# Geometry of the Hitchin component

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in the University of Michigan
2015

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This thesis is dedicated to Sung Eun, for believing in me even when I doubt myself.

## ACKNOWLEDGMENTS

Producing this work would have been impossible without the continuous encouragement from my family and friends. I would like to thank them all for the love and understanding they have so generously provided me with in these last few years.

Also, I would like to express heartfelt gratitude towards Richard Canary for his patience, and the invaluable guidance and support he has provided me with in the course of my PhD.

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# ABSTRACT

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We construct a parameterization of the $PSL(n, \mathbb{R})$ Hitchin component that generalizes the Fenchel-Nielsen coordinates on Teichmüller space. Using this parameterization, we study the degeneration of certain geometric quantities, such as length functions and topological entropy, that are associated to the representations in the Hitchin component.

# CHAPTER 1

# Introduction

Let $S$ be a closed, oriented topological surface of genus at least $2$, and denote its fundamental group by $\Gamma$. The Teichmüller space of $S$, denoted $\mathcal{T}(S)$, is the space of marked hyperbolic metrics on $S$. From a representation theoretic point of view, one can think of $\mathcal{T}(S)$ as a component of the space of conjugacy classes of discrete, faithful representations from $\Gamma$ to $PSL(2,\mathbb{R})$. An advantage of taking this point of view is that it allows one to define a higher rank generalization of $\mathcal{T}(S)$, which was first studied by Hitchin [28]. Presently, this "higher Teichmüller space" is known as the *Hitchin component*, and can be defined as follows. Let $\iota_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ be the unique (up to conjugation) irreducible representation. This induces, via post-composition, an embedding $i_n$ from $\mathcal{T}(S)$ into the character variety

$$\mathcal{X}_n(S) := \mathrm{Hom}(\Gamma, PSL(n,\mathbb{R}))/PSL(n,\mathbb{R}).$$

The *$n$-th Hitchin component of $S$*, denoted $Hit_n(S)$, can then be defined to be the connected component of $\mathcal{X}_n(S)$ that contains the image of $i_n$, which is also known as the *Fuchsian locus*.

It is well-known that $\mathcal{T}(S) = Hit_2(S)$. Also, by the work of Choi-Goldman [10] and Guichard-Wienhard [26], we know respectively that $Hit_3(S)$ is the space of marked convex $\mathbb{RP}^2$ structures on $S$ and that $Hit_4(S)$ is the space of marked convex foliated $\mathbb{RP}^3$ structures on $T^1S$. These realizations of the lower rank Hitchin components as deformations spaces of geometric structures associated to $S$ provide a strong motivation for studying Hitchin representations. Guichard-Wienhard [27] also constructed domains of discontinuities for the image of Hitchin representations in $Hit_n(S)$ for all $n$.

Interestingly, the Hitchin components have many of the desirable properties that $\mathcal{T}(S)$ possess. Hitchin [28] proved using Higgs bundle techniques that $Hit_n(S)$ is a cell of real dimension $(n^2-1)(2g-2)$, where $g$ is the genus of $S$. By understanding the dynamics of the $\Gamma$-action induced by Hitchin representations on the space of complete flags in

$\mathbb{R}^n$, Labourie [30] proved that they are discrete, faithful, and their images consist only of diagonalizable elements with eigenvalues that have pairwise distinct norms. Using the work of Fock-Goncharov [18], Bonahon-Dreyer [6] gave a real-analytic parameterization of $Hit_n(S)$ that is a generalization of Thurston's shear coordinates in $\mathcal{T}(S)$. There is also a parameterization of $Hit_3(S)$ by Goldman [21] which generalizes the Fenchel-Nielsen coordinates on $\mathcal{T}(S)$.

By taking a special case of the parameterization by Bonahon-Dreyer [6] and performing a linear reparameterization, one can obtain another parameterization that is explicitly analogous to the Fenchel-Nielsen coordinates on $\mathcal{T}(S)$. More specifically, if we choose an oriented pants decomposition $\mathcal{P}$ of $S$, then $Hit_n(S)$ can be parameterized by the following parameters:

- $n-1$ *boundary invariants* for each simple closed curve in $\mathcal{P}$.
- $n-1$ *gluing parameters* for each simple closed curve in $\mathcal{P}$.
- $(n-1)(n-2)$ *internal parameters* for each pair of pants given by $\mathcal{P}$.

The boundary invariants take values in $\mathbb{R}^+$ while the gluing and internal parameters take values in $\mathbb{R}$. Here, one should think of the boundary invariants and gluing parameters as analogs of the Fenchel-Nielsen length and twist coordinates respectively. We call this parameterization of $Hit_n(S)$ the *modified shear-triangle parameterization*.

In view of this parameterization, one can ask if there is any geometric meaning behind deforming the internal parameters. One way to approach this question is to study sequences $\{\rho_i\}_{i=1}^\infty$ in $Hit_n(S)$, along which the boundary invariants are held bounded away from $0$ and $\infty$, while the $(n-1)(n-2)$ internal parameters for each pair of pants escape every compact set in the cell where they take values. Such sequences are called *internal sequences*. More informally, the internal sequences are those where we do not change the boundary invariants by much while deforming the internal parameters as much as possible. For these internal sequences, there are no conditions on the gluing parameters.

One can also study Hitchin representations by considering the induced flows on $T^1S$. For any Hitchin representation $\rho$, define the length function $l_\rho : \Gamma \to \mathbb{R}$ by

$$l_\rho(X) = \log\left|\frac{\lambda_n(\rho(X))}{\lambda_1(\rho(X))}\right|,$$

where $\lambda_n(\rho(X))$ and $\lambda_1(\rho(X))$ are the eigenvalues of $\rho(X)$ with largest and smallest norm respectively. These are related to the length functions studied by Dreyer [15]. When $n=2$, $l_\rho(X)$ is the length of the closed geodesic in $S$ corresponding to $X$ in $\Gamma$, measured in the hyperbolic metric on $S$ corresponding to the representation $\rho$.

Sambarino [43] constructed, for each $\rho$, a unique (up to Livšic cohomology) Hölder reparameterization $(\phi_\rho)_t$ of the geodesic flow on $T^1S$, so that the closed orbit of $(\phi_\rho)_t$ corresponding to the conjugacy class $[X]$ in $[\Gamma]$ has period $l_\rho(X)$. In the case when $\rho$ is in $\mathcal{T}(S)$, $(\phi_\rho)_t$ is Livšic equivalent to the geodesic flow of the hyperbolic metric corresponding to $\rho$. This allows us to define the *topological entropy of* $\rho$, $\mathrm{h}_{\mathrm{top}}(\rho)$, to be the topological entropy of the flow $(\phi_\rho)_t$. It then follows from the work of Bowen [7] and Pollicot [40] that

$$\mathrm{h}_{\mathrm{top}}(\rho) = \limsup_{T\to\infty} \frac{1}{T} \log |\{[X] \in [\Gamma] : l_\rho(X) < T\}|.$$

In this paper, we study how the dynamics of these induced flows degenerate along internal sequences. The main goal is to prove the following theorem.

**Theorem** (Theorem 4.4.4)**.** *There exists a continuous function* $\Theta : Hit_n(S) \to \mathbb{R}^+$ *with the following properties:*

- *If* $X$ *in* $\Gamma$ *does not correspond to a curve homotopic to a multiple of a curve in* $\mathcal{P}$*, then* $\Theta(\rho) \le l_\rho(X)$.
- *If* $\{\rho_i\}_{i=1}^\infty$ *is an internal sequence, then*

$$\lim_{i\to\infty} \Theta(\rho_i) = \infty.$$

*Furthermore,*

$$\lim_{i\to\infty} \mathrm{h}_{\mathrm{top}}(\rho_i) = 0.$$

Note that the above theorem is vacuously true for $Hit_2(S)$ because there are no internal sequences in that case. This theorem is a generalization of the results in Zhang [47], where the author proved the same statement for $Hit_3(S)$ using the Goldman parameterization. Nie [38] also has some related results.

One can interpret this theorem as a statement highlighting some stark structural differences between $\mathcal{T}(S)$ and the higher rank Hitchin components. It follows from the work of Bers [4] that there exists a constant $L > 0$ depending only on $S$, with the property that for any $\rho \in \mathcal{T}(S)$, there is some $X \in \Gamma$ so that $l_\rho(X) < L$. Also, it is well known that $\mathrm{h}_{\mathrm{top}}(\rho) = 1$ for any $\rho \in \mathcal{T}(S)$. On the other hand, the main theorem implies, via a simple diagonalization argument, that in $Hit_n(S)$ for $n \ge 3$, there is a sequence of representations $\{\rho_i\}_{i=1}^\infty$ such that $\lim\limits_{i\to\infty} l_{\rho_i}(X) = \infty$ for any $X \in \Gamma \setminus \{\mathrm{id}\}$ and $\lim\limits_{i\to\infty} \mathrm{h}_{\mathrm{top}}(\rho_i) = 0$. In particular, Mumford compactness must fail for $Hit_n(S)$ when $n \ge 3$.

In addition to the consequences mentioned in Zhang [47], the main theorem has several other interesting geometric corollaries. Let $M$ be the $SL(n,\mathbb{R})$ symmetric space. Define the *critical exponent of* $\rho$

$$h_M(\rho) := \limsup_{T\to\infty} \frac{1}{T} \log |\{X \in \Gamma : d_M(o, \rho(X)\cdot o) < T\}|,$$

where $d_M$ is the distance function on $M$ induced by the Riemannian metric, and $o$ is any point in $M$. The main theorem allows us to deduce how the critical exponent degenerates along internal sequences.

**Corollary** (Corollary 4.4.5)**.** *Let $\{\rho_i\}_{i=1}^\infty$ be an internal sequence. Then*

$$\lim_{i\to\infty} h_M(\rho_i) = 0.$$

This corollary has further implications on the minimal immersions that arise from Hitchin representations. For any $\rho$ in $Hit_n(S)$ and any conformal structure $\Sigma$ on $S$, a special case of the work of Corlette [12] or Eells-Sampson [16] implies the existence of a unique (up to $PSL(n,\mathbb{R})$ action) harmonic map

$$f : \Sigma \to \rho(\Gamma)\backslash M.$$

Labourie [31] then proved that for every $\rho$ in $Hit_n(S)$, there are conformal structures $\Sigma$ on $S$ so that $f$ is a branched minimal immersion. Recently, Sanders [45] showed that these branched minimal immersions are in fact always immersions. Furthermore, using the work of Sanders [45], we can deduce the following.

**Corollary** (Corollary 4.4.7)**.** *Let $\{\rho_i\}_{i=1}^\infty$ be an internal sequence in $Hit_n(S)$, and let $\Sigma_i$ be a conformal structure on $S$ for which the harmonic map $f_i : \Sigma_i \to \rho_i(\Gamma)\backslash M$ is a minimal immersion. Then*

$$\lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f_i^* m_i)} \int_{\Sigma_i} \sqrt{-\operatorname{Sec}_i(T_{f_i(p)} f_i(\Sigma_i))}\, \mathrm{d}V_i(p) = 0$$

*and*

$$\lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f_i^* m_i)} \int_{\Sigma_i} ||B_{f_i}(p)||\, \mathrm{d}V_i(p) = 0.$$

*Here, $B_{f_i}$ is the second fundamental form of $f_i$, $m_i$ is the Riemannian metric on $\rho_i(\Gamma)\backslash M$, $\operatorname{Sec}_i$ is the sectional curvature in $\rho_i(\Gamma)\backslash M$, and the integral is taken using the volume measure of $f_i^* m_i$.*

More informally, this corollary says that the minimal immersions corresponding to the Hitchin representations along any internal sequence are on average becoming flatter and more totally geodesic as we move along the sequence. Collier-Li [11] also have results that are similar in flavor to this corollary. For more consequences of the main theorem, see Chapter 4.4.

We will now give a sketch of the proof of the main theorem in three main steps. Choose a hyperbolic metric on $S$ and consider the ideal triangulation on $S$ that is obtained by further subdividing each pair of pants given by $\mathcal{P}$ into two ideal triangles. Also, fix $\rho$ in $Hit_n(S)$. For the first step, we obtain a combinatorial description of every oriented closed geodesic $\gamma$ on $S$ using the intersection pattern of $\gamma$ with the ideal triangulation. Roughly, this combinatorial description keeps track of how many times $\gamma$ "winds around" a collar neighborhood of a simple closed curve in $\mathcal{P}$, and how many times $\gamma$ "crosses between" these collar neighborhoods. By design, two oriented geodesics on $S$ have the same combinatorial description if and only if they are the same oriented geodesic.

In the second step, we find, for any Hitchin representation $\rho$ and any $X$ in $\Gamma$, a lower bound for $l_\rho(X)$ of the form

$$l_\rho(X) \geq r(X) \cdot K(\rho) + s(X) \cdot L(\rho). \tag{1.0.1}$$

Here, $K, L : Hit_n(S) \to \mathbb{R}^+$ are continuous functions, $s(X)$ is the number of times the oriented closed geodesic $\gamma$ corresponding to $X$ "winds around" collar neighborhoods of the simple closed curves in $\mathcal{P}$, and $r(X)$ is the number of times $\gamma$ "crosses between" these collar neighborhoods. Finally, in the third step, we show that $\lim_{i\to\infty} K(\rho_i) = \infty$ for any internal sequence $\{\rho_i\}_{i=1}^\infty$ in $Hit_n(S)$. This fact, combined with a counting argument demonstrated in the appendix, proves the main theorem.

The second and third steps of the proof rely heavily on the work of Labourie [30] and Guichard [25]. Together, they proved that a representation $\rho$ in $\mathcal{X}_n(S)$ is a Hitchin representation if and only if there exists a $\rho$-equivariant Frenet curve $\xi : \partial\Gamma \to \mathcal{F}(\mathbb{R}^n)$. Understanding the way the cross ratio interacts with $\xi$ is of central importance to the arguments used in the second and third steps.

The rest of this thesis is structured as follows. In Chapter 2, we describe a geometric way to think about cross ratios and triple ratios, and investigate how they interact with the $\rho$-equivariant Frenet curve. Then, in Chapter 3, we develop some of the general theory of $(X, G)$-structures, and show how some Hitchin components can be realized as the holonomy of hyperbolic structures and convex $\mathbb{RP}^2$-structures on $S$. Here, we also describe how one can use the hyperbolicity of the fundamental group of $S$ to give purely topological definitions of laminations and pants decompositions. These will be useful for us in the later chapters.

Next, in Chapter 4, we define the Hitchin component and describe several ways one can think of this space of representations. In particular, we give a detailed explanation of the modified shear-triangle parameterization, and how to derive it from the work of Bonahon-Dreyer and Fock-Goncharov. In this chapter, we also present a careful restatement of the main theorem and prove some of its corollaries, including those mentioned above. Chapter 5 contains the first and second steps, i.e. the combinatorial description of the oriented closed curves on $S$, and the proof of the lower bound (1.0.1). Finally, we combine all the proof ingredients and execute the third step of the proof of the main theorem in Chapter 6.

# CHAPTER 2

# Cross ratio and triple ratios

The cross ratio is a classically studied projective invariant that has proven useful in many settings. In this chapter, we will define the cross ratio we use, explain several equivalent definitions, and discuss some of its basic properties. We will also do the same for a related but less well known projective invariant called the triple ratio, and then use the cross ratios and triple ratios to describe a positvity phenomena for Frenet curves.

## 2.1 Projective geometry

We start by developing some terminology that is standard in projective geometry. Let $V$ be a (real, finite dimensional) vector space. The *projectivization* of $V$, denoted $\mathbb{P}(V)$, is the quotient $V/\mathbb{R}^*$, where $\mathbb{R}^*$ acts on $V$ by scaling. A *projective space* is then defined to be the projectivization of a vector space. If the dimension of $V$ is $n$, then $\mathbb{P}(V)$ is a $(n-1)$-dimensional manifold that is orientable when $n$ is even and non-orientable when $n$ is odd. In the case when $V = \mathbb{R}^n$, i.e. $V$ comes equipped with a choice of basis, then we denote $\mathbb{P}(\mathbb{R}^n)$ by $\mathbb{RP}^{n-1}$. The projective space $\mathbb{RP}^{n-1}$ comes equipped with a set of homogeneous coordinates induced by the standard basis on $\mathbb{R}^n$.

In the rest of this section, vectors in $V$ will be denoted by lower case letters and subspaces of $V$ will be denoted by upper case letters. If $l \in V$ is a vector, then $[l]$ will denote the point in $\mathbb{P}(V)$ that is the equivalence class containing $l$. Since there is a natural identification of the line $L = \mathrm{Span}_{\mathbb{R}}(l)$ in $V$ with $[l]$, in the rest of this thesis, we will fudge the difference between lines in $V$ and points in $\mathbb{P}(V)$. This will simplify notation, and it should be clear from the context which we are referring to.

A *projective subspace* of $\mathbb{P}(V)$ is the projectivization of a vector subspace in $V$, and in particular, a *projective line* is the projectivization of a plane in $V$. As before, we will also often fudge the difference between a vector subspace and its projectivization.

If $V$ and $W$ are two vector spaces of the same dimension, then a map $\phi : \mathbb{P}(V) \to \mathbb{P}(W)$

is a *projective transformation* if there is a linear isomorphism $F : V \to W$ such that $\phi([v]) = [F(v)]$ for all $v \in V$. As such, the set of projective automorphisms on $\mathbb{RP}^{n-1}$ can be naturally identified with $PGL(n, \mathbb{R})$. Also, we say that two projective spaces are *isomorphic* if there is a projective transformation between them. Since any two vector spaces are isomorphic if and only if they have the same dimension, the same is true for projective spaces as well.

Let $V$ be an $n$-dimensional vector space. Often, when doing computations, it is convenient to choose a projective transformation to identify $\mathbb{P}(V)$ with $\mathbb{RP}^{n-1}$ and use the coordinates in $\mathbb{RP}^{n-1}$ to perform the required computations. This procedure is often called *choosing coordinates* on $\mathbb{P}(V)$ or *choosing a normalization* for $\mathbb{P}(V)$. A convenient way to specify such a normalization is using a collection of generic points, which we now define.

**Definition 2.1.1.** Let $V$ be a $n$-dimensional vector space. We say a collection of $n + 1$ points in $\mathbb{P}(V)$ (or $n + 1$ lines in $V$) is *generic* if any $n$ of them span all of $V$.

The next proposition is an easy linear algebra exercise, whose proof we omit. It is analogous to the fact that a linear isomorphism $F$ from $\mathbb{R}^n$ to any $n$-dimensional vector space determines, and is determined by, the image of the standard basis of $\mathbb{R}^n$ under $F$.

**Proposition 2.1.2.** *Let $V$, $W$ be $n$-dimensional vector spaces and let $\mathbb{P}(W)_{gen}$ be the set of ordered generic collections of $n + 1$ points in $\mathbb{P}(W)$. Also, let $\mathrm{Isom}(\mathbb{P}(V), \mathbb{P}(W))$ be the space of projective transformations from $\mathbb{P}(V)$ to $\mathbb{P}(W)$, and let $(L_1, \ldots, L_{n+1})$ be an ordered generic collection of $n + 1$ points in $\mathbb{P}(V)$. Then the map*

$$\begin{array}{rcl} \mathrm{Isom}(\mathbb{P}(V), \mathbb{P}(W)) & \to & \mathbb{P}(W)_{gen} \\ \phi & \mapsto & \big(\phi([L_1]), \ldots, \phi([L_{n+1}])\big) \end{array}$$

*is a bijection.*

In other words, a choice of coordinates on $\mathbb{P}(V)$ is precisely an assignment of $n + 1$ generic points in $\mathbb{RP}^{n-1}$ to $n + 1$ generic points in $\mathbb{P}(V)$.

## 2.2 Cross ratio

Next, we will define the cross ratio and mention some of its basic properties.

**Definition 2.2.1.** Let $L_1 = [l_1], \ldots, L_4 = [l_4]$ be four lines in $\mathbb{R}^n$ through the origin, and let $M = \mathrm{Span}\{m_1, \ldots, m_{n-2}\}$ be a $(n-2)$-dimensional subspace of $\mathbb{R}^n$ not containing $L_i$

for any $i = 1, \dots, 4$, so that no three of the four $(n-1)$-dimensional subspaces $M + L_i$ agree. Define the *cross ratio* of the lines $L_1, L_2, L_3, L_4$ based at $M$ by

$$(L_1, L_2, L_3, L_4)_M := \frac{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_3 \cdot m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_2}{m_1 \wedge \cdots \wedge m_{n-2} \wedge l_1 \wedge l_2 \cdot m_1 \wedge \cdots \wedge m_{n-2} \wedge l_4 \wedge l_3}.$$

Here, we choose a linear identification of $\bigwedge^n(\mathbb{R}^n)$ with $\mathbb{R}$ to evaluate the expression on the right as a number in the one point compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$, by adopting the convention that $\frac{c}{0} = \infty$ for any non-zero real number $c$. The condition that no three of the four $M + L_i$ agree ensures that if one of the terms in the numerator is zero, then none of the terms in the denominator can be zero, and if one of the terms in the denominator is zero, then none of terms in the numerator can be zero. Observe also that the cross ratio does not depend on the linear identification we chose, and also depends neither on the choice of basis $\{m_1, \dots, m_{n-2}\}$ for $M$, nor the choice of representatives $l_i$ for $L_i$. Hence, the cross ratio is a well-defined.

The next proposition summarizes some basic properties of this cross ratio.

**Proposition 2.2.2.** *Let $L_1, \dots, L_5$ be pairwise distinct lines in $\mathbb{R}^n$ through the origin. Let $M$, $M'$ be $(n-2)$-dimensional subspaces of $\mathbb{R}^n$ not containing $L_i$ for any $i = 1, \dots, 5$, so that no three of the five $(n-1)$-dimensional subspaces $M + L_i$ agree and no three of the five $(n-1)$-dimensional subspaces $M' + L_i$ agree.*

*(1) For any $X$ in $PGL(n, \mathbb{R})$, $(X \cdot L_1, \dots, X \cdot L_4)_{X \cdot M} = (L_1, \dots, L_4)_M$.*

*(2) For all $i$, let $L'_i$ be a line in $\mathbb{R}^n$ such that $L'_i \subset M + L_i$ and $L'_i \not\subset M$. Then*

$$(L'_1, L'_2, L'_3, L'_4)_M = (L_1, L_2, L_3, L_4)_M.$$

*(3) Suppose $L_1, L_2, L_3, L_4$ lie in a plane. Then*

$$(L_1, L_2, L_3, L_4)_M = (L_1, L_2, L_3, L_4)_{M'}.$$

*(4) If $M + L_1$, $M + L_2$, $M + L_3$ are pairwise distinct, then*

$$(L_1, L_1, L_2, L_3)_M = (L_1, L_2, L_3, L_3)_M = \infty.$$

*(5) If $M + L_1$, $M + L_2$, $M + L_3$ are pairwise distinct, then*

$$(L_1, L_2, L_2, L_3)_M = (L_1, L_2, L_3, L_1)_M = 1.$$

*(6)* $(L_1, L_2, L_3, L_4)_M = (L_4, L_3, L_2, L_1)_M$.

*(7)* $(L_1, L_2, L_3, L_4)_M = 1 - (L_2, L_1, L_3, L_4)_M$.

*(8)* $(L_1, L_2, L_3, L_5)_M \cdot (L_1, L_3, L_4, L_5)_M = (L_1, L_2, L_4, L_5)_M$.

*Proof.* It is clear that (1) holds because each of the four terms in the definition of the cross ratio is a constant multiple of the volume form, so $X$ scales each of these terms by the same amount. (2) follows from the observation that replacing any of the $l_i$ in Definition 2.2.1 with a linear combination of $m_1, \dots, m_{n-2}, l_i$ so that the coefficient of $l_i$ is non-zero does not change the cross ratio. To prove (3), note that we can choose $X$ in $PGL(n, \mathbb{R})$ so that $X \cdot M = M'$ and $X$ fixes $L_1$, $L_2$, $L_3$. Since $L_1$, $L_2$, $L_3$ and $L_4$ lie in a plane, $X$ also fixes $L_4$. The $PGL(n, \mathbb{R})$-invariance of the cross ratio stated in (1) then proves (3). Parts (4), (5), (6) and (8) are immediate from the formula in Definition 2.2.1.

Now, we will prove (7). By (2), we can assume without loss of generality that $L_2$ and $L_3$ lie in $L_1 + L_4$. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$, and choose a normalization so that $M = \mathrm{Span}\{e_1, \dots, e_{n-2}\}$, $L_1 = [e_{n-1}]$, $L_4 = [e_n]$, $L_2 = [e_{n-1} + e_n]$. Then $L_3 = [a \cdot e_{n-1} + (1 - a) \cdot e_n]$ for some $a \in \mathbb{R} \cup \{\infty\}$, where we adopt the convention that $[\infty \cdot e_{n-1} + (1 - \infty) \cdot e_n] = [e_{n-1} - e_n]$. From the cross ratio definition, we can compute that

$$(L_1, L_2, L_3, L_4)_M = \frac{1-a}{a} \text{ and } (L_2, L_1, L_3, L_4) = 1 - \frac{1-a}{a},$$

with the convention that $\dfrac{1-\infty}{\infty} = -1$. $\square$

Parts (1) and (2) of Proposition 2.2.2 allows one to think of the cross ratio as a projective invariant associated to four $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ that intersect along a $(n-2)$-dimensional subspace. Also, in view of (3) of Proposition 2.2.2, we will denote $(L_1, L_2, L_3, L_4)_M$ by $(L_1, L_2, L_3, L_4)$ in the case when $L_1$, $L_2$, $L_3$, $L_4$ lie in the same plane.

Suppose that $L_1, \dots, L_4$ are four points on $\mathbb{RP}^{n-1}$ that lie in the same projective line $P$ (or alternatively, four lines in $\mathbb{R}^n$ that lie in the same plane) so that no three of the four agree. For any projective transformation $\phi : P \to \mathbb{RP}^1$, we can identify each $L_i$ with the extended real number $\frac{a}{b}$, where $\phi(L_i) = [a : b]^T$. This allows us to define the quantity

$$C(L_1, \dots, L_4) = \frac{(L_1 - L_3)(L_4 - L_2)}{(L_1 - L_2)(L_4 - L_3)},$$

with the adopted convention that $\infty - \infty = 0$. The assumption that no three of the four $L_i$ agree ensures that given $\phi$, this quantity is well defined. An easy computation then verifies that it is independent of the choice of $\phi$.

The following proposition summarizes several alternative definitions of the cross ratio.

**Proposition 2.2.3.** *Let $L_1 = [l_1], \dots, L_4 = [l_4]$ be pairwise distinct lines in $\mathbb{R}^n$ through the origin and let $M$ be a $(n-2)$-dimensional subspace of $\mathbb{R}^n$ not containing $L_i$ for any $i = 1, \dots, 4$, so that no three of the four $(n-1)$-dimensional subspaces $M + L_i$ agree.*

*(1) If $L_1, \dots, L_4$ lie on a projective line, then $(L_1, L_2, L_3, L_4) = C(L_1, L_2, L_3, L_4)$.*

*(2) Let $L_i^*$ be the line in $(\mathbb{R}^n)^*$ dual to $M + L_i$ and let $M^*$ be the plane in $(\mathbb{R}^n)^*$ dual to $M$. Then $L_1^*, \dots, L_4^*$ lie in the projective line $M^*$ and*

$$(L_1, L_2, L_3, L_4)_M = (L_1^*, L_2^*, L_3^*, L_4^*).$$

*(3) For $i = 1, \dots, 4$, let $c_i \in (\mathbb{R}^n)^*$ be a linear functional with kernel $M + L_i$. Then*

$$(L_1, L_2, L_3, L_4)_M = \frac{c_1(l_3) \cdot c_4(l_2)}{c_1(l_2) \cdot c_4(l_3)}.$$

*Proof.* In this proof, we will assume that the lines $L_1$, $L_2$ and $L_4$ are pairwise distinct; the other cases are similar.

Proof of (1). Since $L_1, \dots, L_4$ lie on a projective line in $\mathbb{RP}^n$, call it $\mathbb{P}(P)$, we can assume without loss of generality that $l_2 = l_1 + l_4$ and $l_3 = al_1 + (1-a)l_4$ for some $a \in \mathbb{R} \cup \{\infty\}$. One can then compute from the definition of the cross ratio that

$$(L_1, L_2, L_3, L_4) = \frac{1-a}{a}.$$

On the other hand, the linear transformation $F : P \to \mathbb{R}^2$ so that

$$F(l_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, F(l_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

induces a projective transformation $\phi : \mathbb{P}(P) \to \mathbb{RP}^1$ so that

$$\phi(L_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \phi(L_4) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \phi(L_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \phi(L_3) = \begin{bmatrix} a \\ 1 \end{bmatrix}.$$

Using this, one can also compute that $C(L_1, L_2, L_3, L_4) = \dfrac{1-a}{a}$.

Proof of (2). Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$. By a suitable choice of normalization and (2) of Proposition 2.2.2, we can assume without loss of generality that

$M = \mathrm{Span}(e_1, \dots, e_{n-2})$, $l_1 = e_{n-1}$, $l_4 = e_n$, $l_2 = e_{n-1} + e_n$ and $l_3 = ae_{n-1} + (1-a)e_n$ for some $a \in \mathbb{R} \cup \{\infty\}$. Observe that

$$\begin{aligned} L_1^* &= [0 : \cdots : 0 : 0 : -1] \\ L_4^* &= [0 : \cdots : 0 : 1 : 0] \\ L_2^* &= [0 : \cdots : 0 : 1 : -1] \\ L_3^* &= [0 : \cdots : 0 : 1-a : -a] \end{aligned}$$

so the computation in the proof of (1) shows that $(L_1^*, L_2^*, L_3^*, L_4^*) = \dfrac{1-a}{a}$.

Proof of (3). First, note that since the $l_i$ and $c_i$ are well-defined up to scaling, the fraction

$$\frac{c_1(l_3) \cdot c_4(l_2)}{c_1(l_2) \cdot c_4(l_3)}$$

is well-defined. Choose the normalization that we used in the proof of (2), and choose the linear identification $\bigwedge^n \mathbb{R}^n \to \mathbb{R}$ so that $\bigwedge^n e_i \mapsto 1$. By scaling $l_i$ if necessary, we can ensure that under this linear identification, $c_i(l_j) = e_1 \wedge \cdots \wedge e_{n-2} \wedge l_i \wedge l_j$ for $i = 1, 4$ and $j = 2, 3$. $\square$

In particular, Proposition 2.2.3 proves that the cross ratio is a one dimensional phenomenon. More concretely, by taking the dual, the cross ratio in $\mathbb{RP}^n$ can be converted to a projective invariant that describes four points on a projective line up to projective transformations.

There will be two main ways we use the cross ratio. The first is to capture some eigenvalue data of $X$ in $PSL(n, \mathbb{R})$. This is described in the next proposition.

**Proposition 2.2.4.** *Let $X \in PGL(n, \mathbb{R})$ be an element with a representative in $GL(n, \mathbb{R})$ that is diagonalizable with $n$ real, positive, pairwise distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$. Let $X^-$ and $X^+$ be the eigenspaces corresponding to $\lambda_1$ and $\lambda_n$ respectively, and let $M$ be the sum of the eigenspaces corresponding to $\lambda_2, \dots, \lambda_{n-1}$. Then for any line $L$ in $\mathbb{R}^n$ through the origin such that $L \not\subset M + X^-$ and $L \not\subset M + X^+$, we have*

$$(X^-, L, X \cdot L, X^+)_M = \frac{\lambda_n}{\lambda_1}.$$

*Proof.* Let $e_i$ be an eigenvector of $X$ corresponding to the eigenvalue $\lambda_i$, and let $l =$

$\sum_{i=1}^{n} \alpha_i e_i$ be a vector such that $[l] = L$. Then

$$\begin{aligned}
& (X^-, L, X \cdot L, X^+)_M \\
= \quad & \frac{e_2 \wedge \cdots \wedge e_{n-1} \wedge e_1 \wedge (\alpha_n \lambda_n e_n) \cdot e_2 \wedge \cdots \wedge e_{n-1} \wedge e_n \wedge (\alpha_1 e_1)}{e_2 \wedge \cdots \wedge e_{n-1} \wedge e_1 \wedge (\alpha_n e_n) \cdot e_2 \wedge \cdots \wedge e_{n-1} \wedge e_n \wedge (\alpha_1 \lambda_1 e_1)} \\
= \quad & \frac{\lambda_n}{\lambda_1}.
\end{aligned}$$

□

Given three pairwise distinct $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ that intersect along a common $(n-2)$-dimensional subspace $M$, we can also use this cross ratio to parameterize the set of $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ that contain $M$. More precisely, we have the following proposition.

**Proposition 2.2.5.** *Let $M$ be any $(n-2)$-dimensional subspace of $\mathbb{R}^n$, and let $N_1$, $N_2$, $N_3$ be pairwise distinct $(n-1)$-dimensional subspaces in $\mathbb{R}^n$ that contain $M$. For $i = 1, 2, 3$, let $L_i$ be a line through the origin in $N_i$ that does not lie in $M$. Denote the space of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ containing $M$ by $\mathcal{S}$, and for any $N$ in $\mathcal{S}$, let $L_N$ be any line through the origin in $N$ but not in $M$. Then the map*

$$f : \mathcal{S} \to \mathbb{R} \cup \{\infty\}$$

*given by*

$$f(N) = (L_1, L_N, L_2, L_3)_M$$

*is a homeomorphism. Moreover, $f(N_1) = \infty$, $f(N_2) = 1$ and $f(N_3) = 0$.*

*Proof.* First, note that by (2) of Proposition 2.2.2, $f$ is independent of the choice of $L_1$, $L_2$, $L_3$ and $L_N$. For convenience, we choose $L_2 = N_2 \cap (L_1 + L_3)$ and $L_N = N \cap (L_1 + L_3)$. Choose vectors $l_1$, $l_2$, $l_3$, $l_N$ in $\mathbb{R}^n$ so that $L_i = [l_i]$ for $i = 1, 2, 3$, $L_N = [l_N]$. By scaling each $l_i$ and $l_N$ by a real number if necessary, we can assume that $l_2 = l_1 + l_3$ and $l_N = a \cdot l_1 + (1-a) \cdot l_3$ for some $a \in \mathbb{R} \cup \{\infty\}$. One can then compute that

$$(L_1, L_N, L_2, L_3)_M = \frac{a}{1-a},$$

from which the proposition follows immediately. □

As an immediate consequence of Proposition 2.2.5, we have the following interpretation of the sign of the cross ratio.

**Corollary 2.2.6.** *Let $M$ be a $(n-2)$-dimensional subspace of $\mathbb{R}^n$, and let $L_1, \dots, L_4$ be four lines in $\mathbb{R}^n$ that do not lie in $M$, so that no three of the four $M + L_i$ agree. The cross ratio $(L_1, L_2, L_3, L_4)_M$ is positive if and only if the points $L_2$ and $L_3$ in $\mathbb{RP}^{n-1}$ lie in the same connected component of $\mathbb{RP}^{n-1} \setminus \big((M + L_1) \cup (M + L_4)\big)$.*

The main objects we will be using the cross ratio (and later the triple ratio) to study are flags in $\mathbb{R}^n$, which we will define now.

**Definition 2.2.7.**

(1) A *(complete) flag* in $\mathbb{R}^n$ is a nested sequence of $n$ linear subspaces in $\mathbb{R}^n$, each properly contained in its predecessor. Let $\mathcal{F}(\mathbb{R}^n)$ denote the space of flags in $\mathbb{R}^n$, and for any $F$ in $\mathcal{F}(\mathbb{R}^n)$, let $F^{(l)}$ be the $l$-dimensional subspace of $F$.

(2) A collection of $k$ flags $F_1, \dots, F_k$ in $\mathcal{F}(\mathbb{R}^n)$ is *generic* if for all positive integers $n_1, \dots, n_k$ such that $\displaystyle\sum_{i=1}^{k} n_i = n$, we have $\displaystyle\sum_{i=1}^{k} F_i^{(n_i)} = \mathbb{R}^n$.

**For the rest of this section, we will adopt the following notation.** Let $F_1, F_2$ be a generic pair of flags in $\mathcal{F}(\mathbb{R}^n)$ and $L_1$, $L_2$ be a pair of lines in $\mathbb{R}^n$ that do not lie in $F_1^{(k)} + F_2^{(n-k-1)}$ for any $k = 0, \dots, n-1$. We call such a collection $F_1, F_2, L_1, L_2$ a *generic collection*. For all $i = 1, \dots, n$, $T_i := F_1^{(i)} \cap F_2^{(n-i+1)}$ is a line in $\mathbb{R}^n$, and the set of lines $\{T_1, \dots, T_n\}$ span $\mathbb{R}^n$. Hence, for $k = 1, \dots, n-1$,

$$M_k := \sum_{i \neq k, k+1} T_i$$

is a $n-2$ dimensional subspace of $\mathbb{R}^n$, so we can define

$$S_k(F_1, F_2, L_1, L_2) := -(T_k, L_1, L_2, T_{k+1})_{M_k}.$$

Proposition 2.2.5 also has the following consequence, which explains the geometric data that the $n-1$ cross ratios $\{S_k(F_1, F_2, L_1, L_2) : k = 1, \dots, n-1\}$ encode.

**Lemma 2.2.8.**

*(1) Let $r_1, \dots, r_{n-1}$ be any collection of $n-1$ non-zero real numbers, let $F_1$ and $F_2$ be a generic pair of flags in $\mathcal{F}(\mathbb{R}^n)$, and let $L_2$ be a line that does not lie in $F_1^{(i)} + F_2^{(n-i-1)}$ for any $i = 0, \dots, n-1$. Then there is a unique line $L_1$ in $\mathbb{R}^n$ so that $F_1$, $F_2$, $L_1$, $L_2$ is a generic collection and $S_k(F_1, F_2, L_1, L_2) = r_k$ for all $k = 1, \dots, n-1$.*

*(2) Let $F_1, F_2, L_1, L_2$ and $F_1', F_2', L_1', L_2'$ be two generic collections of a pair of flags and a pair of lines, so that $S_k(F_1, F_2, L_1, L_2) = S_k(F_1', F_2', L_1', L_2')$ are non-zero real numbers for all $k = 1, \ldots, n-1$. Then there is a projective transformation $\phi$ so that $\phi(F_i) = F_i'$ and $\phi(L_i) = L_i'$ for $i = 1, 2$.*

*Proof.* Proof of (1). Fix any $k = 1, \ldots, n-1$, let $N$ be a hyperplane in $\mathbb{R}^n$ containing the $(n-2)$-dimensional subspace $M_k$, and let $L_N$ be a line in $N$ that does not lie in $M_k$. Since $F_1$, $F_2$ and $L_2$ determine the hyperplanes $M_k + T_k$, $M_k + T_{k+1}$ and $M_k + L_2$, Proposition 2.2.5 tells us that there is a unique hyperplane containing $M_k$, call it $N_k$, so that $(T_k, L_{N_k}, L_2, T_{k+1})_{M_k} = r_k$. Doing this for all $k = 1, \ldots, n-1$, we have the $n-1$ hyperplanes $N_1, \ldots, N_{n-1}$.

Observe that the intersection $\displaystyle\bigcap_{i=1}^{n-1} N_i$ is at least 1-dimensional, so it contains a line, call it $L_1$. By (2) of Proposition 2.2.2, we can assume without loss of generality that $L_{N_k}$ lies in $T_k + T_{k+1}$ for all $k = 1, \ldots, n-1$. Hence, there are vectors $\{t_1, \ldots, t_n\}$ in $\mathbb{R}^n$ so that $t_i \in T_i$ for all $i = 1, \ldots, n-1$, and $L_{N_k} = [t_k + t_{k+1}]$ for all $k = 1, \ldots, n-1$.

Suppose for contradiction that $L_1 \subset M_j$ for some $j = 1 \ldots, n-1$. Then

$$L_1 \subset \left( \bigcap_{k \neq j} N_k \right) \cap M_j.$$

On the other hand, since $L_{N_{j-1}} = [t_{j-1} + t_j]$, $L_{N_{j+1}} = [t_{j+1} + t_{j+2}]$ and $\{t_1, \ldots, t_n\}$ is a basis for $\mathbb{R}^n$, we see that $N_{j-1} \cap M_j = M_{j-1} \cap M_j$ and $M_j \cap N_{j+1} = M_j \cap M_{j+1}$. This implies that $\displaystyle\left( \bigcap_{k \neq j} N_k \right) \cap M_j = \bigcap_{k=1}^{n-1} M_k$. However, the linear independence of $\{t_1, \ldots, t_n\}$ tells us that $\displaystyle\bigcap_{k=1}^{n-1} M_k = \{0\}$, which is impossible since $\displaystyle L_1 \subset \bigcap_{k=1}^{n-1} M_k$. As such, for all $k = 1, \ldots, n-1$, $L_1$ lies in $N_k$ but not in $M_k$, so $N_k = M_k + L_1$. It is then clear by construction that with this choice of $L_1$, we have $S_k(F_1, F_2, L_1, L_2) = r_k$ for all $k = 1, \ldots, n-1$.

Since $N_k = M_k + L_1$, we see that

$$\bigcap_{k=1}^{n-1} N_k = \bigcap_{k=1}^{n-1} (M_k + L_1) = \left( \bigcap_{k=1}^{n-1} M_k \right) + L_1 = L_1.$$

Hence, any line $L$ that satisfies $S_k(F_1, F_2, L, L_2) = r_k$ for all $k = 1, \ldots, n-1$ must lie in $\displaystyle\bigcap_{k=1}^{n-1} N_k$, and is thus equal to $L_1$. Furthermore, for each $k = 1, \ldots, n-1$, the fact that $r_k$ is a real number implies that $M_k + L_1 \neq M_k + T_{k+1}$ and the fact that $r_k \neq 0$ implies that

$M_k + L_1 \neq M_k + T_k$, so $F_1, F_2, L_1, L_2$ is a generic collection.

Proof of (2). We can assume that $F_1$, $F_2$, $L_1$, $L_2$ is the generic collection constructed in (1). For all $i = 1, \dots, n$, let $T'_i := F_1'^{(i)} \cap F_2'^{(n-i+1)}$, and note that $\{T_1, \dots, T_n, L_2\}$ and $\{T'_1, \dots, T'_n, L'_2\}$ are both collections of $n+1$ generic points in $\mathbb{RP}^{n-1}$. Hence, there is a unique projective transformation $\phi$ that maps each $T'_i$ to $T_i$ and $L'_2$ to $L_2$. In particular, $\phi(F'_i) = F_i$ for $i = 1, 2$ and $\phi(L'_2) = L_2$. Since the cross ratio is a projective invariant, the uniqueness in the statement of (1) implies that $\phi(L'_1) = L_1$. $\square$

## 2.3 Triple ratio

The triple ratio was first introduced by Goncharov [20] in the case of $\mathbb{P}^2$, but was first used by Fock-Goncharov [18] to parameterize flags, to great effectiveness. The triple ratio also appeared in Bonahon-Dreyer [6] in their version of the parameterization of the Hitchin component. We will now define this projective invariant.

**Definition 2.3.1.** Let $M = \mathrm{Span}_{\mathbb{R}}\{m_1, \dots, m_{n-3}\}$ be a $(n-3)$-dimensional subspace of $\mathbb{R}^n$. For $i = 1, 2, 3$, let $L_i = [l_i]$ be a line in $\mathbb{R}^n$ through the origin and let $P_i = \mathrm{Span}_{\mathbb{R}}\{l_i, p_i\}$ be a plane containing $L_i$ so that the following properties holds:

(1) $M + P_i$ is a hyperplane in $\mathbb{R}^n$ for all $i = 1, 2, 3$.

(2) $(M + P_1) \cap (M + P_2) \cap (M + P_3) = M$.

(3) $M + L_1 + L_2 + L_3 = \mathbb{R}^n$.

We call such a collection $(M, L_1, P_1, L_2, P_2, L_3, P_3)$ *ratioable*. For such a collection, we can define the *triple ratio* to be the quantity

$$\begin{aligned}(L_1, P_1, L_2, P_2, L_3, P_3)_M := \frac{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_1 \wedge l_3 \wedge p_3}{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_1 \wedge l_2 \wedge p_2} \cdot \\ \frac{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_1 \wedge p_1 \wedge l_2}{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_2 \wedge l_3 \wedge p_3} \cdot \frac{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_2 \wedge p_2 \wedge l_3}{m_1 \wedge \cdots \wedge m_{n-3} \wedge l_1 \wedge p_1 \wedge l_3}.\end{aligned}$$

As before, by choosing a linear identification between $\bigwedge^n(\mathbb{R}^n)$ and $\mathbb{R}$, we can evaluate the expression on the right as a number in $\mathbb{R} \cup \{\infty\}$. One can then check that this is independent of the linear identification we chose, and of the choice of basis for $M$, $L_i$ and $P_i$ for $i = 1, 2, 3$. The next proposition highlights some basic properties of the triple ratio.

