1.1 Balance Sheet of the Bank . . . . . . . . . . . . . . . . . . . . . . . . . 9

3.1 Balance Sheets . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 109

# Abstract

This dissertation provides a theoretical study of bank and household capital structure and how they shape real allocations in the economy.

The first chapter presents a theory of bank capital structure based on a governance problem between the banker, outside equity investors, and households. The banker determines both the investment level in a project and its financing. A unique mix of equity capital and short-term debt maximizes the project's surplus for the investors. However, if the rents in banking are high and the banker's internal funding is scarce, the equilibrium features lower equity financing than the social optimum, implying a higher likelihood of bank runs and under-investment in the project. A minimum equity capital requirement simultaneously reduces the risk of bank runs and increases the investment level, while the banker is unambiguously worse-off.

The second chapter analyzes a portfolio problem with non-recourse debt. A risk-averse agent finances investment in a risky asset using a loan collateralized by the asset itself. The lenders offer her a competitive menu of interest rates and margin requirements. Her choice depends on her optimism about the asset values relative to the lenders, her risk aversion, and her wealth. The chapter uncovers a complementarity between the demand for the risky asset and the leverage ratio to finance this demand. A more optimistic agent buys a greater quantity of the risky asset, and is more levered. This co-movement result provides key insights into household debt: the mortgage loan-to-value ratio and the mortgage debt-to-income ratio should be pro-cyclical.

The third chapter introduces a general equilibrium framework of credit markets

in which households and banks borrow and lend from each other. The equilibrium determines not only the interest rate and also two leverage ratios for the banks and the households. I find that a positive shock to asset values leads to a credit boom and higher household leverage, amplified by the general equilibrium effect of rising interest rates.

# Chapter 1

# Real Effects of Bank Capital Structure

## 1.1 Introduction

The level of equity capital in banking is under intense scrutiny following the substantial impact of the 2008 financial crisis on the real economy. The large U.S. financial institutions, bank holding companies and investment banks, had $3-5\%$ book equity-to-asset ratios at the beginning of 2007. To limit the damaging effects of a similar systemic bank run in the future, Dodd-Frank Act of 2010 requires that the quantity and the quality of capital held by banks conform to Basel III capital standards. For example, the minimum equity capital requirement as a percentage of risk-weighted assets is raised from 2% to 4.5%. With the additional requirements for large institutions, the minimum book equity-to-asset ratio can reach 9.5%.

Critiques of the new requirements warn that the regulation reduces the cost and the likelihood of bank runs at the expense of lowering bank lending to the real economy[1]. Calomiris and Kahn (1991) and Diamond and Rajan (2000, 2001) point out

[1]For example, Larry Summers voiced this concern in the IMF Conference, November 8, 2013 that "most of what might be done under the aegis of preventing a future crisis would be counterproductive, because it would, in one way or another, raise the cost of financial intermediation."

that in order to limit the moral hazard problem of bankers diverting cash flows away from investors, who are unable to act as effective monitors, it is optimal for the bank to be financed with short-term debt so that the banker is subject to the market discipline of runs. This strand of the literature argues that an equity capital requirement can reduce the associated bankruptcy costs but the investors demand a higher return on equity to compensate for the greater moral hazard problem, which diminishes the bank's ability to raise funds. In another strand of the literature, DeAngelo and Stulz (2014) and Gorton and Winton (2014) focus on the liquidity premium households are willing to pay to hold the bank's short-term debt. A higher equity capital requirement reduces the supply of liquid deposits. Finally, the belief that the equity is the costliest form of financing and that higher equity capital requirements increase the overall cost of funding is pervasive in the banking industry.

What capital structure for banks best serves the interests of the real economy? To address the potential trade-offs between bank lending and the riskiness of the capital structure, I introduce a model in which the two are jointly determined. In the model, an owner/manager banker with limited internal funds supplies credit to a long-term project with uncertain returns. The banker receives a signal of future returns and can liquidate the project early. He faces two agency problems in raising external funds: one with the households and another with the outside equity investors. At the equilibrium, the households hold short-term debt and bank runs occur if there is a bad expectation of cash flows. I characterize the investment level and the short-term debt/equity mix that maximize the project's surplus, and then analyze the equilibrium in which they are chosen by the utility-maximizing banker.

The paper has two main results. First, there is a unique benchmark investment level and a corresponding equity-to-investment ratio that lead to the highest surplus from the project. At this benchmark there is just enough equity in the bank to absorb the anticipated losses from liquidation so that the short-term debt is risk-free. Second, if the banker's internal funds are scarce, the equilibrium features both a riskier capital structure with too many bank runs and lower investment in the project. Therefore, a minimum equity capital requirement to force the banker to issue outside equity can reduce the likelihood of bank runs and increase investment at the same time.

Capital structure decisions are concerned with giving the banker the right incentives. The banker and the outside equity investors are better-skilled at monitoring the cash flows from the project than the households are. The banker cares about both monetary returns and non-monetary private benefits from allowing projects to continue. He may want to avoid liquidating bad projects to protect the private benefits. In this framework short-term debt has a dual disciplining effect on the banker. First, the banker truthfully reveals his information about the future cash flow because trying to divert cash flows by announcing them to be low leads to mass withdrawal resulting in the liquidation of the project and the loss of private benefits. Second, forced liquidations are in accordance with the outside equity investors' interest if the cash flow from continuing these projects is less than their liquidation value.

The uniqueness of benchmark capital structure follows from a hump-shaped relation between the project's surplus and the equity-to-investment ratio. If the banker overly relies on short-term debt to finance the investment, the households suffer a loss whenever the project is liquidated, because the liquidation recovers only a fraction of the investment. For the households to break even in expectation, the banker pays a compensating premium whenever the project is completed. This premium determines a threshold below which the banker is insolvent and runs may occur. The larger the investment and the lower the banker's equity cushion, the greater the likelihood that the banker ends up liquidating a high cash flow project at a loss. The anticipated loss from bank runs depresses the expected return on investment and lowers the willingness to invest ex ante. Likewise, over-reliance on equity capital is undesirable because the banker lowers the expected quality of completed projects by allowing some bad ones to survive for their private benefit. The anticipated drop in the marginally completed project's quality also lowers the willingness to invest in the project ex ante.

The banker treats inside and outside equity differently and his preferred capital structure follows a pecking order: internal funds are preferred to any other external funding and short-term debt is preferred to outside equity. The banker and the outside equity investors share the project's surplus so they both capture rents at the equilibrium. This feature makes outside equity privately costly for the banker because the households earn zero rent on their short-term debt. When the internal funds are scarce, the banker compares the marginal increase in the project's surplus to the marginal decrease in the claim on the new surplus. I show that if the rent is large enough, the banker is better-off leveraging the internal funds by short-term debt and capturing the entire surplus as the only residual claimant. This insight remains valid even as the outside equity investors become perfectly competitive and earn zero rent because the banker's rent includes private benefits that are reduced when the controlling interest is diluted with outside equity.

There are no gains from an equity capital requirement if the banker's internal funds are abundant. In this case there is no need for outside equity to build the necessary buffer against liquidation losses. The banker issues risk-free, short-term debt and only the low cash flow projects are liquidated at the equilibrium. If the private benefits are small, the unregulated equilibrium obtains the benchmark outcomes. Requiring an increase in the equity-to-investment ratio leads to excessive continuation of bad projects, which depresses the investment level.

In my model the motivation for the banker to choose low equity levels come from the loss of rents on issuing new equity. Much of the recent literature focuses instead on the role of regulatory frictions leading to excessively risky capital structure. Admati and Hellwig (2013) argue that the banks perceive equity as costly because bailouts, deposit insurance, and tax shields subsidize short-term debt. Van den Heuvel (2008); Begenau (2014); Harris, Opp and Opp (2014); and Nguyen (2014) analyze the optimal bank capital ratios that mitigate the excessive risk-taking incentives created by mispriced government guarantees. In contrast, I identify a role for capital requirements even if no regulatory friction exists. An insurance/bailout scheme can be introduced in my model as a policy alternative but it cannot improve on a correctly set minimum equity capital requirement, because the policymaker does not observe the banker's cash flow signal and has to design an incentive-compatible mechanism[2] since the households do not run and discipline the banker once their claim is insured.

The paper is organized as follows: Section 1.2 introduces the model, Section 1.3 studies the liquidation rule induced by the capital structure and investment, Section 1.4 analyzes the effect of capital structure on optimal investment level and Section 1.5 presents the unregulated equilibrium outcome. All proofs are relegated to the Appendix.

## 1.2 The Model

This section lays out the model economy. After the main assumptions, I determine the benchmark outcomes. Then, I introduce informational and contractual structure of the model. Last, I provide an intermediate result on short-term debt and bank runs.

The model has three types of agents: a banker, a continuum of outside investors, and another continuum of households. There are three periods $t \in \{0, 1, 2\}$. All agents are risk-neutral, protected by limited liability, and do not discount future gains for simplicity. Opportunity cost of supplying a unit of capital is the same for all agents. The analysis is concerned with a representative banking relationship between the three agents. The regulator's role is suppressed in the baseline model and all taxes on capital are assumed to be zero.

The real economy in this model is passive. The banker has access to a project with investment outlay $I$ subject to diminishing returns and a costly real option to

[2] The bailout literature suggests additional constraints that the policymaker might face, e.g. the time-inconsistent preferences as in Chari and Kehoe (2013), so the second-best surplus of the policymaker's mechanism might be strictly lower.

liquidate early. The level of investment is chosen at $t = 0$ and cannot be altered at later dates. The project yields cash flow $z\phi(I)$ at $t = 2$ for its investors. Here $\phi(I)$ denotes the baseline production function and $z$ is the realization of continuous random variable $Z$. At $t = 1$ the banker observes a non-verifiable signal of $z$ and this signal perfectly reveals the cash flow at $t = 2$. At this point the banker can decide to liquidate the project. Early liquidation at $t = 1$ recovers a fraction $\alpha < 1$ of the initial investment $I$. There are no profitable reinvestment opportunities if the project's capital is sold at $t = 1$. Figure 1.1 illustrates the time line of events.

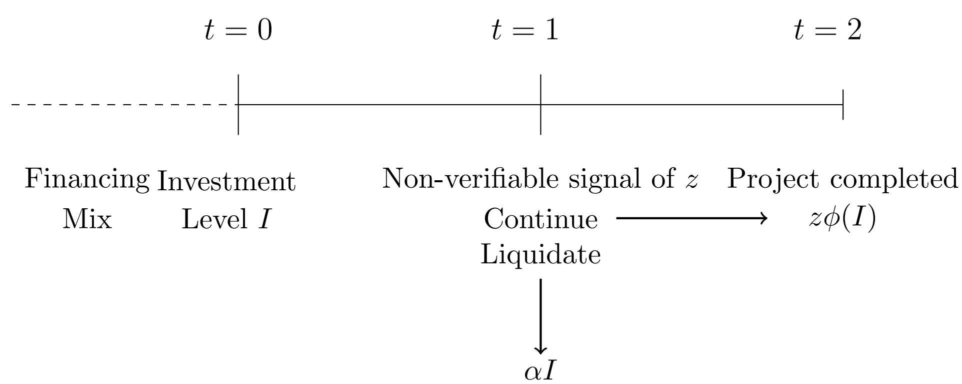



Figure 1.1: The Time Line of Events

I make a standard assumption regarding the production function:

**Assumption 1.1** *$\phi : R_+ \to R_+$ is strictly concave, strictly increasing and $\phi(0) = 0$. $\phi'$ is convex and satisfies:*

$$\lim_{I \to 0} \phi'(I) = \infty, \lim_{I \to \infty} \phi'(I) = 0.$$

$Z$ has a continuous and differentiable probability density function $f$ on $[0, \infty)$ and $F$ denotes the distribution function. $Z$ satisfies the following condition:

**Assumption 1.2** *$Z$ is a log-concave random variable i.e.* $\ln f(z)$ *is concave.*

The log-concave family includes commonly used distributions such as Normal, Exponential, Uniform, Logistic and certain classes of Gamma and Beta. Bagnoli and Bergstorm (2005) provides a comprehensive list of distributions that satisfy various log-concavity properties.

### 1.2.1 The Benchmark

As a benchmark, I show there is a unique investment level and a liquidation rule that maximize the project's surplus. For any given $I > 0$, it is optimal to liquidate the project at $t = 1$ if the signal $z < z^*(I)$ where $z^*$ is determined by

$$z^*\phi(I) = \alpha I \tag{1.1}$$

The function $z^*$ gives a liquidation rule which is increasing in $I$ by Assumption 1.1. Taking this liquidation rule as given, denote $I^*$ as a solution to:

$$\max_I \int_{z^*}^{\infty} z\phi(I)dF(z) + F(z^*)\alpha I - I \tag{1.2}$$

satisfying the first-order condition:

$$\int_{z^*}^{\infty} z\phi'(I)dF(z) + \alpha F(z^*) - 1 = 0 \tag{1.3}$$

**Proposition 1.1.** *There exists a unique benchmark investment level* $I^* \in (0, \infty)$.

Absent Assumption 1.2, (1.2) is not a standard problem. For an incremental rise in investment level, any completed project has lower marginal return by concavity of $\phi(I)$. On the other hand, the expected payoff to completed projects rises because better projects are completed. Log-concavity of the distribution resolves this ambiguity always in favor of diminishing marginal returns. The proof of Proposition 1.1 establishes that (1.2) is quasi-concave in $I$ under Assumption 1.1 and 1.2. In fact it is sufficient for $Z$ to have increasing hazard rates, $h(z) = f(z)/1 - F(z)$ to be an increasing function for all $z \in Z$ which is weaker than Assumption 1.2, for this result to hold.

### 1.2.2 The Governance Problem

I formalize the conflicts of interests between the banker, outside investors, and households by introducing an information asymmetry and private benefits. The banker can finance up to a fraction $\eta < 1$ of the investment $I$ with internal funding and the remainder fraction $(1 - \eta)$ must be financed by issuing claims (external funding). Here the banker is an owner/manager; I assume his incentives are perfectly aligned with bank insiders who supply the inside equity for the banker's use by a managerial compensation contract that is left out of the analysis.

Households represent the general public, who cannot monitor the banker's cash flow. I assume that the banker has superior information about the cash flow at both $t = 1, 2$. The banker can divert cash flows away from the households at $t = 2$ when they accrue. Therefore, any incentive-compatible contract would require the banker to reveal his signal at $t = 1$ and commit to a repayment based on this announcement. Cash flow announcements are publicly verifiable and the banker is held accountable if he reneges on his obligations based on what he reported[3].

Outside investors represent any other financial institution interested in buying long-term, non-controlling claims on cash flows. The simplest of such claims is equity[4]. They observe and enforce their claim on cash flows without friction. Outside

[3] One might argue why the regulator does not monitor the banker on behalf of the households. Although it is true that the banks disclose more substantive information to the regulators, nothing can be disclosed to the public unless the banker is charged with fraud. Not all forms of diverting cash flows are fraudulent e.g. reinvesting cash flows through a subsidiary outside the regulator's control. I abstract away from legal considerations of what is a fraud by assuming that the discount rate after $t = 2$ is infinity so that suing the bank to collect cash flows is not a worthwhile option for the households. Implementation of the contract requires a minimal regulatory role. Since the households cannot learn how much the banker actually has, earnings can be overstated at $t = 1$ and the banker claims insolvency later at $t = 2$. All that is necessary is to freeze the asset and impose a penalty to offset the private benefit.

[4] I do not consider additional compensation schemes because the outside investor, by assumption,

investors are passive shareholders in the sense that they value the monetary returns to their equity taking the banker's decisions as given. As they are more skilled investors than the households, I assume that they are in short supply in the economy and capture some of the project's surplus as rent, whereas the uninformed households are perfectly competitive and earn zero rent.

Table 1.1 illustrates a typical balance sheet. Let $\kappa$ denote the fraction of the investment financed by equity and a fraction $\eta/\kappa$ of this equity is held internally by the banker. In the baseline model, the inside and outside equity earn the same payoff per share. That is, the outside equity investors own a $(1-\eta/\kappa)$ fraction of the bank's equity and they claim $(1-\eta/\kappa)$ of the cash flows. Section 1.5.1 analyzes the limiting case in which the banker can sell outside equity without leaving any rent to the outside investors.

Table 1.1: Balance Sheet of the Bank

| Assets | Liabilities | |
|---|---|---|
| | $\eta I$ | Inside Equity |
| $I$ | $(\kappa-\eta)I$ | Outside Equity |
| | $(1-\kappa)I$ | Household Claims |

A conflict of interest between the banker and the outside investors exists because they have different preferences. The banker cares about the monetary returns to his inside equity and derives utility from control rights in the bank. To capture this idea, the banker receives a non-monetary private benefit proportional to his claim on cash flows. The smaller the banker's ownership, the larger the outside investors' influence in decision-making after $t = 2$. I take it as an axiom that the banker dislikes outside interference. For example, the outside investors may impose a different long-run agenda than the banker wants, alter his compensation contract, or replace him altogether. Private benefit is lost if the project is liquidated and the banker is fired. Therefore, the banker has an incentive to complete projects with cash flows lower

is not interested in controlling the bank at $t = 0$.

than the liquidation value. I formalize these ideas as follows[5]:

$$\text{Private Benefit} = \begin{cases} \frac{\eta}{\kappa} \times B > 0 & \text{if the project survives} \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

This is well-defined for $0 < \eta \leq \kappa$ and set the private benefit equal to $B$ if $\eta = \kappa = 0$. Section 1.5.2 analyzes the case without private benefits which includes $\eta = 0 < \kappa$.

### 1.2.3 Demandable Debt and Bank Runs

I conclude the model section by proving that the only incentive-compatible claim the banker can issue to raise funds from uninformed households is demandable debt and that bank runs can occur at $t = 1$.

The problem between the banker and the households is mechanically similar to Townsend (1979) and Gale and Hellwig (1985) and conceptually similar to Calomiris and Kahn (1991) and Diamond and Rajan (2001). Consider a contract that requires the banker to announce the future cash flow at $t = 1$ and gives the right to withdraw at $t = 1$. The banker needs to liquidate the project early to meet the demand for withdrawals. The project cannot be partially liquidated and so let $\mathbb{1}_{\mathcal{B}}(y)$ be a function taking value 1 when the announced state $y$ is in the set $\mathcal{B}$ in which some households withdraw funds. A bank run is defined as an event in which households exercise the withdraw option en masse. The set $\mathcal{B}$ is referred to as *bank run* states.

The report $y$ at $t = 1$ is contractible so let $R(y)$ denote the repayment to the households if a non-bank run state $y \notin \mathcal{B}$ is announced. The equity of any kind is junior to the claim issued to the households. For a given $(I; \kappa)$, the households' claim is a triplet $\{(1-\kappa)I, R(z;\kappa), \mathcal{B}(\kappa)\}$, the amount borrowed, repayment function, and a set of enforced liquidations, respectively.

[5]In the Appendix, I use a general functional form to illustrate that the insights behind the results are robust to many other specifications.The proportionality is also desirable for tractability. The banker's monetary return to inside equity in the event of completion and liquidation are $z\phi(I)$ and $\alpha I$ times his fractional claim on these cash flows, which is $\eta/\kappa$ in the baseline model. If the private benefit is also proportional to $\eta/\kappa$ then the decision is distorted by a constant $B$.

**Lemma 1.1.** *The unique incentive-compatible claim the banker can issue to the households is demandable debt. Households receive a constant repayment $\bar{R}(\kappa)$ whenever the project is completed. If the cash flow is not enough to repay $\bar{R}(\kappa)$, a bank run occurs and the project is liquidated.*

A flat repayment removes the banker's incentive to misreport his type as long as he can repay $\bar{R}(\kappa)$. For the banker to reveal the truth when $z \notin \mathcal{B}(\kappa)$, first he must expect to pay the same amount had he announced a different $y \notin \mathcal{B}(\kappa)$. Second, the repayment has to be feasible with the project's cash flow $z\phi(I) \geq R(z;\kappa)$. These two assertions lead to a constant repayment on the complement set of $\mathcal{B}(\kappa)$ – non-bank run states.

Demandable claims attract uninformed capital by giving an option to threaten the banker with liquidation. For simplicity of illustration, suppose that $\eta = \kappa = 0$ and the banker is the only residual claimant without private benefits. The banker is always tempted to offer the liquidation value $\alpha I$. If the households commit to withdrawing whenever $\alpha I$ is announced, it would force the banker's hand to liquidate. Liquidation makes him worse off even if the project he had were $z^*(I)$ because he would also lose the private benefit. Pursuing this logic, the threshold below which households withdraw can be pushed up to a high enough level $\bar{R}(\kappa)$ to make it individually rational to buy the claim.

Determining $\mathcal{B}$ at $t = 0$ might appear as if assuming mass pre-commitment to withdrawal decisions at $t = 1$. Following Diamond and Rajan (2001), this issue can be easily resolved by augmenting the claim with a first-come first-served clause on $\mathcal{B}(\kappa)$ to encourage everyone to be among the first $\alpha < 1$ to withdraw and redeem $I$ if the banker announces that he cannot meet the promised repayment $\bar{R}(\kappa)$. When strictly less than $\alpha$ fraction withdraws, they disproportionately reduce the liquidation value $\alpha I$ to be shared among those who wait. When more than $\alpha$ fraction withdraws simultaneously, each take a random place in the queue to share the liquidation value among fewer people and those who wait receive nothing. It is easy to see that withdrawing is the dominant strategy in this game. It is to nobody's benefit to withdraw if the banker announces that he can repay $\bar{R}(\kappa)$ so I assume the first-come first-serve clause can be restricted to the set $\mathcal{B}(\kappa)$.

## 1.3 Equilibrium Liquidation Rule

Taking the investment $I$ and the capital structure decision $\kappa$ as given, I solve for the liquidation rule $\ell(I;\kappa)$. This rule determines the marginally completed project which is comparable to the benchmark $z^*(I)$ in (1.1). Figure 1.3 graphically summarizes the key result of this section. The higher the leverage, the more liquidation takes place at $t = 1$.

Incentive-compatibility constrains the banker to liquidate at least the projects in the set $\mathcal{B}(\kappa)$. Yet the statement of Lemma 1.1 leaves it open that not every project that can repay the debt is necessarily completed. When the project is liquidated, the households have seniority in claiming the liquidation value $\alpha I$. So the critical $\kappa^*$ to consider is

$$\alpha I \equiv (1 - \kappa^*) I \tag{1.5}$$

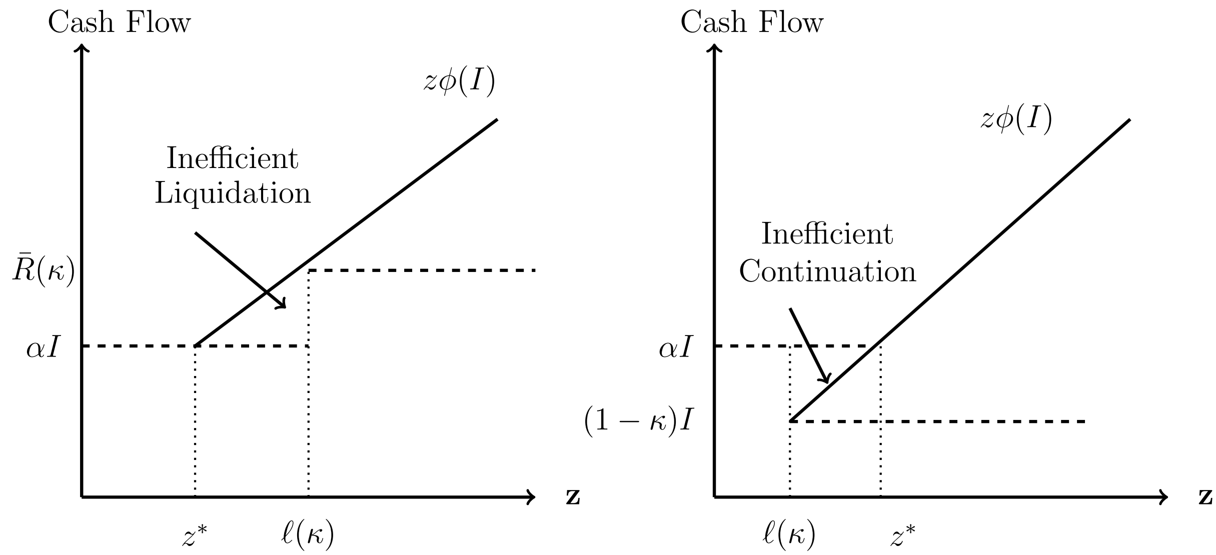
or simply $1 - \alpha$. If $\kappa > 1 - \alpha$, then the bank has so much equity that it can repay the households even in liquidation states. If $\kappa < 1 - \alpha$, then the converse is true; the bank has too little equity to spare the households from liquidation loss. Within the model's context, the cases $\kappa \lesseqgtr 1 - \alpha$ are referred to as under- and over- capitalized bank, respectively.

### 1.3.1 Excessive Liquidation When $\kappa < 1 - \alpha$

Fix $I > 0$ and let $\ell(\kappa)$ denote the marginal project that can repay $\bar{R}(\kappa)$

$$\bar{R}(\kappa) \equiv \ell(\kappa)\phi(I) \tag{1.6}$$

The banker strictly prefers the marginal project to be completed because even though the payoff to equity is zero, he gets a positive private benefit. When the




(a) Under-capitalized $\kappa < 1 - \alpha$ (b) Over-capitalized $\kappa > 1 - \alpha$

Figure 1.2: Payoffs for a given $I > 0$

banker completes any project that is not forced into liquidation by a bank run, the liquidation rule is implicitly given by the households' participation constraint:

$$F(\ell(\kappa))\ \alpha I + (1 - F(\ell(\kappa)))\bar{R}(\kappa) \geq (1 - \kappa)I \tag{1.7}$$

with equality for each $I$ in a competitive capital market.

The left panel of Figure 1.2 illustrates the payoffs for $\kappa < 1 - \alpha$. The areas under the two dashed lines corresponds to the first and second terms in (1.7). For lending to the bank to be individually rational, households should be paid a premium in the completion states to compensate the loss $(\alpha - (1 - \kappa))I < 0$ they make in liquidation states. The banker pays the premium only if failing to do so is more costly. By mere observation $\ell(\kappa) > z^*$ is necessary to satisfy (1.7). Hence, some of the projects with $z\phi(I) > z^*\phi(I) = \alpha I$ must be liquidated at $t = 1$. The intuition is that the bank runs burn surplus to force the banker to pay an interest rate that he would otherwise never pay.

The marked triangle in Figure 1.2a is the deadweight loss from bank runs. Not all liquidations are wasteful; bank runs liquidate all projects $z < z^*(I)$ that would be liquidated in the benchmark. Excessive liquidations $z^* < z < \ell(\kappa)$ can be viewed as an externality. Had the banker financed the same investment $I$ with more equity, the project had more cash flow available to the households in the liquidation states. Thus, the repayment $\bar{R}(\kappa)$ for households to break even could be reduced. Less likely the banker is insolvent and hit by a run, more projects are completed and the deadweight loss triangle in Figure 1.2a gets smaller.

### 1.3.2 Excessive Continuation When $\kappa > 1 - \alpha$

With enough equity, the banker can guarantee households a fixed payoff for all states of the world without jeopardizing incentive-compatibility. Despite this enticing outcome, the liquidation rule is given by:

$$\ell(\kappa)\phi(I) \equiv \max\Big((1-\kappa)I, \alpha I - B\Big) \tag{1.8}$$

and it is always lower than the benchmark $z^*$.

Figure 1.2b illustrates why there is a deviation from the benchmark. Consider the project $z\phi(I) = (1-\kappa)I < \alpha I = z^*\phi(I)$. Outside investors are better served if this project is liquidated because their payoff would be $(1-\eta/\kappa)(\alpha-(1-\kappa))I$, the share in profits after the debt is repaid, positive. Yet the banker might have a large private benefit $(\eta/\kappa)B > (\eta/\kappa)(\alpha-(1-\kappa))I$ such that completing the project is appealing even though it makes nothing for the equity. The earlier discussion insinuates a benign view of bank runs in terms of forcing bad projects into liquidation. Given that the private benefit distorts the banker's incentive to be lenient towards bad projects, the bank runs prevent excessive continuation.

Proposition 1.2 generalizes the observations made so far and gives a full characterization of the equilibrium liquidation function.

**Proposition 1.2.** *The liquidation rule $\ell(I;\kappa)$ is:*

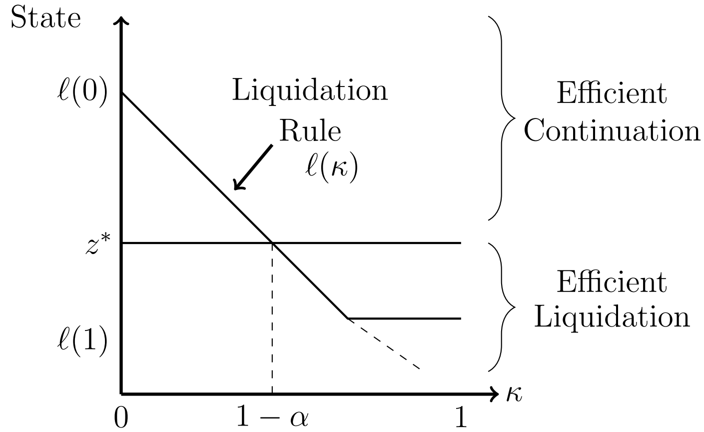



Figure 1.3: Liquidation Rule for a Given $I$

1. *non-increasing in* $\kappa$ *for each* $I$*:* $\ell_\kappa \leq 0$

2. *singly crossing the benchmark at* $1-\alpha$ *for each* $I$*:* $\ell(I; 1-\alpha) \equiv z^*(I)$

3. *increasing in* $I$ *for each* $\kappa$*:* $\ell_I > 0$

4. *submodular in* $(I; \kappa)$*:* $\ell_{I\kappa} \leq 0$

The first two items are visualized in Figure 1.3. Dewatripont and Tirole (1994) obtain a similar result whereby the debt/equity mix can lead to excessive liquidation or continuation. The third item is analogous to $z_I^* > 0$, better projects survive for a higher level of investment. For the fourth item, take $\kappa^L < \kappa^H < 1-\alpha$ and compute the range of projects that be spared from excessive liquidation: $\ell(I; \kappa^L) - \ell(I; \kappa^H)$. Submodularity implies that the difference is rising in $I$. As the size of the bank grows, the lack of equity cushion leads to more excessive liquidations. A parallel statement can be made for the benefit of issuing demandable debt when $\kappa$ is too high.

## 1.4 Investment in the Project

In this section I analyze the investment level and the surplus from the project for any given capital structure the banker may choose. I show that whenever $\ell(\kappa)$ deviates from the benchmark $z^*$, both the investment and the surplus from the project are distorted compared to the benchmark.

Given the liquidation rule $\ell(I;\kappa)$, the surplus from the project captured by the equity-holders is:

$$\begin{aligned} S(I;\kappa) &\equiv \int_{\ell(\kappa)}^{\infty} (z\phi(I) - \bar{R}(\kappa))\, dF(z) + F(\ell(\kappa)) \max(\alpha I - (1-\kappa)I, 0) - \kappa I \\ &\equiv \int_{\ell(\kappa)}^{\infty} z\phi(I)\, dF(z) + F(\ell(\kappa))\alpha I - I \end{aligned} \tag{1.9}$$

The foremost two terms in the first line represent the cash flow after debt and the last term is the opportunity cost of equity capital. The second line substitutes the value of debt using households' participation constraint (1.7). After the substitution (1.9) is directly comparable to the benchmark (1.2).

Two features of $S(I;\kappa)$ need to be emphasized. First, the banker's private benefit is excluded. The aim of this section is to isolate how a given capital structure affects the willingness to invest in the project and the way the benchmark is defined implicitly assumes that the social purpose of banking is to maximize the surplus in the real economy[6]. Second, since the payoff to the outside equity investors is proportional to $S(I;\kappa)$ by their fractional ownership $(1-\eta/\kappa)$, they capture rents at any equilibrium while the households break even.

The surplus-maximizing investment is denoted $I^S(\kappa)$:

$$I^S(\kappa) \equiv \arg\max_I S(I;\kappa) \tag{1.10}$$

[6]The benchmark is not defined as a problem of a benevolent social planner who cares about the banker's private benefit. It is possible to take a firmer stance that the banker's private benefit is an uncompensated transfer from the agents in the unmodeled broader economy and not a positive spillover the policymaker/regulator should internalize.

The surplus $S(I;\kappa)$ coincides with the benchmark only if the liquidation rule $\ell(\kappa)$ corresponds to $z^*$ which occurs at $\kappa = 1-\alpha$ in Figure 1.3. For any $\kappa \neq 1-\alpha$, there will be an additional term that shifts willingness-to-invest. I study under- and over-capitalized cases separately.

### 1.4.1 Under-capitalized Case $\kappa < 1-\alpha$

Decompose $S$ such that $I^S(\kappa)$ solves:

$$\max_I \int_{z^*}^{\infty} z\phi(I)\,dF(z) + F(z^*)\alpha I - I - \int_{z^*}^{\ell(\kappa)} (z\phi(I) - \alpha I)\,dF(z) \tag{1.11}$$

The last term is the deadweight loss triangle in Figure 1.2a and $I^*$ maximizes the rest of (1.11). $I^S(\kappa)$ deviates from $I^*$ if the marginal deadweight loss

$$\ell_I(\kappa)\Big(\ell(\kappa)\phi(I) - \alpha I\Big) f(\ell(\kappa)) + \int_{z^*}^{\ell(\kappa)} (z\phi'(I) - \alpha) dF(z) \tag{1.12}$$

is non-zero evaluated at $I^*$.

The first term of (1.12) is unambiguously positive for all $I$. When investment rises from $I'$ to $I''$, the type $\ell(I',\kappa) < \ell(I'',\kappa)$ is inefficiently liquidated at $t = 1$. This is a disincentive to invest. The sign of the second term is hard to predict. The ambiguity stems from diminishing marginal returns, because while payoff levels are the same $z^*\phi(I) = \alpha I$, the marginals favor liquidation $z^*\phi'(I) < \alpha$. This ambiguity makes it uncertain whether switching from completing the project $z$ to liquidation, $z\phi'(I) - \alpha$, would yield to a total marginal gain or loss. It is possible, in principle, that the banker is content to swap low marginal return projects with $\alpha$ even though his payoff level is going down. Despite this interesting twist, disincentive to invest dominates the second effect for all $I$ regardless of its sign.

