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Growth and Spacing
in an
Even-aged Stand of Douglas-fir

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Introduction

The successful practice of forestry is based on the growth of trees. In agriculture, profitable and successful husbandry is dependent on continuous, maximum crop yields from the land. Just so in forestry: a successful forest business requires a growing forest and the repeated harvest of the growth. Forest growth is imperfectly understood; yet with the practice of more and more intensive management, knowledge of that growth becomes increasingly important.

Much information exists on per-acre growth rates. Yield tables prepared for a great many forest types in the United States are the foremost example. Forest inventory growth data are almost universally expressed as so much per acre per year. Such information, though essential to forest management, really gives only a rudimentary description of growth. Rather than the final answer, as regards growth, it is only the beginning.

A recent and fruitful development in growth studies has been to regard the tree as the unit rather than the acre. Per-acre growth rates, if they are to be completely explained, require that the stand to which they apply be described in terms that are often discovered only after considerable research. For example, one stand grows twice as fast as another in terms of board feet per acre. Why? Assuming that the familiar variables of site and age are eliminated there comes the difficulty of describing such things as the stocking of the stand, diameter distribution, and stand uniformity in terms which will explain the difference in growth rates. Wide variations in mortality, particularly, make interpretation of per-acre growth rates difficult.

When the tree is regarded as the unit the object is to explain the growth of each tree by means of easily recognized characteristics. When this is accomplished the growth of a particular stand may be determined by adding the growth of the component trees. And what is more important, hypothetical stands may be synthesized and comparative growth rates studied from a management or silvicultural standpoint. It is inevitable that in the solution of an individual tree growth problem much will be learned of the growth characteristics of trees. Knowledge of growth so defined is the very backbone of intelligent silvicultural and management practice.

The approach in this study, which is an individual tree proposition, has been to try to explain a tree's growth in terms of the space which it occupies in the forest. If space, or spacing, or competition, can be defined in such a way as to explain a tree's growth the possible applications are legion.

Perhaps the foremost possibility is the study of the perplexing problem of stocking. The best known definition of adequate stocking is that of Prof. Filibert Roth, "room to grow and none to waste ". Probably no better purely qualitative definition will ever be written. Perhaps there are foresters with long experience, highly developed powers of observation, and excellent judgment who can look at a forest and decide whether or not the trees have room to grow while wasting none; foresters who can thin a stand to bring about adequate stocking by this criterion. However, until "room to grow and none to waste" is defined in quantitative terms their art must remain the property of a few.

Thinning practice embraces the idea of the tree as the basic unit. Crop trees are picked, partly for their form and quality, and partly because it is expected that they will make adequate growth between the time of thinning and their harvest. Alternative choices are nearly always encountered. Will two or more small trees after release make more growth than one larger one? Is the larger one capable of more growth if the small ones are removed? Just how much space does each need for maximum growth? The choice must be made for each tree in turn, and optimum densities in per-acre terms, even if they were known, would not make the decision any less difficult. Again, it is evident that what it needed is knowledge of growth-growing space relationships defined, perhaps, by tree diameter and dominance.

Intensive silviculture is likely to borrow a term from the engineer, "quality control" . Lumber grading rules specify grades in terms of size and number of knots and rings per inch, among other things. Both characteristics are largely controlled by stand density, which is under the silviculturist's control. Production of lumber with a specific number of rings per inch, "quality control", certainly will require a complete knowledge of the growth-growing space equation.

In this study an attempt has been made to correlate a tree's diameter growth with the area which the tree occupies, or more specifically, with the competition exerted on it by its neighbors. This difficult problem has by no means been solved and this thesis should be considered only as a progress report. So little is known of the laws of tree growth in terms of competition that what has been done in this study is simply exploratory in nature. The true relationships can only be discovered by trial and error, hence there is a virtual infinity of possibilities and pitifully few of them have been investigated.

Source of data - stand description

Data used in the study came from an even-aged stand of Douglas-fir on the north side of the Olympic Pennisula in Washington. They are taken from the Pacific Northwest Forest and Range Experiment Station's Kugel Creek Plots. The four plots in the series were established in 1939, when the stand was 38 years old. Three were thinned in 1939, and again in 1950. The fourth was left undisturbed as a check plot. Data analyzed for this study came from the .63-acre check plot.

All trees were tagged and measured in 1939 and remeasured in 1944, and again in 1949. In 1950 a stem map was made of the check plot (fig. 1) and two of the thinned plots. Stumps were located and measured in the thinned areas so that the stand can be reconstructed as it was in 1939 before thinning. The stand was chosen for study largely because of these thinning plots in the same general stand. Growth-spacing relations can be developed for natural and thinned plots in the same stand and results compared and studied.

The stand is uniformly well-stocked. Originating after a fire, the trees varied in age from 30 to 50 years in 1939, although the great majority are within a few years of the average (38 years). The stand occupies an average site for Douglas-fir, with a site index of 140 ft. in 1949.

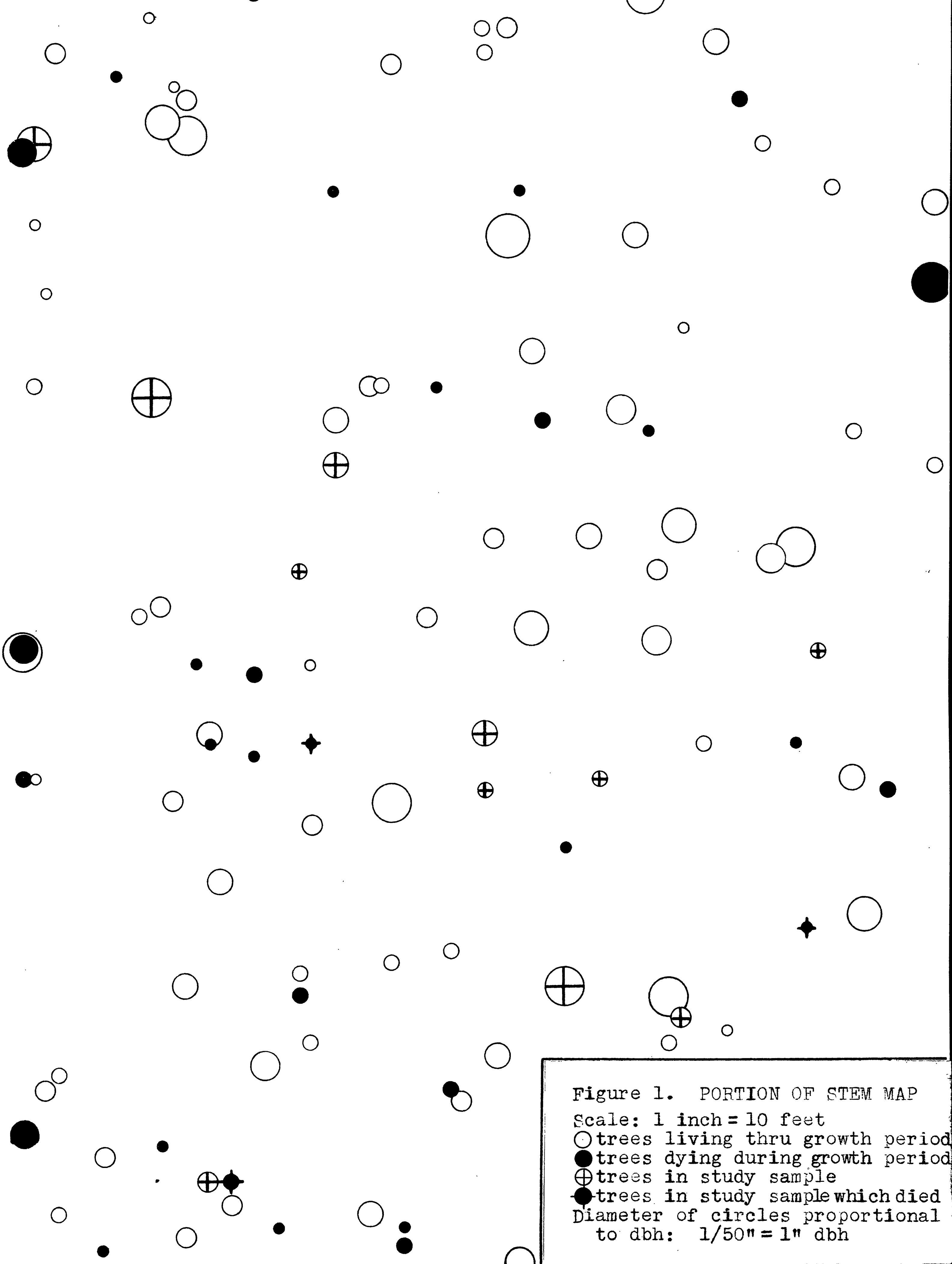

Figure 1. PORTION OF STEM MAP

Scale: 1 inch = 10 feet
○ trees living thru growth period
● trees dying during growth period
⊕ trees in study sample
✦ trees in study sample which died
Diameter of circles proportional
to dbh: 1/50" = 1" dbh

Table 1. Stand table of study plot as it existed in 1939 at beginning of growth period

dbh	trees per acre	B.A. per acre
3	3.2	0.2
4	73.0	6.4
5	87.3	12.3
6	107.9	21.2
7	60.3	16.1
8	82.5	28.8
9	50.8	22.5
10	44.4	24.2
11	22.2	14.7
12	19.0	14.9
13	14.3	13.2
14	7.9	8.4
15	6.3	7.7
16	12.7	17.7
17	3.2	5.0
18	3.2	5.7
19	3.2	6.3
20	-	-
21	-	-
22	1.6	2.8
Total	603.0	228.1

On the check plot average dbh in 1939 was 8.3 inches. The stand table at that time (Table 1) showed 603 trees per acre with a basal area of 228 sq. ft. There were 7334 cu. ft. per acre which is 150 percent of full stocking by yield table standard. Dominant and codominant trees in 1939 averaged 73 ft. in height.