**Proposition 2.3.2.** *Let $(M, L_1, P_1, L_2, P_2, L_3, P_3)$ be ratioable. Then the following hold.*

*(1) For any $X \in PGL(n, \mathbb{R})$,*

$$(L_1, P_1, L_2, P_2, L_3, P_3)_M = (X \cdot L_1, X \cdot P_1, X \cdot L_2, X \cdot P_2, X \cdot L_3, X \cdot P_3)_{X \cdot M}.$$

*(2) For $i = 1, 2, 3$, let $L_i'$ be lines in $\mathbb{R}^n$ so that $M + L_i = M + L_i'$ and let $P_i'$ be planes in $\mathbb{R}^n$ containing $L_i'$ so that $M + P_i = M + P_i'$. Then*

$$(L_1, P_1, L_2, P_2, L_3, P_3)_M = (L_1', P_1', L_2', P_2', L_3', P_3')_M.$$

*(3) Suppose that for $i = 1, 2, 3$, $P_i \subset L_1 + L_2 + L_3$. If $M'$ is another $(n-3)$-dimensional subspace of $\mathbb{R}^n$ so that $(M', L_1, P_1, L_2, P_2, L_3, P_3)$ is ratioable, then*

$$(L_1, P_1, L_2, P_2, L_3, P_3)_M = (L_1, P_1, L_2, P_2, L_3, P_3)_{M'}.$$

*(4) $(L_1, P_1, L_2, P_2, L_3, P_3)_M = (L_3, P_3, L_1, P_1, L_2, P_2)_M = (L_2, P_2, L_3, P_3, L_1, P_1)_M$.*

*(5) $(L_1, P_1, L_2, P_2, L_3, P_3)_M \cdot (L_2, P_2, L_1, P_1, L_3, P_3)_M = 1$.*

*Proof.* The proofs of (1), (2) and (3) are very similar to the proofs of (1), (2) and (3) of Proposition 2.2.2. Parts (4) and (5) are immediate from the definition. □

Part (2) of the above proposition shows that the triple ratio is really a projective invariant associated to the $(n-2)$-dimensional subspaces $M + L_i$ and the hyperplanes $M + P_i$ for $i = 1, 2, 3$. In particular, when computing the triple ratio, we can always choose $L_1$, $L_2$, $L_3$, $P_1$, $P_2$ and $P_3$ so that for all $i = 1, 2, 3$, $P_i$ lies in the three-dimensional subspace $L_1 + L_2 + L_3$ of $\mathbb{R}^n$. Also, in view of (3) of the above proposition, we will denote $(L_1, P_1, L_2, P_2, L_3, P_3)_M$ by $(L_1, P_1, L_2, P_2, L_3, P_3)$ in the case when $P_i \subset L_1 + L_2 + L_3$ for $i = 1, 2, 3$.

In the case when $n = 3$, the pairs $L_i \subset P_i$ for $i = 1, 2, 3$ are flags in $\mathbb{R}^3$ and $M = \{0\}$. The ratioability conditions on the $L_i$ and $P_i$ in Definition 2.3.1 simply ensures that the projective points $L_1$, $L_2$ and $L_3$ are not collinear and the projective lines $P_1$, $P_2$ and $P_3$ do not have a common point of intersection.

Just like the cross ratio, one can simplify the definition of the triple ratio by taking the dual. However, unlike the cross ratio, which can be converted into a one-dimensional projective invariant, the triple ratio can only be converted into a two-dimensional projective invariant.

**Proposition 2.3.3.** *Let $(M, L_1, P_1, L_2, P_2, L_3, P_3)$ be a ratioable, let $M^*$ be the three dimensional subspace in $(\mathbb{R}^n)^*$ dual to $M$, let $L_i^*$ be the planes in $(\mathbb{R}^n)^*$ dual to $M + L_i$ and* *let $P_i^*$ be the lines in $(\mathbb{R}^n)^*$ dual to $M+P_i$. Then*

$$\frac{1}{(L_1,P_1,L_2,P_2,L_3,P_3)_M} = (P_1^*,L_1^*,P_2^*,L_2^*,P_3^*,L_3^*),$$

*where the quantity on the right hand side is the triple ratio in $M^*$.*

*Proof.* Let $\{e_1,\dots,e_n\}$ be the standard basis for $\mathbb{R}^n$ and for $i=1,2,3$, let $l_i$ be a vectors in $\mathbb{R}^n$ so that $L_i=[l_i]$. By (2) of Proposition 2.3.2, we can assume without loss of generality that $P_i \subset L_1+L_2+L_3$ for all $i=1,2,3$. Choose a normalization so that $M=\mathrm{Span}_{\mathbb{R}}\{e_1,\dots,e_{n-3}\}$, $l_i=e_{n-3+i}$ for $i=1,2,3$. We can choose $p_i\in P_i$ so that $P_i=\mathrm{Span}_{\mathbb{R}}\{l_i,p_i\}$ and

$$p_1=al_2+(1-a)l_3,\ p_2=bl_3+(1-b)l_1,\ p_3=cl_1+(1-c)l_2$$

for some $a,b,c\in\mathbb{R}\cup\{\infty\}$. A computation using the definition of the triple ratio gives

$$(L_1,P_1,L_2,P_2,L_3,P_3)_M=\frac{(a-1)(b-1)(c-1)}{abc}.$$

In the dual basis, $\{e_1^*,\dots,e_n^*\}$ to the standard basis of $\mathbb{R}^n$, we can write

$$\begin{aligned}
M^* &= \mathrm{Span}_{\mathbb{R}}\{e_{n-2}^*,e_{n-1}^*,e_n^*\}\\
L_1^* &= \mathrm{Span}_{\mathbb{R}}\{e_{n-1}^*,e_n^*\}\\
L_2^* &= \mathrm{Span}_{\mathbb{R}}\{e_{n-2}^*,e_n^*\}\\
L_3^* &= \mathrm{Span}_{\mathbb{R}}\{e_{n-2}^*,e_{n-1}^*\}\\
P_1^* &= [(a-1)e_{n-1}^*+ae_n^*]\\
P_2^* &= [(b-1)e_n^*+be_{n-2}^*]\\
P_3^* &= [(c-1)e_{n-2}^*+ce_{n-1}^*]
\end{aligned}$$

A direct computation will then show that

$$(P_1^*,L_1^*,P_2^*,L_2^*,P_3^*,L_3^*)_{\{0\}}=\frac{abc}{(a-1)(b-1)(c-1)}.$$

□

As in the case of the cross ratio, the triple ratio can also be used to parameterize the hyperplanes in $\mathbb{R}^n$ that contains a particular $(n-2)$-dimensional subspace or $\mathbb{R}^n$. The next proposition is the analog of Proposition 2.2.5 for triple ratios.

**Proposition 2.3.4.** *Let $M$ be any $(n-3)$-dimensional subspace of $\mathbb{R}^n$ and let $R_1$, $R_2$, $R_3$ be $(n-2)$-dimensional subspaces of $\mathbb{R}^n$ that contain $M$, so that $R_1+R_2+R_3=\mathbb{R}^n$. Also, let $N_1$ and $N_3$ be hyperplanes in $\mathbb{R}^n$ containing $R_1$ and $R_3$ respectively so that $N_1 \neq N_3$. For $i=1,2,3$, let $L_i$ be a line in $R_i$ that is not in $M$, and for $i=1,3$, let $P_i$ be a plane in $N_i$ that is transverse to $M$ and contains $L_i$. Denote the space of hyperplanes in $\mathbb{R}^n$ containing $R_2$ by $\mathcal{S}$, and for any $N$ in $\mathcal{S}$, let $P_N$ be a plane in $N$ that is transverse to $M$ and contains $L_2$. Then the map*

$$f:\mathcal{S}\to\mathbb{R}\cup\{\infty\}$$

*given by*

$$f(N)=(L_1,P_1,L_2,P_N,L_3,P_3)_M$$

*is a homeomorphism with $f(N_1)=\infty$ and $f(N_3)=0$.*

Observe that by (2) of Lemma 2.3.2, the map $f$ in the proposition above is well-defined.

*Proof.* Choose any $N\in\mathcal{S}$. Let $\{m_1,\dots,m_{n-3}\}$ be a basis for $M$ and let $l_1$, $l_2$ and $l_3$ be vectors in $\mathbb{R}^n$ so that $L_i=[l_i]$ for $i=1,2,3$. Observe that $\{m_1,\dots,m_{n-3},l_1,l_2,l_3\}$ is a basis for $\mathbb{R}^n$, and by Lemma 2.3.2, we can assume without loss of generality that $P_1$, $P_3$ and $P_N$ lie in $\mathrm{Span}_{\mathbb{R}}\{l_1,l_2,l_3\}$.

By rescaling $l_1$, $l_2$, $l_3$ if necessary, we can choose vectors $p_1$, $p_N$ and $p_3$ so that $P_1=\mathrm{Span}_{\mathbb{R}}\{l_1,p_1\}$, $P_3=\mathrm{Span}_{\mathbb{R}}\{l_3,p_3\}$, $P_N=\mathrm{Span}_{\mathbb{R}}\{l_2,p_N\}$ and

$$\begin{aligned} p_1 &= l_2+l_3,\\ p_3 &= l_1+l_2.\\ p_N &= al_1+(1-a)l_3 \end{aligned}$$

for some $a\in\mathbb{R}\cup\{\infty\}$. A straight forward computation using the definition of the triple ratio then gives

$$f(N)=\frac{a}{a-1}.$$

The proposition follows from this. $\square$

We will use the triple ratio in the following setting. Let $(F,G,H)$ be a generic triple of flags in $\mathcal{F}(\mathbb{R}^n)$. For any triple of positive integers $x,y,z$ that sum to $n$, define $M_{x,y,z}:=F^{(x-1)}+G^{(y-1)}+H^{(z-1)}$. Also, let $P_{F,x}$, $P_{G,y}$, $P_{H,z}$ be planes in $\mathbb{R}^n$ and $L_{F,x}$, $L_{G,y}$, $L_{H,z}$ be lines in $\mathbb{R}^n$, so that

1. $L_{F,x}\subset P_{F,x}$, $L_{G,y}\subset P_{G,y}$ and $L_{H,z}\subset P_{H,z}$.

2. $F^{(x-1)} + L_{F,x} = F^{(x)}$ and $F^{(x-1)} + P_{F,x} = F^{(x+1)}$.

3. $G^{(y-1)} + L_{G,y} = G^{(y)}$ and $G^{(y-1)} + P_{G,y} = G^{(y+1)}$.

4. $H^{(z-1)} + L_{H,z} = H^{(z)}$ and $H^{(z-1)} + P_{H,z} = H^{(z+1)}$.

Using this, we can define $T_{x,y,z}(F, G, H) := (L_{F,x}, P_{F,x}, L_{G,y}, P_{G,y}, L_{H,z}, P_{H,z})_{M_{x,y,z}}$. For convenience, define

$$\mathcal{A} = \mathcal{A}^n := \{(x, y, z) \in (\mathbb{Z}^+)^3 : x + y + z = n\},$$

which is the indexing set of the triple ratios associated to any triple of flags. Part (4) of Proposition 2.3.2 immediately implies the following.

**Corollary 2.3.5.** *Let $F$, $G$, $H$ be a generic triple of flags in $\mathcal{F}(\mathbb{R}^n)$ and $(x, y, z) \in \mathcal{A}^n$. Then*

$$T_{x,y,z}(F, G, H) = T_{y,z,x}(G, H, F) = T_{z,x,y}(H, F, G).$$

The next lemma gives us a way to understand the triple ratio geometrically.

**Lemma 2.3.6.** *Fix $y = 1, \dots, n-2$ and let $r_1, \dots, r_{n-y-1}$ be non-zero real numbers. Let $F$, $H$ be a generic pair of flags in $\mathbb{R}^n$, let $\{f_1, \dots, f_n\}$ be a basis of $\mathbb{R}^n$ so that $[f_i] = F^{(i)} \cap H^{(n-i+1)}$ for all $i = 1, \dots, n$, and define $h_i = f_{n-i+1}$ for notational convenience. Also, let $G^{(y-1)} \subset G^{(y)}$ be a nested pair of subspaces in $\mathbb{R}^n$ with $\dim(G^{(y-1)}) = y - 1$ and $\dim(G^{(y)}) = y$, that are transverse to the span of any subset of $\{f_1, \dots, f_n\}$. Define $L_{F,k} := [f_k]$, $P_{F,k} := \mathrm{Span}_{\mathbb{R}}\{f_k, f_{k+1}\}$, $L_{H,k} := [h_k]$, $P_{H,k} := \mathrm{Span}_{\mathbb{R}}\{h_k, h_{k+1}\}$, and let $L_{G,y}$ be any line in $G^{(y)}$ that is not in $G^{(y-1)}$. Then there is a unique $(y+1)$-dimensional subspace $G^{(y+1)}$ of $\mathbb{R}^n$ containing $G^{(y)}$, with the property that for any pair of positive integers $x, z$ with $x + y + z = n$, we have*

$$(L_{F,x}, P_{F,x}, L_{G,y}, P_{G,y}, L_{H,z}, P_{H,z}) = r_x,$$

*where $P_{G,y}$ is any plane containing $L_{G,y}$ so that $G^{(y-1)} + P_{G,y} = G^{(y+1)}$. Furthermore, $F^{(x)} + G^{(y+1)} + H^{(n-x-y-1)} = \mathbb{R}^n$ for all $x = 0, \dots, n-y-1$.*

*Proof.* By Proposition 2.3.4, we know that for each pair of positive integers $x, z$ so that $x + y + z = n$, there is a unique hyperplane $N_x$ in $\mathbb{R}^n$ containing $M_x := F^{(x-1)} + G^{(y)} + H^{(z-1)}$ with the property that $(L_{F,x}, P_{F,x}, L_{G,y}, P_{N_x}, L_{H,z}, P_{H,z}) = r_x$. Here, $P_{N_x}$ is any plane containing $L_{G,y}$ so that $F^{(x-1)} + G^{(y-1)} + H^{(z-1)} + P_{N,x} = N_x$. Also, let $v_x$ be a vector in $\mathbb{R}^n$ that spans the line $N_x \cap \mathrm{Span}_{\mathbb{R}}\{f_x, h_z\}$. Observe that $M_x + [v_x] = N_x$, and since $r_x$ is a non-zero real number for all $x = 1, \ldots, n-y-1$, Proposition 2.3.4 implies that $v_x$ does not lie in $M_{x-1}$ or $M_{x+1}$. In particular, $v_x = af_x + bh_z$ for non-zero real numbers $a, b$.

Note that $G^{(y)} \subset \bigcap_{x=1}^{n-y-1} N_x$ and $\dim\left(\bigcap_{x=1}^{n-y-1} N_x\right) \geq y+1$, so there is some line $L$ that lies in $\bigcap_{x=1}^{n-y-1} N_x$ but not in $G^{(y)}$. Suppose for contradiction that $L \subset M_i$ for some $i = 1, \ldots, n-y-1$. Then

$$L \subset M_i \cap \left(\bigcap_{x \neq i} N_x\right).$$

However, $v_{x-1}, v_{x+1}$ do not lie in $M_x$, so

$$N_{x-1} \cap M_x = M_{x-1} \cap M_x \text{ and } M_x \cap N_{x+1} = M_x \cap M_{x+1}$$

for all $x = 1, \ldots, n-y-1$. This implies that $M_i \cap \left(\bigcap_{x \neq i} N_x\right) = \bigcap_{x=1}^{n-y-1} M_x = G^{(y)}$, so $L \subset G^{(y)}$, which is a contradition.

We have thus proven that for all positive integers $x, z$ such that $x+y+z = n$, $L$ does not lie in $M_x$ but lies in $N_x$, so $N_x = M_x + L$. Thus, if $l$ is a vector in $\mathbb{R}^n$ so that $[l] = L$, then $l$ is a linear combination of $f_1, \ldots, f_{x-1}, g_1, \ldots, g_y, h_1, \ldots, h_{z-1}, v_x$, where the coefficient of $v_x$ is non-zero. Define $G^{(y+1)} := G^{(y)} + L$ and note that $F^{(x-1)} + G^{(y+1)} + H^{(z-1)} = N_x$ for all positive integers $x, z$ such that $x+y+z = n$. Since

$$\bigcap_{x=1}^{n-y-1} N_x = \bigcap_{x=1}^{n-y-1} (M_x + L) = \left(\bigcap_{x=1}^{n-y-1} M_x\right) + L = G^{(y)} + L,$$

we see that $G^{(y+1)}$ is the unique $(y+1)$-dimensional subspace of $\mathbb{R}^n$ that satisfy the required triple ratio conditions.

Recall that $v_x = af_x + bh_z$ for non-zero real numbers $a, b$, so it lies in neither $F^{(x)} + G^{(y)} + H^{(n-x-y-1)}$ nor $F^{(x-1)} + G^{(y)} + H^{(n-x-y)}$. Hence,

$$F^{(x)} + G^{(y)} + H^{(n-x-y-1)} \neq N_x \neq F^{(x-1)} + G^{(y)} + H^{(n-x-y)},$$

which means that $l$ lies in neither $F^{(x)} + G^{(y)} + H^{(n-x-y-1)}$ nor $F^{(x-1)} + G^{(y)} + H^{(n-x-y)}$ for $x = 1, \ldots, n-y-1$. It follows that $F^{(x)} + G^{(y+1)} + H^{(n-x-y-1)} = \mathbb{R}^n$ for all $x = 0, \ldots, n-y-1$. $\square$

Using Lemma 2.3.6, we can prove the following, which is the analog of Lemma 2.2.8

for triple ratios.

**Lemma 2.3.7.**

*(1) Let $\{r_{x,y,z} : (x,y,z) \in \mathcal{A}^n\}$ be a collection of non-zero real numbers. Then there is a generic triple of flags $F$, $G$, $H$ in $\mathcal{F}(\mathbb{R}^n)$ so that $T_{x,y,z}(F,G,H) = r_{x,y,z}$ for all $(x,y,z) \in \mathcal{A}^n$.*

*(2) Suppose that $F$, $G$, $H$ and $F'$, $G'$, $H'$ are generic triples of flags in $\mathcal{F}(\mathbb{R}^n)$ such that $T_{x,y,z}(F,G,H) = T_{x,y,z}(F',G',H') \neq 0$ for all $(x,y,z) \in \mathcal{A}^n$. Then there is a projective transformation $\phi$ such that $\phi(F) = F'$, $\phi(G) = G'$ and $\phi(H) = H'$.*

*Proof.* Proof of (1). Choose any generic pair of flags $F$, $H$ in $\mathcal{F}(\mathbb{R}^n)$ and choose any line $G^{(1)}$ that does not lie in $F^{(x)} + H^{(n-x-1)}$ for all $x = 0, \dots, n-1$. Apply Lemma 2.3.6 inductively to construct the rest of the flag $G$.

Proof of (2). Since there is a unique projective transformation that maps $F$ to $F'$, $H$ to $H'$ and $G^{(1)}$ to $G'^{(1)}$, and the triple ratios are projective invariants, it is sufficient to show that for fixed $F$, $H$ and $G^{(1)}$, the flag $G$ is uniquely determined by the collection of real numbers $\{T_{x,y,z}(F,G,H) : (x,y,z) \in \mathcal{A}^n\}$. This follows from the uniqueness statement in Lemma 2.3.6. □

We can also make the following observation, which we record as Proposition 2.3.8.

**Proposition 2.3.8.** *Let $\{(F_i, G_i, H_i)\}_{i=1}^{\infty}$ be a sequence of generic triples of flags in $\mathcal{F}(\mathbb{R}^n)$ such that for all positive integers $i, j$, $G_i^{(1)} = G_j^{(1)}$, $F_i = F_j$ and $H_i = H_j$. Suppose that there is some $y_0 = 1, \dots, n-2$ so that for any $(x,y,z)$ in $\mathcal{A}_n$ with $y < y_0$, $\lim_{i\to\infty} T_{x,y,z}(F_i, G_i, H_i)$ is a non-zero real number. Then for any integers $x_0$, $z_0$ such that $(x_0, y_0, z_0) \in \mathcal{A}_n$, we have the following statements.*

*(1) $\lim_{i\to\infty} T_{x_0,y_0,z_0}(F_i, G_i, H_i) = \infty$ if and only if*

$$\lim_{i\to\infty} F_i^{(x_0-1)} + G_i^{(y_0+1)} + H_i^{(z_0-1)} = \lim_{i\to\infty} F_i^{(x_0)} + G_i^{(y_0)} + H_i^{(z_0-1)}.$$

*(2) $\lim_{i\to\infty} T_{x_0,y_0,z_0}(F_i, G_i, H_i) = 0$ if and only if*

$$\lim_{i\to\infty} F_i^{(x_0-1)} + G_i^{(y_0+1)} + H_i^{(z_0-1)} = \lim_{i\to\infty} F_i^{(x_0-1)} + G_i^{(y_0)} + H_i^{(z_0)}.$$

*Proof.* For any positive integer $i$, let $F^{(k)} := F_i^{(k)}$, $H^{(k)} := H_i^{(k)}$ for all $k = 1, \dots, n$ and $G^{(1)} := G_i^{(1)}$. By Lemma 2.3.6, we can construct a nested sequence of flags

$$G^{(1)} \subset \cdots \subset G^{(y_0)}$$

so that $\dim(G^{(y)}) = y$ for all $y = 1, \dots, y_0$, $T_{x,y,z}(F, G, H) = \lim_{i\to\infty} T_{x,y,z}(F, G_i, H)$ for all $(x, y, z) \in \mathcal{A}^n$ such that $y < y_0$, and $F^{(x)} + G^{(y)} + H^{(n-x-y)} = \mathbb{R}^n$ for $y = 1, \dots, y_0$ and $x = 0, \dots, n - y$. Since the set of nested subspaces

$$\{(K^{(1)}, \dots, K^{(y_0)}) : K^{(1)} \subset \cdots \subset K^{(y_0)} \subset \mathbb{R}^n \text{ and } \dim(K^{(y)}) = y \text{ for all } y = 1, \dots, y_0\}$$

is a compact manifold, every subsequence of $\{(G_i^{(1)}, \dots, G_i^{(y_0)})\}_i$ has a convergent subsequence. The uniqueness in Lemma 2.3.6 and the continuity of the triple ratio ensures that the limit of this convergent subsequence is $(G^{(1)}, \dots, G^{(y_0)})$. The fact that this holds for all subsequences of $\{(G_i^{(1)}, \dots, G_i^{(y_0)})\}_i$ ensures that $G^{(y)} = \lim_{i\to\infty} G_i^{(y)}$. for all $y = 1, \dots, y_0$.

Let $L_{F,x_0}$ be a line in $F^{(x_0)}$ but not in $F^{(x_0-1)}$, let $L_{G,y_0}$ be a line in $G^{(y_0)}$ but not in $G^{(y_0-1)}$ and let $L_{H,z_0}$ be a line in $H^{(z_0-1)}$ but not in $H^{(z_0)}$. Also, let $P_{F,x_0}$ and $P_{H,z_0}$ be planes containing $L_{F,x_0}$ and $L_{H,z_0}$ respectively, so that $P_{F,x_0} + F^{(x_0-1)} = F^{(x_0+1)}$ and $P_{H,z_0} + H^{(z_0-1)} = H^{(z_0+1)}$. Finally, let $L_{G_i,y_0+1}$ be a line in $G_i^{(y_0+1)}$ but not in $G_i^{(y_0)}$ and $G^{(y_0)}$, and let $M := F^{(x_0-1)} + G^{(y_0-1)} + H^{(z_0-1)}$.

By choosing a convergent subsequence, we can assume that the sequence of lines $\{L_{G_i,y_0+1}\}_i$ converges. Since $G_i^{(y)}$ converges to $G^{(y)}$ for all $y = 1, \dots, y_0$, we see that

$$\lim_{i\to\infty} F^{(x_0-1)} + G^{(y_0)} + L_{G_i,y_0+1} + H^{(z_0-1)} = \lim_{i\to\infty} F^{(x_0-1)} + G_i^{(y_0+1)} + H^{(z_0-1)}.$$

In particular,

$$\lim_{i\to\infty} (L_{F,x_0}, P_{F,x_0}, L_{G,y_0}, L_{G,y_0} + L_{G_i,y_0+1}, L_{H,z_0}, P_{H,z_0})_M = \lim_{i\to\infty} T_{x_0,y_0,z_0}(F, G_i, H).$$

Apply Proposition 2.3.4 to finish the proof. □

Unlike the cross ratio, the geometric meaning of the sign of the triple ratio is a little more subtle. In the case when $n = 3$ though, we have the following.

**Lemma 2.3.9.** *Let $F, G, H$ be a generic triple of flags in $\mathbb{R}^3$. Then $T_{1,1,1}(F, G, H)$ is a positive real number if and only if there is a triangle in $\mathbb{RP}^2$ with vertices $F^{(1)}$, $G^{(1)}$, $H^{(1)}$, and whose interior does not intersect the projective lines $F^{(2)}$, $G^{(2)}$ and $H^{(2)}$.*

*Proof.* First, observe that the genericity of $F, G, H$ ensures that $T_{1,1,1}(F, G, H)$ is a non-zero real number by Proposition 2.3.4. Let $\{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, and choose a normalization so that $F^{(1)} = [e_1]$, $F^{(2)} \cap H^{(2)} = [e_2]$, $H^{(1)} = [e_3]$ and $G^{(1)} = [e_1 + e_2 + e_3]$. There are four triangles in $\mathbb{RP}^2$ with vertices $[e_1]$, $[e_1 + e_2 + e_3]$, $[e_3]$, and by our normalization, it is easy to see that the only such triangle whose interior does not

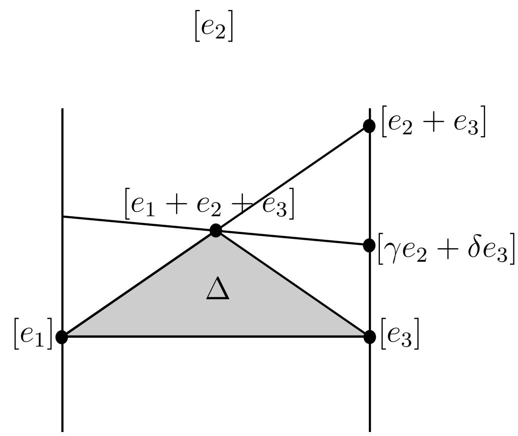


Figure 2.1: Triple ratio in the $n = 3$ case.

intersect $F^{(2)}$ and $H^{(2)}$ is

$$\Delta := \{[(\alpha + 1)e_1 + e_2 + (\beta + 1)e_3] : \alpha, \beta > 0\}.$$

Thus, we need to show that $T_{1,1,1}(F, G, H)$ is a positive real number if and only if $G^{(2)}$ does not intersect $\Delta$.

To do so, consider the point $L := G^{(2)} \cap H^{(2)}$. Since $L$ lies in $H^{(2)}$, there are real numbers $\gamma, \delta \in \mathbb{R}$ so that $L = [\gamma e_2 + \delta e_3]$. Furthermore, $F^{(1)} + G^{(1)}$ intersects $H^{(2)}$ at the point $[e_2 + e_3]$, so $G^{(2)}$ does not intersect $\Delta$ if and only if $0 < \dfrac{\gamma}{\delta} < 1$ (see Figure 2.1). At the same time, we can compute that

$$T_{1,1,1}(F, G, H) = \frac{1}{\frac{\delta}{\gamma} - 1}.$$

This finishes the proof. $\square$

By Proposition 2.3.3, we see that when we take the dual, the positivity of the triple ratio is preserved. Thus, the above lemma also gives a geometric interpretation of the positivity of any triple ratio by taking the dual.

## 2.4 The Frenet curve

To finish the projective geometry background needed to understand Hitchin representations, we will study curves $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$ that have some strong transversality properties as described in Definition 2.4.1. These curves are called Frenet curves, and will play an important role in describing Hitchin representations later. In this section, we will study some features of Frenet curves, the most important of which is a "positivity" property.

**Definition 2.4.1.** A closed curve $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$ is *Frenet* if the following two conditions are satisfied:

1. Let $x_1, \dots, x_k$ be pairwise distinct points in $S^1$ and let $n_1, \dots, n_k$ be positive integers so that $\sum_{i=1}^k n_i = n$. Then
$$\sum_{i=1}^{k} \xi(x_i)^{(n_i)} = \mathbb{R}^n.$$

2. Let $x_1, \dots, x_k$ be pairwise distinct points in $S^1$ and let $n_1, \dots, n_k$ be positive integers so that $m := \sum_{i=1}^k n_i \leq n$. Then for any $x \in S^1$,
$$\lim_{\substack{x_i \to x, \forall i \\ x_i \neq x_j, \forall i \neq j}} \sum_{i=1}^{k} \xi(x_i)^{(n_i)} = \xi(x)^{(m)}.$$

One should think of Frenet curves as having the property that points along the curve are "maximally transverse". Often, we will denote the image of the Frenet curve $\xi$ by $\xi$ as well. We can study a Frenet curve by considering its projections onto some special projective lines. To that end, we develop the following notation.

*Notation* 2.4.2. Let $M_1, \dots, M_k$ be pairwise distinct points along a Frenet curve $\xi$ and let $n_1, \dots, n_k$ be positive integers. For any positive integer $m \leq n - \displaystyle\sum_{i=1}^{k} n_i$ and for any $E$ on $\xi$, define $L_E^{(m)}$ as follows:

- if $E \neq M_i$ for all $i = 1, \dots, k$ then $L_E^{(m)} = E^{(m)}$
- if $E = M_i$ for some $i = 1, \dots, k$, then $L_E^{(m)}$ is a choice of $m$-dimensional subspace in $M_i^{(n_i+m)}$ that is transverse to $M_i^{(n_i)}$.

The notation above depends on a choice of $M_1 \dots, M_k$, which should be clear in any setting where we use this notation. Also, any statement we make involving the above notation is true for all possible choices of the $m$-dimensional subspaces in $M_i^{(n_i+m)}$ that is transverse to $M_i^{(n_i)}$.

**Lemma 2.4.3.** *Let $A, B$ be distinct points along a Frenet curve $\xi$, let $M_1, \dots, M_k$ be pairwise distinct points along $\xi$ and let $n_1, \dots, n_{k+1}$ be positive integers such that* $\displaystyle\sum_{i=1}^{k+1} n_i = n - 1$.

1. *If $n_{k+1} = 1$, the map*
$$f_1 : \xi(S^1) \to \mathbb{P}(L_A^{(1)} + L_B^{(1)})$$
*given by*
$$f_1(E) = \left( \sum_{i=1}^{k} M_i^{(n_i)} + L_E^{(1)} \right) \cap (L_A^{(1)} + L_B^{(1)})$$
*is a homeomorphism with $f_1(A) = L_A^{(1)}$ and $f_1(B) = L_B^{(1)}$.*

2. *Let $s$ be a closed subsegment of $\xi$ with endpoints $A$ and $B$. Then there exists a closed subsegment $t$ of $\mathbb{P}(L_A^{(1)} + L_B^{(1)})$ with endpoints $L_A^{(1)}$ and $L_B^{(1)}$, such that the map*
$$f_{n_{k+1}} : s \to t$$
*given by*
$$f_{n_{k+1}}(E) = \left( \sum_{i=1}^{k} M_i^{(n_i)} + L_E^{(n_{k+1})} \right) \cap (L_A^{(1)} + L_B^{(1)})$$
*is a homeomorphism with $f_{n_{k+1}}(A) = L_A^{(1)}$ and $f_{n_{k+1}}(B) = L_B^{(1)}$.*

*Proof.* Proof of (1). The continuity and well-definedness of $f_1$ is clear by the definition of the Frenet curve. Suppose for contradiction that there exist $E \neq E'$ such that $f_1(E) = f_1(E')$. Since $\xi$ is a Frenet curve, we have

$$\begin{aligned} \sum_{i=1}^{k} M_i^{(n_i)} + L_E^{(1)} &= \sum_{i=1}^{k} M_i^{(n_i)} + f_1(E) \\ &= \sum_{i=1}^{k} M_i^{(n_i)} + f_1(E') \\ &= \sum_{i=1}^{k} M_i^{(n_i)} + L_{E'}^{(1)}. \end{aligned}$$

In particular, $\sum_{i=1}^{k} M_i^{(n_i)} + L_E^{(1)} + L_{E'}^{(1)} \neq \mathbb{R}^n$, which contradicts the fact that $\xi$ is a Frenet curve. This proves that $f_1$ is an injective continuous map between two spaces homeomorphic to $S^1$, so $f_1$ has to be a homeomorphism. It is easy to verify that $f_1(A) = L_A^{(1)}$ and $f_1(B) = L_B^{(1)}$.

Proof of (2). As before, the continuity of $f_{n_{k+1}}$ is clear. We will prove that $f_{n_{k+1}}$ is a homeomorphism by induction. The base case when $n_{k+1} = 1$ follows from (1). For the inductive step, consider the case when $n_{k+1} = m + 1$. Pick any pair of distinct points $E_0$ and $E_1$ in the interior of $s$, and assume without loss of generality that $E_1$ lies between

$E_0$ and $B$ on $s$. Since $\xi$ is a Frenet curve, $f_{m+1}(E_0) \neq L_A^{(1)}, L_B^{(1)}$, so there is a unique subsegment of $\mathbb{P}(L_A^{(1)} + L_B^{(1)})$ with endpoints $L_A^{(1)}, L_B^{(1)}$ that contains $f_{m+1}(E_0)$. Let this subsegment be $t$.

By the inductive hypothesis, the map

$$f_m : s \to t$$

is a homeomorphism. Hence, the point

$$f_m(E_1) = \mathbb{P}\left(\sum_{i=1}^{k} M_i^{(n_i)} + L_{E_0}^{(1)} + L_{E_1}^{(m)}\right) \cap \mathbb{P}(L_A^{(1)} + L_B^{(1)})$$

lies on $t$, strictly between the points

$$f_m(E_0) = \mathbb{P}\left(\sum_{i=1}^{k} M_i^{(n_i)} + L_{E_0}^{(m+1)}\right) \cap \mathbb{P}(L_A^{(1)} + L_B^{(1)}) \text{ and } f_m(B) = L_B^{(1)}.$$

By the base case, the map

$$f_1 : s \to t$$

is a homeomorphism. Thus, we can conclude that the point

$$f_1(E_1) = \mathbb{P}\left(\sum_{i=1}^{k} M_i^{(n_i)} + L_{E_1}^{(m+1)}\right) \cap \mathbb{P}(L_A^{(1)} + L_B^{(1)})$$

lies on $t$, strictly between the points

$$f_1(E_0) = \mathbb{P}\left(\sum_{i=1}^{k} M_i^{(n_i)} + L_{E_1}^{(m)} + L_{E_0}^{(1)}\right) \cap \mathbb{P}(L_A^{(1)} + L_B^{(1)}) \text{ and } f_1(B) = L_B^{(1)}.$$

Since $f_m(E_1) = f_1(E_0)$, $f_{m+1}(E_0) = f_m(E_0)$ and $f_{m+1}(E_1) = f_1(E_1)$, we see in particular that $f_{m+1}(E_0) \neq f_{m+1}(E_1)$, so $f_{m+1}$ is injective. It is clear that $f_{m+1}(A) = L_A^{(1)}$ and $f_{m+1}(B) = L_B^{(1)}$, so the continuity of $f_{m+1}$ implies that it is surjective. This finishes the inductive step. $\square$

The homeomorphisms $f_{n_{k+1}}$ should be thought of as projections of subsegments of $\xi$ (or all of $\xi$ in the case when $n_{k+1} = 1$) onto the projective line $\mathbb{P}(L_A^{(1)} + L_B^{(1)})$ via the "base" $(n - 1 - n_{k+1})$-dimensional subspace $\sum_{i=1}^{k} M_i^{(n_i)}$ of $\mathbb{R}^n$.

Next, we discuss how the cross ratio interacts with a Frenet curve. For that purpose, we introduce the following notation.

*Notation* 2.4.4. Let $A, B, C, D$ be pairwise distinct points along a Frenet curve $\xi$. Let $M_1, \dots, M_k$ be another set of pairwise distinct points along $\xi$ and let $n_1, \dots n_k$ be positive integers such that $\sum_{i=1}^{k} n_i = n - 2$. Let $M := \sum_{i=1}^{k} M_i^{(n_i)}$, and denote

$$(A, B, C, D)_M := (L_A^{(1)}, L_B^{(1)}, L_C^{(1)}, L_D^{(1)})_M,$$

where $L_A^{(1)}$, $L_B^{(1)}$, $L_C^{(1)}$, $L_D^{(1)}$ are as defined in Notation 2.4.2.

By (2) of Proposition 2.2.2, the cross ratio $(A, B, C, D)_M$ is independent of the choices (if any) made to define $L_A^{(1)}$, $L_B^{(1)}$, $L_C^{(1)}$ or $L_D^{(1)}$. Using this notation, we can state the following proposition, which is a collection of useful inequalities involving the cross ratio and the Frenet curve.

**Proposition 2.4.5.** *Let $A, U, B, C, V, D$ be pairwise distinct points along a Frenet curve $\xi$, in that order. Let $M_1, \dots, M_k$ be another collection of pairwise distinct points along $\xi$, let $n_1, \dots, n_k$ be positive integers such that $\sum_{i=1}^{k} n_i = n - 2$ and let $M := \sum_{i=1}^{k} M_i^{(n_i)}$. Then the following inequalities hold:*

*(1)* $(A, B, C, D)_M > 1.$

*(2)* $(A, B, C, D)_M < (U, B, C, D)_M$

*(3)* $(A, B, C, D)_M < (A, U, C, D)_M$

*(4)* $(A, B, C, D)_M < (A, B, V, D)_M$

*(5)* $(A, B, C, D)_M < (A, B, C, V)_M$

By (6) of Proposition 2.2.2, it does not matter if, in the above proposition, $A, U, B, C, V, D$ lie in clockwise or anti-clockwise order along $\xi$.

*Proof.* Part (1) follows from (1) of Lemma 2.4.3 and Proposition 2.2.5. Since the proofs for (2) to (5) are very similar, we will only show the proof for (2).

Proof of (2). Consider the lines

$$\begin{aligned} L_1 &:= (L_U^{(1)} + M) \cap (L_B^{(1)} + L_C^{(1)}) \\ L_2 &:= (L_A^{(1)} + M) \cap (L_B^{(1)} + L_C^{(1)}) \\ L_3 &:= (L_D^{(1)} + M) \cap (L_B^{(1)} + L_C^{(1)}) \end{aligned}$$

By (1) of Lemma 2.4.3, we can choose vectors $l_1, l_2, l_3, l_B, l_C$ in $\mathbb{R}^n$ such that $[l_i] = L_i$ for $i = 1, 2, 3$, $[l_B] = L_B^{(1)}$, $[l_C] = L_C^{(1)}$ and $l_1 = a \cdot l_B + (1-a) \cdot l_C$, $l_2 = b \cdot l_B + (1-b) \cdot l_C$, $l_3 = c \cdot l_B + (1-c) \cdot l_C$ for $0 < c < b < a < 1$. Then

$$\begin{aligned} (A, B, C, D)_M &= (L_2, L_B^{(1)}, L_C^{(1)}, L_3)_M \\ &= \frac{(1-c)b}{(1-b)c} \\ &< \frac{(1-c)a}{(1-a)c} \\ &= (L_1, L_B^{(1)}, L_C^{(1)}, L_3)_M \\ &= (U, B, C, D)_M. \end{aligned}$$

□

Proposition 2.4.5 has the following useful consequence.

**Corollary 2.4.6.** *Let $A, C$ be distinct points along a Frenet curve $\xi$, and let $s$ be a closed subinterval of $\xi$ with endpoints $A$ and $C$. Also, let $\mathcal{S}$ be the space of hyperplanes in $\mathbb{R}^n$ containing the $(n-2)$-dimensional subspace $A^{(n-1)} \cap C^{(n-1)}$. Then there is a closed subinterval $t$ of $\mathcal{S}$ (which is topologically a circle) with endpoints $A^{(n-1)}$ and $C^{(n-1)}$ so that the map*

$$f : s \to t$$

*given by*

$$f : B \mapsto \left(A^{(n-1)} \cap C^{(n-1)}\right) + B^{(1)}$$

*is a homeomorphism onto its image.*

*Proof.* First, observe that $f$ is continuous and well-defined as a map to $\mathcal{S}$ because $\xi$ is continuous and Frenet. Pick any flag $B$ in the interior of $s$ and let $t$ be the unique subsegment of $\mathcal{S}$ with endpoints $A^{(n-1)}$ and $C^{(n-1)}$ that contains $(A^{(n-1)} \cap C^{(n-1)}) + B^{(1)}$. Since $\xi$ is Frenet, it is clear that for all $B$ in $s$,

$$(A^{(n-1)} \cap C^{(n-1)}) + B^{(1)} = A^{(n-1)}$$

if and only if $B = A$, and

$$(A^{(n-1)} \cap C^{(n-1)}) + B^{(1)} = C^{(n-1)}$$

if and only if $B = C$. Thus, $f$ is a continuous surjection.

Also, for any pair of distinct points $B$, $B'$ in $\partial\Gamma$ so that $A$, $B$, $B'$, $C$ lie in $s$ in that order, one can choose a normalization so that $A^{(i)} \cap C^{(n-i+1)} = [e_i]$ and $B^{(1)} = \left[\sum_{i=1}^{n} e_i\right]$, where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Choose a vector $v \in \mathbb{R}^n$ so that $[v] = B'^{(1)}$ and let $\alpha_1, \dots, \alpha_n$ be real numbers so that $v = \left[\sum_{i=1}^{n} \alpha_i e_i\right]$. One can then compute that $(A, B, B', C)_{A^{(k)}+C^{(n-k-2)}} = \dfrac{\alpha_{k+2}}{\alpha_{k+1}}$, which is greater than $1$ by (1) of Proposition 2.4.5. Also, one can compute that $(A^{(1)}, B^{(1)}, B'^{(1)}, C^{(1)})_{A^{(n-1)}\cap C^{(n-1)}} = \dfrac{\alpha_n}{\alpha_1}$, so

$$(A^{(1)}, B^{(1)}, B'^{(1)}, C^{(1)})_{A^{(n-1)}\cap C^{(n-1)}} = \prod_{k=0}^{n-2} (A, B, B', C)_{A^{(k)}+C^{(n-k-2)}} > 1.$$

Thus, by Proposition 2.2.5, $f$ is also injective, and hence is a homeomorphism. $\square$

As mentioned previously, an important feature of Frenet curves is a "positivity" property of the cross ratios and triple ratios associated to these curves. This is more formally described in the next proposition (see Section 7.1 and 7.2 of Fock-Goncharov [18], or Lemma 8.4.2 of Labourie-Mcshane [33] for an alternate proof).

**Proposition 2.4.7.** *Let $\xi : S^1 \to \mathcal{F}(\mathbb{R}^n)$ be a Frenet curve and let $a, b, c, d$ be four points along $S^1$ in that order. Then the following hold.*

1. $S_k\big(\xi(a), \xi(c), \xi(b)^{(1)}, \xi(d)^{(1)}\big) > 0$ *for all* $k = 1, \dots, n-1$.

2. $T_{x,y,z}\big(\xi(a), \xi(b), \xi(c)\big) > 0$ *for all* $(x, y, z) \in \mathcal{A}$.

*Proof.* Proof of (1). For any $x = 1, \dots, n-1$, recall that

$$S_k\big(\xi(a), \xi(c), \xi(b)^{(1)}, \xi(d)^{(1)}\big) = -\big(\xi(a), \xi(b), \xi(d), \xi(c)\big)_{\xi(a)^{(k-1)}+\xi(c)^{(n-k-1)}},$$

which is positive by (1) of Proposition 2.4.5 and (6) and (7) of Proposition 2.2.2.

Proof of (2). Let $L_1$ be a line in $\xi(a)^{(x)}$ but not in $\xi(a)^{(x-1)}$, let $L_2$ be a line in $\xi(b)^{(y)}$ but not in $\xi(b)^{(y-1)}$ and let $L_3$ be a line in $\xi(c)^{(z)}$ but not in $\xi(c)^{(z-1)}$. Then let $M := \xi(a)^{(x-1)} + \xi(b)^{(y-1)} + \xi(c)^{(z-1)}$ and let $N := L_1 + L_2 + L_3$, which is a three-dimensional subspace of $\mathbb{R}^n$ by the Frenet property of $\xi$.

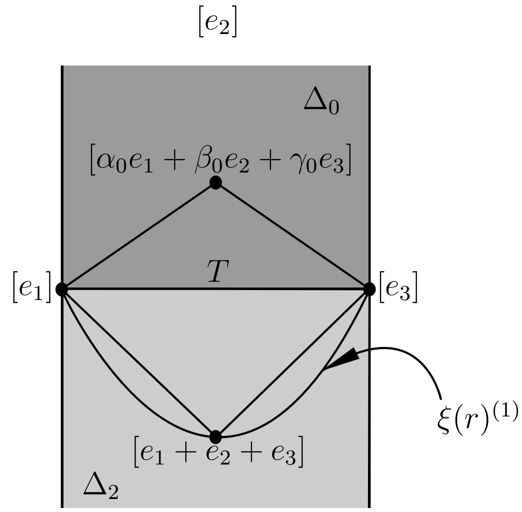


Figure 2.2: Positivity of the Frenet curve in the $n = 3$ case.

