RICHARDS' TRANSFORMATION AND POSITIVE REAL FUNCTIONS*

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1. Introduction. The class of analytic functions which map the right half-plane into itself and take the real axis into the real axis, sometimes called "positive real" functions, have been extensively studied in the literature, originally for their mathematical interest and more recently because of their importance in the synthesis of two-terminal, passive networks (see Brune [2], Cauer [3] and Bott and Duffin [1] in this connection). It is our purpose to present some new results for the class of "positive real" functions which stem from a theorem of Richards [7; Theorem 6]. Besides some incidental results that seem of some interest we prove five main theorems; a representation theorem for "positive real" functions as limits of rational "positive real" functions, Corollary 3.1; an approximation theorem, Theorem 2.5; a necessary and sufficient condition for the existence of a "positive real" function which takes prescribed values on certain prescribed sets, Theorem 2.7; and two generalizations of a theorem of Seshu and Balabanian [9; Theorem 6] concerning the zeros of differences of "positive real" functions, Theorems 2.3 and 2.4.

In section 2 we define the iterated "Richards' Transformation" for functions belonging to the class we define as positive real. If we apply this transformation to a given positive real function, we can construct four sequences of polynomials and two sequences of rational positive real functions. The bulk of our theorems are consequences of the intimate relations that exist between the given function and the polynomials obtained from this transformation.

In section 3 we extend the result of the previous sections to the infinite case and derive two new continued fraction expansions for positive real functions.

Our approach is modelled after the fundamental work of Schur [8] and Nevanlinna [6] for the related class of functions that are bounded in the unit circle.

We write $z = x + iy$, $f(z) = u(z) + iv(z)$, and when there is no possibility of ambiguity, f for $f(z)$, u for $u(z)$ and v for $v(z)$. It is also convenient to abbreviate the phrase, "open right half-plane," by the notation *ORHP*, by which we mean the set of complex numbers with real part positive. We shall have occasion to distinguish between points at which a function vanishes and points at which a function is analytic and vanishes. We reserve the name, "zeros of the function" for the latter case.

Definitions of positive real functions abound in the literature, varying from author to author and from paper to paper. For this reason, if for no other, we conclude the introduction with a definition of the class of functions we wish to call positive real and a statement of a few of their elementary properties. It

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should be observed that the definition we shall give is more general than the usual ones (for instance, Richards [7]). In particular, we do not restrict ourselves to rational functions or even to functions that are necessarily defined on the i-axis. Finally, note that the first part of Theorem 1.1 is not true under the more restrictive definition given by Richards [7].

Definition 1.1. *We say f is positive real and write f is PR if f satisfies the conditions:*

(i) *f is single-valued and analytic in ORHP,*

(ii) *u is positive for z in ORHP,*

(iii) *f is real for real, positive z.*

THEOREM 1.1. *If f and g are PR, then so are $1/f$ and $af + bg$ if $a, b > 0$.*

THEOREM 1.2. *The i-axis zeros and poles of PR functions are simple and the derivative at such zeros is positive.*

2. The Iterated Richards' Transformation. In [7; Theorem 6] Richards proved the following theorem which has become known as Richards' Transformation:

THEOREM 2.1. *If f is PR and if neither $af(z) - zf(a)$ nor $af(a) - zf(z)$ vanish identically for arbitrary choice of $a > 0$, then*

$$f^*(z) = \frac{af(z) - zf(a)}{af(a) - zf(z)} \tag{2.1}$$

is PR.