**Proposition 1.3.** *An under-capitalized bank under-invests in the project: $I^S(\kappa) < I^*$.*

Proposition 1.3 proves that the deadweight loss triangle in Figure 1.2a grows with the investment level. The bigger the size of an under-capitalized bank, the more wasteful bank runs become. To see why under-investment occurs in a simple example, consider the following:

**Example 1.1.** *$Z \sim UNIF(0, p)$ with $f(z) = 1/p$ is log-concave and $F(z) = z/p$ is linear.*

I claim that the first positive term in (1.12), the disincentive to invest, is greater than the largest negative value of the integral term. To show that, replace the integral in (1.12) with its lower bound when the integrand is evaluated at $z^*$ for all $z$ on its support. Since $z^*\phi(I) = \alpha I$ by definition and $z_I^*\phi(I) + z^*\phi'(I) = \alpha$ by chain rule, the marginal deadweight loss is bounded below by:

$$\begin{aligned} &\phi(I)\Big[\ell_I(\kappa)(\ell(\kappa) - z^*)f(\ell(\kappa)) - z_I^*\Big(F(\ell(\kappa)) - F(z^*)\Big)\Big] \\ &= \phi(I)\Big(\frac{\ell(\kappa) - z^*}{p}\Big)(\ell_I(\kappa) - z_I^*) > 0 \end{aligned} \tag{1.13}$$

The sign is positive because for $\kappa < 1 - \alpha$, Proposition 1.2 proves $\ell(\kappa) > z^*$ and $\ell_{I\kappa} < 0$ implies $\ell_I(\kappa) > \ell_I(1 - \alpha) = z_I^*$.

Underinvestment is a typical outcome in asymmetric information models in which the principal forgoes some surplus to make it incentive-compatible for the agent. Here the forgone surplus is the high cash flow projects liquidated at a discount. An under-capitalized bank invests less than the benchmark because the optimal investment has to trade off maximizing the project's surplus with the growing cost of excessive liquidations. Proposition 1.3 does not imply a stronger prediction that investment always rises in $\kappa$. However, since $I^S(\kappa)$ is continuous in $\kappa$ and bounded above by $I^*$ which is obtained as the limit $I^S(\kappa) \to I^*$ as $\kappa \to 1 - \alpha$, the comparative statics must hold locally. A strong comparative static can be made for the surplus from the project.

**Corollary 1.1.** *Optimal investment and the resulting surplus from the project are* *positively correlated to the bank's equity-to-investment ratio.*

1. *There exists a* $\delta > 0$ *such that*

$$\forall \kappa \in [1-\alpha-\delta, 1-\alpha] : I^S(\kappa) \nearrow \kappa$$

2. *For all* $\kappa < 1-\alpha$

$$S(I^S(\kappa);\kappa) \nearrow \kappa$$

### 1.4.2 Over-capitalized Case $\kappa > 1-\alpha$

Throughout this subsection, I make the following assumption:

**Assumption 1.3** *The private benefit satisfies:* $E[z\phi'(B/\alpha)] > 1$

This assumption rules out an unedifying situation in which the private benefit is so large that the banker completes every project and liquidation never takes place at the optimum.

Decompose $S$ such that $I^S(\kappa)$ solves:

$$\max_I \int_{z^*}^{\infty} z\phi(I)\,dF(z) + F(z^*)\alpha I - I - \int_{\ell(\kappa)}^{z^*} (\alpha I - z\phi(I))\,dF(z) \tag{1.14}$$

In this case the last term is the cost of excessive continuation to the outside investors, which is the triangle in Figure 1.2b. To see how the size of the triangle changes with investment, differentiate it in $I$ to get:

$$-\ell_I(\kappa)\Big(\alpha I - \ell(\kappa)\phi(I)\Big)f(\ell(\kappa)) + \int_{\ell(\kappa)}^{z^*} (\alpha - z\phi'(I))\,dF(z) \tag{1.15}$$

Excessive continuation affects the willingness-to-invest through two channels but contrary to the under-capitalized analogue (1.12), they unambiguously go in opposite directions. The first term of (1.15) captures the level effect that encourages more investment. For an increase from $I'$ to $I''$, the type $\ell(I', \kappa)$ which is inefficiently completed at $t = 1$ is instead liquidated with the new and higher threshold $\ell(I'', \kappa)$. When the banker is reducing the quality of completed projects, the first response is to invest more to improve the quality of the marginally completed project. The second term of (1.15) captures the disincentive to invest. For an incremental rise in the investment level, the outside investors forgo $\alpha - z\phi'(I) > 0$ on the margin for each $z \in [\ell(\kappa), z^*]$. Figure 1.4 illustrates the two channels for two distributions.

Which effect dominates at the optimum? Unlike the under-capitalized case where the deadweight loss always grows with investment, here the answer depends on both the distribution of $Z$ and the level of $I$. I prove in the next proposition that the cost of excessive continuation is increasing [decreasing] in $I$ for small [large] $I$ .

**Proposition 1.4.** *For each $\kappa > 1 - \alpha$ there exists a threshold $T(\kappa)$ such that*

1. *If $T(\kappa) > I^*$, there is under-investment: $I^S(\kappa) < I^*$*
2. *If $T(\kappa) < I^*$, there is over-investment: $I^S(\kappa) > I^*$*

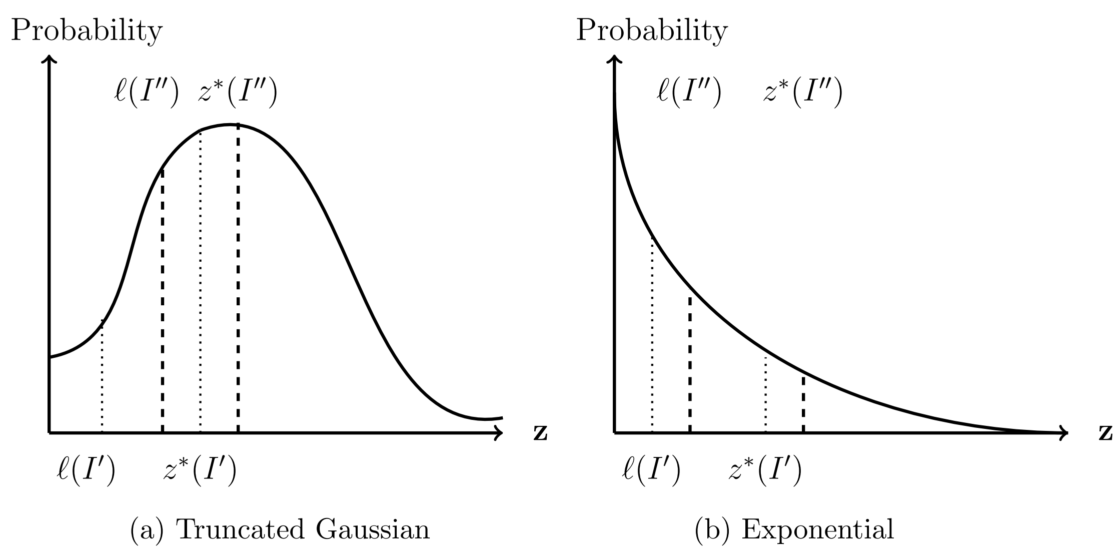



Figure 1.4: Excessive Continuation and Investment $I' < I''$

It is possible to make accurate predictions for specific choices of $Z$. Recall $Z \sim UNIF(0, p)$ in Example 1.1. Following exactly the same steps described before leads to an increasing cost of excessive continuation. Therefore $T(\kappa) \to \infty$ and there is under-investment for any $\kappa$. Consider a second example:

**Example 1.2.** *$Z \sim EXP(p)$ with $f(z) = pe^{-pz}$ decreasing and log-concave, $F(z) = 1 - e^{-pz}$ increasing and concave.*

I claim that as $\kappa \to 1$ so that the bank is almost exclusively financed by equity, there is over-investment. Weigh the largest disincentive to invest to the positive level effect by replacing the integral in (1.15) by the integrand evaluated at $\ell(1)$ for all $z$ on its support. The upper bound on (1.15) is

$$\ell_I(1)\phi(I)\Big[\Big(F(z^*) - F(\ell(1))\Big) - (z^* - \ell(1))f(\ell(1))\Big] \leq 0 \tag{1.16}$$

Since $F$ is concave, the value at a higher point is bounded above by its first-order Taylor approximation at a lower point, $F(z^*) \leq F(\ell(1)) + (z^* - \ell(1))f(\ell(1))$, which proves the cost of excessive continuation is shrinking in $I$. This example corresponds to $T(\kappa) = B/\alpha < I^*$ by Assumption 1.3 for large enough $\kappa$.

It is not possible to accurately predict under- and over-investment when the probability density of $Z$ is non-monotone, such as Truncated Gaussian in Figure 1.4a, and $\kappa$ is relatively low so that the optimum occurs on the falling segment of Figure 1.3. The only unambiguous comparative static in $\kappa$ is the following:

**Corollary 1.2.** *The surplus from the project $S(I^S(\kappa); \kappa)$ decreases in $\kappa$.*

The intuition behind Corollary 1.2 is that when the demandable debt is risk-free, it plays a desirable disciplining role on the banker to cap excessive continuation. Therefore, limiting debt issuance does not increase the surplus in any way because every project liquidated due to insolvency is a low cash flow project that is also liquidated at the benchmark.

## 1.5 The Banker's Optimal Capital Structure

In this last section I characterize the investment level and the capital structure of an unregulated equilibrium taking the limit on the banker's internal funding $\eta$ as given. I show that if $\eta < 1-\alpha$, the banker's internal funding is less than necessary to maximize the surplus from the project, the banker does not issue enough outside equity to close the gap. Therefore, both the investment and the surplus from the project increases if the policymaker requires an equity-to-investment ratio of $1-\alpha$.

The banker's utility for a given $(I;\eta,\kappa)$ is

$$U(I;\eta,\kappa) \equiv \frac{\eta}{\kappa}\Big(S(I;\kappa) + (1 - F(\ell(\kappa))B\Big) = \frac{\eta}{\kappa}V(I;\kappa) \tag{1.17}$$

I have computed the surplus all equity-holders capture in (1.9) denoted by $S(I;\kappa)$. The monetary payoff to the banker's inside equity is proportional to his fractional ownership $\eta/\kappa$. The second term in (1.17) is the expected private benefits which are also proportional to the ownership by (1.4).

I refer to $V$ as the banker's rent. The banker's utility is decomposed in a way that the investment maximizes $V(I;\kappa)$ independent of $\eta$. Denote the utility-maximizing investment by $I^U(\kappa)$.

**Proposition 1.5.** *The banker invests less than the level that maximizes the surplus from the project* $\forall\kappa : I^U(\kappa) < I^S(\kappa)$

Higher investment increases the marginally completed project, $\ell_I > 0$ for every $\kappa$, making it less likely to capture the private benefit ex ante. This intuition would be true even if the private benefit had positive returns to scale in investment. In that case, the expected loss of private benefit from higher investment dominates whatever the marginal increase in private benefit might be for large $I$. I explore this case in the Appendix.

Evaluate the banker's utility (1.17) at the optimal investment $I^U(\kappa)$ so that the only choice variable is $\kappa$:

$$U(I^U(\kappa);\eta,\kappa) = \frac{\eta}{\kappa}V(I^U(\kappa);\kappa) \equiv \frac{\eta}{\kappa}\hat{V}(\kappa) \tag{1.18}$$

I assume $\hat{V}(\kappa) \geq 0$ for all $\kappa$ so that the banker's decision is non-trivial. I provide a key property of $\hat{V}(\kappa)$ to facilitate the analysis.

**Lemma 1.2.** *The banker's rent $\hat{V}(\kappa)$ is weakly increasing in $\kappa$. There exists a $K \in (1-\alpha, 1)$ with*

$$\hat{V}_\kappa(K) = 0 \tag{1.19}$$

*such that $\hat{V}(\kappa)$ is constant for all $\kappa \geq K$.*

Lemma 1.2 yields three intermediate results. First, if the banker had no limits on internal funding, $\eta = 1$, then he would never issue outside equity but he might issue some demandable debt. Let $K \in (1-\alpha, 1)$ denote the threshold above which the liquidation rule is constant in Figure 1.3. The debt does not interfere with the banker's liquidation decision beyond a threshold because even if the banker is insolvent, the cash flow from the project is so low that the banker is willing to liquidate the project despite the private benefit. Second, when the banker's inside equity is constrained $\eta \leq K$, he would use all of it before considering external funding. Third, if the banker issues outside equity when $\eta < K$, the total equity never exceeds $K$ because the rent is constant but the ownership is reduced.

Using the implications of Lemma 1.6 the banker chooses $\kappa^U \in [\eta, K]$ to maximize (1.17). Choosing $\kappa^U = \eta$ means that the banker does not issue outside equity. The choice critically depends on the size of the maximal rent. The banker's rent is *high* if $\hat{V}(K) > K$ or *low* if $\hat{V}(K) \leq K$. Note that $B$ can be chosen large enough to guarantee that the banker's rent is high.

**Proposition 1.6.** *If the banker's rent is high, $\hat{V}(K) > K$, then the optimal capital structure follows a pecking order. Inside equity is preferred to any external funding. Then demandable debt is preferred to outside equity if external financing is used.*

The proof of this Proposition is simple and instructive so it is presented here. The banker compares three options: (i) at the lower bound the banker leverages the inside equity by demandable debt; (ii) at the upper bound the banker capitalizes the bank up to $K$ to capture the maximum rent; and (iii) a potential interior solution. The utility from the options (i) and (ii) are respectively $\hat{V}(\eta)$ and $\eta\hat{V}(K)/K$. For the third option, differentiate the left-hand side of (1.17) in $\kappa$:

$$\Big(\frac{\eta}{\kappa}\hat{V}(\kappa)\Big)' = \frac{\eta}{\kappa}\Big(\hat{V}_\kappa - \frac{\hat{V}(\kappa)}{\kappa}\Big) \tag{1.20}$$

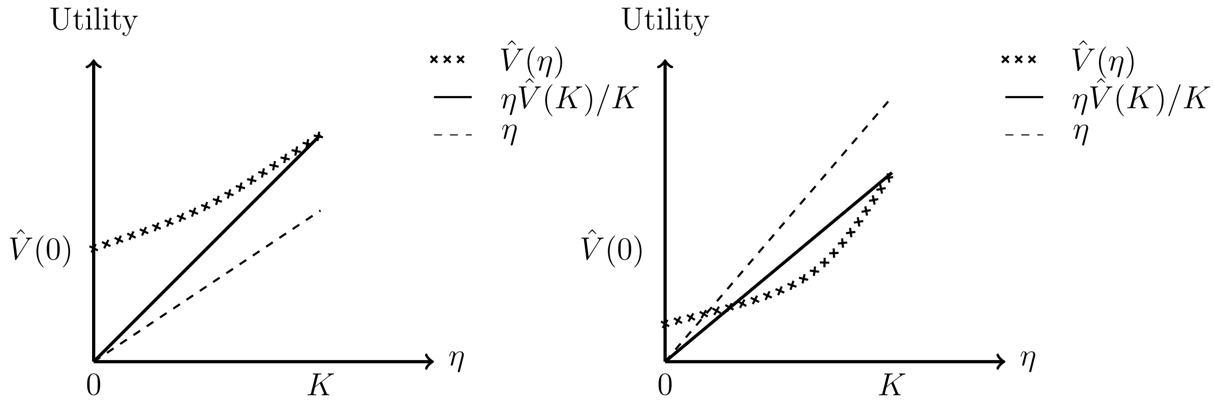
The banker's utility is eventually declining since $\hat{V}_\kappa(K) = 0$. Suppose there exists a local interior maximum $k'$ such that

$$\hat{V}_\kappa(k') = \frac{\hat{V}(k')}{k'} \iff (\ln \hat{V}(\kappa))'\Big|_{\kappa=k'} = (\ln \kappa)'\Big|_{\kappa=k'} \tag{1.21}$$

or simply $\hat{V}(k') = k'$. The banker's utility at the local maximum should be $\eta$. By construction this utility should be larger than the utility at $K$ where it is negatively sloped. However, $\frac{\eta}{K}\hat{V}(K) > \eta$ is a contradiction. Therefore, (1.20) must be negative for all $\kappa$ and the unique solution is the lower bound $\kappa^U = \eta$.

Figure 1.5 plots the banker's utility for each of the three options as a function of $\eta$. The left panel covers the high-rent case in Proposition 1.6. I have proven the inequality $\hat{V}(\eta)/\eta > \hat{V}(K)/K$ for any $\eta < K$ provided that $\hat{V}(K)/K > 1$. Therefore, the banker's options can be ranked and not issuing outside equity dominates all options. I illustrate the low-rent case in Figure 1.5b. Even if the equilibrium surplus from the project and the private benefit are both small, the banker still does not issue outside equity if $\eta$ is low enough.

There are two reasons behind the banker's preference against issuing outside equity even if it increases the investment in the project and likelihood of its survival. One is because the outside investors capture some of the new surplus and the other is because the private benefit is reduced. I isolate the role played by each in the next two subsections. A simple way to illustrate the wedge between what is socially




(a) The banker's rent is high: $\hat{V}(K) > K$ (b) The banker's rent is low: $\hat{V}(K) \leq K$

Figure 1.5: The Banker's Utility for $\eta < K$

desirable and what is privately optimal is by writing the term that determines the sign of (1.20) as

$$\epsilon(\kappa) \equiv \frac{\hat{V}_\kappa(\kappa)\kappa}{\hat{V}(\kappa)} \tag{1.22}$$

I interpret $\epsilon(\kappa)$ as the point elasticity of the banker's rent to new equity capital. The banker computes whether the increase in his rent is more than enough to compensate the loss of the claim on cash flows. If (1.20) is negative, the elasticity $\epsilon(\kappa)$ is less than unity. When $0 < \epsilon(\kappa) < 1$, outside equity is privately costly for the banker and he does not want to issue any. The policymaker wants more equity capital because he is concerned only with the surplus from the project and indifferent about how the surplus is distributed between the banker and the outside equity investors.

### 1.5.1 Perfect Competition among Outside Equity Investors

In this subsection I relax the assumption that the outside equity earn a higher return than the demandable debt. I show that the equilibrium can still exhibit under-capitalization and under-investment even though the pecking order result in Proposition 1.6 does not necessarily hold.

In the baseline model if the banker owns $\eta/\kappa$ of the bank's equity at $t = 0$, he claims $\eta/\kappa$ fraction of the cash flows after repaying the debt. This way the banker offers the same excess return per share that the inside equity earns to the outside equity investors. Suppose instead that the outside equity investors are perfectly competitive and the banker offers a new fractional claim on cash flows $\mathbb{f}$ for the outside investors to break-even:

$$\mathbb{f} \times \Big( \int_{\ell(\kappa)}^{\infty} z\phi(I)dF(z) + F(\ell(\kappa))\alpha I - (1-\kappa)I \Big) = (\kappa - \eta)I \tag{1.23}$$

The right-hand side of (1.23) is the opportunity cost and the parenthetical term is expected cash flows after debt computed earlier in (1.9). I retain the assumption that the private benefit is proportional to the banker's claim on cash flows, therefore $1-\mathbb{f}$ replaces $\eta/\kappa$ in (1.4). With the new specification the banker's utility maximization differs from (1.17):

$$\max_{I,\kappa} U(I;\eta,\kappa) \equiv S(I;\kappa) + (1-\mathbb{f})(1-F(\ell(\kappa))B \tag{1.24}$$

If all the capital markets are competitive, the banker captures the entire surplus from the project $S(I;\kappa)$ for himself. The only reason a different $(I;\kappa)$ is chosen is because of the private benefits. Although $\mathbb{f}$ is a complicated function of $(I;\kappa)$, it is possible to prove an analogue of Proposition 1.5 that the banker wants to invest less than the surplus-maximizing level $I^S(\kappa)$ for each $\kappa$. However, the optimal $\kappa^U$ is no longer tractable so I provide a sufficient condition instead.

**Lemma 1.3.** *If the banker's internal funds are scarce $\eta < 1-\alpha$, both the investment level $I^U$ and the equity-to-investment ratio $\kappa^U$ that solve (1.24) are less than the benchmark $(I^*, 1-\alpha)$ if evaluated at $I^U(\kappa)$, the outside investors' claim on cash flows is not too small:*

$$\lim_{\kappa\to 1-\alpha} \mathbb{f} > \lim_{\kappa\to 1-\alpha} \Big( -\ell_\kappa \frac{f(\ell(\kappa))}{1-F(\ell(\kappa))}(\kappa-\eta) \Big) \tag{1.25}$$

Since the banker extracts more surplus from the project by issuing outside equity, the only downside is how much ownership he has to give up that reduces the private benefits. The right-hand side of (1.25) is the threshold above which the private benefit is more valuable to the banker than the increase in the project's surplus. Therefore, even if the banker has some incentive to issue outside equity, he does not enough have incentive to capitalize up to $1 - \alpha$ provided (1.25) is satisfied.

### 1.5.2 No Private Benefits

In this subsection I analyze the baseline model without the private benefits by setting $B = 0$. The private benefits create a friction at both $t = 0$ and $t = 1$. Without the private benefits, there is no longer excessive continuation at $t = 1$ when $\kappa > 1 - \alpha$. Therefore, 100% equity capital gets the same benchmark outcome as $1 - \alpha$. Then the $\eta \geq 1 - \alpha$ case is uninteresting because the banker's preferences are perfectly aligned with the policymaker and he does not face an external funding constraint. I analyze $\eta < 1 - \alpha$ case and show that the equilibrium again features an under-capitalized bank with low level of investment.

The banker's utility (1.17) is modified to

$$U(I; \eta, \kappa) \equiv \frac{\eta}{\kappa} S(I; \kappa) \tag{1.26}$$

The banker receives a monetary payoff from the project's surplus proportional to his ownership $t = 0$ and the outside equity investors capture the rest. Since $B = 0$, Proposition 1.5 does not apply and the banker's choice of investment corresponds to $I^S(\kappa)$ modeled in Section 1.4. Evaluated at $I^S(\kappa)$, Corollary 1.1 shows that the equilibrium surplus $S(I^S(\kappa); \kappa)$ is rising in $\kappa$ until $1 - \alpha$ and constant thereafter, as argued in the opening paragraph of this subsection. Therefore, the banker chooses $\kappa \in [\eta, 1-\alpha]$ to maximize (1.26) evaluated at $I^S(\kappa)$. I get an analogue of Proposition 1.6:

**Lemma 1.4.** *If the banker's internal funds are scarce $\eta < 1-\alpha$, then both the investment level and the equity-to-investment ratio that solve (1.26) are less than the benchmark $(I^*, 1-\alpha)$.*

The derivative of (1.26) in $\kappa$ evaluated at $I^S(\kappa)$ using the Envelope Theorem is

$$\frac{\eta}{\kappa}\Big(S_\kappa(I^S(\kappa);\kappa) - \frac{S(I^S(\kappa);\kappa)}{\kappa}\Big) \tag{1.27}$$

It suffices to show that as $\kappa \to 1-\alpha$, the marginal surplus is zero at the maximum so that the optimal choice $\kappa^U$ must satisfy $\eta \leq \kappa^U < 1-\alpha$. Proposition 1.3 proves $I^S(\kappa^U) < I^*$ and completes the proof. It is possible to predict pecking order, $\kappa^U = \eta$ and no outside equity, if the maximum surplus from the project $S(I^*, 1-\alpha)$ is larger than $1-\alpha$. This is the analogue of $\hat{V}(K) > K$ in the statement of Proposition 1.6.

## 1.6 Conclusion

I have developed a model of bank capital structure and investment to evaluate the commonly held view that reducing the risk of bank runs with more equity capital trades off the volume of lending to the real economy. The main results of this paper suggest the opposite conclusion: a minimum equity capital requirement can make the bank safer and at the same time, create an incentive to invest more.

In his survey of empirical evidence, Thakor (2014) writes that "in the cross-section of banks, higher capital is associated with higher lending, higher liquidity creation, higher bank values and higher probabilities of surviving crisis". For example, Berrospide and Edge (2010) and Kapan and Minoui (2013) find that well-capitalized banks with a stronger ability to buffer losses cut lending less in response to a negative shock. My model is consistent with these findings, except for liquidity creation left outside the model, and can explain why the banks do not increase their equity capital by themselves.

How should the policymaker set the requirement? This paper offers two insights.

The requirement depends only on $\alpha$, the recovery rate of investment but not size-related parameters $\eta$ or $I$, or the distribution $F$. If the parameter $\alpha$ is interpreted as a measure of asset liquidity, then the more liquid a bank's portfolio is, the higher the leverage the bank can sustain. Recently Brunnermeier et al. (2014) develop a measure of mismatch between the asset liquidity and the funding liquidity as an alternative prudential tool. My model suggests that this is a more promising approach than the standard risk-weights. Second, the policymaker can monitor the riskiness of the bank's short-term debt instead of tracking $\alpha$. At the benchmark of my model, the debt is risk-free. Market signals such as CDS spread on the individual bank's debt might reveal how far that bank is away from the benchmark.

# Appendix 1

**Lemma 1.5.** *$\forall \bar{z} : \partial E(Z - \bar{z}|Z > \bar{z})/\partial \bar{z} = h(\bar{z})E(Z - \bar{z}|Z > \bar{z}) - 1 \leq 0$ where $h$ is the hazard function $f/(1 - F)$ of $Z$.*

*Proof.* $E(Z - \bar{z}|z > \bar{z})$ is known as the mean residual lifetime function. Bagnoli and Bergstorm (2005) Theorem 6 proves that a random variable with an increasing hazard rate has a decreasing mean residual lifetime. Assumption 1.2 guarantees that the hazard rate is increasing. □

*Proof of Proposition 1.1.* Divide the left hand side of (1.3) by $1 - F(z^*)$; the probability that the project is completed. Rearrange terms to get:

$$\phi'(I)E(Z|Z > z^*) - 1 - \frac{1}{1 - F(z^*)}(1 - \alpha)F(z^*) \tag{1.28}$$

(1.28) has the same sign as the left-hand side of (1.3) and I prove that (1.28) singly crosses 0 once from above at an interior $I^*$ which proves that (1.2) is quasi-concave $I$ with a single peak at $I^*$. (1.28) is interesting on its own. The term $\phi'(I)E(Z|Z > z^*) - 1$ is the marginal return from investing a dollar in the project at $t = 0$ conditional on the project being completed. The remainder term is the conditional marginal cost.

Note that $z^* = \alpha I/\phi(I)$ is an increasing function of $I$ by Assumption 1.1 with limits $z^* \to 0$ as $I \to 0$ and $\infty$ as $I \to \infty$. Evaluate (1.28) as $I \to 0$. Marginal return goes to infinity and the marginal cost goes to zero and thus (1.28) goes to infinity as $I \to 0$. To see the limit $I \to \infty$, use the inequality in Lemma 1.5 evaluated at $z^*$ and multiply both sides by $\phi'(I) > 0$. Using $z^*\phi'(I) < \alpha < 1$, obtain an upper bound on the marginal return

$$\phi'(I)E(z|z > z^*) - 1 < \frac{\phi'(I)}{h(z^*)} \tag{1.29}$$

where $h$ is the hazard function.

This upper bound is a decreasing function that converges to 0 as $I \to \infty$. The marginal cost goes to $\infty$ as $F$ goes to 1. So (1.28) goes to $-\infty$ as $I \to \infty$. All terms in the left-hand side of (1.28) are continuous in $I$ so (1.28) crosses 0 at least once. Let $I^*$ denote that point. To establish the uniqueness of $I^*$, I prove that both marginal return and cost are monotone.

The marginal cost $F(z^*)/(1 - F(z^*))$ is increasing in $I$: by chain rule $F$ is increasing in $z^*$ and $z^*$ is increasing in $I$. The slope of marginal return is given by

$$\phi''(I)E(Z|Z > z^*) + \phi'(I)\frac{\partial E(Z|Z > z^*)}{\partial z^*} z_I^* \tag{1.30}$$

The first term is negative and the second is positive so the sign is ambiguous. However, Lemma 1.5 implies that $0 \leq \partial E(Z|Z > z^*)/\partial z^* \leq 1$ and by chain rule $z_I^* = (\alpha - z^*\phi'(I))/\phi(I) > 0$. Using these two inequalities, (1.30) is bounded from above by

$$\begin{aligned}
&\phi''(I)E(Z|Z > z^*) - z^*\frac{\phi'(I)^2}{\phi(I)} + \alpha\frac{\phi'(I)}{\phi(I)} \\
&< z^*\left(\phi''(I) - \frac{\phi'(I)^2}{\phi(I)}\right) + \alpha\frac{\phi'(I)}{\phi(I)} \\
&= z^*\phi(I)\left(\frac{\phi'(I)}{\phi(I)}\right)' + \alpha\frac{\phi'(I)}{\phi(I)} = \alpha\left\{\left(\frac{\phi'(I)}{\phi(I)}\right)' I + \frac{\phi'(I)}{\phi(I)}\right\}
\end{aligned} \tag{1.31}$$

By Assumption 1.1 $\phi'(I)$ and $1/\phi(I)$ are decreasing convex functions so their product is also decreasing convex. Then $\phi'(I)/\phi(I)$ satisfies

$$\left|\left(\frac{\phi'(I)}{\phi(I)}\right)'\right| \geq \frac{1}{I}\left(\frac{\phi'(I)}{\phi(I)}\right)$$

That is, the slope is larger than the average in absolute value. Using this inequality the upper bound (1.31) is non-positive and therefore, the conditional marginal return is decreasing. This suffices to conclude that (1.28) single crosses 0 from above at $I^*$.

Since their signs are the same, (1.3) is uniquely satisfied at $I^*$ as well. $\square$

*Proof of Lemma 1.1.* Fix any $I > 0$, the dependence of functions on $I$ is suppressed throughout. The proof is similar to Townsend (1979) and Gale and Hellwig (1985). Incentive-compatibility can be stated in its most general form as for any $z$

$$\begin{aligned} z = \frac{\eta}{\kappa} \times \arg\max_y \Big[ & \mathbb{1}_{\mathcal{B}(\kappa)}(y) \max\Big((1-\alpha-\kappa)I, 0\Big) \\ & + (1 - \mathbb{1}_{\mathcal{B}(\kappa)}(y)) \max\Big(z\phi(I) - R(y;\kappa) + B, (1-\alpha-\kappa)I, 0\Big)\Big] \end{aligned} \tag{1.32}$$

The first term captures the liquidation payoff to the banker and the second term says the banker receives the maximum of completion and liquidation payoffs if there is no run.

Suppose that $\kappa < 1 - \alpha$ so the equity gets nothing in liquidation. Since $B > 0$ and $z\phi(I) \geq R(z;\kappa)$, (1.32) can be simplified to

$$z = \frac{\eta}{\kappa} \times \arg\max_y (1 - \mathbb{1}_{\mathcal{B}(\kappa)}(y))(z\phi(I) - R(y;\kappa) + B)$$

For any $z, z' \notin \mathcal{B}(\kappa)$, it must be that $R(z;\kappa) = R(z';\kappa) = \bar{R}(\kappa) \leq \inf_{y \notin \mathcal{B}(\kappa)} y\phi(I)$. There is no incentive for $z \notin \mathcal{B}(\kappa)$ to pretend to be $y \in \mathcal{B}(\kappa)$ or for a $z \in \mathcal{B}(\kappa)$ to report another $y \in \mathcal{B}(\kappa)$. However, $z \in \mathcal{B}(\kappa)$ may pretend to be $y \notin \mathcal{B}(\kappa)$. As Footnote 3 explains, there is a non-pecuniary penalty at least as large as $B$ in case the banker reports $y$ which he later fails to repay $R(y;\kappa)$ at $t = 2$. So for this deviation to be unprofitable, it has to be that $\bar{R}_\kappa \geq \sup_{y \in \mathcal{B}(\kappa)} y\phi(I)$. Putting all together, $\bar{R}(\kappa) \equiv \ell(\kappa)\phi(I)$ such that $\mathcal{B}(\kappa) = [0, \ell(\kappa)]$.

Consider now $\alpha I > (1-\kappa)I$ or $\kappa > 1 - \alpha$. Here there are two sub-cases. First suppose that $\alpha I > (1-\kappa)I > \alpha I - B$. The private benefit is so large that whenever $z \notin \mathcal{B}(\kappa)$, the banker is better off completing the project. In this sub-case the proof is identical to the one above; the only difference is that the liquidation payoff is shifted from zero to $(1-\alpha-\kappa)I > 0$. Last, suppose that $\alpha I > \alpha I - B > (1-\kappa)I$. The steps behind constant repayment $\bar{R}(\kappa)$ are identical but in this sub-case there are projects $z \notin \mathcal{B}(\kappa)$ that the banker does not want to complete. The run threshold is lower than the liquidation threshold $\ell(\kappa)$ as the latter is defined $\ell(\kappa)\phi(I) - \bar{R}(\kappa) + B \equiv \alpha I - (1-\kappa)I$ and therefore $\ell(\kappa)\phi(I) > \bar{R}(\kappa)$.

Pinning down $\ell(\kappa)$ is simpler in either sub-case of $\kappa > 1-\alpha$. The equation

$$F(\ell(\kappa))(1-\kappa)I + (1 - F(\ell(\kappa)))\bar{R}(\kappa) = (1-\kappa)I$$

holds when $\bar{R}(\kappa) = (1-\kappa)I$. If $(1-\kappa)I > \alpha I - B$, then $\ell(\kappa)\phi(I) = (1-\kappa)I$. Otherwise $\ell(\kappa)\phi(I) = \alpha I - B$ for all $\kappa$. Put together $\ell(\kappa)\phi(I) = \max((1-\kappa)I, \alpha I - B)$.

***A Note on the General Functional Form for Private Benefits***

Suppose instead the private benefit has a general functional form $B(I, \eta, \kappa)$. As long as $B$ function is positive, the proof is identical for $\kappa \leq 1-\alpha$. In $\kappa > 1-\alpha$ case the $\bar{R}(\kappa) = (1-\kappa)I$ result is also identical; if the banker can repay the face value in both liquidation and completion, he never compensates the households more regardless of how his private benefits work. The only change would be the liquidation rule is replaced by

$$\kappa > 1-\alpha : \ell(\kappa) = \max\left((1-\kappa)I, \alpha I - \frac{\kappa}{\eta}B(I;\kappa,\eta)\right) \tag{1.33}$$

$\square$

*Proof of Proposition 1.2.* I prove $\ell_\kappa \leq 0, \ell_I > 0, \ell_{I\kappa} \leq 0$ for $\kappa < 1-\alpha$ first and then repeat the proofs for $\kappa > 1-\alpha$. The single crossing at $1-\alpha$ follows immediately from the first assertion together with $\ell(0) > z^* > \ell(1)$ which is proved in the text.

$\bar{R}(\kappa)$ and $\ell(\kappa)$ are determined by households' participation constraint (1.7) in the text. Define $H(\ell) : [z^*, \infty) \to \mathcal{R}$ by

$$H(\ell) \equiv F(\ell)\alpha I + (1 - F(\ell))\ell\phi(I) - (1-\kappa)I \tag{1.34}$$

and $H(\ell(\kappa)) = 0$ corresponds to the equilibrium solution. The lower bound is $z^*$ because the lowest the banker can pay out to the households is $\alpha I$ which is less than what he owes $(1-\kappa)I$. So at the lower bound $H(z^*) < 0$. Differentiate $H(\ell)$ to get

$$H'(\ell) = (\phi(I) - (\ell\phi(I) - \alpha I)h(\ell))(1 - F(\ell)) \tag{1.35}$$

The sign of $H'(\ell)$ is determined by $\phi(I) - (\ell\phi(I) - \alpha I)h(\ell)$. Since $z^*\phi(I) = \alpha I$, $\lim_{\ell \to z^*} H'(\ell) > 0$. The hazard rate $h(\ell)$ is increasing by Assumption 1.2, so $\phi(I) - (\ell\phi(I) - \alpha I)h(\ell)$ is a decreasing function of $\ell$ with limit $-\infty$ as $\ell$ goes to infinity. Therefore there is a unique interior maximum of $H$ at $\ell^{\max} > z^*$ such that $H'(\ell^{\max}) = 0$. The banker can never raise capital from households if $H(\ell^{\max}) \leq 0$. Otherwise there exists a unique $\ell(\kappa) < \ell^{max}$ such that $H(\ell(\kappa)) = 0$ at which $H$ is positively sloped $H'(\ell(\kappa)) > 0$.