Define the map $\eta : S^1 \to \mathcal{F}(N)$ by

$$\eta : t \mapsto \big((M + L^{(1)}_{\xi(t)}) \cap N, (M + L^{(2)}_{\xi(t)}) \cap N\big),$$

(use Notation 2.4.2 with $M_1 = \xi(a)$, $M_2 = \xi(b)$ and $M_3 = \xi(c)$) and it is clear from the Frenet property of $\xi$ that $\eta$ is also Frenet. Furthermore, by (2) of Proposition 2.3.2, we have that

$$T_{x,y,z}\big(\xi(a), \xi(b), \xi(c)\big) = T_{1,1,1}\big(\eta(a), \eta(b), \eta(c)\big).$$

This reduces the problem to the case when $n = 3$.

To prove the $n = 3$ case, choose a normalization so that $\xi(a)^{(1)} = [e_1]$, $\xi(a)^{(2)} \cap \xi(c)^{(2)} = [e_2]$, $\xi(c)^{(1)} = [e_3]$, $\xi(d)^{(1)} = [e_1+e_2+e_3]$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. Let $\alpha_0, \beta_0, \gamma_0 \in \mathbb{R}$ be such that $\xi(b)^{(1)} = [\alpha_0 e_1 + \beta_0 e_2 + \gamma_0 e_3]$. The Frenet property of $\xi$ ensures that $\alpha_0$, $\beta_0$ and $\gamma_0$ are non-zero. In this normalization, one can compute

$$S_1\big(\xi(a), \xi(c), \xi(b)^{(1)}, \xi(d)^{(1)}\big) = -\frac{\alpha_0}{\beta_0} \text{ and } S_2\big(\xi(a), \xi(c), \xi(b)^{(1)}, \xi(d)^{(1)}\big) = -\frac{\beta_0}{\gamma_0},$$

which are both positive by (1).

There are four triangles in $\mathbb{RP}^2$ with vertices $[e_1]$, $[e_2]$, $[e_3]$. The interiors of these three triangles, denote them by $\Delta_i$ for $i = 0, 1, 2, 3$ can be described in the following way.

$$\begin{aligned}
\Delta_0 &= \{[\alpha e_1 + \beta e_2 + \gamma e_3] : \alpha, \beta, \gamma > 0\} \\
\Delta_1 &= \{[\alpha e_1 + \beta e_2 + \gamma e_3] : \alpha < 0, \beta, \gamma > 0\} \\
\Delta_2 &= \{[\alpha e_1 + \beta e_2 + \gamma e_3] : \beta < 0, \alpha, \gamma > 0\} \\
\Delta_3 &= \{[\alpha e_1 + \beta e_2 + \gamma e_3] : \gamma < 0, \alpha, \beta > 0\}
\end{aligned}$$

Thus, the computation in the previous paragraph shows that $\xi(d)^{(1)}$ lies in $\Delta_0$ and $\xi(b)^{(1)}$ lies in $\Delta_2$.

Furthermore, since $\xi$ is Frenet, we know that if $r$ is the open subinterval of $S^1$ with endpoints $a$, $c$ and containing $d$, then $\xi(r)^{(1)}$ is a continuous injective curve in $\mathbb{RP}^2$ which lies entirely in $\Delta_2$. Hence, any projective line in $\mathbb{RP}^2$ through $[\alpha_0 e_1 + \beta_0 e_2 + \gamma_0 e_3]$ intersects the common edge $T$ between $\Delta_0$ and $\Delta_2$ if and only if it intersects $\xi(r)^{(1)}$ (see Figure 2.2). This implies that the line $\xi(b)^{(2)}$ cannot intersect $T$. In particular, $\xi(b)^{(2)}$ does not intersect the triangle in $\Delta_0$ with vertices $\xi(a)^{(1)}$, $\xi(b)^{(1)}$, $\xi(c)^{(1)}$. Applying Lemma 2.3.9 finishes the proof. $\square$

# CHAPTER 3

# Closed hyperbolizable surfaces

In this section, we will establish some basic facts and terminology about hyperbolizable surfaces, the geometric structures on them, and their fundamental groups. This provides motivation for approaching the Hitchin component from a geometric point of view. **For the rest of this thesis, let $S = S_g$ be a closed oriented smooth surface of genus $g > 1$ and let $\Gamma := \pi_1(S)$.**

## 3.1 Geometric structures and representations

First, we will describe what we mean by a geometric structure on a general real-analytic manifold $M$, and then specialize to the case when $M = S$. The material discussed in this section can also be found in Goldman [23]. For an alternative treatment, see Chapter I.1 of Canary-Epstein-Marden [9].

A geometric structure on $M$ is a way to model $M$ locally on a space $X$ with sufficiently many symmetries. More precisely, we have the following definition.

**Definition 3.1.1.** Let $X$ be a real-analytic manifold and $G$ a Lie group that acts real analytically and transitively on $X$. An $(X, G)$-*structure* on a closed real-analytic manifold $M$ is a maximal collection

$$\Phi := \{\phi_\alpha : U_\alpha \to X\},$$

where $\{U_\alpha\}$ is an open cover of $M$ consisting of connected open sets, and each $\phi_\alpha$ is a diffeomorphism onto its image with the following property. For any $\phi_\alpha, \phi_\beta \in \Phi$ and for any connected component $C$ of $U_\alpha \cap U_\beta$, there is some $g \in G$ so that the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(C) \to \phi_\alpha(C)$$

is given by $\phi_\alpha \circ \phi_\beta^{-1}(x) = g \cdot x$ for all $x \in \phi_\beta(C)$. The maps in $\Phi$ are called *charts*, and we denote the space of $(X, G)$-structures on $M$ by $\mathcal{D}_{(X,G)}(M)$.

In the above definition, one can think of the geometry of the space $X$ as being specified by the $G$-action, i.e. the symmetries, on $X$. Observe that in order for $M$ to admit an $(X, G)$-structure, it is necessary that $\dim(M) = \dim(X)$. Also, the real analyticity of the $G$-action on $X$ ensures that if $g \in G$ fixes every point in an open set $U \subset X$, then $g$ fixes every point in $X$. This is commonly known as the *unique extension property*.

In the case when $X = \mathbb{RP}^{n-1}$ and $G = PGL(n, \mathbb{R})$, $G$ acts transitively on $X$ via projective transformations. We call a $\big(\mathbb{RP}^{n-1}, PGL(n, \mathbb{R})\big)$-structure on an $n$-dimensional manifold $M$ a *real projective structure*. Also, when $X = \mathbb{H}^2$ and $G = PGL(2, \mathbb{R})$, $G$ acts transitively on $X$ via Möbius transformations, which are exactly the isometries of $\mathbb{H}^2$ equipped with the hyperbolic metric. We then call an $\big(\mathbb{H}^2, PGL(2, \mathbb{R})\big)$-structure on $S$ a *hyperbolic structure*.

We will now define a natural notion of equivalence between $(X, G)$-structures.

**Definition 3.1.2.** Let $\Phi$ and $\Phi'$ be two $(X, G)$-structures on a closed smooth manifold $M$.

(1) A diffeomorphism $f : M \to M$ is an *isomorphism* from $\Phi$ to $\Phi'$ if the following hold. For all charts $\phi : U \to X$ and $\phi' : U' \to X$ in $\Phi$ and $\Phi'$ respectively so that $f(U) = U'$, there is some $g \in G$ so that $g \circ \phi(x) = \phi' \circ f(x)$ for all $x \in U$.

(2) $\Phi$ and $\Phi'$ are *isotopic* if the identity map on $M$ is isotopic to an isomorphism between $\Phi$ and $\Phi'$.

(3) The *deformation space* of $(X, G)$-structures on $M$, denoted $\mathcal{T}_{(X,G)}(M)$, is the set of isotopy classes of $(X, G)$-structures on $M$.

More informally, two $(X, G)$-structures on $M$ are isotopic if one can be "perturbed" to obtain the other. The deformation space $\mathcal{T}_{(X,G)}(M)$ has a natural topology on it, which we will describe later.

**For the rest of this section, fix a universal covering map $\Pi : \widetilde{M} \to M$. Also, for every point $p \in M$, choose once and for all a point $\widetilde{p} \in \widetilde{M}$ so that $\Pi(\widetilde{p}) = p$.** An important consequence of the unique extension property of the action of $G$ on $X$ is the existence of a "globalization" for any $(X, G)$-structure on $M$. This is known as the *developing pair*, which consists of a group homomorphism $\rho : \pi_1(M) \to G$ and a $\rho$-equivariant local diffeomorphism $d : \widetilde{M} \to X$. The homomorphism $\rho$ is called the *holonomy representation* and the local homeomorphism $d$ is known as the *developing map*. We will now give a brief description of $\rho$ and $d$.

**Definition 3.1.3.** Let $\Phi$ be an $(X, G)$-structure on $M$, and let $\Phi_p$ be the charts in $\Phi$ whose domain contains $p$. Then define

$$\mathrm{germ}_p(\Phi) := \Phi_p/\sim,$$

where two charts $\phi : U \to X$ and $\phi' : U' \to X$ in $\Phi_p$ are equivalent under $\sim$ if $\phi|_{U\cap U'} = \phi'|_{U\cap U'}$. The elements in $\mathrm{germ}_p(\Phi)$ are called *germs*, and denote the germ containing the chart $\phi$ by $[\phi]$.

Observe that $\Phi$ induces an $(X, G)$-structure $\widetilde{\Phi}$ on $\widetilde{M}$ by pre-composing each chart in $\Phi$ by restrictions of $\Pi$ to suitable open sets. Thus, for any germ $[\phi] \in \mathrm{germ}_p(\Phi)$, we can define the *lift* $\widetilde{[\phi]}$ of $[\phi]$ to be the germ $[\phi \circ \Pi] \in \mathrm{germ}_{\widetilde{p}}(\widetilde{\Phi})$.

Given an $(X, G)$-structure $\Phi$ on $M$, a point $p \in M$ and a germ $[\phi] \in \mathrm{germ}_p(\Phi)$, we can define a map $d : \widetilde{M} \to X$ in the following way. For any point $\widetilde{q} \in \widetilde{M}$, choose a path $\alpha$ in $\widetilde{M}$ between $\widetilde{p}$ and $\widetilde{q}$ and a finite collection of charts $\{\widetilde{\phi}_i : \widetilde{U}_i \to X\}_{i=1}^k$ in $\widetilde{\Phi}$ so that $\widetilde{[\phi_1]} = \widetilde{[\phi]}$ as germs in $\mathrm{germ}_{\widetilde{p}}(\widetilde{\Phi})$, and $\widetilde{U}_1, \dots, \widetilde{U}_k$ is a chain of open sets that cover $\alpha$. In other words, $\widetilde{q} \in \widetilde{U}_k$, $\alpha \subset \bigcup_{i=1}^k \widetilde{U}_i$, $\widetilde{U}_i \cap \widetilde{U}_j$ is nonempty if and only if $j = i-1, i, i+1$, and $\widetilde{U}_i \cap \widetilde{U}_j$ is connected when it is nonempty.

For all $j = 1, \dots, k-1$, let $g_j$ be the unique element in $G$ such that $g_j \cdot x = \widetilde{\phi}_j \circ \widetilde{\phi}_{j+1}^{-1}(x)$ for all $x \in \widetilde{\phi}_{j+1}(\widetilde{U}_j \cap \widetilde{U}_{j+1})$. (The uniqueness of $g_j$ is a consequence of the unique extension property.) Then define $d(\widetilde{q}) := g_1 \dots g_{k-1} \cdot \widetilde{\phi}_k(\widetilde{q})$. One needs to check that $d(\widetilde{q})$ is independent of the choice of $\alpha$ between $\widetilde{p}$ and $\widetilde{q}$, and the choice of charts $\{\widetilde{\phi}_1, \dots, \widetilde{\phi}_k\}$. In particular, $d$ is a well-defined map that depends only on the $(X, G)$-structure $\widetilde{\Phi}$ and the choice of initial germ $[\phi] \in \mathrm{germ}_p(\Phi)$. By construction, $d$ restricted to each chart is a diffeomorphism onto its image, so $d$ is a local diffeomorphism. This is a developing map for $\Phi$ mentioned above.

Next, we will define the holonomy representation $\rho$ from the same initial germ $[\phi] \in \mathrm{germ}_p(M)$. Let $\phi : U \to X$ be a chart in $\Phi$ that lies in $[\phi]$. By shrinking $U$ if necessary, we can ensure that each component of $\Pi^{-1}(U)$ is diffeomorphic to $U$ via $\Pi$. Let $\widetilde{U}$ be the connected component of $\Pi^{-1}(U)$ containing $\widetilde{p}$, let $\gamma \in \pi_1(M)$, and define $\widetilde{U}' := \gamma \cdot \widetilde{U}$, where $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations. Then $\widetilde{\phi} := \phi \circ (\Pi|_{\widetilde{U}}) : \widetilde{U} \to X$ and $\widetilde{\phi}' := \phi \circ (\Pi|_{\widetilde{U}'}) : \widetilde{U}' \to X$ are charts of $\widetilde{\Phi}$ with the same image. If $d$ is the developing map for $\Phi$ constructed with initial germ $[\phi]$, then $d|_{\widetilde{U}} = \phi \circ \Pi|_{\widetilde{U}} : \widetilde{U} \to X$ and by the unique extension property, there is a unique $g \in G$ so that $d|_{\widetilde{U}'} = g \circ \phi \circ (\Pi|_{\widetilde{U}'}) : \widetilde{U}' \to X$. This allows us to define a map $\rho : \pi_1(M) \to G$ by $\rho(\gamma) \mapsto g$. It is easy to see that $\rho$ is in fact a group homomorphism, and that $d$ is $\rho$-equivariant. The pair $(\rho, d)$ constructed above is a developing pair for the $(X, G)$-structure $\Phi$, with developing map $d$ and holonomy representation $\rho$.

To further discuss developing pairs, it is convenient to have the following definitions.

**Definition 3.1.4.**

1. Define

$$\mathcal{P}'_{(X,G)}(M) := \left\{ (\rho, d) : \begin{array}{l} \rho : \pi_1(M) \to G \text{ is a group homomorphism,} \\ d : \widetilde{M} \to X \text{ is a } \rho\text{-equivariant local diffeomorphism} \end{array} \right\}.$$

2. Fix $p \in M$, and define

$$\mathcal{D}'_{(X,G)}(M) := \big\{ (\Phi, [\phi]) : \Phi \text{ is an } (X,G)\text{-structure on } M \text{ and } [\phi] \in \operatorname{germ}_p(\Phi) \big\}.$$

Let $f : M \to M$ be a diffeomorphism that is isotopic to the identity via the isotopy $F : [0,1] \times M \to M$ so that $F(0, \cdot)$ is the identity map and $F(1, \cdot) = f(\cdot)$. Let $\widetilde{F} : [0,1] \times \widetilde{M} \to \widetilde{M}$ be the lift of $F$ so that $\widetilde{F}(0, \cdot)$ is the identity map on $\widetilde{M}$, and let $\widetilde{f}(\cdot) := \widetilde{F}(1, \cdot)$. Since $f_* : \pi_1(M) \to \pi_1(M)$ is the identity map, we see that if $d : \widetilde{M} \to X$ is a local diffeomorphism that is $\rho$-equivariant for some representation $\rho : \pi_1(M) \to G$, then $d \circ \widetilde{f}^{-1}$ is also a diffeomorphism that is $\rho$-equivariant. Hence, we can define a $\operatorname{Diff}_0(M)$ action on $\mathcal{P}'_{(X,G)}(M)$ by $f \cdot (\rho, d) := (\rho, d \circ \widetilde{f}^{-1})$.

Let $d$ be any developing map for the $(X,G)$-structure $\Phi$. For any point $p \in M$, let $\widetilde{U}$ be a sufficiently small neighborhood of $p$ so that $\Pi_{\widetilde{U}}$ is a diffeomorphism onto its image. Then $[d \circ (\Pi|_{\widetilde{U}})^{-1}] \in \operatorname{germ}_p(\Phi)$ and the developing map constructed using initial germ $[d \circ (\Pi|_{\widetilde{U}})^{-1}]$ is exactly $d$. This allows us to define an action of $\operatorname{Diff}_0(M)$, the group of diffeomorphisms from $M$ to itself that are isotopic to the identity map, on $\mathcal{D}'_{(X,G)}(M)$ by $f \cdot (\Phi, [\phi]) := (\Phi', [d_f \circ (\Pi|_{\widetilde{U}})^{-1}])$, where $\Phi' := \{\phi \circ f^{-1} : \phi \in \Phi\}$ and $d_f$ is the developing map constructed with initial germ $[\phi \circ f^{-1}] \in \operatorname{germ}_{f(p)}(\Phi')$.

**Definition 3.1.5.** Let $\operatorname{Diff}_0(M)$ be the connected component of the group of diffeomorphisms from $M$ to itself that contains the identity map.

1. Define

$$\mathcal{P}_{(X,G)}(M) := \mathcal{P}'_{(X,G)}(M) / \operatorname{Diff}_0(M),$$

where the action of $\operatorname{Diff}_0(M)$ on $\mathcal{P}'_{(X,G)}(M)$ is as defined above.

2. Define

$$\mathcal{D}_{(X,G)}(M) := \mathcal{D}'_{(X,G)}(M) / \operatorname{Diff}_0(M),$$

where the action of $\operatorname{Diff}_0(M)$ on $\mathcal{D}'_{(X,G)}(M)$ is as defined above.

More explicitly, two elements $(\Phi, [\phi])$ and $(\Phi', [\phi'])$ in $\mathcal{D}'_{(X,G)}(M)$ correspond to the same point in $\mathcal{D}_{(X,G)}(M)$ if there is some isomorphism $f : M \to M$ from $\Phi$ to $\Phi'$ that is isotopic to the identity, so that the developing maps constructed using the initial germs $[\phi'] \in \mathrm{germ}_p(\Phi')$ and $[\phi \circ f^{-1}] \in \mathrm{germ}_{f(p)}(\Phi')$ agree.

From the way we constructed developing pairs for any $(X, G)$-structures, we can define the map

$$\widetilde{dev} : \mathcal{D}'_{(X,G)}(M) \to \mathcal{P}'_{(X,G)}(M)$$

which sends each pair $(\Phi, [\phi])$ to the developing pair for $\Phi$ constructed with the initial germ $[\phi]$. Also, it is easy to see that $\widetilde{dev}$ is equivariant with respect to the action of $\mathrm{Diff}_0(M)$ on $\mathcal{D}'_{(X,G)}(M)$ and $\mathcal{P}'_{(X,G)}(M)$, so it descends to a map

$$dev : \mathcal{D}_{(X,G)}(M) \to \mathcal{P}_{(X,G)}(M).$$

Furthermore, $G$ acts on $\mathcal{D}'_{(X,G)}(M)$ and on $\mathcal{P}'_{(X,G)}(M)$ by $g \cdot (\Phi, [\phi]) := (\Phi, [g \circ \phi])$ and $g \cdot (\rho, d) := (g\rho(\cdot)g^{-1}, g \circ d)$. Note that the $\mathrm{Diff}_0(M)$ and $G$ actions on both $\mathcal{D}'_{(X,G)}(M)$ and $\mathcal{P}'_{(X,G)}(M)$ commute, so the $G$ actions on $\mathcal{D}'_{(X,G)}(M)$ and $\mathcal{P}'_{(X,G)}(M)$ descend to $G$-actions on $\mathcal{D}_{(X,G)}(M)$ and $\mathcal{P}_{(X,G)}(M)$. Moreover, the map $dev$ is also equivariant with respect to these $G$-actions, so it further descends to a map

$$\overline{dev} : \mathcal{D}_{(X,G)}(M)/G \to \mathcal{P}_{(X,G)}(M)/G.$$

**Proposition 3.1.6.**

1. *The maps $\widetilde{dev}$, $dev$ and $\overline{dev}$ defined above are bijections.*

2. *The map*

$$\mathcal{D}_{(X,G)}(M)/G \to \mathcal{T}_{(X,G)}(M)$$

*that sends each equivalence class $[\Phi, [\phi]]$ in $\mathcal{D}_{(X,G)}(M)/G$ to the equivalence class $[\Phi]$ in $\mathcal{T}_{(X,G)}(M)$ is a bijection.*

*Proof.* Proof of (1). It is sufficient to show that $\widetilde{dev}$ is a bijection. Suppose we can show that for any given pair $(\rho, d)$ in $\mathcal{P}'_{(X,G)}(M)$, there is an $(X, G)$-structure $\Phi$ on $M$ for which $(\rho, d)$ is a developing pair. Then $\Phi$ is unique because for every $p \in M$, $G$ acts transitively on $\mathrm{germ}_p(\Phi)$, and there is a sufficiently small open set $\widetilde{U}$ containing $\widetilde{p}$ so that $[d|_{\widetilde{U}} \circ (\Pi|_{\widetilde{U}}^{-1})] \in \mathrm{germ}_p(\Phi)$. Also, the bijection between $\mathrm{germ}_p(\Phi)$ and the developing maps for $\Phi$ ensures that $\widetilde{dev}$ is a bijection. Thus, it is sufficient to construct $\Phi$ from the pair $(\rho, d)$.

Let $\mathcal{U}$ be a cover of $M$ so that the following hold.

- For any $U \in \mathcal{U}$, $\Pi^{-1}(U)$ is a disjoint union of connected opens sets in $\widetilde{M}$, each of which is diffeomorphic to $U$ via $\Pi$.

- For any $U \in \mathcal{U}$, $d$ restricted to each connected component of $\Pi^{-1}(U)$ is a diffeomorphism.

- For any $U_1, U_2 \in \mathcal{U}$, $U_1 \cap U_2$ is connected (but possibly empty).

For any $U \in \mathcal{U}$ and any connected component $\widetilde{U}$ of $\Pi^{-1}(U)$, define

$$\phi_{U,\widetilde{U}} := d|_{\widetilde{U}} \circ (\Pi|_{\widetilde{U}})^{-1} : U \to X$$

and consider the set

$$\Psi := \{\phi_{U,\widetilde{U}} : U \in \mathcal{U}, \widetilde{U} \in \Pi^{-1}(U)\}.$$

Now, suppose that $U_1$ and $U_2$ are open sets in $\mathcal{U}$ that have nonempty intersection, and let $\widetilde{U}_1$ and $\widetilde{U}_2$ be connected components in $\Pi^{-1}(U_1)$ and $\Pi^{-1}(U_2)$ respectively. Then there is some $\gamma \in \pi_1(M)$ so that $(\gamma \cdot \widetilde{U}_1) \cap \widetilde{U}_2$ is non-empty. This means that

$$\rho(\gamma) \circ d|_{\widetilde{U}_1 \cap (\gamma^{-1} \cdot \widetilde{U}_2)} = d|_{(\gamma \cdot \widetilde{U}_1) \cap \widetilde{U}_2} \circ \gamma : \widetilde{U}_1 \cap (\gamma^{-1} \cdot \widetilde{U}_2) \to X,$$

and in particular,

$$\begin{aligned}
\rho(\gamma) \circ \phi_{U_1,\widetilde{U}_1}|_{U_1 \cap U_2} &= \rho(\gamma) \circ d|_{\widetilde{U}_1 \cap (\gamma^{-1} \cdot \widetilde{U}_2)} \circ (\Pi|_{\widetilde{U}_1 \cap (\gamma^{-1} \cdot \widetilde{U}_2)})^{-1} \\
&= d|_{(\gamma \cdot \widetilde{U}_1) \cap \widetilde{U}_2} \circ \gamma \circ (\Pi|_{\widetilde{U}_1 \cap (\gamma^{-1} \cdot \widetilde{U}_2)})^{-1} \\
&= d|_{(\gamma \cdot \widetilde{U}_1) \cap \widetilde{U}_2} \circ (\Pi|_{(\gamma \cdot \widetilde{U}_1) \cap \widetilde{U}_2})^{-1} \\
&= \phi_{U_2,\widetilde{U}_2}|_{U_1 \cap U_2}
\end{aligned}$$

as maps from $U_1 \cap U_2$ to $X$. Hence, there is some $(X, G)$-structure $\Phi$ on $M$ so that $\Psi \subset \Phi$.

For any open set $U \in \mathcal{U}$ containing $p$, let $\widetilde{U}$ be the connected component of $\Pi^{-1}(U)$ containing $\widetilde{p}$. It is clear that the pair $(\rho, d)$ is a developing pair for $\Phi$ constructed with initial germ $[\phi_{U,\widetilde{U}}]$. This proves (1).

Part (2) follows immediately from the observation that $G$ acts transitively on $\mathrm{germ}_p(\Phi)$. $\square$

Proposition 3.1.6 allows us to endow all the spaces we have defined with a natural topology in the following way. The set $\mathcal{P}'_{(X,G)}(M)$ can be topologized by the $C^\infty$-topology on the local diffeomorphisms $d$ in the pairs $(\rho, d)$. This allows us to endow $\mathcal{P}_{(X,G)}(M)$ and $\mathcal{P}_{(X,G)}(M)/G$ with the respective quotient topologies, which in turn induce topologies on $\mathcal{D}'_{(X,G)}(M)$, $\mathcal{D}_{(X,G)}(M)$ and $\mathcal{D}_{(X,G)}(M)/G$ by (1) of Proposition 3.1.6. The bijection between $\mathcal{D}_{(X,G)}(M)/G$ and $\mathcal{T}_{(X,G)}(M)$ in (2) of Proposition 3.1.6 then endows a topology on the latter. With these identifications, we will no longer distinguish between the spaces $\mathcal{P}_{(X,G)}(M)/G$, $\mathcal{D}_{(X,G)}(M)/G$ and $\mathcal{T}_{(X,G)}(M)$.

There is an obvious map from $\mathcal{P}'_{(X,G)}(M) \to \mathrm{Hom}\big(\pi_1(M), G\big)$ given by $(\rho, d) \mapsto \rho$. By precomposing this map with $\widetilde{dev}$, we get a map

$$\widetilde{hol} : \mathcal{D}'_{(X,G)}(M) \to \mathrm{Hom}\big(\pi_1(M), G\big).$$

Now, suppose that $(\Phi, [\phi])$ and $(\Phi', [\phi'])$ in $\mathcal{D}'_{(X,G)}(M)$ are identified in $\mathcal{D}_{(X,G)}(M)$. By definition, this means that there is some isomorphism $f : M \to M$ from $\Phi$ to $\Phi'$ that is isotopic to the identity map on $M$. From the way we constructed the developing pairs $(\rho, d) = \widetilde{dev}(\Phi, [\phi])$ and $(\rho', d') = \widetilde{dev}(\Phi', [\phi'])$, it is clear that $\rho' = \rho$. Hence, the map $\widetilde{hol}$ further descends to the maps

$$hol : \mathcal{D}_{(X,G)}(M) \to \mathrm{Hom}\big(\pi_1(M), G\big)$$

and

$$\overline{hol} : \mathcal{D}_{(X,G)}(M)/G \to \mathrm{Hom}\big(\pi_1(M), G\big)/G,$$

where $G$ acts on $\mathrm{Hom}\big(\pi_1(M), G\big)$ by conjugation. If we topologize $\mathrm{Hom}\big(\pi_1(M), G\big)$ with the compact-open topology and $\mathrm{Hom}\big(\pi_1(M), G\big)/G$ with the quotient topology, then we have the following theorem.

**Theorem 3.1.7.** *Suppose that $M$ is compact, then the maps $hol$ and $\overline{hol}$ are local homeomorphisms.*

*Proof.* See Chapter I.1.7 of Canary-Epstein-Green [9] or Section 3 of Goldman [23]. □

It is important to note that the topology we have equipped $\mathrm{Hom}\big(\pi_1(M), G\big)/G$ with is in general not Hausdorff. As such, it is common for many authors to consider only the conjugacy classes of reductive representations in $\mathrm{Hom}\big(\pi_1(M), G\big)/G$. However, for the $(X, G)$-structures we will be considering, the image of $\overline{hol}$ in $\mathrm{Hom}\big(\pi(M), G\big)/G$ is Hausdorff, so we will not bother to do so.

The developing map $d$ of an $(X, G)$-structure is typically not a diffeomorphism onto its image. For the rest of this thesis though, we will only consider $(X, G)$-structures with this property.

**Definition 3.1.8.** An $(X, G)$-structure is *Kleinian* if its developing map is a diffeomorphism onto its image.

If any $(X, G)$-structure on a manifold $M$ is Kleinian, then it is immediate that its holonomy representation is discrete and faithful, and the obvious map $M \simeq d(\widetilde{M})/\rho(\Gamma)$ is an isomorphism of $(X, G)$-structures.

## 3.2 Hyperbolic structures on $S$.

We will now specialize the general set up developed in Section 3.1 to the case when $M = S$, which we recall is a closed oriented smooth surface of genus at least 2. As mentioned previously, one such example is a hyperbolic structure, i.e. a $\big(\mathbb{H}^2, PGL(2,\mathbb{R})\big)$-structure.

The isometries of the hyperbolic metric on $\mathbb{H}^2$ is the group of Möbius transformations on $\mathbb{H}^2$, which can be naturally identified with $PGL(2,\mathbb{R})$. Hence, a hyperbolic structure on $S$ is equivalent to equipping $S$ with a hyperbolic metric. An important feature of hyperbolic structures is that they are *complete*, i.e. the developing map is a diffeomorphism onto all of $X$. This is a consequence of a Riemannian geometry fact that if $f : M \to N$ is a local isometry from a complete Riemannian manifold to another Riemannian manifold, then $f$ is a covering map. In particular, these structures are Kleinian, so they have discrete and faithful holonomy representations.

The group $PGL(2,\mathbb{R})$ has two connected components characterized by the sign of the determinant, and $PSL(2,\mathbb{R}) \subset PGL(2,\mathbb{R})$ is the connected component containing the identity element. In terms of the action on $\mathbb{H}^2$, $PSL(2,\mathbb{R})$ is exactly the subgroup of elements in $PGL(2,\mathbb{R})$ that act by orientation-preserving Möbius transformations on $\mathbb{H}^2$. Since $S$ is an orientable surface, this implies that the image of the holonomy representation of any hyperbolic structure on $S$ has to lie in $PSL(2,\mathbb{R})$. As such,

$$\mathcal{D}'_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}(S) = \mathcal{D}'_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}(S)$$

and

$$\mathcal{D}_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}(S) = \mathcal{D}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}(S).$$

However, since $PSL(2,\mathbb{R}) \subset PGL(2,\mathbb{R})$ is an index 2 subgroup, there is a natural two-to-one map from $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}$ to $\mathcal{T}_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}$. Intuitively, if the developing maps $[d, \rho]$ and $[d', \rho']$ are developing pairs in $\mathcal{D}_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}$ so that $\rho$ is conjugate to $\rho'$ by an element in $PGL(2,\mathbb{R})$, then they are equivalent in $\mathcal{T}_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}$, but will be equivalent in $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}$ if and only if $d^{-1} \circ d'$ is orientation preserving. As such, one can think of the elements in $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}$ as hyperbolic structures on $S$, together with a choice of orientation for the developing map.

**Proposition 3.2.1.** *The maps*

$$hol : \mathcal{D}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S) \to \mathrm{Hom}\big(\Gamma, PSL(2,\mathbb{R})\big),$$

$$\overline{hol} : \mathcal{T}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S) \to \mathrm{Hom}\big(\Gamma, PSL(2,\mathbb{R})\big)/PSL(2,\mathbb{R})$$

*and*

$$\overline{hol}' : \mathcal{T}_{(\mathbb{H}^2, PGL(2,\mathbb{R}))}(S) \to \mathrm{Hom}\big(\Gamma, PGL(2,\mathbb{R})\big)/PGL(2,\mathbb{R})$$

*are homeomorphisms onto their images. Moreover, the images of* $\overline{hol}$ *and* $\overline{hol}'$ *are the conjugacy classes of discrete and faithful representations in* $\mathrm{Hom}\big(\Gamma, PSL(2,\mathbb{R})\big)/PSL(2,\mathbb{R})$ *and* $\mathrm{Hom}\big(\Gamma, PGL(2,\mathbb{R})\big)/PGL(2,\mathbb{R})$ *respectively.*

*Proof.* By Theorem 3.1.7, it is sufficient to show that $hol$ is a bijection onto the set of discrete and faithful representations. The completeness of hyperbolic structures ensures that every representation in the image of $hol$ is discrete and faithful.

On the other hand, given any discrete and faithful representation $\rho : \Gamma \to PSL(2,\mathbb{R})$, the quotient $\mathbb{H}^2/\rho(\Gamma)$ is a hyperbolic surface diffeomorphic to $S$. Moreover, $\rho$ gives us an isomorphism $\rho : \Gamma \to \rho(\Gamma) = \pi_1\big(\mathbb{H}^2/\rho(\Gamma)\big)$. This allows us to build a homeomorphism from $f : S \to \mathbb{H}^2/\rho(\Gamma)$ so that $f_* = \rho$, which can be isotoped to a diffeomorphism $g : S \to \mathbb{H}^2/\rho(\Gamma)$. We have seen previously that this data defines a point in $\mathcal{D}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S)$, so $hol$ surjects onto the set of discrete and faithful representations.

To prove injectivity, suppose that $\big[\Phi, [\phi]\big]$ and $\big[\Phi', [\phi']\big]$ are elements in $\mathcal{D}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S)$ with developing pairs $[\rho, d]$ and $[\rho', d']$ respectively, so that $\rho = \rho'$. Since $d$ and $d'$ are diffeomorphisms onto $X$, $\widetilde{f} := d^{-1} \circ d' : \widetilde{S} \to \widetilde{S}$ is a diffeomorphism that is $\rho^{-1} \circ \rho'$-equivariant. The assumption that $\rho = \rho'$ thus ensures that $\widetilde{f}$ descends to a map $f : S \to S$ which is homotopic to the identity map. By a classical theorem of Baer [1, 2] (also, see Epstein [17]), $f$ is isotopic to the identity map. Also, since $d \circ \widetilde{f} = d'$, it follows from the definition of the action of $\mathrm{Diff}_0(S)$ on $\mathcal{D}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S)$ that $\big[\Phi, [\phi]\big] = \big[\Phi', [\phi']\big]$. $\square$

As a consequence of Proposition 3.2.1, we will no longer distinguish between the geometric structures in $\mathcal{T}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S)$ or $\mathcal{T}_{(\mathbb{H}^2, PGL(2,\mathbb{R}))}(S)$ and the conjugacy classes of their holonomy representations. We will also often abuse notation by denoting a conjugacy class of representations $[\rho]$ in $\mathcal{T}_{(\mathbb{H}^2, PSL(2,\mathbb{R}))}(S)$ or $\mathcal{T}_{(\mathbb{H}^2, PGL(2,\mathbb{R}))}(S)$ by a representative $\rho$.

There are two other ways one can think of a hyperbolic structure on $S$. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ equipped with the Poincaré metric. Then the action of $PU(1,1)$ on $\mathbb{D}$ by Möbius transformations preserves the Poincaré metric, and the subgroup of $PU(1,1)$ that acts by orientation preserving isometries is $PSU(1,1)$. Furthermore, all isometries of $\mathbb{D}$ can be realized this way. Since $\mathbb{H}^2$ equipped with the hyperbolic metric is isometric to $\mathbb{D}$

with the Poincaré metric, $\big(\mathbb{H}^2, PGL(2,\mathbb{R})\big)$-structures and $\big(\mathbb{H}^2, PSL(2,\mathbb{R})\big)$-structures on $S$ are exactly $\big(\mathbb{D}, PU(1,1)\big)$-structures and $\big(\mathbb{D}, PSU(1,1)\big)$-structures on $S$ respectively.

Alternatively, there is a unique (up to post composition by an isometry) isometric embedding $i : \mathbb{H}^2 \to \mathbb{R}^{1,2}$ whose image $\mathcal{H}$ is one of the sheets of a two-sheeted hyperboloid in Minkowski space $\mathbb{R}^{1,2}$. The subgroup $O(1,2)^+$ of the isometry group $O(1,2)$ of $\mathbb{R}^{1,2}$ is the subgroup that preserves $\mathcal{H}$, and hence acts via pull-back as isometries on $\mathbb{H}^2$. In fact, all isometries of $\mathbb{H}^2$ can be realized this way. Furthermore, the identity component of $O(1,2)^+$, denoted $SO(1,2)^+$, preserves both $\mathcal{H}$ and the orientation on $\mathcal{H}$. Thus, it acts via orientation preserving isometries on $\mathbb{H}^2$ as well. Hence, $\big(\mathbb{H}^2, PGL(2,\mathbb{R})\big)$-structures and $\big(\mathbb{H}^2, PSL(2,\mathbb{R})\big)$-structures on $S$ are equivalent to $\big(\mathcal{H}, O(1,2)^+\big)$-structures and $\big(\mathcal{H}, SO(1,2)^+\big)$-structures on $S$ respectively.

As such, we will freely switch between these different descriptions of hyperbolic structures on $S$, and no longer distinguish between the spaces $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}(S)$, $\mathcal{T}_{(\mathbb{D},PSU(1,1))}(S)$ and $\mathcal{T}_{(\mathcal{H},SO(1,2)^+)}(S)$. Similarly, the deformation spaces $\mathcal{T}_{(\mathbb{H}^2,PGL(2,\mathbb{R}))}(S)$, $\mathcal{T}_{(\mathbb{D},PU(1,1))}(S)$ and $\mathcal{T}_{(\mathcal{H},O(1,2)^+)}(S)$ will be considered equal.

The deformation space $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R}))}(S)$ has two connected components, each of which is isomorphic to $\mathcal{T}_{(\mathbb{H}^2,PGL(2,\mathbb{R})}(S)$ (see Theorem D of Goldman [24]). One of the components of $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R})}(S)$ consists of $\big(\mathbb{H}^2, PSL(2,\mathbb{R})\big)$-structures whose developing maps are orientation preserving, while the other consists of those whose developing maps are orientation reversing. For the purposes of understanding deformations, we only need to focus on understanding one of these components.

**Definition 3.2.2.** The *Teichmüller space* of $S$, denoted $\mathcal{T}(S)$, is a connected component of $\mathcal{T}_{(\mathbb{H}^2,PSL(2,\mathbb{R})}(S)$ that have orientation preserving developing maps. The $\big(\mathbb{H}^2, PSL(2,\mathbb{R})\big)$-structures in $\mathcal{T}(S)$ are called *oriented hyperbolic structures*.

As a consequence of Theorem 3.2.1, we have the following corollary.

**Corollary 3.2.3.** *The map $\overline{hol}$ restricted to $\mathcal{T}(S)$ is a homeomorphism onto a connected component of $Hom\big(\Gamma, PSL(2,\mathbb{R})\big)/PSL(2,\mathbb{R})$.*

*Proof.* Theorem 3.2.1 implies that $\overline{hol}$ restricted to $\mathcal{T}(S)$ is a homeomorphism onto its image, so the image of $\overline{hol}$ is open in $Hom\big(\Gamma, PSL(2,\mathbb{R})\big)/PSL(2,\mathbb{R})$. Also, by the Margulis lemma, the image of $\overline{hol}$ is closed in $Hom\big(\Gamma, PSL(2,\mathbb{R})\big)/PSL(2,\mathbb{R})$ (see Theorem 5.3 of [37]). □

This description of $\mathcal{T}(S)$ will be important as we try to generalize this theory to other Lie groups.

## 3.3 Convex real projective structures

Another kind of $(X, G)$-structure on $S$ mentioned previously are the real projective structures. Unlike hyperbolic structures however, real projective structures are in general not complete or even Kleinian. However, some properties of hyperbolic structures hold in this setting if we restrict the types of projective structures we consider. This motivates the next definition.

**Definition 3.3.1.**

(1) A domain $\Omega \subset \mathbb{RP}^2$ is *properly convex* if the following hold

- The closure $\overline{\Omega}$ of $\Omega$ in $\mathbb{RP}^2$ is contained in an affine chart of $\mathbb{RP}^2$,
- For any two points $a, b \in \Omega$, there is a projective line segment in $\Omega$ whose endpoints are $a$ and $b$

(2) A domain $\Omega \subset \mathbb{RP}^2$ is *strictly convex* if it is properly convex and its boundary, denoted $\partial\Omega$, does not contain any projective line segments.

(3) A real projective structure on $S$ is *convex* if its developing map is a diffeomorphism onto a properly convex subset of $\mathbb{RP}^2$. Denote the deformation space of convex real projective structures on $S$ by $\mathcal{C}(S)$.

A theorem of Benoist (Théorème 1.1 of [3]) implies that the image of the developing map of any convex projective structure on $S$ is in fact strictly convex. Convex real projective structures on $S$ are clearly Kleinian, so the holonomy representations for such structures are also discrete and faithful. Furthermore, Choi-Goldman [10] proved the following theorem, which is an analog of Corollary 3.2.3 for convex real projective structures on $S$.

**Theorem 3.3.2** (Choi-Goldman)**.** *The map* $\overline{hol} : \mathcal{C}(S) \to Hom\big(\Gamma, PSL(3, \mathbb{R})\big)/PSL(3, \mathbb{R})$ *is a homeomorphism onto a connected component of* $Hom\big(\Gamma, PSL(3, \mathbb{R})\big)/PSL(3, \mathbb{R})$.

The image of $\overline{hol}$ in the theorem above is an example of a Hitchin component, which we will define formally in the next chapter. These Hitchin components are the main object we study in this thesis, and are a generalization of $\mathcal{T}(S)$.

On any properly convex domain $\Omega \subset \mathbb{RP}^2$, one can define a canonical Finsler metric known as the *Hilbert metric* in the following way. For any two points $a, b \in \Omega$, let $l$ be the projective line segment in $\Omega$ through $a$ and $b$, and let $p$ and $q$ be the endpoints of $l$ in $\partial\Omega$ so that $p, a, b, q$ lie in $l$ in that order. Then the Hilbert distance between $a$ and $b$ is given by

$$d_\Omega(a, b) = \log(p, a, b, q),$$

where $(p, a, b, q)$ is the cross ratio defined in Section 2.2. One can verify that this metric is geodesically complete. Since this metric is defined by the cross ratio which is a projective invariant, the subgroup of $PGL(3, \mathbb{R}) = PSL(3, \mathbb{R}) = SL(3, \mathbb{R})$ which preserves $\Omega$ are isometries of $(\Omega, d_\Omega)$. In the case when $\Omega$ is strictly convex, this metric is uniquely geodesic, and the geodesic between any two points is the projective line segment between them. See Section 2 of [47] for more details.

The Hilbert distance is a Finsler metric, i.e. it is induced by an infinitesimal norm on the tangent space of every point in $\Omega$. In fact, there is a known explicit formula for its infinitesimal norm, $||\cdot||$. To describe this formula, choose an affine chart $U$ in $\mathbb{RP}^2$ containing $\overline{\Omega}$, and equip $U$ with the Euclidean metric, $|\cdot|$. This induces a norm $|\cdot|_q$ on the tangent space at every point $q \in U$. For any tangent vector $v$ at $q$, let $\gamma$ be the line through $q$ so that its tangent vector at $q$ is $v$, and let $q^-$ and $q^+$ be the points where $\gamma$ intersects $\partial\Omega$. Then

$$||v||_q = \frac{|v|_q}{2}\left(\frac{1}{|q - q^+|} + \frac{1}{|q - q^-|}\right).$$

Next, we will describe a natural relationship between $(\mathbb{H}^2, PSL(2, \mathbb{R}))$-structure and convex real projective structures on $S$. This motivates our definition of the Hitchin component, which we will give later. Consider the embedding of $\mathbb{H}^2$ as one of the sheets of a two-sheeted hyperboloid in $\mathbb{R}^{1,2}$. Each line in $\mathbb{R}^{1,2}$ intersects this hyperboloid at a unique point, so the projectivization $\mathbb{R}^{1,2} \to \mathbb{RP}^2$ induces an embedding

$$f : \mathbb{H}^2 \to \mathbb{RP}^2,$$

One can verify that the image of $f$ is a strictly convex subset of $\mathbb{RP}^2$, and that $f$ is in fact an isometry between $\mathbb{D}$ and its image equipped with the Hilbert metric. Hence, realizing oriented hyperbolic structures on $S$ as convex real projective structures gives an embedding $\mathcal{T}(S) \to \mathcal{C}(S)$.