This theorem has found important applications in certain synthesis methods (see, for instance, Bott and Duffin [1]). We are lead naturally, then, to consider the consequences of repeated applications of this algorithm. Two alternatives present themselves. For any given function, it may happen that after some finite number of iterations of (2.1) we obtain a function of the form Az or A/z, in which case we can proceed no further. Or alternately, this never happens and we can iterate indefinitely. Thus, associated with each PR function and each set of positive constants $\{a_i\}$, is a sequence of positive real functions (possibly finite) obtained by repeated application of (2.1). We collect these comments in

Definition 2.1. *Suppose f is PR and $\{a_i\}$ is a sequence of positive constants. Set $f_1 = f$ and define $f_{n+1}(z)$ by*

$$f_{n+1}(z) = \frac{a_n f_n(z) - zA_n}{a_n A_n - zf_n(z)} \tag{2.2}$$

where $f_n(a_n) = A_n$ and $n \geqq 1$. The sequence of functions so defined is called the associated sequence of f at $\{a_i\}$, and f_n is called the n^{th} associated function. The associated sequence is terminated at f_n if either $a_n f_n(z) - A_n z$ or $a_n A_n - zf_n(z)$ is identically zero.

We shall often omit reference to the constants $\{a_i\}$ when there is no likelihood of ambiguity. It is an immediate consequence of Theorem 2.1 that if the n^{th} associated function exists, it is PR. It is also evident that the n^{th} associated function may be expressed directly in terms of f and the constants $\{a_i\}_1^{n-1}$ and $\{A_i\}_1^{n-1}$. Towards this end we introduce four sequences of polynomials. These polynomials play a fundamental role in our analysis.

First we suppose given two sequences of arbitrary constants, $\{a_i\}_1^\infty$ and $\{A_i\}_1^\infty$, not necessarily related to any *PR* function, nor for that matter, necessarily positive. Set $P_0(z) = R_0(z) = 1$ and $Q_0(z) = S_0(z) = 0$. Then the relationships

$$\begin{aligned} P_n &= a_n P_{n-1} + zA_n S_{n-1}, \qquad & Q_n &= a_n Q_{n-1} + zA_n R_{n-1}, \\ R_n &= a_n A_n R_{n-1} + zQ_{n-1}, \qquad & S_n &= a_n A_n S_{n-1} + zP_{n-1} \end{aligned} \tag{2.3}$$

for $n \geqq 1$, define four sequences of polynomials with $P_n = P_n(z)$, $Q_n = Q_n(z)$, $R_n = R_n(z)$ and $S_n = S_n(z)$. If the constants $\{a_i\}$ are all positive and $A_i = f_i(a_i)$, where f is *PR* and $i = 1, 2, \cdots$, then we call the polynomials of (2.3) the *associated polynomials* of f at $\{a_i\}$. It is now a simple matter to prove by induction that if f_n is the n^{th} associated function of the *PR* function f and if P_{n-1}, Q_{n-1}, R_{n-1} and S_{n-1} are the $n - 1^{\text{st}}$ associated polynomials of f at $\{a_i\}_1^{n-1}$ then,

$$f_n = \frac{P_{n-1}f - Q_{n-1}}{R_{n-1} - S_{n-1}f}, \qquad n \geqq 1. \tag{2.4}$$

We reserve for a later section a more detailed study of these polynomials. At this point we collect a few simple properties that we shall find of immediate use.

The identity,

$$\begin{vmatrix} P_n & Q_n \\ R_n & S_n \end{vmatrix} = A_n(a_n^2 - z^2)\begin{vmatrix} P_{n-1} & Q_{n-1} \\ R_{n-1} & S_{n-1} \end{vmatrix}$$

follows from the defining equations, (2.3). This yields

$$P_n(z)R_n(z) - S_n(z)Q_n(z) = \textstyle\prod_{i=1}^n A_i(a_i^2 - z^2), \tag{2.5}$$

valid for all $n \geqq 1$.

LEMMA 2.1. *For any complex constant, θ, and any $n \geqq 1$, the polynomials $\theta P_n(z) - Q_n(z)$ and $R_n(z) - \theta S_n(z)$ can vanish simultaneously only if $z = \pm a_i$ $(i = 1, 2, \cdots, n)$.*

PROOF. Suppose z_0 is a zero of $\theta P_n - Q_n$ and $\theta \neq 0$. Then the left-hand side of (2.5) may be written as $P_n(z_0)[R_n(z_0) - \theta S_n(z_0)]$. But if z_0 is also a zero of $R_n(z) - \theta S_n(z)$ then the right-hand side of (2.5) implies that $z_0 = \pm a_i$, for some i. A similar argument completes the proof in event $\theta = 0$.

