BOUNDS OF THE LEGENDRE FUNCTION

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When one attempts to use inequalities involving Legendre functions in analyses involved in scattering problems, one finds that most of the existent upper bounds are functions of both the degree and the argument of the Legendre function. This paper exhibits inequalities which are functions of only the degree of the Legendre function.

It is well known that when n is an integer

$$P_n(\cos \theta) \leqq 1 \tag{1}$$ (Ref. 1, p. 52)

One can obtain this result directly from the Leplace Integral $n > 0$

$$P_n(\cos \theta) = (1/\pi) \int_0^\pi (\cos \theta + \sqrt{-1} \sin \theta \cos \phi)^n \, d\phi \tag{2}$$

$$P_n(\cos \theta) \leqq (1/\pi) \int_0^\pi d\phi = 1, \qquad \text{q.e.d.} \tag{3}$$

One may use the bounds for the Legendre polynomials to obtain the bounds on their derivatives.

This can be done upon application of the following recursion formula,

$$dP_{n+1}(x)/dx - dP_{n-1}(x)/dx = (2n + 1)P_n(x), \tag{4}$$

which is readily obtained from the recursion formulas appearing on pages 53 and 54 of Ref. 1.

The form of (4) was chosen so that the bound obtained depends only on n and not on x.

By application of (1) with (4), one obtains*

$$dP_n(x)/dx \leqq \tfrac{1}{2}n(n + 1) \tag{5}$$

When n is not an integer the problem becomes more complex.

Let us consider the definition of P_{n_i}, $i = 0, 1, 2, 3, \cdots$,

$$P_{n_i}(x) = {}_2F_1(-n_i, n_i + 1; 1; \tfrac{1}{2}(1 - x)). \tag{6}$$

When x is equal to -1 and the n_i are not integers

$$P_{n_i}(-1) = {}_2F_1(-n_i, n_i + 1; 1; 1) \tag{7}$$

and it is well known that (7) is without bound. In fact the two singular points of $P_{n_i}(x)$ are at -1 and ∞.

By use of (4) (which applies whether or not n is an integer) we observe that the derivative of the Legendre function of non-integral degree also has a singular point at $x = -1$.

* The proof is by induction on n. Prof. R. V. Churchill obtains the same result by a different method [see p. 190 of Ref. 6].

When one attempts to find the field for the scattering of either sound or electromagnetic waves from one or more cones one may have to satisfy the following type boundary conditions:

$$P_{n_i}(x) \mid_{x=x_0} = 0, \tag{8}$$

$$dP_{n_i}(x)/dx \mid_{x=x_0} = 0, \tag{9}$$

or combinations of the two. In this case the field can sometimes be expressed in a form

$$\Phi(r, x) = \textstyle\sum_i A_i f(n_i, r) P_{n_i}(x). \tag{10}$$

Usually the A_i do not dominate (10) to such an extent that convergence is forced. In most cases when (10) appears the $f(n_i, r)$ are in the form of spherical Bessel functions and these functions force convergence *if the* $P_{n_i}(x)$ *are bounded.* The purpose of this paper is to show that bounds exist for the $P_{n_i}(x)$ under the condition (8) or (9).

This problem is of physical interest for the following reason. Consider an observer located very close to the cone and then consider making the cone smaller and smaller with the observer remaining close to the cone. Does the field seem to increase without limit? A priori, mathematically, one might expect this since the n_i are not integers; and if one takes the limit as x approaches -1, the Legendre functions increase without limit, that is

$$\lim_{x \to -1} P_{n_i}(x) = \infty. \tag{11}$$

The above is without physical sense because the fields do not increase without limit, and it will be shown that the above conclusion does not make mathematical sense either.

To show this let us move the observer right to the surface of the cone; then we must show that

$$\lim_{x_0 \to -1} P_{n_i}(x_0) \neq \infty. \tag{12}$$

Since the boundary conditions (8) or (9) make the x_0 a function of n_i and vice versa, then to take the limit described in (12) one must take a limit in degree and argument simultaneously, i.e.,

$$\lim_{x_0 \to -1} P_{g(x_0)}(x_0).$$

In order to show that (12) is a correct inequality it is only necessary to show that, under boundary condition (8) or (9),

$$\lim_{x_0 \to -1} g(x_0)$$

is equal to an integer. Then the upper bound to the absolute value of the Legendre function in the limit ($x_0 = -1$) would be unity.