In 1939, the year for which competition was measured, average spacing was 8.5 feet for trees 2.6 inches and over.

For trees 5.0 inches and over average dbh was 9.5 inches and average spacing, 10.0 feet (smaller trees ignored).

The plot is practically level except for one corner on a ten percent slope.

Growth and dbh data for the study were taken from the permanent sample plot record sheets. Spacing figures (for measuring competition) were taken from the stem map. The record sheets show the dbh of each numbered tree by 1/10 inch classes for each of the three measurements, 1939, 1944, and 1949. Crown classes are given and the heights of several trees. Also shown, of course, is the record of the trees which died. Diameter growth for the trees chosen for the study were taken from these sheets.

The stem map was carefully plotted on cross section paper, and each tree designated by its tagged number corresponding to the plot record sheet. A celluloid overlay with concentric circles one foot apart (on the stem-map scale) was used to determine distances to competing trees for the trees whose growth was studied.

Factors affecting tree growth

It is well known that, other things being equal, open-grown trees make rapid diameter growth. Those growing in closed stands make slower growth. Often the density is so great that the stand stagnates and individual trees make almost no growth, though here poor site as well as density is often a contributing factor. How individual tree growth changes between these extremes is not known. Schematically, growth can be expected to increase as growing space increases up to the point where the tree can no longer utilize the increased room (Fig. 2). Definition of such a curve is the real crux of the growth-growing space problem. Very little is known about it; yet, where thinning can be practiced, growing space or spacing is the thing most immediately under the forester's control.

More fundamentally, why does a tree in a crowded stand grow slowly? What are the basic factors controlling growth? F. S. Baker in his book, "Principles of Silviculture"1/ lists four factors controlling the gross growth of a tree in any specific year of its life.

1. The site factors, which are virtually fixed and can be but slightly modified by the forester's art.
2. The inherent capacity of the leaves to carry on photosynthesis (tolerance is involved here)
3. The input of light energy, water, and nutrients
4. The photosynthetic area

The first two of Baker's factors are assumed to be constant in this study since it is confined to one species in a single stand.

1/ Baker, F. S., 1950. Principles of Silviculture. McGraw-Hill. New York. p.283.

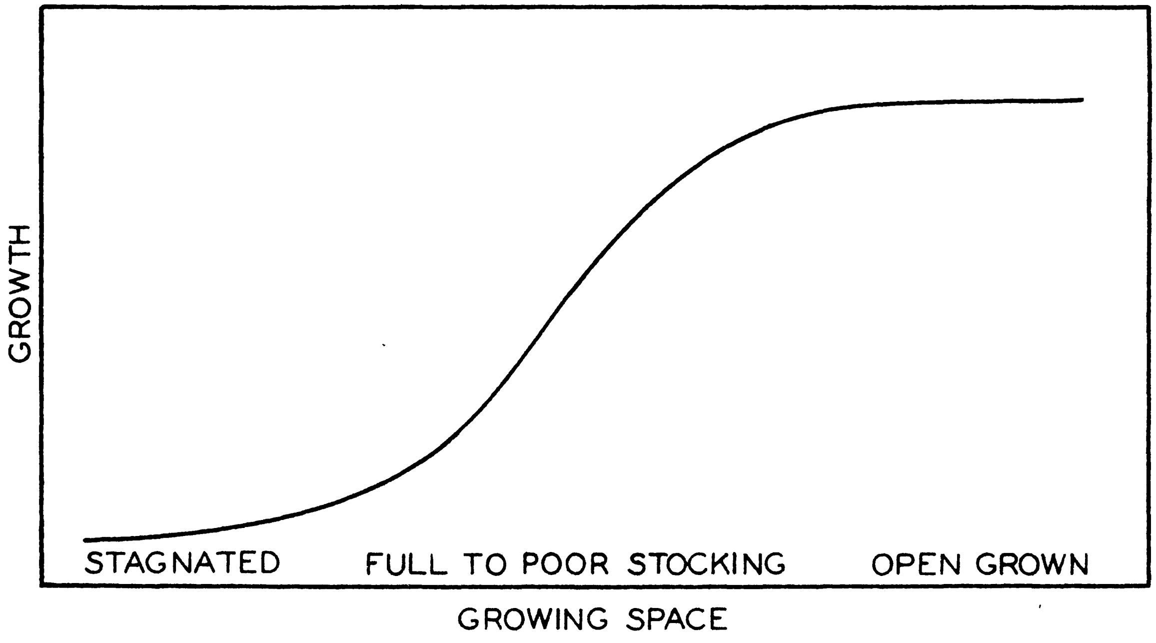

Fig. 2. Schematic representation of the relation of individual tree growth to growing space.

The third factor, input of light energy, water, and nutrients may be related to spacing. If the available supply is critical in the growth of a tree it is possible that there is a difference whether it is shared by many trees or by a few. Hence, an increase in spacing might mean increased growth (per tree) even if there were no increase in the photosynthetic area of the tree, Baker's fourth controlling factor.

It is readily apparent that this fourth factor is directly related to spacing. Photosynthetic area, meaning leaf surface, has some direct relation to crown area. Open-grown trees have large, long crowns. In closed, even-aged stands of intolerant species the crown becomes greatly reduced both in length and width so that intermediate trees become "tassel topped". And even dominants may have less than one-third of their length in live crowns.

Osborne, in his thinning study of southern pine,[2] found that growth of individual trees could be closely predicted if the diameter and height of the tree and its crown width and length are known. That is to say he had a correlated measure of photosynthetic area. In this study the assumption is made that crown area is closely controlled by spacing and dominance. If these factors could be adequately described, crown area, and hence photosynthetic area, would also be defined. If all trees were of the same height, spacing alone should control crown area (if we eliminate genetic differences between individuals as a factor). Since they are not, the added factor of dominance or relative height must also be considered. Two trees may have exactly the same growing space but if one is taller it has a crown above the general level of the canopy and so makes faster growth.

A more direct measure of photosynthetic area--such as crown width and length--could undoubtedly be used to predict growth with more precision than can spacing and dominance. The use of such a measure in Douglas-fir, however, is impractical since the trees are so tall and the stands so dense that accurate crown estimates are very difficult to make. It is also exceedingly difficult to find measures of spacing and dominance that are correlated to growth, but once they are found their application should be relatively simple.

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[2] Osborne, J. G., 1939. A design for experiments in thinning forest stands. <u>Jour. Forestry</u> 37: 296-304.

This discussion of growth factors should not be construed to mean that total growth per acre is affected by spacing, at least for stands not stagnated. Rather, it means that a given total growth may be put on many or few trees. Again, it should be emphasized that in this basic study the tree is the unit and it is individual tree growth which is being investigated.

Carl Mar:Möller's European work on the relation of thinning to dry matter reported on in the Journal of Forestry,[3] indicates that total leaf surface per acre cannot be increased by thinning. Regardless of stand density, except for very open stands, an acre of trees supports just so much leaf surface when measured in dry weight of foliage. Hence, photosynthetic activity and total gross growth is about constant.

Formulation of spacing hypotheses to explain growth

The basic thought behind the hypotheses tested is that a tree's growth varies inversely to the competition which it receives from neighboring trees. This, as explained earlier, is brought about because of a reduction in crown area by competition and perhaps also because of increased competition for water and nutrients.

Just what the relations are have never been worked out in quantitative terms. It is known that growth varies with

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3/ Möller, C. M., 1947. The effect of thinning, age, and site on foliage, increment, and loss of dry matter. Jour. Forestry 45:393-404.

competition, but how shall this competition be expressed to explain the growth? What is the role of dominance and how shall it be expressed? In this study the underlying assumption is that competition is directly proportional to some function of the diameter of competing trees and inversely proportional to some function of the distance to competing trees.

The effect of dominance, the second major factor controlling crown area, is even more difficult to define and visualize. To account for this, it is assumed that dominance (in a particular stand) is some function of the diameter of the tree studied. This is logical because in an even-aged stand curves of height over dbh are fairly well defined: the larger a tree is the taller it is. The larger, taller trees are the dominants and codominants. This reasoning is slightly fallacious since a tree may be larger in diameter because it had abundant room to grow while it is no taller than smaller trees growing under more competition. In a uniformly well-stocked stand this is probably of little importance.