We return our attention to the associated functions. If (2.4) is solved for f we obtain

$$f = \frac{Q_{n-1} + R_{n-1}f_n}{P_{n-1} + S_{n-1}f_n}. \tag{2.6}$$

Thus if f_n, $\{a_i\}_1^{n-1}$ and $\{A_i\}_1^{n-1}$ are given, then the associated polynomials may be computed and used in (2.6) to determine f. In fact we can prove much more.

THEOREM 2.2. *If g is PR and $\{a_i\}_1^{n-1}$ and $\{A_i\}_1^{n-1}$ are arbitrary positive constants, then there exists a PR function, f, such that $f_n = g$ and $f_k(a_k) = A_k$ for each $k = 1, 2, \cdots, n$.*

PROOF. Consider the function f_{n-1} defined by

$$f_{n-1}(z) = A_{n-1}\frac{z + a_{n-1}\,g(z)}{a_{n-1} + zg(z)}$$
$$= A_{n-1}[(g(z) + a_{n-1}/z)^{-1} + a_{n-1}(z + a_{n-1}/g(z))^{-1}].$$

It is PR by Theorem 1.1. If we write $g(z)$ in terms of $f_{n-1}(z)$ we see that $g(z)$ is the second associated function of $f_{n-1}(z)$ at a_{n-1} and $f_{n-1}(a_{n-1}) = A_{n-1}$. Put another way, if f_{n-1} is thought of as a term in the associated sequence of some function, then $g(z)$ is the next term in the sequence. If we repeat this process using f_{n-1} in the place of g and defining the resulting PR function as f_{n-2}, we see that $f_{n-2}(a_{n-2}) = A_{n-2}$. It is now easy to prove by induction that the function f_1, obtained after $n - 1$ steps, is the desired PR function.

It is instructive to observe that f could have been computed directly by (2.6) using g in the place of f_{n-1}. In fact the proof given in the previous paragraph may be thought of as another method of deriving (2.6). If we had used (2.6) to compute f, however, we would still need a proof that the function so obtained was PR.

We have g at our disposal; two special choices yield

COROLLARY 2.1. *Suppose P_n, Q_n, R_n, S_n are defined by* (2.3), *where* $\{a_i\}_1^n$ *and* $\{A_i\}_1^n$ *are arbitrary positive constants. Set*

$$E(z; n) = Q_n(z)/P_n(z), \qquad H(z; n) = R_n(z)/S_n(z). \tag{2.7}$$

Then $E(z; n)$ and $H(z; n)$ are PR and their i^{th} associated functions, $E_i(z; n)$ and $H_i(z; n)$, respectively, have the property that $E_i(a_i; n) = H_i(a_i; n) = A_i$ for $i = 1, 2, \cdots, n$.

PROOF. The choice $g(z) = A_n z/a_n$ in Theorem 2.2 implies the existence of a PR function, call it $E(z; n)$, with the property that $E_n(z; n) = A_n z/a_n$. Now (2.6) combined with (2.3) proves the first of the equalities in (2.7). The choice, $g(z) = a_n A_n/z$ proves the second.

This corollary has many ramifications which we proceed to explore. First of all we may resolve the issue of when the associated sequence terminates. Clearly, if $f(z) = E(z; n)$ or $H(z; n)$ then f_n terminates the associated sequence. Conversely, if the sequence terminates, then $f = Bz$ or C/z $(B, C > 0)$ for some n. In this event we have $f_n(a_n) = A_n = Ba_n$ which implies that $B = A_n/a_n$ and $A_n = C/a_n$ which implies that $C = a_n A_n$. Now Corollary 2.1 and (2.6) shows that either $f = E(z; n)$ or $f = H(z; n)$. One last remark on this matter is pertinent. If f is PR and has the additional property that $f(-z) = -f(z)$ with no singularities other than poles on the i-axis, we say that f is IPR. Call the degree of a rational function the maximum of the degrees of the numerator and denominator polynomials. Richards [7; Corollary 6.1 (*b*)] has shown that an application of (2.1) to a rational IPR function reduces the degree by one. Noting that E and H are rational IPR functions we may summarize our comments by stating that a necessary and sufficient condition that an associated sequence terminates after n terms is that f is a rational IPR function of degree n.