Consider the recursion formula (Ref. 2, formula 6)

$$P_n^m(-x) = \cos[(n + m)\pi] P_n^m(x) - (2/\pi)\sin[(n + m)\pi] Q_n^m(x). \tag{13}$$

Letting $m = 1$ and replacing the associated Legendre function by its definition in terms of the derivative of the ordinary Legendre function and then dividing through by $\sqrt{1 - x^2}$ we obtain the following recursion formula:

$$(14) \qquad -\frac{dP_n(-x)}{dx} = -\cos(n\pi)\frac{dP_n(x)}{dx} + \frac{2}{\pi}\sin(n\pi)\frac{dQ_n(x)}{dx}.$$

Now letting $x = -x_0$, $n = n_i$ and considering boundary condition (9) we obtain

$$(15) \qquad \tan(n_i \pi) = \frac{\pi}{2}\cdot\frac{dP_{n_i}(x)/dx\,|_{x=-x_0}}{dQ_{n_i}(x)/dx\,|_{x=-x_0}}.$$

Now taking the limit as $x_0 \to -1$ and $n_i = g(x_0)$ we observe

$$(16) \qquad \lim_{x_0 \to -1} dQ_{g(x_0)}(x)/dx\,|_{x=-x_0} = \infty$$

since the derivative of the Legendre function of the second kind has singularities at $x = \pm 1$ independent of whether the n_i are integers or not.

Since the Legendre function of the first kind does not have a singularity when $x = 1$ it is clear by (9), (15) and (16) that

$$17) \qquad \lim_{x_0 \to -1} \tan(n_i \pi) = 0$$

Thus the n_i must become integers as $x_0 \to -1$.

Since

$$\lim_{x_0 \to -1} P_{n_i}(x_0) = P_i(-1) = (-1)^i$$

we have shown that (12) is true and that (11) does not make mathematical sense if boundary condition (9) is applied simultaneously.

The same proof as the above applies for boundary condition (8) except that one lets $m = 0$ in equation (13). In that case a jump occurs as one goes from cone to no cone, that is

$$(18) \qquad |P_{n_i}(x_0)| = \delta_{n_i, i} = \delta_{\mu_0, -1}.$$

Under boundary condition (9) (which determines the n_i) the value of $P_{n_i}(x_0)$ for $n_i \to 0$ will be a relative maximum or minimum. This is easily shown because the derivative at the point x_0 is equal to zero and the second derivative is not equal to zero. [The first derivative is equal to zero by the boundary condition. The second derivative is not equal to zero by an indirect proof, which follows:

If the second derivative were equal to zero ($x_0 \neq \pm 1$) then by the recursion formula the function and all its derivatives would be equal to zero. Since we are dealing with an analytic function, then the function is identically zero, but this latter statement is known to be false; thus the second derivative of the function is not zero.]

In the region where $0 \leqq x \leqq 1$, one may use the following equation (Ref. 1, p. 67)*

* The region of validity of (19) is $0 \leqq \theta \leqq \pi/2$ rather than $0 \leqq \theta < \pi$. See discussion (Ref. 3, p. 313).

$$\cos(\mu\pi)P_{n_i}^{\mu}(\cos\theta) - (2/\pi)\sin(\mu\pi)Q_{n_i}^{\mu}(\cos\theta) = \frac{\Gamma(\mu+n_i+1)(\sin\theta)^{\mu}2^{-\mu}}{\Gamma(n_i-\mu+1)\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})}\int_0^{\pi}\frac{\sin^{2\mu}\psi}{B^{\mu-n_i}}\,d\psi \tag{19}$$

where $B = (\cos\theta \pm \sqrt{-1}\sin\theta\cos\psi)$, $\operatorname{Re}(\mu+\frac{1}{2}) > 0$, $0 \leqq \theta \leqq \pi/2$. Let $\mu = 0$ in the above formula:

$$P_{n_i}(\cos\theta) = (1/\pi)\int_0^{\pi}(\cos\theta \pm \sqrt{-1}\sin\theta\cos\psi)^{n_i}\,d\psi, \qquad 0 < \theta < \pi/2. \tag{20}$$

Therefore

$$P_{n_i}(x) \leqq 1. \tag{21}$$

The point $x = 0$ can be handled separately.