Use the implicit function theorem to get

$$\ell_\kappa = -\frac{I}{H'(\ell(\kappa))} < 0 \tag{1.36}$$

and

$$\ell_I = -\frac{F(\ell(\kappa))\alpha + (1 - F(\ell(\kappa)))\ell(\kappa)\phi'(I) - (1-\kappa)}{H'(\ell(\kappa))} \tag{1.37}$$

where $H'(\ell(\kappa)) > 0$. To sign the nominator of (1.37), first divide both sides of (1.7) by $I > 0$ to write $F(\ell)\alpha + (1 - F(\ell))\ell\phi(I)/I = (1-\kappa)$. By strict concavity of $\phi$, $\phi(I)/I > \phi'(I)$. This suffices to show that the nominator is negative and thus the sign of (1.37) positive.

To prove submodularity, differentiate (1.36) to get:

$$\ell_{I\kappa} = -\Big[\frac{1}{H'(\ell(\kappa))} - \frac{H''(\ell(\kappa))\ell_I I}{H'^2(\ell(\kappa))}\Big] \tag{1.38}$$

The only previously unstudied term in (1.38) is $H''(\ell(\kappa))$. $H'(\ell)$ is defined in (1.35) as a product of two decreasing functions. The second is always positive, the first decreasing function alternates sign but I have proved that it is positive at the optimum $\ell(\kappa)$. This suffices to argue that $H''(\ell(\kappa)) < 0$. All other terms in the square bracket of (1.38) are positive and therefore the overall sign is negative.

Proving these assertions for $\kappa > 1-\alpha$ is easier since $\ell(\kappa)\phi(I) = \max((1-\kappa)I, \alpha I - B)$ gives an explicit function in $(I;\kappa)$. If $(1-\kappa)I > \alpha I - B$ then $\ell_\kappa = -I/\phi(I) < 0$. In the other sub-case, $\ell(\kappa)\phi(I) = \alpha I - B$ so $\ell$ is independent of $\kappa$ and $\ell_\kappa = 0$.

In either sub-case $\ell(\kappa)$ is increasing in $I$ since $I/\phi(I)$ is increasing by strict concavity of $\phi(I)$. Finally if $(1-\kappa)I > \alpha I - B$, then $\ell_{I\kappa} = -\left(\frac{I}{\phi(I)}\right)' < 0$ and in the other subcase it is trivially submodular in $(I;\kappa)$ as it does not depend on $\kappa$.

***A Note on the General Functional Form for Private Benefits***

I derived the liquidation rule for a general functional form for private benefits in (1.33). The only change in results is in $\kappa > 1-\alpha$ case. Add a restriction

$$\frac{\partial}{\partial\kappa}\Big(\kappa B(I;\eta,\kappa)\Big) = \kappa B_\kappa + B \geq 0 \tag{1.39}$$

which is an elasticity condition. If in addition $\lim_{\kappa\to1}(\alpha I - \frac{\kappa}{\eta}B(I;\kappa,\eta)) > 0$, then once more there exists a unique $\kappa' > 1-\alpha$ such that the liquidation rule does not depend on $B$ function for $\kappa \leq \kappa'$ and I can only focus on $\kappa > \kappa'$. Under the premise of (1.39), $\ell_\kappa \leq 0$. The restriction for $\ell_I \geq 0$ is another elasticity condition

$$\frac{\partial}{\partial I}\Big(\frac{B(I;\eta,\kappa)}{\phi(I)}\Big) \leq 0 \Rightarrow \frac{B_I}{B} \leq \frac{\phi'(I)}{\phi(I)} \tag{1.40}$$

For $\ell_{I\kappa} \leq 0$ to go through, a third and the last restriction is

$$B_{I\kappa}B \leq B_\kappa B_I \tag{1.41}$$

The baseline form (1.4) satisfies all the restrictions: (1.39) always holds with equality and $B_I = B_{I\kappa} = 0$. $\square$

*Proof of Proposition 1.3.* The arguments of functions $\ell$ and $z^*$ are suppressed throughout the proof. The optimality condition is

$$\int_{z^*}^{\infty} z\phi'(I)dF(z)+F(z^*)\,\alpha-1-\int_{z^*}^{\ell}(z\phi'(I)-\alpha)dF(z)-\ell_I(\ell\phi(I)-\alpha I)f(\ell)=0 \quad (1.42)$$

The first three terms correspond to (1.3) and Proposition 1.1 shows that its limit is $\infty$ as $I\to 0$. The last two terms are the marginal loss. $\ell$ is implicitly defined by (1.7). Both $\ell$ and $z^*$ go to 0 for $I\to 0$ and the marginal loss disappears. Thus the left hand side of (1.54) evaluated as $I\to 0$ is $\infty$.

The last term of the marginal loss is unambiguously positive but the sign of the integral component is unknown in general. I now prove that it is bounded below by zero. A lower bound is proposed in the text (1.13). This lower bound requires a precise relationship between $\ell_I$ to $z_I^*$ that I obtain from (1.7).

$$\begin{aligned} F(\ell)\alpha I+(1-F(\ell))\ell\phi(I)&=(1-\kappa)I \\ F(\ell)z^*+(1-F(\ell))\ell=(1-\kappa)\frac{I}{\phi(I)}&=\frac{1-\kappa}{\alpha}z^* \\ F(\ell)z_I^*+(1-F(\ell))\ell_I-\ell_I(\ell-z^*)f(\ell)&=\frac{1-\kappa}{\alpha}z_I^* \qquad (1.43)\end{aligned}$$

Now rewrite the lower bound as

$$\phi(I)\Big[-z_I^*(F(\ell)-F(z^*))+\ell_I(\ell-z^*)f(\ell)\Big] \qquad (1.44)$$

$$=\phi(I)\Big[(1-F(\ell))\ell_I-\Big(\frac{1-\kappa}{\alpha}-F(z^*)\Big)z_I^*\Big] \qquad (1.45)$$

and thus the lower bound is non-negative if

$$\frac{\ell_I}{z_I^*}\geq\frac{(1-\kappa)/\alpha-F(z^*)}{1-F(\ell)} \qquad (1.46)$$

Notice that while Proposition 1.2 proves that the ratio of two derivatives is larger than unity, the right-hand-side of (1.46) is strictly larger than unity. The exact ratio can be inferred from (1.43).

$$\frac{\ell_I}{z_I^*} = \frac{(1-\kappa)/\alpha - F(\ell)}{1 - F(\ell) - (\ell - z^*)f(\ell)} = \frac{(1-\kappa)/\alpha - F(z^*) - (F(\ell) - F(z^*))}{1 - F(\ell) - (\ell - z^*)f(\ell)} \tag{1.47}$$

To simplify notation, let the letters $a$ through $d$ denote:

$$a = (1-\kappa)/\alpha - F(z^*)\ ,\ b = 1 - F(\ell)\ ,\ c = F(\ell) - F(z^*)\ ,\ d = (\ell - z^*)f(\ell)$$

All letters are positive. I now claim that

$$\frac{\ell_I}{z_I^*} = \frac{a-c}{b-d} > \frac{a}{b} \tag{1.48}$$

which is the sufficient condition (1.46) to prove that the lower bound is non-negative. (1.48) is equivalent to $a/b \geq c/d$. There are two cases for $F$ being convex/concave on $[z^*, \ell]$. If $F$ is convex on that interval:

$$\frac{a}{b} = \frac{(1-\kappa)/\alpha - F(z^*)}{1 - F(\ell)} > \frac{1 - F(z^*)}{1 - F(\ell)} > 1 \geq \frac{F(\ell) - F(z^*)}{(\ell - z^*)f(\ell)} = \frac{c}{d} \tag{1.49}$$

As observed at the beginning of the proof, the lower bound is easily signed in the convex portion of $F$ , if any. If $F$ is concave on $[z^*, \ell]$, then

$$\frac{a}{b} = \frac{(1-\kappa)/\alpha - F(z^*)}{1 - F(\ell)} > \frac{1 - F(z^*)}{1 - F(\ell)} \geq \frac{f(z^*)}{f(\ell)} \geq \frac{F(\ell) - F(z^*)}{(\ell - z^*)f(\ell)} = \frac{c}{d} \tag{1.50}$$

The middle inequality is increasing hazard rate and the last one is the concavity of $F$. Lastly, suppose that $F$ is convex at $z^*$ and concave at $\ell$ such that $F(\ell) - F(z^*) > (\ell - z^*)\max(f(\ell), f(z^*))$. Then there must exists $\bar{z} \in (z^*, \ell)$ such that

$F(\ell) - F(z^*) = (\bar{z} - z^*)f(\bar{z})$ and

$$\frac{a}{b} > \frac{1 - F(z^*)}{1 - F(\ell)} > \frac{1 - F(\bar{z})}{1 - F(\ell)} \geq \frac{f(\bar{z})}{f(\ell)} > \frac{(\bar{z} - z^*)f(\bar{z})}{(\ell - z^*)f(\ell)} = \frac{F(\ell) - F(z^*)}{(\ell - z^*)f(\ell)} = \frac{c}{d} \tag{1.51}$$

This exhausts all cases an arbitrary investment level $I$ can map onto a convex-concave distribution function $F$ via functions $\ell$ and $z^*$ and in all of them the marginal loss is bounded above zero. So the left-hand-side (1.42) evaluated at $I^*$ must be negative and this suffices to argue that (1.42) is satisfied at a lower $I^S(\kappa) < I^*$, concluding the proof. $\square$

*Proof of Corollary 1.1.* The first part of the corollary is proven in the text. The second part is an application of Envelope Theorem. The function $S(I^S(\kappa); \kappa)$ is a well-defined continuous function of $\kappa$ by Proposition 1.3. Differentiate in $\kappa$ to get

$$S_\kappa(I^S(\kappa); \kappa) = \left( S_I \Big|_{(I^S(\kappa),\kappa)} \frac{\partial I^S}{\partial \kappa} + S_\kappa \Big|_{(I^S(\kappa),\kappa)} \right) = -\ell_\kappa(\kappa)\Big(\ell(\kappa)\phi(I^S(\kappa)) - \alpha I^S(\kappa)\Big) > 0 \tag{1.52}$$

The first term vanishes at the optimum and the second term is positive by Proposition 1.2. $\square$

**Lemma 1.6.** *For $i \in \{S, U\}$ there exists $K^i \in (1 - \alpha, 1)$ such that at the optimum*

$$\alpha I^i(\kappa) - B \gtreqless (1 - \kappa)I^i(\kappa) \text{ for } \kappa \gtreqless K^i \tag{1.53}$$

*Proof of Lemma 1.6.* I present the proof for $I^S$, the steps are identical for $I^U$. The critical investment level is $B/(\kappa - (1 - \alpha))$. The claim is $I^S(\kappa)$ single crosses $B/(\kappa - (1 - \alpha))$ from below. $I^S(\kappa)$ starts from below because as $\kappa \to 1 - \alpha$, $I^S = I^* <$

$\lim_{\kappa\to 1-\alpha} B/(\kappa - 1 - \alpha) = \infty$. The critical level $B/(\kappa - (1-\alpha))$ is decreasing and convex in $\kappa$ with lower limit $B/\alpha$. Suppose for some $\kappa^L < \kappa^H$, $I^S(\kappa^L) \geq B/(\kappa^L - 1 - \alpha)$. Then for all $I \geq B/(\kappa^L - (1-\alpha)) > B/(\kappa^H - (1-\alpha))$, the thresholds are identical $\ell(I;\kappa^L) = \ell(I;\kappa^H)$ as $\ell\phi(I) = \alpha I - B$ is independent of $\kappa$. Therefore the optimal investments must be identical. This implies that if $I^S(\kappa)$ crosses $B/(\kappa - (1-\alpha))$ once, it is constant in $\kappa$ thereafter.

Take $\kappa = 1$. For $I < B/\alpha$, $\ell(I;1) = 0$ so the left-hand-side of (1.54) collapses to $E[z\phi'(I)] - 1$. By Assumption 1.3, $E[z\phi'(B/\alpha)] - 1 > 0$ and $\phi'' < 0$ so the optimal investment has to be larger than $B/\alpha$. Hence there exists $K^S \in (1-\alpha, 1)$ such that

$$\forall \kappa \geq (>) K^S : I^S(\kappa) = \frac{B}{K^S - (1-\alpha)} \geq (>) \frac{B}{\kappa - (1-\alpha)}$$

independent of $\kappa$, whereas for all $\kappa < K^S : \ell(\kappa)\phi(I^S(\kappa)) = (1-\kappa)I^S(\kappa)$. □

*Proof of Proposition 4.* The arguments of functions $\ell$ and $z^*$ are suppressed though-out. The optimality condition is

$$\int_{z^*}^{\infty} z\phi'(I)dF(z) + F(z^*)\alpha - 1 - \int_{\ell}^{z^*} (\alpha - z\phi'(I))dF(z) + \ell_I(\alpha I - \ell\phi(I))f(\ell) = 0 \tag{1.54}$$

Once more the first three terms correspond to the benchmark problem and the last two terms, the first of which is negative and the second is positive, represent the marginal loss. For $\kappa > 1 - \alpha$, the liquidation rule is $\ell\phi(I) = \max(\alpha I - B, (1-\kappa)I)$. As $I \to 0$ the marginal loss becomes zero and Proposition 1.1 proves that the rest of the terms tend to $\infty$. The aim is to sign the marginal loss

$$-\ell_I(\alpha I - \ell\phi(I))f(\ell) + \int_{\ell}^{z^*} (\alpha - z\phi'(I))\, dF(z) \tag{1.55}$$

Let $\zeta$ be the unique mode of $Z$ by log-concavity. I proceed case-by-case.

**Case 1.1.** *$\ell(I) < z^*(I) \leq \zeta$ i.e. $F$ is convex on the interval $[\ell(I), z^*(I)]$.*

(1.55) is bounded below by

$$\begin{aligned} LB(I;\kappa) &\equiv -\ell_I(z^* - \ell)\phi(I)f(\ell) + (\alpha - z^*\phi'(I))(F(z^*) - F(\ell)) \\ &= \phi(I)\Big[z_I^*(F(z^*) - F(\ell)) - \ell_I(z^* - \ell)f(\ell)\Big] \geq 0 \end{aligned} \tag{1.56}$$

The lower bound $LB(I;\kappa)$ is non-negative as $z_I^* \geq \ell_I$ by Proposition 1.2 and $F(z^*) - F(\ell) \geq (z^* - \ell)f(\ell)$ by convexity of $F$.

**Case 1.2.** *$z^*(I) > \ell(I) \geq \zeta$ or $F$ is concave on the interval $[\ell(l), z^*(I)]$, and $\kappa \geq K$ defined in Lemma 1.6.*

Whenever $F$ is concave on an interval, the lower bound (1.56) has an ambiguous sign. Using Lemma 1.6 (1.55) can be bounded above by

$$UB^{\kappa\geq K}(I;\kappa) = \phi(I)\ell_I\Big[F(z^*) - F(\ell) - (z^* - \ell)f(\ell)\Big] \leq 0 \tag{1.57}$$

It is redundant to analyze $I \leq B/\alpha$ since $I^S(\kappa) > B/\alpha$ so totally differentiate $\ell\phi(I) = \alpha I - B > 0$ and replace $\alpha - \ell\phi'(I)$ to simplify the expression. The inequality follows from concavity of $F$.

**Case 1.3.** *$z^*(I) > \ell(I) \geq \zeta$ or $F$ is concave on the interval $[\ell(l), z^*(I)]$, and $\kappa < K$ defined in Lemma 1.6.*

Neither the lower nor the upper bound can be unambiguously signed in this case. Define

$$z^*/\ell = \alpha/(1-\kappa) = \lambda > 1 \tag{1.58}$$

and observe that the equality also holds for the ratio of their derivatives $z_I^*/\ell_I$. Now the lower bound (1.56) can be rewritten

$$
\begin{aligned}
LB^{\kappa<K}(I;\kappa) &\equiv \ell_I\phi(I)\Big[\lambda(F(\lambda\ell)) - F(\ell)) - (\lambda-1)\ell f(\ell)\Big] \\
&= \ell_I\phi(I)(1-F(\ell))\Big[\lambda(1-\frac{1-F(\lambda\ell)}{1-F(\ell)}) - (\lambda-1)\ell h(\ell)\Big]
\end{aligned} \tag{1.59}
$$

Increasing hazard rate implies that $1-F(\lambda\ell)/1-F(\ell)$ is a decreasing function of its argument, which is itself increasing in $I$. Therefore the first term in the square brackets of (1.59) is an increasing function of $I$ ranging $(0,\lambda)$. The second term is increasing in $I$ ranging $(0,\infty)$ by the very same assumption. Hence there exists an $\bar{I}_1$ such that $LB^{\kappa<K}(I;\kappa) < 0$ whenever $I > \bar{I}_1$. To prove it starts positive, $F(\lambda\ell) - F(\ell) \geq (\lambda-1)\ell f(\lambda\ell)$ by concavity of $F$ so the lower bound itself satisfies

$$
LB^{\kappa<K}(I;\kappa) \geq \ell_I\phi(I)(\lambda-1)\ell\Big[\lambda f(\lambda\ell) - f(\ell\Big] \tag{1.60}
$$

Notice that the square bracket term is $\partial\Big(F(\lambda\ell) - F(\ell)\Big)/\partial\ell$. Under the log-concavity assumption, this derivative alternates sign from positive to negative once.

**Claim 1.1.** *For any $\lambda > 1$, $F(\lambda z) - F(z)$ is unimodal at $\bar{\zeta} \geq \zeta$.*

The claim proves that the lower bound starts positive, crosses zero at some $\hat{I}_1$ and remains negative thereafter. I conclude without loss of generality that

$$
LB^{\kappa<K}(I;\kappa) \gtrless 0 \Longleftrightarrow I \lessgtr \bar{I}_1
$$

The same analysis can be repeated for the upper bound:

$$
\begin{aligned}
UB^{\kappa<K}(I;\kappa) &\equiv LB^{\kappa<K}(I;\kappa) + z^*\phi'(I)(F(z^*) - F(\ell))) \\
&= LB^{\kappa<K}(I;\kappa) + \frac{\alpha I\phi'(I)}{\phi(I)}(F(\lambda\ell) - F(\ell))
\end{aligned} \tag{1.61}
$$

Whenever $LB^{\kappa<K}(I;\kappa) \leq 0$, I have argued that $\partial\Big(F(\lambda\ell) - F(\ell)\Big)/\partial\ell \leq 0$. In

addition, $\phi(I)/I \geq \phi'(I)$ by concavity of $\phi(I)$. Therefore the additional term in (1.61) is less than or equal to $\alpha(F(\lambda\ell) - F(\ell))$, a decreasing function that converges to 0 as $I \to \infty$. This suffices to conclude that $\exists \bar{I}_2 > \bar{I}_1$ such that

$$UB^{\kappa<K}(I;\kappa) \gtrless 0 \Longleftrightarrow I \lessgtr \bar{I}_2$$

Put together, the marginal loss is positive for small $I \leq \bar{I}_1$ and negative for large $I \geq \bar{I}_2$ so there must exist a threshold $T : \bar{I}_1 < T < \bar{I}_2$ such that (1.55) is zero. I assume without loss of generality that it singly crosses zero.

All cases are covered for $\kappa > 1 - \alpha$, in each case there exists $T(\kappa)$ such that if $I^* < T(\kappa)$, the benchmark is in the increasing segment of marginal loss and therefore $I^S(\kappa) < I^*$. Otherwise if $I^* > T(\kappa)$ so that the benchmark is in the decrasing segment of marginal loss, then $I^S(\kappa) > I^*$. The proof of Claim 1.1 is presented below.

*Proof of Claim 1.1.* The proof of this claim uses the following result. See Ramos and Diaz (2001) for a proof.

**Result 1.1** (Theorem 1.C.29 in Shaked and Shantikumar (2007))**.** *Let $Z$ be a non-negative, absolutely continuous random variable with density function $f(z)$ on $(0, \infty)$. $\lambda Z$ likelihood ratio-dominates $Z$ for all $\lambda > 1$ if and only if $f(e^z)$ is log-concave.*

The slope of $F(\lambda z) - F(z)$ is $\lambda f(\lambda z) - f(z)$. Let $\zeta$ denote the mode of $Z$ such that $f'(z) \gtrless 0$ whenever $z \lessgtr \zeta$. Consider first $z < \zeta$. Since $F''(z) = f'(z)$ this is the convex portion of $F$. Now the slope can be unambiguously signed as $f(\lambda z) \geq f(z)$ and $\lambda > 1$.

Suppose $z \geq \zeta$ so that $F$ is concave at $z$ or $f'(z) \leq 0$. Notice that if $f(z)$ is the probability density of $Z$ at point $z$, then $g(z) = f(z/\lambda)$ is the probability density of $\lambda Z$. By Result 1.1 $f(z/\lambda)/f(z)$ is increasing, or equivalently $f(\lambda z)/f(z)$ is decreasing, if and only if $f(e^z)$ is log-concave. I now claim that for $z \geq \zeta$, $f(e^z)$ is log-concave.

$$(\ln f(e^z))'' = e^z \frac{f'(e^z)}{f(e^z)} + e^{2z}\Big(\frac{f'(e^z)}{f(e^z)}\Big)' \leq 0 \tag{1.62}$$

The non-positive sign follows from the fact that $f'(e^z) \leq 0$ as $e^z$ is an increasing function of $z$ so $e^z > z \geq \zeta$ and the second is log-concavity of $f(z)$. Likelihood ratio order is preserved under integration, therefore

$$\frac{f(\lambda z)}{f(z)} \searrow z \Longrightarrow \frac{F(\lambda z)}{F(z)} \searrow z \Longrightarrow \frac{F(\lambda z)}{F(z)} - 1 > \frac{\lambda f(\lambda z)}{f(z)} - 1 \tag{1.63}$$

The second inequality completes the proof. The limit of left-hand side is 0 since $\lim_{z\to\infty} F(\lambda z) = \lim_{z\to\infty} F(z) = 1$ and therefore $\lambda f(\lambda z)/f(z)$ must be below1 as $z \to \infty$. The ratio of densities is decreasing in $z$ so either it single crosses 1 from above or it is always below 1. In either case $\exists\bar{\zeta} \geq \zeta$ such that $\lambda f(\lambda z) - f(z)$ single crosses 0 from above at $\bar{\zeta}$. This proves that $F(\lambda z) - F(z)$ is increasing for $z \leq \bar{\zeta}$ and decreasing for $z \geq \bar{\zeta}$ concluding the proof. □

□

*Proof of Corollary 1.2.* The derivative of $S(I^S(\kappa);\kappa)$ in $\kappa$ using envelope theorem is

$$S_\kappa(I^S(\kappa);\kappa) = \Big(S_I\Big|_{(I^S(\kappa),\kappa)} \frac{\partial I^S}{\partial \kappa} + S_\kappa\Big|_{(I^S(\kappa),\kappa)}\Big) = -\ell_\kappa(\kappa)\Big(\ell(\kappa)\phi(I^S(\kappa)) - \alpha I^S(\kappa)\Big) \tag{1.64}$$

When $\kappa > 1-\alpha$, the parenthetical term is negative. $\ell_\kappa(\kappa) \leq 0$ by Proposition 1.2 concludes that the derivative is non-negative. □

*Proof of Proposition 1.5.* By definition in (1.17), $I^U$ must satisfy $V_I(I^U;\eta,\kappa) = 0$ and Proposition 1.3 and 1.4 proves $\exists\ I^S$ that satisfies $S_I(I^S;\kappa) = 0$. Then

$$V_I(I^S;\eta,\kappa) = \frac{\eta}{\kappa}\Big(S_I(I^S;\kappa) - f(\ell(\kappa))\ell_I(I^S;\kappa)B\Big) = -\frac{\eta}{\kappa}f(\ell(\kappa))\ell_I(I^S;\kappa)B < 0 \quad (1.65)$$

since $\lim_{I\to 0} V_I(I;\kappa,\xi) \to \infty$ as $\lim_{I\to 0} S_I(I;\kappa) \to \infty$, there exists $I^U(\kappa) < I^S(\kappa)$ that satisfies the optimality condition.

Consider a general form for private benefits $B(I;\eta,\kappa)$ such that $B_I \geq 0$ and the condition (1.40) that is sufficient for all earlier results to hold. The marginal effect of $I$ on the expected private benefit would be

$$-f(\ell(\kappa)\ell_I B + (1 - F(\ell(\kappa))B_I = (1 - F(\ell(\kappa))B_I\Big(1 - h(\ell(\kappa))\ell_I\frac{B}{B_I}\Big)$$

The parenthetical term determines the sign and is bounded above by $1-h(\ell(\kappa))\ell_I\phi(I)/\phi'(I)$. It starts positive as $I \to 0$ and since hazard rate $h$ and $\phi/\phi'$ are increasing with limit infinity, it crosses zero and stays negative provided $\ell_I$ is also well-behaved. Let $\bar{I}$ denote the crossing point. Proposition 1.5 is valid as long as $I^S(\kappa) > \bar{I}$. □

*Proof of Lemma 1.2.* Compute the marginal rent in $\kappa$ using Envelope Theorem:

$$\begin{aligned}\hat{V}_\kappa &= \Big(V_I\Big|_{(I^U(\kappa),\kappa)} + V_\kappa\Big|_{(I^U(\kappa),\kappa)}\Big) \\ &= -\ell_\kappa\Big(\ell(\kappa)\phi(I^U(\kappa)) + B - \alpha I^U(\kappa)\Big) \geq 0 \end{aligned} \quad (1.66)$$

In Lemma 1.6 I proved that there exists a $K$ such that for $\kappa < K$ the optimum satifies

$$\ell(\kappa)\phi(I^U) > \alpha I^U(\kappa) - B$$

and by Proposition 1.2 $\ell_\kappa < 0$ whenever this is the case. Therefore $\hat{V}_\kappa$ is positive for $\kappa < K$ and zero for $\kappa \geq K$. □

*Proof of Lemma 1.3.* First I show that $I^U(\kappa) < I^S(\kappa)$ for all $\kappa$ where $I^S(\kappa)$ maximizes $S(I;\kappa)$ and $I^U(\kappa)$ maximizes (1.24). Since $1 - F(\ell(\kappa))$ is decreasing in $I$ with the slope $-f(\ell(\kappa))\ell_I(\kappa) < 0$, I study the sign of $\partial \mathbb{f}/\partial I$ and prove it is positive evaluated at $I^S(\kappa)$. Differentiate $\mathbb{f}$ is $I$ to get

$$\frac{\partial \mathbb{f}}{\partial I} \equiv \frac{\partial}{\partial I}\Big(\frac{(\kappa-\eta)I}{S(I;\kappa)+\kappa I}\Big) = \frac{\kappa-\eta}{S(I;\kappa)+\kappa I}\Big(1 - \frac{(S_I(I;\kappa)+\kappa)I}{S(I;\kappa)+\kappa I}\Big) \tag{1.67}$$

Evaluated at $I^S(\kappa)$ the derivative inside the parenthesis vanishes and the remaining terms are unambiguously positive. Therefore:

$$U_I(I^S(\kappa);\eta,\kappa) = S_I(I^S(\kappa);\kappa) + \frac{\partial}{\partial I}\Big((1-\mathbb{f})(1-F(\ell(\kappa))B\Big)\Big|_{I^S(\kappa)} < 0 \tag{1.68}$$

I have proven in Proposition 1.3 that $I \to 0$, $S_I(I;\kappa) \to \infty$. Then $U_I(I;\eta,\kappa)$ equals zero for some $I^U(\kappa) < I^S(\kappa)$.

For the second part of the result, evaluate (1.24) at $I^U(\kappa)$ and use Envelope Theorem to get

$$U_\kappa(I^U(\kappa);\eta,\kappa) = -\ell_\kappa\Big(\ell(\kappa)\phi(I^U(\kappa)) - \alpha I^U(\kappa) + (1-\mathbb{f})B\Big)f(\ell(\kappa)) - \mathbb{f}_\kappa(1-F(\ell(\kappa))B \tag{1.69}$$

The term $\mathbb{f}_\kappa$ denotes the derivative of $\mathbb{f}$ in $\kappa$ evaluated at $I^U(\kappa)$. To compute this derivative, first I get:

$$\frac{\partial \mathbb{f}}{\partial \kappa} \equiv \frac{\partial}{\partial \kappa}\Big(\frac{(\kappa-\eta)I}{S(I;\kappa)+\kappa I}\Big) = \frac{I}{S(I;\kappa)+\kappa I}\Big(1 - \frac{(S_\kappa(I;\kappa)+I)(\kappa-\eta)}{S(I;\kappa)+\kappa I}\Big) \tag{1.70}$$

Consider $\kappa \to 1-\alpha$. Since $\ell(\kappa) \to z^*$ and by definition of $z^*$ in (1.1), $\lim_{\kappa\to 1-\alpha} S_\kappa(I;\kappa) = 0$ for all $I$. Now $\partial \mathbb{f}/\partial \kappa$ simplifies to

$$\lim_{\kappa\to 1-\alpha}\frac{\partial \mathbb{f}}{\partial \kappa} = \frac{\mathbb{f}}{1-\alpha-\eta}(1-\mathbb{f}) \tag{1.71}$$

for all $I$. Likewise the limit of $U_\kappa$ simplifies to

$$\lim_{\kappa \to 1-\alpha} U_\kappa(I^U(\kappa); \eta, \kappa) = \lim_{\kappa \to 1-\alpha} (1 - \mathbb{f}) B\Big( - \ell(\kappa) f(\ell(\kappa)) - \frac{\mathbb{f}}{\kappa - \eta}(1 - F(\ell(\kappa))\Big) < 0 \tag{1.72}$$

under condition (1.25). Therefore, even if $\lim_{\kappa \to \eta} U_\kappa > 0$ so that the banker has an incentive to issue outside equity, he does not have enough incentive to capitalize up to $1 - \alpha$. $\square$

# Chapter 2

# Savings Portfolio Problem with Collateralized Debt[1]

## 2.1 Introduction

One of the striking developments of the Great Recession of 2008–9 was the rise and fall of household leverage before and after the Financial Crisis. For instance, Justiniano et al (2013) report that the ratio of mortgages to GDP rose an unprecedented 30 percentage points in the period 2000-2007 and declined 10 percentage points after the crisis. This has often been cited as a contributing factor to the severity and duration of the Great Recession. In fact, the fraction that the households put down in buying homes has steadily fallen during 1981-2007 except briefly around the 2001 stock market crash.

This paper seeks insights into this trend, by introducing a simple partial equilibrium model of household portfolio and leverage. One antecedent of this paper is Geanakoplos (2003, 2010), who explores a rich general equilibrium environment with endogenous leverage and belief disagreement. In his two-state model, the loans are collateralized for the worst state and because default happens only at the worst state,

[1]This chapter presents joint work with Lones A. Smith

there is no value-at-risk at the equilibrium. Simsek (2013) addresses this problem but preserves the risk-neutrality assumption. His agent does not hold any risk-free asset and the relationship between the demand for the risky asset and leverage is therefore mechanically dictated by the budget constraint.

I generalize Simsek (2013), exploring the portfolio problem of a risk-averse agent with one risky and a risk-free asset. I assume that he may borrow against the risky asset, using it as collateral on the loan. Unlike Simsek (2013), there are wealth effects on the optimal leverage decision. As in Geanakoplos, belief disagreement about the asset values creates gains from trade between optimistic borrowers and competitive lenders. Given the competitive assumption, when the agent makes a bigger down payment, she secures a lower interest on the loan because the lender assumes less risk. If the promised payment on the loan exceeds the asset value, the agent defaults and consumes her risk-free savings, while the lender seizes the asset as collateral. This framework captures a typical household portfolio decision with real estate and mortgage.

The agent optimally chooses both the portfolio allocation and its leverage ratio. As a result, belief disagreement in the credit market leads to a more leveraged portfolio with more of both risky and risk-free assets, compared to the standard portfolio problem without collateralized borrowing. I also uncover a complementarity between the agent's demand for the risky asset and the optimally chosen leverage. For example, the agent prefers to make a smaller down payment if risky assets figure more prominently in his portfolio. Conversely, the demand for the risky asset is higher with more leverage (a lower loan margin). This complementarity induces to a key comparative statics result: a more optimistic agent buys more risky asset with bigger credit. Provided the agent's beliefs about the asset value arise from a business cycle, I predict that the household leverage ratio is pro cyclical.

My approach to leverage cycles differs from the credit cycles literature that follows Kiyotaki and Moore (1997). In that paper, the agent faces a collateral constraint that limits the borrowing capacity by a fraction of the collateral value. This requirement amplifies the real effect of a drop in asset values by making it harder the borrow against low-value collateral. This mechanism has appeared in Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011) and Midrigan and Phillippon (2011), who introduce credit supply shocks to the collateral constraint to study changes in household debt. In my model, the margin is an endogenous collateral constraint determined by both the agent's demand for credit and the lenders' supply of it. my agent optimally curbs his credit demand in response to a negative shock to asset values. By contrast, their agents are forced to de-leverage by the lender's unwillingness to supply credit, which can be addressed by the monetary authority by increasing liquidity for the lender.

The paper is organized as follows: Section 2 describes the model, Section 3 shows how the endogenous choice of margin changes the standard portfolio problem, Section 4 presents the solution to the decision problem, and Section 5 presents the comparative statics result.

## 2.2 The Model

In this section I lay out the preliminary assumptions and formalize the decision problem. There is only one period, the decision is made at the beginning and the payoff is realized at the end. The agent is endowed with initial wealth $w > 0$ and derives utility solely from her ending wealth $W$. She allocates her initial wealth between one risky and one risk-free asset neither of which can be sold short. The price of the risky asset is normalized to unity. Let $x \geq 0$ denote the investment in the risky asset. Each unit of the asset yields a dollar payoff $s \geq 0$ at the end of the period. This is the source of all uncertainty in the model, so ex ante is denoted $S$. The risk-free asset earns a known gross return $R_f$.

### 2.2.1 Preferences

The agent is risk-averse. Let $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a strictly increasing, continuous, differentiable and strictly concave utility function on $\{u > -\infty\}$ representing the agent's preferences. Negative wealth is not allowed: $u(W) = -\infty$ whenever $W < 0$. At the extreme wealths, the marginal utility satisfies

$$\lim_{W \to \infty} u'(W) = 0 \ , \ \lim_{W \to 0} u'(W) = \infty$$

I restrict the agent's risk aversion in the following ways. Let $A$ and $R$ denote the coefficient of absolute and relative risk aversion respectively,

$$A(W) = -\frac{u''(W)}{u'(W)} \ , \ R(W) = \frac{-Wu''(W)}{u'(W)} \tag{2.1}$$

**Assumption 2.1** *Absolute risk aversion is non-increasing:* $A'(W) \leq 0$, *and relative risk aversion is non-decreasing:* $R'(W) \geq 0$.