To summarize, three factors have been considered in the attempt to explain the growth of individual trees in this even-aged, well-stocked stand:

1. Size of competing trees
2. Distance to competing trees
3. Size of the tree being studied (for its definition of dominance)

In the absence of concrete knowledge of the role of these factors any number of hypotheses might be formulated to explain their effect on a tree's growth. A controlled experiment presumably would involve the study of a single competing tree's effect on another's growth and the change in growth as the competing tree was made larger or smaller and the distance between trees increased or decreased. Then the experiment would have to be repeated using different sized subject trees. And repeated again with two, three, and more competing trees. No such data exist, hence, a tree's growth must be studied in the situation in which it is found. Assumptions must be made as to which trees are competitors, the function of distance, the function of size of competitors, and the function of diameter of the subject tree. Then using any one set of assumptions the total competition must be computed and correlated with the growth of the tree studied. Since total competition must be the parameter the effect of any single tree can never be positively assessed. All that can be done is to correlate the various measures of competition and to accept that measure with the highest correlation as the one which best explains the tree's growth. Competition can then be broken down to an individual tree basis and one can see if the thing looks logical, but it still can't be said that the effect of a single tree has been isolated. It can only be known that all the competing trees taken together explain the growth of the subject tree.

Variations of two major hypotheses concerning competition have been tested and are presented in the following sections. None appears entirely satisfactory but the results are encouraging in that they indicate that the basic assumptions are sound. Competition, and hence growth, is some function of the three factors listed earlier.

Multiple correlations were run using 40 trees drawn at random from the plot record sheets. The 12-year diameter growth which occurred (expressed in two different ways) was the dependent variable. Competition (expressed in a variety of ways) was the independent variable. Competition ~~came~~ was computed from data on the trees surrounding each of the subject trees. Of the 40 trees chosen for study, eight died during the growing period. At first, they were included in the sample in hopes that their death might be explained on the basis of intense competition. It became evident, however, that correlations could be greatly improved if these eight trees were omitted. Hence, most of the results discussed are based on the 32 trees which survived the growth period. Competing trees were included if they were alive at the beginning of the period, although many later died.

Area of overlap hypothesis

Essentially, this hypothesis states that trees require a circular area to grow in, a circle whose diameter varies with the dbh of the tree. Two trees which are so close together that their circles overlap are competing with each other. The growth of any tree is inversely proportional to the amount of overlapping from competing trees.

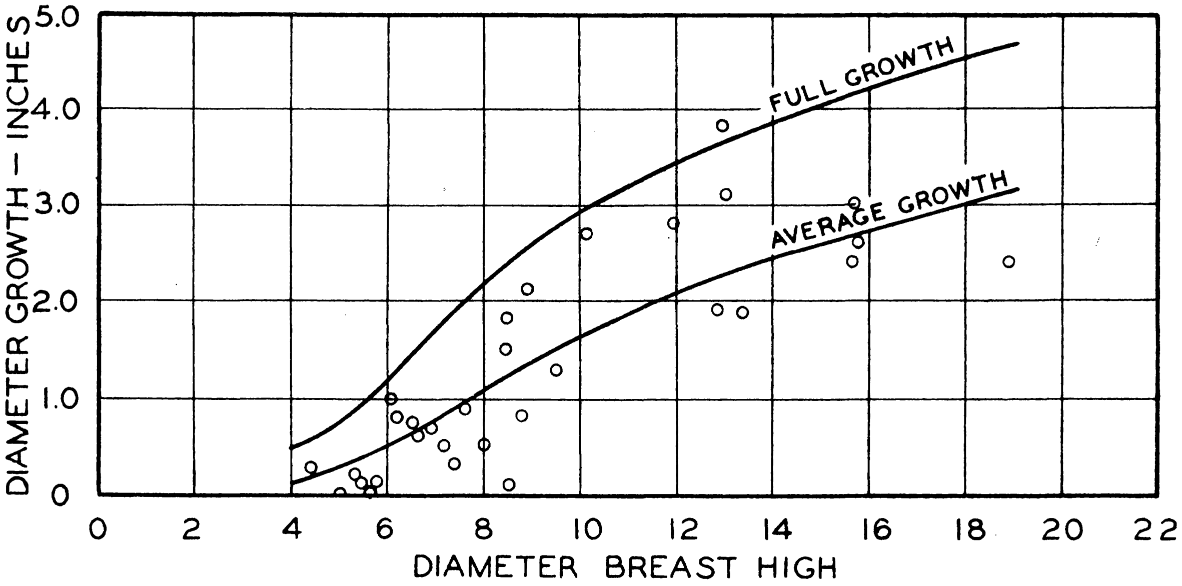

Figure 3. Diameter growth (12-year) by dbh. Plotted points indicate growth of trees studied. Average growth curve fitted to 270 trees which survived the growth period. Full growth curve is two standard deviations from regression above average growth curve.

The dependent variable

The residual from a curve of diameter growth over dbh is the growth variable to be correlated with the competition measures. A free hand curve was fitted to the growth data for all trees on the plot which survived the 12-year growth period (Fig. 3). Residuals from this curve, or the difference between actual and curved growth, were read for the trees used in the correlations. The curve of diameter growth over dbh has a correlation coefficient of about .81. In other words, such a curve alone accounts for 65 percent of the variation in growth rate for the trees in the stand. Correlations to be worked out test the hypothesis that those trees which grow faster than the average of trees of the same diameter have less competition and those that grow slower have more.

Use of a growth curve based on dbh provides a reference which largely accounts for dominance as a growth factor. Reading from the curve in Figure 3, 16-inch trees averaged about 2.7 inches diameter growth in the 12-year period, while 6-inch trees (those which survived the period) grew only .5 inch. Why the difference? On the average the 16-inch trees enjoyed a dominant position with much crown exposed to full light well above the level of surrounding trees, while the 6-inch tree was lucky if it had the very top of its crown in a position to receive full light.

Use of a residual off such a growth curve in effect nullifies this effect of dominance. The 6-inch tree has as good a chance of growing faster than the 6-inch average as does the 16-inch tree of growing faster than the average for its class.

Measuring competition

Development of the competition measure can best be understood by reference to the diagrams in Figure 4. The tree whose growth is being studied is shown at A. An assumed area it would like to have to itself, if it is to make full growth, is shown by the solid circle. B, C, and D are competing trees and the areas they need for full

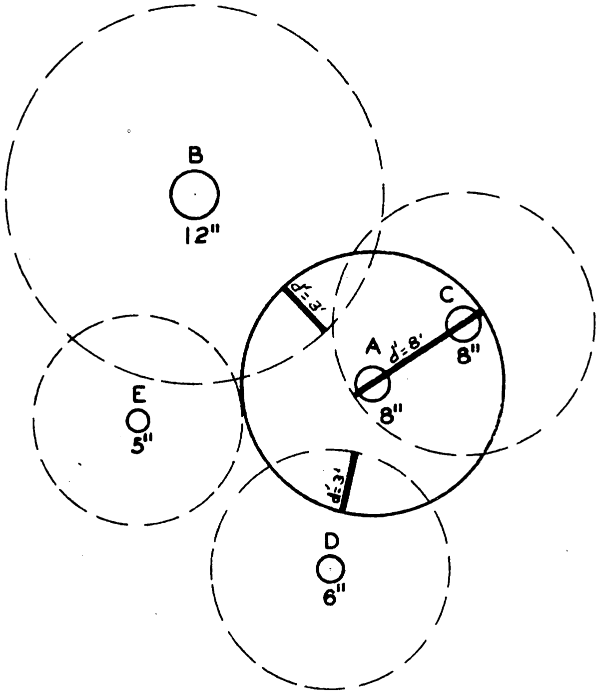

Figure 4. Illustration of length of overlap, d', used as a measure of competition. Diameters of large circles are 1.2(dbh)+5. Dbh of subject tree, A, is 8 inches.

growth are shown by the dashed circles. E is just far enough from A so that the area it requires does not overlap A's circle. The heavy lines between A and B, C, and D represent the length of overlap-- B, C, D are too close to A by this amount.

(It is probable that competition is best represented by the area of overlap. However, the expression giving this area in terms of distance between centers of circles and diameters of circles is extremely complicated. In fact, all that is feasible is a graphical solution. Hence, the distance between the edges of the overlapping circles has been used for simplicity.)

Now if these overlap distances, or some function of them, are to represent competition the first question needing an answer is: How big are the required circles?

In the trials the assumption has been that the diameter of the circle in feet is some constant times the tree's dbh in inches plus another constant or, diameter of circle equals $a(D)+k$, where $D = \text{dbh}$, and a and k are constants. This is a combination of the D times and D plus relationships widely used as rules of thumb in thinning. The combination seems logical in that if a D times constant could be solved for experimentally the result would be of the form, $k+aD$. That is, if the equation were not specifically restricted to passing through the origin ($k=0$), it almost certainly would not, and k would have some finite value.

In the trial correlations "a" was arbitrarily set equal to 1.6, 1.2, and .8 and "k" to 7, 3, and 5 feet. Six of the nine possible combinations were tested.

In spite of the fact that the form as set up amounts to a test of D plus and D times rules no particular brief is held for them as such, although competition based on them is correlated with growth. Other functions of dbh, notably an exponential function, might well be better. They have not been tested.

The maximum distance at which two trees can compete with each other becomes just less than $\frac{1}{2}a(D_O+D) + k$, where $D_O = \text{dbh}$ of the subject tree and $D = \text{dbh}$ of competing tree. In other words, at this maximum distance their circles are tangent.

For example, if "a"= 1.2 and "k" = 5, a 10-inch tree requires a circle 17 feet in diameter, equivalent to 1.2 x 10 + 5. A 6-inch tree requires a circle 12.2 feet in diameter. At $\frac{1}{2}$ x 1.2 (10+6) + 5, or 14.6 feet between the two trees, their circles are tangent.