Bounds on $P_{n_i}(x)$, non-integer n_i, $x_0 \leqq x \leqq 1$.

The first value of n_i for which the Neumann condition (9) holds is $n_0 = 0$; the second is for $n_i > 1$. None of the $n_i (i > 0)$ is integral (Ref. 4 & 5) if the cone does not have either a 0°, 90°, or 180° half cone angle (all of which are not really cones). Since $P_0(x) = 1$, we will only consider $n_i > 1$: By applying the formulas

$$z\Gamma(z) = \Gamma(z+1) \tag{22}$$

$$P_{n_i}(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}n_i+1)\Gamma(-\frac{1}{2}n_i+\frac{1}{2})} \qquad \text{(Ref. 1, p. 63)} \tag{23}$$

we obtain

$$P_{n_i}(0) = \frac{\sqrt{\pi}\left(\frac{n_i}{2}+1\right)}{\Gamma\left(\frac{n_i}{2}+2\right)\Gamma\left(\frac{-n_i}{2}+\frac{1}{2}\right)}; \tag{24}$$

but since

$$\frac{\cos(n_i\pi/2)}{\pi} = \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}n_i)\Gamma(\frac{1}{2}-\frac{1}{2}n_i)} \qquad \text{(Ref. 1, p. 1)},$$

$$P_{n_i}(0) < \frac{1}{\sqrt{\pi}}\left|\cos n_i\frac{\pi}{2}\right|\left(\frac{n_i+2}{n_i+1}\right). \tag{25}$$

By applying the Neumann condition, one finds $n_i = 2i$

$$P2_i(0) = \frac{(-1)^i(2i)!}{2^{2i}i!i!} \leqq \frac{1}{2} \tag{26}$$

Thus the only bound in the range of physical interest for $P_{n_i}(x)$ which remains to be found is for $x_0 \leqq x < 0$.

A very general bound (which does not use the boundary condition) for this region is (Ref. 1, p. 67)

(27) $$P_{n_i}(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos[(n_i+\frac{1}{2})\phi]}{(\cos\phi-\cos\theta)^{\frac{1}{2}}}\,d\phi$$

$$P_{n_i}(\cos\theta) \leqq \frac{\sqrt{2}}{\pi}\left|\int_0^\theta \frac{d\phi}{(\cos\phi-\cos\theta)^{\frac{1}{2}}}\right|$$

which states that

(28) $$P_{n_i}(\cos\theta) \leqq |P_{-\frac{1}{2}}(\cos\theta)| \leqq (2/\pi)\,|K|,$$

where K is the complete elliptic integral of modulus $\sqrt{\frac{1}{2}(1-x)}$.

When $0 < x \leqq 1$ the derivative of the Legendre function of non-integral degree has a bound similar to the derivative of the polynomial.

Using equation (19) and letting $\mu = 1$,

$$P'_{n_i}(\cos\theta) = \frac{-n_i(n_i+1)\sin\theta}{\pi}\int_0^\pi \frac{\sin^2\psi\,d\psi}{(\cos\theta \pm \sqrt{-1}\sin\theta\cos\psi)}\,1-n_i$$

(29) $$\frac{dP_{n_i}(\cos\theta)}{d(\cos\theta)} = \frac{n_i(n_i+1)}{\pi}\int_0^\pi \frac{\sin^2\psi\,d\psi}{(\cos\theta \pm \sqrt{-1}\sin\theta\cos\psi)}\,1-n_i$$

$$|dP_{n_i}(x)/dx| \leqq \tfrac{1}{2}n_i(n_i+1).$$

When $x = 0$ (Ref. 1, p. 64)

$$\frac{dP_{n_i}(0)}{dx} = \frac{2\sin(\pi n_i/2)\Gamma(\frac{1}{2}n_i+1)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}n_i+\frac{1}{2})}$$

(30) $$\frac{dP_{n_i}(0)}{dx} = \frac{n_i}{\sqrt{\pi}}\sin\left(\frac{n_i\pi}{2}\right)\frac{\Gamma(\frac{1}{2}n_i)}{\Gamma(\frac{1}{2}n_i+\frac{1}{2})}, \qquad \frac{dP_{n_i}(0)}{dx} \leqq \frac{n_i+1}{\sqrt{\pi}}\left|\sin\left(\frac{n_i\pi}{2}\right)\right|.$$