### 2.2.2 Borrowing and Lending

There is a continuum of identical and risk-neutral lenders willing to finance the risky asset with simple debt contracts. All borrowing in this economy has to be backed by collateral. I model the collateral constraints similar to Geanakoplos (2003, 2010) and Simsek (2013). Lenders offer a menu of contracts specifying per unit investment: (1) the fraction put down by the agent at the beginning of the period, denote $m$ for margin, (2) the promised payment at the end of the period, denote $i$ for gross interest. In this specification $x(1-m)$ is the size of the loan, $xi$ is the total interest payment on the loan, the implied interest rate is

$$\iota = \frac{i}{1-m} \tag{2.2}$$

and the agent's *leverage ratio* is $1/m$. All loans are non-recourse; if the risky asset is worth less than $I$, the agent can walk away and the lenders foreclose the risky asset but they are not allowed to seize the agent's risk-free asset.

### 2.2.3 Beliefs

The lenders and the agent have different beliefs about the distribution of $S$ and they agree to disagree about it. In particular, the agent's optimism measure ($\theta$) exceeds the lenders's (0). Let $f(s|\theta), f(s|0)$ denote two continuous and differentiable probability density functions of subjective beliefs on full support $[0, \infty)$. $F(s|\theta)$ and $F(s|0)$ denote the cumulative distribution functions.

**Assumption 2.2** *The cumulative distribution function of the agent's beliefs $F(s|\theta)$ is log-concave.*

Log-concavity of the distribution function is a mild assumption; in addition to all random variables with a log-concave density, e.g. Normal, Exponential, Uniform, it includes random variables such as Pareto and Log-Normal with log-convex and mixed density respectively[2].

The expectation of the subjective beliefs are such that

$$E[S|\theta] > R_f > E[S|0] \tag{2.3}$$

Subjective beliefs represent the different preferences towards the risky asset. The agent is the natural buyer of the asset and the lenders are not interested unless the agent is willing to contribute towards the purchase.

More generally, the beliefs are ranked in the likelihood ratio order for any $\theta^H > \theta^L$:

$$\frac{f(s|\theta^H)}{f(s|\theta^L)} \nearrow \; s \tag{2.4}$$

[2]See Bagnoli and Bergstorm (2005) for a comprehensive list of log-concavity properties of commonly used distributions.

or $f(s|\theta)$ is log-supermodular in $(s, \theta)$.

### 2.2.4 Margin-Interest Menu of Loans

With a competitive supply of lenders, each lenders must earn zero profit on each loan. That is, $(m, i)$ must satisfy:

$$E[\min(S, i)|0] = (1 - m)R_f \tag{2.5}$$

The right-hand side is the expected payoff from lending against one unit of risky asset promising $i$ in return. Since the agent puts down $m$ for that one unit, the left-hand side is the opportunity cost of lending $(1 - m)$. Since $E[S|0] > R_f$, the lowest margin the lenders is willing to accept $\underline{m} > 0$ is

$$E[S|0] = (1 - \underline{m})R_f \tag{2.6}$$

I characterize the margin-interest menu that the competition among lenders determines and study its comparative statics in the next lemma. Choosing $m = 1$ subsumes the case the agent does not borrow.

**Lemma 2.1.** *For each margin $m \in [\underline{m}, 1]$, the lenders offers a unique $i(m)$ solving (2.5). The loans satisfy:*

1. *$i(m)$ is decreasing and convex in $m$.*

2. *Implied interest rate $\iota(m)$ is decreasing in $m$*

3. *A Second-order Stochastically dominant increase in the lenders' beliefs from* 0 *to* $0'$

$$\forall \bar{s} \geq 0 : \int_0^{\bar{s}} F(s|0')ds \leq \int_0^{\bar{s}} F(s|0)ds \tag{2.7}$$

*lowers $i$ and $\iota$ for each $m$.*

4. *A decrease in the opportunity cost of funds lowers $i$ and $\iota$ for each $m$.*

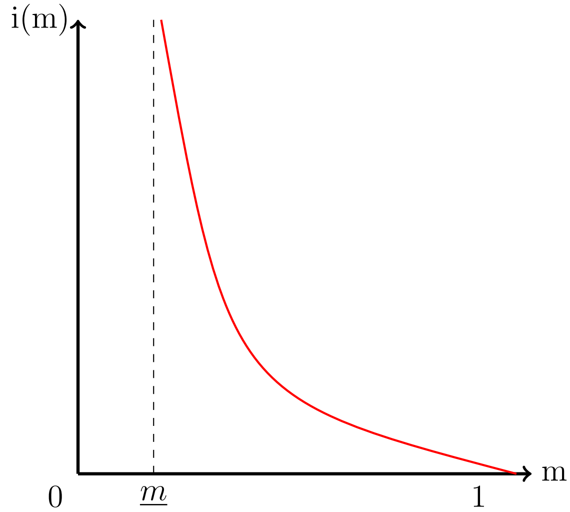



Figure 2.1: Margin-Interest Menu

Figure 2.1 plots the menu of loans available to the agent. The more the agent puts down, the lower the default risk and the compensation to the lenders. Convexity rules out any gains from forming a portfolio of loans to finance the risky asset. Suppose that the agent wants to buy 2 units financing each unit separately by $(m_1, i_1)$ and $(m_2, i_2)$ such that $m_1 < m_2$. The agent puts down $(m_1 + m_2)$ and owes $i_1 + i_2$. If 2 units are financed by a single loan putting down $m_3 = (m_1 + m_2)/2$, the agent makes the same down payment with lower interest $i_3 < (i_1 + i_2)/2$. The corollary of Lemma 2.1 formalizes this insight.

**Corollary 2.1.** *The agent picks a single loan* $(m, i(m))$ *from the menu in Figure 2.1 for each investment level* $x$*.*

I say that the credit market are *looser*, as opposed to *tighter*, if the lenders charge lower interest $I$ and $i$ for every margin $m$. There are two lead causes of loose credit: the lenders' perception of the risk summarized by the distribution $F(S|0)$ and the risk-free rate that determines the opportunity cost of each loan in (2.5). The special case of (2.7) is a mean-preserving contraction; if $E[S|0'] = E[S|0]$ and the lenders perceive a lower spread of risky asset values, the competition drives the interest down at all margins. The same intuition works for a decrease in the risk-free rate as the the lenders break-even at smaller interest rate at every margin.

### 2.2.5 The Decision Problem(s)

The agent's wealth after settling the debt at the end of the period is:

$$xs + (w - mx)R_f - x\min(s, i(m)) = w_f + x(\max(s - i(m), 0) - mR_f) \tag{2.8}$$

Here $w_f = wR_f$ is the sure wealth. The central object of studying the margin is the function $\ell(S, m)$

$$\ell(S, m) \equiv \max(S - i(m), 0) - mR_f \tag{2.9}$$

I refer to $\ell(s, m)$ as the levered excess return on a unit of risky asset purchased at margin $m$ when the asset is worth $s \in S$. The agent defaults and hands over the risky asset to the lenders if $S < i(m)$, the agent keeps the risky asset by paying $i(m)$ if $S \geq i(m)$. The term $mR_f$ is the opportunity cost of the down payment. The structure of the payoffs resembles a call option on the risky asset with strike price $i(m)$ that costs $mR_f$ ex ante.

The agent solves

$$\max_{(x,m)\in[0,w/m]\times[\underline{m},1]} U(x, m|\theta) \equiv \int u(w_f + x\ell(s, m))f(s|\theta)\, ds \tag{2.10}$$

There is an equivalent formulation of the problem whereby the agent chooses how much money to spend on the risky asset, rather than how many to buy at the unit price. Denote the down payment on the risky asset

$$y \equiv mx \tag{2.11}$$

so that $x - y$ is the size of the loan and $w - y$ is the investment in the risk-free asset.

The levered excess return on a dollar invested in the risky asset at margin $m$ when the asset is worth $s \in S$ is

$$\lambda(S, m) \equiv \frac{1}{m}\ell(S, m) = \frac{1}{m}(\max(S - i(m), 0) - R_f \tag{2.12}$$

Notice that $\ell$ and $\lambda$ are related by the leverage ratio because a dollar invested at margin $m$ buys $1/m$ units of risky asset each earning $\max(S - i(m), 0)$ minus the opportunity cost $R_f$. Solving (2.10) is equivalent to solving

$$\max_{(y,m)\in[0,w]\times[\underline{m},1]} V(y, m|\theta) \equiv \int u(w_f + y\lambda(s, m))f(s|\theta)\, ds \tag{2.10'}$$

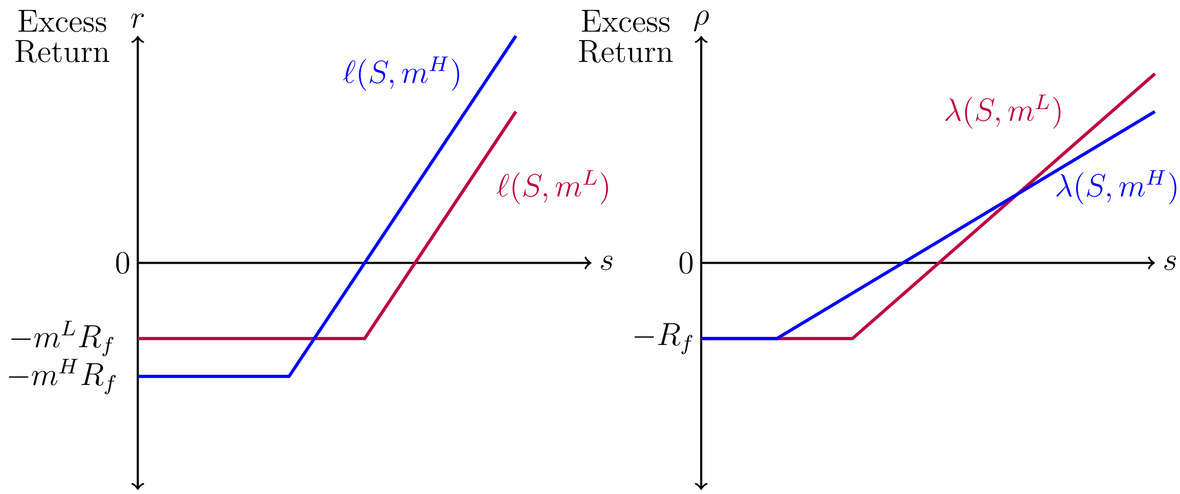
Both formulations of the agent's problem (2.10) and (2.10') collapse to the standard portfolio problem if the margin is fixed to $m = 1$.

## 2.3 Levered Excess Returns

In this section I show that choosing the margin is tantamount to choosing the distribution of excess returns. I induce a stochastic order on a subset of these distributions and show that they are never chosen at the optimum. Figure 2.2 plots $\ell$ and $\lambda$ for two different margins. Since $i(m^H) < i(m^L)$ by Lemma 2.1, the probability of default is higher for the low margin. More strongly, Lemma 2.1 proves that $i(m)+mR_f$ is decreasing in $m$. This implies that the agent investing at a lower margin requires a higher state $s$ to break even at since $i(m^L) + m^L R_f > i(m^H) + m^H R_f$.

Figure 2.2a illustrates a key insight. The agent is better off at the bad states of the world had she purchased $x$ units at a low margin. This is because if she defaults on the loan, she does not lose as much. This insight is reversed if the units of levered excess return is $y$ instead. Figure 2.2b illustrates that a more leveraged (low margin) agent is better off at the high states of the world because if she does not default, her excess return on the same $y$ dollars is bigger.

To demonstrate how the choice of margin affects the riskiness of excess returns, I




(a) Per $x$ units

(b) Per $y$ units

Figure 2.2: Levered Excess Returns for Margins $m^H > m^L$

directly compute the distribution of excess returns induced by $m$ by inverting $\ell$ and $\lambda$ in $S$ whenever $S \geq i(m)$. Denote

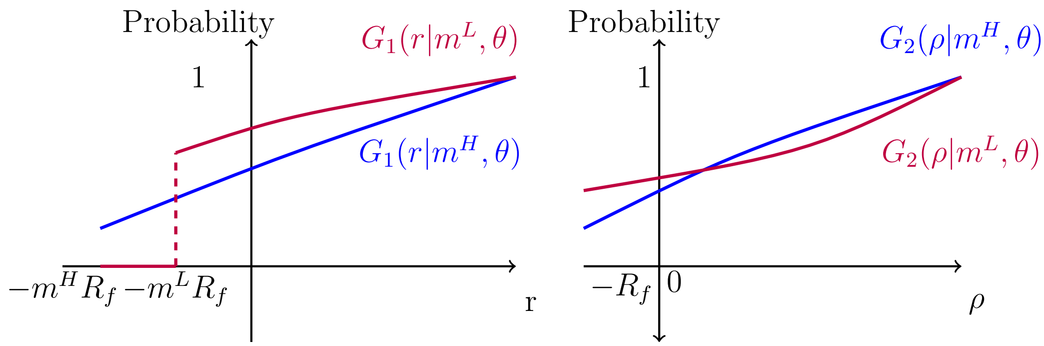
$$r \equiv \ell(s, m) \ , \ \rho = \lambda(s, m) \tag{2.13}$$

for the excess return when the state is $s$ at margin $m$. Let $G_1$ and $G_2$ be the respective distributions of $r$ and $\rho$ and $g_1$ and $g_2$ the density functions, whenever they exist. $G_1$ has an atom at $-mR_f$ with size $F(i(m)|\theta)$. Also, for all $r > -mR_f$, the distribution and its density are given by

$$G_1(r|m, \theta) = F(\ell^{-1}(r, m)|\theta) \ , \ g_1(r|m, \theta) = f(\ell^{-1}(r, m)|\theta) \tag{2.14}$$

Figure 2.3a illustrates $G_1$ for two margins $m^H > m^L$. The single crossing is flipped compared to Figure 2.2a due to the inverse operation. The distribution $G_2(\rho|m, \theta)$ is obtained by integrating $\lambda^{-1}(\rho, m)$ instead.

Single crossing distributions are related to *increase in risk* a la Diamond and Stiglitz (1974). Buying one unit of risky asset at a low margin generates a lower




(a) $G_1(r|m,\theta) = F(\ell^{-1}(r,m)|\theta)$ (b) $G_2(\rho|m,\theta) = F(\lambda^{-1}(\rho,m)|\theta)$

Figure 2.3: Distribution of Levered Returns

spread of excess returns so intuitively, it is safer. On the contrary, investing one dollar in the risky asset at a low margin generates a bigger spread of excess returns and thus, it is riskier. The next lemma computes the expected levered excess returns in margin $m$ to illustrate the compensation for bigger spreads per $x$ or $y$ units.

**Lemma 2.2.** *$E[\ell(S,m)|\theta]$ rises in the margin $m$ and singly crosses 0 from below. $E[\lambda(S,m)|\theta] \equiv 1/m \times E[\ell(S,m)|\theta]$ is unimodal in the margin with the mode $\overline{m}(\theta) \in (0,1)$ for which $E[\lambda(S,\overline{m})|\theta] > 0$.*

Figure 2.4 plots $E[\ell(S,m)|\theta]$ and $E[\lambda(S,m)|\theta]$. Lemma 2.2 has two implications. If the agent were risk-neutral, she would take the levered position in the risky asset that yields the highest expected return regardless its variance. Since the levered expected return is higher than $R_f$ and the agent does not care about risk, she would put all her money on the risky asset. This extreme case corresponds to Simsek (2013).

**Corollary 2.2** (Simsek (2013))**.** *If the utility function $u$ were linear and strictly increasing, then the unique solution to (2.10) or (2.10') is to invest at $\overline{m}(\theta)$ and to buy $x^*(\theta) = w/\overline{m}(\theta)$ units of risky asset.*

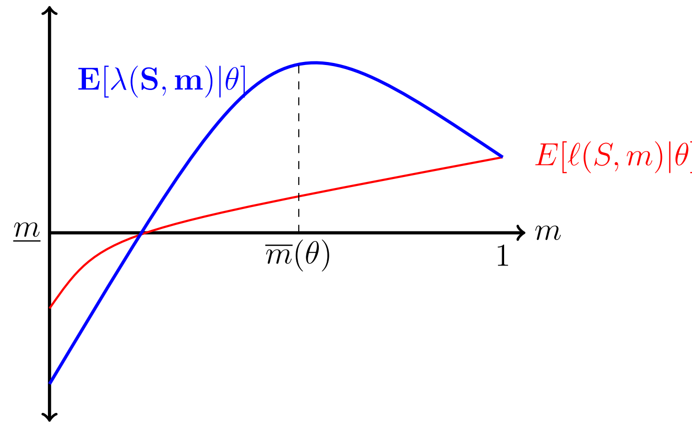



Figure 2.4: Expected Levered Excess Returns in $m$

Second, Lemma 2.2 suggests that the higher margin yields a higher expected return for each unit of risky asset to compensate the bigger spread of excess returns. For $m \geq \overline{m}(\theta)$, higher margin yields a lower spread per dollar invested and has lower expected return. For $m < \overline{m}(\theta)$ however, it is possible to Second-order stochastically rank the distributions $G_2$. A risk averse agent never invests at a margin less than $\overline{m}(\theta)$ because

$$\forall y : V(y, m^L|\theta) \leq V(y, m^H|\theta) \tag{2.15}$$

and strict inequality for some $y$. Since the solution $m^*(\theta)$ to (2.10) and (2.10') must coincide, I conclude that Simsek (2013) constitutes a lower bound on the margins and an upper bound on risky investment for a risk averse agent.

## 2.4 Solution to the Decision Problem

I characterize the solution to my decision problem (2.10) and (2.10').

**Proposition 2.1.** *There exists unique interior solutions* $(x^*(\theta), m^*(\theta))$ *and* $(y^*(\theta), m^*(\theta))$ *where* $y^*(\theta) = m^*(\theta)x^*(\theta)$.

I present this result in a sequence of lemmata. The agent has two controls $(x, m)$ or $(y, m)$ and I analyze her optimal choice of one control at a time keeping the other fixed. The goal of this approach is to illustrate the potential complementarities among controls. Intuitively, higher $x$ or $y$ implies that the agent is taking on bigger risk in her portfolio. The main point of the previous section was that whether higher margin (lower leverage) is riskier depends on the unit of choice $x$ or $y$. The general insight of this section is the following: a risk-averse agent tries to hedge the increase in risk coming from a higher value of one control by reducing the other. For example, higher level of investment in the risky asset i.e. higher $x$ induces the agent to lower the margin she is invest at. Conversely, investing at higher margin reduces the optimal level of investment. The direction of the hedging complementarity is flipped in $(y, m)$: higher $y$ induces higher $m$ which in return induces higher $y$ again.

### 2.4.1 Demand for Leverage

The optimal margin for a given $(x, \theta)$ is

$$m^*(x,\theta) \equiv \min\left(\arg \max_{m\in[\overline{m}(\theta),\min(w/x,1)]} U(x,m|\theta)\right) \tag{2.16}$$

Likewise, the optimal margin for a given $(y, \theta)$ is

$$m^{**}(y,\theta) \equiv \min\left(\arg \max_{m\in[\overline{m}(\theta),1]} V(y,m|\theta)\right) \tag{2.16'}$$

The maximization problems (2.16) and (2.16') are well-defined as $U$ and $V$ are continuous functions maximized over a compact intervals $[\overline{m}(\theta), \min(w/x, 1)]$ and $[\overline{m}(\theta), 1]$, respectively. If the agent wants to buy $x > w$, buying at the highest margin such that $mx = w$ risks default with a positive probability. The marginal utility of wealth is infinity in these states of the world because the agent has no risk-free savings. When $x \leq w$ so that the agent can afford the risky asset without leverage, it is still optimal to borrow some because of the belief disagreement. The lenders earns zero profit on the loan according to his pessimistic beliefs but the optimistic agent believes that the lenders is leaving money on the table by asking a low payment $D$. These arguments guarantee that $m^*(x,\theta)$ is never the upper bound.

The solution to (2.16) or (2.16') are not necessarily unique. If there are multiple margins the agent is indifferent to invest $x$ or spend $y$ at, I invoke a selection rule that picks the lowest margin so that the agent has the maximum risk-free asset in her portfolio. I are interested in the comparative statics in $(x,\theta)$ and $(y,\theta)$.

**Lemma 2.3.** *The optimal margin $m^*(x,\theta)$ is non-increasing in $x$ and non-decreasing in $\theta$. However, the optimal margin $m^{**}(y,\theta)$ is non-decreasing in $y$ and non-increasing in $\theta$.*

The agent wants to put down smaller fraction to buy more risky asset. Milgrom and Shannon (1994) establish that Single Crossing Property is necessary and sufficient for the solution set to rise in an appropriate set order, which guarantees that the minimum also rises. This property corresponds to the difference

$$U(x,m^L|\theta)-U(x,m^H|\theta)=-x\Big(\int_{-m^H R_f}^{\infty}u'(w_f+xr)[G_1(r|m^L,\theta)-G_1(r|m^H,\theta)]dr\Big) \tag{2.17}$$

which is obtained by integration by parts on (2.56), singly crosses in $x$. If the agent prefers the low mean-low spread distribution $G_1$ induced by $m^L$ over $m^H$ for some $x^L$, the preference is preserved for $x^H > x^L$. Intuitively, more investment is riskier and the agent hedges by switching to a safer distribution of excess returns.

In Figure 2.3a the distribution $G_1(r|m^L,\theta)$ crosses $G_1(r|m^H,\theta)$ once from below at $r=-m^L R_f$. Karlin and Rubin (1956) show that integration over $r$ carries the single crossing over $x$ if the fictional density $u'(w_f+xr)$ is log-supermodular in $(x,-r)$. I argue that the log-supermodularity premise is guaranteed by Assumption 2.1.

**Remark 2.1.** *Non-increasing absolute risk aversion and non-decreasing relative risk aversion jointly imply that $u'(w_f+xr)$ is log-supermodular in $(x,-r)$.*

Then $U(x,m|\theta)$ satisfies the Single Crossing Property in $(m,-x)$ for each $\theta$ and the minimum of the optimal solutions $m^*(x,\theta)$ falls in $x$.

The optimal margin $m^{**}(y,\theta)$ has the opposite sign in both $y$ because the single crossing in excess return distributions is flipped in Figure 2.3b. The analogue of (2.17) is

$$V(y,m^H|\theta) - V(y,m^L|\theta) = -y\Big(\int_{-R_f}^{\infty} u'(w_f + y\rho)[G_2(\rho|m^H,\theta) - G_2(\rho|m^L,\theta)]d\rho\Big) \tag{2.17'}$$

which singly crosses in $y$ by the Remark 2.1.If the agent prefers low mean-low spread distribution $G_2$ induced by $m^H$ over $m^L$ for some $y^L$, the preference is preserved for $y^H > y^L$. Spending more on the risky asset is riskier and the agent hedges the risk by switching to a safer distribution of excess returns.

A more optimistic agent buys the same $x$ at a higher margin. In Figure 2.2a the agent is better off investing at high margin in the high states of the world because she gets to keep the risky asset by paying less. Therefore, a shift in beliefs that increases the likelihood of high states of the world reinforces the high margin. Formally, for any $\theta^H > \theta^L$ :

$$U_m(x,m|\theta^L) \geq 0 \Longrightarrow U_m(x,m|\theta^H) > 0 \tag{2.18}$$

suffices to conclude that the optimal margin rises in $\theta$ keeping $x$ fixed.

However, a more optimistic agent wants to increase the leverage keeping the down payment $y$ fixed, instead of keeping $x$ fixed. In Figure 2.3b the agent is better of investing at a low margin in the high states of the world and a shift in beliefs that increases the likelihood of these states reinforces the low margin. Formally, for any $\theta^H > \theta^L$ :

$$V_m(y,m|\theta^L) \leq 0 \Longrightarrow V_m(x,m|\theta^H) < 0 \tag{2.18'}$$

suffices to conclude that the optimal margin falls in $\theta$ keeping $y$ fixed.

### 2.4.2 Demand for the Risky Asset

The optimal investment for a given $(m, \theta)$ is

$$x^*(m, \theta) \equiv \arg \max_{x \in [0, w/m]} U(x, m|\theta) \tag{2.19}$$

Likewise, the optimal down payment on the risky asset is

$$y^*(m, \theta) \equiv \arg \max_{y \in [0, w]} V(y, m|\theta) \tag{2.19'}$$

$U(x, m|\theta)$ is strictly concave in $x : U_{xx} \leq 0$ so $x^*(m, \theta)$ is well-defined and unique. When $x \to 0$, $U_x(0, m|\theta) = u'(w)E[\ell(S, m)|\theta] > 0$. When $x \to w/m$, the agent has negative infinite marginal utility at the states of the world she defaults (that have positive probability unless $m = 1$) because she has no risk-free asset. When $m = 1$, (2.10) collapses to the standard portfolio problem and is known to have a unique solution. I assume without loss of generality that $x^*(1, \theta) < w$. To conclude, $U(x, m|\theta)$ is hump-shaped in $x$ for each $(m, \theta)$ with a unique peak $x^*(m, \theta)$.

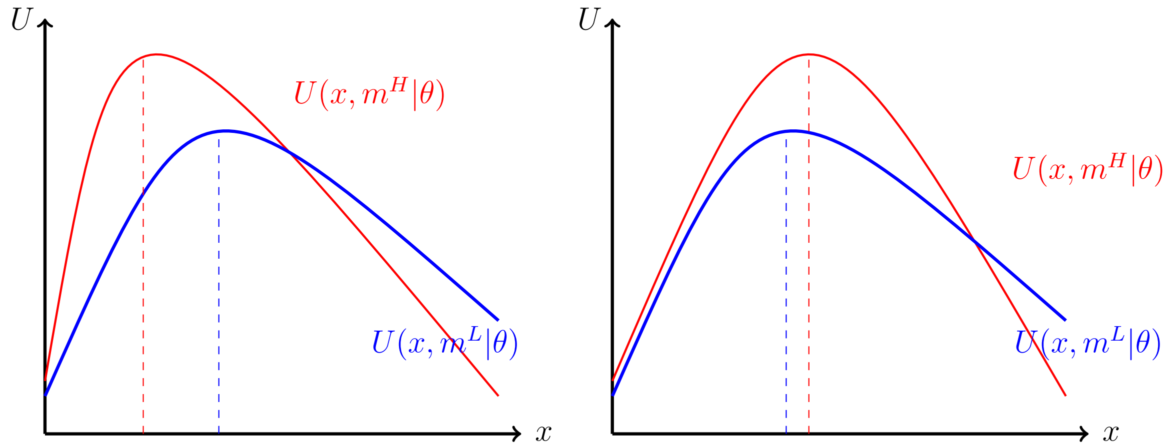
The derivatives in $y$ and $x$ are related by the identity

$$V_y(y, m|\theta) = \frac{1}{m} U_x(x, m|\theta) \tag{2.20}$$

Therefore, $y^*(m, \theta) = mx^*(m, \theta)$.

The comparative statics in $m$ and $\theta$ are similar because they both shift the distribution of excess return, albeit in different ways. The margin generates a single-crossing shift illustrated in Figure 2.3, which is weaker than monotone likelihood ratio shift that $\theta$ generates. Athey (2002) proves that the latter is not only sufficient for the demand for the risky asset to rise, but also necessary if the risk-free rate $R_f$ is arbitrary. For this reason it is natural to expect that $x^*(m, \theta)$ or $y^*(m, \theta)$ do not have monotone comparative statics for all $m$.

To circumvent this ambiguity, I filter the margins with a necessary condition for optimality. It turns out the margins that fail this necessary condition also lead to




(a) $x^*(m^H, \theta) < x^*(m^L, \theta)$ (b) $x^*(m^H, \theta) > x^*(m^L, \theta)$

Figure 2.5: Margins that Violate the Necessary Condition for Optimality (2.21)

ambiguous comparative statics of $x^*(m, \theta)$ and $y^*(m, \theta)$ in $m$.

**Remark 2.2.** *Any margin $m$ that can be a solution to (2.10) satisfies*

$$\forall m' > m : U(x^*(m, \theta), m|\theta) \geq (>) \; U(x^*(m, \theta), m'|\theta) \tag{2.21}$$

*Any margin $m$ that can be a solution to (2.10') satisfies*

$$\forall m > m'' : V(y^*(m, \theta), m|\theta) \geq (>) \; V(y^*(m, \theta), m''|\theta) \tag{2.21'}$$

Lemma 2.3 proves that hump $U(x, m^L|\theta)$ crosses the hump $U(x, m^H|\theta)$ in $x$ once from below. Figure 2.5 illustrates a margin $m^L$ that fails (2.21). If there exists a $m^H > m^L$ that guarantees a higher expected utility for the best portfolio allocation of $m^L$, then $m^L$ is never selected when $m^H$ is available. However, nothing can be said whether $x^*(m^H, \theta)$ is bigger or smaller than $x^*(m^L, \theta)$.

**Lemma 2.4.** *Unique interior $x^*(m, \theta)$ is non-increasing in $m$ for all $m$ that satisfies*

*(2.21) and non-decreasing in $\theta$. However, unique interior $y^*(m, \theta)$ is non-decreasing in both $m$ that satisfies (2.21') and $\theta$. Since $y^*(m, \theta) = mx^*(m, \theta)$, the elasticities of $x^*(m, \theta)$ and $y^*(m, \theta)$ in $m$ are both less than unity.*

For all $m$ and for any $\theta^H > \theta^L$, a likelihood ratio shift yields

$$U_x(x, m|\theta^L) \geq 0 \Longrightarrow U_x(x, m|\theta^H) > 0 \tag{2.22}$$

and this is sufficient for $x^*(m, \theta)$ to rise in $\theta$.

The complementarity insight is true for the set of margins that satisfy (2.21): the lower the margin, the higher the investment in the risky asset. Formally, I prove

$$U_x(x, m^L|\theta) \leq 0 \Longrightarrow U_x(x, m^H|\theta) < 0 \tag{2.23}$$

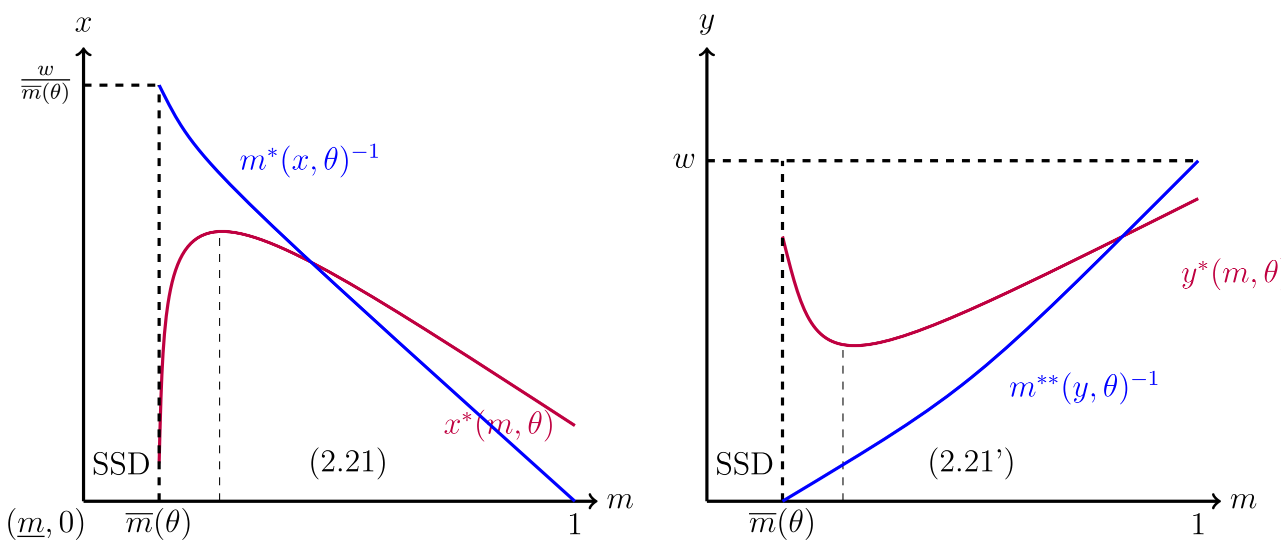
Since $U$ is hump-shaped, this condition suffices to argue that the unique peak of $U(x, m)$ in $x$ falls in $m$. Intuitively, high margin generates a riskier distribution of excess returns with high spread and high mean. The agent is risk-averse and hedges the increasing riskiness of excess returns by investing less in the risky asset.

Once more the sign in $m$ is flipped for $y^*(m, \theta)$ compared to $x^*(m, \theta)$. Since the hump $V(y, m^H|\theta)$ crosses the hump $V(y, m^L|\theta)$ once from below, if $m^L < m^H$ violate (2.21') then $m^H$ is never selected when $m^L$ is available. The lower margin generates a high spread-high mean distribution of excess returns on wealth, therefore the risk-averse agent hedges the increasing risk by spending less money on the risky asset.

This corollary exploits the sign-switch of optimal investments in $m$. One percent rise in the margin must reduce the investment in the risky asset $x^*(m, \theta)$ by less than one percent, so that the down payment on the risky asset $y^*(m, \theta)$ rises but less than one percent.

### 2.4.3 The Optimal Portfolio and the Margin

I complete the proof of Proposition 2.1 by combining the thought experiments of Lemma 2.3 and 2.4. Figure 2.6a plots $m^*(x,\theta)^{-1}$ and $x^*(m,\theta)$ on $(m,x)$ space whereby the points on the first locus satisfy $U_m(x,m|\theta) = 0$ and on the second $U_x(x,m|\theta) = 0$. Figure 2.6b does the same on $(m,y)$ space instead. The label *SSD* marks the margins ruled out by Second-order Stochastic Dominance in Corollary 2.2. $m^*(x,\theta)^{-1}$ starts above and ends below $x^*(m,\theta)$ and has to cross $x^*(m,\theta)$ at least once. Lemma 2.4 proves that the optimal margin has to occur at the decreasing segment of $x^*(m,\theta)$.




(a) Uniqueness in $(x,m)$ space

(b) Uniqueness in $(y,m)$ space

Figure 2.6: Points that Satisfy the First-Order Conditions given $\theta$

In principle the two loci can cross any odd number of times. I prove in the Appendix that the locus $m^*(x,\theta)^{-1}$ is steeper than $x^*(m,\theta)$ locus at any crossing point. That is,

$$\left|\left(m^*(x,\theta)^{-1}\right)'\right|_{(x^*(\theta),m^*(\theta))} = \frac{-U_{mm}}{-U_{xm}} \geq \frac{-U_{xm}}{-U_{xx}} = \left|x^{*\prime}(m,\theta)\right|_{(x^*(\theta),m^*(\theta))} \tag{2.24}$$

This inequality also proves the quasi-concavity of $U(x,m|\theta)$ in $(x,m)$. The Hessian matrix of (2.10) evaluated at the optimum is negative semi-definite if and only if (2.24) holds. Hence the interior solution is unique. If the solution to (2.10) is unique, so is $(y^*(\theta), m^*(\theta))$ given by $y^*(\theta) = m^*(\theta)x^*(\theta)$ as a solution to (2.10').