We can also describe this embedding algebraically. Via the embedding $f$, the hyperbolic isometries of $\mathbb{H}^2$ can be realized as projective transformations of $\mathbb{RP}^2$. In other words, we have an irreducible representation

$$\iota_3 : PGL(2, \mathbb{R}) \to PSL(3, \mathbb{R}).$$

Restricting to the connected component of the identity in $PGL(2,\mathbb{R})$, we get an irreducible representation

$$\iota_3 : PSL(2,\mathbb{R}) \to PSL(3,\mathbb{R}).$$

The classification of the finite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ implies that $\iota_3$ is in fact the unique (up to conjugation in $PSL(3,\mathbb{R})$) such irreducible representation. Post-composing the holonomy representations in $\mathcal{T}(S)$ with $\iota_3$ thus induces a map

$$i_3 : \mathcal{T}(S) \to Hom\big(\Gamma, PSL(3,\mathbb{R})\big)/PSL(3,\mathbb{R}),$$

whose image lies in the component consisting of the holonomies of real projective structures on $S$. Hence, the image of the map $\overline{hol}$ in Theorem 3.3.2 is the component of $Hom\big(\Gamma, PSL(3,\mathbb{R})\big)/PSL(3,\mathbb{R})$ that contains the image of $i_3$.

## 3.4 Hyperbolicity properties of $\Gamma$

In this section, we will describe a coarse notion of hyperbolicity for $\Gamma := \pi_1(S)$, and show how one can use that to define a boundary of $\Gamma$. This will be important for us as we will use this boundary in the next section to construct descriptions of geodesics on $S$ that are independent of any choice of metric on $S$.

Choose a finite generating set $F$ for $\Gamma = \pi_1(S)$ so that $F = F^{-1}$. Using this, we can build a graph where the vertices are the elements in $\Gamma$, and two vertices $a$ and $b$ are connected by an edge if and only if $ab^{-1} \in F$. This graph is called the *Cayley graph* of $\Gamma$ corresponding to the generating set $F$, and is also denoted by $\Gamma(F)$. Declaring the length of each edge in $\Gamma(F)$ to be $1$ endows $\Gamma(F)$ with the structure of a geodesic metric space. Note that the left action of $\Gamma$ on itself induces a proper action by isometries on $\Gamma(F)$.

Although this construction associates to $\Gamma$ a geodesic metric space on which it acts on by isometries, it has the disadvantage of depending on the choice of a finite generating set for $\Gamma$. Hence, we would like an equivalence relation on metric spaces that allows us to ignore the choice of generating set. The right notion of equivalence is called a quasi-isometry, which we define below.

**Definition 3.4.1.** Let $X$ and $X'$ be two metric spaces.

(1) A map $f : X \to X'$ is a *quasi-isometric embedding* if there is some $c > 0$ so that

$$\frac{1}{c}d_X(x,y) - c \leq d_{X'}\big(f(x), f(y)\big) \leq c d_X(x,y) + c$$

for all $x, y \in X$.

(2) A quasi-isometric embedding is a *quasi-isometry* if there is some $c > 0$ so that for every $x' \in X'$, there is some $x \in X$ so that $d_{X'}\big(x', f(x)\big) < c$.

(3) $X$ and $X'$ are *quasi-isometric* if there is a quasi-isometry $f : X \to X'$.

Given a quasi-isometry $f : X \to X'$, one can construct a *quasi-inverse*, i.e. a quasi-isometry $g : X' \to X$ with the property that there is some $c > 0$ so that $d_X\big(g \circ f(x), x\big) < c$ for all $x \in X$ and $d_{X'}\big(f \circ g(x'), x'\big) < c$ for all $x' \in X'$. It is also easy to check that the composition of two quasi-isometries is a quasi-isometry, and the identity map is clearly a quasi-isometry. As such, being quasi-isometric is an equivalence relation.

The Švarc-Milnor Lemma (See Proposition 8.19 of [5]) states that if $\Gamma$ acts properly discontinuously, cocompactly, and by isometries on a proper metric space $X$, then any orbit map $\Gamma \to X$ induces a quasi-isometry $\Gamma(F) \to X$. In particular, it implies that if $F'$ is another finite generating set for $\Gamma$, then $\Gamma(F)$ is quasi-isometric to $\Gamma(F')$. Hence, in this setting, we will often drop the finite generating set $F$ and simply denote $\Gamma(F)$ by $\Gamma$.

If we choose a hyperbolic structure on $S$ then the developing map for the hyperbolic structure is an isometry from $\widetilde{S}$ to the Poincarè disc $\mathbb{D}$ equipped with the Poincarè metric. Since the action of $\Gamma$ on $\widetilde{S}$ is properly discontinuous, cocompact and by isometries, the Švarc-Milnor lemma also implies that $\Gamma$ is quasi-isometric to $\mathbb{D}$. In particular, the metric on $\Gamma$ should have some "coarse hyperbolic behavior". This is described more formally below.

**Definition 3.4.2.**

(1) A geodesic metric space $X$ is *Gromov hyperbolic* if there is some $\delta > 0$ so that the following property holds. For any point $a$ on any edge of any geodesic triangle in $X$, there is a point $b$ in the union of the other two edges such that $d_X(a, b) < \delta$.

(2) A group is *Gromov hyperbolic* if its Cayley graph (with respect to some/any finite generating set) is Gromov hyperbolic.

(3) Let $X$ be a Gromov hyperbolic metric space. The *Gromov boundary* of $X$ is

$$\partial X := \{\text{unit speed geodesic rays in } X\}/\sim,$$

where $\gamma_1 \sim \gamma_2$ if $d_X\big(\gamma_1(t), \gamma_2(t)\big)$ is uniformly bounded above for all $t \geq 0$. If $\gamma$ is a unit speed geodesic ray in $X$, we denote its equivalence class in $X$ by $[\gamma]$.

It is easy to verify that the Gromov hyperbolicity of a metric space is preserved by quasi-isometry, so the Gromov hyperbolicity of a group is well-defined. Also, the boundary $\partial X$ comes equipped with a natural topology, which we will now describe. For any three points $a, b, c \in X$, define

$$(a, b)_c := d_X(a, c) + d_X(b, c) - d_X(a, b).$$

Choose a point $o \in X$. For any $\alpha \in \partial X$ and any $r > 0$, define

$$V(\alpha, r) := \big\{[\gamma] \in \partial X : \text{there exists } \gamma_1 \text{ with } [\gamma_1] = \alpha \text{ and } \liminf_{t\to\infty} \big(\gamma_1(t), \gamma(t)\big)_o \geq r\big\}.$$

One should think of $V(\alpha, r)$ as the set of "end points" of a cone of geodesic rays emanating from $o$. Topologize $\partial X$ by declaring the collection $\{V(\alpha, r) : \alpha \in \partial X, r > 0\}$ to be a basis, and check that this topology is independent of the choice of $o$.

The next proposition tells us that the Gromov boundary behaves well under quasi-isometry.

**Theorem 3.4.3.** *Let $X$, $X'$ be geodesic metric spaces and $f : X \to X'$ be a quasi-isometry. If $X$ is Gromov hyperbolic, then $f$ extends uniquely to a homeomorphism $\partial f : \partial X \to \partial X'$.*

*Proof.* See Theorem 3.9 of [5]. □

We will now apply this technology to $\Gamma$. An elementary exercise in hyperbolic geometry allows one to prove that $\mathbb{D}$ is Gromov hyperbolic, and that the Gromov boundary of $\mathbb{D}$ is topologically a circle. Theorem 3.4.3 then implies that $\Gamma$ is also Gromov hyperbolic, and $\partial\Gamma$ is also topologically a circle. If we choose a hyperbolic structure on $S$, then the holonomy representation $\rho$ induces a quasi-isometry $f : \Gamma \to \mathbb{D}$ that is $\rho$-equivariant. Theorem 3.4.3 then implies that $f$ further induces a $\rho$-equivariant homeomorphism $\partial f : \partial\Gamma \to \partial\mathbb{D}$. In particular, $\partial\Gamma$ is topologically a circle.

Furthermore, it is a standard fact in hyperbolic geometry that if $\Gamma' \subset PSL(2, \mathbb{R})$ is a discrete subgroup isomorphic to $\Gamma$, then every nonidentity element $g \in \Gamma$ has exactly two fixed points $g^-, g^+ \in \partial\mathbb{D}$, so that for any $x \in \partial\Gamma \setminus \{g^-\}$, $\lim_{n\to\infty} g^n \cdot x = g^+$. Hence, by Theorem 3.4.3, we also know that every non-identity element $X \in \Gamma$ has exactly two fixed points $X^-, X^+ \in \partial\Gamma$, with the property that for any $x \in \partial\Gamma \setminus \{X^-\}$, $\lim_{n\to\infty} X^n \cdot x = X^+$. The points $X^-$ and $X^+$ are the *repelling fixed point* and *attracting fixed point* for $X$ respectively.

## 3.5 Ideal triangulations and pants decompositions

In this section, we will use $\partial\Gamma$ defined in the previous section to give a topological description of geodesics, ideal triangulations and pants decompositions of $S$. This point of view is necessary for our purposes because we need a description of these objects in a way that is independent of a choice of hyperbolic metric on $S$. We start with the space of geodesics on $S$.

**Definition 3.5.1.**

(1) The *space of (undirected) geodesics in* $S$ is $\partial\Gamma^{[2]}/\Gamma$, where $\partial\Gamma^{[2]}$ is the set of unordered distinct pairs of points in $\partial\Gamma$.

(2) The *space of directed geodesics in* $S$ is $\partial\Gamma^{(2)}/\Gamma$, where $\partial\Gamma^{(2)}$ is the set of ordered distinct pairs of points in $\partial\Gamma$.

For any choice of a hyperbolic structure on $S$, the developing map $d : \widetilde{S} \to \mathbb{D}$ induces a quasi-isometry $f : \Gamma \to \mathbb{D}$ by precomposing $d$ with any orbit map $\Gamma \to \widetilde{S}$. This then induces a homeomorphism $\partial f : \partial\Gamma \to \partial\mathbb{D}$. Let $\gamma$ be any oriented geodesic in $S$, let $\widetilde{\gamma}$ be a lift of $\gamma$ to $\widetilde{S}$, and let $\widetilde{\gamma}^-$ and $\widetilde{\gamma}^+$ be the points in $\partial\Gamma$ so that $\partial f(\widetilde{\gamma}^-)$ and $\partial f(\widetilde{\gamma}^+)$ are the forward and backward endpoints of the geodesic $d(\widetilde{\gamma})$. This defines a bijection

$$\begin{array}{rcl}\{\text{directed geodesics in } S\} & \to & \partial\Gamma^{(2)}/\Gamma \\ \gamma & \mapsto & [\widetilde{\gamma}^-, \widetilde{\gamma}^+],\end{array}$$

which induces a bijection

$$\begin{array}{rcl}\{\text{(undirected) geodesics in } S\} & \to & \partial\Gamma^{[2]}/\Gamma \\ \gamma & \mapsto & \big[\{\widetilde{\gamma}^-, \widetilde{\gamma}^+\}\big].\end{array}$$

Thus, the definitions of the space of geodesics and the space of directed geodesics given in Definition 3.5.1 agree with the usual definitions once we choose an oriented hyperbolic structure on $S$.

Next, we will give a description of ideal triangulations of $S$ in terms of $\partial\Gamma$. If we choose a hyperbolic structure on $S$, then our definition of an ideal triangulation of $S$ is equivalent to a geodesic lamination on $S$ with finitely many leaves. (See figure 3.1 for an ideal triangulation of a pair of pants.) We say that the geodesics $\{a, b\}$ and $\{c, d\}$ in $\partial\Gamma^{[2]}$ *intersect* if neither of the closed subsegments of $\partial\Gamma$ with endpoints $a$, $b$ contain both $c$ and $d$. With this, we can define an ideal triangulation on $S$.

**Definition 3.5.2.**

(1) An (undirected) $\Gamma$-*invariant ideal triangulation* of the universal cover $\widetilde{S}$ of $S$ is a maximal $\Gamma$-invariant pairwise nonintersecting subset $\widetilde{\mathcal{T}}$ of $\partial\Gamma^{[2]}$ such that for any geodesic $\{a, b\}$ in $\widetilde{\mathcal{T}}$, either one of the following must hold:

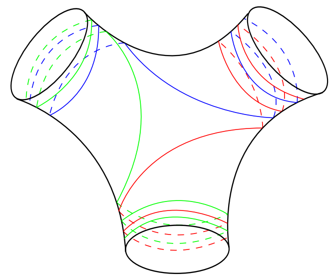

Figure 3.1: Ideal triangulation of a pair of pants.

- There is some $c$ in $\partial\Gamma$ such that $\{b, c\}$ and $\{c, a\}$ both lie in $\widetilde{\mathcal{T}}$.
- There is some $X$ in $\Gamma$ such that $\{a, b\}$ is the set of repelling and attracting fixed points of $X$.

(2) An *ideal triangulation* of $S$ is the quotient of a $\Gamma$-invariant ideal triangulation of $\widetilde{S}$ by $\Gamma$. If $\widetilde{\mathcal{T}}$ is an ideal triangulation of $\widetilde{S}$, then we denote by $\mathcal{T}$ its quotient by $\Gamma$.

If $\{a, b\}$ in $\widetilde{\mathcal{T}}$ is the set of repelling and attracting fixed points of some $X$ in $\Gamma$, then we call $\{a, b\}$ a *closed leaf*. Also, $[a, b]$ in $\mathcal{T}$ is called a *closed leaf* if some (or equivalently, all) of its representatives in $\widetilde{\mathcal{T}}$ are closed leaves. By a *triangle* in $\widetilde{\mathcal{T}}$, we mean a subset of $\widetilde{\mathcal{T}}$ that is of the form $\big\{\{a, b\}, \{b, c\}, \{c, a\}\big\}$, where $a, b, c$ are points in $\partial\Gamma$. Each of the three pairs in any triangle is called an *edge* of that triangle, and a point in any edge is called a *vertex* of that edge. Also, we say that two triangles in $\widetilde{\mathcal{T}}$ are *adjacent* if they share a common edge. We will denote by $\widetilde{\Delta} = \widetilde{\Delta}_{\widetilde{\mathcal{T}}}$ the set of triangles in $\widetilde{\mathcal{T}}$. There is an obvious $\Gamma$-action on $\widetilde{\Delta}$, so we can consider

$$\Delta = \Delta_{\mathcal{T}} := \widetilde{\Delta}/\Gamma$$

and call any element in $\Delta$ a *triangle* in $\mathcal{T}$. An *edge* of a triangle $T$ in $\Delta$ is an element $e$ in $\mathcal{T}$ so that $T$ has a representative (in $\widetilde{\Delta}$) which has a representative (in $\widetilde{\mathcal{T}}$) of $e$ as an edge. As before, we say two triangles in $\Delta$ are *adjacent* if they share an edge, or equivalently, if they have adjacent representatives in $\widetilde{\Delta}$.

If we choose a hyperbolic structure on $S$, then the notion of ideal triangulation defined in Definition 3.5.2 then gives us an ideal triangulation of $\widetilde{S}$ (in the classical sense) by assigning to each pair $\{a, b\}$ in $\widetilde{\mathcal{T}}$ to the unique geodesic $\widetilde{\gamma}$ in $\widetilde{S}$ so that $d(\widetilde{\gamma})$ is the geodesic in $\mathbb{D}$ with end points $\partial d(a)$ and $\partial d(b)$. Moreover, this ideal triangulation is $\Gamma$-invariant, so $\mathcal{T}$ can be thought of as a finite leaf geodesic lamination of $S$ equipped with the hyperbolic structure.

In the same spirit, we can define a pants decomposition in the following way.

**Definition 3.5.3.**

(1) An (undirected) *pants decomposition* $\widetilde{\mathcal{P}}$ on $\widetilde{S}$ is a maximal $\Gamma$-invariant collection of pairwise nonintersecting closed leaves in $\partial\Gamma^{[2]}$.

(2) A *pants decomposition* of $S$ is the quotient of a pants decomposition of $\widetilde{S}$ by $\Gamma$. If $\widetilde{\mathcal{P}}$ is a pants decomposition, we denote by $\mathcal{P}$ its quotient by $\Gamma$.

Just as for ideal triangulations, once we choose a hyperbolic structure on $S$, then the $\Gamma$-equivariant homeomorphism between $\partial\Gamma$ and $\partial\mathbb{D}$ causes the definition of pants decomposition on $S$ given in Definition 3.5.3 to agree with the classical definition of a pants decomposition of a hyperbolic surface.

# CHAPTER 4

# The Hitchin component

In this chapter, we will explain several different ways one can think of the Hitchin component, which is the main object of study in this thesis. At the end, we will state our main theorem and some of its implications.

## 4.1 Definition and origins

Before we formally define the Hitchin component, we need to describe the unique (up to conjugation) irreducible representation from $PSL(2,\mathbb{R})$ to $PSL(n,\mathbb{R})$ for any $n \geq 2$. There is a linear identification of $\mathbb{R}^n$ with $\mathcal{K}'$, the space of homogeneous degree $n-1$ polynomials with real coefficients in the two variables $X$ and $Y$. This induces an identification between $\mathbb{RP}^{n-1}$ and $\mathcal{K} := \mathcal{K}'/\mathbb{R}^+$, which identifies the point $[a_0 : \cdots : a_n] \in \mathbb{RP}^{n-1}$ with the equivalence class of the homogeneous polynomial $\displaystyle\sum_{i=0}^{n} a_i X^{n-i} Y^i$. Define a $PSL(2,\mathbb{R})$ action on $\mathcal{K}$ by

$$[a_{i,j}] \cdot [h(X,Y)] := [h(a_{1,1}X + a_{1,2}Y, a_{2,1}X + a_{2,2}Y)]$$

for any $[a_{i,j}] \in PSL(2,\mathbb{R})$ and any $[h(X,Y)] \in \mathcal{K}$.

It is easy to check that this action induces an action of $PSL(2,\mathbb{R})$ on $\mathbb{RP}^{n-1}$ as projective transformations, and this action does not preserve any projective subspaces. Thus, we have an irreducible homomorphism

$$\iota_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R}).$$

By the classification of the irreducible finite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$, this is the unique (up to conjugation in $PSL(n,\mathbb{R})$) irreducible representation of $PSL(2,\mathbb{R})$ into $PSL(n,\mathbb{R})$. Also, for any integer $n \geq 2$, we can define an embedding $f : \mathbb{RP}^1 \to \mathbb{RP}^{n-1}$

by

$$f : [a : b] \mapsto [a^{n-1} : a^{n-2}b : \cdots : ab^{n-2} : b^{n-1}].$$

One can verify that this embedding is $\iota_n$-equivariant, and is the identity map when $n = 2$.

The irreducible representation $\iota_n$, induces an embedding

$$i_n : \mathcal{T}(S) \to \mathcal{X}_n(S) := \mathrm{Hom}\left(\Gamma, PSL(n, \mathbb{R})\right)/PSL(n, \mathbb{R})$$

defined by $i_n : [\rho] \mapsto [\iota_n \circ \rho]$ for any conjugacy class of representations $[\rho] \in \mathcal{T}(S)$. Using this, we can define a generalization of the Teichmüller space $\mathcal{T}(S)$.

**Definition 4.1.1.**

(1) The image of the map $i_n$ defined above is the *Fuchsian locus*.

(2) The *$n$-th Hitchin component of $S$*, denoted $Hit_n(S)$, is the connected component of $\mathcal{X}_n(S)$ that contains the Fuchsian locus.

As mentioned in Chapter 3, $Hit_2(S) = \mathcal{T}(S)$ and $Hit_3(S) = \mathcal{C}(S)$. In the $n = 4$ case, Guichard-Wienhard [26] proved that via the holonomy representation, the deformation space of *convex foliated* $\mathbb{RP}^3$ *structures* on $T^1S$ can be naturally identified with $Hit_4(S)$. For any $\rho \in Hit_n(S)$, Guichard-Wienhard [27] also constructed domains of discontinuities for $\rho$ in $\mathcal{F}(\mathbb{R}^n)$.

The Hitchin component was first studied by Hitchin [28], who then called it the Teichmüller component of $\mathcal{X}_n(S)$. Using Higgs bundle techniques, he proved, among many other things, that the Hitchin component is diffeomorphic to a cell of dimension $(6g - 6)(n^2 - 1)$.

The tools that Hitchin used were complex analytic in nature. In a nutshell, Simpson [46], generalizing the work of Hitchin [29], proved that the moduli space of reductive representations from $\pi_1(S)$ to $PSL(n, \mathbb{C})$ considered up to conjugation, is homeomorphic to the moduli space of semistable Higgs bundles, considered up to gauge transformation. Given a choice of a base conformal structure on $S$, Hitchin [28] realized this moduli space as a fibration over the *Hitchin base*, which is the vector space of sums of holomorphic differentials on $S$ of degree $2, \dots, n$. Furthermore he constructed a natural section of this fibration whose image is $Hit_n(S)$.

A main step of Hitchin's argument involves solving a system of partial differential equations that are today known as the *Hitchin equations*. Corlette [12], generalizing the work of Eells-Sampson [16] and Donaldson [14], proved that solving these equations is equivalent to specifying a harmonic map from $S$ (equipped with the base conformal structure) to the $SL(n,\mathbb{R})$ symmetric space $M := SL(n,\mathbb{R})/SO(n)$, and thus obtained the following theorem.

**Theorem 4.1.2** (Corlette)**.** *Choose a conformal structure $\Sigma$ on $S$, and let $\widetilde{\Sigma}$ be $\widetilde{S}$ equipped with the lifted conformal structure. For any reductive representation $\rho : \Gamma \to PSL(n,\mathbb{R})$, there is a harmonic map*

$$h : \widetilde{\Sigma} \to M$$

*that is $\rho$-equivariant. Furthermore, if $\rho$ is irreducible, then $h$ is unique up to post-composition by $PSL(n,\mathbb{R})$.*

Later, Labourie (Corollary 1.0.4 of [31]) proved that in the case of Hitchin representation, one can in fact make "good" choices of conformal structures on $S$.

**Theorem 4.1.3** (Labourie)**.** *For any $\rho \in Hit_n(S)$, there is a conformal structure $\Sigma$ on $S$ so that the $\rho$-equivariant harmonic map $h : \widetilde{\Sigma} \to M$ is a conformal immersion away from a (possibly empty) discrete set of branched points.*

In the case when $n = 3$, Labourie [32] and Loftin [35] independently proved that for any Hitchin representation in $Hit_3(S)$, such a choice of conformal structure is in fact unique.

Given any representation $\rho \in Hit_n(S)$, we can also define the following invariant, which is commonly known as the critical exponent.

**Definition 4.1.4.** Let $\rho \in Hit_n(S)$. The *critical exponent* is the quantity

$$h_M(\rho) := \limsup_{T\to\infty} \frac{1}{T} \log \left| \Big\{ X \in \Gamma : d_M\big(o, \rho(X)\cdot o\big) < T \Big\} \right|,$$

where $o$ is a choice of base point in $M$ and $d_M$ is the distance function induced by the Riemannian metric on $M$.

It is easy to see that this quantity is independent of the choice of $o$. Less formally, the critical exponent is the exponential growth rate of the number of points in the $\Gamma$-orbit of $o$ that are contained in a ball of growing radius centered about $o$. Sanders [45] proved the following statement relating the branched minimal immersions in Labourie's Theorem and the critical exponent.

**Theorem 4.1.5** (Sanders)**.** *For any $\rho$ in $Hit_n(S)$, choose a conformal structure $\Sigma$ on $S$, so that the harmonic map*

$$f : \Sigma \to \rho(\Gamma)\backslash M$$

*is a branched minimal immersion. Then $f$ is a minimal immersion (without branched points) and satisfies the following inequality:*

$$\frac{1}{\mathrm{Vol}(f^*m)}\int_\Sigma \sqrt{-\operatorname{Sec}\left(T_{f(p)}f(\Sigma)\right)+\frac{1}{2}||B_f(p)||^2}\,\mathrm{d}V(p)\le h_M(\rho),$$

*where* $\operatorname{Sec}$ *is the sectional curvature in $\rho(\Gamma)\backslash M$, $m$ is the Riemannian metric on $\rho(\Gamma)\backslash M$,* $\mathrm{d}V$ *is the volume measure of $f^*m$ and $B_f$ is the second fundamental form of $f$.*

Although the Hitchin's techniques gives a complete description of the global topology of the Hitchin component, almost nothing could be said about the geometric properties of the representations in the Hitchin component at that time. This was remedied by Labourie [30], who proved the following theorem using dynamical techniques.

**Theorem 4.1.6** (Labourie)**.** *Let $\rho \in Hit_n(S)$. Then for any $p \in M$, the map $f : \Gamma \to M$ given by $f(X) = \rho(X)\cdot p$ is a quasi-isometric embedding. In particular, $\rho$ is discrete and faithful. Furthermore, for any $X \in \Gamma \setminus \{\mathrm{id}\}$, $\rho(X)$ has a lift to $SL(n,\mathbb{R})$ that is diagonalizable with pairwise distinct positive eigenvalues.*

We will often abuse terminology by referring to the positive eigenvalues of the lift of $\rho(X)$ as the eigenvalues of $\rho(X)$. By using combinatorial methods, Fock-Goncharov [18] also arrived at similar conclusions. In that same paper, Labourie showed that every Hitchin representation $\rho$ preserves a unique $\rho$-equivariant Frenet curve $\xi : \partial\Gamma \to \mathcal{F}(\mathbb{R}^n)$. Shortly after, Guichard [25] also proved that the existence of such an equivariant Frenet curve is a sufficient condition for a representation to be Hitchin. Hence, we have the following useful characterization of the representations in $Hit_n(S)$.

**Theorem 4.1.7** (Guichard, Labourie)**.** *A representation $\rho \in \mathcal{X}_n(S)$ lies in $Hit_n(S)$ if and only if there exists a $\rho$-equivariant Frenet curve $\xi : \partial\Gamma \to \mathcal{F}(\mathbb{R}^n)$. If $\xi$ exists, then it is Hölder continuous, and is uniquely determined (up to post-composition by $PSL(n,\mathbb{R})$) by $\rho$.*

This theorem gives us the option of understanding Hitchin representations via their corresponding Frenet curves. Given any Hitchin representation, we can also define a notion of length for each $X \in \Gamma$.

**Definition 4.1.8.** For any representation $\rho$ in $Hit_n(S)$, and any $X$ in $\Gamma$, define the *length* of $X$ to be the quantity

$$l_\rho(X) := \log\left(\frac{\lambda_n}{\lambda_1}\right),$$

where $\lambda_1 < \cdots < \lambda_n$ are the eigenvalues of $\rho(X)$.

Observe that $l_\rho : \Gamma \to \mathbb{R}$ is a non-negative function that is invariant under conjugation in $PSL(n, \mathbb{R})$. Furthermore, when $n = 2$, then $l_\rho(X)$ is the hyperbolic length of the closed geodesic in $S$ corresponding to $X$, measured in the hyperbolic metric with holonomy representation $\rho$. Similarly, when $n = 3$, then $l_\rho(X)$ is the Hilbert length of the closed geodesic in $S$ corresponding to $X$, measured in the Hilbert metric induced by the convex projective structure on $S$ with holonomy representation $\rho$.

These length functions play an important role in parameterizing the Hitchin component, as well as studying the Hitchin component using symbolic dynamics. We will spend the next two sections describing these two ways to understand the Hitchin component.

## 4.2 Dynamics and topological entropy

One of the main result in this thesis involves some dynamics one can associate to Hitchin representations. In this section, we will demonstrate how one can construct a flow on the unit tangent bundle of a surface from the Frenet curve of any representation in $Hit_n(S)$, and study a dynamical invariant of this flow called the topological entropy. This in fact holds for a much more general class of representations called *projectively Anosov representations*, but we will not discuss them here. The material in this section is the work of Sambarino, and can be found in Section 2 and 3 of Sambarino [44].

Let $\rho$ be a representation in $Hit_n(S)$ and $\xi$ the corresponding Frenet curve. Choose norms on $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$, which we denote by $||\cdot||$ and $||\cdot||^*$ respectively. Consider the map

$$\begin{aligned} c_\rho : \quad \Gamma \times \partial\Gamma &\to \mathbb{R} \\ (X, x) &\mapsto \log\left(\frac{||\rho(X)\cdot v|| \cdot ||\rho(X)\cdot w||^*}{||v||\cdot||w||^*}\right) \end{aligned}$$

where $v \in \xi(x)^{(1)}$, $w \in (\mathbb{R}^n)^*$ is any linear functional with kernel $\xi(x)^{(n-1)}$. It is clear that $c_\rho$ does not depend on the choice of $v$ or $w$, and $c_\rho(X, \cdot)$ is Hölder continuous because $\xi$ is. Also, a quick computation shows that if $X^+$ is the attracting fixed point of $X$ in $\partial\Gamma$, then $c_\rho(X, X^+) = l_\rho(X)$. Furthermore, $c_\rho$ satisfies the following *cocycle condition*:

$$c_\rho(XY, x) = c_\rho(X, Y\cdot x) + c_\rho(Y, x).$$

Hence, $c_\rho$ is an example of a Hölder cocycle, which we will now define.

**Definition 4.2.1.** A *Hölder cocycle* is a function $c : \Gamma \times \partial\Gamma \to \mathbb{R}$ such that the following hold:

1. $c(XY, x) = c(X, Y \cdot x) + c(Y, x)$,

2. $c(X, \cdot) : \partial\Gamma \to \mathbb{R}$ is Hölder continuous.

The *period* of $c$ is the map $l_c : \Gamma \setminus \{\mathrm{id}\} \to \mathbb{R}$ defined by $l_c(X) = c(X, X^+)$, where $X^+$ is the attracting fixed point of $X$.

Given such a Hölder cocycle $c$, a theorem of Ledrappier (Théorème 3 in Section II of [34]) allows us to construct Hölder continuous functions on $T^1S$ that recovers the periods of $c$.

**Theorem 4.2.2** (Ledrappier)**.** *Let $\Sigma$ be a hyperbolic surface diffeomorphic to $S$, let $\psi_t$ be the geodesic flow on $T^1\Sigma$ and let $c$ be a Hölder cocycle. Then there exists a Hölder continuous function $F : T^1\Sigma \to \mathbb{R}$ with the following property. If $x$ is a point in $T^1\Sigma$ that lies on a closed orbit of the geodesic flow on $T^1\Sigma$ with period $t_0$ (i.e. $\psi_{t_0}(x) = x$ and $\psi_t(x) \neq x$ for all $0 < t < t_0$), then*

$$\int_0^{t_0} F\big(\psi_t(x)\big)dt = l_c(X),$$

*where $X \in \Gamma$ corresponds to the closed orbit of $\psi_t$ that contains $x$.*

Applying this theorem to our setting, we have the following.

**Corollary 4.2.3.** *Let $\Sigma$ be a hyperbolic surface. Then for any $\rho \in Hit_n(S)$, there is some positive Hölder continuous function $F_\rho$ on $T^1\Sigma$ such that for any point $x$ on any closed orbit of the geodesic flow on $T^1\Sigma$ with period $t_0$,*

$$\int_0^{t_0} F_\rho\big(\psi_t(x)\big)dt = l_\rho(X),$$

*where $X \in \Gamma$ corresponds to the closed orbit that contains $x$.*

Next, define the map $\kappa_\rho : \mathbb{R} \times T^1\Sigma \to \mathbb{R}$ by $\kappa_\rho(s, x) := \displaystyle\int_0^s F_\rho\big(\psi_t(x)\big)dt$. Observe that $\kappa_\rho$ is differentiable and satisfies

$$\kappa_\rho(s + t, x) = \kappa_\rho\big(s, \psi_t(x)\big) + \kappa_\rho(t, x)$$

for all $s, t \in \mathbb{R}$ and $x \in T^1\Sigma$. Since $T^1\Sigma$ is compact, $F_\rho$ has a positive minimum, so from the definition, it is easy to see that $\kappa_\rho(\cdot, x)$ is strictly increasing and surjective. Hence, $\kappa_\rho$ has an inverse $\alpha_\rho : \mathbb{R} \times T^1\Sigma \to \mathbb{R}$, i.e. $\alpha_\rho$ satisfies

$$\alpha_\rho\big(\kappa_\rho(t, x), x\big) = t = \kappa_\rho\big(\alpha_\rho(t, x), x\big).$$

Using this, define a flow $(\phi_\rho)_t : T^1\Sigma \to T^1\Sigma$ by $(\phi_\rho)_t(x) = \psi_{\alpha(t,x)}(x)$. Clearly, the periodic orbits of $(\phi_\rho)_t$ agree with the periodic orbits of $\psi_t$. Furthermore, it is an easy computation to see that the periodic orbit of $(\phi_\rho)_t$ corresponding to $X \in \Gamma$ has period $l_\rho(X)$.

We have thus produced a new flow $(\phi_\rho)_t$ on $T^1\Sigma$ via reparameterizing the geodesic flow $\psi_t$ by a Hölder continuous function, so that the periods of the reparameterized flow agree with the length function $l_\rho$. Although this flow is not the unique flow on $T^1\Sigma$ whose periods agree with $l_\rho$, the Livšic theorem for flows says that such a reparameterization is unique up to *Livšic cohomology*. We will not define this notion here, as we will not need this notion in the rest of this thesis. Understanding Hitchin representations from this point of view has been very fruitful though; for example, Bridgeman-Canary-Labourie-Sambarino [8] used this to define a Riemannian metric on $Hit_n(S)$ that is invariant under the action of the mapping class group, and restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus.

Associated to any dynamical system, there is an important dynamical quantity known as the topological entropy, which is defined as follows.

**Definition 4.2.4.** Let $X$ be a compact metrizable space, and choose a metric $d$ on $X$. Let $f : X \to X$ be a continuous map, and for any $n \in \mathbb{Z}^+$, define a new metric $d_n$ on $X$ by

$$d_n(x, y) = \max\big\{d\big(f^i(x), f^i(y)\big) : i = 1, \ldots, n\big\}.$$

For any $n \in \mathbb{Z}^+$ and any $\epsilon > 0$, let $N(n, \epsilon)$ be the largest number of points in $X$ so that the $d_n$ distance between any pair of them is at least $\epsilon$. Then the *topological entropy* of $f$ is the quantity

$$\mathrm{h}_{\mathrm{top}}(f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} N(n, \epsilon).$$

If $\phi_t$ is a flow on $X$, then the *topological entropy* of $\phi_t$, denoted by $\mathrm{h}_{\mathrm{top}}(\phi_t)$, is the topological entropy of the continuous map $\phi_1(\cdot) : X \to X$.

Roughly, the topological entropy is a measurement of the amount of data needed to coarsely track the dynamics on a dynamical system. It thus gives an indication of how complicated or chaotic a dynamical system is.

If $\rho$ is a Hitchin representation, we argued previously that the flow $(\phi_\rho)_t$ is a Hölder reparameterization of the geodesic flow of a hyperbolic surface. Hence, by standard arguments in symbolic dynamics (Theorem 8 of Parry-Pollicot [39] or the proof of Theorem 2 of Pollicot [40]), the topological entropy of $(\phi_\rho)_t$ is the exponential growth rate with $T$ of the number of closed orbits of $(\phi_\rho)_t$ that have period at least $T$, i.e.

$$\mathrm{h_{top}}\left((\phi_\rho)_t\right) = \lim_{T\to\infty} \frac{1}{T}\left|\left\{[X] \in [\Gamma] : l_\rho(X) < T\right\}\right|,$$

where $[\Gamma]$ is the set of conjugacy classes of $\Gamma$. The above discussion motivates the next definition.

**Definition 4.2.5.** Let $\rho$ be a Hitchin representation. Then the *topological entropy* of $\rho$ is

$$\mathrm{h_{top}}(\rho) := \mathrm{h_{top}}\left((\phi_\rho)_t\right).$$

Pollicot-Sharp proved (Theorem 1.3 of [41]) that $\mathrm{h_{top}}$ is a real analytic function on $Hit_n(S)$. Also, it is a consequence of a famous result by Manning [36] that for a hyperbolic surface, the topological entropy of the geodesic flow is equal to the volume entropy, which one can easily calculate to be $1$. In other words, $\mathrm{h_{top}}(\rho) = 1$ for all $\rho \in Hit_2(S)$. From this, an easy calculation will show that $\mathrm{h_{top}}(\rho) = \dfrac{1}{n-1}$ for all $\rho$ in the Fuchsian locus of $Hit_n(S)$.

As special case of a theorem by Crampon [13] tells us that on $Hit_3(S)$, the range of the function $\mathrm{h_{top}}$ is bounded above, and attains its maximal value exactly on the Fuchsian locus of $Hit_3(S)$. Using Coxeter group techniques, Nie [38] then constructed sequences in $Hit_3(S)$ along which the value of $\mathrm{h_{top}}$ converges to $0$. Combining these two results, we thus have the following.

**Theorem 4.2.6** (Crampon, Nie)**.** *The image of the function $\mathrm{h_{top}} : Hit_3(S) \to \mathbb{R}$ is the interval $\left(0, \dfrac{1}{2}\right]$, and $\mathrm{h_{top}}(\rho) = \dfrac{1}{2}$ if and only if $\rho$ lies in the Fuchsian locus.*

A version of Crampon's result was recently proven by Potrie-Sambarino [42] for all Hitchin representations. More precisely, we have the following.

**Theorem 4.2.7** (Potrie-Sambarino)**.** *For all $\rho \in Hit_n(S)$, $\mathrm{h_{top}}(\rho) \leq \frac{1}{n-1}$ and equality holds if and only if $\rho$ lies in the Fuchsian locus.*

## 4.3 Shear-triangle parameterization

In the first parts of this section, we will briefly describe a particular case of what we call the shear-triangle parameterization of $Hit_n(S)$ given by Bonahon-Dreyer [6]. A version of this parameterization can also be found in the monumental work of Fock-Goncharov [18], though in a much less explicit form. We will also give a geometric interpretation of the parameters in terms of flags. After that, we slightly modify this parameterization to obtain a parameterization of $Hit_n(S)$ that is more explicitly analogous to the Fenchel-Nielsen coordinates on $Hit_2(S)$ and the Goldman parameterization [21] on $Hit_3(S)$.

To specify the shear-triangle parameterization, one needs to first choose an ideal triangulation of the surface $S$. For our purposes, we will only be considering this parameterization for a particular ideal triangulation, which we will now describe.

**For the rest of this paper, fix a (undirected) pants decomposition $\mathcal{P}$ for $S$.** This pants decomposition cuts $S$ into finitely many pairs of pants, $P_1, \dots, P_{2g-2}$. For each of these pairs of pants $P_j$, let $A_j$, $B_j$ and $C_j := A_j^{-1}B_j^{-1}$ be three elements in $\pi_1(P_j)$ that correspond to the three boundary components of $P_j$, oriented so that $P_j$ lies on the left of each of these boundary components. Let $f_j : P_j \to S$ be the obvious inclusion, and let $(f_j)_* : \pi_1(P_j) \to \Gamma$ be the induced injection on fundamental groups. The map $(f_j)_*$ is well-defined up to conjugation by elements in $\Gamma$. However, it will be clear that the statements and constructions we make involving $(f_j)_*$ will not depend on the choice of representative in the conjugacy class of $(f_j)_*$. To simplify notation, we will also denote the images of $A_j$, $B_j$ and $C_j$ under $(f_j)_*$ by $A_j$, $B_j$ and $C_j$ respectively.

Recall that the action of any non-identity element $X \in \Gamma$ on $\partial\Gamma$ has a repelling and attracting fixed point. Let $a_j$, $b_j$, $c_j$ be the repelling fixed points and $a_j^+$, $b_j^+$, $c_j^+$ be the attracting fixed points of $A_j$, $B_j$, $C_j$ respectively. Let $\widetilde{\mathcal{Q}}_j$ and $\widetilde{\mathcal{P}}_j$ be the subsets of $\partial\Gamma^{[2]}$ defined by

$$\widetilde{\mathcal{Q}}_j := \bigcup_{X\in\Gamma} \{X\cdot\{b_j,a_j\}, X\cdot\{a_j,c_j\}, X\cdot\{c_j,b_j\}\},$$
$$\widetilde{\mathcal{P}}_j := \bigcup_{X\in\Gamma} \{X\cdot\{a_j,a_j^+\}, X\cdot\{b_j,b_j^+\}, X\cdot\{c_j,c_j^+\}\},$$

and let

$$\widetilde{\mathcal{Q}} := \bigcup_{j=1}^{2g-2} \widetilde{\mathcal{Q}}_j,$$
$$\widetilde{\mathcal{P}} := \bigcup_{j=1}^{2g-2} \widetilde{\mathcal{P}}_j.$$

One can check that $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{P}}$ are disjoint, and $\widetilde{\mathcal{Q}} \cup \widetilde{\mathcal{P}}$ is an ideal triangulation of $\widetilde{S}$. **For the rest of this paper, we denote this particular ideal triangulation by $\widetilde{\mathcal{T}}$ and let $\mathcal{T}$ be the quotient of $\widetilde{\mathcal{T}}$ by $\Gamma$.** Since $\widetilde{\mathcal{Q}}$ and $\widetilde{\mathcal{P}}$ are $\Gamma$-invariant, we can define $\mathcal{P} := \widetilde{\mathcal{P}}/\Gamma$ and $\mathcal{Q} := \widetilde{\mathcal{Q}}/\Gamma$. It is easy to see that $\mathcal{P}$ is the pants decomposition we chose for $S$, $\mathcal{T} = \mathcal{P} \cup \mathcal{Q}$

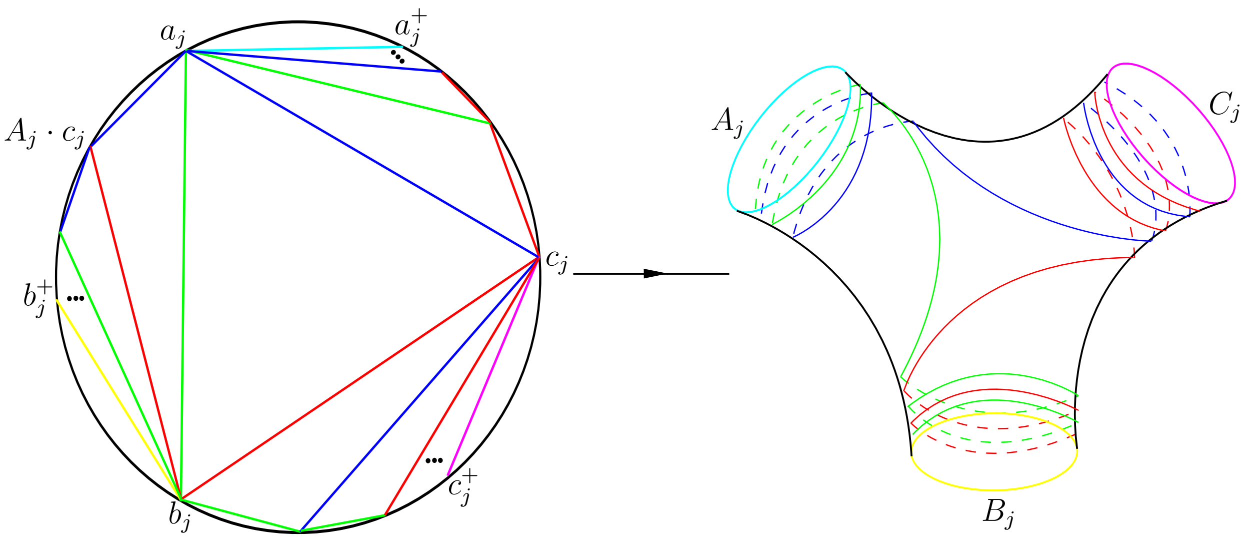

Figure 4.1: $a_j$, $b_j$, $c_j$, $a_j^+$, $b_j^+$, $c_j^+$, $A_j \cdot c_j$ in $\partial\Gamma$.

and $\mathcal{P}$ is exactly the set of closed leaves in $\mathcal{T}$. (See Figure 4.1 for a picture of $\mathcal{T}$ restricted to a pair of pants given by $\mathcal{P}$.)

Define $\mathcal{Q}_j := \widetilde{\mathcal{Q}}_j/\Gamma$ and observe that for any $j = 1, \dots, 2g-2$, $\mathcal{Q}_j$ has exactly three elements, which are the $\Gamma$ orbits of $\{b_j, a_j\}$, $\{a_j, c_j\}$ and $\{c_j, b_j\}$. Moreover, a triangle in $\Delta$ has an edge in $\mathcal{Q}_j$ if and only if all its edges lie in $\mathcal{Q}_j$. Furthermore, there are exactly two triangles in $\Delta$ with edges in $\mathcal{Q}_j$, and we can describe them explicitly. One of them, denoted $T_j$, is the $\Gamma$-orbit of the triangle $\big\{\{b_j, a_j\}, \{a_j, c_j\}, \{c_j, b_j\}\big\}$ and the other, denoted $T_j'$, is the $\Gamma$-orbit triangle $\big\{\{b_j, a_j\}, \{a_j, A_j \cdot c_j\}, \{A_j \cdot c_j, b_j\}\big\}$. (See Figure 4.1.) The triangles $T_j$ and $T_j'$ share all their edges, and any adjacent pair of triangles in $\Delta$ is the pair $T_j$, $T_j'$ for some $j$.