When one has the Neumann boundary conditions, not only is $|P_{n_i}(x_0)|$ a relative maximum with respect to x, but also it can be shown by application of the recursion formulas that

$$P_{n_i-1}(x_0) = P_{n_i+1}(x_0) = x_0 P_{n_i}(x_0).$$

Therefore

(31) $$|P_{n_i}(x_0)| > P_{n_i+1}(x_0).$$

Rewriting equation (14) with x replaced by $-x$, we obtain

(32) $$\frac{dP_{n_i}(x)}{dx} = -\frac{2}{\pi}\sin(n_i\pi)\frac{dQ_{n_i}(-x)}{dx} + \cos(n_i\pi)\frac{dP_{n_i}(-x)}{dx}.$$

Letting $x = x_0$ and applying the Neumann boundary condition and inequality (29) we obtain

(33) $$\frac{dQ_{n_i}(-x_0)}{dx} = \frac{\pi}{2}\cot(n_i\pi)\frac{dP_{n_i}(-x_0)}{dx} \leqq \frac{\pi}{2}\,|\cot(n_i\pi)|\,\frac{n_i(n_i+1)}{2}.$$

The usual bound on the Legendre Function good for all θ for $P_{n_i}(x)$ is

(34) $$P_{n_i}(\cos) < \frac{2}{\sqrt{n_i \pi \sin \theta}} \qquad \text{(Ref. 1, p. 71)}$$

Now applying the non-linear relationship

(35) $$P_{n_i}(x) \frac{dQ_{n_i}(x)}{dx} - Qn_i(x) \frac{dP_{n_i}(x)}{dx} = \frac{1}{1-x^2} \qquad \text{(Ref. 1, p. 63)}$$

and equation (13), we obtain

(36) $$P_{n_i}(-x) \frac{dP_{n_i}(-x)}{dx} - P_{n_i}(x) \frac{dP_{n_i}(-x)}{dx} = \frac{2}{\pi} \frac{\sin n_i \pi}{1-x^2}$$

Now applying the Neumann Condition at $x = x_0$ where $-1 < x_0 < 0$, we obtain

(37) $$P_{n_i}(x_0) = -\frac{2}{\pi} \frac{\sin n_i \pi}{(1-x_0^2) \left. \dfrac{dP_{n_i}(-x)}{dx} \right|_{x = x_0}}$$

We have already observed that as $x_0 \to -1$ (subject to the Neumann Condition which makes n_i a function of x_0), that the left-hand side approaches $(-1)^i$. The right-hand side becomes an indeterminate form. The

$$\left. \frac{dP_{n_i}(-x)}{dx} \right|_{x = x_0 \to -1} \quad \text{approaches} \; \frac{i(i+1)}{2}.$$

This latter function decreases with decreasing $| x_0 |$ becoming zero at $x_0 = 0$. The $\sin n_i\pi$ is zero at $x_0 = -1$ because there $n_i = i$ and at $x_0 = 0$, as there $n_i = 2i$. It's minimum at $n_1 = 1.5$ occurs for $x_0 = \cos 115°$.

The numerical values of $P_{n_1}(\cos \theta_0)$ are monotonically increasing from -1 at $\theta_0 = 180°$ to -0.5 at $\theta_0 = 90°$. In-between values are $P_{1.5}(\cos 115°) = -0.6$ and $P_{1.03}(\cos 165°) = -0.9$. The upper hound then is $| P_{n_i}(x_0) | \leqq 1$, when the function obeys the Neumann Condition. The author, despite much effort, has not been able to find a short proof of this statement. The original analysis involved the substitution of the series obtained from the Neumann Condition into the series for the Legendre function and bounds obtained from this series. This analysis seems too lengthy and uninspiring to present here.

It is also true that $| P_{n_1}(x_0) | \leqq 1$ implies that $| P_{n_i}(x_0)| \leqq 1$, $(i = 1, 2, \cdots)$. Again the author fails to find other than a lengthy series proof as no one has been able to find a recursion formula for the Legendre functions when the successive n_i differ by other than an integer.

The Legendre function inequalities obtained in this paper are in all cases stated in forms which are explicitly independent of their argument.

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