I conclude this section by comparing Proposition 2.1 with the solution to the standard portfolio problem which is the special case of (2.10) when $m = 1$. Figure 2.6 illustrates that at the unique optimum the agent holds a levered portfolio with more of both risky and the risk-free asset i.e. $x^*(\theta) > x^*(1,\theta)$ and $w - y^*(\theta) > w - y^*(1,\theta)$. The idea that borrowing against the asset using it as collateral increases the demand for that asset goes back to Kiyotaki and Moore (1997). I offer a different intuition based on how the risk premium, the outperformance of one unit of risky asset over risk-free return on down payment, changes with leverage. At the optimum

$$E[\ell(S, m^*(\theta))|\theta)] = -\frac{COV(\ell(S, m^*(\theta)), u'(w_f + x^*(\theta)\ell(S, m^*(\theta)))|\theta)}{E[u'(w_f + x^*(\theta)\ell(S, m^*(\theta)))|\theta]} < E[S - R_f|\theta] \tag{2.25}$$

which is obtained by re-expressing $U_x(x^*(\theta), m^*(\theta)|\theta) = 0$ and the inequality follows from Lemma 2.2 and $m^*(\theta) < 1$. The smaller the premium needed to hold the risky asset, the bigger the demand for it. Similarly, the risk premium on 1$ down payment is larger for $m^*(\theta)$ and hence lower the down payment compared to $m = 1$.

## 2.5 Comparative Statics

The main comparative statics is good news about the asset values. Despite the complementarity between the controls $(x,m)$, I cannot appeal to the multi-variable results of Milgrom and Shannon (1994). Define a partial order on $(x,m)$ space such that $(x'', m'') \geq (x', m')$ if $x'' \geq x'$ and $m'' \leq m'$. Lemma 2.3 and 2.4 imply that $U$ is *quasi-supermodular* in $(x,m)$ in a neighborhood of the optimum. Yet it does not satisfy single crossing property in $(x,m|\theta)$ with respect to this order because a more optimistic agent invests at a higher margin for any $x$. Likewise, $V(y,m|\theta)$ is quasi-supermodular in $(y, m)$ with respect to the usual order but it violates single crossing property in $(y, m|\theta)$. Despite this shortcoming I provide a monotone comparative statics result locally.

**Proposition 2.2.** *A more optimistic agent invests more in the risky asset, $x^*(\theta) \nearrow \theta$, makes a larger down payment and buys less risk-free asset, $y^*(\theta) \nearrow \theta$, and is more levered, $m^*(\theta) \searrow \theta$.*

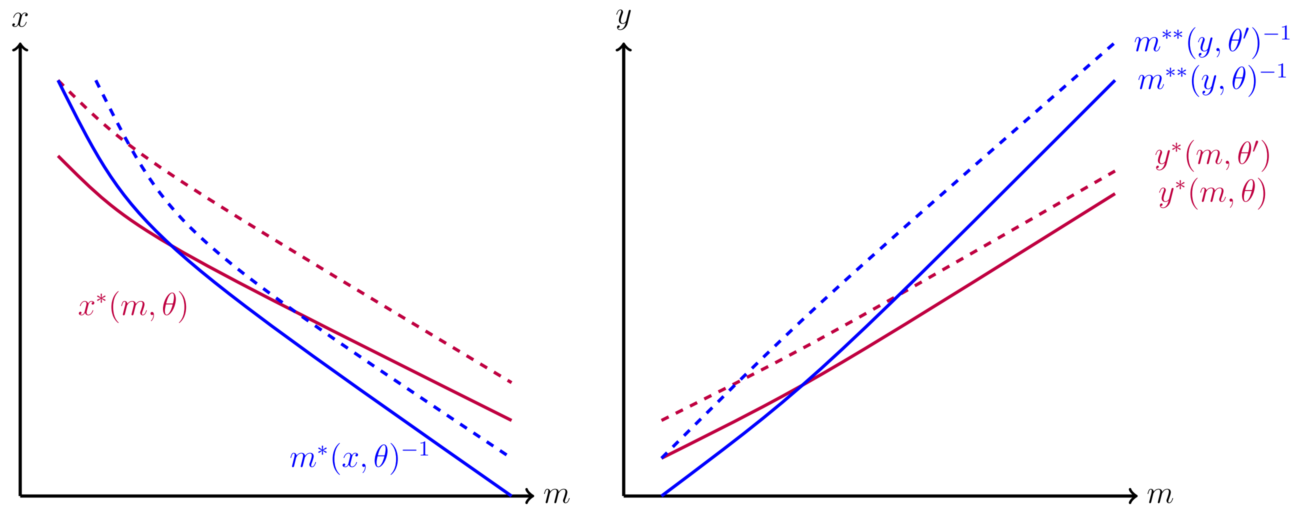
The key inequality I prove in Proposition 2.2 is

$$\frac{U_{x\theta}}{-U_{xx}} \geq \frac{U_{m\theta}}{-U_{xm}} \tag{2.26}$$

where all derivatives are evaluated at the optimum. The left-hand side of this inequality is the vertical shift of the $x^*(m, \theta)$ locus in $\theta$ at $(x^*(\theta), m^*(\theta))$ in Figure 2.6a. The right-hand side is $U_{m\theta}/-U_{mm}$, the horizontal shift of $m^*(x, \theta)^{-1}$ on $(m, x)$ space, multiplied by the slope of the locus in absolute value $-U_{mm}/-U_{xm}$ to convert the horizontal shift units into vertical shift units. Ranking vertical shifts in (2.26) lead to monotone comparative statics locally despite the lack of single crossing property. In principal I are comparing the direct positive effect of a shift in beliefs on the optimal margin against the substitution effect of higher investment. Figure 2.7a illustrates more vertical shift of $x^*(m, \theta)$ locus guarantees that the new crossing has to occur at a higher $x$ and lower $m$. Notice in Figure 2.7a that more vertical shift is always coupled with more horizontal shift, which I interpret as $x^*(m, \theta)$ being more elastic all around.

The analysis in $(m, y)$ space is similar; I prove $V_{y\theta}/-V_{yy} \leq -V_{m\theta}/-V_{ym}$ whenever (2.26) holds. This ranking suggests that $y^*(m, \theta)$ locus shifts vertically less than $m^{**}(y, \theta)^{-1}$. In Figure 2.7b illustrates that the new crossing occurs at a higher $y$ and lower $m$.

my motivating story is an household investing in real estate by using the home itself as collateral on the mortgage. Provided the household's beliefs are based on the news about the real estate values that are subject to macroeconomic fluctuations,




(a) $x^*(\theta) \nearrow \theta$ and $m^*(\theta) \searrow \theta$ (b) $y^*(\theta) \nearrow \theta$ and $m^{**}(\theta) \searrow \theta$

Figure 2.7: Comparative Statics in $\theta$

then Proposition 2.2 predicts pro-cyclical demand for new homes and loan-to-value ratios on mortgages. Common to most studies of household (mortgage) debt is the assumption that leverage ratio is determined exogenously through a collateral constraint a la Kiyotaki and Moore (1997-8). I provide a more powerful mechanism: an increase in the asset values leads to more asset purchased by pledging a smaller fraction as collateral. Decompose the total change in risky asset demand by

$$\frac{\partial x^*(\theta)}{\partial \theta} = \frac{\partial x^*(m,\theta)}{\partial m}\Big|_{m^*(\theta)} \frac{\partial m^*(\theta)}{\partial \theta} + \frac{\partial x^*(m,\theta)}{\partial \theta}\Big|_{m^*(\theta)} > 0 \tag{2.27}$$

Both terms are positive and the first captures the added amplification in asset demand due to endogenous increase in the leverage.

Proposition 2.2 has another insightful corollary. The demand for loan is given by

$$q^*(\theta) = x^*(\theta)(1 - m^*(\theta)) \tag{2.28}$$

which is the investment minus the down payment: $x^*(\theta) - y^*(\theta)$. It immediately follows from the definition that a more optimistic agent has a bigger demand for loan, $q^*(\theta) \nearrow \theta$, and pays a higher interest rate on the loan $i^*(\theta) \nearrow \theta$ as the margin gets smaller.

Not only the down payment $y^*(\theta)$, the investment $x^*(\theta)$ and the loan $q^*(\theta)$ all increase with optimism, I can rank their growth rates with respect to the same change in beliefs. Since $y^*(\theta) = m^*(\theta)x^*(\theta)$ and $m^{*\prime}(\theta) \leq 0$, the log-change in the down payment must be less than the log-change in the investment. Likewise (2.28) implies that the log-change in loan demand must be larger than that of the investment.

**Corollary 2.3.** *The elasticities of $y^*(\theta), x^*(\theta)$ and $q^*(\theta)$ satisfy:*

$$\frac{y^{*\prime}(\theta)}{y^*(\theta)} \leq \frac{x^{*\prime}(\theta)}{x^*(\theta)} \leq \frac{q^{*\prime}(\theta)}{q^*(\theta)} \tag{2.29}$$

Continuing with the interpretation of $\theta$ as a macroeconomic fluctuation in the real estate values, this corollary implies that not only the down payments, the investments and loan demand are pro-cyclical but their growth rates can be ranked. If $\theta$ itself is uncertain and governed by a stochastic process, then the corollary ranks the expected growth rates and volatilities i.e. loan demand is more volatile than the investment in the underlying asset and than the down payment on the asset.

### 2.5.1 Risk Aversion

Suppose instead I would like to compare two agents with different risk preferences indexed by a parameter $\alpha$. I say an agent $\alpha^H$ with the utility function $u(\cdot|\alpha^H)$ is less risk averse in the Arrow-Pratt sense than $\alpha^L$ if the preferences satisfy:

$$\forall W : \frac{-u''(W|\alpha^H)}{u'(W|\alpha^H)} \leq \frac{-u''(W|\alpha^L)}{u'(W|\alpha^L)} \tag{2.30}$$

or $u'(W|\alpha)$ is log-supermodular in $(W, \alpha)$. Let $x^*(\alpha)$ and $m^*(\alpha)$ denote the solutions to (2.10) indexed by preference parameter $\alpha$ rather than optimism parameter $\theta$. Nor surprisingly, the comparative statics in $\alpha$ is very similar to $\theta$.

**Proposition 2.3.** *A less risk averse agent invests more in the risky asset, $x^*(\alpha) \nearrow \alpha$, makes a larger down payment and buys less risk-free asset, $y^*(\alpha) \nearrow \alpha$, and is more levered, $m^*(\alpha) \searrow \alpha$.*

A change in risk aversion works identical to a change in beliefs because a less risk averse $\alpha^H$ agent can be thought as the same agent $\alpha^L$ who weighs the likelihood of states by the ratio of marginal utilities:

$$\forall s \in S : f(s|\theta^H) \equiv \frac{u'(w_f + x\ell(s,m)|\alpha^H)}{u'(w_f + x\ell(s,m)|\alpha^L)} f(s|\theta^L) \tag{2.31}$$

By the definition of Arrow-Pratt comparison of risk aversion, $u'(W|\alpha)$ is log-supermodular in $(W, \alpha)$ and thus the ratio of marginal utilities is rising.

### 2.5.2 Initial Wealth

Proposition 2.3 lends itself to a monotone comparative statics of initial wealth. The preferences of a wealthier agent $w^H > w^L$ can be expressed by a utility function

$$u(w_f^H + x\ell(S,m)) \equiv u(w_f^L + x\ell(S,m) + (w_f^H - w_f^L)) \tag{2.32}$$

The agent with utility function $u(W+c)$ is less risk averse than $u(W)$ for $c > 0$ if and only if $u$ satisfies non-increasing absolute risk aversion, which is a part of Assumption 2.1. Therefore, the comparative statics of initial wealth follows from Proposition 2.3.

**Proposition 2.4.** *A wealthier agent invests more in the risky asset, $x^*(w) \nearrow w$, makes a larger down payment and buys less risk-free asset, $y^*(w) \nearrow w$, and is more levered, $m^*(w) \searrow w$. Even though a wealthier agent borrows more $q^*(w) \nearrow w$, she has a smaller debt-to-income ratio: $q^*(w)/w \searrow w$.*

The first part of Proposition 2.1 is concerned with the absolute quantities and the second with the relative. A more general statement would be that whenever the preferences satisfy non-decreasing relative risk aversion, the second half of Assumption 2.1, a wealthier agent invests a smaller fraction of its wealth in the risky asset, $x^*(w)/w \searrow w$ and a larger fraction on the risk-free asset $1 - y^*(w)/w \nearrow w$. Its implication on relative indebted-ness is that the poorer the agent gets, the debt constitutes a larger fraction of her wealth. In some sense higher wealth leads to a controversially higher degree of safety due to smaller exposure of one's wealth to default risk despite bigger investment in the risky with higher leverage.

### 2.5.3 The Lenders' Perception of Risk

A change in the lenders' valuation of the risky asset is modeled as shift in their belief distribution $F(s|0)$. Lemma 2.1 proves that a Second-order Stochastically dominant shift of this distribution, a decrease in risk a la Diamond and Stiglitz (1974) leads to a loose credit market with lower interest at every margin. I are agnostic about the source of this shift; it might be good news about asset values, possibly shared by the agent, or it might reflect an implicit government guarantee in case the loan defaults. A common narrative for the causes of financial crisis of 2008 is that the mortgage lenders enjoyed implicit and explicit insurance from the government to lend against real estate. Although a change in borrowing conditions is often labeled as a credit supply shock, within the confines of my model this is a comparative statics of the agent's demand in $I$. Let $x^*(m, i)$ and $m^*(x, i)$ replace the optimal solution loci of (2.10) for a given menu such as Figure 2.1.

**Proposition 2.5.** *If there is a second-order stochastically dominance increase in the lenders' belief, then the interest rate $i$ falls, and thus the agent's investment in the risky asset $x^*(i)$, her down payment $y^*(i)$ and the leverage ratio $1/m^*(i)$ all increase.*

This is the first result that log-concavity of $F(s|\theta)$ plays a key role. To see how this comparative statics is related to the comparative statics of $\theta$, translate the decrease in $I$ for each $m$ into a shift in the distribution of excess returns. If the state

$s \leq i^H(m)$, then either the return before and after the shift is the same $-mR_f$ or the agent is better off because $i^L(m) < s$. Whenever $s \geq i^H(m)$:

$$\frac{G_1(r|m, i^L, \theta)}{G_1(r|m, i^H, \theta)} = \frac{G_1(r + i^L - i^H|m, i^H, \theta)}{G_1(r|m, i^H, \theta)} \tag{2.33}$$

where $r = \ell(s, m)$. The ratio of distributions $G_1(r - c|m, i, \theta)/G_1(r|m, i, \theta)$ is increasing in $r$ for any $c > 0$ if and only if $G_1$ is log-concave and $G_1$ is log-concave if and only if the $F$ is. Therefore, a decrease in $i$ yields a monotone probability ratio shift.

## 2.6 Conclusion

When the agent chooses both the optimal allocation and the leverage ratio of the portfolio, the belief disagreement in the credit market leads to a more leveraged portfolio with more of both risky and the risk-free assets, compared to the standard portfolio problem without default and collateral. Moreover, I have shown that both the borrower's optimism and the lender's risk perception lend themselves to more risky asset bought at lower margins. The model is agnostic about whether the shifts in these beliefs are a result of rational expectations about the economy's fundamentals, or the investors are irrationally exuberant a la Shiller (2005) about risky assets during booms, or the banks feel secure because of an anticipated government insurance a la Admati and Hellwig (2013). The positive theoretical predictions of borrower and lender-driven shocks have the identical signs, therefore the three popular narratives are not distinguishable from one another if judged by the single merit that they predict higher leverage.

# Appendix 2

*Proof of Lemma 1.* Using integration by parts, (2.5) can be expressed as

$$E[\min(S,I)|0] = \int_0^I (1 - F(s|0))\,ds = (1-m)R_f \tag{2.34}$$

The left-hand side of this equation is an increasing function of $I$ with limits 0 as $I \to 0$ and $E[S|0]$ as $I \to \infty$. The lowest margin the lenders are willing to accept is given in the text (2.6) and denoted $\underline{m}$. Fix any $m > \underline{m}$, the function $\int_0^I (1-F(s|0))\,ds$ of $I$ crosses the constant $(1-m)R_f < E[S|0]$ at a unique $i(m) < \infty$. At the limit as $m \to 1$, $I \to 0$.

The partial derivatives can be obtained by using implicit function theorem on (2.34).

$$i'(m) = -\frac{R_f}{1 - F(i(m))|0)} < 0 \tag{2.35}$$

$$I''(m) = -\frac{f(i(m)|0)}{1 - F(i(m)|0)} R_f i'(m) > 0 \tag{2.36}$$

Moreover,

$$(i(m) + mR_f)' = i'(m) + R_f = -\frac{F(i(m)|0)}{1 - F(i(m)|0)} R_f < 0 \tag{2.37}$$

which is referred to in the text as the break-even state for a levered position in the risky asset.

I now solve $\iota(m)$ as an implicit function of $m$. Expand and rearrange (2.34) to get

$$\iota - R_f = \frac{1}{1-m} \int_0^{(1-m)\iota} F(s|0)\,ds \tag{2.38}$$

Fix any $\underline{m} < m < 1$, I analyze the left and right-hand sides of (2.38) as a function of $\iota$ on $[R_f, \infty)$. Two solid lines in Figure 2.8 illustrate these functions. $\iota - R_f$ is a linear function from 0 to $\infty$ with a unit slope. Whereas the right-hand side begins strictly positive at $\iota = R_f$ and has slope $F((1-m)\iota|0) \in (0,1)$ for any $\iota < \infty$. Thus

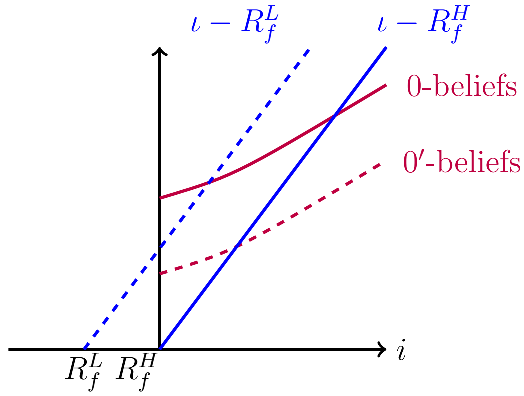



Figure 2.8: Comparative Statics of Lenders' Belief and $R_f$

there exists a unique $\iota(m) > R_f$ for any such $m$. To prove $\iota'(m) \leq 0$, differentiate the right-hand side in $m$:

$$\frac{1}{1-m}\Big[\frac{1}{1-m}\int_0^{(1-m)\iota} F(s|0)\,ds - \iota F((1-m)\iota|0)\Big] < 0 \tag{2.39}$$

The right-hand side shifts down for all values of $\iota$ for a higher $m$ while the left-hand side is not affected and therefore, the new crossing is at a lower $i$. At the limit as $m \to \underline{m}$, $I \to \infty$ and hence $\iota \to \infty$. As $m \to 1$, $I \to 0$ and using l'Hopital's Rule, $\iota \to R_f$.

The comparative statics are illustrated by dashed lines in Figure 2.8. A Second-order Stochastic dominant shift 0 to $0'$ point-wise lowers the left-hand side, therefore the new crossing occurs at a lower $\iota$. A decrease in the risk-free rate point-wise lowers the right-hand side and the new crossing occurs again at a lower $i$. Since $i(m) = \iota(1-m)$, a decrease in $\iota$ leads to a decrease in $I$. $\square$

*Proof of Lemma 2.2.* Expand the first expectation:

$$E[\ell(S,m)|\theta] = \int_{i(m)}^{\infty} (s - i(m)) f(s,\theta) ds - mR_f \tag{2.40}$$

Differentiating in $m$:

$$\partial E[\ell(S,m)|\theta]/\partial m = \int_{i(m)}^{\infty} -i'(m) f(s,\theta) ds - R_f = \Big(\frac{1-F(i(m)|\theta)}{1-F(i(m)|0)} - 1\Big) R_f > 0 \tag{2.41}$$

where $i'(m)$ is taken from Lemma 2.1 (2.35). The ratio of survival functions is always greater than unity by (2.4). As $i(m) \nearrow \infty$, I have $E[\ell(S,\underline{m})|\theta] = -\underline{m}R_f < 0$ and $\lim_{m\to 1} E[\ell(S,m)|\theta] = E[S|\theta] - R_f > 0$ by (2.4), concluding the first part of the proof.

For the second half, expand the expectation of $E[\lambda(S,m)|\theta)]$ as

$$E[\lambda(S,m)|\theta] + R_f = \frac{1}{m}\int_{i(m)}^{\infty} (s - i(m)) f(s|\theta) ds \tag{2.42}$$

Differentiating in $m$:

$$\partial E[\lambda(S,m)|\theta]/\partial m = \frac{1}{m}\Big[\frac{1-F(i(m)|\theta)}{1-F(i(m)|0)} R_f - \frac{1}{m}\int_{i(m)}^{\infty} (s - i(m)) f(s|\theta) ds\Big] \tag{2.43}$$

$$= \frac{1}{m}\Big[\Big(\frac{1-F(i(m)|\theta)}{1-F(i(m)|0)} - 1\Big) R_f - E[\lambda(S,m)|\theta]\Big] \tag{2.44}$$

The sign is determined by the term in brackets. The likelihood ratio order (2.4) implies that the first term is decreasing in its argument $I$ and non-negative. The argument $i(m)$ is a decreasing function of $m$. Therefore, the first term is a decreasing function of $m$ with lower bound zero at $m = 1$.

Consider the limits of $E[\lambda(S,m)|\theta]$. I have $E[\lambda(S,\underline{m})|\theta] = -R_f < 0$ and at $E[\lambda(S,1)|\theta] = E[S|\theta] - R_f > 0$ by (2.4). This proves that the function $E[\lambda(S,m)|\theta]$ crosses the ratio of survival functions, which decreases from some positive number to zero in $m$, at least once. Let $\overline{m}$ denote the crossing point:

$$E[\lambda(S,\overline{m})|\theta] = \Big(\frac{1-F(i(\overline{m})|\theta)}{1-F(i(\overline{m})|0)} - 1\Big) R_f > 0 \tag{2.45}$$

Applying this to (2.44):

$$E[\lambda(S,m)|\theta] \lesseqgtr \left(\frac{1-F(i(m)|\theta)}{1-F(i(m)|0)} - 1\right) R_f \Longleftrightarrow \frac{\partial E[\lambda(S,m)|\theta]}{\partial m} \gtreqless 0 \tag{2.46}$$

Figure 2.9 illustrates the unimodality of $E[\lambda(S,m)|\theta]$ at $\overline{m}$. $E[\lambda(S,m)|\theta]$ starts below below the ratio of survival functions and increases in $m$ until it crosses it at $\overline{m}$. After the cross, $E[\lambda(S,m)|\theta]$ decreases in $m$ while staying above the ratio of survival functions.

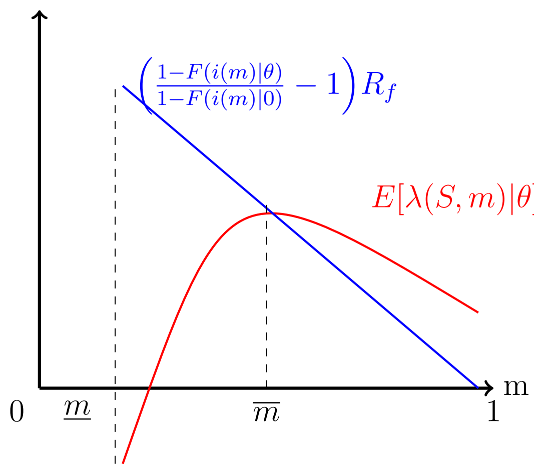



Figure 2.9: Proof of Lemma 2.2

□

*Proof of Remark 2.1.* Assumption 2.1 amounts to

$$A'(W) \leq 0 \ , \ R'(W) = A(W) + WA'(W) \geq 0 \tag{2.47}$$

Denote $r = \ell(s,m)$, the function $u'(w_f + xr)$ is log-supermodular in $(x, -r)$ if

$$\frac{\partial \ln u'(w_f + xr)}{\partial x} = \frac{ru''(w_f + xr)}{u'(w_f + xr)} = -rA(w_f + xr) \tag{2.48}$$

is decreasing in $r$. The derivative of (2.48) in $r$ is

$$
\begin{aligned}
-A(w_f + xr) - xrA'(w_f + xr) &\leq (w_f + xr)A'(w_f + xr) - xrA'(w_f + xr) \\
&= A'(w_f + xr)w_f \leq 0 \qquad (2.49)
\end{aligned}
$$

and both inequalities follows from (2.47). Log-supermodularity implies that

$$
\frac{x\ell(s,m)u''(w_f + x\ell(s,m))}{u'(w_f + x\ell(s,m))} = \frac{y\lambda(s,m)u''(w_f + y\lambda(s,m))}{u'(w_f + y\lambda(s,m))} \searrow s \qquad (2.50)
$$

$\square$

*Proof of Lemma 3.* $U(x, m|\theta)$ is a continuous and differentiable function of $m$ on a compact set $[\overline{m}(\theta), \min(w/x, 1)]$. The derivative in $m$ is:

$$
\begin{aligned}
U_m(x, m|\theta) \equiv x\Big[&\frac{F(i(m)|0)}{1 - F(i(m)|0)} R_f \int_{i(m)}^{\infty} u'(w_f + x\ell(s,m)) f(s|\theta)\, ds \\
&- R_f u'(w_f - xmR_f) F(i(m)|\theta)\Big] \qquad (2.51)
\end{aligned}
$$

By Extreme Value Theorem, there is at least one $m^*(x,\theta)$ satisfying $U_m(x, m^*(x,\theta)|\theta) = 0$ and $U_{mm}(x, m^*(x,\theta)|\theta) < 0$. To prove that the solution does not occur at the upper bound, consider separately $w \gtreqless x$ cases. If $x \geq w$, then $\lim_{m \to w/x} U_m(x, m|\theta) < 0$ as $w_f = wR_f$ and $u'(0) \to \infty$. If $x < w$, then

$$
\begin{aligned}
&\lim_{m \to 1} U_m(x, m|\theta) < \\
&\lim_{m \to 1} xR_f \Big[\frac{F(i(m)|0)}{1 - F(i(m)|0)} (1 - F(i(m)|\theta)) - F(i(m)|\theta)\Big] u'(w_f - xmR_f) = 0 \quad (2.52)
\end{aligned}
$$

Since $u'$ decreasing in $s$ by strict concavity of $u$, I bound the integral in (2.51) from above. Since $x < w \leq w/m$ and $F(s|0)/(1 - F(s|0)) \geq F(s|\theta)/(1 - F(s|\theta))$

by (2.4), the upper bound (2.52) is non-negative. $m \to 1$ implies $I(1) \to 0$ and $F(0|0) = F(0|\theta) = 0$, concluding the proof.

Consider now $m^{**}(y,\theta)$ that maximizes $V(y,m|\theta)$ in $m$. The first derivative in $m$ is

$$V_m(y,m|\theta) = x\Big(\int_{i(m)}\Big(\frac{R_f}{1-F(i(m)|0)} - \frac{s-i(m)}{m}\Big)u'(w_f + y\lambda(s,m))f(s|\theta)ds\Big) \tag{2.53}$$

By Extreme Value Theorem, there is at least one $m^{**}(y,\theta)$ satisfying $V_m(y,m^{**}(y,\theta)|\theta) = 0$ and $V_{mm}(y,m^{**}(y,\theta)|\theta) < 0$. To prove that the solution does not occur at the lower bound, note that

$$V_m(y,m|\theta) \geq xu'(w_f + y\lambda(\tilde{s},m))\int_{i(m)}\Big(\frac{R_f}{1-F(i(m)|0)} - \frac{s-i(m)}{m}\Big)f(s|\theta)\,ds \tag{2.54}$$

where $\tilde{s} = i(m) + mR_f/(1-F(i(m)|0))$ is the point the integrand singly crosses from above. The lower bound (2.54) follows from integrating the decreasing function $u'$ out at $\tilde{s}$. The expectation in (2.54) is zero as $m \to \overline{m}(\theta)$ by definition of (2.45). Therefore,

$$\lim_{m\to\overline{m}(\theta)} V_m(y,m|\theta) > 0 \tag{2.55}$$

concluding the proof.

**Result 2.1.** *$m^*(x,\theta)$ falls in $x$*

To prove the assertion, express $U(x,m|\theta)$ in (2.10) as:

$$U(x,m|\theta) \equiv u(w_f - xmR_f)G_1(-mR_f|m,\theta) + \int_{-mR_f}^{\infty} u(w_f + xr)g_1(r|m,\theta)\,dr \tag{2.56}$$

so that choosing the margin is identical to choosing the distribution of excess returns. Use integration by parts twice to obtain (2.17) in the text.

$$
\begin{aligned}
U(x, m^L|\theta) - U(x, m^H|\theta) = x\Big[ \int_{-m^L R_f}^{\infty} & u(w_f + xr)[g_1(r|m^L, \theta) - g_1(r|m^H, \theta)]dr \\
+u(w_f - xm^L R_f)G_1(-m^L R_f|m^L, \theta) & - u(w_f - xm^H R_f)G_1(-m^H R_f|m^H, \theta) \\
& - \int_{-m^H R_f}^{-m^L R_f} u(w_f + xr)g_1(r|m^H, \theta)dr\Big] \\
& = -x\Big[ \int_{-m^H R_f}^{\infty} u'(w_f + xr)[G_1(r|m^L, \theta) - G_1(r|m^H, \theta)]\Big]dr
\end{aligned} \tag{2.57}
$$

I want to show that the difference (2.57) singly crosses 0 in $x$. Since $x > 0$ the integral alone determines the sign. $G_1(r|m^L, \theta) - G_1(r|m^H, \theta)$ singly crosses 0 in $r$ as plotted in Figure 2.3a. To preserve single-crossing under integration, it suffices that $-u'(w_f + xr)$ is log-supermodular in $(x, r)$ by Karlin and Rubin (1956) and Athey (2002). Remark 2.1 guarantees the sufficient condition under Assumption 2.1.

**Result 2.2.** *$m^*(x, \theta)$ rises in $\theta$*

I prove a property similar to Quah and Stroluvici (2009). By (2.4), the likelihood ratio $f(s|\theta^H)/f(s|\theta^L)$ is rising in $s$ and $f(s|\theta^H)/f(s|\theta^L) \geq F(s|\theta^H)/F(s|\theta^L)$. Using these two inequalities

$$
\begin{aligned}
U_m(x, m|\theta^H) > x\Big[ & \frac{f(i(m)|\theta^H)}{f(i(m)|\theta^L)} \frac{F(i(m)|0)}{1 - F(i(m)|0)} R_f \int_{i(m)}^{\infty} u'(w_f + x\ell(s, m))f(s|\theta^L)\, ds \\
& - R_f u'(w_f - xmR_f) \frac{F(i(m)|\theta^H)}{F(i(m)|\theta^L)} F(i(m)|\theta^L)\Big] \\
& \geq \frac{F(i(m)|\theta^H)}{F(i(m)|\theta^L)} U_m(x, m|\theta^L)
\end{aligned} \tag{2.58}
$$

which implies that $m^*(x, \theta)$ must rise in $\theta$.

**Result 2.3.** *$m^{**}(y, \theta)$ rises in $y$ and $\theta$*

I follow identical steps to the comparative statics of $m^*(x,\theta)$ in $(x,\theta)$. The analogue of the difference (2.57) is given in the text (2.17'). $G_2(\rho|m^H,\theta)-G_2(\rho|m^L,\theta)$ singly crosses 0 in $\rho$, and the log-supermodularity of $u'(w_f+y\rho)$ in $(y,-\rho)$ implies preservation of single-crossing in $y$. Therefore, $m^{**}(y,\theta)$ rises in $y$.

The analogue of (2.58) is

$$V_m(y,m|\theta^H) < \frac{f(\tilde{s}|\theta^H)}{f(\tilde{s}|\theta^L)} V_m(y,m|\theta^L) \tag{2.59}$$

where $\tilde{s} = i(m) + mR_f/(1-F(i(m)|0))$ is the point the integrand singly crosses from above and hence the inequality is reversed. (2.59) implies that $m^{**}(y,\theta)$ falls in $\theta$. $\square$

*Proof of Remark 2.2.* Suppose that $m^L$ does not satisfy (2.21) and there exists $m^H > m^L$ such that

$$U(x^*(m^L,\theta),m^L|\theta) < U(x^*(m^L,\theta),m^H|\theta) \leq \max_x U(x,m^H|\theta) \tag{2.60}$$

Then $m^L$ is never selected when $m^H$ is available.

I now generalize (2.21) to a differentiable form. Since $U(x,m^L|\theta) - U(x,m^H|\theta)$ crosses 0 in $x>0$ for once, (2.21) can be stated instead as

$$U_x(x,m^L|\theta) \leq 0 \Longrightarrow U_m^+(x,m^L|\theta) \leq 0 \tag{2.61}$$

where $U_m^+$ denotes the right derivative at $m^L$, which is $\lim_{m^H\to m^L} U(x,m^H|\theta) - U(x,m^L|\theta)/(m^H-m^L)$ non-positive for all $x \geq x^*(m^L,\theta)$.

Consider now (2.21'). Suppose that $m^H$ does not satisfy (2.21') and there exists $m^L < m^H$ such that

$$V(y^*(m^H,\theta),m^H|\theta) < V(y^*(m^H,\theta),m^L|\theta) \leq \max_y V(y,m^L|\theta) \tag{2.62}$$

Then $m^H$ is never selected when $m^L$ is available.

I similarly generalize (2.21') to a differentiable form. For convenience of Lemma 2.4, I provide two versions. Since $V(y, m^H|\theta) - V(y, m^L|\theta)$ crosses 0 in $y > 0$ for once, and $V(0, m|\theta)$ is the same for all $m$, (2.21') can be stated first as

$$V_y(y, m^L|\theta) \geq 0 \Longrightarrow V_m^+(y, m^L|\theta) \leq 0 \tag{2.63}$$

where $V_m^+$ denotes the right derivative at $m^L$, which is $\lim_{m^H \to m^L} V(y, m^H|\theta) - V(y, m^L|\theta)/(m^H - m^L)$ non-positive for all $y \leq y^*(m^L, \theta)$.

Alternatively, (2.21') can be generalized for $y \geq y^*(m^H, \theta)$ using the same logic as

$$V_y(y, m^H|\theta) \leq 0 \Longrightarrow V_m^-(y, m^H|\theta) \geq 0 \tag{2.64}$$

where $V_m^-$ denotes the left derivative at $m^H$, which is $\lim_{m^L \to m^H} V(y, m^H|\theta) - V(y, m^L|\theta)/(m^H - m^L)$ non-negative for all $y \geq y^*(m^H, \theta)$.