For trees closer together than the maximum distance the length of overlap becomes $[\frac{1}{2}a(D_o+D) + k]-d$, where d = distance between trees. This length of overlap in the paragraphs to follow is designated d'. In the example just cited, if the 10- and 6-inch trees were actually 10 feet apart, d' would equal 14.6-10 or 4.6.

In reality d' is a coded value for the distance between any pair of trees. The length of the overlap, d', increases as the distance between subject and competing tree decreases. It also increases with the size of the competing tree, and with the size of the subject tree. This will be explained more fully later.

The sum of the overlap distances, Sd', is the first variable representing competition. The second variable chosen is the sum of the squared distances, $S(d')^2$. This is to give more weight to a single long overlap than to several short overlaps whose sum might be equal to the one long one. The third variable is Sd'D, or the sum of the products of overlap and corresponding tree dbh. In Figure 4 note that trees B and D have the same length of overlap, although B is a much larger tree than D. It seems logical that because B is larger and higher in the scale of dominance than D that it exerts more competition, even though it is farther away. It is to give trees like B more weight that the third variable, $Sd'D$, was adopted. For good measure a fourth variable, $S(d')^2D$, was added, the sum of the products of the squared overlap distances and the size of the corresponding competing tree.

Remember that d' is a coded distance between trees depending on an assumed maximum distance at which trees will compete with each other.

This index of competition has the effect of changing when the size of the subject tree changes even if competing trees keep their position. Figure 4 represents an 8-inch subject tree and competition is computed from the assumption that all trees closer than $.6(D_0+D) + 5$ are competitors, hence have a measurable overlap. Figure 5 shows the same set-up except that the subject tree is a 14-inch tree. Note how much greater is the competition. Tree E now becomes a competitor and the effect of the other trees is increased. This is not entirely illogical since a 14-inch tree ought to require more room than an 8-inch tree. An inspection of the d' variables when computed for 40 trees, however, showed that on the average the large trees had much more competition than the small ones. Competition was so much greater that the relation to the dependent variable obviously changed over the diameter range. Apparently this choice of variables more than offset the effect of dominance. Larger trees might have

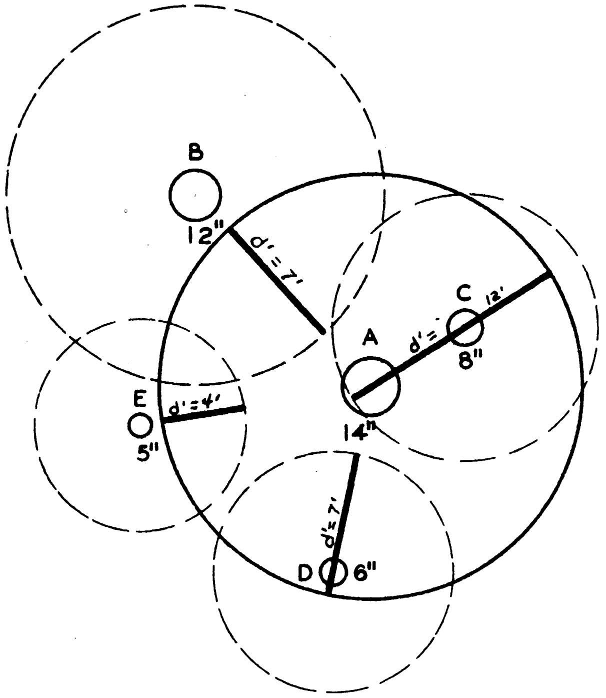

Figure 5. Illustration of changes in competition with change in size of subject tree while distance to and size of competing trees remain constant (see Fig. 3). Dbh of subject tree, A, is 14 inches.

more competition, by the assumed criteria, but it was less effective since they were more dominant.

Hence, an adjusting set of assumptions was called for. The average sized tree in the stand at the beginning of the growth period (when competition was measured) was 8.3 inches. If the average competing tree is 8.3 inches this can be substituted for D in the maximum distance expression, $\frac{1}{2}a(Do+D) + k$. For various values of D_o the radius of the circle which includes the average competing tree can be computed. For example, if the maximum distance expression is $.6(D_o+D) + 5$, substitution of the average-sized competing tree, 8.3, for D gives $.6D_o + 10$. For 12-inch subject trees this means that all trees 8.3 inches and larger within a radius of .6 x 12 + 10, or 17.2 feet, are included as competitors. For six-inch subject trees all trees within an average of 13.6 feet are competitors. These radius figures squared are proportional to the area of what is considered an average circle. The squared figure divided by 10 and rounded to the nearest whole number gives an area proportional factor, designated F. The competition variables as computed were divided by these factors, a different factor for each sized subject tree and for each assumption of maximum competition distance, $\frac{1}{2}a(D_o+D) + k$. The factors are given in Table 2.

The final observation equation now becomes:

$$Y = a + b_1Sd'/F + b_2S(d')^2/F + b_3Sd'D/F + b_4S(d')^2D/F$$

Y is the residual from the diameter growth over dbh curve. The term "a" is a constant to be solved for. The other symbols have already been explained.

It should be noted how purely arbitrary are the assumptions made. Dozens of substitutions might be made while still staying within the general area-of-overlap hypothesis. The problem becomes one of trying everything that looks reasonable, and a lot of things that don't, in an attempt to find out what the controlling laws are (if there are any). It is essentially a trial and error proposition.

Table 2. Factors for adjusting competition to size of subject tree

Subject tree dbh	a=.8	a=1.2			a=1.6	
	k=7	k=3	k=5	k=7	k=3	k=5
			Factor (F)			
4	14	11	15	21	16	22
5	15	12	17	22	18	24
6	16	13	18	24	21	27
7	17	15	20	26	23	30
8	18	16	22	28	26	32
9	19	18	24	30	28	35
10	20	20	26	32	31	38
11	22	21	28	35	34	42
12	23	23	30	37	37	45
13	24	25	32	39	40	48
14	25	27	34	42	43	52
15	27	29	36	44	47	56
16	28	31	38	47	50	60
17	29	33	41	49	54	64
18	31	35	43	52	58	68
19	32	38	46	55	62	72

a and k are constants in the maximum distance expression $a/2(D_o+D)+k$

Correlation results

Results are inconclusive but encouraging. Table 3 gives the prediction equations, and the coefficients of correlation for the various assumed sizes of required circle. These correlations are all based on samples of 32 trees which survived the growth period. Diameters ranged from 4 inches to 19 inches.

Table 3. Prediction equations for the area-of-overlap hypothesis

Assumed size of circle required	Prediction equation	Coefficient of multiple correlation
(1) .8D+7	$Y' = .508 + 1.1033Sd'/F - .0822Sd'^2/F - .1876Sd'D/F + .0123Sd'^2D/F$	.464
(2) 1.2D+3	$Y' = .353 + 0.7595Sd'/F - .0683Sd'^2/F - .1158Sd'D/F + .0080Sd'^2D/F$	.386
(3) 1.2D+5	$Y' = .710 + 0.8808Sd'/F - .0400Sd'^2/F - .1809Sd'D/F + .0086Sd'^2D/F$	.473
(4) 1.2D+7	$Y' = .576 + 1.9015Sd'/F - .1586Sd'^2/F - .2507Sd'D/F + .0061Sd'^2D/F$	.575
(5) 1.6D+3	$Y' = .663 + 0.7389Sd'/F - .0552Sd'^2/F - .1208Sd'D/F + .0061Sd'^2D/F$	.391
(6) 1.6D+5	$Y' = .594 + 1.0433Sd'/F - .0683Sd'^2/F - .1332Sd'D/F + .0061Sd'^2D/F$	.439

Four-variable multiple correlation in a sample this small cannot be expected to give fully reliable results. In these early trials, however, a test of methods and assumptions was more of an objective than workable prediction equations. For this purpose they serve sufficiently well.

The highest correlation, .575, is found when it is assumed that a tree requires a circle whose diameter in feet is equivalent to 1.2 times the dbh of the tree in inches plus 7. For example, it is assumed that a 10-inch tree requires a 19-foot circle for best possible growth in this stand. Restrictions in this circle, as measured by the d' variables, is significantly correlated with growth. None of the other tests show a significant correlation when sample size and numbers of variables are taken into account although they all approach significance closely.

The coefficient of multiple correlation, .575, indicates that for this particular assumption the d' variables explain 33 percent of the variation existing in the dependent variable. Put this way, the prediction equation can hardly be considered a useful tool in predicting growth of individual trees. The expected error would still be high.

Further, it cannot be assumed that full growth would result if trees were no closer than .6 times the sum of their diameters plus 7 feet, in spite of the fact that this is the underlying reference point for computing competition, competition which turned out to be significantly correlated to growth. Actually no trees studied were free of competition or even close to it by these requirements. All that can be said is that growth is related to this measure of competition over the range of competition studied. Whether or not the equation can be used to predict growth with zero competition is not known. That the trees in this well-stocked stand are competing strongly for space is evident from the part of the stand map reproduced in Figure 6. This is the same part of the plot that is shown in Figure 1, but with the desired circular growth areas drawn in. Small wonder that the equation can't quite be sure how much is "full" growth !