Apart from choosing a the pants decomposition $\mathcal{P}$, in order to specify the shear-triangle parameterization, one needs to make some additional choices for each edge of $\mathcal{P} \subset \mathcal{T}$. Any edge in $\mathcal{P}$ is the $\Gamma$-orbit of the edge $\{u, v\} \in \widetilde{\mathcal{P}} \subset \widetilde{\mathcal{T}}$. For this edge in $\widetilde{\mathcal{P}}$, we choose two points $p, q \in \partial\Gamma$ so that $\{u, p\}$ and $\{v, q\}$ are edges in $\widetilde{\mathcal{Q}}$. Observe that $\partial\Gamma \setminus \{u, v\}$ has two connected components, one of which contains $p$ and the other contains $q$. Using the $\Gamma$-action, we have thus chosen two points in $\partial\Gamma$ for each edge in the $\Gamma$-orbit of $\{u, v\}$. Doing this for every edge in $\mathcal{P}$ assigns two points in $\partial\Gamma$ to each edge in $\widetilde{\mathcal{P}}$. These additional choices are needed to specify the "gluing parameters" later.

Since we can realize $\mathcal{T}$ as an ideal triangulation (in the classical sense) of $S$ by choosing a hyperbolic metric on $S$, the Gauss-Bonnet theorem tells us that the cardinalities of $\Delta$, $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{Q}$ are $4g-4$, $9g-9$, $3g-3$ and $6g-6$ respectively.

In the rest of this section, we will demonstrate how one can construct a parameterization of $Hit_n(S)$ once the choices discussed above are made. Using the chosen topological data, we will associate some cross ratios to the edges in $\mathcal{T}$ and some triple ratios to the triangles in $\Delta_{\mathcal{T}}$. Together, these will parameterize the Hitchin component.

We start by describing the cross ratios associated to the edges in $\mathcal{T}$. Choose any $\rho$ in $Hit_n(S)$ and let $\xi$ be the Frenet curve for $\rho$. First, consider the edge $[a_j, c_j] \in \mathcal{Q}_j$ which lifts to the edge $\{a_j, c_j\} \in \widetilde{\mathcal{Q}}_j$. For any $x = 1, \dots, n-1$, recall that we defined the quantity

$$S_x\big(\xi(a_j), \xi(b_j), \xi(c_j), \xi(A_j \cdot c_j)\big) = -\big(\xi(a_j), \xi(c_j), \xi(A_j \cdot c_j), \xi(b_j)\big)_{\xi(a_j)^{(x-1)}+\xi(b_j)^{(y-1)}}.$$

Since $a_j, c_j, b_j, A \cdot c_j$ lie along $\partial\Gamma$ in that order (see Figure 4.1), (1) of Proposition 2.4.7 implies that for any $x = 1, \dots, n-1$,

$$S_x\big(\xi(a_j), \xi(b_j), \xi(c_j), \xi(A_j \cdot c_j)\big) > 0.$$

Hence, we can define

$$\sigma_{(x,n-x,0),j}(\rho) := \log\Big(S_x\big(\xi(a_j), \xi(b_j), \xi(c_j), \xi(A_j \cdot c_j)\big)\Big),$$

for all $x = 1, \dots, n-1$. These are called the *shear invariants* along $[a_j, b_j]$, and are the projective invariants we associate to the edge $[a_j, b_j]$ mentioned above. Geometrically, these $n-1$ shear invariants determine the pair of flags $\xi(a_j)$, $\xi(b_j)$ and the pair of lines $\xi(c_j)^{(1)}$, $\xi(A \cdot c_j)^{(1)}$ up to $PSL(n, \mathbb{R})$ action. This is an obvious consequence of Lemma 2.2.8.

Similarly, we can also define the shear invariants along $[c_j, a_j]$ and $[b_j, c_j]$ respectively by

$$\sigma_{(n-z,0,z),j}(\rho) := \log\Big(S_z\big(\xi(c_j), \xi(a_j), \xi(b_j), \xi(C_j \cdot b_j)\big)\Big),$$
$$\sigma_{(0,y,n-y),j}(\rho) := \log\Big(S_y\big(\xi(b_j), \xi(c_j), \xi(a_j), \xi(B_j \cdot a_j)\big)\Big).$$

for all $y, z = 1, \dots, n-1$. Hence, we have defined the shear invariants for all the edges in $\mathcal{Q}_j$. Doing this for all $j = 1, \dots, 2g-2$ then defines the shear invariants for all the edges in $\mathcal{Q}$. We will use the set

$$\mathcal{C} := \{(x, y, z) \in (\mathbb{Z}_{\geq 0})^3 : x + y + z = n \text{ and exactly one of } x, y, z \text{ is } 0\}$$

to label the shear invariants for the edges in $\mathcal{Q}_j$, i.e. the shear invariants associated to $\mathcal{Q}_j$

is the set

$$\mathcal{C}_j := \{\sigma_{(x,y,z),j} : (x,y,z) \in \mathcal{C}\}.$$

Next, we define the cross ratios associated to the edges in $\mathcal{P}$. Pick any edge $\eta$ in $\mathcal{P}$, and let $\{u,v\}$ be the edge in $\widetilde{\mathcal{P}}$ so that $\eta$ is the $\Gamma$-orbit of $\{u,v\}$. Let $p,q \in \partial\Gamma$ be the points that we previously chose for the edge $\{u,v\}$, so that $v,p,v,q$ lie on $\partial\Gamma$ in clockwise order. As before, it follows from (1) of Proposition 2.4.7 that for any $k = 1,\dots,n-1$,

$$S_k\big(\xi(u),\xi(v),\xi(p),\xi(q)\big) > 0.$$

This allows us to define the *gluing parameter* for the edge $e$ by

$$\sigma_{k,\eta}(\rho) := \log\Big(S_k\big(\xi(u),\xi(v),\xi(p),\xi(q)\big)\Big).$$

In Bonahon-Dreyer [6], the shear invariants along the edges in $\mathcal{Q}$ are called the *shear invariants along infinite leaves* while the gluing parameters along the edges in $\mathcal{P}$ are called the *shear invariants along closed leaves*. For our purposes, these two kinds of shear invariants play very different roles, hence the renaming. By allowing $\rho$ to vary over $Hit_n(S)$, we can view each shear invariant and each gluing parameter as a real valued function on $Hit_n(S)$.

Now, we will describe the triple ratios associated to each triangle in $\Delta_{\mathcal{T}}$. As before, let $\rho$ be a representation in $Hit_n(S)$ and let $\xi$ be the corresponding Frenet curve. By (2) of Proposition 2.4.7, we can define, for each $j = 1,\dots,2g-2$ and each $(x,y,z) \in \mathcal{A}$, the real numbers $\tau_{(x,y,z),j}(\rho)$ and $\tau'_{(x,y,z),j}(\rho)$ given by the formulas

$$\begin{aligned}
\tau_{(x,y,z),j}(\rho) &:= \log\Big(T_{x,z,y}\big(\xi(a_j),\xi(c_j),\xi(b_j)\big)\Big),\\
\tau'_{(x,y,z),j}(\rho) &:= \log\Big(T_{x,y,z}\big(\xi(a_j),\xi(b_j),\xi(A\cdot c_j)\big)\Big).
\end{aligned}$$

(Recall that $\mathcal{A} := \{(x,y,z) \in (\mathbb{Z}^+)^3 : x+y+z = n\}$.) These are called the *triangle invariants* for $P_j$. We can do this for every $\rho$ in $Hit_n(S)$, so $\tau'_{(x,y,z),j}$ and $\tau_{(x,y,z),j}$ can be viewed as real valued functions on $Hit_n(S)$. We will also use the notation

$$\mathcal{A}_j := \{\tau_{(x,y,z),j} : (x,y,z) \in \mathcal{A}\} \text{ and } \mathcal{A}'_j := \{\tau'_{(x,y,z),j} : (x,y,z) \in \mathcal{A}\}.$$

By Lemma 2.3.6, $\mathcal{A}_j$ determines the triple of flags $\big(\xi(a_j),\xi(c_j),\xi(b_j)\big)$ and $\mathcal{A}'_j$ determines the triple of flags $\big(\xi(a_j),\xi(b_j),\xi(A\cdot c_j)\big)$ up to the action of $PSL(n,\mathbb{R})$.

To obtain their parameterization of $Hit_n(S)$, Bonahon-Dreyer found linear relations between these shear and triangle invariants, which we will now describe. As before, let $\rho$ be a representation in $Hit_n(S)$ and $\xi$ the corresponding Frenet curve. Recall that for every non-identity element $X \in \Gamma$, there is a lift of $\rho(X)$ (which lies in $PSL(n,\mathbb{R})$) to $SL(n,\mathbb{R})$ so that all the eigenvalues of the lift are positive. We abuse terminology by referring to these positive eigenvalues as the eigenvalues of $\rho(X)$.

Let $\lambda_{1,j} < \cdots < \lambda_{n,j}$ be the eigenvalues for $\rho(A_j)$, $\mu_{1,j} < \cdots < \mu_{n,j}$ be the eigenvalues for $\rho(B_j)$ and $\nu_{1,j} < \cdots < \nu_{n,j}$ be the eigenvalues for $\rho(C_j)$. Then for any $k = 1, \dots, n-1$, Bonahon-Dreyer established the following equalities (see Proposition 13 of [6]).

$$\log\left(\frac{\lambda_{k+1,j}}{\lambda_{k,j}}\right) = \sigma_{(k,n-k,0),j} + \sigma_{(k,0,n-k),j} + \sum_{i=1}^{n-k-1} \left(\tau_{(k,i,n-i-k),j} + \tau'_{(k,i,n-i-k),j}\right), \quad (4.3.1)$$

$$\log\left(\frac{\mu_{k+1,j}}{\mu_{k,j}}\right) = \sigma_{(0,k,n-k),j} + \sigma_{(n-k,k,0),j} + \sum_{i=1}^{n-k-1} \left(\tau_{(n-i-k,k,i),j} + \tau'_{(n-i-k,k,i),j}\right), \quad (4.3.2)$$

$$\log\left(\frac{\nu_{k+1,j}}{\nu_{k,j}}\right) = \sigma_{(n-k,0,k),j} + \sigma_{(0,n-k,k),j} + \sum_{i=1}^{n-k-1} \left(\tau_{(i,n-i-k,k),j} + \tau'_{(i,n-i-k,k),j}\right). \quad (4.3.3)$$

The sum on the right hand side of Equations (4.3.1), (4.3.2) and (4.3.3) is the sum of the numbers assigned to all the points in $\mathcal{A} \cup \mathcal{C}$ that lie on the $x = k$, $y = k$ and $z = k$ plane respectively. (See Figure 4.2.) These equations immediately imply that the sums on the right hand side have to be positive. Doing this over every pair of pants given by the pants decomposition gives us $(3n-3)(2g-2)$ linear inequalities involving the shear and triangle invariants. These inequalities are called the *closed leaf inequalities*.

Now, pick any edge $\eta$ in $\mathcal{P}$ and let $P_1$, $P_2$ be the two pairs of pants on either side of $\eta$. Assume without loss of generality that $A_1$ and $A_2$ are the two elements in $\pi_1(P_1)$ and $\pi_1(P_2)$ that correspond to $\eta$. It is clear that the orientations on $\eta$ corresponding to $A_1$ and $A_2$ are opposite, so $\log\left(\frac{\lambda_{k+1,1}}{\lambda_{k,1}}\right) = \log\left(\frac{\lambda_{n-k+1,2}}{\lambda_{n-k,2}}\right)$ for all $k = 1, \dots, n-1$. This implies the equality

$$\begin{aligned}\sigma_{(k,n-k,0),1} + \sigma_{(k,0,n-k),1} + \sum_{i=1}^{n-k-1} \left(\tau_{(k,i,n-i-k),1} + \tau'_{(k,i,n-i-k),1}\right) \\ = \sigma_{(n-k,k,0),2} + \sigma_{(n-k,0,k),2} + \sum_{i=1}^{k-1}\left(\tau_{(n-k,i,k-i),2} + \tau'_{(n-k,i,k-i),2}\right).\end{aligned}$$

Doing this for each curve in $\mathcal{P}$, we have $(n-1)(3g-3)$ linear equations involving the shear and triangle invariants. These equations are known as the *closed leaf equalities*.

Putting all of these together, Bonahon-Dreyer specified $(2g-2)(n^2-1)$ parameters as-

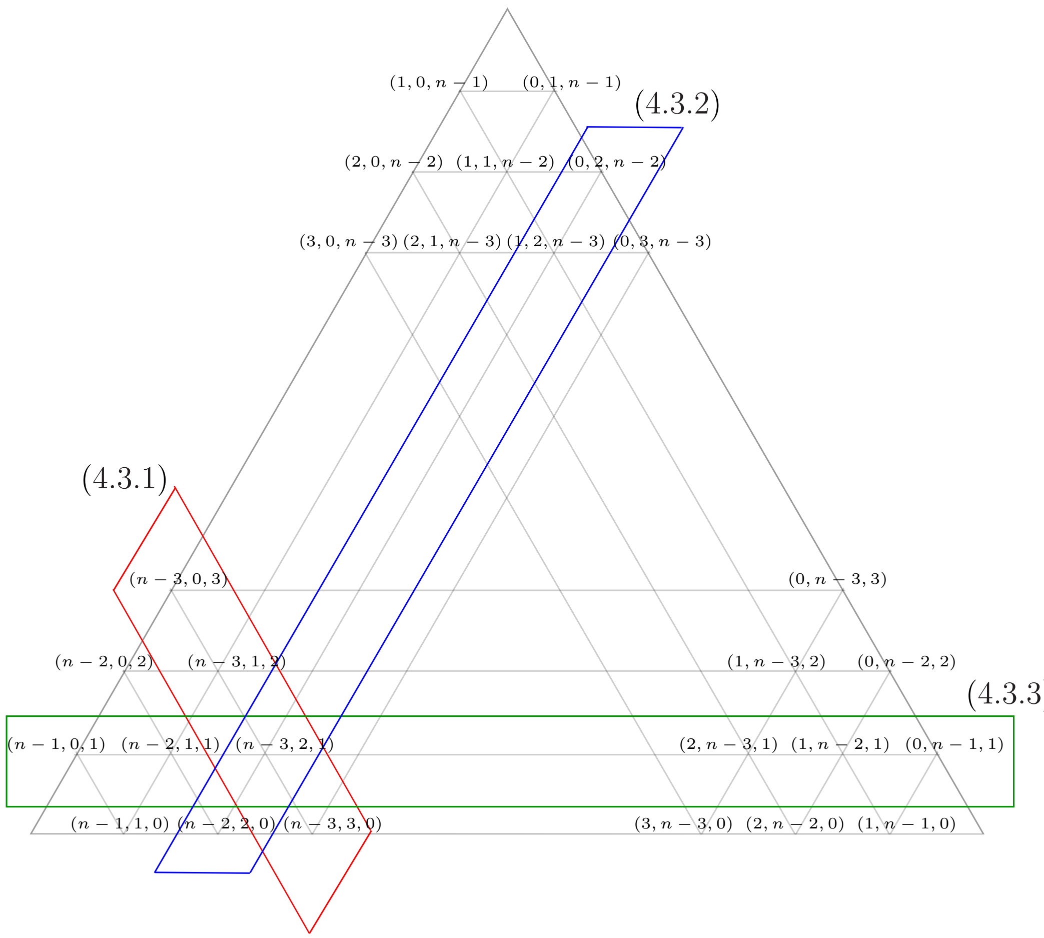


Figure 4.2: Shear and triangle invariants

sociated to the pairs of pants given by $\mathcal{P}$ (the shear and triangle invariants), $(3g-3)(n-1)$ parameters associated to each simple closed curve in $\mathcal{P}$ (the gluing parameters), $(2g-2)(3n-3)$ closed leaf inequalities, and finally $(3g-3)(n-1)$ closed leaf equalities. They then proved (Theorem 2 of [6]) that one can use this information to obtain a parameterization of $Hit_n(S)$, which we call the *shear-triangle parameterization*.

**Theorem 4.3.1** (Bonahon-Dreyer)**.** *The shear invariants, triangle invariants and gluing parameters give a real analytic parameterization of $Hit_n(S)$ by a convex polytope in $\mathbb{R}^{(g-1)(2n^2+3n-5)}$ of dimension $(2g-2)(n^2-1)$ that is cut out by the $(3g-3)(n-1)$ closed leaf equalities and $(2g-2)(3n-3)$ closed leaf inequalities described above.*

Finally, we will give a linear reparameterization of the shear-triangle parameterization. This reparameterization will have exactly $(2g-2)(n^2-1)$ parameters (instead of $(g-1)(2n^2+3n-5)$ parameters in the shear-triangle parameterization), and will be more explicitly analogous to the Fenchel-Nielsen coordinates for $Hit_2(S)$, or the Goldman parameters for $Hit_3(S)$.

To specify this parameterization, we make the same choices as we did to specify the shear-triangle parameterization. On top of that, we choose an orientation on each curve in $\mathcal{P}$. **Henceforth, $\mathcal{P}$ will be an oriented pants decomposition.**

*Notation* 4.3.2. Denote the set of group elements in $\Gamma$ corresponding to oriented closed curves in $\mathcal{P}$ by $\Gamma_{\mathcal{P}}$.

We will have three different kinds of parameters. The first kind is the eigenvalue information of the holonomy about each of the oriented simple closed curves in $\mathcal{P}$. More specifically, for any $\eta$ in $\mathcal{P}$, choose any $X$ in $\Gamma$ that corresponds to $\eta$. Then for any $\rho$ in $Hit_n(S)$, let $\alpha_{1,\eta} < \cdots < \alpha_{n,\eta}$ be the eigenvalues of $\rho(X)$ and define

$$\beta_{k,\eta}(\rho) := \log\left(\frac{\alpha_{k+1,\eta}}{\alpha_{k,\eta}}\right)$$

for any $k = 1, \ldots, n-1$. These quantities are called the *boundary invariants*, and there are $n-1$ of them for each of the $3g-3$ simple closed curves in $\mathcal{P}$. Since we can do this for every $\rho$ in $Hit_n(S)$, we can view these boundary invariants as functions $\beta_{k,\eta} : Hit_n(S) \to \mathbb{R}^+$.

The second kind of parameters are what we will call the *internal parameters*, which are functions associated to each pair of pants $P_j$. In fact, these are a specially chosen subset of the shear and triangle invariants used in the shear-triangle parameterization. For all $j = 1, \ldots, 2g-2$, these parameters are

- $\tau_{(x,y,z),j}$ for all positive integers $x, y, z$ such that $x + y + z = n$,

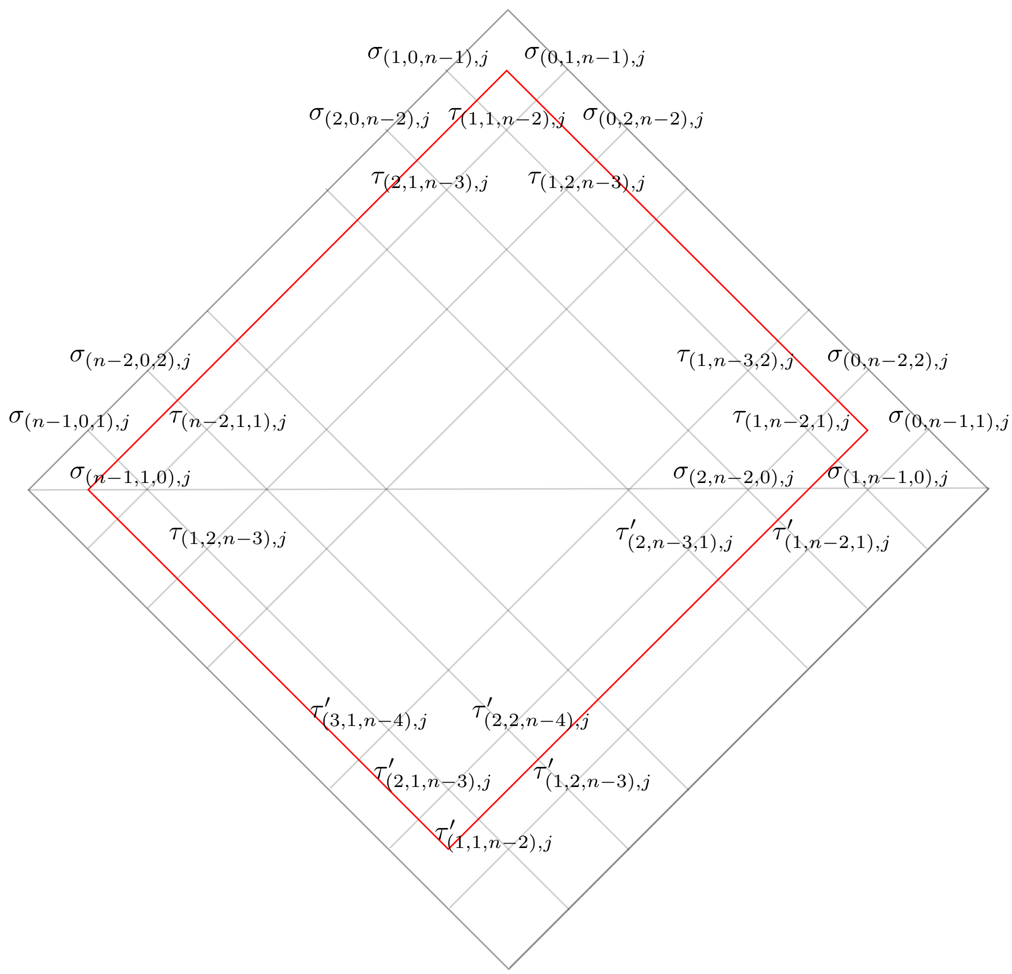


Figure 4.3: Invariants that label points in the red box are the internal parameters.

- $\tau'_{(x,y,z),j}$ for all positive integers $x, y, z$ such that $x + y + z = n$ and $x > 1$,
- $\sigma_{(x,y,0),j}$ for all positive integers $x, y$ such that $x + y = n$ and $x > 1$.

One can easily verify that there are $(n-1)(n-2)$ internal parameters for each of the $2g-2$ pairs of pants given by $\mathcal{P}$. (See Figure 4.3.)

The third and final kind of parameters are the gluing parameters used in the shear-triangle parameterization. As mentioned before, there are $(3g-3)(n-1)$ of them. Together, the boundary invariants, internal parameters and gluing parameters give us $(2g-2)(n^2-1)$ functions on $Hit_n(S)$. We claim that these quantities in fact give us a parameterization of $Hit_n(S)$.

**Proposition 4.3.3.** *The $(3g-3)(n-1)$ boundary invariants, $(2g-2)(n-1)(n-2)$ internal parameters and $(3g-3)(n-1)$ gluing parameters described above define a real analytic parameterization*

$$\Xi : Hit_n(S) \to (\mathbb{R}^+)^{(3g-3)(n-1)} \times \mathbb{R}^{(2g-2)(n-1)(n-2)} \times \mathbb{R}^{(3g-3)(n-1)}.$$

*Proof.* Let $ST(n)$ be the convex polytope used to parameterize $Hit_n(S)$ in the shear-triangle parameterization (see Theorem 4.3.1). We will prove this proposition by showing that the map

$$\Xi' : ST(n) \to (\mathbb{R}^+)^{(3g-3)(n-1)} \times \mathbb{R}^{(2g-2)(n-1)(n-2)} \times \mathbb{R}^{(3g-3)(n-1)}$$

induced by $\Xi$ is a real-analytic bijection. Observe that Equations (4.3.1), (4.3.2) and (4.3.3) imply that $\Xi'$ is the restriction of a linear map to $ST(n)$. Since the dimensions of the domain and range of $\Xi'$ are equal, it is thus sufficient to show that $\Xi'$ is surjective, i.e. we need to show that given a tuple

$$\left(b_{v,u}\right)_{\{v=1,\dots,n-1;u=1,2,3\}} \in (\mathbb{R}^+)^{3n-3}$$

and a tuple

$$\left(\left(t_{(x,y,z)}\right)_{\{(x,y,z)\in\mathcal{A}\}}, \left(t'_{(x,y,z)}\right)_{\{(x,y,z)\in\mathcal{A},x>1\}}, \left(s_{(x,y,0)}\right)_{\{(x,y,0)\in\mathcal{C},;x>1\}}\right) \in \mathbb{R}^{(n-1)(n-2)}$$

we can find

$$\left(\left(t_{(x,y,z)}\right)_{\{(x,y,z)\in\mathcal{A}\}}, \left(t'_{(x,y,z)}\right)_{\{(x,y,z)\in\mathcal{A}\}}, \left(s_{(x,y,z)}\right)_{\{(x,y,z)\in\mathcal{C},\}}\right) \in \mathbb{R}^{(n-1)(n+1)}$$

so that

$$b_{v,1} = s_{(v,n-v,0)} + s_{(v,0,n-v)} + \sum_{i=1}^{n-v-1} (t_{(v,i,n-i-v)} + t'_{(v,i,n-i-v)}), \tag{4.3.4}$$

$$b_{v,2} = s_{(0,v,n-v)} + s_{(n-v,v,0)} + \sum_{i=1}^{n-v-1} (t_{(n-i-v,v,i)} + t'_{(n-i-v,v,i)}), \tag{4.3.5}$$

$$b_{v,3} = s_{(n-v,0,v)} + s_{(0,n-v,v)} + \sum_{i=1}^{n-v-1} (t_{(i,n-i-v,v)} + t'_{(i,n-i-v,v)}). \tag{4.3.6}$$

Here, Equations (4.3.4), (4.3.5) and (4.3.6) are simply Equations (4.3.1), (4.3.2) and (4.3.3) restated using the parameters.

From Equations (4.3.4), (4.3.5), (4.3.6), we can obtain the relation

$$\sum_{v=1}^{n-1} v \cdot b_{v,1} + \sum_{v=1}^{n-1} v \cdot b_{v,2} + \sum_{v=1}^{n-1} (v-n) \cdot b_{v,3} = n \cdot \sum_{k=1}^{n-1} s_{(k,n-k,0)}. \tag{4.3.7}$$

To see that this equality holds, observe that $t_{(x,y,z)}$ is a term in the right hand side of

- Equation (4.3.4) if and only if $v = x$,
- Equation (4.3.5) if and only if $v = y$,
- Equation (4.3.6) if and only if $v = z$.

Hence, $t_{(x,y,z)}$ will appear $x$ times in the sum $\sum_{v=1}^{n-1} v \cdot b_{v,1}$, $y$ times in the sum $\sum_{v=1}^{n-1} v \cdot \beta_{v,2}$ and $z - n$ times in the sum $\sum_{v=1}^{n-1}(v-n) \cdot \beta_{v,3}$. Since $x + y + z - n = 0$, this implies that $t_{(x,y,z)}$ does not appear as a term on the right hand side of Equation (4.3.10). The same inspection argument for $s_{(x,y,0)}$, $s_{(x,0,z)}$ and $s_{(0,y,z)}$ will yield Equation (4.3.10). Similarly, we can also show that

$$\sum_{v=1}^{n-1} (v-n) \cdot b_{v,1} + \sum_{v=1}^{n-1} v \cdot b_{v,2} + \sum_{v=1}^{n-1} v \cdot b_{v,3} = n \cdot \sum_{k=1}^{n-1} s_{(0,k,n-k)}, \tag{4.3.8}$$

$$\sum_{v=1}^{n-1} v \cdot b_{v,1} + \sum_{v=1}^{n-1} (v-n) \cdot b_{v,2} + \sum_{v=1}^{n-1} v \cdot b_{v,3} = n \cdot \sum_{k=1}^{n-1} s_{(n-k,0,k)}. \tag{4.3.9}$$

Now, observe that from the data we are given, Equation (4.3.7) determines $s_{(1,n-1,0)}$ and Equation (4.3.4) determine $s_{(k,0,n-k)}$ for all $k > 1$. By using Equation (4.3.9), we can also find $s_{(1,0,n-1)}$.

Next, we will show that from the given data, we can also find $s_{(0,n-k,k)}$ for $k = 1, \ldots, n-1$ and $t'_{(1,n-k-1,k)}$ for $k = 1, \ldots, n-2$. We will proceed by induction on $k$. For the base case, note that Equation (4.3.5) determines $s_{(0,n-1,1)}$ because we have already found $s_{(1,n-1,0)}$. Then knowing $s_{(0,n-1,1)}$ and $s_{(n-1,0,1)}$ allows us to use Equation (4.3.6) to find $t'_{(1,n-2,1)}$.

For the inductive step, suppose we already know $s_{(0,n-k,k)}$ and $t'_{(1,n-k-1,k)}$ for $k < l$. We need to demonstrate how to find $s_{(0,n-l,l)}$ and $t'_{(1,n-l-1,l)}$. To find $s_{(0,n-l,l)}$, use Equation (4.3.5). Once we have $s_{(0,n-l,l)}$, we can then use Equation (4.3.6) to obtain $t'_{(1,n-l-1,l)}$. $\square$

We call this new parameterization of $Hit_n(S)$ the *modified shear-triangle parameterization*.

In the proof above, we found some useful relations between the shear parameters for a pair of pants and the eigenvalues of the holonomy about the boundary components of that pair of pants. These were stated as Equations (4.3.7), (4.3.9) and (4.3.8). Since we will be using these relations later, we will restate them here in terms of the modified shear-triangle parameterization.

**Lemma 4.3.4.** *Let $\eta_1$, $\eta_2$, $\eta_3$ be the three boundary components of $P_j \subset S$ corresponding to $A_j$, $B_j$, $C_j \in \pi_1(P_j)$ respectively. For $i = 1, 2, 3$, let $(\beta_{v,i})_{\{v=1,\ldots,n-1\}}$ be the boundary invariants for $Hit_n(S)$ corresponding to $\eta_i$. Then*

$$\sum_{v=1}^{n-1} v \cdot \beta_{v,1} + \sum_{v=1}^{n-1} v \cdot \beta_{v,2} + \sum_{v=1}^{n-1} (v-n) \cdot \beta_{v,3} = n \cdot \sum_{k=1}^{n-1} \sigma_{(k,n-k,0),j}, \tag{4.3.10}$$

$$\sum_{v=1}^{n-1} (v-n) \cdot \beta_{v,1} + \sum_{v=1}^{n-1} v \cdot \beta_{v,2} + \sum_{v=1}^{n-1} v \cdot \beta_{v,3} = n \cdot \sum_{k=1}^{n-1} \sigma_{(0,k,n-k),j}, \tag{4.3.11}$$

*and*

$$\sum_{v=1}^{n-1} v \cdot \beta_{v,1} + \sum_{v=1}^{n-1} (v-n) \cdot \beta_{v,2} + \sum_{v=1}^{n-1} v \cdot \beta_{v,3} = n \cdot \sum_{k=1}^{n-1} \sigma_{(n-k,0,k),j}. \tag{4.3.12}$$

## 4.4 Main results

Using the modified shear-triangle parameterization of $Hit_n(S)$, we can now state the main results in this thesis. First, we define a type of sequence in $Hit_n(S)$ that one can think of as being "transverse" to the Fuchsian locus. These are called the *internal sequences*.

*Notation* 4.4.1. Define $\pi_j : Hit_n(S) \to \mathbb{R}^{(n-1)(n-2)}$ to be the projection given by

$$\pi_j(\rho) = \Big( \big(\tau_{(x,y,z),j}(\rho)\big)_{x+y+z=n}, \big(\tau'_{(x,y,z),j}(\rho)\big)_{x+y+z=n,x>1}, \big(\sigma_{(x,y,0),j}(\rho)\big)_{x+y=n,x>1} \Big).$$

More informally, the map $\pi_j$ sends each Hitchin representation to its internal parameters for the $j$th pair of pants.

**Definition 4.4.2.** A sequence $\{\rho_i\}_{i=1}^{\infty}$ is an *internal sequence* if

1. The boundary invariants $\beta_{v,u}(\rho_i)$ are uniformly bounded away from $0$ and $\infty$, i.e. there are constants $K_0, K_1 > 0$ so that $K_0 < \beta_{v,u}(\rho_i) < K_1$ for all $v \in \{1, \dots, n-1\}$, $u \in \{1, \dots, 3g-3\}$, $i \in \mathbb{Z}^+$.

2. For each $j = 1, \dots, 2g-2$ and for any compact subset $K \subset \mathbb{R}^{(n-1)(n-2)}$, there is some integer $N$ so that $\pi_j(\rho_i)$ is not in $K$ for $i > N$.

One should think of the internal sequences as sequences where we hold the boundary invariants "essentially fixed" and deform the internal parameters "as much as possible". In this definition, we do not impose any conditions on the gluing parameters because we do not require them for our main theorem, Theorem 4.4.4.

We are interested in how geometric properties of Hitchin representations degenerate along these internal sequences. Of particular interest are two geometric quantities, $\Theta$ and $\mathrm{h}_{\mathrm{top}}$. We have previously defined $\mathrm{h}_{\mathrm{top}}$ (see Definition 4.2.5), and we will define $\Theta$ now.

**Definition 4.4.3.** Let $\Theta : Hit_n(S) \to \mathbb{R}_{\geq 0}$ be the function given by

$$\Theta(\rho) := \min\{l_\rho(X) : X \neq A^k \text{ for any } A \in \Gamma_{\mathcal{P}} \text{ and any } k \in \mathbb{Z}\}.$$

Recall that $\Gamma_{\mathcal{P}}$ is the subset of $\Gamma$ corresponding to the closed curves in $\mathcal{P}$.

Observe that $\Theta$ depends only on the conjugacy class of $\rho$, and is well-defined because of the fact that we can find a lower bound for $l_\rho(X)$ which depends only on the combinatorics of how a closed curve in $S$ corresponding to $X$ interacts with $\mathcal{T}$ (see Lemma 6.2.2). More informally, $\Theta(\rho)$ is the length (which depends on $\rho$) of the shortest closed curve in $S$ that is not a multiple of a simple closed curve in $\mathcal{P}$. With this, we can now state the main theorem of this paper.

**Theorem 4.4.4.** *Let $\{\rho_i\}_{i=1}^{\infty}$ be an internal sequence. Then the following hold:*

*1.* $\lim\limits_{i\to\infty} \Theta(\rho_i) = \infty$.

2. $\lim_{i\to\infty} \mathrm{h}_{\mathrm{top}}(\rho_i) = 0$.

Before we proceed to the proof of Theorem 4.4.4, we will highlight some corollaries.

**Corollary 4.4.5.** *Let $\{\rho_i\}_{i=1}^{\infty}$ be an internal sequence in $Hit_n(S)$. Then*

$$\lim_{i\to\infty} h_M(\rho_i) = 0,$$

*where $h_M$ is the critical exponent (see Definition 4.1.4).*

*Proof.* Choose a Cartan decomposition of the lie algebra $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is a compact Lie algebra. Then let $\mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ in $\mathfrak{p}$ and $\mathfrak{a}^+$ a choice of positive Weyl chamber in $\mathfrak{a}$. For example, we can take $\mathfrak{k}$ to be the Lie algebra of $SO(n) \subset SL(n,\mathbb{R})$, $\mathfrak{p}$ to be the Lie algebra of the group of upper triangular matrices in $SL(n,\mathbb{R})$, $\mathfrak{a}$ to be the set of traceless $n \times n$ diagonal matrices and $\mathfrak{a}^+$ to be the subset of $\mathfrak{a}$ where the diagonal entries are strictly decreasing down the diagonal.

Let $a : PSL(n,\mathbb{R}) \to \overline{\mathfrak{a}^+}$ be the Cartan projection, i.e. the map satisfying the property that for all $g$ in $PSL(n,\mathbb{R})$, $g = k \cdot \exp\big(a(g)\big) \cdot l$, where $k, l$ are elements in the maximal compact subgroup $K$ of $PSL(n,\mathbb{R})$ with Lie algebra $\mathfrak{k}$. One can then verify that if $o$ is the point in $M$ whose stabilizer is $K$, then for any $g$ in $PSL(n,\mathbb{R})$,

$$d_M(o, g\cdot o) = ||a(g)|| := c_n \sqrt{\sum_{i=1}^{n} \lambda_i\big(a(g)\big)^2}$$

where $\lambda_1\big(a(g)\big) \leq \lambda_2\big(a(g)\big) \leq \cdots \leq \lambda_n\big(a(g)\big)$ are the eigenvalues of $a(g)$ and $c_n$ is a constant depending only on $n$. Thus, for any $\rho$ in $Hit_n(S)$, we have

$$h_M(\rho) = \limsup_{T\to\infty} \frac{1}{T} \log \Big|\big\{X \in \Gamma : \big|\big|a\big(\rho(X)\big)\big|\big| < T\big\}\Big|.$$

If $X_1, X_2 \in \Gamma$ are conjugate, then $a(X_1) = a(X_2)$. Thus, for any $\rho$ in $Hit_n(S)$,

$$\mathrm{h}_{\mathrm{top}}(\rho) = \lim_{T\to\infty} \frac{1}{T} \log \left|\left\{X \in \Gamma : \lambda_n\Big(a\big(\rho(X)\big)\Big) - \lambda_1\Big(a\big(\rho(X)\big)\Big) < T\right\}\right|.$$

Using the fact that $\sum_{i=1}^{n} \lambda_i\big(a(g)\big) = 0$ for all $g$ in $PSL(n, \mathbb{R})$, we can also deduce

$$\begin{aligned} ||a(g)|| &\geq \frac{c_n}{n}\Big(n\lambda_n\big(a(g)\big)\Big) \\ &\geq \frac{c_n}{n}\Big(\lambda_n\big(a(g)\big) + \sum_{i=2}^{n} \lambda_i\big(a(g)\big)\Big) \\ &= \frac{c_n}{n}\Big(\lambda_n\big(a(g)\big) - \lambda_1\big(a(g)\big)\Big), \end{aligned}$$

which implies that for any $\rho$ in $Hit_n(S)$,

$$\begin{aligned} \mathrm{h}_{\mathrm{top}}(\rho) &= \frac{c_n}{n} \limsup_{T\to\infty} \frac{1}{T} \log \left|\left\{X \in \Gamma : \frac{c_n}{n}\Big(\lambda_n\Big(a\big(\rho(X)\big)\Big) - \lambda_1\Big(a\big(\rho(X)\big)\Big)\Big) < T\right\}\right| \\ &\geq \frac{c_n}{n} \limsup_{T\to\infty} \frac{1}{T} \log \left|\left\{X \in \Gamma : \big|\big|a\big(\rho(X)\big)\big|\big| < T\right\}\right| \\ &= \frac{c_n}{n} h_M(\rho). \end{aligned}$$

The corollary follows immediately from Theorem 4.4.4. □

As a consequence of Corollary 4.4.5, Theorem 4.2.7 and the continuity of of the critical exponent as a function on $Hit_n(S)$ we immediately have the following.

**Corollary 4.4.6.** *The image of the map $h_M : Hit_n(S) \to \mathbb{R}$ has image $(0, 1]$.*

Using Corollary 4.4.5, we can also deduce some properties of the minimal immersions in Labourie's theorem (see Theorem 4.1.3). By the Gauss equation, one sees (see Section 6.1 of Sanders [45]) that if $f$ is a minimal immersion, then for all points $p$ in $\Sigma$,

$$\mathrm{Sec}\left(T_{f(p)}f(\Sigma)\right) \leq 0.$$

This, together with Sander's theorem (see Theorem 4.1.5) and Corollary 4.4.5, allows us to conclude the following.

**Corollary 4.4.7.** *Let $\{\rho_i\}_{i=1}^{\infty}$ be an internal sequence in $Hit_n(S)$, and let $\Sigma_i$ be a conformal structure on $S$ for which the harmonic immersion $f_i : \Sigma_i \to \rho_i(\Gamma)\backslash M$ is minimal. Then*

$$\lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f_i^* m_i)} \int_{\Sigma_i} \sqrt{-\mathrm{Sec}_i\left(T_{f_i(p)}f_i(\Sigma_i)\right)}\, \mathrm{d}V_i(p) = 0$$

*and*

$$\lim_{i\to\infty} \frac{1}{\mathrm{Vol}(f_i^* m_i)} \int_{\Sigma_i} ||B_{f_i}(p)||\, \mathrm{d}V_i(p) = 0,$$

*where $B_{f_i}$ is the second fundamental form of $f_i$, $m_i$ is the Riemannian metric on $\rho_i(\Gamma)\backslash M$, $\mathrm{Sec}_i$ is the sectional curvature in $\rho_i(\Gamma)\backslash M$, and the integral is taken using the volume measure of $f_i^* m_i$.*

More informally, this corollary tells us that when we deform along an internal sequence, the minimal immersions that are equivariant with respect to the Hitchin representations in the internal sequence are on average becoming flatter and more totally geodesic.

The next corollary gives a positive answer to a question posted by Crampon and Marquis (Question 13 of [48]). They asked if there is, for any number $\alpha \in [0, \frac{1}{2}]$, a diverging sequence $\{\rho_i\}_{i=1}^{\infty}$ in $Hit_3(S)$ such that $\lim_{i\to\infty} h_{top}(\rho_i) = \alpha$. In fact, we give an answer to the analogous question for all Hitchin components.

**Corollary 4.4.8.** *Let $n$ be a positive integer that is at least $3$. For any number $\alpha \in \left[0, \frac{1}{n}\right]$, there is a diverging sequence $\{\rho_i\}_{i=1}^{\infty}$ in $Hit_n(S)$ such that $\lim_{i\to\infty} h_{top}(\rho_i) = \alpha$.*

*Proof.* The case when $\alpha = 0$ is immediate from our Theorem 4.4.4, so we will assume that $\alpha > 0$. Consider any diverging sequence $\{\rho_i'\}_{i=1}^{\infty}$ in the Fuchsian locus of $Hit_n(S)$ corresponding to pinching all the curves in the pants decomposition of $S$, i.e. all the boundary invariants are converging to $0$ along $\{\rho_i'\}_{i=1}^{\infty}$.

Now, for each $i$, Theorem 4.4.4 implies that there is an internal sequence $\{\rho_{i,j}\}_{j=1}^{\infty}$ in $Hit_n(S)$ so that $\rho_{i,1} = \rho_i'$, $\lim_{j\to\infty} \mathrm{h}_{\mathrm{top}}(\rho_{i,j}) = 0$, and all the boundary invariants for $\rho_{i,j}$ are constant over all $j$. Since $\mathrm{h}_{\mathrm{top}}(\rho_i') = 1$, the continuity of $\mathrm{h}_{\mathrm{top}}$ implies that there is some $\rho_i$ with the same boundary invariants as $\rho_i'$ so that $\mathrm{h}_{\mathrm{top}}(\rho_i) = \alpha$. Furthermore, note that $\{\rho_i\}_{i=1}^{\infty}$ is diverging because all the boundary invariants are converging to $0$ along $\{\rho_i\}_{i=1}^{\infty}$. □

By Corollary 4.4.5, we also have a similar statement, with the topological entropy replaced with the critical exponent.

We will end this chapter with one final corollary, which demonstrates a striking structural difference between $Hit_2(S)$ and $Hit_n(S)$ for $n \geq 3$ due to the existence of these internal sequences.

**Corollary 4.4.9.** *Let $n$ be a positive integer that is at least $3$. Then there is a sequence $\{\rho_i\}_{i=1}^{\infty}$ so that $\lim_{i\to\infty} \mathrm{h}_{\mathrm{top}}(\rho_i) = 0$ and $\lim_{i\to\infty} l_{\rho_i}(X) = \infty$ for any $X \in \Gamma \setminus \{\mathrm{id}\}$.*

*Proof.* Choose a sequence $\{\rho_i'\}_{i=1}^{\infty}$ in the Fuchsian locus of $Hit_n(S)$ so that $l_{\rho_i'}(A) > i$ for all $A \in \Gamma_{\mathcal{P}}$. For each $i$, let $\{\rho_{i,j}\}_{j=1}^{\infty}$ be an internal sequence so that $\rho_{i,1} = \rho_i'$. By Theorem 4.4.4, there is some $j(i) \in \mathbb{Z}^+$ such that $\mathrm{h}_{\mathrm{top}}(\rho_{i,j(i)}) < \frac{1}{i}$ and $\Theta(\rho_{i,j(i)}) > i$. Let $\rho_i := \rho_{i,j(i)}$, and observe that the sequence $\{\rho_i\}_{i=1}^{\infty}$ has the required properties. □

# CHAPTER 5

# Lower bound for lengths of closed curves

In this section, we will explain the first crucial step of the proof of the main theorem. Fix $\rho$ in $Hit_n(S)$, a non-identity element $X$ in $\Gamma$, and let $\eta$ be the closed curve in $S$ corresponding to $X$. The goal of this section is to prove Theorem 5.3.7, which states that we have the length lower bound

$$l_\rho(X) \geq r\big(\psi(X)\big) \cdot \frac{K(\rho)}{11} + s\big(\psi(X)\big) \cdot \frac{L(\rho)}{11}, \tag{5.0.1}$$

were $K$ and $L$ are a pair of positive functions on $Hit_n(S)$ and $\psi(X)$ is some combinatorial data associated to $\rho(X)$ that we will define later. Informally, one should think of $s(\psi(X))$ as the number of times $\eta$ "winds around" collar neighborhoods of the simple closed curves in $\mathcal{P}$, while $r(\psi(X))$ is the number of times $\eta$ "crosses between" these collar neighborhoods. This inequality is then roughly saying that whenever $\eta$ winds around one of these collar neighborhoods, it picks up at least $\frac{L(\rho)}{11}$ amount of length, and whenever it crosses between these collar neighborhoods, it picks up at least $\frac{K(\rho)}{11}$ amount of length.