□

*Proof of Lemma 4.* The derivative of (2.10) in $x$ for a given $m \in [\overline{m}(\theta), 1]$ is

$$U_x(x, m|\theta) \equiv \int_0^\infty \ell(s, m)u'(w_f + x\ell(s, m))f(s, \theta)ds \tag{2.65}$$

Evaluated at $x = 0$, $U_x(0, m|\theta) > 0$ since $E[\ell(s, m)|\theta] > 0$. As $x \to w/m$, $u'(0)F(i(m)) \to \infty$ as long as $m \neq 1$, thus $U(w/m, m|\theta) < 0$. The second derivative is

$$U_{xx}(x, m|\theta) = \int_0^\infty \ell(s, m)^2 u''(w_f + x\ell(s, m))f(s, \theta)ds < 0 \tag{2.66}$$

by strict concavity of $u$. Therefore, there is a unique interior solution $x^*(m, \theta)$ such that (2.65) is zero at this point. When $m = 1$, I assume without loss of generality that $x^*(1, \theta) \leq w$.

Consider instead choosing $y^*(m, \theta)$ that maximizes $V(y, m|\theta)$ in $y$. The derivative in $y$ is

$$V_y(y,m|\theta) = \int_0^\infty \lambda(s,m)u'(w_f + y\lambda(s,m))f(s|\theta)ds = \frac{1}{m}U_x(x,m|\theta) \tag{2.67}$$

since $\lambda(s,m) = \ell(s,m)/m$ and $y = mx$. Evaluated at $y^*(m,\theta) = mx^*(m,\theta)$, (2.67) vanishes to zero and therefore, $y^*(m,\theta)$ is a solution. Since $V_{yy} \leq 0$, this solution is unique.

**Result 2.4.** *$x^*(m,\theta)$ falls in $m$ at the optimum*

Compute the cross-partial derivative $U_{xm}$

$$\begin{aligned}U_{xm}(x,m|\theta) \equiv &\int_{i(m)}^\infty (-i'(m) - R_f)u'(w_f + x(s - i(m) - mR_f))f(s|\theta)ds \\ &+ \int_{i(m)}^\infty x(-i'(m) - R_f)(s - i(m) - mR_f)u''(w_f + x(s - i(m) - mR_f))f(s|\theta)ds \\ &- R_f u'(w_f - xmR_f)F(i(m)|\theta) - mR_f(-xR_f)u''(w_f - xmR_f)F(i(m)|\theta)\end{aligned} \tag{2.68}$$

Regroup the terms to get

$$\begin{aligned}U_{xm}(x,m|\theta) = &\frac{F(i(m)|0)}{1 - F(i(m)|0)}R_f \\ &\int_{i(m)}^\infty \Big(1 + \frac{x\ell(s,m)u''(w_f + x\ell(s,m))}{u'(w_f + x\ell(s,m))}\Big)u'(w_f + x\ell(s,m))f(s|\theta)ds \\ &- R_f u'(w_f - xmR_f)F(i(m)|\theta)\Big(1 - xmR_f\frac{u''(w_f - xmR_f)}{u'(w_f - xmR_f)}\Big)\end{aligned} \tag{2.69}$$

Since (2.50) by Remark 2.1, $U_{xm}$ is bounded above by

$$
\begin{aligned}
U_{xm}(x,m|\theta) &< \Big(1 - xmR_f\frac{u''(w_f - xmR_f)}{u'(w_f - xmR_f)}\Big)R_f\times \\
&\times \Big(\frac{F(i(m)|0)}{1-F(i(m)|0)}\int_{i(m)}^{\infty} u'(w_f + x\ell(s,m))f(s|\theta)ds - u'(w_f - xmR_f)F(i(m)|\theta)\Big) \\
&= \frac{1}{x}\Big(1 - xmR_f\frac{u''(w_f - xmR_f)}{u'(w_f - xmR_f)}\Big)U_m(x,m|\theta) \qquad (2.70)
\end{aligned}
$$

which follows from the definition of $U_m(x,m|\theta)$ in (2.51).

The proof is completed using the differentiable version of (2.21) formulated in Remark 2.2 in (2.61). Whenever $U_x(x,m^L|\theta) \leq 0$, the cross partial (2.70) is negative as $U_m(x,m^L) \leq 0$ and hence, $U_x(x,m^H) < 0$.

**Result 2.5.** *$x^*(m,\theta)$ rises in $\theta$*

Suppose that $U_x(x,m|\theta^L) \geq 0$. I have the chain of inequalities:

$$
\begin{aligned}
U_x(x,m|\theta^H) \geq & \frac{f(i(m)+mR_f)|\theta^H)}{f(i(m)+mR_f|\theta^L)}\int_{i(m)}^{\infty}(s - i(m) - mR_f)u'(w_f + x(\ell(s,m))f(s|\theta^L)ds \\
& - mR_f u'(w_f - xmR_f)\frac{F(i(m)|\theta^H)}{F(i(m)|\theta^L)}F(i(m)|\theta^L) \\
\geq & \frac{F(i(m)|\theta^H)}{F(i(m)|\theta^L)}U_x(x,m|\theta^L) \geq 0 \qquad (2.71)
\end{aligned}
$$

using the implication of (2.4) that

$$
\frac{f(i(m)+mR_f|\theta^H)}{f(i(m)+mR_f|\theta^L)} \geq \frac{f(i(m)|\theta^H)}{f(i(m)|\theta^L)} \geq \frac{F(i(m)|\theta^H)}{F(i(m)|\theta^L)} \qquad (2.72)
$$

This inequality proves that at the unique optimum $U_x(x^*(m,\theta^L),m|\theta^L) = 0$, the marginal utility is non-negative for $\theta^H$, therefore $x^*(m,\theta)$ rises in $\theta$.

**Result 2.6.** *$y^*(m,\theta)$ rises in $m$ (at the optimum) and $\theta$*

The comparative statics in $\theta$ is identical to $x^*(m,\theta)$ since $V_y$ is proportional to $U_x$ by $1/m$ which does not effect the sign in $\theta$. For the comparative statics in $m$, compute $V_{ym}$

$$V_{ym}(y,m|\theta) \equiv \frac{1}{m}\int_{i(m)}^{\infty}\Big(\frac{R_f}{1-F(i(m)|0)} - \frac{s-i(m)}{m}\Big)u'(w_f + y\lambda(s,m)) \\ \times\Big(1+\frac{y\lambda(s,m)u''(w_f+y\lambda(s,m))}{u'(w_f+y\lambda(s,m))}\Big)f(s|\theta)ds \tag{2.73}$$

Denote the unique point the first term in (2.74) crosses 0 in $s$ from above by $\tilde{s}$. By (2.50) the last term is also decreasing and crosses 0 in $s$ from above. Regardless whether $1 + y\lambda(s,m)u''(w_f + y\lambda(s,m))/u'(w_f + y\lambda(s,m))$ evaluated at $\tilde{s}$ is positive or negative, $V_{ym}(y,m|\theta)$ is bounded below by

$$V_{ym}(y,m|\theta) \geq \frac{1}{y}\Big(1+\frac{y\lambda(\tilde{s},m)u''(w_f+y\lambda(\tilde{s},m))}{u'(w_f+y\lambda(\tilde{s},m))}\Big)V_m(y,m|\theta) \tag{2.74}$$

The proof is completed using the differentiable versions of (2.21') formulated in Remark 2.2 in (2.63) and (2.64) depending on the sign of the first term. If the first term of the lower bound in (2.74) is negative, then whenever $V_y(y,m^L|\theta) \geq 0$, (2.63) implies $V_m(y,m^L) \leq 0$ and the cross partial (2.70) is positive. Therefore, $V_y(y,m^L|\theta) \geq 0$ implies $V_y(y,m^H) > 0$, concluding the proof. If instead the first term of the lower bound in (2.74) is positive, then whenever $V_y(y,m^H|\theta) \leq 0$, (2.64) implies $V_m(y,m^H) \geq 0$ and the cross partial (2.70) is positive. Therefore, $V_y(y,m^H|\theta) \leq 0$ implies $V_y(y,m^L|\theta) < 0$, concluding the proof.

□

*Proof of Proposition 1.* Any bivariate optimization problem can be solved sequentially. In Lemma 2.4 I prove that there exists a unique interior maximizer $x^*(m,\theta)$. Choose $m \in [\overline{m}(\theta), 1]$ to maximize $U(x^*(m,\theta), m|\theta)$. By the Envelope Theorem, the optimum is determined by the solution to

$$U_m(x^*(m,\theta), m|\theta) = V_m(y^*(m,\theta), m|\theta) = 0 \tag{2.75}$$

Evaluate at the lower limit $m \to \overline{m}(\theta)$

$$\lim_{m\to\overline{m}(\theta)} V_m(y^*(m,\theta), m|\theta) = \lim_{m\to\overline{m}(\theta)} U_m(x^*(m,\theta), m|\theta) > 0 \tag{2.76}$$

as Lemma 2.3 proves $\lim_{m\to\overline{m}(\theta)} V_m(y, m|\theta) > 0$ for any $y > 0$ and $y^*(\overline{m}(\theta), \theta) > 0$. Evaluate at the upper limit $m \to 1$

$$\lim_{m\to 1} U_m(x^*(m), \theta), m)|\theta) < 0 \tag{2.77}$$

as Lemma 2.3 proves $\lim_{m\to 1} U_m(x, m|\theta) < 0$ for all $mx < w$ and $x^*(1, \theta) < w$.

By continuity of $U_m$ in $m$, there must exists an interior $m^*(\theta) \in (\overline{m}(\theta), 1)$ such that the optimality condition $U_m(x^*(m^*(\theta), \theta), m^*(\theta)|\theta) = 0$. Denote $x^*(\theta) = x^*(m^*(\theta), \theta)$ and $(x^*(\theta), m^*(\theta))$ satisfies the first-order conditions of (2.10). Likewise $(y^*(\theta), m^*(\theta)) = (x^*(\theta)m^*(\theta), m^*(\theta))$ satisfies the first-order conditions of (2.10').

I prove that $U$ is quasi-concave in $(x, m)$ and hence the interior solution I found must be unique. The Hessian matrix evaluated at $(x^*(\theta), m^*(\theta))$ is

$$H \equiv \begin{vmatrix} U_{xx} & U_{xm} \\ U_{xm} & U_{mm} \end{vmatrix} \tag{2.78}$$

I aim to show that $H$ is negative semi-definite whenever $U_m = U_x = 0$. Since the first leading-minors are negative, it suffices to show that determinant satisfies:

$$\det H \equiv U_{mm}U_{xx} - U_{xm}^2 \geq 0 \tag{2.79}$$

**Claim 2.1.** *There exists a constant $K \in (0, 1]$ such that at the optimum $U_m = U_x =$*

0:

$$\frac{-U_{mm}}{-U_{xm}} \geq K\frac{x^*(\theta)}{m^*(\theta)} \geq \frac{-U_{xm}}{-U_{xx}} \tag{2.80}$$

If the claim were true, then reorganizing (2.80) implies (2.79) and concludes the proof. Now I prove Claim 2.1. The derivatives of $U$ and $V$ in $m$ are related by the identity

$$V_m(y, m|\theta) = U_m(x, m|\theta) - \frac{x}{m}U_x(x, m|\theta) \tag{2.81}$$

Evaluated at the optimum, partial derivatives of $V_m$ satisfies

$$V_{ym} = \frac{1}{m^*(\theta)}\Big(U_{xm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xx}\Big) = \frac{1}{m^*(\theta)}\psi_1 \geq 0 \tag{2.82}$$

$$V_{mm} = \Big(U_{mm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xm}\Big) - \frac{x^*(\theta)}{m^*(\theta)}\Big(U_{xm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xm}\Big) \tag{2.83}$$

$$= \psi_1 - \frac{x^*(\theta)}{m^*(\theta)}\psi_2 < 0 \tag{2.84}$$

Lemma 2.3 proves that $V_{ym} \geq 0$ and $V_{mm} < 0$ is the optimality condition for $V_m = 0$. Denote the parenthetical term in $V_{ym}$ by $\psi_1$ and first parenthetical term of $V_{mm}$ by $\psi_2$. I have

$$\psi_1 \leq \frac{x^*(\theta)}{m^*(\theta)}\psi_2$$

Since $\psi_1 \geq 0$, if $\psi_2 \leq 0$ then the proof of the claim is complete with $K = 1$.

Suppose not i.e. $\psi_2 > 0$. To prove the claim for $K < 1$ I create a family of auxiliary variables indexed by $k \in (0, 1]$ as

$$y_k = m^k x \tag{2.85}$$

where $y_1 = y = mx$. The auxiliary variable $y_k$ does not have an economic meaning but this does not prohibit maximizing a familiy of auxiliary objective functions

$V_{[k]}(y_k, m|\theta)$. Let $\lambda_k(S, m)$ denote the corresponding excess levered return on $y_k$:

$$\lambda_k(S, m) \equiv \frac{1}{m^k}\ell(S, m) \tag{2.86}$$

The agent's problem is to choose $(y_k, m)$ to solve

$$V_{[k]}(y_k, m|\theta) \equiv \max \int u(w_f + y_k\lambda_k(s, m))f(s|\theta)ds \tag{2.87}$$

where the first derivatives are given by

$$V_{[k]y_k}(y_k, m|\theta) = m^{-k}U_x(x, m|\theta) \tag{2.88}$$

$$V_{[k]m}(y_k, m|\theta) = U_m(x, m|\theta) - k\frac{x}{m}U_x(x, m|\theta) \tag{2.89}$$

Evaluated at the optimum $U_x = U_m = 0$, both first-order conditions of $V_{[k]}$ also vanishes. I prove an intermediate result to complete the proof of the claim.

**Result 2.7** (**Family of Auxiliary Controls in** $k$)**.** *The optimal solution locus* $m_k^{**}(y_k, \theta)$ *is non-increasing in* $y_k$ *and* $\theta$ *for all* $k \in (0, 1]$

All I need to show is the isomorphism with the problem studied in Lemma 2.3, which is to show that $\lambda_k(s, m^L) - \lambda_k(s, m^H)$ singly crosses 0 in $s$ and $\lambda_k(s, m)$ singly crosses 0 in $s$ for each $m$. Both premises are readily observable in Figure 2.10. When both default, the opportunity cost $-m^{-k-1}R_f$ is rising in $m$, whereas when both pay off, lower margin makes more explosive gains than high margin. Then I apply Lemma 2.3: $V_{[k]}(y_k, m^H|\theta) - V_{[k]}(y_k, m^L|\theta)$ singly crosses in $y_k$ and $V_{[k]y_k}(y_k, m|\theta^L) \geq 0$ implies $V_{[k]y_k}(y_k, m|\theta^H) \geq 0$. The two implications prove the two assertions made.

Due to this intermediate result, the partial derivatives of $V_{[k]m}$ evaluated at the

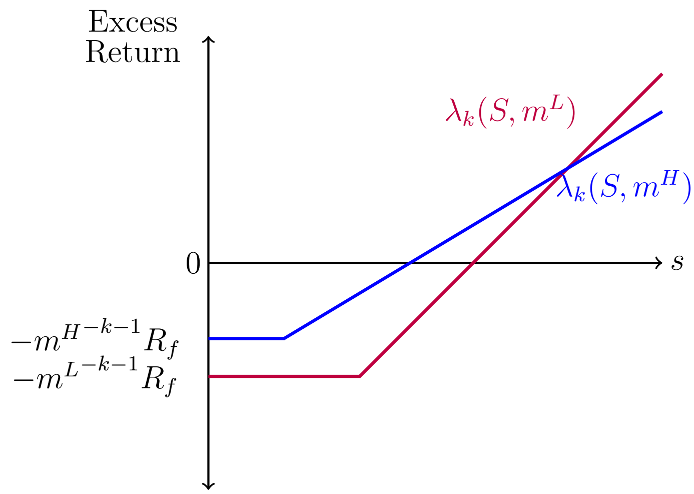



Figure 2.10: Levered Excess Returns $\lambda_k(S, m)$ for Margins $m^H > m^L$

optimum, which coincide with $(x^*(\theta), m^*(\theta))$ are given by

$$V_{[k]y_k m} = \frac{1}{m^*(\theta)}\Big(U_{xm} - k\frac{x^*(\theta)}{m^*(\theta)}U_{xx}\Big) = \frac{1}{m^*(\theta)}\psi_1(k) \geq 0 \tag{2.90}$$

$$V_{[k]mm} = \Big(U_{mm} - k\frac{x^*(\theta)}{m^*(\theta)}U_{xm}\Big) - k\frac{x^*(\theta)}{m^*(\theta)}\Big(U_{xm} - k\frac{x^*(\theta)}{m^*(\theta)}U_{xm}\Big) \tag{2.91}$$

$$= \psi_2(k) - k\frac{x^*(\theta)}{m^*(\theta)}\psi_1(k) < 0 \tag{2.92}$$

The second inequality gives

$$\psi_2(k) \leq k\frac{x^*(\theta)}{m^*(\theta)}\psi_1(k)$$

Take the limits of both sides as $k \to 0$

$$\lim_{k\to 0}\psi_2(k) = U_{mm} < 0 = \lim_{k\to 0} k\frac{x^*(\theta)}{m^*(\theta)}\psi_1(k) \tag{2.93}$$

The slope of $\psi_2(k)$ in $k$ is

$$\psi_2'(k) = -\frac{x^*(\theta)}{m^*(\theta)} U_{xm} \geq 0 \tag{2.94}$$

as $U_{xm} \leq 0$ at the optimum. Since by assumption $\psi_2(1) > 0$, there exists a unique $\bar{k} \in (0, 1)$ such that $\psi_2(\bar{k}) = 0$. Now pick any $K \in (0, \bar{k})$, I prove that

$$\psi_2(k) = U_{mm} - K\frac{x^*(\theta)}{m^*(\theta)} U_{xm} \leq 0 \Longleftrightarrow \frac{-U_{mm}}{-U_{xx}} \geq K\frac{x^*(\theta)}{m^*(\theta)} \tag{2.95}$$

and

$$\psi_1(k) = U_{xm} - K\frac{x^*(\theta)}{m^*(\theta)} U_{xx} \geq 0 \Longleftrightarrow \frac{-U_{xm}}{-U_{xx}} \leq K\frac{x^*(\theta)}{m^*(\theta)} \tag{2.96}$$

Combining the two inequalities completes the proof of Claim 2.1. □

*Proof of Proposition 2.* I prove the proposition in five steps. First I prove that $x^*(\theta) \nearrow \theta$ but $m^*(\theta)$ can rise or fall depending on an inequality relating the horizontal shift of optimal loci in Figure 2.6. At the second step I introduce a third variation of the decision problem taking $q = x(1-m)$ as the control variable replacing $x$ and $y$ and provide a similar comparative statics result: $q^*(\theta) \nearrow \theta$ and the sign of $m^{*\prime}(\theta)$ is ambiguous. Then I take an unconventional step and formulate a fictitious family of decision problems taking $q_n = x(1-m)^n$ as a control variable. I generalize the monotone comparative statics result $q_n(\theta) \nearrow \theta$ by induction. I use the implication of this result to rule out $m^{*\prime} > 0$ in the first step of the proof, resolving the ambiguity. The proof is completed by repeating the first step of the analysis on $(m, y)$ space instead with the additional knowledge of intermediate steps.

**Step 1: Analysis on $(m, x)$ Space**

Apply Implicit Function Theorem to two first-order conditions of $U(x, m|\theta)$ and obtain two equations with two unknowns

$$U_{xx}x^{*\prime}(\theta) + U_{xm}m^{*\prime}(\theta) + U_{x\theta} = 0 \tag{2.97}$$

$$U_{xm}x^{*\prime}(\theta) + U_{mm}m^{*\prime}(\theta) + U_{m\theta} = 0 \tag{2.98}$$

where all partial derivatives are evaluated at the optimum and therefore suppressed in the notation. Express $x^{*\prime}(\theta)$ as linear functions of $m^{*\prime}(\theta)$

$$x^{*\prime}(\theta) = \frac{U_{x\theta}}{-U_{xx}} - \frac{U_{xm}}{U_{xx}}m^{*\prime}(\theta) \tag{2.99}$$

$$x^{*\prime}(\theta) = \frac{U_{m\theta}}{-U_{xm}} - \frac{U_{mm}}{U_{xm}}m^{*\prime}(\theta) \tag{2.100}$$

Both lines are decreasing in $m^{*\prime}(\theta)$ since $U_{xm}, U_{mm}, U_{xx} \leq 0$. In (2.80) in Proposition 2.1 reproduced here for convenience:

$$\frac{-U_{xm}}{-U_{xx}} \leq \frac{x^*(\theta)}{m^*(\theta)} \leq \frac{-U_{mm}}{-U_{xm}} \tag{2.80}$$

This inequality implies that (2.99) is flatter than (2.100).

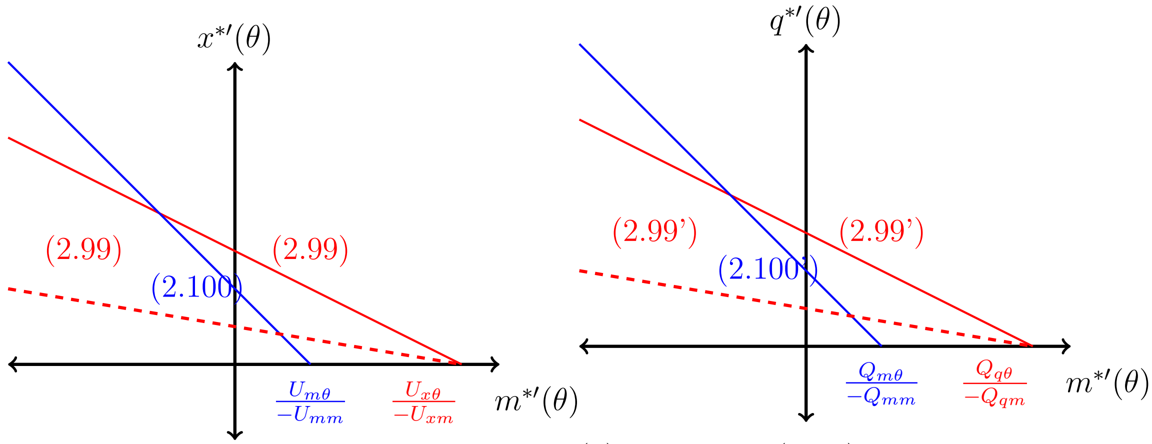
I claim that (2.99) has a bigger $m^{*\prime}(\theta)$ intercept than (2.100). Lemma 2.3 proves that $m^{**}(y,\theta)$ is non-increasing in $\theta$ and evaluated at the optimum $y^*(\theta) = x^*(\theta)m^*(\theta)$

$$V_{m\theta} = U_{m\theta} - \frac{x^*(\theta)}{m^*(\theta)}U_{x\theta} \leq 0 \Longrightarrow \frac{U_{m\theta}}{U_{x\theta}} \leq \frac{x^*(\theta)}{m^*(\theta)} \tag{2.101}$$

provided $U_{x\theta} > 0$. Combined with (2.80), I obtain

$$\frac{-U_{m\theta}}{-U_{mm}} \leq \frac{U_{x\theta}}{-U_{xm}} \tag{2.102}$$

Figure 2.11a plots (2.99) and (2.100) on $(m^{*\prime}(\theta), x^{*\prime}(\theta))$ space. The unique intersection of two lines has to occur either on the first or the second quadrant depending on whichever has the higher $x^{*\prime}(\theta)$ intercept. However, the comparison of $x^{*\prime}(\theta)$ intercepts, $U_{x\theta}/-U_{xx}$ and $U_{m\theta}/-U_{xm}$, is ambiguous. Therefore, $x^{*\prime}(\theta) \geq 0$ but the




(a) Equations (2.99) and (2.100)

(b) Equations (2.99') and (2.100')

Figure 2.11: Unambiguous Comparative Statics of $x^*(\theta)$ and $q^*(\theta)$

sign of $m^{*\prime}(\theta)$ is ambiguous.

In Figure 2.6 in the text, the inequality (2.102) implies that $x^*(m, \theta)$ shifts horizontally more than $m^*(x, \theta)^{-1}$ in $\theta$. This shift guarantees the intersection of the optimal locus has higher $x^*(\theta)$ but cannot predict $m^*(\theta)$.

**Step 2: Analysis on** $(m, q)$ **Space for** $q = 1 - m$

The variable $q = x(1 - m)$ stands for the demand for loan. The corresponding excess levered return on a unit wealth borrowed is

$$\varrho(S, m) \equiv \frac{1}{1-m}\ell(S, m) = \frac{m}{1-m}\lambda(S, m) \tag{2.103}$$

The agent's problem is to choose $(q, m) \in [0, w(1/m - 1)] \times [\overline{m}(\theta), 1]$ to solve

$$Q(q, m|\theta) \equiv \max \int u(w_f + q\varrho(s, m))f(s|\theta)ds \tag{2.104}$$

The analysis of $Q(q, m|\theta)$ is very similar to $U(x, m|\theta)$. Since $\ell(s, m^H) - \ell(s, m^L)$ singly crosses in $s$ and $1/(1 - m^H) > 1/(1 - m^L) > 0$, $\varrho(s, m^H) - \varrho(s, m^L)$ also singly crosses in $s$. Denote $q^*(m, \theta)$ and $m^{***}(x, \theta)$ the single variable solution loci to (2.104).

By following the identical steps as in Lemma 2.3, $m^{***}(q,\theta)$ is non-increasing in $q$ and non-decreasing in $\theta$ as

$$Q_{m\theta} = U_{m\theta} + U_{x\theta}\frac{q^*(\theta)}{(1-m^{***}(\theta))^2} \geq 0 \tag{2.105}$$

and $U_{m\theta}, U_{x\theta} \geq 0$ at the optimum. Likewise,

$$q^*(m,\theta) = x^*(m,\theta)(1-m) = x^*(m,\theta) - y^*(m,\theta) \tag{2.106}$$

is non-increasing in $m$ at the optimum because $x^*(m,\theta)$ is non-decreasing and $y^*(m,\theta)$ is non-decreasing in $m$. Finally, $q^*(m,\theta)$ is non-decreasing in $\theta$ as $\varrho(s,m)$ is single-crossing in $s$ for each $m$ and $\theta$ generates a likelihood ratio shift.

$m^{***}(q,\theta)^{-1}$ and $q^*(m,\theta)$ loci look similar to Figure 2.6a. The only difference on $(m,q)$ space is that both loci converge to the point $(1,0)$ eventually. At the unique intersection $m^{***}(q,\theta)^{-1}$ is steeper than $q^*(m,\theta)$, that is $-Q_{mm}/-Q_{qm} \geq -Q_{qm}/-Q_{qq}$. Apply Implicit Function Theorem to the first-order conditions of (2.104):

$$q^{*\prime}(\theta) = \frac{Q_{q\theta}}{-Q_{qq}} - \frac{Q_{qm}}{Q_{qq}}m^{***\prime}(\theta) \tag{2.99'}$$

$$q^{*\prime}(\theta) = \frac{-Q_{m\theta}}{Q_{qm}} - \frac{Q_{mm}}{Q_{qm}}m^{***\prime}(\theta) \tag{2.100'}$$

Both lines are decreasing and (2.99') is flatter than (2.100'). Suppose the $m^{***\prime}(\theta)$ intercepts satisfy: $Q_{q\theta}/-Q_{qm} < Q_{m\theta}/-Q_{mm}$. Then the unique intersection has to occur at the fourth quadrant where $q^{*\prime}(\theta) < 0, m^{***\prime}(\theta) > 0$. However, the premise implies the $q^{*\prime}(\theta)$ intercepts satisy

$$\frac{Q_{q\theta}}{-Q_{qq}} = \frac{\frac{1}{1-m^*(\theta)}U_{x\theta}}{\left(\frac{1}{1-m^*(\theta)}\right)^2 U_{xx}} > \frac{U_{m\theta} + \frac{x^*(\theta)}{1-m^*(\theta)}U_{x\theta}}{\frac{1}{1-m^*(\theta)}\left(U_{xm} + \frac{x^*(\theta)}{1-m^*(\theta)}U_{xx}\right)} = \frac{Q_{m\theta}}{-Q_{qm}} \tag{2.107}$$

This inequality is equivalent to $U_{x\theta}/-U_{xx} > U_{m\theta}/-U_{xm}$, which guarantees in Figure 2.11a that $m^{*\prime}(\theta) = m^{***\prime}(\theta) \leq 0$, a contradiction. Therefore, the plot of (2.99') and (2.100') on $(m^{***\prime}(\theta), q^{*\prime}(\theta))$ space must look like Figure 2.11b with the intersection occurring either in the first or the second quadrant where $q^{*\prime}(\theta) \geq 0$ and the sign of $m^{***\prime}(\theta)$ is ambiguous.

**Step 3: A Generalization by Induction**

Let subscript $n \in \{2, 3, ..., N\}$ denote an auxiliary variable

$$q_n(m, \theta) = q_{n-1}(m, \theta)(1 - m) = x(1 - m)^n \tag{2.108}$$

and $q_1(m, \theta) = q = x(1 - m)$ is the loan demand studied in the previous section.

Denote the corresponding excess levered return on the auxiliary variable $q_n$ as

$$\varrho_n(S, m) \equiv \frac{1}{1-m}\varrho_{n-1}(S, m) = \frac{1}{(1-m)^n}\ell(S, m) \tag{2.109}$$

and $\varrho_1(S, m)$ is defined in the previous section. For each $n$, the agent's problem is to choose $(q_n, m) \in [0, w(1 - m)^n/m] \times [\overline{m}(\theta), 1]$ to solve

$$Q_{[n]}(q_n, m|\theta) \equiv \max \int u(w_f + q_n\varrho_n(s, m))f(s|\theta)ds \tag{2.110}$$

Denote the single variable solution loci of (2.110) by $q_n^*(m, \theta)$ and $m_n^{***}(q_n, \theta)$. First I study $q_n^*(m, \theta)$ whose properties do not require induction on $n$. Since

$$Q_{[n]q_n}(q_n, m|\theta) = \frac{1}{(1-m)^n}U_x(x, m|\theta) \tag{2.111}$$

it follows that $q_n^*(m, \theta) = x^*(m, \theta)(1 - m)^n$. Therefore at the optimum, $q_n^*(m, \theta)$ is non-increasing in $m$ and non-increasing in $m$ and non-decreasing in $\theta$ because $x^*(m, \theta)$ is.

**Claim 2.2** (Induction Hypothesis). *$m_{n-1}^{***}(q_{n-1}, \theta)$ is non-increasing in $q_{n-1}$, non-decreasing in $\theta$. Comparative statics of (2.110) satisfy: $q_{n-1}^*(\theta) \nearrow \theta$ and $m_{n-1}^{***}(\theta)$*

*has an ambiguous sign in $\theta$.*

Now I study $m_n^{***}(q_n, \theta)$. If $\varrho_{n-1}(s, m^H) - \varrho_{n-1}(s, m^L)$ singly crosses in $s$, then the difference $\varrho_n(s, m^H) - \varrho_n(s, m^L)$ also singly crosses in $s$ since $1/(1 - m^H) > 1/(1 - m^L) > 0$. Lemma 2.3 proves that $m_n^{***}(q_n, \theta)$ is non-increasing in $q_n$ whenever $\varrho_n(S, m)$ is singly crosses in $(S, -m)$. $m_n^{***}(q_n, \theta)$ is non-decreasing in $\theta$ if

$$Q_{[n]m\theta} = Q_{[n-1]m\theta} + Q_{[n-1]q_{n-1}\theta} \frac{q_n^*(\theta)}{(1 - m_n^{***}(\theta))^2} \geq 0 \tag{2.112}$$

Since $Q_{[n]x\theta} \geq 0$ for all $n$, $Q_{[n]m\theta} \geq 0$ whenever $Q_{[n-1]m\theta} \geq 0$ which holds by the induction hypothesis.

In sum, I have proven that the solution to the decision problem (2.110) is identical to the one described in the previous section for $n = 1$ on the $(m, q)$ space. Apply Implicit Function Theorem to the first-order conditions of (2.110):

$$q_n^{*\prime}(\theta) = \frac{Q_{[n]q_n\theta}}{-Q_{[n]q_nq_n}} - \frac{Q_{[n]q_nm}}{Q_{[n]q_nq_n}} m_n^{***\prime}(\theta) \tag{2.99n}$$

$$q_n^{*\prime}(\theta) = \frac{-Q_{[n]m\theta}}{Q_{[n]q_nm}} - \frac{Q_{[n]mm}}{Q_{[n]q_nm}} m_n^{***\prime}(\theta) \tag{2.100n}$$

Both lines are decreasing and (2.99n) is flatter than (2.100n) because at the unique intersection of $m_n^{***}(q_n, \theta)^{-1}$ and $q_n^*(m, \theta)$, the former must be steeper than the latter i.e. $-Q_{[n]mm}/-Q_{[n]q_nm} \geq -Q_{[n]q_nm}/-Q_{[n]q_nq_n}$.

Suppose the $m_n^{***\prime}(\theta)$ intercepts satisfy: $Q_{[n]q_n\theta}/-Q_{[n]q_nm} < Q_{[n]m\theta}/-Q_{[n]mm}$. Then the unique intersection has to occur at the fourth quadrant where $q_n^{*\prime}(\theta) < 0, m_n^{***\prime}(\theta) > 0$. However, the premise implies that the $q_n^{*\prime}(\theta)$ intercepts satisfy:

$$
\begin{aligned}
\frac{Q_{[n]q_n\theta}}{-Q_{[n]q_nq_n}} &= \frac{\frac{1}{1-m^*(\theta)}Q_{[n-1]q_{n-1}\theta}}{\left(\frac{1}{1-m^*(\theta)}\right)^2 Q_{[n-1]q_{n-1}q_{n-1}}} \\
&> \frac{Q_{[n-1]m\theta} + \frac{q_n^*(\theta)}{1-m^*(\theta)}Q_{[n-1]q_{n-1}\theta}}{\frac{1}{1-m^*(\theta)}\left(Q_{[n-1]q_{n-1}m} + \frac{q_n^*(\theta)}{1-m^*(\theta)}Q_{[n-1]q_{n-1}q_{n-1}}\right)} = \frac{Q_{[n]m\theta}}{-Q_{[n]q_nm}} \qquad (2.113)
\end{aligned}
$$

This inequality is equivalent to $Q_{[n-1]q_{n-1}\theta}/-Q_{[n-1]q_{n-1}q_{n-1}} > Q_{[n-1]m\theta}/-Q_{[n-1]q_{n-1}m}$ and it guarantees that $m_{n-1}^{***\prime}(\theta) \leq 0$, a contradiction with the induction hypothesis.

Therefore, the plot of (2.99n) and (2.100n) on $(m_n^{***\prime}(\theta), q_n^{*\prime}(\theta))$ space looks similar to Figure 2.11b with the intersection occurring either in the first or the second quadrant where $q_n^{*\prime}(\theta) \geq 0$ and the sign of $m_n^{***\prime}(\theta)$ is ambiguous.