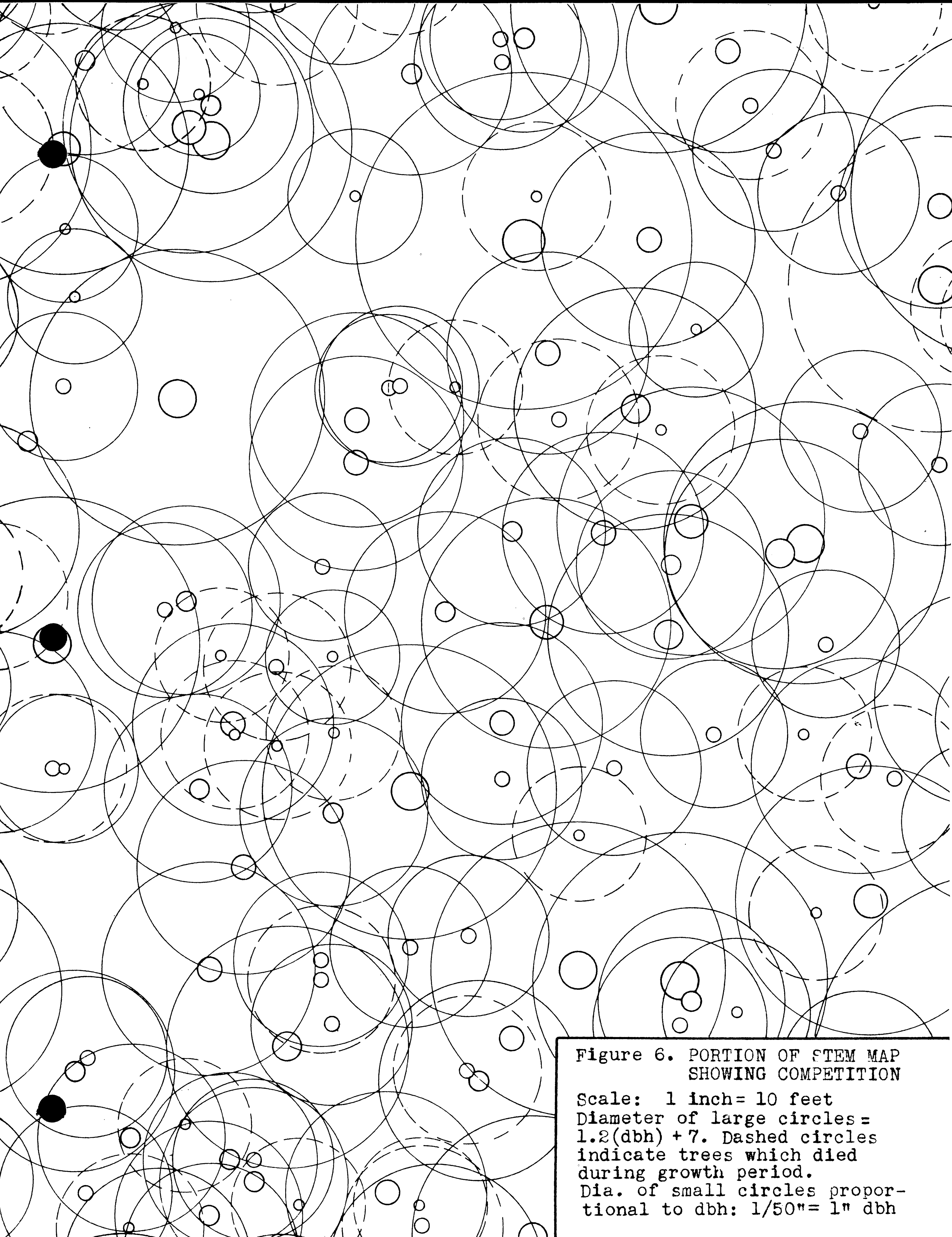

Figure 6. PORTION OF STEM MAP SHOWING COMPETITION

Scale: 1 inch = 10 feet
Diameter of large circles = 1.2(dbh) + 7. Dashed circles indicate trees which died during growth period.
Dia. of small circles proportional to dbh: 1/50" = 1" dbh

In spite of the fact that even the best equation is too poor for growth prediction purposes, it is well to analyze it carefully to learn how it operates and to set the stage for the hypothesis discussed in the next section.

All four variables taken together explain 33 percent of the variation in Y, deviations from a growth curve. The diameter growth curve, used as the reference point for the residuals, explained 65 percent of the variation in the original growth rates, hence competition accounts for .33 x (1-.65) or 12 percent of the total variation, and altogether 65+12, or 77 percent has now been accounted for.

Coefficients of partial correlation show that the arbitrary choice of competition functions actually wasn't too bad. All four variables are of sufficient importance to be included in an equation of this type--but that is not to say that better ones can't be found. $S(d')^2$ and $Sd'D$ account for somewhat more variation than do the other two variables (Table 4).

The equations in Table 3 can be simplified for plotting by substituting appropriate values for F and D. First, an equation may be written for any sized subject tree by substituting the proper value of F from Table 2 and clearing. Equations derived from equation (4), Table 3, have been written for 8- and 12-inch subject trees (Table 5). Next, an equation for a particular sized competing tree may be written by substituting its dbh for D whenever D occurs in

Table 4. Summary of correlation coefficients for equation (4), Table 3

Variables	R^2	Remaining Variable	r^2
Sd'/F, Sd'^2/F, $Sd'D/F$, SD'^2D/F	.331	--	--
Sd'/F, Sd'^2/F, $Sd'D/F$	.189	Sd'^2D/F	.175
Sd'/F, Sd'^2/F, Sd'^2D/F	.100	$Sd'D/F$	.257
Sd'/F, $Sd'D/F$, Sd'^2D/F	.085	Sd'^2/F	.268
Sd'^2/F, $Sd'D/F$, Sd'^2D/F	.159	Sd'/F	.205
$Sd'D/F$, Sd'^2D/F	.139	Sd'/F, Sd'^2/F	.223

R = coefficient of multiple correlation
r = partial correlation coefficient
Squared coefficients are equivalent to amount of variation explained by the chosen variables.

a specific subject tree equation. This gives an equation in d'. Since d' is a coded value of distance the equation may be written in terms of distance directly by clearing the formula $[\frac{1}{2}a(D_o+D) + k] - d$, now that all terms in the bracket are known. The result is a set of equations for each sized subject tree, each equation in the set being for a particular size of competing tree. Each equation predicts Y for the subject tree in terms of distance between it and the competing tree. It should be noted that the constant term, "a", in the subject tree equation drops out in the individual competing tree equations. The competing tree equations are for one tree only and the "a" term

Table 5. Growth Prediction Equations for 8- and 12-inch trees

Subject tree dbh	Competing tree dbh	Prediction equation
All	All	$Y' = .756 + 1.901487Sd'/F - .158599S(d')^{2}/F -250680Sd'D/F + .016024S(d')^{2}D/F$
8	All	$Y' = .756 + .067910Sd' - .005664S(d')^{2} -008953Sd'D + .000572S(d')^{2}D$
	6	$Y' = -.310 + .0546d - .00223d^{2}$
	8	$Y' = -.362 + .0398d - .00109d^{2}$
	10	$Y' = -.367 + .0196d + .00006d^{2}$
	12	$Y' = -.318 - .0061d + .00120d^{2}$
	14	$Y' = -.204 - .0373d + .00234d^{2}$
12	All	$Y' = .756 + .051392Sd' - .004286S(d')^{2} -.006775Sd'D + .000433S(d')^{2}D$
	6	$Y' = -.344 + .0494d - .00169d^{2}$
	8	$Y' = -.350 + .0340d - .00082d^{2}$
	10	$Y' = -.312 + .0146d + .00004d^{2}$
	12	$Y' = -.223 - .0090d + .00091d^{2}$
	14	$Y' = -.075 - .0368d + .00178d^{2}$

in the subject tree equation applies only to the sums of all competing trees.

With the equations in this form they may be shown graphically (Fig. 7). This puts the relationships out where they can be more easily visualized and incidentally shows up glaring weaknesses in the equations. The down bend at the end of the 6-inch curves is illogical; the curve should level off at some distance but not fall. And of course the sharply defined minimum in the 14-inch curves is hardly plausible. According to the curve a 14-inch competing tree has less effect on a subject tree when it is 2 feet away than it does when it is 10 feet away.

Throughout almost the entire range of the curves, the effect is to subtract from the average growth; that is, the predicted deviation from the growth curve is minus. It should be remembered that after values are read from the curves for the competing trees around a subject tree the values must be summed, and "a", which is 0.756 inches, added to get total effect on the subject tree. The large relative size of "a" is of course another indication of the poor correlation existing between the dependent variable and the chosen independents.

For the portion of any of the curves which show a plus deviation in growth, the anomaly apparently exists that the more competing trees there are at that distance the higher will be the predicted growth of the subject tree.