We will denote the Frenet curve corresponding to $\rho$ by $\xi$, and the attracting and repelling fixed points of $X$ by $x^+$ and $x^-$ respectively. Choose once and for all an orientation on $\partial\Gamma$, and let $s_0$ and $s_1$ be the two closed subsegments of $\partial\Gamma$ with endpoints $x^-$ and $x^+$ that are oriented from $x^-$ to $x^+$ so that the orientation on $s_0$ agrees with the orientation on $\partial\Gamma$.

## 5.1 Finite combinatorial description of closed curves

Now, we give a complete description of $\rho(X)$ by finitely many pieces of combinatorial data. As mentioned in the introduction, if we choose an oriented hyperbolic structure on $S$, the combinatorial data associated to $\rho(X)$ needs to capture how the directed geodesic in $S$ associated to $X$ "winds around" collar neighborhoods of the curves in $\mathcal{P}$ and how it "crosses between" these collar neighborhoods. Since we want to use this combinatorial description in the lower bound (5.0.1), it must also behave well with respect to the length, i.e. if $\rho(X)$ and $\rho(X')$ correspond to curves $\gamma$ and $\gamma'$ so that $\gamma$ "winds around" a collar neighborhood many more times than an otherwise similar curve $\gamma'$, then $l_\rho(X)$ must be much longer then $l_\rho(X')$.

The ideal triangulation is good enough to describe the "crossing", but not the "winding". In fact, we cannot have a purely topological description of the "winding" that behaves well with respect to the length. This can be seen even in $\mathcal{T}(S)$. Suppose that $\rho$ and $\rho'$ are two Fuchsian representations so that $\rho$ is obtained from $\rho'$ by performing many Dehn twists about a simple closed curve $\eta$ in $\mathcal{P}$. Then one can find a pair of curves $\gamma$ and $\gamma$' in $S$ transverse to $\eta$ so that the geodesic representatives of $\gamma$ is much longer than that of $\gamma'$ in the hyperbolic structure corresponding to $\rho$, but the reverse holds in the hyperbolic structure corresponding to $\gamma'$. Hence, for an appropriate description of the "winding", we need to define some additional structure on $\partial\Gamma$ which depends on $\rho$.

For any oriented closed curve in $\mathcal{P}$, and choose any $A$ in $\Gamma_{\mathcal{P}}$ corresponding to this closed curve. (Recall that $\Gamma_{\mathcal{P}}$ is the subset of $\Gamma$ corresponding to the closed curves in $\mathcal{P}$.) Let $a^-$ and $a^+$ be the repelling and attracting fixed points of $A$ in $\partial\Gamma$ respectively, and let $r_0$ and $r_1$ be the two closed subsegments of $\partial\Gamma$ with endpoints $a^-$ and $a^+$, oriented from $a^-$ to $a^+$, and so that the orientation on $r_0$ agrees with the orientation on $\partial\Gamma$.

Observe that there is some $x$ in $r_0\backslash\{a^+\}$ such that $\{x, a^-\}$ lies in $\widetilde{\mathcal{T}}$. Choose any such $x$, and choose a normalization so that for all $k = 1, \dots, n$, we have $\xi(a^-)^{(k)} \cap \xi(a^+)^{(n-k+1)} = [e_k]$ and $\xi(x)^{(1)} = [e_1 + \dots + e_n]$. Here, $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. This implies that $\rho(A)$ is a diagonal matrix, and that if we denote the eigenvalue of $\rho(A)$ corresponding to $e_i$ by $\lambda_i$, then $0 < \lambda_1 < \dots < \lambda_n$.

Since

$$\mathcal{S} := \{\text{hyperplanes in } \mathbb{R}^n \text{ containing } \xi(a^-)^{(n-1)} \cap \xi(a^+)^{(n-1)}\}$$

is topologically a circle, there are two closed subsegments of $\mathcal{S}$ with endpoints $\xi(a^-)^{(n-1)}$ and $\xi(a^+)^{(n-1)}$. Pick any $z$ in the interior of $r_1$ and let $t_1$ be the unique subsegment of $\mathcal{S}$ with endpoints $\xi(a^-)^{(n-1)}$ and $\xi(a^+)^{(n-1)}$ that contains $\big(\xi(a^-)^{(n-1)} \cap \xi(a^+)^{(n-1)}\big) + \xi(z)^{(1)}$. By Corollary 2.4.6, the map

$$\begin{aligned} f : r_1 &\to t_1 \\ z &\mapsto \big(\xi(a^-)^{(n-1)} \cap \xi(a^+)^{(n-1)}\big) + \xi(z)^{(1)} \end{aligned}$$

is a homeomorphism.

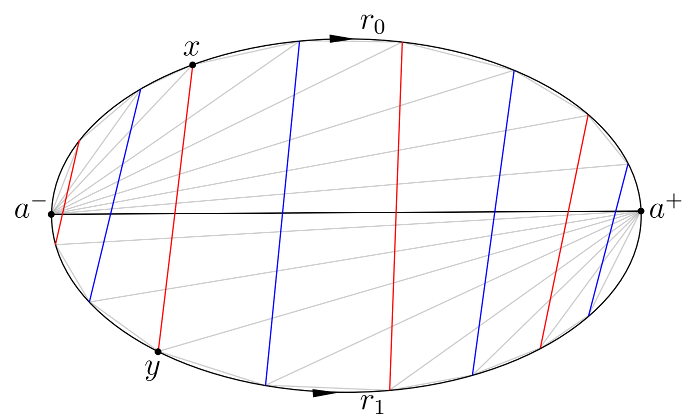


Figure 5.1: Two possible meshes for $A$, in blue and red, depending on the choice of $x$.

Using our normalization, we can write

$$\xi(z)^{(1)} = [\delta_1(z) : \cdots : \delta_n(z)]$$

for all $z$ in $r_1$. Since $f$ is a homeomorphism, the map

$$\begin{aligned} g : r_1 &\to \mathbb{R}_{\geq 0} \cup \{\infty\} \\ z &\mapsto \left|\frac{\delta_n(z)}{\delta_1(z)}\right| \end{aligned}$$

is also a homeomorphism. Moreover, $g(A \cdot z) = \dfrac{\lambda_n}{\lambda_1} g(z)$ for all $z$ in $r_1$. Hence, if we let $y$ in $r_1 \setminus \{a^-\}$ be the point so that $g(y)$ is minimized subject to the conditions that $\{y, a^+\}$ lies in $\widetilde{\mathcal{T}}$ and $g(y) \geq 1$, then we can conclude that

$$1 \leq g(y) < \frac{\lambda_n}{\lambda_1}. \tag{5.1.1}$$

Using $x$ and $y$ as described, we can define some additional structure on $\partial\Gamma$.

**Definition 5.1.1.** Let $A$ be an element in $\Gamma_{\mathcal{P}}$, and choose any $x$ in $r_0$ so that $\{x, a^-\}$ is an edge in $\widetilde{\mathcal{T}}$. Let $y$ be the point in $r_1$ described as above. A *mesh* (see Figure 5.1) of $A$ is the set of pairs $\big\{\{A^k \cdot x, A^k \cdot y\} : k \in \mathbb{Z}\big\}$.

One can check that if we use any $x'$ in $\langle A\rangle \cdot x$ in place of $x$ and perform this construction, then the mesh we obtain will be the same. This implies that there are only two possible meshes of $A$, because the set

$$\big\{x \in \partial\Gamma : \{x, a^-\} \in \widetilde{\mathcal{T}}\big\}$$

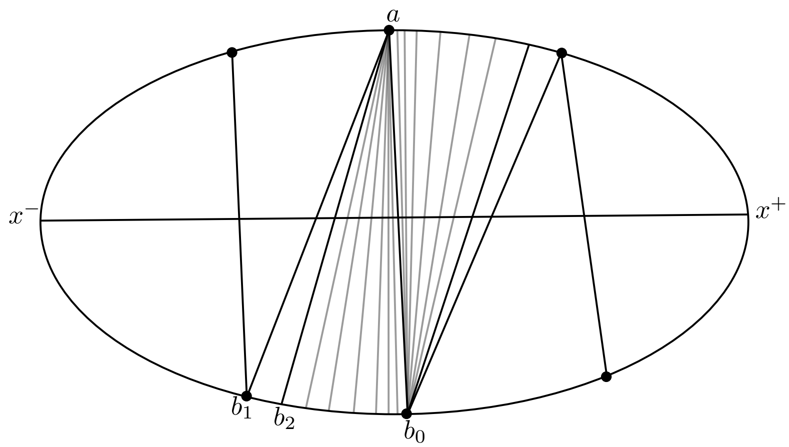


Figure 5.2: $\mathcal{N}_a$ contains the vertices of the grey lines.

is the union of two $\langle A\rangle$-orbits. Once and for all, choose one of these meshes, denoted $\mathcal{E}_A$, for each $A$ in $\Gamma_{\mathcal{P}}$, so that if $A' = YAY^{-1}$ for some $Y$ in $\Gamma$, then

$$\mathcal{E}_{A'} = Y \cdot \mathcal{E}_A.$$

There is a natural ordering on $\mathcal{E}_A$ induced by the action of $A$.

Next, we will use the pair $(x^-, x^+)$ and the subsegments $s_0$, $s_1$ to define several subsets of $\partial\Gamma$ and $\partial\Gamma^{[2]}$ that we use to give the combinatorial description of $X$. Some of these are summarized in Notation 5.1.2 below.

*Notation* 5.1.2.

- Let $\widetilde{\mathcal{I}}'_X = \widetilde{\mathcal{I}}'$ be the set of edges in $\widetilde{\mathcal{T}}$ that intersect $\{x^-, x^+\}$ and let $\widetilde{\mathcal{I}}_X = \widetilde{\mathcal{I}}$ be the subset of $\widetilde{\mathcal{I}}'$ that are not closed leaves. Observe that both $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}}'$ are $\langle X\rangle$-invariant, so we can define $\mathcal{I} := \widetilde{\mathcal{I}}/\langle X\rangle$ and $\mathcal{I}' := \widetilde{\mathcal{I}}'/\langle X\rangle$.

- A vertex in $\partial\Gamma$ is a *node* if it is the common vertex of two distinct edges in $\widetilde{\mathcal{T}}$ that intersect $\{x^-, x^+\}$. We call the edge $\{a, b\}$ in $\widetilde{\mathcal{I}}'$ *binodal* if $a$ and $b$ are both nodes. Denote the set of binodal edges in $\widetilde{\mathcal{I}}$ by $\widetilde{\mathcal{B}}_X = \widetilde{\mathcal{B}}$ and let $\mathcal{B}_X = \mathcal{B} := \widetilde{\mathcal{B}}/\langle X\rangle$.

- Let $\mathcal{V}'_i$ be the set of vertices of the edges in $\widetilde{\mathcal{I}}'$ that lie in $s_i$.

Observe that $\mathcal{B}$ is finite, and is empty if and only if $\{x^-, x^+\}$ is a closed leaf in $\widetilde{\mathcal{T}}$. For the rest of this section, we will assume that $\mathcal{B}$ is non-empty. Also, the orientations on $s_0$ and $s_1$ induce orderings $\leq$ on $\mathcal{V}'_0$ and $\mathcal{V}'_1$, which in turn induce an ordering $\preceq$ on $\widetilde{\mathcal{I}}$ defined as follows. Suppose $\{a, b\}$ and $\{a', b'\}$ are edges in $\widetilde{\mathcal{I}}$ so that $a, a'$ lie in $s_0$ and $b, b'$ lie in $s_1$. Then $\{a, b\} \preceq \{a', b'\}$ if and only if $a \leq a'$ and $b \leq b'$. Since the accumulation points of $\widetilde{\mathcal{I}}'$ are exactly the closed leaves, we can define a bijective successor map $\mathrm{suc} : \widetilde{\mathcal{I}} \to \widetilde{\mathcal{I}}$.

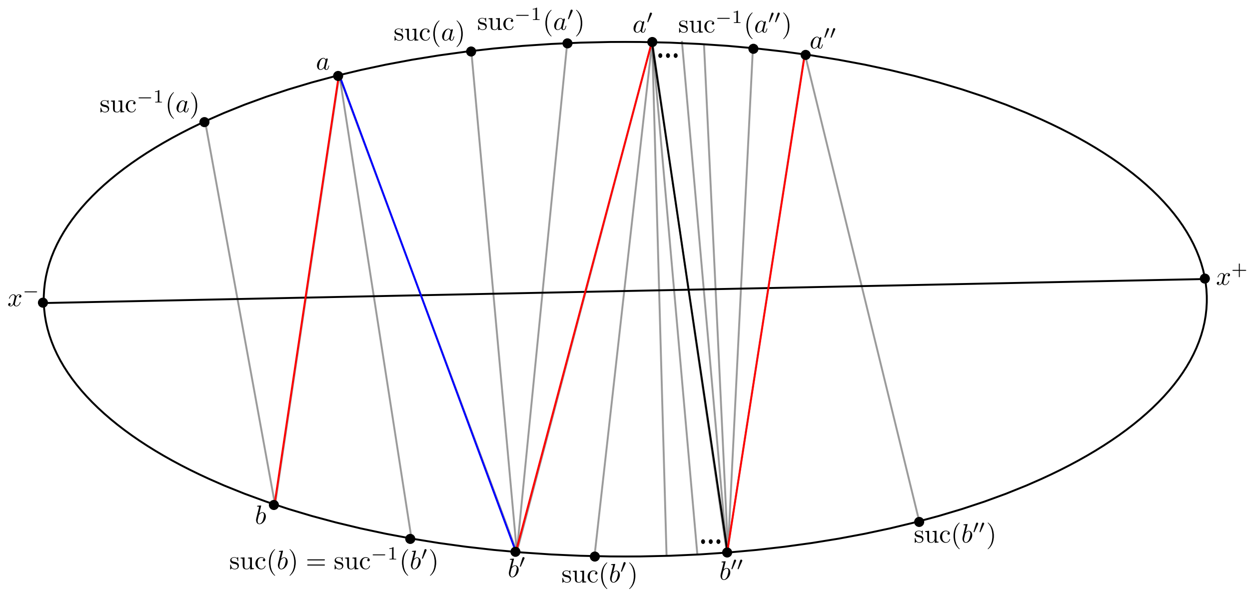


Figure 5.3: $\widetilde{\mathcal{I}}'$ partially drawn. The closed leaf is $\{a', b''\}$, the S-type binodal edges are $\{a, b\}$, $\{a', b'\}$, $\{a'', b''\}$ and the Z-type binodal edge is $\{a, b'\}$.

Moreover, the ordering $\preceq$ induces a cyclic order on $\mathcal{I}$, and the successor map $\mathrm{suc} : \widetilde{\mathcal{I}} \to \widetilde{\mathcal{I}}$ descends to a successor map $\mathrm{suc} : \mathcal{I} \to \mathcal{I}$.

From the way $\widetilde{\mathcal{T}}$ was defined, it is easy to see that every closed leaf in $\widetilde{\mathcal{I}}'$ is binodal. Also, any node is the vertex of exactly two distinct binodal edges, and at most one of these binodal edges is a closed leaf. Thus, for any vertex $a$ of any closed leaf $\{a, b_0\}$ in $\widetilde{\mathcal{I}}'$, there is a unique binodal edge $\{a, b_1\}$ that is not a closed leaf and has $a$ as a vertex. Let $\{a, b_2\}$ be the unique edge in $\widetilde{\mathcal{I}}'$ that is adjacent to $\{a, b_1\}$ and also has $a$ as a vertex. Define

$$\mathcal{N}_a := \left\{ b \in \partial\Gamma : \{a, b\} \in \widetilde{\mathcal{I}}', b \neq b_i \text{ for } i = 0, 1, 2 \right\},$$

and let $\mathcal{W}$ be the set of vertices for the closed leaves in $\widetilde{\mathcal{I}}'$ (see Figure 5.2). Then define

$$\mathcal{V}_i := \mathcal{V}_i' \setminus \bigcup_{a \in \mathcal{W}} \mathcal{N}_a.$$

The main payoff we gain from considering $\mathcal{V}_i$ instead of $\mathcal{V}_i'$ is that $\mathcal{V}_i$ is discrete, which allows us to define bijective successor functions $\mathrm{suc} : \mathcal{V}_i \to \mathcal{V}_i$ for both $i = 0, 1$.

**Definition 5.1.3.** Let $\{a, b\}$ be an edge in $\widetilde{\mathcal{B}}$ and assume without loss of generality that $a$ lies in $s_0$ and $b$ lies in $s_1$. We say $\{a, b\}$ is

- *Z-type* if $\mathrm{suc}\{a, b\} = \{\mathrm{suc}(a), b\}$ and $\mathrm{suc}^{-1}\{a, b\} = \{a, \mathrm{suc}^{-1}(b)\}$,

- *S-type* if $\operatorname{suc}\{a,b\} = \{a, \operatorname{suc}(b)\}$ and $\operatorname{suc}^{-1}\{a,b\} = \{\operatorname{suc}^{-1}(a), b\}$.

(See Figure 5.3.) Let $\widetilde{\mathcal{Z}}$ be the edges in $\widetilde{\mathcal{B}}$ that are Z-type and $\widetilde{\mathcal{S}}$ be the edges in $\widetilde{\mathcal{B}}$ that are S-type. Since $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{S}}$ are $\langle X\rangle$-invariant, we can define $\mathcal{Z} := \widetilde{\mathcal{Z}}/\langle X\rangle$ and $\mathcal{S} := \widetilde{\mathcal{S}}/\langle X\rangle$.

Note that $\mathcal{Z} \cup \mathcal{S} = \mathcal{B}$, and the cyclic order on $\mathcal{I}$ induces cyclic orders on $\mathcal{Z}$, $\mathcal{S}$ and $\mathcal{B}$. Let $e$ and $e'$ be consecutive edges in $\mathcal{B}$ with $e$ preceding $e'$, and observe the following remark (see Figure 5.3).

*Remark* 5.1.4.

1. If $e$ and $e'$ are not of the same type, then in $\widetilde{\mathcal{B}}$, there are representatives $\widetilde{e}$, $\widetilde{e}'$ of $e$, $e'$ respectively so that $\widetilde{e} \prec \widetilde{e}'$ and $\widetilde{e}$, $\widetilde{e}'$ share a common vertex.
2. If $e$ and $e'$ are of the same type, then in $\widetilde{\mathcal{B}}$, there are representatives $\widetilde{e}$, $\widetilde{e}'$ of $e$, $e'$ respectively so that $\widetilde{e} \prec \widetilde{e}'$ and there is exactly one closed leaf between them (in $\widetilde{\mathcal{I}}'$).

If $e$ and $e'$ are not of the same type, choose a pair $\widetilde{e}, \widetilde{e}'$ as described in (1) and let $A(\widetilde{e}, \widetilde{e}')$ be the element in $\Gamma_{\mathcal{P}}$ that has the common vertex of $\widetilde{e}$ and $\widetilde{e}'$ as a fixed point. If $e$, $e'$ are of the same type, choose a pair $\widetilde{e}$, $\widetilde{e}'$ as described in (2) and let $A(\widetilde{e}, \widetilde{e}')$ be the element in $\Gamma_{\mathcal{P}}$ whose attracting and repelling fixed points are the initial and terminal vertices of the oriented closed leaf between $\widetilde{e}$ and $\widetilde{e}'$. In either case, consider $\mathcal{E}_{A(\widetilde{e},\widetilde{e}')}$.

*Notation* 5.1.5. Let $t(e,e')$ be the signed number of edges in $\mathcal{E}_{A(\widetilde{e},\widetilde{e}')}$ that intersect $\{x^+, x^-\}$, where the sign is positive if the ordering on these edges induced by the ordering on $\mathcal{E}_{A(e,e')}$ is the same as the ordering induced by the orientation on $s_0$ and $s_1$, and negative otherwise.

Observe that $t(e,e')$ does not depend on the choice of $\widetilde{e}$ and $\widetilde{e}'$. Cyclically enumerate $\mathcal{B} = \{e_{m+1} = e_1, e_2 \dots, e_m\}$, and for each $i = 1, \dots, m$, let $T_i$ be the type (Z or S) of $e_i$. Then define the cyclic sequence of tuples

$$\psi_\rho(X) = \psi(X) := \Big\{ \big(\operatorname{suc}^{-1}(e_i), e_i, \operatorname{suc}(e_i), T_i, t(e_i, e_{i+1})\big) \Big\}_{i=1}^m.$$

This is the combinatorial data we associate to each $X$ in $\Gamma$. If we choose a hyperbolic metric on $S$ and let $\gamma$ be the closed geodesic in $S$ associated to $\rho(X)$, then the cyclic sequence $\Big\{ \big(\operatorname{suc}^{-1}(e_i), e_i, \operatorname{suc}(e_i), T_i\big) \Big\}_{i=1}^m$ tells us how $\gamma$ "crosses between" the collar neighborhoods of curves in $\mathcal{P}$ and the cyclic sequence $\big\{t(e_i, e_{i+1})\big\}_{i=1}^m$ tells us how $\gamma$ "winds around" these collar neighborhoods.

**Proposition 5.1.6.** *Let $X_0, X_1$ be elements in $\Gamma$. Then $\psi(X_0) = \psi(X_1)$ if and only if $X_0$ and $X_1$ are conjugate.*

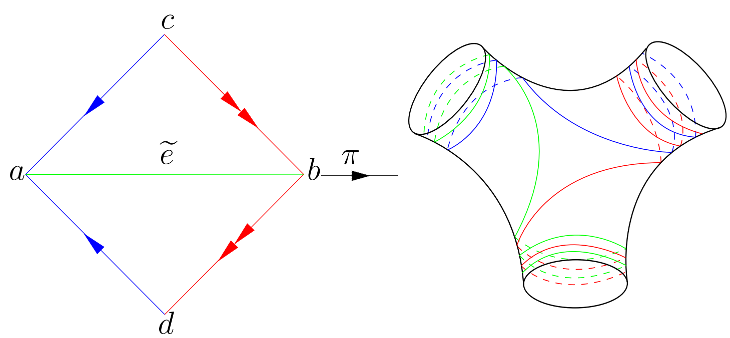


Figure 5.4: $Q_{\widetilde{e}}$ is mapped via $\pi$ to a pair of pants.

*Proof.* It is clear that if $X_0$ and $X_1$ are conjugate, then $\psi(X_0) = \psi(X_1)$. We will now show the converse. Choose a hyperbolic metric on $S$. Then the ideal triangulation $\widetilde{\mathcal{T}}$ can be viewed as a $\Gamma$-invariant ideal triangulation of the Poincaré disc $\mathbb{D}$, so $\mathcal{T}$ is an ideal triangulation of the hyperbolic surface $S$. Also, the union of meshes

$$\widetilde{\mathcal{E}} := \bigcup_{A \in \Gamma_{\mathcal{P}}} \mathcal{E}_A$$

can be viewed as a $\Gamma$-invariant collection of geodesics in $\mathbb{D}$, so the quotient

$$\mathcal{E} := \widetilde{\mathcal{E}}/\Gamma$$

is a collection of $3g-3$ geodesics in the hyperbolic surfaces $S$. Observe that $\gamma$ in $\mathcal{E}$ has a lift to $\widetilde{\mathcal{E}}$ that lies in $\mathcal{E}_A$ if and only if $\gamma$ intersects the closed geodesic corresponding to $A$. Moreover, $\gamma$ intersects $\mathcal{P}$ only at this closed geodesic.

Let $\gamma_{X_0}$, $\gamma_{X_1}$ be the oriented closed geodesics in $S$ that correspond to $X_0$, $X_1$ respectively. It is sufficient to show that if

$$\psi(X_0) = \psi(X_1) = \Big\{ \big( \mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i), T_i, t(e_i, e_{i+1}) \big) \Big\}_{i=1}^{m},$$

then $\gamma_{X_0}$ and $\gamma_{X_1}$ are homotopic as oriented curves. We will do this by constructing polygons in $S$ along the paths of $\gamma_{X_0}$ and $\gamma_{X_1}$, and show that we can homotope the subsegments of $\gamma_{X_0}$ and $\gamma_{X_1}$ that lie in these polygons relative to the edges of the polygons.

Let $\pi : \widetilde{S} \to S$ be the covering map. For any non-closed leaf $\widetilde{e} = \{a, b\}$ in $\widetilde{\mathcal{T}}$, let $c$ and $d$ be points in $\partial\mathbb{D}$ so that $\{a, c\}$, $\{b, c\}$, $\{a, d\}$, $\{b, d\}$ are in $\widetilde{\mathcal{T}}$. Then let $Q_{\widetilde{e}}$ be the closed convex quadrilateral in $\mathbb{D}$ with vertices $a$, $b$, $c$, $d$. Observe that $\pi$ restricted to the interior of $Q_{\widetilde{e}}$ is injective. (See Figure 5.4.)

Pick any $A$ in $\Gamma_{\mathcal{P}}$ and let $a^-$, $a^+$ be the repelling and attracting fixed points of $A$ respectively. Also, let $r_0$ and $r_1$ be two oriented subsegments of $\partial\Gamma = \partial\mathbb{D}$ with endpoints $a^-$ and $a^+$, oriented from $a^-$ to $a^+$, and such that the orientation on $r_0$ agrees with the clockwise orientation on $\partial\mathbb{D}$. Let $l := \{b_0, b_1\}$ and $l' := \{b_0', b_1'\}$ be two consecutive geodesics in $\mathcal{E}_A$, with $b_0$ and $b_0'$ in $r_0$ and $l$ preceding $l'$. In $r_0$, there is a unique $c_0$ strictly between $b_0$ and $b_0'$ so that $\{c_0, a^-\}$ lies in $\widetilde{\mathcal{T}}$. Similarly, in $r_1$, there is a unique $c_1$ strictly between $b_1$ and $b_1'$ so that $\{c_1, a^+\}$ lies in $\widetilde{\mathcal{T}}$. Let $H_{l,l'}$ be the closed convex hexagon in $\mathbb{D}$ with vertices $b_0$, $c_0$, $b_0'$, $b_1'$, $c_1$, $b_1$, and observe that $\pi$ restricted to the interior of $H_{l,l'}$ is also injective. (See Figure 5.5.)

For $j = 0, 1$, let $\widetilde{\gamma}_{X_j}$ be the axis of $X_j$ and let $\widetilde{e}_i$ be a lift of $e_i$ that intersects $\widetilde{\gamma}_{X_j}$. Then define the points

$$\widetilde{p}_{i,-,X_j} := \mathrm{suc}^{-1}(\widetilde{e}_i) \cap \widetilde{\gamma}_{X_j},\ \widetilde{p}_{i,+,X_j} := \mathrm{suc}(\widetilde{e}_i) \cap \widetilde{\gamma}_{X_j},$$

and let $p_{i,\pm,X_j} = \pi(\widetilde{p}_{i,\pm,X_j})$. Let $\alpha_{i,X_j}$ be the oriented closed subsegment of $\gamma_{X_j}$ containing $\pi(\widetilde{e}_i \cap \widetilde{\gamma}_{X_j})$ and with endpoints $p_{i,-,X_j}$, $p_{i,+,X_j}$, oriented from $p_{i,-,X_j}$ to $p_{i,+,X_j}$. Also, let $\beta_{i,X_j}$ be the closed subsegment of $\gamma_{X_j}$ containing $\pi(\widetilde{e}_i \cap \widetilde{\gamma}_{X_j})$ and with endpoints $p_{i,-,X_j}$, $p_{i+1,+,X_j}$, oriented from $p_{i,-,X_j}$ to $p_{i+1,+,X_j}$. Observe that $\gamma_{X_j}$ can be written as the cyclic concatenation

$$\alpha_{1,X_j}^{-1} \cdot \beta_{1,X_j} \cdot \alpha_{2,X_j}^{-1} \cdot \beta_{2,X_j} \cdot \dots \cdot \alpha_{m,X_j}^{-1} \cdot \beta_{m,X_j}$$

where $\cdot$ is concatenation and the inverse is reversing the parameterization. Since $\psi(X_0) = \psi(X_1)$, we know that the initial and terminal endpoints of $\alpha_{i,X_0}$ lie on the same edges of $\mathcal{T}$ as those of $\alpha_{i,X_1}$ respectively. For the same reasons, the initial and terminal endpoints of $\beta_{i,X_0}$ lie on the same edges of $\mathcal{T}$ as those of $\beta_{i,X_1}$. It is thus sufficient to show that for all $i = 1, \dots, m$,

1. $\alpha_{i,X_0}$ is homotopic to $\alpha_{i,X_1}$ and
2. $\beta_{i,X_0}$ is homotopic to $\beta_{i,X_1}$

as oriented curves relative to the edges in $\mathcal{T}$ containing their endpoints.

First, we will show that (1) holds. Observe that $\alpha_0 := \alpha_{i,X_0}$ and $\alpha_1 := \alpha_{i,X_1}$ lie in $\pi(Q_{\widetilde{e}_i})$ for some lift $\widetilde{e}_i$ of $e_i$. Also, for each vertex of $\widetilde{e}_i$, the two edges of $Q_{\widetilde{e}_i}$ adjacent to this vertex are mapped via $\pi$ to the same edge in $\mathcal{T}$. (See Figure 5.4.) Since we know $e_i$ is the same type (Z or S) for both $X_0$ and $X_1$, the lifts $\widetilde{\alpha}_0$, $\widetilde{\alpha}_1$ of $\alpha_0$, $\alpha_1$ respectively that lie in $Q_{\widetilde{e}_i}$ have their initial endpoints in a common edge of $Q_{\widetilde{e}_i}$ and their terminal endpoints in a common edge of $Q_{\widetilde{e}_i}$. It is thus clear that (1) holds.

To show that (2) holds, further partition each $\beta_j := \beta_{i,X_j}$ in the following way. Let $\{q_{1,j}, \dots, q_{|t(e_i,e_{i+1})|,j}\}$ be the $|t(e_i, e_{i+1})|$ points of intersection of $\beta_j$ with the mesh $\mathcal{E}_{A(e_i,e_{i+1})}$,

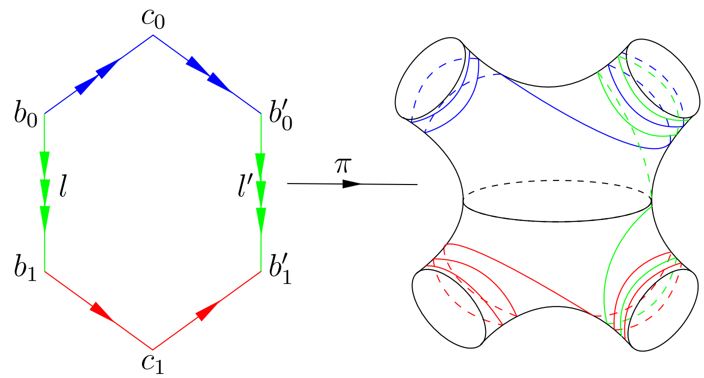


Figure 5.5: $H_{l,l'}$ is mapped via $\pi$ to two pairs of pants.

ordered according to the orientation of $\beta_j$. For $k = 0, \ldots, |t(e_i, e_{i+1})|$, let $\beta_{k,j}$ be the subsegment of $\beta_j$ with endpoints

- $p_{i,-,X_j}$ and $p_{i+1,+,X_j}$ if $t(e_i, e_{i+1}) = 0$, oriented from $p_{i,-,X_j}$ to $p_{i+1,+,X_j}$,
- $p_{i,-,X_j}$ and $q_{1,j}$ if $|t(e_i, e_{i+1})| > 0$ and $k = 0$, oriented from $p_{i,-,X_j}$ to $q_{1,j}$,
- $q_{k,j}$ and $q_{k+1,j}$ if $|t(e_i, e_{i+1})| > 0$ and $0 < k < |t(e_i, e_{i+1})|$, oriented from $q_{k,j}$ to $q_{k+1,j}$,
- $q_{|t(e_i,e_{i+1})|,j}$ and $p_{i+1,+,X_j}$ if $|t(e_i, e_{i+1})| > 0$ and $k = |t(e_i, e_{i+1})|$, oriented from $q_{|t(e_i,e_{i+1})|,j}$ to $p_{i+1,+,X_j}$.

We now need to show that for $k = 0, \ldots, |t(e_i, e_{i+1})|$, the segments $\beta_{k,0}$ and $\beta_{k,1}$ are homotopic relative to the edges in $\mathcal{E}_{A(e_i,e_{i+1})}$ and $\mathcal{T}$ that contain their endpoints. Observe that $\beta_{k,0}$ and $\beta_{k,1}$ lie in $\pi(H_{l,l'})$ for any consecutive pair $l$, $l'$ in $\mathcal{E}_{A(e_i,e_{i+1})}$, with $l$ preceding $l'$.

Consider the lift $\widetilde{\beta}_{0,j}$ of $\beta_{0,j}$ that lies in $H_{l,l'}$. Since the initial endpoint of $\widetilde{\beta}_{0,j}$ is $\widetilde{p}_{i,-,X_j}$, the triple $(\mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i))$ determines the edge of $H_{l,l'}$ that $\widetilde{p}_{i,-,X_j}$ lies in. Observe then that

- if $t(e_i, e_{i+1}) < 0$, the terminal endpoint $q_{1,j}$ of $\widetilde{\beta}_{0,j}$ lies in $l$,
- if $t(e_i, e_{i+1}) > 0$, the terminal endpoint $q_{1,j}$ of $\widetilde{\beta}_{0,j}$ lies in $l'$,
- if $t(e_i, e_{i+1}) = 0$, the triple $(\mathrm{suc}^{-1}(e_{i+1}), e_{i+1}, \mathrm{suc}(e_{i+1}))$ determines the edge of $H_{l,l'}$ containing $\widetilde{p}_{i,-,X_j}$, which is the terminal endpoint of $\widetilde{\beta}_{0,j}$.

In any case, this proves that $\widetilde{\beta}_{0,0}$ and $\widetilde{\beta}_{0,1}$ have initial endpoints on the same edge of $H_{l,l'}$ and terminal endpoints on the same edge in $H_{l,l'}$. Similar arguments show the same for $\widetilde{\beta}_{k,0}$ and $\widetilde{\beta}_{k,1}$ for $k = 1, \ldots, |t(e_i, e_{i+1})|$, so (2) holds. $\square$

## 5.2 Crossing and winding $(p)$-subsegments of $X$

In the rest of this section, we will use the combinatorial description $\psi(X)$ of $\rho(X)$ to obtain a lower bound for $l_\rho(X)$. Let $H$ be the plane $\xi(x^-)^{(1)} + \xi(x^+)^{(1)}$ in $\mathbb{R}^n$. The next two definitions describe two kinds of subsegments of $\mathbb{P}(H)$ that we will use to obtain our lower bound. These are the crossing $(p)$-subsegments and the winding $(p)$-subsegments

**Definition 5.2.1.** Let $\widetilde{e} = \{a, b\}$ be an element in $\widetilde{\mathcal{B}}$. Assume without loss of generality that $a$ lies in $s_0$ and $b$ lies in $s_1$. For $p = 0, \dots, n-1$, define the projective points $L_{p,+}(\widetilde{e})$, $L_p(\widetilde{e})$, $L_{p,-}(\widetilde{e})$ as follows:

- $L_p(\widetilde{e}) := \mathbb{P}(\xi(a)^{(p)} + \xi(b)^{(n-p-1)}) \cap \mathbb{P}(H)$
- If $\widetilde{e}$ is in $\widetilde{\mathcal{Z}}$, let

$$L_{p,+}(\widetilde{e}) := \mathbb{P}\big(\xi(\mathrm{suc}(a))^{(p)} + \xi(b)^{(n-p-1)}\big) \cap \mathbb{P}(H),$$
$$L_{p,-}(\widetilde{e}) := \mathbb{P}\big(\xi(a)^{(p)} + \xi(\mathrm{suc}^{-1}(b))^{(n-p-1)}\big) \cap \mathbb{P}(H).$$

- If $\widetilde{e}$ is in $\widetilde{\mathcal{S}}$, let

$$L_{p,+}(\widetilde{e}) := \mathbb{P}\big(\xi(a)^{(p)} + \xi(\mathrm{suc}(b))^{(n-p-1)}\big) \cap \mathbb{P}(H),$$
$$L_{p,-}(\widetilde{e}) := \mathbb{P}\big(\xi(\mathrm{suc}^{-1}(a))^{(p)} + \xi(b)^{(n-p-1)}\big) \cap \mathbb{P}(H).$$

**Definition 5.2.2.** Let $\widetilde{e} = \{a, b\}$ and $\widetilde{e}' = \{a', b'\}$ be two consecutive elements in $\widetilde{\mathcal{B}}$, with $\widetilde{e}$ preceding $\widetilde{e}'$.

- The *crossing* $(p)$-*subsegment* of $\mathbb{P}(H)$ corresponding to $\widetilde{e}$, denoted $c_p(\widetilde{e})$, is the closed subsegment of $\mathbb{P}(H)$ containing $L_p(\widetilde{e})$ with endpoints $L_{p,-}(\widetilde{e})$ and $L_{p,+}(\widetilde{e})$.
- The *winding* $(p)$-*subsegment* of $\mathbb{P}(H)$ corresponding to $\widetilde{e}$ and $\widetilde{e}'$, denoted $w_p(\widetilde{e}, \widetilde{e}')$, is the closed subsegment of $\mathbb{P}(H)$ containing $L_p(\widetilde{e})$ and $L_p(\widetilde{e}')$, with endpoints $L_{p,-}(\widetilde{e})$ and $L_{p,+}(\widetilde{e}')$.

If we apply (2) of Lemma 2.4.3 to this setting for a fixed $p = 0, \dots, n-1$, we see that there is some subsegment $\gamma$ of $\mathbb{P}(H)$ with endpoints $\xi(x^-)^{(1)}$, $\xi(x^+)^{(1)}$ that contains $L_{p,+}(\widetilde{e})$, $L_{p,-}(\widetilde{e})$ for all $\widetilde{e}$ in $\widetilde{\mathcal{B}}$. In particular, $c_p(\widetilde{e})$ and $w_p(\widetilde{e})$ lie in $\gamma$ for all $\widetilde{e}$ in $\widetilde{\mathcal{B}}$. In fact, if we orient $\gamma$ from $\xi(x^-)^{(1)}$ to $\xi(x^+)^{(1)}$, then we also have orientations induced on $c_p(\widetilde{e})$ and $w_p(\widetilde{e})$. Part (2) of Lemma 2.4.3 then tells us that the orientation on $c_p(\widetilde{e})$ is from $L_{p,-}(\widetilde{e})$ to $L_{p,+}(\widetilde{e})$ and the induced orientation on $w_p(\widetilde{e}, \widetilde{e}')$ is from $L_{p,-}(\widetilde{e})$ to $L_{p,+}(\widetilde{e}')$.

Next, we define a notion of length for subsegments of $\gamma$. This gives us a notion of length for each crossing $(p)$-subsegment and winding $(p)$-subsegment, which we will use to obtain a lower bound for $l_\rho(X)$.

**Definition 5.2.3.** Let $\gamma$ be a subsegment of $\mathbb{P}(H)$ with endpoints $\xi(x^-)^{(1)}$ and $\xi(x^+)^{(1)}$. Let $y$, $z$ be two points in $\gamma$ so that $\xi(x^-)^{(1)}, y, z, \xi(x^+)^{(1)}$ lie on $\gamma$ in that order. Then let $\eta$ be the closed subsegment of $\gamma$ with endpoints $y$, $z$. The length of $\eta$, denoted $l(\eta)$, is given by

$$l(\eta) := \log\big(\xi(x^-)^{(1)}, y, z, \xi(x^+)^{(1)}\big).$$

We will now obtain a lower bound for the length of $X$ in terms of the lengths of the crossing $(p)$-subsegments and winding $(p)$-subsegments of $H$. Choose an edge $\widetilde{z}$ in $\widetilde{\mathcal{Z}}$ and let $\widetilde{z}' = X \cdot \widetilde{z}$. Observe that the set of elements in $\widetilde{\mathcal{Z}}$ between $\widetilde{z}$ and $\widetilde{z}'$ is finite, so we can enumerate them according to the ordering on $\widetilde{\mathcal{Z}}$. In other words, the set of elements between $\widetilde{z}$ and $\widetilde{z}'$ can be written as

$$\{\widetilde{z}_1, \dots, \widetilde{z}_{|\mathcal{Z}|+1}\}$$

where

$$\widetilde{z} = \widetilde{z}_1 \prec \widetilde{z}_2 \prec \dots \prec \widetilde{z}_{|\mathcal{Z}|+1} = \widetilde{z}'.$$

**Lemma 5.2.4.** *Fix any* $p = 0, \dots, n-1$. *Then*

$$l_\rho(X) \geq \sum_{i=1}^{|\mathcal{Z}|} l\big(c_p(\widetilde{z}_i)\big).$$

*Proof.* By Proposition 2.2.4, we know

$$l_\rho(X) = \log\big(\xi(x^-)^{(1)}, L_{p,-}(\widetilde{z}_1), L_{p,-}(\widetilde{z}_{|\mathcal{Z}|+1}), \xi(x^+)^{(1)}\big).$$

Moreover, since $\operatorname{suc}(\widetilde{z}_i) \preceq \operatorname{suc}^{-1}(\widetilde{z}_{i+1})$, we can use (2) of Lemma 2.4.3 to see that

$$\big(\xi(x^-)^{(1)}, L_{p,+}(\widetilde{z}_i), L_{p,-}(\widetilde{z}_{i+1}), \xi(x^+)^{(1)}\big) \geq 1$$

for $i = 1, \dots, |\mathcal{Z}|$. Appying (8) of Proposition 2.2.2, we then have

$$\begin{aligned}
& \big(\xi(x^-)^{(1)}, L_{p,-}(\widetilde{z}_1), L_{p,-}(\widetilde{z}_{|\mathcal{Z}|+1}), \xi(x^+)^{(1)}\big) \\
= & \prod_{i=1}^{|\mathcal{Z}|} \big(\xi(x^-)^{(1)}, L_{p,-}(\widetilde{z}_i), L_{p,+}(\widetilde{z}_i), \xi(x^+)^{(1)}\big) \cdot \big(\xi(x^-)^{(1)}, L_{p,+}(\widetilde{z}_i), L_{p,-}(\widetilde{z}_{i+1}), \xi(x^+)^{(1)}\big) \\
\geq & \prod_{i=1}^{|\mathcal{Z}|} \big(\xi(x^-)^{(1)}, L_{p,-}(\widetilde{z}_i), L_{p,+}(\widetilde{z}_i), \xi(x^+)^{(1)}\big).
\end{aligned}$$

Taking the logarithm gives us the lemma. $\square$

Similarly, if we choose an edge $\widetilde{s}$ in $\widetilde{\mathcal{S}}$ and let $\widetilde{s}' = X \cdot \widetilde{s}$, then we can label the edges in $\widetilde{\mathcal{S}}$ between $\widetilde{s}$ and $\widetilde{s}'$ by

$$\{\widetilde{s}_1, \dots, \widetilde{s}_{|\mathcal{S}|+1}\}$$

where

$$\widetilde{s} = \widetilde{s}_1 \prec \widetilde{s}_2 \prec \cdots \prec \widetilde{s}_{|\mathcal{S}|+1} = \widetilde{s}'.$$

The same proof as in Lemma 5.2.4 will give the following lemma.