In [2] Brune showed that $N(z) = f(z) - zL$ $(L > 0)$ can have no multiple zeros in *ORHP*. Seshu and Balabanian [9] showed that if N is rational then it could have at most one zero in *ORHP*, and this zero must be real. If $z = a$ is a zero of N, then $L = f(a)/a$ and N is reminiscent of the numerator of (2.1); it is just this similarity that we exploit to prove some extensive generalizations of this result. We note for emphasis that no assumption of rationality is involved in the hypothesis of the following theorems. Hence, even in the case $n = 1$, these theorems generalize the results of Brune, and Seshu and Balabanian.

THEOREM 2.3. *If f is PR and f_{n+1} exists and if $E(z; n)$ and $H(z; n)$ are constructed as in* (2.7) *with $A_i = f_i(a_i)$ for $i = 1, 2, \cdots, n$, then $f - E$ and $f - H$ have only $\{a_i\}_1^n$ as zeros in ORHP. Duplication of values among the a_i's are to be counted as multiple zeros according to the number of repetitions.*

PROOF. Since $P_n(f - E) = P_n f - Q_n$ and $-S_n(f - H) = R_n - S_n f$, the zeros of $f - E$ and $f - H$ are counted among the zeros of $P_n f - Q_n$ and $R_n - S_n f$, respectively. By hypothesis f_{n+1} is *PR* and by (2.1) and property (ii) of Definition 1.1 we see that $P_n f - Q_n$ and $R_n - S_n f$ must vanish simultaneously, if at all, for z in *ORHP*. But Lemma 2.1 restricts such zeros to the set $\{a_i\}_1^n$. It now remains to show that $f - E$ and $f - H$ do indeed vanish at $\{a_i\}_1^n$. To this end we first show that S_n/P_n is *PR*. We do this by induction. It is apparent that S_1/P_1 is *PR*. Suppose as the induction hypothesis that S_i/P_i is *PR*. Then P_i/S_i is *PR*. We use (2.3) to write

$$S_{i+1}/P_{i+1} = \frac{a_{i+1} A_{i+1}}{a_{i+1} P_i/S_i + A_{i+1} z} + \frac{1}{a_{i+1}/z + A_{i+1} S_i/P_i}$$

and conclude from Theorem 1.1 that both S_{i+1}/P_{i+1} and P_{i+1}/S_{i+1} are *PR*. This completes the induction. Now consider $f_{n+1} + P_n/S_n$. It is *PR* because it is the sum of *PR* functions. But from (2.1) and (2.5) we find

$$f_{n+1} + P_n/S_n = \frac{P_n R_n - S_n Q_n}{R_n - S_n f} = \frac{\prod_{i=1}^n A_i(a_i^2 - z^2)}{R_n - S_n f}.$$

Since the left-hand side is *PR* it cannot vanish in *ORHP*. Thus a_i is a zero of $R_n - S_n f$ and hence of $P_n f - Q_n$ also. Finally, since P_n and S_n take positive values when z is real and positive, $f - E$ and $f - H$ have zeros at $\{a_i\}_1^n$ counting multiplicities.

THEOREM 2.4. *Under the hypothesis of Theorem* 2.3, *$f - E$ and $f - H$ can have only simple zeros on the i-axis. Furthermore, if $z = iy_0$ is a zero of $f - E$ and $z = iy_1$ is a zero of $f - H$, then*

$$f'(iy_0) > E'(iy_0; n) \quad \text{and} \quad f'(iy_1) < H'(iy_1; n).$$

PROOF. Since f_{n+1} is *PR* it has only simple zeros and poles on the i-axis from Theorem 1.2. Thus any multiple zero of either $P_n f - Q_n$ or $R_n - S_n f$ must be a zero of the other. But Lemma 2.1 prohibits such simultaneous zeros on the i-axis, and so prohibits multiple zeros of $f - E$ and $f - H$.