**Step 4: Proof of $m^{*\prime}(\theta) \leq 0$ by Contradiction**

I have proven that for each $n \geq 1$

$$
\begin{aligned}
q_n^{*\prime}(\theta) &= x^{*\prime}(\theta)(1-m^*(\theta))^n - x^*(\theta)n(1-m^*(\theta))^{n-1}m^{*\prime}(\theta) \geq 0 \qquad (2.114) \\
&\Rightarrow x^{*\prime}(\theta) \geq n\Big(\frac{x^*(\theta)}{1-m^*(\theta)}\Big)m^{*\prime}(\theta) \qquad (2.115)
\end{aligned}
$$

Suppose that on the $(m, x)$ space of Figure 2.11a, the vertical intercepts satisfy

$$
\frac{U_{x\theta}}{-U_{xx}} < \frac{U_{m\theta}}{-U_{xm}} \iff m^{*\prime}(\theta) > 0 \qquad (2.116)
$$

This implies that the intersection of (2.99) and (2.100) has to lie above the (2.115) for all $n \geq 1$. The contradiction is illustrated in Figure 2.12.Whatever the value of $x^*(\theta)/(1-m^*(\theta))$ is, there must exists a large enough $N \geq 1$ such that the ray originating from $(0, 0)$ with slope $Nx^*(\theta)/(1-m^*(\theta))$ lies above the intersection. Otherwise as $n \to \infty$, (2.115) implies $x^{*\prime}(\theta) \to \infty$, contradicting that the optimal solutions are finite.

**Step 5: Analysis on $(m, y)$ Space**

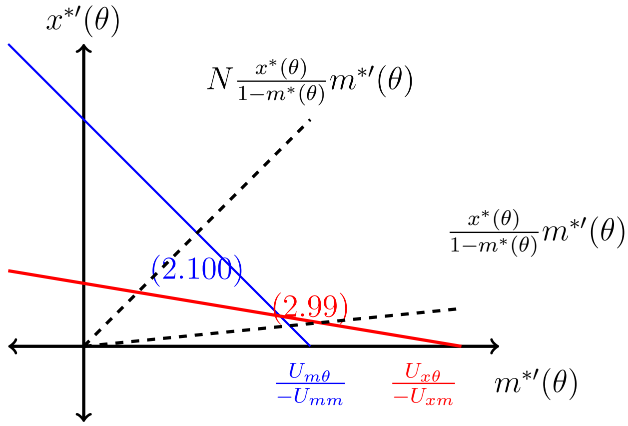



Figure 2.12: Contradiction when $m^{*\prime}(\theta) > 0$

Apply Implicit Function Theorem to two first-order conditions of $V(y, m|\theta)$ and obtain the analogues of (2.99) and (2.100)

$$y^{*\prime}(\theta) = \frac{V_{y\theta}}{-V_{yy}} - \frac{V_{ym}}{V_{yy}}m^{*\prime}(\theta) \tag{2.99”}$$

$$y^{*\prime}(\theta) = \frac{-V_{m\theta}}{V_{ym}} - \frac{V_{mm}}{V_{ym}}m^{*\prime}(\theta) \tag{2.100”}$$

Both functions are increasing as $V_{ym} \geq 0, V_{mm}, V_{yy} \leq 0$. At the crossing point of $m^{**}(y,\theta)^{-1}$ and $y^*(m,\theta)$ in Figure 2.6b, the first locus is steeper than the second, which implies $-V_{mm}/V_{ym} \geq V_{ym}/-V_{yy} \geq 0$. Therefore, (2.99”) is flatter than (2.100”).

The $m^{*\prime}(\theta)$ intercepts are negative and satisfy

$$\frac{V_{m\theta}}{-V_{mm}} = \frac{U_{m\theta} - \frac{x^*(\theta)}{m^*(\theta)}U_{x\theta}}{-(U_{mm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xm}) + \frac{x^*(\theta)}{m^*(\theta)}(U_{xm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xx})} \tag{2.117}$$

$$\geq \frac{-\frac{1}{m^*(\theta)}U_{m\theta}}{\frac{1}{m^*(\theta)}(U_{xm} - \frac{x^*(\theta)}{m^*(\theta)}U_{xx})} = \frac{-V_{y\theta}}{V_{ym}} \tag{2.118}$$

since $V_{m\theta}, V_{mm} \leq 0, V_{ym} \geq 0$ and $U_{mm} - x^*(\theta)/m^*(\theta)U_{xm} \leq 0$ by (2.80).

The $y^{*\prime}(\theta)$ intercepts satisfy

$$\frac{V_{y\theta}}{-V_{yy}} \leq (>) \frac{-V_{m\theta}}{V_{ym}} \iff \frac{U_{x\theta}}{-U_{xx}} \geq (<) \frac{U_{m\theta}}{-U_{xm}} \tag{2.119}$$

I have proven that (2.116) leads to a contradiction in Step 4, therefore the $V_{y\theta}/-V_{yy} \leq -V_{m\theta}/V_{ym}$.

Consider the plot of (2.99") and (2.100") on $(m^{*\prime}(\theta), y^{*\prime}(\theta))$ space. Two increasing lines originate from the second quadrant, satisfying the inequalities relating their intercepts must cross at the second quadrant where $y^{*\prime}(\theta) \geq 0$ and $m^{*\prime}(\theta) \leq 0$.

□

*Proof of Proposition 2.3.* I prove that this comparative statics problem is isomorphic to Proposition 2.2 by proving that the comparative statics of all four optimal loci have identical signs in $\alpha$ and in $\theta$. Then all partial derivatives in $\theta$ in Proposition 2.2 can be replaced by partial derivatives in $\alpha$.

**Step 1: $x^*(m,\alpha)$ and $y^*(m,\alpha)$ rises in $\alpha$**

Suppose that $U_x(x, m|\alpha^L) \geq 0$. Since the ratio of marginal utilities are rising: $u'(W|\alpha^H)/u'(W|\alpha^L) \nearrow W$, I have a chain of inequalities:

$$\begin{aligned} U_x(x, m|\alpha^H) \geq & \frac{u'(w_f|\alpha^H)}{u'(w_f|\alpha^L)} \int_{i(m)}^{\infty} \ell(s,m) u'(w_f + x\ell(s,m)|\alpha^L) f(s|\theta) ds \qquad (2.120)\\ & - mR_f u'(w_f - xmR_f|\alpha^L) \frac{u'(w_f - xmR_f|\alpha^H)}{u'(w_f - xmR_f|\alpha^L)} F(i(m)|\theta) \\ \geq & \frac{u'(w_f - xmR_f|\alpha^H)}{u'(w_f - xmR_f|\alpha^L)} U_x(x, m|\alpha^L) \geq 0 \qquad (2.121) \end{aligned}$$

This inequality proves that at the unique optimum $U_x(x^*(m,\alpha^L), m|\theta^L) = 0$, the marginal utility is non-negative for $\alpha^H$, therefore $x^*(m,\alpha)$ rises in $\alpha$. Since $y^*(m,\alpha) = mx^*(m,\alpha)$, the comparative statics of $y^*(m,\alpha)$ in $\alpha$ is identical to the proof above.

**Step 2:** $m^*(x,\alpha)$ **rises in** $\alpha$

Following identical steps as above, I get a chain of inequalities:

$$
\begin{aligned}
U_m(x,m|\alpha^H) \geq x\Big[&\frac{u'(w_f - xmR_f|\alpha^H)}{u'(w_f - xmR_f|\alpha^L)}\frac{F(i(m)|0)}{1-F(i(m)|0)}R_f\int_{i(m)}^{\infty} u'(w_f + x\ell(s,m)|\alpha^L)f(s|\theta)\,ds \\
&- R_f u'(w_f - xmR_f|\alpha^L)\frac{u'(w_f - xmR_f|\alpha^H)}{u'(w_f - xmR_f|\alpha^L)}F(i(m)|\theta)\Big] \\
&\geq \frac{u'(w_f - xmR_f|\alpha^H)}{u'(w_f - xmR_f|\alpha^L)}U_m(x,m|\alpha^L)
\end{aligned} \tag{2.122}
$$

This inequality proves that whenever $U_m(x, m^*(x,\alpha^L)|\alpha^L) = 0$, the marginal utility is non-negative for $\alpha^H$, therefore $m^*(x,\alpha)$ rises in $\alpha$.

**Step 3:** $m^{**}(y,\alpha)$ **falls in** $\alpha$

$V_m$ is given in (2.53). Denote the point the integrand singly crosses from above by $\tilde{s} = i(m) + mR_f/(1-F(i(m)|0))$. Then the analogue of the inequality above is reversed by

$$
V_m(y,m|\alpha^H) < \frac{u'(w_f + y\lambda(\tilde{s},m)|\alpha^H)}{u'(w_f + y\lambda(\tilde{s},m)|\alpha^L)}V_m(y,m|\alpha^L) \tag{2.123}
$$

This inequality proves that whenever $V_m(y, m^{**}(y,\alpha^L)|\alpha^L) = 0$, the marginal utility is non-positive for $\alpha^H$, therefore $m^{**}(y,\alpha)$ falls in $\alpha$.

□

*Proof of Proposition 2.4.* For the first half of the proof, I show that this comparative statics problem is isomorphic to that of risk aversion. Following (2.32), define

$$
u'(w_f^L + x\ell(s,m)|\alpha^H) = u'(w_f^L + x\ell(s,m) + (w_f^H - w_f^L)) \tag{2.32'}
$$

and let $c = w_f^H - w_f^L > 0$. The ratio of high wealth and low wealth marginal utilities is rising whenever $u'$ satisfies non-increasing absolute risk aversion i.e. $u'(W)$ is log-convex:

$$\frac{u'(w_f^H + x\ell(s,m))}{u'(w_f^L + x\ell(s,m))} \nearrow s \Longleftrightarrow \frac{u'(W+c)}{u'(W)} \nearrow W \Longleftrightarrow \frac{u'(w_f^L + x\ell(s,m)|\alpha^H)}{u'(w_f^L + x\ell(s,m)|\alpha^L)} \nearrow s \tag{2.124}$$

Given this equivalence the comparative statics of $w$ follows from Proposition 2.3.

For the second half of the proof concerning the relative indebtedness $q^*(w)/w$, refine the auxiliary objective function $Q(q, m|\theta)$ in (2.104) by defining

$$\bar{q} = \frac{q}{w} \tag{2.125}$$

as the debt-to-income ratio. The agent chooses $(\bar{q}, m)$ to maximize

$$\bar{Q}(\bar{q}, m|w) \equiv \max \int u\Big(w(R_f + \bar{q}\varrho(s,m))\Big) f(s|\theta) ds \tag{2.126}$$

Clear from the definition of (2.126) that $\bar{q}^*(m,w) = q^*(m,w)/w$, thus non-increasing in $m$ at the optimum since $q^*(m,w)$ is. To prove that $\bar{q}(m,w)$ is non-increasing in $w$

$$\begin{aligned}
&\bar{Q}_{\bar{q}}(\bar{q}^*(m,w), m|w) = w\Big[\int \varrho(s,m) u'\Big(w(R_f + \bar{q}\varrho(s,m))\Big) f(s|\theta) ds\Big] = 0 \\
&\Rightarrow\; Q_{\bar{q}w} = w\Big[\int \varrho(s,m)(R_f + \bar{q}\varrho(s,m) u''\Big(w(R_f + \bar{q}\varrho(s,m))\Big) f(s|\theta) ds\Big] \\
&\Big[\int \varrho(s,m) w(R_f + \bar{q}\varrho(s,m)) \frac{u''\Big(w(R_f + \bar{q}\varrho(s,m))\Big)}{u'\Big(w(R_f + \bar{q}\varrho(s,m))\Big)} u'\Big(w(R_f + \bar{q}\varrho(s,m))\Big) f(s|\theta) ds\Big]
\end{aligned} \tag{2.127}$$

$$\leq \frac{w_f u''(w_f)}{u'(w_f)} \bar{Q}(\bar{q}^*(m,w), m|w) = 0 \tag{2.128}$$

The equation (2.127) follows from non-decreasing relative risk aversion which is a part of Assumption 2.1 and the fact that $\varrho(s,m)$ singly crosses in $s$ for each $m$.

This chain of inequalities prove that $\bar{q}^*(m,w)$ falls in $w$ for each $m$.

The last desired information is the sign of $m^{****}(\bar{q},w)$ in $w$ at the optimum. Note that

$$\bar{Q}_m(\bar{q},m|w) = Q_m(q/w,m|w) \tag{2.129}$$

$$\bar{Q}_{mw}(\bar{q},m|w) = Q_{mq}\frac{-q}{w^2} + Q_{mw} \geq 0 \tag{2.130}$$

The inequality follows from $Q_{mq} \leq 0$ at the optimum and $m^{***}(q,w)$ is non-decreasing in $w$ for fixed $q$ since the comparative statics of $w$ is identical to $\theta$ and $\alpha$.

I complete the proof by the following observation:

$$\Big(\frac{q^*(w)}{w}\Big)' = (\bar{q}^*(m^{***}(w),w))' = \bar{q}^*_m m^{****}_w + \bar{q}^*_w \leq 0 \tag{2.131}$$

both terms are non-positive. □

*Proof of Proposition 2.5.* Similar to Proposition 2.3, the four solution loci have the identical sign structure in $i$.

**Step 1: $x^*(m,i)$ and $y^*(m,i)$ falls in $i$**

The first derivative in $x$ is given by (2.65) in Lemma 2.4. The cross-partial in $i$ is given by

$$U_{xi}(x,m|\theta) \equiv \int_i^\infty \Big( - u'(w_f + x\ell(s,m)) - x\ell(s,m)u''(w_f + x\ell(s,m))\Big) f(s|\theta)ds \tag{2.132}$$

To relate this cross-partial to $U_x$, I re-express (2.65) by integrating it by parts.

$$U_x(x,m|\theta) \equiv \int_i^\infty \Big(-u'(w_f+x\ell(s,m))-x\ell(s,m)u''(w_f+x\ell(s,m))\Big) F(s|\theta)ds \tag{2.133}$$

Denote the common term in parenthesis by

$$\upsilon_1(s,x,m) = \Big(-1 - \frac{x\ell(s,m)u''(w_f + x\ell(s,m))}{u'(w_f + x\ell(s,m))}\Big)u'(w_f + x\ell(s,m)) \tag{2.134}$$

I prove that $\upsilon_1(s,x,m)$ singly crosses in $s$ for each $(x,m)$. The sign is determined by the first term as $u' \geq 0$ for any finite $s$. As $s \to i(m)$, $\ell(i(m),m) = -mR_f < 0$ and $u'' < 0$ so $\upsilon_1(i(m),x,m) < 0$. Since (2.50) by Remark 2.1, $x\ell(s,m)u''(w_f + x\ell(s,m))/u'(w_f+x\ell(s,m))$ is decreasing in $s$. Therefore, the first term starts negative and increases in $s$. if it never crosses 0 in $s$, then $U_{xi} \leq 0$ since $f(s|\theta)$ is non-negative. Otherwise it singly crosses in $s$.

Log-concavity of $F$ implies that the reverse hazard rate $f(s|\theta)/F(s|\theta)$ is a decreasing function of $s$. I exploit these two properties to obtain:

$$U_{xi}(x,m|\theta) = \int_i^\infty \upsilon_1(s,x,m)\frac{f(s|\theta)}{F(s|\theta)}F(s|\theta)ds \leq \frac{f(\tilde{s}|\theta)}{F(\tilde{s}|\theta)}U_x(x,m|\theta) \tag{2.135}$$

where $\tilde{s}$ is the point $\upsilon_1(s,x,m)$ is zero. This inequality proves that

$$U_x(x,m|\theta) \leq 0 \Longrightarrow U_{xi}(x,m|\theta) \leq 0 \tag{2.136}$$

completing the proof of $x^*(m,i) \searrow i$. Since $y^*(m,i) = mx^*(m,i)$, $y^*(m,i)$ also falls in $i$.

Now I turn onto $m^*(x,i)$. Here the sign in $i$ is ambiguous; it is possible to show that $U_{mi}$ is bounded above by a positive function and below by a negative function whenever $U_m = 0$. I consider two cases

**Case 2.1.** $U_x = U_m = 0 \Longrightarrow U_{mi}(x,m|\theta) \geq 0$

If this is the case, then the final result can be illustrated on Figure 2.6a directly. A downward shift of $x^*(m,i)$ and an upward shift of $m^*(x,i)$ individually reinforce higher $x^*(i)$ and lower $m^*(i)$.

**Case 2.2.** $U_x = U_m = 0 \Longrightarrow U_{mi}(x,m|\theta) < 0$

In this case I complete the proof by establishing the isomorphism between the comparative statics of $i$ and $-\theta$, and apply Proposition 2.2. The only unstudied loci is $m^{**}(y, i)$.

**Step 3:** $m^{**}(y, i)$ **rises in** $i$

The first derivative in $m$ is given by (2.53) in Lemma 2.3. The cross-partial in $i$ is given by

$$V_{mi}(y, m|\theta) \equiv x\Big[-\frac{f(i|\theta)}{1-F(i|0)}u'(w_f - yR_f) + \frac{f(i|0)}{(1-F(i|0))^2}\int_i^\infty u'(w_f + y\lambda(s,m))f(s|\theta)ds$$
$$\int_i^\infty \frac{1}{m}\Big(u'(w_f + y\lambda(s,m)) - y\Big(\frac{R_f}{1-F(i|0)} - \frac{s-i}{m}\Big)u''(w_f + y\lambda(s,m))\Big)f(s|\theta)ds\Big] \tag{2.137}$$

To relate this cross-partial to $V_m$, I re-express (2.53) by integrating it by parts.

$$V_m(y, m|\theta) \equiv x\Big[-\frac{F(i|\theta)}{1-F(i|0)}u'(w_f - yR_f) +$$
$$\int_i^\infty \frac{1}{m}\Big(u'(w_f + y\lambda(s,m)) - y\Big(\frac{R_f}{1-F(i|0)} - \frac{s-i}{m}\Big)u''(w_f + y\lambda(s,m))\Big)F(s|\theta)ds\Big] \tag{2.138}$$

Reorganize the common integrand term in (2.137) and (2.138) as

$$\upsilon_2(s, x, m) = \frac{u'(w_f + y\lambda(s,m))}{m}$$
$$\times\Big(1 + \frac{y\lambda(s,m)u''(w_f + y\lambda(s,m))}{u'(w_f + y\lambda(s,m))} + \frac{F(i|0)}{1-F(i|0)}R_f A(w_f + y\lambda(s,m))\Big) \tag{2.139}$$

where $A$ denotes the coefficient of absolute risk aversion.

I prove that $\upsilon_2(s, x, m)$ singly crosses in $s$ from above for each $(x, m)$. The sign is determined by the parenthetical term as $u' \geq 0$ for any finite $s$. As $s \to i(m)$, $\lambda(i(m), m) = -R_f < 0$, $u'' < 0$ and $A > 0$ so $\upsilon_2(i(m), x, m) > 0$. Since (2.50) by Remark 2.1 $y\lambda(s,m)u''(w_f + y\lambda(s,m))/u'(w_f + y\lambda(s,m))$ is decreasing in $s$. Moreover, $A$ is decreasing in $s$ by Assumption 2.1. Therefore, the parenthetical term starts positive and decreases in $s$. if it never crosses 0 in $s$, then $U_{xi} \leq 0$ since $f(s|\theta)$ is non-negative. Otherwise it singly crosses in $s$.

Log-concavity of $F$ implies that the reverse hazard rate $f(s|\theta)/F(s|\theta)$ is a decreasing function of $s$. I exploit these two properties to obtain:

$$V_{mi}(y, m|\theta) \geq \frac{f(\tilde{s}|\theta)}{F(\tilde{s}|\theta)} V_m(y, m|\theta) + \frac{f(i|0)}{(1 - F(i|0))^2} \int_i^\infty u'(w_f + y\lambda(s, m)) f(s|\theta) ds \tag{2.140}$$

where $\tilde{s}$ is the point $v_2(s, x, m)$ is zero. Since the second term is always positive, this inequality proves that

$$V_m(y, m|\theta) \geq 0 \Longrightarrow V_{mi}(x, m|\theta) > 0 \tag{2.141}$$

completing the proof of $m^{**}(y, i) \nearrow i$.

$\square$

# Chapter 3

# A General Equilibrium Model of Leverage[1]

## 3.1 Introduction

In the decade running up to the 2008 crisis there was a rapid growth in leverage (the ratio of assets to equity) both in the household and in the banking sector which supplies the mortgages that make up the bulk of the household debt. If the household and bank balance sheets expanded similarly prior to the recession, is there a structural relationship governing leverage across the two sectors? This chapter introduces a general equilibrium model of credit markets based on belief disagreements between lenders and borrowers to seek insights into this question. At the epicenter of this theory is the observation that banks not only lend mortgages to the households, but also borrow from them in the form of deposits. The equilibrium of this model features a risk-free interest rate on deposits and two leverage ratios.

A positive shock to asset values raises the equilibrium interest rate and the household leverage. Compared to the partial equilibrium model of Chapter 2, I find that the general equilibrium effect of rising, i.e. pro-cyclical interest rates amplifies the

[1]This chapter presents joint work with Lones A. Smith

counter-cyclicality of equilibrium loan margins, the down payment on a unit invest, but curbs the pro-cyclicality of debt-to-income ratio. Similar to Chapter 2, risk-averse households invest their savings in a portfolio of one risky asset and safe deposits in banks, and decide how much down payment to make on the risky asset taking as given the banks' valuation of the collateral as given. As long as banking is profitable, competitive lenders with limited equity capital enter the credit market to raise deposits and lend them back to the households. I summarize the equilibrium of this model by a standard supply and demand curve for equity in the deposit interest rate. The demand for equity capital is motivated by *deposit insurance*.

My theory illuminates a novel role the interest on deposits play in transmitting a positive shock to asset values. The households' elevated optimism increases the loan demand to buy more risky asset while putting down less at the same time, competing lenders push the interest on deposits up to create more funds to meet this new demand. The interest rate on deposits is the opportunity cost of down payment for the borrower; the higher the opportunity cost, even lower the incentive to make a down payment. I find that the loan demand is more elastic in asset values but less elastic in interest rate compared to the deposit demand. Therefore, the marginal entrant needs more equity capital to bridge the widening gap between the loan demand she is facing and the deposits she can collect at the prevailing interest rate. At the new equilibrium the lender shifts her portfolio more towards risky loans and less towards treasury bills, but she is less leveraged.

The empirical evidence suggests that leverage in both sectors is pro-cyclical. On the household balance sheet front, Mian and Sufi (2011), Mian et al. (2013) and Justiniano et al. (2014) report an unprecedented rise in mortgage-to-income ratios during 2001-2007. Mian and Sufi (2010) furthers the claim that household leverage is an early statistical predictor of the recession. The main result of this chapter is consistent with these findings, however agnostic about the popular *cheap credit* narrative. The positive shock to asset values might be according to the households' or banks' belief and may not be *rationally* governed by fundamentals, perhaps due to a behavioral bias or implicit government subsidies. All stories lead to the same positive prediction of a credit-fueled expansion into risky assets.

A key assumption of my model is that the deposit insurance is correctly priced; the banks pay the full insurance fee on the deposits they raise without the possibility of a government subsidy. This assumption contributes to a rather counter-factual prediction that debt-to-equity ratio falls with the business cycle: Adrian and Shin (2010, 2014) find that almost all the growth in bank balance sheet comes from the changes in debt. Suppose the interest rate rises just enough to offset the fall in deposit demand due to households' optimism. Since the bank's value of debt has gone up, insuring them is costlier and the bank has less funds to lend out. However, the loan demand has gone up due to the difference in elasticities. The consequence is more equity entering the market and nudging the interest rate down, leading to more equity and less deposits in the bank balance sheet. This line argument hints at *mispriced* deposit insurance increases the bank's leverage, yet it remains ambiguous whether it can explain pro-cyclical leverage.

The chapter is organized as follows. Section 1 lays out the preliminaries, Section 2 studies how the household portfolio problem behaves in the deposit interest rates that might arise at the equilibrium. Section 3 characterizes the unique equilibrium and Section 4 presents the comparative statics. Section 5 concludes the chapter.

## 3.2 Model

There are two types of identical agents in the economy: a unit mass of borrowers (households) and a large number of potential lenders (banks). There is only one period. All decisions are made at the beginning and the payoffs are realized at the end. All agents derive utility from the terminal wealth $W$ and negative wealth is not allowed. The borrowers' preferences on terminal wealth are modeled by a exponential utility function

$$u(W) \equiv -e^{-\alpha W} \tag{3.1}$$

with a constant absolute risk aversion coefficient $\alpha$ and increasing relative risk aversion. Lenders are risk neutral.

Each borrower is endowed with initial wealth $w > 0$. She allocates her initial wealth between one risky and one risk-free asset. Neither of these assets can be sold short. Let $x$ denote the investment in the risky asset whose price is normalized to unity. Each unit of this asset yields a dollar payoff $s \geq 0$ at the end of the period. This is the source of all uncertainty in the model, so ex ante is denoted $S$.

The borrowers and lenders have different subjective beliefs about the distribution of $S$ and they agree to disagree about it. The differences in beliefs are governed by two parameters $\theta$ and $\sigma$. Borrower's optimism measure ($\theta$) exceeds the lender's (0), and $\sigma$ denotes the lender's measure of risk. Let two continuous and differentiable probability density functions $f(s|\theta), f(s|0,\sigma)$ represent the subjective beliefs on full support $s \in [0,\infty)$. $F(s|\theta)$ and $F(s|0,\sigma)$ denote their cumulative distribution functions and I assume that they are *log-concave* functions of $s$.

I formalize optimism and risk measures by stochastic orders in $\theta$ and $\sigma$ respectively. A borrower with $\theta^H > \theta^L$ is more optimistic if her beliefs satisfy *monotone likelihood ratio property*:

$$\frac{f(s|\theta^H)}{f(s|\theta^L)} \nearrow s \tag{3.2}$$

or $f(s|\theta)$ is log-supermodular in $(s,\theta)$. The lender with $\sigma^H > \sigma^L$ deems $S$ riskier if her beliefs are *Second-order stochastically ordered* in $\sigma$:

$$\forall \bar{s} \geq 0 : \int_0^{\bar{s}} F(s|0,\sigma^L)ds \leq \int_0^{\bar{s}} F(s|0,\sigma^H)ds \tag{3.3}$$

Unless the comparative statics of $\sigma$ is studied explicitly, I simply write $f(s|0)$ and $F(s|0)$ to represent the lender's belief.

Potential lenders decide whether to open a bank in this economy. Each new lender brings equity capital. Let $\kappa \in [0,\infty)$ denote the mass of equity capital in the market, which is determined at the equilibrium. Its supply reflects the available outside options to the equity holders, which is holding the treasury bill and earn $R_f$

Lenders collect deposits from the borrowers by offering a fixed return $R_d$, also to be determined at the equilibrium, and invest their funds by making risky loans to the borrower and holding treasury bills with a fixed return $R_f$ determined by the monetary authority outside the model. The deposits are legally required to be fully insured and that the deposit insurance has to be provided privately, which restricts the lender's investment opportunities. Due to this insurance, the deposits are the risk-free asset for the borrowers.

I assume that the expectation of the subjective beliefs satisfy

$$E[S|\theta] > R_f > E[S|0] \tag{3.4}$$

so that the optimistic borrower is always the natural buyer of the asset. If the deposit interest rate $R_d$ lenders offer to the borrower is less than $R_f$, the borrowers hold only the treasury bills as the risk-free asset in their portfolio. Therefore, $R_d \geq R_f$ must hold at the equilibrium for lenders to leverage their equity with deposits. (3.4) puts a ceiling on deposit interest rates. If $R_d \geq E[S|\theta]$, the borrower has no demand for the risky asset and prefers to hold deposits alone. The equilibrium interest on deposits must satisfy:

$$R_d \in (R_f, E[S|\theta]) \tag{3.5}$$

The lenders supply loans backed by collateral to finance the borrower's demand for the risky asset. Each lender presents a menu of loans specifying per unit investment: (1) the fraction put down by the borrower at the beginning of the period, denote $m$ for margin, (2) the promised payment at the end of the period, denote $i$ for gross interest. All loans are non-recourse; if the risky asset is worth less than $i$, the lender seizes the risky asset as collateral but cannot touch the borrower's deposits. If the borrower buys $x$ units of risky asset, then

$$y \equiv mx \ , \ d \equiv w - y \ , \ q \equiv (1 - m)x \tag{3.6}$$

$y$ is the down payment, $d$ is the investment in the risk-free deposits, $q$ is the size of the loan. Finally, $1/m$ is the borrower's leverage ratio.

Table 3.1 illustrates the balance sheet of a representative borrower on right and a lender on the left. The lender's leverage ratio would be $d/\kappa + 1$.

Table 3.1: Balance Sheets

(a) Lender

| | |
|---|---|
| Loan | $\kappa$ - Equity |
| T-Bills | $d$ - Deposit |

(b) Borrower

| | |
|---|---|
| y - Down payment | q - Loan |
| d - Deposit | (w-q) - Net Worth |

### 3.2.1 The Supply of Loans

The borrowers choose from a menu of loans indexed by $(m, i)$. Lenders screen the borrower and charge a fixed fee $\phi > 0$ per unit loan to cover their screening cost left outside the model. Competition among lenders drive the excess return on each loan to their fee:

$$E[\min(S, i)|0] = (1 + \phi)(1 - m)R_f \tag{3.7}$$

The left-hand side is the expected payoff from lending against one unit of risky asset promising $i$ in return. Since the borrower puts down $m$ for that one unit, $(1-m)R_f$ is the monetary opportunity cost of lending. Dividing both sides of (3.7) by $(1-m)$, the expected return on each loan is $(1 + \phi)R_f$.

Since $E[S|0] < R_f$, the lowest margin the lenders are willing to accept is:

$$E[S|0] = (1 + \phi)(1 - \underline{m})R_f \tag{3.8}$$

For any margin larger than $\underline{m}$, I solve and plot the competitive interest $i(m)$ solving (3.7).

**Lemma 3.1.** *There exists a unique, decreasing and convex $i(m)$ satisfying (3.7) for all $m \in [\underline{m}, 1]$. A decrease in $R_f$ or $\phi$ lowers $i$ for each $m$. An increase in lender's risk measure $\sigma$ increases $i$ for each $m$.*

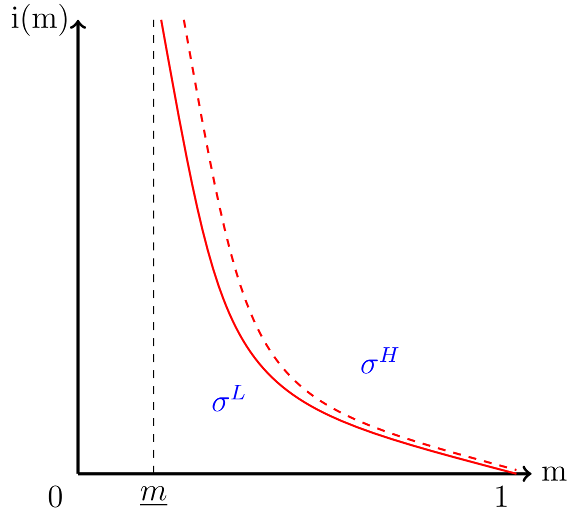



Figure 3.1: Margin-Interest Menu

### 3.2.2 The Borrower's Problem in Partial Equilibrium

The borrower takes the menu in Figure 3.1 and the deposit interest rate $R_d$ as given. She chooses her portfolio allocation and the margin. The main result of subsection is borrowed from Chapter 2. The borrower's objective function is

$$\max_{(x,m)\in[0,w/m]\times[\underline{m},1]} U(x,m|R_d,\theta) \equiv \int -e^{\alpha(wR_d + x\ell(s,m|R_d))} f(s|\theta)\, ds \tag{3.9}$$

where the function

$$\ell(S,m|R_d) \equiv \max(S - i(m), 0) - mR_d \tag{3.10}$$

is the levered excess return on a unit of risky asset purchased at margin $m$ when the asset is worth $s \in S$.

**Proposition 3.1.** *For any $(R_d,\theta)$ the borrower chooses a unique portfolio of $x(\theta,R_d)$ invested in the risky asset by putting down a fraction $m(\theta,R_d) < 1$, and $d(\theta,R_d)$ invested in a risk-free deposit account. A more optimistic borrower invests more in* *the risky asset ($x(\theta, R_d) \nearrow \theta$) with a higher leverage ratio ($m(\theta, R_d) \searrow \theta$) and invests less in the deposit ($d(\theta, R_d) \searrow \theta$).*

Even though a positive shock to asset values separately reinforces higher investment in the risky asset and investing at higher margins, the two choices are substitutes. I find that the borrower re-balances her portfolio more than she adjusts its leverage. Therefore, the effect of higher $x$ dominates at the optimum. I rank the elasticities of $q, x$ and $y$ in $\theta$ using $m'(\theta) \leq 0$.

$$\frac{q'(\theta)}{q(\theta)} \geq \frac{x'(\theta)}{x(\theta)} \geq \frac{y'(\theta)}{y(\theta)} \tag{3.11}$$

I conclude this section with a reminder on the comparative statics of $\sigma$. An increase in the lender's perception of risk, a point-wise higher interest menu leads to smaller demand for the risky asset and the loan, and higher demand for deposits. Essentially the comparative statics of $\sigma$ is isomorphic to that of $-\theta$.

## 3.3 Deposit Interest Rate

This section is dedicated to the analysis of the borrower's portfolio and leverage decisions in the deposit interest rate $R_d$. The return on the safe asset affects not only the demand for that asset, i.e. deposits $d$, but also the risky asset and the loan demand $q$. Since the price of the risky asset is normalized to one, $1/R_d$ is the relative price of deposits. The comparative statics of deposit interest rate is the cross-price elasticity of risky asset demand. Earlier literature offers limited guidance regarding the sign of this elasticity. As early as Fishburn and Porter (1976) noted that in the standard portfolio problem without leverage ($m = 1$), the risky asset demand changes ambiguously in the face of higher risk-free return. The ambiguity is because $R_d$ operates through two opposing channels: lower excess return and higher sure wealth.

Levered excess return $\ell(S, m|R_d)$ is lower at every state $s \in S$ because the opportunity cost of down payment is bigger when the safe return is higher. I guarantee that lower excess return per unit investment lowers the demand for the risky asset if the distribution of $S$ is log-concave. The second channel operates through risk aversion. Higher return on safe asset increases the reference sure wealth which makes the investor less risk averse and demand more risky asset.

My motivation for exponential utility is to shut down the second channel. Constant absolute risk aversion subdues the wealth effect and I guarantee that $x(m, R_d, \theta)$ falls in $R_d$ for all $(m, \theta)$. Formally

$$U_x(x, m|R_d, \theta) \leq 0 \Longrightarrow U_{xR_d}(x, m|R_d, \theta) \leq 0 \tag{3.12}$$

By their definition this inequality also guarantees that down payment $y(m, R_d, \theta)$ and loan demand $q(m, R_d, \theta)$ fall in $R_d$ at every margin.