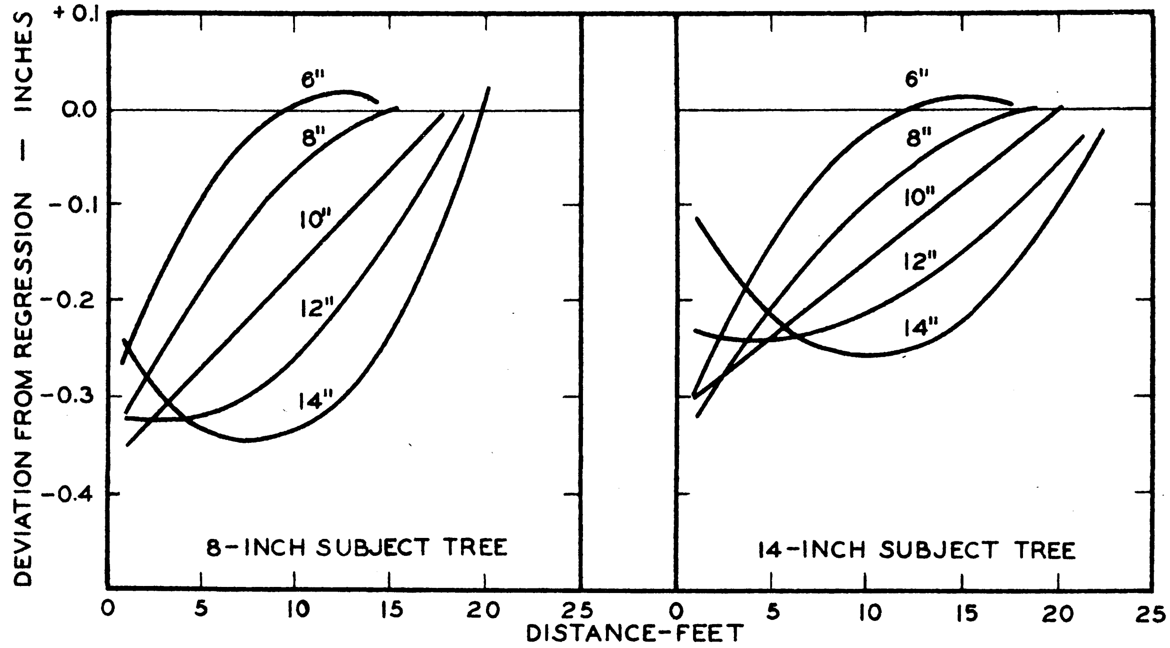

Figure 7. The effect of distance to and size of competing tree on growth of subject tree. This is a partial graphical presentation of equation 4, Table 3.

Trees, however, just did not occur that way in the samples so that this situation is beyond the range of the data. It points up perhaps that the growth curve residual, as such, is not a good dependent variable.

The arbitrarily chosen independent variables turned out to be significantly correlated to growth but when the equations are reduced to their elemental form it becomes obvious that what must be the basic relationships have not been properly expressed. Hence, the very fact that correlation exists, even with these shortcomings, is evidence that there is a relation between growth and the factors of dominance, size of and distance to competing trees. It remains to express all the factors properly.

Competition curve hypothesis

In the hypothesis just explained it was shown how the final equation, when reduced to individual tree form, failed to yield curves of logical shape (Fig. 7). That fact suggests the desirability of starting with a curve form which appears logical, adopting such variables as are needed to describe it, and then running test correlations.

That has been the approach in the competition curve hypothesis. It is assumed that a curve of competition over distance is fairly simple in form, a form which may be guessed at, and that a family of curves all of the same general form and each representing a different sized competing tree may be drawn. Various curve forms were chosen as being possibly descriptive of the curve family.

The method has not been entirely successful and correlations are no higher than for the overlap hypothesis. However, it is felt that it has merit because of its less complicated approach and because it is somewhat more easily justified biologically. Failure to achieve better correlation supposedly results from failure to choose the correct equation forms.

The dependent variable

If competition is to be greatest when trees are close together and lessen as distance between trees increases and finally to disappear at some distance, it must be inversely proportional to growth. In fact, competition can never be measured, we only measure its effect on growth. The problem is to choose some function of growth which is directly proportional to competition in order to fit the hypothesized curve forms which will be discussed later.

The diameter growth over dbh curve again is the starting point (Fig. 3). If thousands of trees from this same stand could have been plotted they would form a scatter of dots which would have a more or less well defined "top". A curve drawn along this top edge of the scatter would represent the growth of trees which had all the room they needed to grow in--assuming the correctness of the hypothesis that deviations from average growth is due to more or less than average growing space.

How shall such a curve of "full growth" be drawn? The standard deviation from regression was computed for each 2-inch diameter class along the curve and points plotted at two standard deviations above the basic curve. The points defined fairly well a curve of "full growth". This curve, it is assumed, is close to one which might be drawn along the top edge of a scatter of dots (Fig. 3).

The amount by which a tree fails to make full growth measured as a percent of full growth is called competition. For example, full growth for a 10-inch tree is 2.95 inches.

For a tree that makes 2.00 inches growth, competition is (2.95 - 2.00)/2.95, or 32 percent.

As in the overlap hypothesis, the use of a changing reference with change in dbh is assumed to level the difference in dominance between trees of different sizes.

The superiority of such a dependent variable over deviations from a growth curve is evident. The curve of average growth must change from stand to stand as average stocking or spacing changes (age and site remaining constant). Hence, where deviation from a specific growth curve is the dependent variable, it is questionable if prediction equations (even if perfected) could be applied to stands other than the one for which they were computed. In contrast a curve of full growth, if identified by age of the stand and site quality, ought to be a uniform reference point for any stand. Failure of a tree to make full growth, expressed as percent, should be a perfectly general dependent variable.

It is by no means certain that the curve of full growth constructed as described does in reality describe growth under optimum conditions. Further work should include study and accurate definition of such a curve.

Measuring competition

It might easily be imagined that a competing tree exerts maximum competition on another when it is zero, or one, foot away from the subject tree. Its effect should fall off slowly, then more and more rapidly as distance between the trees is increased until at some distance its effect disappears (Fig. 8). If this competing tree is, for example, a 10-inch tree, a 14-inch tree ought to exert more competition but with its curve having essentially the same form and with competition becoming zero at some distance greater than that for the 10-inch tree. Similarly, a 6-inch tree ought to have less effect than a 10-inch tree. Further, it seems logical that the difference between the 10-and the 14-inch trees ought to be greater than the difference between the 6- and 10-inch trees (Fig. 9).

A curve like that in Figure 8 is an exponential form, $Y = a + bd^p$, where $Y =$ competition, $d =$ distance, and p is some exponent. The curve has a maximum at $d = 0$ (the sign of b becomes minus). Now, if a family of curves is to be defined (Fig. 9), each curve in the family representing a competing tree of different dbh, the function just given should be multiplied by dbh or, $Y = a + b_1D + b_2Dd^p$ where D equals dbh. If the difference between curves is to progressively increase with increasing dbh, D must be raised to a power, or, $Y = a + b_1D^p + b_2D^pd^p$. By substituting different values for D a family of curves may be drawn.

As pointed out before, no individual tree information exists. Hence, it is necessary to work with all competing trees around a subject tree and to correlate the sum total

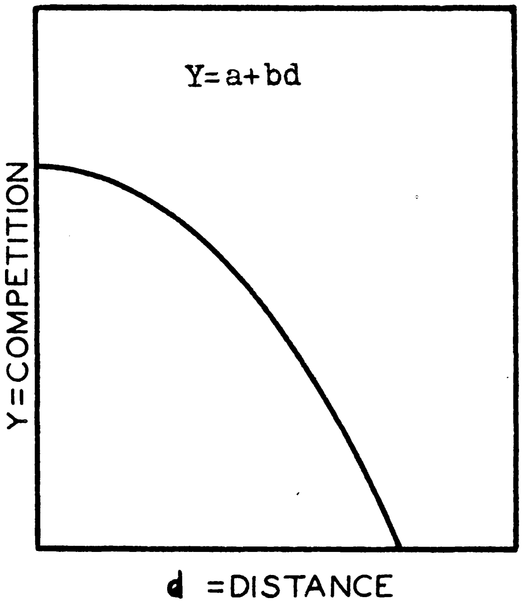

Figure 8. Hypothetical curve of competition over distance for a single tree.

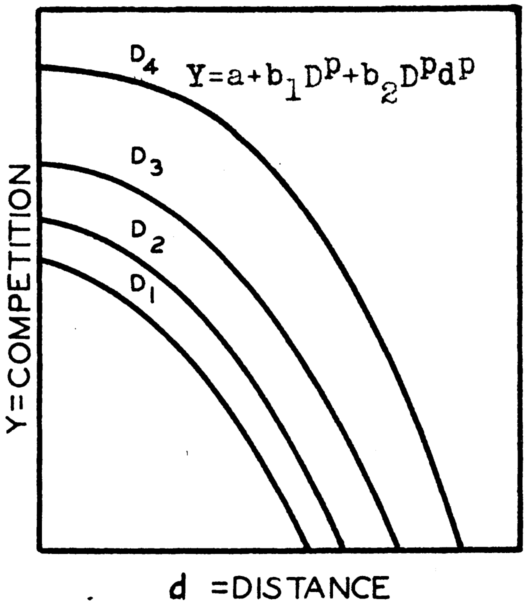

Figure 9. Hypothetical curves of competition over distance for trees of varying dbh.

of their effect with the growth (competition) of the subject tree. This suggests that the number of trees competing also be used as a variable, called N. The observation equation becomes:

$$Y = a + b_1N + b_2SD^p + b_3SD^pd^p$$

Before outlining the attempts made to solve this equation, it is well to point out the obvious advantages of the hypothesis as illustrated in Figure 9. If such a family of curves could be constructed it would show how much reduction in full growth is caused by any sized tree at any distance. Adding the effects of all the trees close enough to compete would result in a percentage figure which is the reduction from full growth caused by competition. Further, the curves would immediately show the distance beyond which a tree is no longer a competitor.