The stated inequalities follow from the last part of Theorem 1.2 applied to the PR functions, f_{n+1} and $1/f_{n+1}$. For,

$$0 < f'_{n+1}(iy_0) = \frac{P_n(iy_0)}{S_n(iy_0)} \frac{f'(iy_0) - E'(iy_0\,; n)}{H(iy_0\,; n) - f(iy_0)}$$
$$= P_n^2(iy_0) \frac{f'(iy_0) - E'(iy_0\,; n)}{P_n R_n - S_n Q_n},$$

because we have assumed that $f(iy_0) = E(iy_0\,; n)$. From (2.5) we see that the denominator is positive; from the fact that P_n is a polynomial in z^2 we see that $P_n^2(iy_0)$ is positive. This establishes the first of the inequalities. The second inequality is established in the same manner using $1/f_{n+1}$ in place of f_{n+1} and iy_1 in place of iy_0.

These last two theorems may be interpreted as restrictions on the values that can be assumed by PR functions. In Theorem 2.6 (below) we exploit this restriction to prove an interpolation theorem. At present we consider the consequences of $f(z) - E(z; n)$ small at $z = z_0$ with $z_0 \neq a_i$, $1 \leqq i \leqq n$. We prove that this assumption forces $f(z) - E(z;n)$ small throughout $ORHP$. Preliminary to this we need a lemma.

LEMMA 2.2. *If* $\{f_i\}$ *is a sequence of PR functions (not necessarily an associated sequence) which assumes the same positive value at one point in ORHP then either*

(i) $\lim_{i\to\infty} f_i$ *exists and is a PR function, or*

(ii) *there exists two subsequences of* $\{f_i\}$ *converging to two distinct PR functions. The convergence is uniform in any compact subset of ORHP.*

PROOF. In [4; Theorem 3] Goldberg proved an inequality for a class of functions containing the class of PR functions. If z and z_0 are two points in $ORHP$ and f is PR then

$$|f(z)| \leqq |f(z_0)| \frac{r^2 + r_0^2 + 3r_0 r}{x_0\, x}$$

where $r = |z|$. If z_0 is the point where each of the functions f_i takes on the same positive value then it is apparent that the sequence $\{f_i\}$ is bounded independent of f_i, z and z_0 as long as z and z_0 are restricted to $ORHP$. Thus $\{f_i\}$ is a normal sequence (Nehari [5; pp. 140–143]) and either (i) or (ii) hold with "PR" replace by "analytic" in the conclusion. The limit functions take real values when z is real and positive because each f_i does. None of the limit functions can be the identically zero function because we have assumed that $f_i(z_0)$ is a positive constant for each i. Finally, if there were any point in $ORHP$ where any of the possible limit functions had a zero real part, there would be a neighboring point in $ORHP$ where the function would have a negative real part. This is impossible and hence all functions in the conclusion are PR.

We use this lemma to prove two theorems, Theorem 2.5 and Theorem 3.1. In both instances the point of the proof is to eliminate the second alternative in the lemma by introducing an appropriate hypothesis.

THEOREM 2.5. *Suppose* $\{f_i\}$ *is an arbitrary sequence of PR functions with the property that for each* $k = 1, 2, \cdots, n$, $f_i(a_k) = c_k > 0$ *where* $a_k > 0$. *Suppose*

$E(z; n)$ is constructed from the constants $\{a_k\}_1^n$ and $\{c_k\}_1^n$ with c_k playing the role of A_k. Set $\lim_{i\to\infty} z_i = z_0 \neq a_k$ for each $k(x_0 > 0)$. If $\lim_{i\to\infty} f_i(z_i) = E(z_0 ; n)$ then $\lim_{i\to\infty} f_i(z) = E(z; n)$ uniformly in any compact subset of ORHP.

PROOF. The lemma applies and either (i) or (ii) holds. Assume (ii) holds. Then there exist two PR functions agreeing with $E(z; n)$ at the $n + 1$ points, $a_1, a_2, \cdots, a_n$ and z_0, counting multiplicities. This is in direct contradiction to Theorem 2.3. Thus not only does (i) hold but $\lim_{i\to\infty} f_i = E(z; n)$.