How the borrower would adjust the margin $m(x, R_d, \theta)$ in the face of higher safe return is unstudied to the best of my knowledge. I argue that it follows a similar logic to $x(m, R_d, \theta)$, albeit stronger. The higher the opportunity cost of down payment $mR_d$ per unit investment, the less the borrower is willing to put down. However, higher sure wealth reinforces the opposite strategy through risk aversion. Constant absolute risk aversion guarantees that the wealth effect is muted and therefore, the optimal margin falls in $R_d$.

$$U_m(x, m|R_d, \theta) \geq 0 \Longrightarrow U_{mR_d}(x, m|R_d, \theta) < 0 \tag{3.13}$$

The stronger implication is that it is also optimal to lower the margin holding $y$ or $q$ fixed, rather than $x$ fixed. In Chapter 2 I have proven that the comparative statics of $m$ in $x$ and $y$ are different because depending on whether I assess the risk per unit or per dollar basis, higher margin can be riskier or safer. While $(m, x)$ are *substitutes*, the higher the margin the lower the demand for the risky asset and vice versa, $(m, y)$ are *complements*. The choice of units is irrelevant in the comparative statics of $R_d$ because the opportunity cost of down payment always rises $R_d$. An increase in deposit interest rate lowers $y$ for all $m$ and $m$ for all $y$, then $y(R_d, \theta)$ and $m(R_d, \theta)$ jointly fall in $R_d$. This suggests an upward sloping supply of deposits; the higher the return the lender offers, the more deposits it collects.

**Proposition 3.2.** *The higher the interest rate on deposits $R_d$, the borrower spends less on the risky asset ($y(R_d, \theta) \searrow R_d$), more on deposits ($d(R_d, \theta) \nearrow R_d$) and she is more leveraged ($m(R_d, \theta) \searrow R_d$). She also has a smaller demand for the risky asset ($x(R_d, \theta) \nearrow R_d$) and the loan ($q(R_d, \theta) \searrow R_d$).*

The comparative statics of $x(R_d, \theta)$ is subtle because it is a matter of whether $x(R_d, \theta)$ or $m(R_d, \theta)$ shifts down more in $R_d$. Figure 3.2 illustrates the comparative statics result. In response to the same interest rate change, the borrower adjusts the margin more than she re-balances her portfolio. This assertion is the polar opposite of the elasticities in $\theta$, following Proposition 3.1 I have argued that the margin is *less elastic than* investment facing a positive shock on asset values.

Formally, I prove the inequalities

$$\left|\frac{U_{mR_d}}{-U_{xm}}\right| \geq \left|\frac{U_{xR_d}}{-U_{xx}}\right| \text{ and } \left|\frac{U_{mR_d}}{-U_{mm}}\right| \leq \left|\frac{U_{xR_d}}{-U_{xm}}\right| \tag{3.14}$$

The first inequality ranks the vertical shifts. If $m(R_d, \theta)$ shifts vertically down more than $x(R_d, \theta)$, then the new crossing is guaranteed to occur at a lower $m$ but $x$ can be higher or lower. The second inequality ranks the horizontal shifts which guarantees that $x(R_d, \theta)$ is lower. The comparative statics of $q(R_d, \theta)$ is similar, in fact this is what I establish in the Appendix and deduce the comparative statics statement for $x(R_d, \theta)$ from it.

I conclude this section with a ranking of elasticities in $R_d$. Given that $m'(R_d) \leq 0$,

$$\left|\frac{q'(R_d)}{q(R_d)}\right| \leq \left|\frac{x'(R_d)}{x(R_d)}\right| \leq \left|\frac{y'(R_d)}{y(R_d)}\right| \tag{3.15}$$

Compare (3.15) to (3.11) the ranking of elasticities are reversed; the loan demand shifts the least with respect to a change in $R_d$. This reversion has two implications.

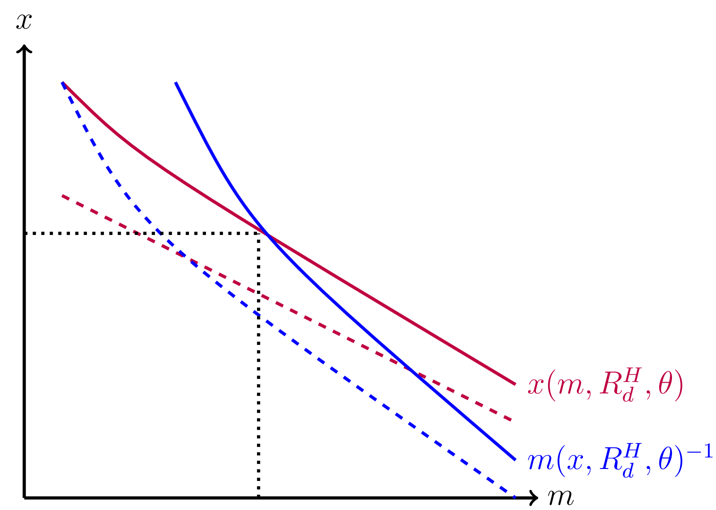



Figure 3.2: $x(R_d, \theta) \searrow R_d$ and $m(R_d, \theta) \searrow R_d$

If there is a joint increase in $(R_d, \theta)$, then the elasticity of loan demand $q$ is still larger than that of asset demand $x$ and down payment $y$. Consider the elasticity of substitution between $\theta$ and $R_d$ and how much the deposit interest rate must fall to compensate an increase in optimism. Since $d' = -y'$ for both $R_d$ and $\theta$, the elasticity of substitution for $q$, $x$ and deposits $d$ can be ranked as

$$\frac{q_\theta(R_d, \theta)}{-q_{R_d}(R_d, \theta)} \geq \frac{x_\theta(R_d, \theta)}{-x_{R_d}(R_d, \theta)} \geq \frac{-d_\theta(R_d, \theta)}{d_{R_d}(R_d, \theta)} \tag{3.16}$$

## 3.4 The Unique Equilibrium

The equilibrium is characterized by two conditions: (i) free-entry determines how much equity flows into the market until expected profits from banking vanish for a given deposit interest rate $R_d$, and (ii) $R_d$ clears the loan market for a given mass of equity $\kappa$.

What remains to be established is each lender's investment plan given her total funds $\kappa + d(R_d, \theta)$, her equity and deposits. Since the lender is risk-neutral and earns the fee $(1+\phi)$ on risky loans, she strictly prefers loans over holding treasury bills. Lenders cannot make only risky loans because they cannot promise a risk-free return on deposits without holding treasury bills against bad states of the world. Let $L$ denote the loans, the minimum amount $T$ each lender invests in treasury bills equates the returns

$$L \cdot 0 + TR_f = d(R_d, \theta)R_d \tag{3.17}$$

so that she can repay the deposits even in the extremely rare worst-case scenario in which the loan entirely busts. The balance sheet equates assets to liabilities, or $L + T = \kappa + d$. Since the bank faces a demand for loans $L = q(R_d^*(\theta), \theta)$ at the equilibrium, the *derived demand for bank capital* is:

$$\kappa^D(R_d, \theta) \equiv q(R_d, \theta) + d(R_d, \theta)\Big(\frac{R_d}{R_f} - 1\Big) \tag{3.18}$$

Next, consider supply. Free entry into banking implies that there is no excess return from banking at the equilibrium. Thus, the supply of loanable funds $\kappa^S(R_d, \theta)$ satisfies the arbitrage condition:

$$(1+\phi)\frac{q(R_d, \theta)}{\kappa^S(R_d, \theta)} = 1 \tag{3.19}$$

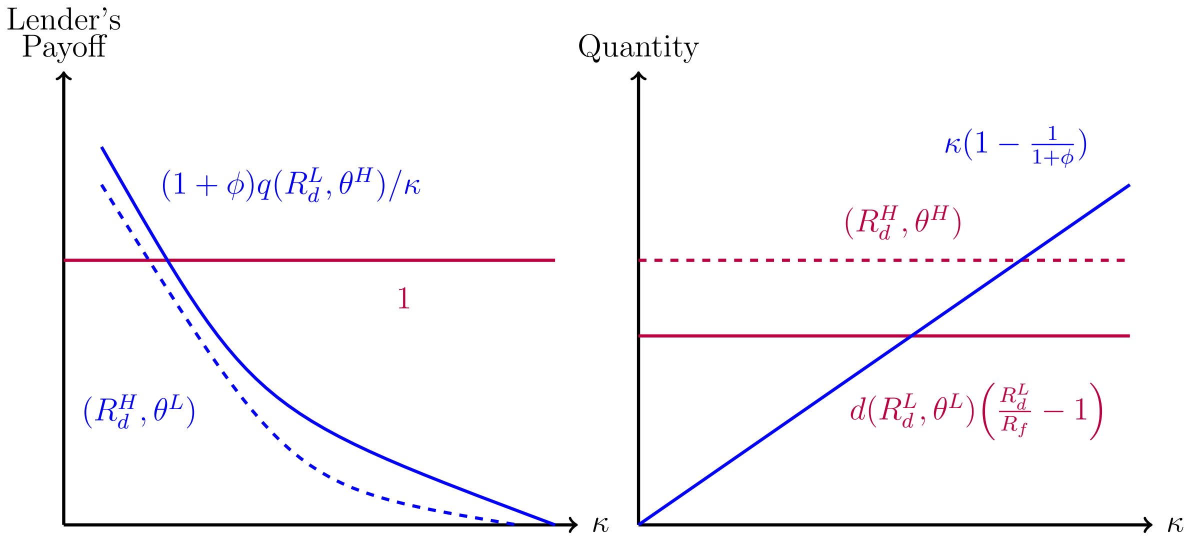
$(1+\phi)R_f \times q/\kappa$ is the return on marginal equity, and the opportunity cost of that equity is $R_f$. The market clears at the equilibrium when

$$\kappa^D(R_d, \theta) = \kappa^S(R_d, \theta) \tag{3.20}$$

This system (3.18) and (3.19) gives two non-linear equations with two unknowns.

**Proposition 3.3.** *There exists a unique equilibrium with $\kappa^*(\theta) \in (0, \infty)$ and $R_d^*(\theta) \in (R_f, E[S|\theta])$.*

The equilibrium conditions (3.19) and (3.20) are solved graphically in Figure 3.3. $\kappa^S(R_d, \theta)$ falls in the interest rate $R_d$. Intuitively, higher deposit interest rates deter




(a) Free-Entry Condition (3.19) (b) $\kappa^D(R_d,\theta)$ when (3.20) holds

Figure 3.3: Equilibrium Conditions

investing in banks. An increase in $R_d$ reduces the loan demand $q$, making banking less attractive. At the limit $R_d \to R_f$, the loan demand is $(1+\phi)q(R_f,\theta) > 0$, so some equity enters the market. When $R_d \to E[S|\theta]$, the loan demand is zero so the supply of equity is zero.

Consider next, $\kappa^D(R_d,\theta)$. It is hard to sign the equity demand in $(R_d,\theta)$ in general, yet it suffices to do so at the equilibrium. At the limit $R_d \to R_f$, the lenders either do not collect deposits because the households invest in the treasury bill directly. Then the loan demand $q(R_f,\theta)$ is satisfied directly out of equity. However, $\kappa^D(R_f,\theta) = q(R_f,\theta)$ is strictly less than $\kappa^S(R_f,\theta) = (1+\phi)q(R_f,\theta)$. This is the crucial role the fees play, if the expected return on loan is identical to treasury bills, the equilibrium interest rate is $R_f$ and deposits play no role. When $R_d \to E[S|\theta]$, the households invest all their savings into deposits and some positive level of equity is needed to insure them, even though $\kappa^S = 0$ since there is no demand for loan. Therefore, (3.20) must hold at least once at an interior interest rate.

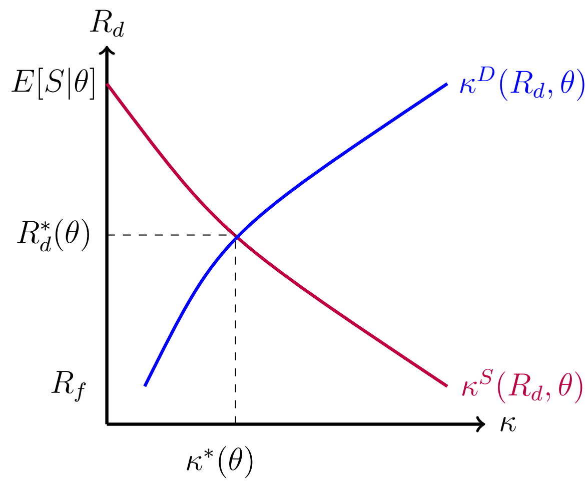



Figure 3.4: Unique Equilibrium

To prove uniqueness, I show that whenever (3.20) holds, $\kappa^D(R_d,\theta)$ given by

$$\kappa^D\left(1-\frac{1}{1+\phi}\right)=d(R_d,\theta)\left(\frac{R_d}{R_f}-1\right) \tag{3.21}$$

is positively sloped in $R_d$. Figure 3.3b plots the two sides of this equation. The left-hand side is a linear increasing function of $\kappa$. Since the amount of treasury bills rises in $R_d$, the new intersection occurs at a higher $\kappa$, concluding the assertion. Figure 3.4 puts together $\kappa^D(R_d,\theta)$ and $\kappa^S(R_d,\theta)$. Finally, the unique equilibrium determines the lender's debt-to-equity ratio from (3.20) by

$$\frac{d(R_d^*(\theta),\theta)}{\kappa^*(\theta)}=\left(1-\frac{1}{1+\phi}\right)/\left(\frac{R_d^*(\theta)}{R_f}-1\right) \tag{3.22}$$

whereas, the borrower's leverage ratio is $1/m(R_d^*(\theta),\theta)$. Note that the two leverage ratios have the opposite sign in $R_d$; the higher the interest rate, the lower the borrower's margin and therefore the higher her leverage ratio, yet the the converse is true the lender.

## 3.5 Comparative Statics

I are interested in the general equilibrium effect of a shock to the asset values $\theta$. Keeping the supply of loans fixed, the borrower puts down a smaller fraction of the risky investment in the portfolio if she is more optimistic. Suppose that the deposit interest rate rises with optimism, consistent with the pro-cyclical behavior of interest rates. Then the general equilibrium effect further reinforces the increase in the borrower's leverage ratio but dampens her demand for the risky asset.

**Proposition 3.4.** *An increase in the borrower's optimism $\theta$ leads to an equilibrium with higher interest on deposits ($R_d^*(\theta) \nearrow \theta$) and more equity flowing into the market ($\kappa^*(\theta) \nearrow$). As a result, the borrowers are more levered, while the lenders are less.*

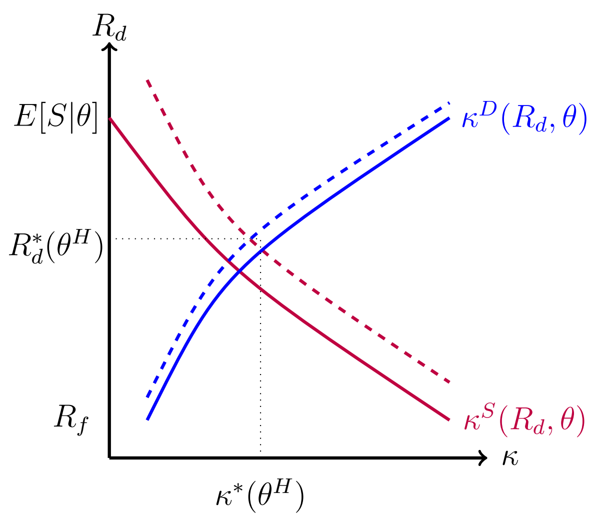



Figure 3.5: Comparative Statics in $\theta$

I illustrate the comparative statics graphically in Figure 3.5. The higher the optimism, the bigger the loan demand $q$ and so more equity to enters to take advantage of lending opportunities. However, optimism lowers the deposit demand $d$ so less equity is needed to insure them according to (3.18). Both shifts separately reinforce higher deposit interest rate.

Facing higher equilibrium interest rate as a result of both forces, the borrower further lowers her margin since

$$\frac{\partial}{\partial\theta}\Big(m^*(R_d^*(\theta),\theta)\Big) = m_\theta^* + m_{R_d}^* R_d^{*\prime}(\theta) < 0 \tag{3.23}$$

The first negative term captures the partial equilibrium effect and the general equilibrium amplifies low margins by increasing the opportunity cost of down payment through higher deposit interest rate.

The net effect on $\kappa^*(\theta)$ is governed by the vertical shifts of (3.19) and (3.20). The ranking of elasticity of substitutions in (3.16) determines which locus shifts more. The vertical shift of (3.19), the change in $R_d$ holding $\kappa$ constant, is given by

$$\frac{q_\theta(R_d^*(\theta),\theta)}{-q_{R_d}(R_d^*(\theta),\theta)} \tag{3.24}$$

and the vertical shift of (3.20) is

$$\frac{-d_\theta(R_d^*(\theta),\theta)(\frac{R_d^*(\theta)}{R_f}-1)}{d_{R_d}(R_d^*(\theta),\theta)(\frac{R_d^*(\theta)}{R_f}-1)+\frac{d(R_d^*(\theta),\theta)}{R_f}} \tag{3.25}$$

Since $q_\theta/-q_{R_d} \geq -d_\theta/d_{R_d}$ and $d/R_f > 0$, (3.25) is less than (3.24) and (3.19) shifts vertically more than (3.20). This ranking guarantees that the new equilibrium has a bigger mass of lenders in the market.

The immediate implication of an higher $\kappa^*(\theta)$ is that the equilibrium quantity of loans is higher

$$\frac{\partial}{\partial\theta}\Big(q(R_d^*(\theta),\theta)\Big) > 0 \tag{3.26}$$

by (3.19) despite the higher interest rate. However, the new equilibrium features less deposits. Re-organize (3.18) such that

$$1 = \frac{q(R_d^*(\theta),\theta)}{\kappa^*(\theta)} + \frac{T^*(\theta)}{\kappa^*(\theta)} - \frac{d(R_d^*(\theta),\theta)}{\kappa^*(\theta)} \tag{3.27}$$

where $T^*(\theta) = d(R_d^*(\theta), \theta)R_d^*(\theta)/R_f$ is the treasury bills. At any equilibrium the ratio $q/\kappa$ is a constant, therefore the change in treasury holdings per equity equals the change in debt-to-equity ratio. Since $R^{*\prime}(\theta) > 0$, debt-to-equity ratio must be falling in $\theta$ and so is treasury holdings of the lender. To summarize, I find that a rise in the borrower's optimism pushes the deposit interest rate not high enough to lower the loan demand, nor to increase the deposit demand. The lender's balance sheet shifts towards more risky loans and less treasury bills, and is less leveraged.

I conclude this section with a corollary of Proposition 3.4.

**Corollary 3.1.** *A decrease in lender's risk measure $\sigma$ leads to an equilibrium with higher interest on deposits ($R_d^*(\theta) \nearrow \theta$) and more equity flowing into the market ($\kappa^*(\theta) \nearrow \theta$).*

In terms of signs, the predictions of $\sigma$ are identical to that of $-\theta$, the borrower getting more pessimistic. The only distinction is the mechanism through which $\sigma$-shock works. If the lender deems $S$ less risky, the menu of loans shift downwards as illustrated in Figure 3.1. Intuitively, this is *cheap credit*. I have established in Chapter 2 that cheap credit has the identical predictions to $\theta$, and this corollary argues that the isomorphism extends to general equilibrium.

## 3.6 Conclusion

This chapter contributes to two literatures, household finance and commercial bank capital structure, by explicitly modeling the reciprocal interaction between banks and the households in mortgage markets. I find that the household's loan margins and debt-to-income ratios are pro-cyclical and the general equilibrium effect of interest rates amplifies this pro-cyclicality. A correctly-priced deposit insurance leads to counter-cyclical bank leverage. I aim to explore the implications of mis-priced deposit insurance scheme in future work.

# Appendix 3

*Proof of Lemma 3.1.* See the proof of Lemma 1 in Chapter 2. The comparative statics of $\phi$ is identical to the comparative statics of $R_f$. □

*Proof of Proposition 3.1.* See Chapter 1 the proof of Proposition 1 for the existence and uniqueness, and Proposition 2 for the comparative statics of $\theta$. □

*Proof of Proposition 3.2.* Consider the objective function $V(y,m)$ defined in Chapter 2 by

$$V(y,m|R_d,\theta) \equiv \max_{y,m} \int u(wR_d + y\lambda(s,m|R_d))f(s|\theta)ds \tag{3.28}$$

where $y = mx$, $\lambda(s,m|R_d) = m\ell(s,m|R_d)$ and utility function $u$ is given by (3.1). The optimal margin given $(y, R_d, \theta)$ is denoted by $m^{**}(y, R_d)$ that satisfies

$$V_m(y,m|R_d,\theta) = x \int_{I(m)} \Big(\frac{R_f}{1-F(I(m)|0)} - \frac{s-I(m)}{m}\Big) u'(wR_d + y\lambda(s,m|R_d))f(s|\theta)\,ds = 0 \tag{3.29}$$

The cross-partial derivate in $R_d$ is

$$V_{mR_d}(y,m|R_d,\theta) = x \int_{I(m)} \Big(\frac{R_f}{1-F(I(m)|0)} - \frac{s-I(m)}{m}\Big)(w-y)u''(w_d + y\lambda(s,m))f(s|\theta)ds \tag{3.30}$$

$$= -x(w-y)\alpha \int_{I(m)} \Big(\frac{R_f}{1-F(I(m)|0)} - \frac{s-I(m)}{m}\Big)u'(w_d + y\lambda(s,m))f(s|\theta)ds \tag{3.31}$$

$$= -(w-y)\alpha V_m(y,m|R_d,\theta) \tag{3.32}$$

where the second line is exploiting the constant absolute risk aversion property of (3.1). Therefore I prove

$$V_m(y,m|R_d,\theta) = 0 \Longleftrightarrow V_{mR_d}(y,m|R_d,\theta) = 0 \tag{3.33}$$

Hence, $m^{**}(y, R_d, \theta)$ is independent of $R_d$.

Denote $m^*(x, R_d, \theta)$ as the solution to (3.9) that satisfies

$$U_m(x, m|\theta) = x\Big[\int_{I(m)}^{\infty}\Big(\frac{R_f}{1-F(I(m)|0)} - R_d\Big)u'(wR_d + x\ell(s, m|R_d)f(s|\theta)\,ds \\ - R_d u'(wR_d - xmR_d)F(I(m)|\theta)\Big] = 0 \tag{3.34}$$

concluding the proof.

The cross-partial derivate in $R_d$ is

$$\begin{aligned} U_{mR_d}(x, m|R_d, \theta) &= -xE[u'(wR_d + x\ell(s, m|R_d)|\theta] \\ &+ x\Big[(w - mx)\int_{I(m)}^{\infty}\Big(\frac{R_f}{1-F(I(m)|0)} - R_d\Big)u''(wR_d + x\ell(s, m|R_d))f(s|\theta)\,ds \\ &- (w - mx)R_d u''(wR_d - xmR_d)F(I(m)|\theta)\Big] && (3.35) \\ &= -xE[u'(wR_d + x\ell(s, m|R_d)|\theta] - (w - mx)\alpha U_m(x, m|R_d, \theta) && (3.36) \end{aligned}$$

The second and the third line exploit the constant absolute risk aversion property of (3.1). Therefore I prove

$$U_m(x, m|R_d, \theta) = 0 \Longleftrightarrow U_{mR_d}(x, m|R_d, \theta) < 0 \tag{3.37}$$

Hence, $m^*(x, R_d, \theta)$ falls in $R_d$.

Since $V_y = 1/m \times U_x$, the solutions $x^*(m, R_d, \theta)$ and $y^*(m, R_d, \theta)$ have the same sign in $R_d$. Observe that at the optimum

$$V_m(y, m|R_d, \theta) = U_m(x, m|R_d, \theta) - \frac{x^*(R_d, \theta)}{m^*(R_d, \theta)}U_x(x, m|R_d, \theta) \tag{3.38}$$

Therefore

$$V_{mR_d} = U_{mR_d} - \frac{x^*(R_d, \theta)}{m^*(R_d, \theta)}U_{xR_d} = 0 \tag{3.39}$$

and both $x^*(m, R_d, \theta)$ and $y^*(m, R_d, \theta)$ falls in $R_d$ as $\operatorname{sgn} U_{xR_d} = \operatorname{sgn} U_{mR_d} < 0$.

I have proven in Chapter 2 that $V$ is quasi-supermodular in $(y, m)$ with respect to the usual order. Now I have shown that $V$ satisfies single-crossing property in $(y, m| - R_d)$ with respect to the same order. By Milgrom and Shannon (1994),

$$y^*(R_d, \theta) \searrow R_d \text{ and } m^*(R_d, \theta) \searrow R_d$$

since $d^*(R_d, \theta) = w - y^*(R_d, \theta)$, deposits increase in the return offered.

Instead of analyzing the comparative statics of $(x, m)$ in $R_d$, I do so in $(q, m)$ are prove that $q^*(R_d, \theta)$ falls in $R_d$. Since $q^*(R_d, \theta) = x^*(R_d, \theta)(1 - m^*(R_d, \theta)$, the suggested signs indicate $x^*(R_d, \theta)$ falls in $R_d$ as well.

The objective function taking $(q, m)$ as the controls is defined in Proposition 2 of Chapter 2 by

$$Q(q, m|R_d, \theta) \equiv \max_{q,m} \int u(wR_d + q\varrho(s, m|R_d))f(s|\theta)ds \tag{3.40}$$

$$= V(qm/(1-m), m|R_d, \theta) \tag{3.41}$$

where $\varrho(S, m) = 1/(1 - m) \times \ell(S, m)$. I relate the derivatives evaluated at the optimum as

$$Q_q = \frac{m}{1-m} V_y \tag{3.42}$$

$$Q_m = V_m + \frac{q}{(1-m)^2} V_y \tag{3.43}$$

$$Q_{qm} = \frac{m}{1-m}\Big(V_{ym} + \frac{q}{(1-m)^2} V_{yy}\Big) \leq 0 \tag{3.44}$$

$$Q_{mm} = \Big(V_{mm} + \frac{q}{(1-m)^2} V_{ym}\Big) + \frac{q}{(1-m)^2}\Big(V_{ym} + \frac{q}{(1-m)^2} V_{yy}\Big) \leq 0 \tag{3.45}$$

and by the algorithm I develop in Proposition 1 of Chapter 2,

$$\Big(V_{mm} + \frac{q}{(1-m)^2} V_{ym}\Big) \leq 0 \tag{3.46}$$

I use these relations as follows. Since $V_{mR_d} = 0$, differentiating (3.42) and (3.43) in $R_d$ and evaluating them at the optimum

$$\frac{-Q_{mR_d}}{-Q_{qR_d}} = \frac{q^*}{(1-m^*)m^*} = \frac{x^*}{m^*} \tag{3.47}$$

Since $q^*(m, R-d, \theta)m/(1-m) = y^*(m, R_d, \theta)$ by (3.42) and $y^*$ is increasing in $m$, I get

$$\frac{-Q_{qm}}{-Q_{qq}} = -q^{*\prime}(m, R_d, \theta) \leq \frac{x^*}{m^*} \tag{3.48}$$

Lastly, by (3.46) I have the inequality

$$Q_{mm} - \frac{x^*}{m^*}Q_{qm} = V_{mm} + \frac{q}{(1-m)^2}V_{ym} \leq 0 \tag{3.49}$$

Now combine (3.47), (3.48) and (3.49) to a single chain of inequalities:

$$\frac{-Q_{qm}}{-Q_{qq}} \leq \frac{x^*}{m^*} = \frac{-Q_{mR_d}}{-Q_{qR_d}} \leq \frac{-Q_{mm}}{-Q_{qm}} \tag{3.50}$$

The rest of the proof follows from applying Implicit Function Theorem to two first-order conditions of $Q(q, m|\theta)$ and solving two equations with two unknowns

$$Q_{qq}q^{*\prime}(R_d) + Q_{qm}m^{*\prime}(R_d) + Q_{qR_d} = 0 \tag{3.51}$$

$$Q_{qm}q^{*\prime}(R_d) + Q_{mm}m^{*\prime}(R_d) + Q_{mR_d} = 0 \tag{3.52}$$

where all partial derivatives are evaluated at the optimum and therefore suppressed in the notation. Express $q^{*\prime}(R_d)$ as linear functions of $m^{*\prime}(R_d)$

$$q^{*\prime}(R_d) = \frac{Q_{qR_d}}{-Q_{qq}} - \frac{Q_{qm}}{Q_{qq}}m^{*\prime}(R_d) \tag{3.53}$$

$$q^{*\prime}(R_d) = \frac{Q_{mR_d}}{-Q_{qm}} - \frac{Q_{mm}}{Q_{qm}}m^{*\prime}(R_d) \tag{3.54}$$

By the chain of inequalities (3.50), (3.53) and (3.54) are decreasing functions such

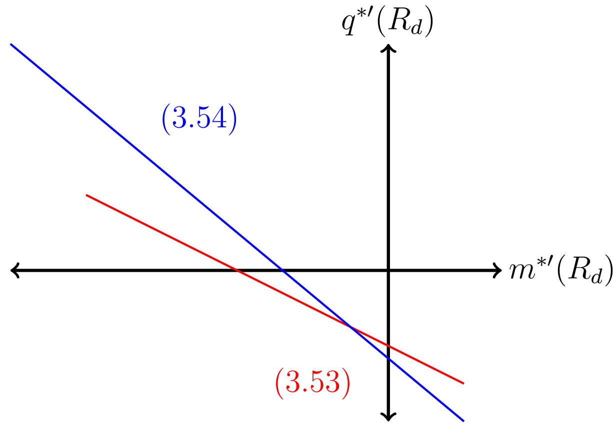


Figure 3.6: Equations (3.53) and (3.54)

that

1. (3.53) is flatter than (3.54)
2. The $x'(R_d)$-intercept of (3.53) is bigger than that of (3.54)
3. The $m'(R_d)$-intercept of (3.53) is smaller than that of (3.54)

Figure 3.6 plots (3.53) and (3.54) consistent with the properties listed above. The lines cross uniquely at the third quadrant where

$$q^*(R_d, \theta) \searrow R_d \text{ and } m^*(R_d, \theta) \searrow R_d$$

concluding the proof.

$\square$

*Proof of Proposition 3.3.* The equilibrium $R_d^*(\theta)$ satisfies $\kappa^D(R_d^*(\theta), \theta) = \kappa^S(R_d^*(\theta), \theta)$. Consider the difference as $R_d \to R_f$:

$$\lim_{R_d \to R_f} \kappa^D(R_d, \theta) - \kappa^S(R_d, \theta) = (1 + \phi)q(R_f, \theta) - q(R_f, \theta) > 0 \tag{3.55}$$

Evaluate the difference as $R_d \to E[S|\theta]$:

$$\lim_{R_d \to E[S|\theta]} \kappa^D(R_d, \theta) - \kappa^S(R_d, \theta) = 0 - (\frac{w\left(\frac{E[S|\theta]}{R_f} - 1\right)}{1 - \frac{1}{\phi}}) < 0 \tag{3.56}$$

Therefore, there must exist at least one interior $R_d^*(\theta)$ such that the difference is zero.

To prove uniqueness, observe that evaluated at (3.20):

$$\begin{aligned} \frac{\partial}{\partial R_d}\left(\kappa^D(R_d, \theta) - \kappa^S(R_d, \theta)\right) &= (1+\phi) q_{R_d} \\ &\quad - \frac{d_{R_d}\left(\frac{R_d}{R_f} - 1\right) + \frac{d}{R_f}}{1 - \frac{1}{1+\phi}} < 0 \end{aligned} \tag{3.57}$$

as $q_{R_d} \leq 0$ and $d_{R_d} \geq 0$. Therefore, $R_d^*(\theta)$ must be unique, and so is $\kappa^*(\theta)$. □

*Proof of Proposition 3.4.* Totally differentiate (3.18) and (3.19) evaluated at (3.20) in $\theta$.

$$\kappa^{*\prime}(\theta) = (1+\phi)\left(q_{R_d}(R_d^*, \theta) R_d^{*\prime}(\theta) + q_\theta(R_d^*, \theta)\right) \tag{3.58}$$

$$\kappa^{*\prime}(\theta) = \frac{1}{1 - \frac{1}{1+\phi}}\left((d_{R_d}(R_d^*, \theta) R_d^{*\prime}(\theta) + d_\theta(R_d^*, \theta))(\frac{R_d^*(\theta)}{R_f} - 1) + \frac{d(R_d^*, \theta)}{R_f} R_d^{*\prime}(\theta)\right) \tag{3.59}$$

The intersection of the linear functions on $(\kappa^{*\prime}(\theta), R_d^{*\prime}(\theta))$ space determines both signs. (3.58) is a decreasing function of $R_d^{*\prime}(\theta)$ as $q_{R_d} \leq 0$. (3.59) is a increasing function of $R_d^{*\prime}(\theta)$ as all the terms of the coefficient $d_{R_d}(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1) + \frac{d(R_d^*, \theta)}{R_f}$ are positive.

Consider the $\kappa^{*\prime}(\theta)$-intercepts when $R_d^{*\prime}(\theta) = 0$. (3.58) has a positive intercept since $q_\theta > 0$. However, (3.59) has a negative intercept since $d_\theta(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1) < 0$

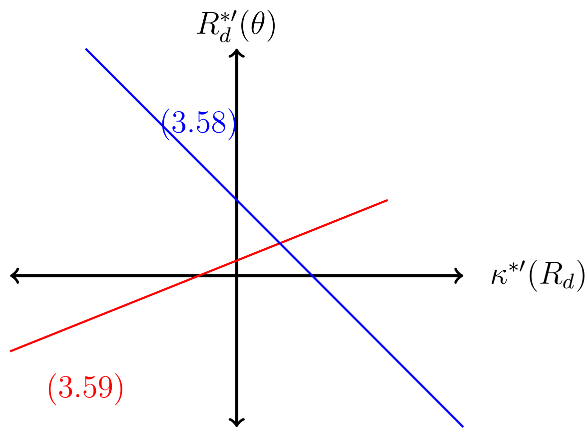



Figure 3.7: Equations (3.58) and (3.59)

Finally, consider the $R_d^{*\prime}(\theta)$-intercepts. (3.58) crosses the point

$$\left(0, \frac{q_\theta(R_d^*, \theta)}{-q_{R_d}(R_d^*, \theta)}\right) \tag{3.60}$$

whereas (3.59) crosses the point

$$\left(0, \frac{-d_\theta(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1)}{d_{R_d}(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1) + \frac{d(R_d^*, \theta)}{R_f}}\right) \tag{3.61}$$

By the ranking of elasticity of substitution in (3.16):

$$\frac{q_\theta}{-q_{R_d}} \geq \frac{-d_\theta}{d_{R_d}} = \frac{-d_\theta(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1)}{d_{R_d}(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1)} \tag{3.62}$$

$$\geq \frac{-d_\theta(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1)}{d_{R_d}(R_d^*, \theta)(\frac{R_d^*(\theta)}{R_f} - 1) + \frac{d(R_d^*, \theta)}{R_f}} \tag{3.63}$$

I plot (3.58) and (3.59) according to these specifications. The unique intersection of (3.58) and (3.59) occurs at the first quadrant where $R_d^{*\prime}(\theta) > 0$ and $\kappa^{*\prime}(\theta) > 0$.

$\square$

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