**Lemma 5.2.5.** *Fix any* $p = 0, \dots, n-1$. *Then*

$$l_\rho(X) \geq \sum_{i=1}^{|\mathcal{S}|} l\big(c_p(\widetilde{s}_i)\big).$$

We now want a similar lower bound for $l_\rho(X)$ in terms of the lengths of the winding $(p)$-subsegments. As before, let $\widetilde{b}$ be any edge in $\widetilde{\mathcal{B}}$, let $\widetilde{b}' = X \cdot \widetilde{b}$ and label the edges in $\widetilde{\mathcal{B}}$ between $\widetilde{b}$ and $\widetilde{b}'$ by

$$\{\widetilde{b}_1, \dots, \widetilde{b}_{|\mathcal{B}|+1}\}$$

where

$$\widetilde{b} = \widetilde{b}_1 \prec \widetilde{b}_2 \prec \cdots \prec \widetilde{b}_{|\mathcal{B}|+1} = \widetilde{b}'.$$

Also, define

$$\begin{aligned}
\widetilde{\mathcal{D}_1} &:= \big\{(\widetilde{b}_i, \widetilde{b}_{i+1}) : \widetilde{b}_i, \widetilde{b}_{i+1} \text{ that are of the same type}\big\}, \\
\widetilde{\mathcal{D}_2} &:= \big\{(\widetilde{b}_i, \widetilde{b}_{i+1}) : \widetilde{b}_i, \widetilde{b}_{i+1} \text{ that are not of the same type}\big\}.
\end{aligned}$$

**Lemma 5.2.6.** *Fix* $p = 0, \ldots, n-1$. *Then*

$$3 \cdot l_\rho(X) \geq \sum_{i=1}^{|\mathcal{B}|} l\big(w_p(\widetilde{b}_i, \widetilde{b}_{i+1})\big).$$

*Proof.* Label the set

$$\big\{L_{p,-}(\widetilde{b}_i), L_p(\widetilde{b}_i), L_{p,+}(\widetilde{b}_i), L_{p,-}(\widetilde{b}_{i+1}), L_p(\widetilde{b}_{i+1}), L_{p,+}(\widetilde{b}_{i+1})\big\}$$

by $\{d_{i,1}, \ldots, d_{i,k_i}\}$ with $d_{i,1}, \ldots, d_{i,k_i}$ arranged in this order along $w_p(\widetilde{b}_i, \widetilde{b}_{i+1})$. For each $j = 1, \ldots, k_i - 1$, let $s_{i,j}$ be the closed subsegment of $w_p(\widetilde{b}_i, \widetilde{b}_{i+1})$ with endpoints $d_{i,j}$ and $d_{i,j+1}$, and let

$$\mathcal{S} := \{s_{i,j} : i = 1, \ldots, |\mathcal{B}|; j = 1, \ldots, k_i - 1\}.$$

Note that either $s_{i,j} = s_{i',j'}$ or $s_{i,j} \cap s_{i',j'}$ is at most a single point. Also, it is clear that

$$\bigcup_{j=1}^{k_i-1} s_{i,j} = w_p(\widetilde{b}_i, \widetilde{b}_{i+1}).$$

Observe also that any point in the subsegment between $L_{p,-}(\widetilde{b}_1)$ and $L_{p,+}(\widetilde{b}_{|\mathcal{B}|+1})$ is contained in the interior of at most three different winding $(p)$-segments. Hence,

$$\begin{aligned}
\sum_{i=1}^{|\mathcal{B}|} l(w_p(\widetilde{b}_i, \widetilde{b}_{i+1})) &= \sum_{i=1}^{|\mathcal{B}|} \sum_{j=1}^{k_i-1} l(s_{i,j}) \\
&\leq 3 \sum_{s \in \mathcal{S}} l(s) \\
&= 3 \cdot l_\rho(X).
\end{aligned}$$

□

Now, we can give a lower bound for $l_\rho(X)$ in terms of the lengths of the crossing $(p)$-subsegments and the winding $(p)$-subsegments.

**Proposition 5.2.7.**

$$\begin{aligned}
l_\rho(X) \quad \geq \quad & \frac{1}{11n} \sum_{p=0}^{n-1} \Bigg( \sum_{i=1}^{|\mathcal{Z}|} l\big(c_p(\widetilde{z}_i)\big) + \sum_{i=1}^{|\mathcal{S}|} l\big(c_p(\widetilde{s}_i)\big) + \sum_{\widetilde{\mathcal{D}}_1} l\big(w_p(\widetilde{b}_i, \widetilde{b}_{i+1})\big) \\
& + \sum_{\widetilde{\mathcal{D}}_2} \Big( l\big(w_1(\widetilde{b}_i, \widetilde{b}_{i+1})\big) + l\big(w_{n-2}(\widetilde{b}_i, \widetilde{b}_{i+1})\big)\Big)\Bigg).
\end{aligned}$$

*Proof.* Lemma 5.2.6 implies that

$$6 \cdot l_\rho(X) \geq \sum_{\widetilde{\mathcal{D}}_2} \Big( l\big(w_1(\widetilde{b}_i, \widetilde{b}_{i+1})\big) + l\big(w_{n-2}(\widetilde{b}_i, \widetilde{b}_{i+1})\big) \Big)$$

and for all $p = 0, \ldots, n-1$,

$$3 \cdot l_\rho(X) \geq \sum_{\widetilde{\mathcal{D}}_1} l\big(w_p(\widetilde{b}_i, \widetilde{b}_{i+1})\big).$$

Sum these two inequalities with the inequalities in Lemma 5.2.4 and Lemma 5.2.5, and then take average over $p$. $\square$

## 5.3 Lower bound for the length of a closed curve

Let $\widetilde{e}, \widetilde{e}'$ in $\widetilde{\mathcal{B}}$ be any consecutive pair with $\widetilde{e}$ preceding $\widetilde{e}'$, and let $e$, $e'$ be the equivalence classes in $\mathcal{B}$ containing $\widetilde{e}$ and $\widetilde{e}'$ respectively. We now want to define numbers $K = K(\rho)$ and $L = L(\rho)$ which depend only on $\rho$, so that

$$\frac{1}{n} \sum_{p=0}^{n-1} l\big(c_p(\widetilde{e})\big) \geq K,$$

$$\frac{1}{n} \sum_{p=0}^{n-1} l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \max\big\{0, |t(e,e')| - 2\big\} \cdot L$$

when $(\widetilde{e}, \widetilde{e}')$ is in $\widetilde{\mathcal{D}}_1$, and

$$l\big(w_1(\widetilde{e}, \widetilde{e}')\big) + l\big(w_{n-2}(\widetilde{e}, \widetilde{e}')\big) \geq \max\big\{0, |t(e,e')| - 2\big\} \cdot L$$

when $(\widetilde{e}, \widetilde{e}')$ is in $\widetilde{\mathcal{D}}_2$. These estimates, together with Proposition 5.2.7, will allow us to obtain a lower bound for $l_\rho(X)$ in Theorem 5.3.7. Let us start with $K$.

*Notation* 5.3.1. Let $\{a, b\}$ be any edge in $\widetilde{\mathcal{T}}$ that is not a closed leaf and let $c$ be a point in $\partial\Gamma$ such that $\{a, c\}$ and $\{b, c\}$ are edges of $\widetilde{\mathcal{T}}$. For $p = 1, \ldots, n-1$, define

$$\mathcal{M}_p^\xi(a,b,c) = \mathcal{M}_p(a,b,c) := \big\{\xi(a)^{(p-r)} + \xi(b)^{(n-p-1)} + \xi(c)^{(r-1)} : r = 1, \ldots, p\big\}$$

**Lemma 5.3.2.** *Let $\widetilde{e} = \{a, b\}$ be an edge in $\widetilde{\mathcal{Z}}$, and suppose $a$ lies in $s_0$ and $b$ lies in $s_1$. Then the following hold.*

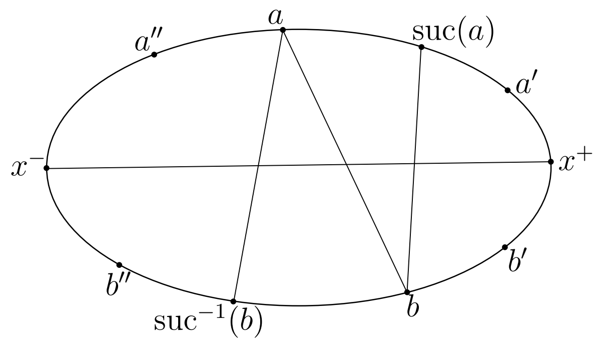

Figure 5.6: Z-type binodal edge

1. *For all* $p=1,\ldots,n-1$ *and for all* $M$ *in* $\mathcal{M}_p\big(a,b,\mathrm{suc}(a)\big)$*, we have*

$$l\big(c_p(\widetilde{e})\big)\geq\log\Big(\xi\big(\mathrm{suc}^{-1}(b)\big),\xi(a),\xi\big(\mathrm{suc}(a)\big),\xi(b)\Big)_M.$$

2. *For all* $p=0,\ldots,n-2$ *and for all* $M$ *in* $\mathcal{M}_{n-p-1}\big(b,a,\mathrm{suc}^{-1}(b)\big)$*, we have*

$$l\big(c_p(\widetilde{e})\big)\geq\log\Big(\xi\big(\mathrm{suc}^{-1}(b)\big),\xi(a),\xi\big(\mathrm{suc}(a)\big),\xi(b)\Big)_M.$$

*Proof.* Proof of (1). Let $s_0'$, $s_0''$ be the closed subintervals of $s_0$ with endpoints $\mathrm{suc}(a)$ and $x^+$, $x^-$ and $a$ respectively. By (2) of Lemma 2.4.3, we know that for any $p=1,\ldots,n-1$ and any $r=1,\ldots,p$, there exists $a'$ in $s_0'$ and $a''$ in $s_0''$ (see Figure 5.6) so that

$$\begin{aligned}L_{p,+}(\widetilde{e}) &= \mathbb{P}\Big(\xi(b)^{(n-p-1)}+\xi\big(\mathrm{suc}(a)\big)^{(p)}\Big)\cap\mathbb{P}(H)\\ &= \mathbb{P}\Big(\xi(a)^{(p-r)}+\xi(b)^{(n-p-1)}+\xi\big(\mathrm{suc}(a)\big)^{(r-1)}+\xi(a')^{(1)}\Big)\cap\mathbb{P}(H),\\ L_{p,-}(\widetilde{e}) &= \mathbb{P}\Big(\xi\big(\mathrm{suc}^{-1}(b)\big)^{(n-p-1)}+\xi(a)^{(p)}\Big)\cap\mathbb{P}(H)\\ &= \mathbb{P}\Big(\xi(a)^{(p-r)}+\xi(b)^{(n-p-1)}+\xi\big(\mathrm{suc}(a)\big)^{(r-1)}+\xi(a'')^{(1)}\Big)\cap\mathbb{P}(H).\end{aligned}$$

These imply that

$$\begin{aligned}&\big(\xi(x^-)^{(1)},L_{p,-}(\widetilde{e}),L_{p,+}(\widetilde{e}),\xi(x^+)^{(1)}\big)\\ =\;&\big(\xi(x^-)^{(1)},\xi(a'')^{(1)},\xi(a')^{(1)},\xi(x^+)^{(1)}\big)_{\xi(a)^{(p-r)}+\xi(b)^{(n-p-1)}+\xi(\mathrm{suc}(a))^{(r-1)}}\\ \geq\;&\Big(\xi\big(\mathrm{suc}^{-1}(b)\big),\xi(a),\xi\big(\mathrm{suc}(a)\big),\xi(b)\Big)_{\xi(a)^{(p-r)}+\xi(b)^{(n-p-1)}+\xi(\mathrm{suc}(a))^{(r-1)}}\end{aligned}$$

where the final inequality is a consequence of Proposition 2.4.5.

Proof of (2). Similarly, let $s_1'$, $s_1''$ be the closed subinterval of $s_1$ with endpoints $b$ and

$x^+$, $x^-$ and $\mathrm{suc}^{-1}(b)$ respectively. For $p = 0, \ldots, n-2$ and any $r = 1, \ldots, n-p-1$, there exists $b'$ in $s_1'$ and $b''$ in $s_1''$ (see Figure 5.6) so that

$$\begin{aligned}
L_{p,+}(\widetilde{e}) &= \mathbb{P}\Big(\xi(b)^{(n-p-1)} + \xi\big(\mathrm{suc}(a)\big)^{(p)}\Big) \cap \mathbb{P}(H) \\
&= \mathbb{P}\Big(\xi(a)^{(p)} + \xi(b)^{(n-p-r-1)} + \xi\big(\mathrm{suc}^{-1}(b)\big)^{(r-1)} + \xi(b')^{(1)}\Big) \cap \mathbb{P}(H), \\
L_{p,-}(\widetilde{e}) &= \mathbb{P}\Big(\xi\big(\mathrm{suc}^{-1}(b)\big)^{(n-p-1)} + \xi(a)^{(p)}\Big) \cap \mathbb{P}(H) \\
&= \mathbb{P}\Big(\xi(a)^{(p)} + \xi(b)^{(n-p-r-1)} + \xi\big(\mathrm{suc}^{-1}(b)\big)^{(r-1)} + \xi(b'')^{(1)}\Big) \cap \mathbb{P}(H).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \big(\xi(x^-)^{(1)}, L_{p,-}(\widetilde{e}), L_{p,+}(\widetilde{e}), \xi(x^+)^{(1)}\big) \\
= \; & \big(\xi(x^-)^{(1)}, \xi(b'')^{(1)}, \xi(b')^{(1)}, \xi(x^+)^{(1)}\big)_{\xi(a)^{(p)}+\xi(b)^{(n-p-r-1)}+\xi(\mathrm{suc}^{-1}(b))^{(r-1)}} \\
\geq \; & \Big(\xi(a), \xi\big(\mathrm{suc}^{-1}(b)\big), \xi(b), \xi\big(\mathrm{suc}(a)\big)\Big)_{\xi(a)^{(p)}+\xi(b)^{(n-p-r-1)}+\xi(\mathrm{suc}^{-1}(b))^{(r-1)}} \\
= \; & \Big(\xi\big(\mathrm{suc}^{-1}(b)\big), \xi(a), \xi\big(\mathrm{suc}(a)\big), \xi(b)\Big)_{\xi(a)^{(p)}+\xi(b)^{(n-p-r-1)}+\xi(\mathrm{suc}^{-1}(b))^{(r-1)}}.
\end{aligned}$$

where the last equality follows from (6) and (7) of Proposition 2.2.2. □

A proof similar to the one for Lemma 5.3.2 gives the following lemma.

**Lemma 5.3.3.** *Let $\widetilde{e} = \{a, b\}$ be an edge in $\widetilde{\mathcal{S}}$, and suppose $a$ lies in $s_0$ and $b$ lies in $s_1$. Then we have the following.*

1. *For all $p = 1, \ldots, n-1$ and for all $M \in \mathcal{M}_p\big(a, b, \mathrm{suc}^{-1}(a)\big)$, we have*

$$l\big(c_p(\widetilde{e})\big) \geq \log\Big(\xi(b), \xi\big(\mathrm{suc}^{-1}(a)\big), \xi(a), \xi\big(\mathrm{suc}(b)\big)\Big)_M.$$

2. *For all $p = 0, \ldots, n-2$ and for all $M \in \mathcal{M}_{n-p-1}\big(b, a, \mathrm{suc}(b)\big)$, we have*

$$l\big(c_p(\widetilde{e})\big) \geq \log\Big(\xi(b), \xi\big(\mathrm{suc}^{-1}(a)\big), \xi(a), \xi\big(\mathrm{suc}(b)\big)\Big)_M.$$

Now, we will define the quantity $K$. For any $[a, b]$ in $\mathcal{T}$ that is not a closed leaf, choose a lift $\{a, b\}$ in $\widetilde{\mathcal{T}}$ of $[a, b]$. Let $c$, $d$ be points in $\partial\Gamma$ such that $\{a, c\}$, $\{b, c\}$, $\{a, d\}$, $\{b, d\}$ lie in $\widetilde{\mathcal{T}}$. For $p = 1, \ldots, n-2$ define

$$\begin{aligned}
K_p' &:= \max\Big\{ \log\big(\xi(d), \xi(a), \xi(c), \xi(b)\big)_M : M \in \mathcal{M}_p(a, b, c) \cup \mathcal{M}_{n-p-1}(b, a, d)\Big\} \\
K_p'' &:= \max\Big\{ \log\big(\xi(b), \xi(d), \xi(a), \xi(c)\big)_M : M \in \mathcal{M}_p(a, b, d) \cup \mathcal{M}_{n-p-1}(b, a, c)\Big\}
\end{aligned}$$

and define

$$
\begin{aligned}
K'_0 &:= \max\Big\{ \log\big(\xi(d),\xi(a),\xi(c),\xi(b)\big)_M : M \in \mathcal{M}_{n-1}(b,a,d)\Big\}\\
K''_0 &:= \max\Big\{ \log\big(\xi(b),\xi(d),\xi(a),\xi(c)\big)_M : M \in \mathcal{M}_{n-1}(b,a,c)\Big\}\\
K'_{n-1} &:= \max\Big\{ \log\big(\xi(d),\xi(a),\xi(c),\xi(b)\big)_M : M \in \mathcal{M}_{n-1}(a,b,c)\Big\}\\
K''_{n-1} &:= \max\Big\{ \log\big(\xi(b),\xi(d),\xi(a),\xi(c)\big)_M : M \in \mathcal{M}_{n-1}(a,b,d)\Big\}
\end{aligned}
$$

Finally, define

$$K[a,b] := \min\left\{\frac{1}{n}\sum_{p=0}^{n-1} K'_p, \frac{1}{n}\sum_{p=0}^{n-1} K''_p\right\}.$$

Note that if we switch the roles of $a$ with $b$, then the quantities $K'_p$ and $K''_{n-p-1}$ are switched. Also, switching $c$ with $d$ causes the quantities $K'_p$ and $K''_p$ to be switched. Thus, permuting $a$ and $b$ or permuting $c$ and $d$ leaves $K[a,b]$ invariant. Moreover, the $PSL(n,\mathbb{R})$ invariance of the cross ratio implies that $K[a,b]$ does not depend on the choice of lift $\{a,b\}$ of $[a,b]$. This allows us to define

$$K(\rho) = K := \min_{[a,b]\in\mathcal{Q}} K[a,b].$$

Recall that $\mathcal{Q}$ is the set of non-closed leaves in $\mathcal{T}$, as defined in Section 4.3.

**Proposition 5.3.4.** *For any $\widetilde{e}$ in $\widetilde{\mathcal{B}}$,*

$$\frac{1}{n}\sum_{p=0}^{n-1} l\big(c_p(\widetilde{e})\big) \geq K.$$

*Proof.* Let $\widetilde{e} = [a,b]$ with $a$ in $s_0$ and $b$ in $s_1$. By Lemma 5.3.2, we see that when $\widetilde{e}$ is in $\mathcal{Z}$, by taking $d = \mathrm{suc}^{-1}(b)$, $c = \mathrm{suc}(a)$, we have $l(c_p(\widetilde{e})) \geq K'_p$ for all $p = 0, \dots, n-1$. Thus,

$$\frac{1}{n}\sum_{p=0}^{n-1} l(c_p(\widetilde{e})) \geq \frac{1}{n}\sum_{p=0}^{n-1} K'_p \geq K.$$

Similarly, by Lemma 5.3.3, we see that when $\widetilde{e}$ is in $\mathcal{S}$, by taking $d = \mathrm{suc}^{-1}(a)$, $c = \mathrm{suc}(b)$, we have

$$\frac{1}{n}\sum_{p=0}^{n-1} l(c_p(\widetilde{e})) \geq \frac{1}{n}\sum_{p=0}^{n-1} K''_p \geq K.$$

□

Next, we want to define the lower bound $L$ for the lengths of the winding $(p)$-subsegments.

For any oriented curve $\gamma$ in $\mathcal{P}$, choose any element $A$ in $\Gamma$ corresponding to $\gamma$. Then define

$$L(\gamma) := l_\rho(A),$$

and observe that $L(\gamma)$ does not depend on the choice of $A$. Thus, we can define

$$L(\rho) = L := \frac{1}{n} \min_{\gamma \in \mathcal{P}} L(\gamma).$$

For the remainder of this section, let $\widetilde{e}$ and $\widetilde{e}'$ be consecutive elements in $\widetilde{\mathcal{B}}$ with $\widetilde{e}$ preceding $\widetilde{e}'$, and let $e$, $e'$ be the equivalence classes in $\mathcal{B}$ that contain $\widetilde{e}$ and $\widetilde{e}'$. Also, let $a^-$ and $a^+$ be the repelling and attracting fixed points for $A = A(\widetilde{e}, \widetilde{e}')$ respectively.

**Lemma 5.3.5.** *If $e$ and $e'$ are not of the same type, then*

$$l\big(w_1(\widetilde{e}, \widetilde{e}')\big) + l\big(w_{n-2}(\widetilde{e}, \widetilde{e}')\big) \geq \max\{0, |t(e, e')| - 1\} \cdot L.$$

*Proof.* This inequality clearly holds when $t(e, e') = -1, 0, 1$, so for the rest of the proof, we will assume that $|t(e, e')| \geq 2$. Let $r_0$, $r_1$ be the oriented subsegments of $\partial\Gamma$ with endpoints $a^-$, $a^+$, oriented from $a^-$ to $a^+$, so that the orientation on $r_0$ agrees with the clockwise orientation on $\partial\Gamma$. Since $e$ and $e'$ are not of the same type, either both of $x^-$, $x^+$ lie in $r_0$ or both of $x^-$, $x^+$ lie in $r_1$ (see Remark 5.1.4). If the former holds, we will show that $l\big(w_1(\widetilde{e}, \widetilde{e}')\big) \geq \max\{0, |t(e, e')| - 1\} \cdot L$ and if the latter holds, we will show that $l\big(w_{n-2}(\widetilde{e}, \widetilde{e}')\big) \geq \max\{0, |t(e, e')| - 1\} \cdot L$.

By taking inverses, we can assume without loss of generality that $a^-$, $x^-$, $x^+$, $a^+$ lie on $r_0$ in that clockwise order. In this case, $t(e, e') \geq 0$, and we need to show that $l\big(w_1(\widetilde{e}, \widetilde{e}')\big) \geq \max\{0, |t(e, e')| - 1\} \cdot L$. Let $c$ be the point in $r_0$ so that $\{c, c'\}$ is an edge in $\mathcal{E}_A$ and $A^{-1} \cdot c$, $x^-$, $c$ lie in $r_0$ in that order. This then implies that $A^{t(e,e')-1} \cdot c$, $x^+$, $A^{t(e,e')} \cdot c$ lie in $r_0$ in that order. (See Figure 5.7.) For any $p = 0, \dots, n-2$, define

$$\alpha_p := \mathbb{P}\big(\xi(a^-)^{(p)} + \xi(a^+)^{(n-p-2)} + \xi(c)^{(1)}\big) \cap \mathbb{P}(H),$$
$$\beta_p := \mathbb{P}\big(\xi(a^-)^{(p)} + \xi(a^+)^{(n-p-2)} + \xi(A^{t(e,e')-1} \cdot c)^{(1)}\big) \cap \mathbb{P}(H).$$

Now, let $b$ be the vertex of $\widetilde{\mathcal{T}}$ that lies on $r_0$ between $A^{-1} \cdot c$ and $c$, so that $\{a^-, b\}$ is an edge of $\widetilde{\mathcal{T}}$. Observe that any edge in $\widetilde{\mathcal{T}}$ with $a^-$ as an endpoint has its other endpoint in $\langle A \rangle \cdot c \cup \langle A \rangle \cdot b$. Hence, if $b$ lies between $A^{-1} \cdot c$ and $x^-$, then

$$\widetilde{e} = \{a^-, c\},\ \mathrm{suc}^{-1}(\widetilde{e}) = \{b, c\},\ \mathrm{suc}(\widetilde{e}) = \{a^-, A \cdot b\},$$

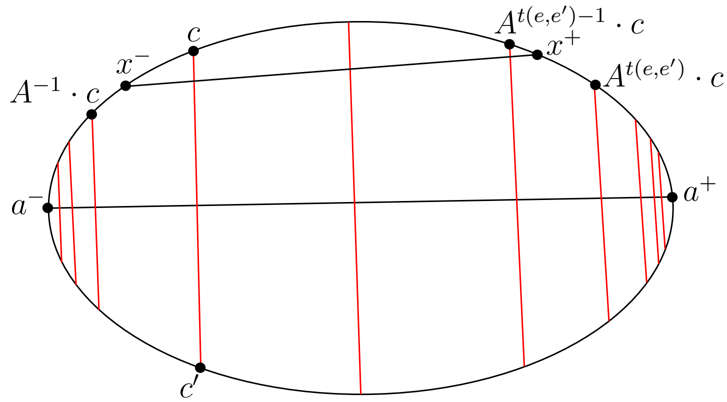


Figure 5.7: $e$, $e'$ are not of the same type.

which implies that $L_{1,-}(\widetilde{e}) = \mathbb{P}(\xi(b)^{(n-2)} + \xi(c)^{(1)}) \cap \mathbb{P}(H)$. On the other hand, if $b$ lies between $x^-$ and $c$, then

$$\widetilde{e} = \{a^-, b\},\ \mathrm{suc}^{-1}(\widetilde{e}) = \{A^{-1} \cdot c, b\},\ \mathrm{suc}(\widetilde{e}) = \{a^-, c\},$$

which implies that $L_{1,-}(\widetilde{e}) = \mathbb{P}\big(\xi(b)^{(1)} + \xi(A^{-1} \cdot c)^{(n-2)}\big) \cap \mathbb{P}(H)$.

By a similar reasoning, if $A^{t(e,e')} \cdot b$ lies between $A^{t(e,e')-1} \cdot c$ and $x^+$, then $L_{1,+}(\widetilde{e}') = \mathbb{P}\big(\xi(A^{t(e,e')} \cdot b)^{(1)} + \xi(A^{t(e,e')} \cdot c)^{(n-2)}\big) \cap \mathbb{P}(H)$ and if $A^{t(e,e')} \cdot b$ lies between $x^+$ and $A^{t(e,e')} \cdot c$, then $L_{1,+}(\widetilde{e}') = \mathbb{P}\big(\xi(A^{t(e,e')} \cdot b)^{(n-2)} + \xi(A^{t(e,e')-1} \cdot c)^{(1)}\big) \cap \mathbb{P}(H)$.

In any case, by (2) of Lemma 2.4.3, we see that $L_{1,-}(\widetilde{e})$, $\alpha_p$, $\beta_p$, $L_{1,+}(\widetilde{e})$ lie in the same subsegment of $\mathbb{P}(H)$ with endpoints $\xi(x^-)^{(1)}$, $\xi(x^+)^{(1)}$, in that order. Proposition 2.2.5 thus implies that for all $p = 0, \dots, n-2$, we have

$$l\big(w_1(\widetilde{e}, \widetilde{e}')\big) \geq \log\big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big).$$

Also, if we let $0 < \lambda_1 < \cdots < \lambda_n$ be the eigenvalues for $\rho(A)$, then for any $p = 0, \dots, n-2$, one can compute that

$$\begin{aligned}
& \big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \\
= \ & \big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big)_{\xi(a^-)^{(p)} + \xi(a^+)^{(n-p-2)}} \\
\geq \ & \big(\xi(a^-), \xi(c), \xi(A^{t(e,e')-1} \cdot c), \xi(a^+)\big)_{\xi(a^-)^{(p)} + \xi(a^+)^{(n-p-2)}} \\
= \ & \left(\frac{\lambda_{p+2}}{\lambda_{p+1}}\right)^{t(e,e')-1},
\end{aligned}$$

where the inequality is a consequence of Proposition 2.4.5, and the last equality is a computation similar to that done in Proposition 2.2.4.

By taking the product of these inequalities over $p = 0, \ldots, n-2$, and then taking logarithm, we obtain

$$(n-1) \cdot l\big(w_1(\widetilde{e}, \widetilde{e}')\big) \geq \big(t(e,e') - 1\big) \cdot \log\left(\frac{\lambda_n}{\lambda_1}\right).$$

This implies the lemma. $\square$

**Lemma 5.3.6.** *If $e$ and $e'$ are of the same type, then*

$$\frac{1}{n}\sum_{p=0}^{n-1} l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \max\{0, |t(e,e')| - 2\} \cdot L.$$

*Proof.* This inequality clearly holds when $t(e,e') = -2, -1, 0, 1, 2$, so for the rest of the proof, we will assume that $|t(e,e')| \geq 3$. As before, let $r_0$, $r_1$ be the two subsegments of $\partial\Gamma$ with endpoints $a^-$ and $a^+$, oriented from $a^-$ to $a^+$, and such that the orientation on $r_0$ agrees with the orientation on $\partial\Gamma$.

By taking inverses, we can assume without loss of generality that $x^-$ lies in $r_0$ and $x^+$ lies in $r_1$. Let $\{c_0, c_1\}$ be the edge in $\mathcal{E}_A$ so that $c_0$ lies in $s_0$ and $A^{-1} \cdot c_0$ lies in $s_1$. This implies that $A^{t(e,e')-1} \cdot c_1$ lies in $s_1$ and $A^{t(e,e')} \cdot c_1$ lies in $s_0$. For any $p \in \{0, \ldots, n-1\}$, let

$$\begin{aligned}
\alpha_p &:= \mathbb{P}\big(\xi(A^{-1} \cdot c_0)^{(n-p-1)} + \xi(c_0)^{(p)}\big) \cap \mathbb{P}(H),\\
\beta_p &:= \mathbb{P}\big(\xi(A^{t(e,e')-1} \cdot c_1)^{(n-p-1)} + \xi(A^{t(e,e')} \cdot c_1)^{(p)}\big) \cap \mathbb{P}(H).
\end{aligned}$$

Using a similar argument as in the proof of Lemma 5.3.5, we have

$$l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \log\big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big).$$

Choose a normalization so that $\xi(a^-)^{(k)} \cap \xi(a^+)^{(n-k+1)} = [e_k]$ for all $k = 1, \ldots, n$ and $\xi(c_0)^{(1)} = [e_1 + \cdots + e_n]$. In this normalization, $\rho(A)$ is the projectivization of a diagonal matrix, and if $\lambda_i$ is the eigenvalue corresponding to $e_i$, then $0 < \lambda_1 < \cdots < \lambda_n$. Write

$$\xi(c_1)^{(1)} = \left[\sum_{i=1}^{n} \delta_i e_i\right]$$

for some real numbers $\delta_i$.

From here, the proof will proceed in two cases, depending on the sign of $t(e,e')$.

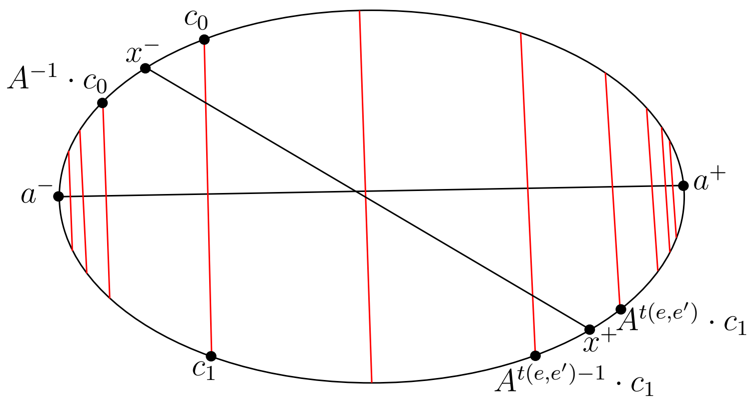


Figure 5.8: $e$, $e'$ are of the same type and $t(e,e') > 0$.

**Case 1:** Suppose that $t(e,e') > 0$. (See Figure 5.8.) For $p = 1, \dots, n-1$, define

$$\begin{aligned}
\alpha'_p &:= \mathbb{P}\big(\xi(a^-)^{(n-p-1)} + \xi(a^+)^{(p-1)} + \xi(c_0)^{(1)}\big) \cap \mathbb{P}(H), \\
\beta'_p &:= \mathbb{P}\big(\xi(a^-)^{(n-p-1)} + \xi(a^+)^{(p)}\big) \cap \mathbb{P}(H).
\end{aligned}$$

By (2) of Lemma 2.4.3, we see that $\alpha_p$, $\alpha'_p$, $\beta'_p$, $\beta_p$ lie on $\mathbb{P}(H)$ in that order. Hence, we can apply Proposition 2.4.5 to see that

$$\begin{aligned}
& \big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \\
\geq \; & \big(\xi(x^-)^{(1)}, \alpha'_p, \beta'_p, \xi(x^+)^{(1)}\big) \\
= \; & \big(\xi(x^-)^{(1)}, \alpha'_p, \beta'_p, \xi(x^+)^{(1)}\big)_{\xi(a^-)^{(n-p-1)}+\xi(a^+)^{(p-1)}} \\
\geq \; & \big(\xi(A^{-1} \cdot c_0), \xi(c_0), \xi(a^+), \xi(A^{t(e,e')-1} \cdot c_1)\big)_{\xi(a^-)^{(n-p-1)}+\xi(a^+)^{(p-1)}} \\
= \; & \frac{1}{1 - \frac{\lambda_{n-p}}{\lambda_{n-p+1}}} \cdot \left(1 - \frac{\delta_{n-p+1}}{\delta_{n-p}} \cdot \left(\frac{\lambda_{n-p+1}}{\lambda_{n-p}}\right)^{t(e,e')-1}\right).
\end{aligned}$$

Observe that for any integer $k$, (1) of Proposition 2.4.5 says that

$$\begin{aligned}
& \big(\xi(A^{-1} \cdot c_0), \xi(c_0), \xi(a^+), \xi(A^{k-1} \cdot c_1)\big)_{\xi(a^-)^{(n-p-1)}+\xi(a^+)^{(p-1)}} \\
= \; & \frac{1}{1 - \frac{\lambda_{n-p}}{\lambda_{n-p+1}}} \cdot \left(1 - \frac{\delta_{n-p+1}}{\delta_{n-p}} \cdot \left(\frac{\lambda_{n-p+1}}{\lambda_{n-p}}\right)^{k-1}\right)
\end{aligned}$$

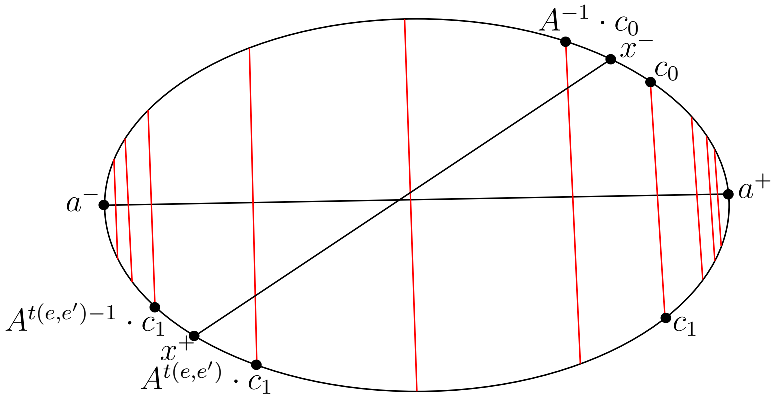


Figure 5.9: $e$, $e'$ are of the same type and $t(e,e') < 0$.

is at least 1. Since $\dfrac{\lambda_{n-p}}{\lambda_{n-p+1}} < 1$, this then implies that $\frac{\delta_{n-p+1}}{\delta_{n-p}} < 0$. Thus, we have

$$\big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \geq \left|\frac{\delta_{n-p+1}}{\delta_{n-p}}\right| \cdot \left(\frac{\lambda_{n-p+1}}{\lambda_{n-p}}\right)^{t(e,e')-1}.$$

By doing this for all $p = 1, \ldots, n-1$ and taking the product, we get that

$$\prod_{p=1}^{n-1} \big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \geq \left|\frac{\delta_n}{\delta_1}\right| \cdot \left(\frac{\lambda_n}{\lambda_1}\right)^{t(e,e')-1},$$

which implies

$$\sum_{p=1}^{n-1} l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \log\left|\frac{\delta_n}{\delta_1}\right| + \big(t(e,e')-1\big)\cdot\log\left(\frac{\lambda_n}{\lambda_1}\right). \tag{5.3.1}$$

**Case 2:** Suppose that $t(e,e') < 0$. (See Figure 5.9.) For all $0 \leq p \leq n-2$, define

$$\alpha''_p := \mathbb{P}\big(\xi(a^-)^{(n-p-2)} + \xi(a^+)^{(p)} + \xi(A^{-1}\cdot c_0)^{(1)}\big) \cap \mathbb{P}(H),$$
$$\beta''_p := \mathbb{P}\big(\xi(a^-)^{(n-p-1)} + \xi(a^+)^{(p)}\big) \cap \mathbb{P}(H).$$

As before, (2) of Lemma 2.4.3, implies that $\alpha_p$, $\alpha''_p$, $\beta''_p$, $\beta_p$ lie on $\mathbb{P}(H)$ in that order, thus allowing us to use Proposition 2.2.5 and Proposition 2.4.5 to compute

$$
\begin{aligned}
& \big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \\
\geq\ & \big(\xi(x^-)^{(1)}, \alpha_p'', \beta_p'', \xi(x^+)^{(1)}\big) \\
=\ & \big(\xi(x^-)^{(1)}, \alpha_p'', \beta_p'', \xi(x^+)^{(1)}\big)_{\xi(a^-)^{(n-p-2)}+\xi(a^+)^{(p)}} \\
\geq\ & \big(\xi(c_0), \xi(A^{-1}\cdot c_0), \xi(a^-), \xi(A^{t(e,e')}\cdot c_1)\big)_{\xi(a^-)^{(n-p-2)}+\xi(a^+)^{(p)}} \\
=\ & \frac{1}{1-\frac{\lambda_{n-p-1}}{\lambda_{n-p}}}\cdot\left(1-\frac{\delta_{n-p-1}}{\delta_{n-p}}\cdot\left(\frac{\lambda_{n-p-1}}{\lambda_{n-p}}\right)^{t(e,e')+1}\right).
\end{aligned}
$$

As before, $\frac{\delta_{n-p-1}}{\delta_{n-p}} < 0$, so

$$
\big(\xi(x^-)^{(1)}, \alpha_p, \beta_p, \xi(x^+)^{(1)}\big) \geq \left|\frac{\delta_{n-p-1}}{\delta_{n-p}}\right|\cdot\left(\frac{\lambda_{n-p-1}}{\lambda_{n-p}}\right)^{t(e,e')+1}.
$$

If we take the product of all these inequalities for $p = 0, \ldots, n-2$ and then take logarithm, we obtain the inequality

$$
\sum_{p=0}^{n-2} l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \log\left|\frac{\delta_1}{\delta_n}\right| + \big(-t(e,e')-1\big)\cdot\log\left(\frac{\lambda_n}{\lambda_1}\right) \tag{5.3.2}
$$

By the way we defined a mesh (see (5.1.1)), we have that $1 \leq \left|\frac{\delta_n}{\delta_1}\right| \leq \frac{\lambda_n}{\lambda_1}$. Thus, both inequalities (5.3.1) and (5.3.2) imply that

$$
\sum_{p=0}^{n-1} l\big(w_p(\widetilde{e}, \widetilde{e}')\big) \geq \big(|t(e,e')|-2\big)\cdot\log\left(\frac{\lambda_n}{\lambda_1}\right).
$$

The inequality in the lemma follows immediately from this. $\square$

In order to emphasize that the lower bound in Theorem 5.3.7 depends only on the combinatorial description $\psi(X)$ of $X$, we will use the notation

$$
r\big(\psi(X)\big) := |\mathcal{B}| \text{ and } s\big(\psi(X)\big) := \sum_{(e,e')\in\widetilde{\mathcal{D}}_1\cup\widetilde{\mathcal{D}}_2} \max\big\{0, |t(e,e')|-2\big\}.
$$

As a corollary of the estimates in Proposition 5.2.7, Proposition 5.3.4, Lemma 5.3.5 and Lemma 5.3.6, we obtain the following theorem.

**Theorem 5.3.7.** *Pick any* $\rho$ *in* $Hit_n(S)$ *and any* $X$ *in* $\Gamma$ *such that* $r\big(\psi(X)\big) \neq 0$. *Then*

$$l_\rho(X) \geq r\big(\psi(X)\big) \cdot \frac{K(\rho)}{11} + s\big(\psi(X)\big) \cdot \frac{L(\rho)}{11}.$$

# CHAPTER 6

# Degeneration along internal sequences

In this section, we will use the analysis in Chapter 5 to prove Theorem 4.4.4. The key fact on which the proof relies on is Theorem 6.1.1. Essentially, this says that when one deforms along an internal sequence, the lengths of the crossing segments of any closed curve must grow to infinity. As a consequence, the lengths of many closed curves must grow to infinity as well when we deform along an internal sequence. A further computation then shows this also forces the topological entropy to converge to $0$ along an internal sequence.

## 6.1 Proof of (1) of main theorem

Observe that $X$ is an element of $\Gamma$ such that $X \neq A^k$ for any $A$ in $\Gamma_{\mathcal{P}}$ and any integer $k$ if and only if $r\big(\psi(X)\big) \neq 0$. Theorem 5.3.7 then implies that for any $\rho$ in $Hit_n(S)$, $K(\rho)$ is a lower bound for $\Theta(\rho)$. Thus, to prove (1) of Theorem 4.4.4, it is sufficient to prove the following theorem.

**Theorem 6.1.1.** *Let $\{\rho_i\}_{i=1}^\infty$ be an internal sequence in $\mathfrak{C}(M)$. Then*

$$\lim_{i\to\infty} K(\rho_i) = \infty.$$

We start by using the closed leaf equalities (4.3.1), (4.3.2) and (4.3.3) to prove the following lemma.

**Lemma 6.1.2.** *Let $\{\rho_i\}_{i=1}^\infty$ be an internal sequence such that the shear and triangle invariants converge (possibly to $\infty$ or $-\infty$) along $\{\rho_i\}_{i=1}^\infty$. Then for any $j = 1, \dots, 2g-2$, one of the following hold:*

1. *There is a pair of shear invariants, call them $\sigma_{1,[a_j,b_j]}$ and $\sigma_{2,[a_j,b_j]}$ for the edge $[a_j, b_j]$, so that*

$$\lim_{i\to\infty} \sigma_{1,[a_j,b_j]}(\rho_i) = \infty \text{ and } \lim_{i\to\infty} \sigma_{2,[a_j,b_j]}(\rho_i) = -\infty.$$

2. *There is some number* $z_0 \in \{1, \ldots, n-2\}$ *so that there are triangle invariants* $\tau_1, \tau_2$ *in* $\{\tau_{(x,y,z),j} : z = z_0\} \cup \{\tau'_{(x,y,z),j} : z = z_0\}$ *satisfying*

$$\lim_{i\to\infty} \tau_1(\rho_i) = \infty \text{ and } \lim_{i\to\infty} \tau_2(\rho_i) = -\infty,$$

*and the triangle invariants in* $\{\tau_{(x,y,z),j} : z < z_0\} \cup \{\tau'_{(x,y,z),j} : z < z_0\}$ *are bounded above and below along* $\{\rho_i\}_{i=1}^{\infty}$.

*Proof.* First, we prove the claim that if (1) does not hold, then there must be some triangle invariant $\tau_2$ in $\mathcal{A}_j \cup \mathcal{A}'_j$ so that

$$\lim_{i\to\infty} \tau_2(\rho_i) = -\infty.$$

Suppose that every triangle invariant in $\mathcal{A}_j \cup \mathcal{A}'_j$ is bounded below by a real number when evaluated along $\{\rho_i\}_{i=1}^{\infty}$. Equations (4.3.1), (4.3.2), (4.3.3) and the definition of an internal sequence then imply that there is some shear invariant $\sigma$ that is converging to $-\infty$ along $\{\rho_i\}_{i=1}^{\infty}$. We will use this to show that for each of the three edges $[a_j, b_j]$, $[b_j, c_j]$ and $[c_j, a_j]$, there are shear parameters associated to that edge that converge to $\infty$ and $-\infty$.

Assume without loss of generality that the shear invariant $\sigma$ is associated to the edge $[b_j, c_j]$, and denote it by $\sigma_{2,[b_j,c_j]}$. By Equation (4.3.11), we know that there is some other shear invariant $\sigma_{1,[b_j,c_j]}$ associated to the edge $[b_j, c_j]$ so that

$$\lim_{i\to\infty} \sigma_{1,[b_j,c_j]}(\rho_i) = \infty.$$

Then, by Equation (4.3.3) and the assumption that every triangle invariant for $P_j$ is bounded below by a real number when evaluated along $\{\rho_i\}_{i=1}^{\infty}$, we can deduce that there is some shear invariant $\sigma_{2,[c_j,a_j]}$ associated to the edge $[c_j, a_j]$ so that

$$\lim_{i\to\infty} \sigma_{2,[c_j,a_j]}(\rho_i) = -\infty.$$

Equation (4.3.12) now implies that there is some other shear invariant $\sigma_{1,[c_j,a_j]}$ associated to the edge $[c_j, a_j]$ so that

$$\lim_{i\to\infty} \sigma_{1,[c_j,a_j]}(\rho_i) = \infty.$$

Using the same arguments as above, Equations (4.3.1) and (4.3.10) together imply that there are shear invariants $\sigma_{1,[a_j,b_j]}$ and $\sigma_{2,[a_j,b_j]}$ associated to the edge $[b_j, c_j]$ so that

$$\lim_{i\to\infty} \sigma_{1,[a_j,b_j]}(\rho_i) = \infty \text{ and } \lim_{i\to\infty} \sigma_{2,[a_j,b_j]}(\rho_i) = -\infty.$$

We have thus proven that under the hypothesis that every triangle invariant in $\mathcal{A}_j \cup \mathcal{A}'_j$

is bounded below by a real number when evaluated along $\{\rho_i\}_{i=1}^{\infty}$, the three edges $[a_j, b_j]$, $[b_j, c_j]$ and $[a_j, c_j]$ each have a pair shear invariants associated to them with the property that one of them converges to $-\infty$ while the other converges to $\infty$. In particular, (1) holds. The contrapositive of this is the claim.