Four different substitutions were made for the p's in the equation and multiple correlations run. In all four attempts D^2 was used in the b_2 term. In the b_3 term the substitutions were:

(1) $D^{-1}d^2$, equivalent to d^2/D
(2) $D^{-1}d^3$, equivalent to d^3/D
(3) $D^{-\frac{1}{2}}d^2$, equivalent to $d^2/\sqrt{D}$
(4) D^2d^2

Competing trees were chosen on the basis of one of the assumptions made in the earlier hypothesis. All trees within a distance of $.6(D_0+D) + 5$ of the subject tree were called competitors but here an average sized tree, 8.3 inches, was substituted for D_0 so that the maximum distance became $.6D + 10$, depending on the size of the competing tree.

Correlation results

The results were just good enough to keep an investigator thinking that the elegant solution lay just around the corner. And indeed it might. There are dozens of possible solutions. The necessity of working with sums makes it impossible to graph out the relations so that prior knowledge of the curve form is impossible.

The prediction equations and their coefficients of multiple correlation are:

(1) $Y' = 1.79 + 6.133N + .049SD^2 - .213S(d^2/D) \quad R = .55$

(2) $Y' = -1.79 + 4.666N + .054SD^2 - .012S(d^3/D) \quad R = .54$

(3) $Y' = 1.19 + 2.480N + .049SD^2 - .000S(d^2D^2) \quad R = .52$

(4) $Y' = 2.23 + 5.121N + .056SD^2 - .065S(d^2/\sqrt{D}) \quad R = .54$

These equations were fitted to data for 40 trees including eight trees which died during the growth period. No. (1) was reworked to include only the 32 surviving trees with the result:

(5) $Y' = 1.48 + 3.305N + .048SD^2 - .167\,S(d^2/D) \quad R = .60$

The correlation coefficients are all significant at the five percent level. However, even the best equation explains only about 30 percent (equivalent to R^2) of the variation existing in the dependent variable--competition as a function of growth.

Figure 10 shows equation (1) converted to a family of curves according to the original objective. One was substituted for N so that the equation for one tree becomes

$$Y' = 6.133 + .0493D^2 - .2126d^2/D.$$

(The constant 1.79 must only be added after the effect of all competing trees are summed.) By substituting a given value for D, an equation in d alone may be written. This is what is plotted. The limits of the data used are shown by the solid portion of the curves while the dotted part shows the

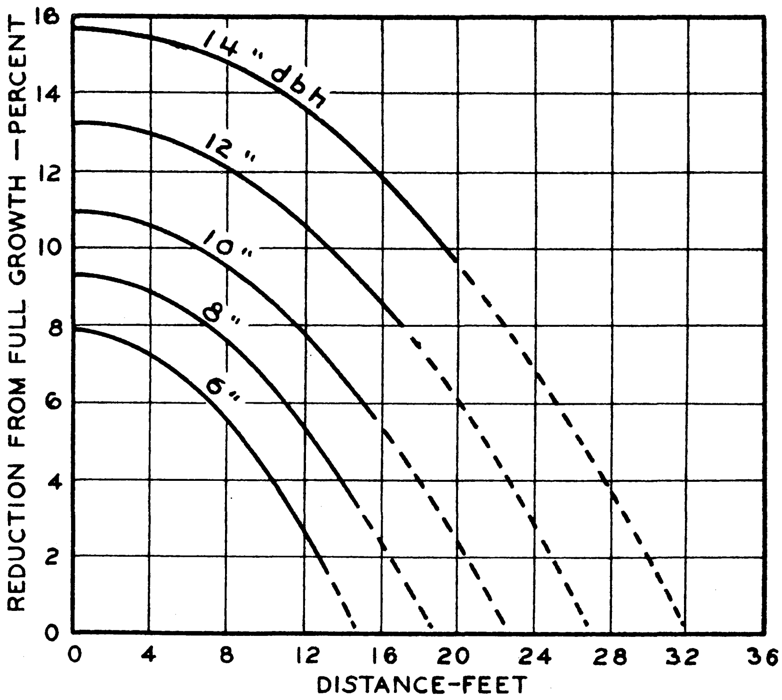

Figure 10. Curves of competition over distance for trees of varying dbh. This is a graphical presentation of equation (1), page 39.

complete curve to the point where it crosses the abscissa. It will be noted that the distance at which the curves cross is illogically great. It is difficult to believe that a 14-inch tree will compete with another tree up to a distance of 32 feet, or that a 10-inch tree is a competitor up to 23 feet.

The problem would seem to be to choose an equation form which would cause the curve to drop faster--to increase the effect of increasing distance. The other three forms chosen were attempts to correct this defect, but none were so successful as the one described.

Another defect in the equations as solved was that distance had entirely too little effect. Coefficients of partial correlation showed that the b_3 term was explaining only about 3 percent of the total variation after the effect of number of trees and diameter was accounted for. This is in contrast to the overlap equations where the two terms containing only distance (in its coded form) accounted for about 19 percent of the total variation after allowing for the effect of the variables containing diameter and distance.

The plan was to solve the equation for the maximum effective distance (the crossing of the curve on the distance axis) and if that differed from the distances actually chosen to solve the equation again for the new limiting distances. And then to repeat the process until the limits of the data used corresponded to the solution. However, it is felt that this is not worthwhile until a more logical curve form is hit upon.

The hypothesis, I feel, holds promise but needs much more trial. The most obvious next step is to try other curve forms. Another probable need is to introduce the size of the subject tree as a variable. The curves as shown apply to subject trees of any diameter. In the overlap equations the size of the subject tree is more directly accounted for.

Conclusions and suggestions for further study

The inevitable conclusion from the study is that a satisfactory explanation of individual tree growth based on spacing has not yet been found. The presence of even poor to moderate correlation, however, is evidence that a relation exists and waits only to be discovered.

Several possible defects exist in the data and in the analysis that may account for failure to get better results.

The data are from a well-stocked, uniform stand. Permanent sample plots are usually chosen for these very characteristics. All trees were crowded and even though the competition measures computed seemed to vary greatly from tree to tree, still the range is not nearly so great as would exist in a stand varying from poor to good stocking. The dependent variable was also affected. Growth rates deviated from those which were expected on the basis of tree diameter, from one-half to three-fourths inch on the average. Perhaps, we should be trying to explain variations two or three times that great, at least in an exploratory study designed to find out what the relations are as well as to evaluate them.

It is possible that the growth rates taken from the sample plot sheets are too crude for careful analysis. Trees were measured to the nearest tenth-inch, but, in converting growth rates to the forms used in the dependent variables, curves were read to the nearest hundredth-inch. In the lower diameters where expected growth rates are small, errors of .05 inch (possible in tenth-inch classes) are serious, especially when expressed in percent, as was done in the competition curve hypothesis.

It is possible, of course, that fundamental errors exist in setting up the hypotheses for test. Genetic differences between trees may partially obscure other factors affecting growth. Some trees are simply better growers. Spacing and dominance may not control crown area and, hence, photosynthetic area as closely as supposed, though it seems hard to believe. A study of the relationship is perhaps required. It might give some clue to the functions of spacing needed to explain growth.

As for the analysis itself, the most obvious handicap is the necessity for solving the problem by trial and error. How to account for dominance? for distance between trees? for size of trees? The methods used were undoubtedly crude, certainly very arbitrary.

One idea suggests itself as a means of learning more of the effects of these factors. Future analyses should be based on much more extensive data, so that separate correlations could be run for groups based on diameter of subject tree. Then within each subject-tree diameter group, competing trees should be grouped by diameter, each group with its own set of variables. A study of the constants derived from such a solution ought to throw light on the effects of dbh.

The final conclusion is that here is a most challenging study. Solution of the problem would yield information and knowledge fundamental to scientific management. It undoubtedly requires a silviculturist thoroughly grounded in silvics and plant physiology, plus a statistician of real ability. The problem is beyond one or the other working alone and equally unsolvable to one who is only half proficient in both fields.

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APPENDIX

Subject tree		Competing Trees			
No.	dbh	Quad	No.	dbh	Dist.
128	12	NE	119*	5	17
			118	5	15
			117*	4	19
		SE	116	6	19
			122	6	17
			121*	7	8
			126	13	7
			123	10	20
		SW	127	9	6
			125*	7	16
			136	9	14
			139*	5	16
			135	9	19
		NW	134	8	18
			133	6	20
			132	7	18
			131	7	16
			130*	7	14
			129	7	14
237	16	NE	239	7	18
		SE	236	16	9
			235	8	10
			234	7	10
			233	5	15
			328	6	19
		SW	326	10	8
			325	9	14
			324*	5	14
		NW	243	6	15
			242	6	10
			240	6	18
			238*	4	12

Sample of basic data as taken from stem map. Competing trees include all trees within 20 ft. of subject tree, separated by quadrants in which they occur. Starred trees (*) died during growth period. "Distance" is distance between competing tree and subject tree.