It is clear that $E(z; n)$ may be replaced by $H(z; n)$ and a similar theorem would result.

Now consider two PR functions f and g. Suppose $f_i(a_i) = g_i(a_i)$, where the subscript identifies the i^{th} associated function and $1 \leqq i \leqq n$ and $a_i > 0$. Since $E(z; n)$ depends only on $\{a_i\}_1^n$ and $\{A_i\}_1^n$ we see at once that $E(z; n)$ is the same for both f and g. Therefore, by Theorem 2.3, $f(a_i) = g(a_i)$ for each i, $1 \leqq i \leqq n$. The converse is also true. It is trivial that $f_1(a_1) = g_1(a_1)$ for $f(a_1) = g(a_1)$. Assume that the converse is true for $i = k$ and use (2.4) to obtain

$$f_{k+1}(z) - g_{k+1}(z) = \frac{(f - g)(P_k R_k - S_k Q_k)}{(R_k - S_k f)(R_k - S_k g)}.$$

A simple count of the order of the zero of the numerator at $z = a_{k+1}$ compared to the order of this zero of the denominator shows that the left-hand side vanishes at $z = a_{k+1}$. These comments prove,

THEOREM 2.6. *If f and g are two PR functions whose n^{th} associated functions exist, then a necessary and sufficient condition that $f(a_i) = g(a_i)$ is that $f_i(a_i) = g_i(a_i)$, where $1 \leqq i \leqq n$.*

We are now in a position to investigate the possibility of constructing PR functions taking prescribed values at prescribed points. In preparation for this we introduce a few definitions for the purpose of simplifying the statement of the results. First we define a counting function for the sequence $\{a_i\}$. Let $n(i)$ be the number of a_k's having the same value, a_i, $k < i$. Set $n(1) = 0$. Thus $n(i) = 0$ if all the a_i's are distinct and $n(i) = i - 1$ if all the a_i's are equal.

Definition 2.2. *The PR function f is called an interpolating function for $\{w_i\}_1^n$ at $\{a_i\}_1^n (a_i > 0)$ if*

$$f^{(n)}(a_i) = w_i, \qquad i = 1, 2, \cdots, n. \tag{2.8}$$

Here, and in the following, $n = n(i)$ is the n^{th} derivative.

We now construct an auxillary sequence of constants $\{B_i\}_1^n$ from the sequences $\{A_i\}_1^n$ and $\{a_i\}_1^n$, by the following recursive device:

(i) $w_1 = B_1$.

(ii) Suppose $\{B_i\}_1^k$ are defined. Set $A_i = B_i$, $i = 1, 2, \cdots, k$ and construct P_k, Q_k, R_k and S_k from (2.3). Set $E(z; k) = Q_k(z)/P_k(z)$ and $H(z; k) = R_k(z)/S_k(z)$, as usual. Now define B_{k+1} by

$$B_{k+1} = \frac{P_k(a_{k+1})}{S_k(a_{k+1})} \cdot \frac{w_{k+1} - E^{(n)}(a_{k+1} ; k)}{H^{(n)}(a_{k+1} ; k) - w_{k+1}}, \tag{2.9}$$

where $n = n(k + 1)$.

THEOREM 2.7. *A necessary and sufficient condition that there exist an interpolating function f for $\{w_i\}_1^n$ at $\{a_i\}_1^n$ with the property that f_{n+1} exists, is that* $0 < B_i < \infty$ *for* $1 \leqq i \leqq n$.