Next, we prove that if (1) does not hold, then (2) must hold. Suppose (1) does not hold. By Equation (4.3.10), we see that every shear invariant for $[a_j, b_j]$ is bounded below along $\{\rho_i\}_{i=1}^{\infty}$ if and only if every shear invariant for $[a_j, b_j]$ is bounded above along $\{\rho_i\}_{i=1}^{\infty}$. Hence, the shear invariants for the edge $[a_j, b_j]$ are bounded both above and below.

Also, we can assume without loss of generality that $\tau_2$ in the above claim (with the property that $\lim_{i\to\infty} \tau_2(\rho_i) = -\infty$) lies in

$$\{\tau_{(x,y,z),j} : z = z_0\} \cup \{\tau'_{(x,y,z),j} : z = z_0\}$$

for some $z_0$ so that the triangle invariants in

$$\{\tau_{(x,y,z),j} : z < z_0\} \cup \{\tau'_{(x,y,z),j} : z < z_0\}$$

are bounded below along $\{\rho_i\}_{i=1}^{\infty}$. Equations (4.3.1) and (4.3.2) then imply respectively that the sequences

$$\{\sigma_{(n-z_0,0,z_0),j}(\rho_i)\}_{i=1}^{\infty} \text{ and } \{\sigma_{(0,n-z_0,z_0),j}(\rho_i)\}_{i=1}^{\infty}$$

are bounded above. Thus, by Equation (4.3.3), there exists a triangle invariant $\tau_1$ in $\{\tau_{(x,y,z),j} : z = z_0\} \cup \{\tau'_{(x,y,z),j} : z = z_0\}$ with

$$\lim_{i\to\infty} \tau_1(\rho_i) = \infty.$$

Finally, suppose for contradiction that there is some $z_0' < z_0$ with the property that there is some triangle invariant $\tau_1'$ in $\{\tau_{(x,y,z),j} : z = z_0'\} \cup \{\tau'_{(x,y,z),j} : z = z_0'\}$ such that

$$\lim_{i\to\infty} \tau_1'(\rho_i) = \infty.$$

We can assume that $z_0'$ is the minimal such number. A similar proof as the one given above then implies that there is some $\tau_2'$ in $\{\tau_{(x,y,z),j} : z = z_0'\} \cup \{\tau'_{(x,y,z),j} : z = z_0'\}$ with

$$\lim_{i\to\infty} \tau_2'(\rho_i) = -\infty.$$

However, this contradicts the definition of $z_0$. $\square$

Armed with Lemma 6.1.2, we are now ready to prove Theorem 6.1.1.

*Proof of Theorem 6.1.1.* For any subsequence of $\{\rho_i\}_{i=1}^{\infty}$, choose a further subsequence, denoted $\{\rho_{i_k}\}_{k=1}^{\infty}$, so that the shear and triangle invariants converge (possibly to $\infty$ or $-\infty$) along $\{\rho_i\}_{i=1}^{\infty}$. It is sufficient to show that

$$\lim_{k\to\infty} K(\rho_{i_k}) = \infty.$$

For the rest of the proof, we will simplify notation by relabeling the sequence $\{\rho_{i_k}\}_{k=1}^{\infty}$ as $\{\rho_i\}_{i=1}^{\infty}$.

Let $\{a, b\}$ be any edge in $\widetilde{\mathcal{T}}$ that is not a closed leaf, and let $c, d$ be the unique pair of points in $\partial\Gamma$ such that $\{a, c\}$, $\{b, c\}$, $\{a, d\}$, $\{b, d\}$ are all edges in $\widetilde{\mathcal{T}}$. By the definition of $K(\rho)$, it is sufficient to show that both of the following hold:

(a) There is a sequence $\{M_i\}_{i=1}^{\infty}$ of $(n-2)$-dimensional subspaces in $\mathbb{R}^n$ so that $M_i$ is an element of
$$\left(\bigcup_{p=1}^{n-1} \mathcal{M}_p^{\xi_i}(a, b, c)\right) \cup \left(\bigcup_{p=0}^{n-2} \mathcal{M}_{n-p-1}^{\xi_i}(b, a, d)\right)$$
for all $i$, and
$$\lim_{i\to\infty} \log\big(\xi_i(d), \xi_i(a), \xi_i(c), \xi_i(b)\big)_{M_i} = \infty.$$

(b) There is a sequence $\{M_i\}_{i=1}^{\infty}$ of $(n-2)$-dimensional subspaces in $\mathbb{R}^n$ so that $M_i$ is an element of
$$\left(\bigcup_{p=1}^{n-1} \mathcal{M}_p^{\xi_i}(a, b, d)\right) \cup \left(\bigcup_{p=0}^{n-2} \mathcal{M}_{n-p-1}^{\xi_i}(b, a, c)\right)$$
for all $i$, and
$$\lim_{i\to\infty} \log\big(\xi_i(b), \xi_i(d), \xi_i(a), \xi_i(c)\big)_{M_i} = \infty.$$

Using Corollary 2.3.5, we can assume without loss of generality that the edge $a = a_j$, $b = b_j$, $c = c_j$ and $d = A_j \cdot c_j$ for some pair of pants $P_j$. In this setting, either (1) or (2) of Lemma 6.1.2 must hold.

Suppose (1) of Lemma 6.1.2 holds. Let $p, q \in \{1, \dots, n-2\}$ be such that

$$\lim_{i\to\infty} \sigma_{(q,n-q,0),j}(\rho_i) = \infty \text{ and } \lim_{i\to\infty} \sigma_{(p,n-p,0),j}(\rho_i) = -\infty.$$

Then

$$\begin{aligned}
& \big(\xi_i(d),\xi_i(a),\xi_i(c),\xi_i(b)\big)_{\xi_i(a)^{(p-1)}+\xi_i(b)^{(n-p-1)}} \\
= \;& 1-\big(\xi_i(a),\xi_i(d),\xi_i(c),\xi_i(b)\big)_{\xi_i(a)^{(p-1)}+\xi_i(b)^{(n-p-1)}} \\
= \;& 1-\frac{1}{\big(\xi_i(a),\xi_i(c),\xi_i(d),\xi_i(b)\big)_{\xi_i(a)^{(p-1)}+\xi_i(b)^{(n-p-1)}}} \\
= \;& 1+e^{-\sigma_{(p,n-p,0),j}(\rho_i)} \to \infty \text{ as } i\to\infty
\end{aligned}$$

and

$$\begin{aligned}
& \big(\xi_i(b),\xi_i(d),\xi_i(a),\xi_i(c)\big)_{\xi_i(a)^{(q-1)}+\xi_i(b)^{(n-q-1)}} \\
= \;& 1-\big(\xi_i(b),\xi_i(d),\xi_i(c),\xi_i(a)\big)_{\xi_i(a)^{(q-1)}+\xi_i(b)^{(n-q-1)}} \\
= \;& 1+e^{\sigma_{(q,n-q,0),j}(\rho_i)} \to \infty \text{ as } i\to\infty
\end{aligned}$$

so (a) and (b) hold.

Next, suppose that (1) of Lemma 6.1.2 does not hold, then (2) of Lemma 6.1.2 must hold. Define $z_0$ as in (2) of Lemma 6.1.2. Let $\{e_1,\dots,e_n\}$ be the standard basis for $\mathbb{R}^n$ and choose normalizations so that for all $i$, $\xi_i(a_j)^{(k)}\cap\xi_i(b_j)^{(n-k+1)}=[e_k]$ for $k=1,\dots,n$ and $\xi_i(c_j)^{(1)}=[e_1+e_2+\cdots+e_n]$. By Theorem 4.3.1, there is some representation $\rho$ in $Hit_n(S)$ with corresponding Frenet curve $\xi$, so that

- $\tau_{(x,y,z),j}(\rho)=\lim\limits_{i\to\infty}\tau_{(x,y,z),j}(\rho_i)$ for all $(x,y,z)\in\mathcal{A}$ such that $z<z_0$,
- $\tau'_{(x,y,z),j}(\rho)=\lim\limits_{i\to\infty}\tau'_{(x,y,z),j}(\rho_i)$ for all $(x,y,z)\in\mathcal{A}$ such that $z<z_0$,
- $\sigma_{(x,y,z),j}(\rho)=\lim\limits_{i\to\infty}\sigma_{(x,y,z),j}(\rho_i)$ for all $(x,y,z)\in\mathcal{C}$.

If we choose the normalization for $\xi$ so that $\xi(a_j)^{(k)}\cap\xi(b_j)^{(n-k+1)}=[e_k]$ for $k=1,\dots,n$ and $\xi(c_j)^{(1)}=[e_1+e_2+\cdots+e_n]$, then Lemma 2.2.8 implies that $\lim\limits_{i\to\infty}\xi_i(A\cdot c_j)^{(1)}=\xi(A\cdot c_j)^{(1)}$. Also, Lemma 2.3.7, implies that $\lim\limits_{i\to\infty}\xi_i(c_j)^{(k)}=\xi(c_j)^{(k)}$ and $\lim\limits_{i\to\infty}\xi_i(A\cdot c_j)^{(k)}=\xi(A\cdot c_j)^{(k)}$ for all $k=1,\dots,z_0$. In particular, for any $(x,y,z_0)$ in $\mathcal{A}$, the triple

$$\begin{aligned}
&\lim_{i\to\infty}\xi_i(A_j\cdot c_j)^{(1)}+\xi_i(a_j)^{(x-1)}+\xi_i(b_j)^{(y-1)}+\xi_i(c_j)^{(z_0)},\\
&\lim_{i\to\infty}\xi_i(a_j)^{(x)}+\xi_i(b_j)^{(y-1)}+\xi_i(c_j)^{(z_0)} \text{ and}\\
&\lim_{i\to\infty}\xi_i(a_j)^{(x-1)}+\xi_i(b_j)^{(y)}+\xi_i(c_j)^{(z_0)}
\end{aligned}\tag{6.1.1}$$

and the triple

$$\begin{aligned}
&\lim_{i\to\infty} \xi_i(c_j)^{(1)} + \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y-1)} + \xi_i(A_j \cdot c_j)^{(z_0)},\\
&\lim_{i\to\infty} \xi_i(a_j)^{(x)} + \xi_i(b_j)^{(y-1)} + \xi_i(A_j \cdot c_j)^{(z_0)} \text{ and}\\
&\lim_{i\to\infty} \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y)} + \xi_i(A_j \cdot c_j)^{(z_0)}
\end{aligned} \tag{6.1.2}$$

are both pairwise distinct triples of hyperplanes.

By Proposition 2.3.8, we see that $\lim\limits_{i\to\infty} \tau_{(x,y,z_0),j}(\rho_i) = \infty$ if and only if

$$\lim_{i\to\infty} \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y-1)} + \xi_i(c_j)^{(z_0+1)} = \lim_{i\to\infty} \xi_i(a_j)^{(x)} + \xi_i(b_j)^{(y-1)} + \xi_i(c_j)^{(z_0)}$$

and $\lim_{i\to\infty} \tau_{(x,y,z_0),j}(\rho_i) = -\infty$ if and only if

$$\lim_{i\to\infty} \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y-1)} + \xi_i(c_j)^{(z_0+1)} = \lim_{i\to\infty} \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y)} + \xi_i(c_j)^{(z_0)}.$$

Then (4) of Proposition 2.2.2 together with the fact that the triple of hyperplanes (6.1.1) are pairwise distinct imply that if $M = \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y-1)} + \xi_i(c_j)^{(z_0)}$, then

$$\begin{aligned}
\lim_{i\to\infty} \tau_{(x,y,z_0),j}(\rho_i) = \infty \quad &\Longleftrightarrow \quad \lim_{i\to\infty} \big(\xi_i(b_j), \xi_i(A_j \cdot c_j), \xi_i(a_j), \xi_i(c_j)\big)_M = \infty.\\
\lim_{i\to\infty} \tau_{(x,y,z_0),j}(\rho_i) = -\infty \quad &\Longleftrightarrow \quad \lim_{i\to\infty} \big(\xi_i(A_j \cdot c_j), \xi_i(a_j), \xi_i(c_j), \xi_i(b_j)\big)_M = \infty,
\end{aligned}$$

The same argument, using the hyperplanes (6.1.2) in place of (6.1.1) proves that if $M = \xi_i(a_j)^{(x-1)} + \xi_i(b_j)^{(y-1)} + \xi_i(A \cdot c_j)^{(z_0)}$, then

$$\begin{aligned}
\lim_{i\to\infty} \tau'_{(x,y,z_0),j}(\rho_i) = -\infty \quad &\Longleftrightarrow \quad \lim_{i\to\infty} \big(\xi_i(A_j \cdot c_j), \xi_i(a_j), \xi_i(c_j), \xi_i(b_j)\big)_M = \infty.\\
\lim_{i\to\infty} \tau'_{(x,y,z_0),j}(\rho_i) = \infty \quad &\Longleftrightarrow \quad \lim_{i\to\infty} \big(\xi_i(b_j), \xi_i(A_j \cdot c_j), \xi_i(a_j), \xi_i(c_j)\big)_M = \infty,
\end{aligned}$$

Hence, (a) and (b) also hold. $\square$

## 6.2 Proof of (2) of main theorem

Let $h : \Gamma \to \mathbb{Z}_{\geq 0}$ be the function give by $h(X) = |\mathcal{B}_X|$. Observe that $h$ is invariant under conjugation because the triangulation $\widetilde{\mathcal{T}}$ is $\Gamma$-invariant. Hence, we can define

$$\Gamma_0 := h^{-1}(0), \Gamma_1 := h^{-1}(\mathbb{Z}_+),$$

and let $[\Gamma_i]$ be the set of conjugacy classes in $\Gamma_i$ for $i = 0, 1$. Observe also that a conjugacy class $[X]$ lies in $[\Gamma_0]$ if and only if $X = \mathrm{id}$ or $X = A^k$ for some integer $k$ and some $A$ in $\Gamma_{\mathcal{P}}$.

We want to produce an upper bound for the function $\mathrm{h}_{\mathrm{top}}$. To do so, we will first find, for any $\rho \in Hit_n(S)$, upper bounds for the size of the sets

$$\{[X] \in [\Gamma_0] : l_\rho(X) \leq T\} \text{ and } \{[X] \in [\Gamma_1] : l_\rho(X) \leq T\}$$

for some fixed $T > 0$. These will give an upper bound on the size of

$$\{[X] \in [\Gamma] : l_\rho(X) \leq T\},$$

which we can then use to control $\mathrm{h}_{\mathrm{top}}(\rho)$.

**Lemma 6.2.1.** *Let $T > 0$ and $\rho$ be a representation in $Hit_n(S)$. Then*

$$|\{[X] \in [\Gamma_0] : l_\rho(X) \leq T\}| \leq (6g - 6)\left\lfloor \frac{T}{L(\rho)} \right\rfloor + 1,$$

*where $g$ is the genus of $S$.*

Recall that $L(\rho)$ is the minimum of $l_\rho(X)$ over all $X$ in $\Gamma_{\mathcal{P}}$, divided by $n$.

*Proof.* Choose group elements $A_1, \dots, A_{3g-3}$ in $\Gamma$ corresponding to the $3g - 3$ oriented simple closed curves in $\mathcal{P}$. Observe that any conjugacy class in $[\Gamma_0]$ has a unique representative of the form $A_i^k$ for some $i = 1, \dots, 3g - 3$ and some integer $k$. Moreover, for any representation $\rho$ in $Hit_n(S)$, we have

$$l_\rho(A_i^k) = |k| \cdot l_\rho(A_i) \geq |k| \cdot L(\rho).$$

These observations imply that

$$\begin{aligned} |\{[X] \in [\Gamma_0] : l_\rho(X) \leq T\}| &\leq \left| \left\{ A_i^k \in \Gamma : i = 1, \dots, 3g-3; |k| \leq \frac{T}{L(\rho)} \right\} \right| \\ &= (3g - 3)\left(2\left\lfloor \frac{T}{L(\rho)} \right\rfloor\right) + 1. \end{aligned}$$

□

**Lemma 6.2.2.** *Let $T > 0$ and $\rho$ be a representation in $Hit_n(S)$. Then*

$$|\{[X] \in [\Gamma_1] : l_\rho(X) \leq T\}| \leq \sum_{a=1}^{\lfloor \frac{11T}{K(\rho)} \rfloor} \frac{(120g-120)^a}{a} \cdot \binom{\lfloor \frac{11T - a\cdot K(\rho)}{L(\rho)} \rfloor + a}{a},$$

*where $g$ is the genus of $S$.*

*Proof.* Let $\Psi_\rho := \{\psi_\rho(X) : X \in \Gamma_1\}$, where $\psi_\rho(X)$ is the combinatorial data defined in Section 5.1. By Proposition 5.1.6, the map $\psi_\rho : \Gamma_1 \to \Psi_\rho$ descends to a bijection $\widehat{\psi}_\rho : [\Gamma_1] \to \Psi_\rho$. Hence, Theorem 5.3.7 implies that

$$\begin{aligned} |\{[X] \in [\Gamma_1] : l_\rho(X) \leq T\}| &\leq |\{\sigma \in \Psi_\rho : r(\sigma)\cdot K(\rho) + s(\sigma)\cdot L(\rho) \leq 11T\}| \\ &= \sum_{a=1}^{\lfloor \frac{11T}{K(\rho)} \rfloor} \left| \left\{ \sigma \in \Psi_\rho : r(\sigma) = a, s(\sigma) \leq \left\lfloor \frac{11T - a\cdot K(\rho)}{L(\rho)} \right\rfloor \right\} \right|. \end{aligned} \tag{6.2.1}$$

For any cyclic sequence $\sigma = \left\{ \left( \mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i), T_i, t(e_i, e_{i+1}) \right) \right\}_{i=1}^{r(\sigma)} \in \Psi_\rho$, let $\pi_0(\sigma)$ be the cyclic sequence

$$\pi_0(\sigma) := \left\{ \left( \mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i), T_i \right) \right\}_{i=1}^{r(\sigma)}$$

and define $\Psi_{\rho,0} := \{\pi_0(\sigma) : \sigma \in \Psi_\rho\}$. For any $i$, let $j \in \{1, \dots, 2g-2\}$ be the number such that $e_i$ lies in $\mathcal{Q}_j$. There are exactly two other edges in $\mathcal{Q}_j$, call them $e_i'$ and $e_i''$. This means that there are at most four possibilities for what $\left( \mathrm{suc}^{-1}(e_i), e_i, \mathrm{suc}(e_i), T_i \right)$ can be, namely

$$(e_i'', e_i, e_i', Z), (e_i'', e_i, e_i', S), (e_i', e_i, e_i'', Z) \text{ or } (e_i', e_i, e_i'', S).$$

Hence, for any $a \geq 1$,

$$|\{\pi_0(\sigma) \in \Psi_{\rho,0} : r(\sigma) = a\}| \leq \frac{(4\cdot(6g-6))^a}{a}. \tag{6.2.2}$$

Next, we make two easy observations. First, consider the map $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $f(a) = \max\{0, |a| - 2\}$. Then for any non-negative integer $b$, observe that $|f^{-1}(b)| \leq 5$. Second, observe that for all positive integers $a$ and $k$, the size of the set $\{(t_1, \dots, t_a) \in (\mathbb{Z}_{\geq 0})^a : \sum_{i=1}^a t_i \leq k\}$ is $\binom{k+a}{a}$. These two observations, together with the inequality

(6.2.2), allow us to conclude that

$$\begin{aligned}
&\left|\left\{\sigma \in \Psi_\rho : r(\sigma) = a, s(\sigma) \leq \left\lfloor \frac{11T - a \cdot K(\rho)}{L(\rho)} \right\rfloor \right\}\right| \\
\leq\ & \frac{(4 \cdot (6g-6))^a}{a} \cdot 5^a \cdot \binom{\lfloor \frac{11T - a\cdot K(\rho)}{L(\rho)} \rfloor + a}{a} \\
=\ & \frac{(120g - 120)^a}{a} \cdot \binom{\lfloor \frac{11T - a\cdot K(\rho)}{L(\rho)} \rfloor + a}{a}.
\end{aligned}$$

The above inequality together with inequality (6.2.1) imply the lemma. □

**Proposition 6.2.3.** *Let $\rho$ be a representation in $Hit_n(S)$. Then*

$$\mathrm{h}_{\mathrm{top}}(\rho) \leq 11 \limsup_{T \to \infty} \frac{1}{T} \log \binom{\lfloor \frac{T - Q \cdot K(\rho)}{L(\rho)} \rfloor + Q}{Q} + \frac{11 \log(120g - 120)}{K(\rho)},$$

*where $Q \in \{0, 1, \dots, \lfloor \frac{T}{K(\rho)} \rfloor\}$ is the integer so that*

$$\binom{\lfloor \frac{T - Q \cdot K(\rho)}{L(\rho)} \rfloor + Q}{Q} = \max_{a \in \{0,1,\dots,\lfloor \frac{T}{K(\rho)} \rfloor\}} \binom{\lfloor \frac{T - a \cdot K(\rho)}{L(\rho)} \rfloor + a}{a}.$$

*Proof.* Since $[\Gamma] = [\Gamma_0] \cup [\Gamma_1]$, Lemma 6.2.1 and Lemma 6.2.2 imply that

$$\begin{aligned}
&\frac{1}{T} \log |\{[X] \in [\Gamma] : l_\rho(X) \leq T\}| \\
\leq\ & \frac{1}{T} \log \left( (6g-6) \left\lfloor \frac{T}{L(\rho)} \right\rfloor + 1 + \sum_{a=1}^{\lfloor \frac{11T}{K(\rho)} \rfloor} \frac{(120g-120)^a}{a} \cdot \binom{\lfloor \frac{11T - a \cdot K(\rho)}{L(\rho)} \rfloor + a}{a} \right) \\
\leq\ & \frac{1}{T} \log \left( (6g-6) \left\lfloor \frac{T}{L(\rho)} \right\rfloor + 1 \right) + \frac{1}{T} \log \left( \frac{(120g-120)^{\lfloor \frac{11T}{K(\rho)} \rfloor}}{\lfloor \frac{11T}{K(\rho)} \rfloor} \right) \\
& + \frac{1}{T} \log \left\lfloor \frac{11T}{K(\rho)} \right\rfloor + \frac{1}{T} \log \binom{\lfloor \frac{11T - R \cdot K(\rho)}{L(\rho)} \rfloor + R}{R}.
\end{aligned}$$

where $R \in \{0, 1, \dots, \lfloor \frac{11T}{K(\rho)} \rfloor\}$ is the integer so that

$$\binom{\lfloor \frac{11T - R \cdot K(\rho)}{L(\rho)} \rfloor + R}{R} = \max_{a \in \{0,1,\dots,\lfloor \frac{11T}{K(\rho)} \rfloor\}} \binom{\lfloor \frac{11T - a \cdot K(\rho)}{L(\rho)} \rfloor + a}{a}.$$

Since

$$\lim_{T \to \infty} \frac{1}{T} \log \left( (6g-6) \left\lfloor \frac{T}{L(\rho)} \right\rfloor + 1 \right) = 0,$$

$$\lim_{T\to\infty}\frac{1}{T}\log\left\lfloor\frac{11T}{K(\rho)}\right\rfloor=0,$$

$$\lim_{T\to\infty}\frac{1}{T}\log\left(\frac{(120g-120)^{\lfloor\frac{11T}{K(\rho)}\rfloor}}{\lfloor\frac{11T}{K(\rho)}\rfloor}\right)=\frac{11\log(120g-120)}{K(\rho)},$$

we have

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T}\log|\{[X]\in[\Gamma]:l_\rho(X)\leq T\}|\\
\leq\ &\limsup_{T\to\infty}\frac{1}{T}\log\binom{\lfloor\frac{11T-R\cdot K(\rho)}{L(\rho)}\rfloor+R}{R}+\frac{11\log(120g-120)}{K(\rho)}\\
=\ &11\limsup_{T\to\infty}\frac{1}{T}\log\binom{\lfloor\frac{T-Q\cdot K(\rho)}{L(\rho)}\rfloor+Q}{Q}+\frac{11\log(120g-120)}{K(\rho)}.
\end{aligned}$$

□

By Proposition 6.2.3 and Theorem 6.1.1, to finish the proof of (2) of Theorem 4.4.4, it is now sufficient to prove the following proposition.

**Proposition 6.2.4.** *Let $\{\rho_i\}_{i=1}^\infty$ be an internal sequence in $Hit_n(S)$. Then*

$$\lim_{i\to\infty}\limsup_{T\to\infty}\frac{1}{T}\log\binom{\lfloor\frac{T-Q\cdot K(\rho_i)}{L(\rho_i)}\rfloor+Q}{Q}=0,$$

*where $Q\in\{0,1,\dots,\lfloor\frac{T}{K(\rho_i)}\rfloor\}$ is the integer so that*

$$\binom{\lfloor\frac{T-Q\cdot K(\rho_i)}{L(\rho_i)}\rfloor+Q}{Q}=\max_{a\in\{0,1,\dots,\lfloor\frac{T}{K(\rho_i)}\rfloor\}}\binom{\lfloor\frac{T-a\cdot K(\rho_i)}{L(\rho_i)}\rfloor+a}{a}.$$

*Proof.* By the definition of an internal sequences, we know that the sequence $\{L(\rho_i)\}_{i=1}^\infty$ is bounded away from $0$ and $\infty$. This means in particular that

$$L_0:=\inf_i\{L(\rho_i)\}$$

is a positive number. Also, by Theorem 6.1.1, we know that $\lim\limits_{i\to\infty}K(\rho_i)=\infty$. This proposition thus follows if we can prove the following statement:
For any pair of sequences of positive numbers $\{T_j\}_{j=1}^\infty$ and $\{K_i\}_{i=1}^\infty$ such that $\lim\limits_{j\to\infty}T_j=\infty$

and $\lim_{i\to\infty} K_i = \infty$, we have that

$$\lim_{i\to\infty} \limsup_{j\to\infty} \frac{1}{T_j} \log \binom{\lfloor \frac{T_j - Q_{i,j}\cdot K_i}{L_0} \rfloor + Q_{i,j}}{Q_{i,j}} = 0,$$

where $Q_{i,j} \in \{0, 1, \ldots, \lfloor \frac{T_j}{K_i} \rfloor\}$ is the integer so that

$$\binom{\lfloor \frac{T_j - Q_{i,j}\cdot K_i}{L_0} \rfloor + Q_{i,j}}{Q_{i,j}} = \max_{a\in\{0,1,\ldots,\lfloor \frac{T_j}{K_i} \rfloor\}} \binom{\lfloor \frac{T_j - a\cdot K_i}{L_0} \rfloor + a}{a}.$$

This is an elementary but long computation involving Stirling's Formula, which we include in the appendix. $\square$

# Appendix A

# Computation involving Stirling's Formula

The purpose of this appendix is to perform the long but elementary computation to show the following fact.

**Proposition A.0.5.** *For any pair of sequences of positive numbers $\{T_j\}_{j=1}^{\infty}$ and $\{K_i\}_{i=1}^{\infty}$ such that $\lim_{j\to\infty} T_j = \infty$ and $\lim_{i\to\infty} K_i = \infty$, we have that*

$$\lim_{i\to\infty} \limsup_{j\to\infty} \frac{1}{T_j} \log \binom{\lfloor \frac{T_j - Q_{i,j}\cdot K_i}{L} \rfloor + Q_{i,j}}{Q_{i,j}} = 0,$$

*where $Q_{i,j} \in \{0, 1, \dots, \lfloor \frac{T_j}{K_i} \rfloor\}$ is the integer so that*

$$\binom{\lfloor \frac{T_j - Q_{i,j}\cdot K_i}{L} \rfloor + Q_{i,j}}{Q_{i,j}} = \max_{a \in \{0,1,\dots,\lfloor \frac{T_j}{K_i} \rfloor\}} \binom{\lfloor \frac{T_j - a\cdot K_i}{L} \rfloor + a}{a}.$$

*and $L$ is a fixed positive number.*

First, we will fix $K_j$ to be $K \gg L$ and compute

$$\lim_{j\to\infty} \frac{1}{T_j} \log \binom{\lfloor \frac{T_j - Q_j K}{L} \rfloor + Q_j}{Q_j}, \tag{A.0.1}$$

where $Q_j$ is a number in $\{1, \dots, \lfloor \frac{T_j}{K} \rfloor\}$ such that $\binom{\lfloor \frac{T_j - Q_j K}{L} \rfloor + Q_j}{Q_j} \geq \binom{\lfloor \frac{T_j - mK}{L} \rfloor + m}{m}$ for all $m \in \{1, \dots, \lfloor \frac{T_j}{K} \rfloor\}$. The main tool to compute the above limit is the asymptotic equality commonly known as Stirling's Formula, which we state here.

**Theorem A.0.6** (Stirling's Formula)**.** $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$*, i.e.* $\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n \sqrt{2\pi n}} = 1.$

However, to use Stirling's formula, we need to know how $\left\lfloor \frac{T_j - Q_j K}{L} \right\rfloor$ and $Q_j$ vary with $j$. Hence the following lemma.

**Lemma A.0.7.** *Let $K, L$ be fixed, with $K \gg L$. Then the following hold:*

*(1)* $\lim_{j\to\infty} \left\lfloor \frac{T_j - Q_jK}{L} \right\rfloor = \infty.$

*(2)* $\lim_{j\to\infty} Q_j = \infty.$

*(3)* $0 \leq \liminf_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \leq \limsup_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \leq 1.$

*(4) There exists $\alpha > 0$ such that* $\alpha \leq \liminf_{j\to\infty} \frac{Q_j}{T_j} \leq \limsup_{j\to\infty} \frac{Q_j}{T_j} \leq \frac{1}{L+K}.$

*Proof.* Observe that for sufficiently large $j$ (so that $T_j \gg K$),

$$1 \geq \binom{\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j}{Q_j} \Bigg/ \binom{\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + Q_j + 1}{Q_j+1}$$
$$= \frac{(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j)(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j - 1)\dots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + Q_j + 2)(Q_j+1)}{(\lfloor \frac{T_j-Q_jK}{L} \rfloor)(\lfloor \frac{T_j-Q_jK}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + 1)}$$

which implies

$$\frac{Q_j+1}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \leq \frac{(\lfloor \frac{T_j-Q_jK}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + 1)}{(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j)\dots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + Q_j + 2)} \leq 1 \tag{A.0.2}$$

Similarly, for sufficiently large $j$,

$$1 \leq \binom{\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 1}{Q_j-1} \Bigg/ \binom{\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j}{Q_j}$$
$$= \frac{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 1)(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 2)\dots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j + 1)(Q_j)}{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor)(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + 1)}$$

which implies

$$\frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor + 1} \geq \frac{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor)\cdots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + 2)}{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 1)\dots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j + 1)}. \tag{A.0.3}$$

Proof of (1). Suppose for contradiction that $\liminf_{j\to\infty} \left\lfloor \frac{T_j - Q_jK}{L} \right\rfloor < \infty$. This implies that $\limsup_{j\to\infty} Q_j = \infty$, so we have

$$\limsup_{j\to\infty} \frac{Q_j+1}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} = \infty.$$

However, that contradicts (A.0.2).

Proof of (2). Suppose for contradiction that $\liminf_{j\to\infty} Q_j < \infty$, then $\limsup_{j\to\infty} \left\lfloor \frac{T_j - Q_jK}{L} \right\rfloor = \infty$, so we have

$$\limsup_{j\to\infty} \frac{\lfloor \frac{T_j-Q_jK}{L} \rfloor + 1}{Q_j} = \infty.$$

However, if $\liminf_{j\to\infty} Q_j < \infty$, then $\limsup_{j\to\infty} \left\lfloor \frac{T_j - Q_jK}{L} \right\rfloor = \infty$, then the right hand side of the inequality (A.0.3) converges to $1$ as $j \to \infty$. This then implies that

$$\limsup_{j\to\infty} \frac{\lfloor \frac{T_j-Q_jK}{L} \rfloor + 1}{Q_j} \leq 1$$

which is a contradiction.

Proof of (3). This follows immediately from (1), (2) and the inequality (A.0.2).

Proof of (4). By (A.0.2), we know that $\frac{Q_j+1}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \leq 1$, so $\frac{Q_j}{T_j} \leq \frac{1}{L+K}\left(1 - \frac{L}{T_j}\right)$. Since $\lim_{j\to\infty} T_j = \infty$, this proves $\limsup_{j\to\infty} \frac{Q_j}{T_j} \leq \frac{1}{L+K}$. It is clear that $\liminf_{j\to\infty} \frac{Q_j}{T_j} \geq 0$, so suppose for contradiction that $\liminf_{j\to\infty} \frac{Q_j}{T_j} = 0$. By taking a subsequence, we can assume that $\lim_{j\to\infty} \frac{Q_j}{T_j} = 0$, which implies that $\lim_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} = 0$.

Since we know (1) and (2) hold, (A.0.2) and (A.0.3) imply that

$$\begin{aligned}
&\limsup_{j\to\infty} \frac{(\lfloor \frac{T_j-Q_jK}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + 1)}{(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j)(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j - 1)\dots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + Q_j + 2)} \\
&\geq \limsup_{j\to\infty} \frac{Q_j+1}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \\
&= \limsup_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor} \\
&= \limsup_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j-Q_jK}{L} \rfloor + 1} \\
&\geq \limsup_{j\to\infty} \frac{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor)(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + 2)}{(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 1)(\lfloor \frac{T_j-(Q_j-1)K}{L} \rfloor + Q_j - 2)\dots(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j + 1)} \\
&= \limsup_{j\to\infty} \frac{(\lfloor \frac{T_j-Q_jK}{L} \rfloor - 1)\cdots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + 1)}{(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j)(\lfloor \frac{T_j-Q_jK}{L} \rfloor + Q_j - 1)\dots(\lfloor \frac{T_j-(Q_j+1)K}{L} \rfloor + Q_j + 2)}
\end{aligned}$$

so in particular,

$$\lim_{j\to\infty} \frac{Q_j}{\lfloor \frac{T_j - Q_j K}{L} \rfloor} \quad = \quad \limsup_{j\to\infty} \frac{(\lfloor \frac{T_j-Q_jK}{L}\rfloor - 1)\cdots(\lfloor \frac{T_j-(Q_j+1)K}{L}\rfloor + 1)}{(\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)\ldots(\lfloor \frac{T_j-(Q_j+1)K}{L}\rfloor + Q_j + 2)}.$$

However, this is not possible because $\lim\limits_{j\to\infty} \dfrac{Q_j}{\lfloor \frac{T_j-Q_jK}{L}\rfloor} = 0$ implies that the left hand side of the above equation is $0$, but the right hand side is $1$. $\square$

With Lemma A.0.7, we can now explicitly compute (A.0.1).

**Proposition A.0.8.** *Let $K, L$ be fixed, with $K \gg L$. Let $\{T_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $\lim\limits_{j\to\infty} T_j = \infty$ and $H := \lim\limits_{j\to\infty} \dfrac{Q_j}{T_j}$ exists. Then*

$$\lim_{j\to\infty} \frac{1}{T_j} \log \binom{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}{Q_j} = H \log\left(1 - \frac{K}{L} + \frac{1}{HL}\right) + \frac{1-HK}{L} \log\left(1 + \frac{HL}{1-HK}\right).$$

*Proof.* By (3) and (4) of Lemma A.0.7, we know that $0 \le \limsup\limits_{j\to\infty} \dfrac{Q_j}{\lfloor \frac{T_j-Q_jK}{L}\rfloor} \le 1$ and $0 < H \le \dfrac{1}{L+K}$. Also, (1) and (2) of Lemma A.0.7 allow us to apply Stirling's formula to obtain

$$\begin{aligned}
\binom{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}{Q_j} &= \frac{(\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)!}{(\lfloor \frac{T_j-Q_jK}{L}\rfloor)!(Q_j)!} \\
&\sim \frac{((\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)\cdot \frac{1}{e})^{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}\sqrt{2\pi(\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)}}{((\lfloor \frac{T_j-Q_jK}{L}\rfloor)\cdot \frac{1}{e})^{\lfloor \frac{T_j-Q_jK}{L}\rfloor}\sqrt{2\pi(\lfloor \frac{T_j-Q_jK}{L}\rfloor)}(\frac{Q_j}{e})^{Q_j}\sqrt{2\pi Q_j}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{(\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)^{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}}{Q_j^{Q_j}\cdot(\lfloor \frac{T_j-Q_jK}{L}\rfloor)^{\lfloor \frac{T_j-Q_jK}{L}\rfloor}} \cdot \sqrt{\frac{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}{Q_j \cdot \lfloor \frac{T_j-Q_jK}{L}\rfloor}},
\end{aligned}$$

i.e.

$$\lim_{j\to\infty}\left(\binom{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}{Q_j}\cdot\sqrt{2\pi}\cdot\frac{Q_j^{Q_j}\cdot(\lfloor \frac{T_j-Q_jK}{L}\rfloor)^{\lfloor \frac{T_j-Q_jK}{L}\rfloor}}{(\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j)^{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}}\cdot\sqrt{\frac{Q_j\cdot\lfloor \frac{T_j-Q_jK}{L}\rfloor}{\lfloor \frac{T_j-Q_jK}{L}\rfloor + Q_j}}\right) = 1$$

By taking the logarithm, we get

$$\lim_{j\to\infty}\left(\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{Q_j}\right)+\frac{1}{2}\log(2\pi)-\frac{1}{2}\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor\cdot Q_j}\right)\right. \tag{A.0.4}$$
$$\left.-Q_j\cdot\log\left(1+\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor}{Q_j}\right)-\left\lfloor\frac{T_j-Q_jK}{L}\right\rfloor\cdot\log\left(1+\frac{Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor}\right)\right)=0.$$

To compute $\lim_{j\to\infty}\frac{1}{T_j}\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{Q_j}\right)$, it is now sufficient to compute

$$\lim_{j\to\infty}\frac{1}{2T_j}\left(\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor\cdot Q_j}\right)\right)$$

and

$$\lim_{j\to\infty}\frac{1}{T_j}\left(Q_j\cdot\log\left(1+\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor}{Q_j}\right)+\left\lfloor\frac{T_j-Q_jK}{L}\right\rfloor\cdot\log\left(1+\frac{Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor}\right)\right).$$

The proposition thus follows from Lemma A.0.9 and A.0.10. □

**Lemma A.0.9.**
$$\lim_{j\to\infty}\frac{1}{2T_j}\left(\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor\cdot Q_j}\right)\right)=0.$$

*Proof.* Note that

$$\log\left(\frac{\lfloor\frac{T_j-Q_jK}{L}\rfloor+Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor\cdot Q_j}\right)=\log\left(\frac{1}{Q_j}\right)+\log\left(1+\frac{Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor}\right).$$

Since $Q_j\geq 1$, we know that $\frac{\log(\frac{1}{Q_j})}{T_j}\leq 0$. Also, since $Q_j\leq\frac{T_j}{K}$, we have

$$\liminf_{j\to\infty}\frac{\log(\frac{1}{Q_j})}{T_j}\geq\lim_{j\to\infty}\frac{-\log(\frac{T_j}{K})}{T_j}=0,$$

which implies
$$\lim_{j\to\infty}\frac{\log(\frac{1}{Q_j})}{T_j}=0.$$

Also, it is clear that $\frac{1}{T_j}\log\left(1+\frac{Q_j}{\lfloor\frac{T_j-Q_jK}{L}\rfloor}\right)\geq 0$ for sufficiently large $j$, and (3) of

Lemma A.0.7 implies

$$\limsup_{i\to\infty} \frac{1}{T_j} \log\left(1 + \frac{Q_j}{\lfloor \frac{T_j - Q_j K}{L} \rfloor}\right) \le \lim_{j\to\infty} \frac{\log(2)}{T_j} = 0.$$

Thus,

$$\lim_{j\to\infty} \frac{1}{T_j} \log\left(1 + \frac{Q_j}{\lfloor \frac{T_j - Q_j K}{L} \rfloor}\right) = 0.$$

Putting all these together, we get the equality in the lemma. □

**Lemma A.0.10.**

$$\lim_{j\to\infty} \frac{1}{T_j} \left( Q_j \cdot \log\left(1 + \frac{\lfloor \frac{T_j - Q_j K}{L} \rfloor}{Q_j}\right) + \left\lfloor \frac{T_j - Q_j K}{L} \right\rfloor \cdot \log\left(1 + \frac{Q_j}{\lfloor \frac{T_j - Q_j K}{L} \rfloor}\right)\right)$$
$$= H \log\left(1 - \frac{K}{L} + \frac{1}{HL}\right) + \frac{1 - HK}{L} \log\left(1 + \frac{HL}{1 - HK}\right)$$

*Proof.* Observe that

$$\lim_{j\to\infty} \frac{1}{T_j} \left( Q_j \cdot \log\left(1 + \frac{\lfloor \frac{T_j - Q_j K}{L} \rfloor}{Q_j}\right) + \left\lfloor \frac{T_j - Q_j K}{L} \right\rfloor \cdot \log\left(1 + \frac{Q_j}{\lfloor \frac{T_j - Q_j K}{L} \rfloor}\right)\right)$$
$$= \lim_{j\to\infty} \left( \frac{Q_j}{T_j} \cdot \log\left(1 - \frac{K}{L} + \frac{T_j}{LQ_j}\right) + \frac{1}{L}\left(1 - K\frac{Q_j}{T_j}\right) \cdot \log\left(1 + \frac{L}{\frac{T_j}{Q_j} - K}\right)\right).$$

By (4) of Lemma A.0.7, we have $0 < H = \lim_{i\to\infty} \frac{Q_j}{T_j} \le \frac{1}{K+L}$ so the required equality follows. □

We are now ready to proof Proposition A.0.5.

*Proof of Proposition A.0.5.* For each $i$ such that $K_i \gg L$, choose a subsequence $\{T_{j_k}\}_{k=1}^{\infty}$ of $\{T_j\}_{j=1}^{\infty}$ such that $H_i := \lim_{k\to\infty} \frac{Q_{i,j_k}}{T_{j_k}}$ exists and

$$\limsup_{j\to\infty} \frac{1}{T_j} \log\left(\frac{\lfloor \frac{T_j - Q_{i,j} K_i}{L} \rfloor + Q_{i,j}}{Q_{i,j}}\right) = \lim_{k\to\infty} \frac{1}{T_{j_k}} \log\left(\frac{\lfloor \frac{T_{j_k} - Q_{i,j_k} K_i}{L} \rfloor + Q_{i,j_k}}{Q_{i,j_k}}\right)$$

(we can do this by (4) of Lemma A.0.7). Proposition A.0.8 then tells us that

$$\begin{aligned}&\limsup_{j\to\infty}\frac{1}{T_j}\log\binom{\lfloor\frac{T_j-Q_{i,j}K_i}{L}\rfloor+Q_{i,j}}{Q_{i,j}} \\ &= H_i\log\left(1+\frac{1-H_iK_i}{H_iL}\right)+\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right)\end{aligned} \tag{A.0.5}$$

Since $0 < Q_{i,j} \leq \left\lfloor\frac{T_j}{K_i}\right\rfloor \leq \frac{T_j}{K_i}$, we know that $0 < \frac{Q_{i,j}}{T_j}K_i \leq 1$. Thus, by choosing a subsequence of $\{K_i\}_{i=1}^{\infty}$, we can assume that

$$0 \leq \lim_{i\to\infty}(H_iK_i) =: V \leq 1.$$

Since $\lim_{i\to\infty}K_i = \infty$, this implies that $\lim_{i\to\infty}H_i = 0$. This means

$$\limsup_{i\to\infty}H_i\log\left(1+\frac{1-H_iK_i}{H_iL}\right) \leq \lim_{i\to\infty}H_i\log\left(1+\frac{1}{H_iL}\right) = 0$$

Also, $H_i\log\left(1+\frac{1-H_iK_i}{H_iL}\right) \geq 0$ for all $i$, so

$$\lim_{i\to\infty}H_i\log\left(1+\frac{1-H_iK_i}{H_iL}\right) = 0 \tag{A.0.6}$$

Next, we show that $\lim_{i\to\infty}\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right) = 0$. Since $\lim_{i\to\infty}H_i = 0$, this is clear in the case when $V < 1$. In the case when $V = 1$,

$$\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right) \leq \frac{1-H_iK_i}{L}\log\left(1+\frac{L}{1-H_iK_i}\right).$$

for large enough $i$. By taking limit supremum,

$$\limsup_{i\to\infty}\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right) \leq \lim_{j\to\infty}\frac{1-H_iK_i}{L}\log\left(1+\frac{L}{1-H_iK_i}\right) = 0.$$

Since $\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right) \geq 0$ for all $i$, we have that

$$\lim_{i\to\infty}\frac{1-H_iK_i}{L}\log\left(1+\frac{H_iL}{1-H_iK_i}\right) = 0. \tag{A.0.7}$$

Together, (A.0.5), (A.0.6) and (A.0.7) imply that

$$\lim_{i\to\infty} \limsup_{j\to\infty} \frac{1}{T_j} \log \binom{\lfloor \frac{T_j - Q_{i,j} K_i}{L} \rfloor + Q_{i,j}}{Q_{i,j}} = 0.$$

□

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