PROOF. Assume that f is an interpolating function and that f_{n+1} exists. Then, using (2.4) to compute $f_i(a_i) = A_i$, we see that (2.9) is, in fact, the definition of A_i, which is always positive and finite. The proof of the sufficiency consists in first observing that $E(z; n)$ and $H(z; n)$ as defined in (ii) above are PR from Corollary 2.1, since we are assuming that each B_i is finite and positive. Moreover,

$$E_{k+1}(z; n) = \frac{P_k(z)}{Q_k(z)} \frac{E(z; n) - E(z; k)}{H(z; k) - E(z; n)} \tag{2.10}$$

from (2.4) and (2.7). If $z = a_{k+1}$, then $E_{k+1}(a_{k+1}; n) = B_{k+1}$ and (2.10) reduces to

$$B_{k+1} = \frac{P_k(a_{k+1})}{Q_k(a_{k+1})} \cdot \frac{E^{(n)}(a_{k+1}; n) - E^{(n)}(a_{k+1}; k)}{H^{(n)}(a_{k+1}; k) - E^{(n)}(a_{k+1}; n)}.$$

But this last equation combined with the definition of B_{k+1} in (2.9) yields $E(a_i; k) = w_i$ for $i = 1, 2, \cdots, n$, which was to be proved.

It is instructive to consider an example. Construct, if possible, a PR function taking the values 3, 0, 3 at 1, 1, 3. We have, $n(1) = 0$, $n(2) = 1$, $n(3) = 0$, so that f must satisfy $f(1) = 3$, $f'(1) = 0$ and $f(2) = 3$. We find $B_1 = 3$, $B_2 = 1$ and $B_3 = 1$, so that an interpolating function does exist. We construct one such interpolating function by setting $A_i = B_i$ $(i = 1, 2, 3)$ and $a_1 = 1$, $a_2 = 1$, $a_3 = 3$ and forming $E(z; 3) = 3z(5 + z^2)(4z^2 + 2)$. A simple computation shows that this PR function is indeed an interpolating function as claimed. It should be noted that, in general, interpolating functions are not unique on finite sets. In the next section, it will be seen that the extension of Theorem 2.7 to the infinite set $\{w_i\}_1^\infty$ will lead to unique PR interpolating functions. The exceptional cases of unique interpolating functions on the set $\{w_i\}_1^n$ occur only when, for some $k \leqq n + 1$, f_k does not exist. In this event it can be shown that $B_k = 0$ or ∞ and that if an interpolating function exists it will be either Q_k/P_k or R_k/S_k. These cases are expressly prohibited in Theorem 2.7 by the hypothesis that f_{n+1} exists.

3. The Sequence, $E(z; n)$, $E(z; 2)$, $\cdots$. Throughout this section we shall assume that $\{a_i\}$ is an infinite sequence of positive constants with one limit point in *ORHP* and no others. We shall also assume that $\{A_i\}$ is a sequence of positive constants not related to any PR function unless explicitly stated otherwise. We write $\lim_{i\to\infty} a_i = a_0$.

THEOREM 3.1. *There exists a unique PR function f such that $f_i(a_i) = A_i$ for all $i \geqq 1$. Here f_i is the i^{th} associated function of f.*

PROOF. From the sequences $\{a_i\}$ and $\{A_i\}$ we construct the functions $\{E(z; n)\}$ in the usual manner. We apply Theorem 2.3 to the PR function $E(z; n)$ to conclude that $E(z; n) = E(z; k)$ for each k, $1 \leqq k \leqq n$, at the points $a_1, a_2, \cdots, a_k$.

Put in somewhat different form, $E(a_i\,;\,i) = E(a_i\,;\,k)$ for each i and every $k \geqq i$. Thus the sequence $\{E(z;\,n)\}$ satisfies the hypothesis of Lemma 2.2 and either alternative (i) or (ii) holds. We shall show that alternative (ii) cannot hold. Suppose it did. If there exists an infinite subsequence of $\{a_i\}$ with distinct values then the two distinct PR functions would agree on this infinite set. But this infinite set has limit point in $ORHP$ because the larger set $\{a_i\}$ has. Hence Vitali's theorem would imply that these two functions were identical. We are left with the case that (ii) of Lemma 2.2 holds and after some N all $a_n = a_0$ with $n > N$. Theorem 2.3 implies that repeated values among the a_i are to be counted as multiple zeros to the order of their repetitions among the preceding a_i. This can be interpreted as saying that the PR functions arising in (ii) have the same Taylor series about a_0. Hence alternative (i) holds. The same reasoning with the sequence $\{E_k(z;\,n)\}_{n=1}^{\infty}$ for each k shows that $\lim_{n\to\infty} E_k(z;\,n)$ exists and is PR. Let $f(z) = \lim_{n\to\infty} E(z;n)$, then

$$f_{k+1}(z) = \frac{P_k(z)}{S_k(z)}\frac{f(z) - E(z;k)}{H(z;k) - f(z)}$$

for each k. Also

$$\lim_{n\to\infty} E_{k+1}(z;n) = \frac{P_k(z)}{S_k(z)}\frac{\lim_{n\to\infty} E(z;n) - E(z;k)}{H(z;k) - \lim_{n\to\infty} E(z;n)}$$

Hence the i^{th} associated function of the limit of $\{E(z;\,n)\}$ is the limit of the sequence $\{E_i(z;\,n)\}$. Therefore $f_i(a_i) = \lim_{n\to\infty} E_i(a_i\,;\,n) = E_i(a_i\,;\,i) = A_i$. This completes the proof.

If f is given then the constants $\{A_i\}$ may be picked as $f_i(a_i) = A_i$. In this event the uniqueness of the limit function guaranties that $\lim_{n\to\infty} E(z;n) = f(z)$. It is easily seen that $H(z;\,n)$ could have been used in place of $E(z;\,n)$ in this construction. We have proved

COROLLARY 3.1. *If f is a PR function with non-terminating associated sequence, then*

$$\lim_{n\to\infty} E(z;\,n) = \lim_{n\to\infty} H(z;\,n) = f(z)$$

uniformly in any compact subset of ORHP.

We may cast Theorem 3.1 and Corollary 3.1 in another form which has some independent interest.

COROLLARY 3.2. *Under the hypothesis of Theorem* 3.1 *the continued fractions*

$$\frac{1}{z}\left[a_1 A_1 + A_1 \frac{z^2 - a_1^2}{a_1 + a_2 A_2 +} A_2 \frac{z^2 - a^2}{a_2 + a_3 A_3 +} \cdots\right] \tag{3.1}$$

$$\frac{A_1}{a_1}\left[z + \frac{a_1^2 - z^2}{z} + \frac{1}{A_1}\frac{a_1 a_2}{z} + \frac{a_2^2 - z^2}{z} + \frac{1}{A_2}\frac{a_2 a_3}{z} + \cdots\right] \tag{3.2}$$

converge uniformly to a PR function f, in any compact subset of ORHP. For each i, with f_i as the i^{th} associated function of f we have $f_i(a_i) = A_i$. If, on the other hand, f is PR and has a non-terminating associated sequence then (3.1) *and* (3.2) *are continued fraction representations for f in ORHP.*

PROOF. The convergents of (3.1) and (3.2) are easily seen to be $H(z; n)$ and $E(z; n)$ respectively when we recall from the proof of Corollary 2.1 that $E_n(z; n) = A_n z/a_n$ and that $H_n(z; n) = a_n A_n/z$. For,

$$E_{k-1}(z; n) = A_{k-1}\left[\frac{z}{a_{k-1}} + \frac{a_{k-1} - z^2/a_{k-1}}{z + a_{k-1}/E_k(z; n)}\right]$$

and

$$H_{k-1}(z; n) = A_{k-1}\left[\frac{a_{k-1}}{z} + \frac{z - a_{k-1}^2/z}{a_{k-1} + zH_k(z; n)}\right].$$

As a final corollary to Theorem 3.1 we generalize Theorem 2.7.

COROLLARY 3.3. *A necessary and sufficient condition that there exist an interpolating function for* $\{w_i\}_1^\infty$ *at* $\{a_i\}_1^\infty$ *with a non-terminating associated sequence is that* $0 < B_i < \infty$ *for all* i. *Here* $\{B_i\}_1^\infty$ *is constructed as in Theorem 2.7 and the interpolating function is unique.*

PROOF. Immediate from Theorems 2.7 and 3.1.

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