Subject Tree		Competing trees				$d' = [a/2(D_o+D)+k] - d$					
						a=1.6		a=1.2			a=.8
dbh (D_o)	No.	quad	No.	dbh (D)	dist. (d)	k=5	k=3	k=7	k=5	k=3	k=7
						d'	d'	d'	d'	d'	d'
9	257	NE	256*	5	12	4	2	3	1		1
			255	6	14	3	1	2			
			253	11	14	7	5	5	3	1	1
		SE	300	6	6	11	9	10	8	6	7
			301	5	13	3	1	2			
			299	8	14	5	3	3	1		
		SW	295*	5	12	4	2	3	1		1
			296	7	16	2		1			
			294	8	16	3	1	1			
			259	9	7	12	10	11	9	7	7
			260	7	8	10	8	9	7	5	5
			258*	5	5	11	9	10	8	6	8
		NW	263	5	14	2		1			
			262	8	14	5	3	3	1		
			261	18	14	13	11	9	7	5	4
				Sd'		95	65	73	46	30	34
				$S(d')^2$		821	501	555	320	172	206
				Sd'D		811	585	609	403	265	273
				$S(d')^2D$		7687	4895	4787	2783	1473	1539

Sample page of data showing organization for computation of d' variables used in area-of-overlap hypothesis. Only those trees are recorded which are closer than the maximum distance expression: $a/2(D_o+D)+k$. Competing trees marked * died during growth period.

Subject Tree		Dia. Growth	z-2.00			$\frac{Sd'}{F}$	$\frac{Sd'^2}{F}$	$\frac{Sd'D}{F}$	$\frac{Sd'^2D}{F}$
No.	dbh	(in.)	Y	Y'	Y-Y'	X_1	X_2	X_3	X_4
158	4.4	0.3	2.17	1.78	+.39	2.3	25.8	30.4	397
265	5.0	0.0	1.78	2.03	-.25	2.2	13.7	17.5	103
93	5.3	0.2	1.90	2.58	-.68	1.0	4.0	7.8	32
218	5.5	0.1	1.74	1.93	-.19	1.4	9.0	15.4	112
328	5.6	0.0	1.61	1.52	+.09	1.0	5.0	14.9	87
240	5.8	0.1	1.65	1.74	-.09	1.5	13.3	17.2	159
122	6.1	1.0	2.46	2.06	+.40	1.4	8.8	12.7	76
255	6.2	0.8	2.23	1.67	+.56	1.7	14.9	17.0	144
284	6.5	0.7	2.04	2.26	-.22	1.5	12.2	12.8	112
314	6.6	0.6	1.91	2.24	-.33	3.2	23.4	22.6	173
239	6.9	0.7	1.92	1.90	+.02	1.5	7.5	14.2	65
317	7.2	0.5	1.63	1.85	-.22	3.4	29.2	23.7	200
298	7.4	0.3	1.37	2.19	-.82	2.5	17.8	17.7	121
197	7.6	0.9	1.92	1.94	-.02	1.7	16.7	14.3	136
268	8.0	0.5	1.40	1.84	-.44	2.4	20.4	19.7	168
294	8.4	1.5	2.28	2.38	-.10	2.1	13.6	14.5	89
235	8.5	0.1	0.85	1.18	-.33	2.5	31.7	31.6	413
185	8.5	1.8	2.55	1.95	+.60	2.2	20.2	17.6	164
259	8.8	0.8	1.46	1.60	-.14	3.0	27.0	23.5	207
257	8.9	2.1	2.73	1.86	+.87	2.4	18.5	20.3	160
183	8.9	1.2	1.83	1.90	-.07	2.2	18.9	17.1	140
213	9.5	1.3	1.76	1.79	-.03	1.8	16.1	18.9	181
91	10.1	2.7	3.01	2.44	+.57	1.8	16.1	14.0	145
128	11.9	2.8	2.72	2.15	+.57	1.6	14.9	14.8	151
114	12.8	1.9	1.67	1.61	+.06	2.2	25.7	21.5	258
357	13.0	3.8	3.53	2.59	+.94	3.3	23.6	19.6	138
20	13.1	3.1	2.81	2.09	+.72	1.5	17.8	15.7	202
107	13.4	1.9	1.57	1.81	-.24	2.7	25.7	24.6	260
112.	15.7	2.4	1.76	2.05	-.33	2.4	28.0	24.4	330
100	15.7	3.0	2.32	2.57	-.25	1.2	7.0	8.5	48
237	15.8	2.6	1.90	2.20	-.30	2.3	23.4	21.8	265
66	18.9	2.4	1.28	2.04	-.76	1.6	18.2	15.9	194

Observations set up for computation of extensions needed for solution of normal equations. The d' variables are based on the maximum distance expression, $.6(D_o + D)+7$. Y is the residual from the growth over dbh curve plus 2.00. The constant, 2.00, is added to avoid negative values for Y. It is subtracted from the constant "a" in the prediction equation to give the predicted deviation directly. Y' is the predicted value of Y from the prediction equation.

SX_1	65.5	X_1	=	2.046875			
SX_2	568.1	X_2	=	17.753125			
SX_3	582.2	X_3	=	18.193750			
SX_4	5,430.	X_4	=	169.687500			
SY	63.72	Y	=	1.991250			
SX_1^2	146.85	-	134.070312	=	Sx_1^2	=	12.779688
SX_1X_2	1,284.32	-	1,162.829688	=	Sx_1x_2	=	121.490312
SX_1X_3	1,268.60	-	1,191.690625	=	Sx_1x_3	=	76.909375
SX_1X_4	11,960.3	-	11,114.531250	=	Sx_1x_4	=	845.768750
SX_1Y	129.732	-	130.426875	=	Sx_1y	=	-0.694875
SX_2^2	11,742.21	-	10,085.550312	=	Sx_2^2	=	1,656.659688
SX_2X_3	11,408.36	-	10,335.869375	=	Sx_2x_3	=	1,072.490625
SX_2S_4	113,529.7	-	96,399.468750	=	Sx_2x_4	=	17,130.231250
SX_2Y	1,105.140	-	1,131.229125	=	Sx_2y	=	-26.089125
SX_3^2	11,484.64	-	10,592.401250	=	Sx_3^2	=	892.238750
SX_3X_4	112,253.2	-	98,792.062500	=	Sx_3x_4	=	13,461.137500
SX_3Y	1,127.487	-	1,159.305750	=	Sx_3y	=	-31.818750
SX_4^2	1,171,721.	-	921,403.125000	=	Sx_4^2	=	250,317.875000
SX_4Y	10,340.53	-	10,812.487500	=	Sx_4y	=	-471.957500
SY^2	136.6432	-	126.882450	=	Sy^2	=	9.760750

Extensions computed from observational data on preceding page. Corrected for departure from mean.

Normal Equations:

I $b_1 Sx_1^2 + b_2 Sx_1x_2 + b_3 Sx_1x_3 + b_4 Sx_1x_4 = Sx_1y$
II $b_1 Sx_1x_2 + b_2 Sx_2^2 + b_3 Sx_2x_3 + b_4 Sx_2x_4 = Sx_2y$
III $b_1 Sx_1x_3 + b_2 Sx_2x_3 + b_3 Sx_3^2 + b_4 Sx_3x_4 = Sx_3y$
IV $b_1 Sx_1x_4 + b_2 Sx_2x_4 + b_3 Sx_3x_4 + b_4 Sx_4^2 = Sx_4y$

I $b_1\ 12.779688 + b_2\ 121.490312 + b_3\ 76.909375 +$
$b_4\ 845.768750 = -0.694875$
II $b_1\ 121.490312 + b_2\ 1{,}656.659688 + b_3\ 1{,}072.490625 +$
$b_4 17{,}130.231250 = -26.089125$
III $b_1\ 76.909375 + b_2\ 1{,}072.490625 + b_3\ 892.238750 +$
$b_4 13{,}461.137500 = -31.818750$
IV $b_1\ 845.768750 + b_2 17{,}130.231250 + b_3\ 13{,}461.137500 +$
$b_4 250{,}317.875000 = -471.957500$

	X_1	X_2	X_3	X_4	Y
I	12.779688	121.490312	76.909375	845.768750	-0.694875
I'	-1.000000	-9.506516	-6.018095	-66.180704	+0.054373
II		1656.659688	1072.490625	17,130.231250	-26.089125
I(-9.506516)		-1154.949595	-731.140204	-8,040.314154	+6.605840
S_2		501.710093	341.350421	9,089.917096	-19.483285
II'		-1.000000	-0.680374	-18.117868	+0.038834
III			892.238750	13,461.137500	-31.818750
I (-6.018095)			-462.847925	-5,089.916686	+4.181824
S_2(-0.680374)			-232.245951	-6,184.543254	-13.255921
S_3			197.144874	2,186.677560	-14,381005
III'			-1.000000	-11.091729	+0.072946
IV				250,317.875000	-471.957500
I (-66.180704)				-55,973.571296	+45.987317
S_2(-18.117868)				-164,689.918076	+352.995586
S_3(-11.091729)				-24,254.034906	+159.510210
S_4				5,400.350722	+86.535613
IV'				-1.000000	-0.016024
Back solution:					
	-0.054373	-0.038834	-0.072946	+0.016024	
	-1.060480	-0.290321	-0.177734		
	+1.508616	+0.170556			
	+1.507241				
	+1.901487	-0.158599	-0.250680	+0.016024	

Solution of normal equations by the Doolittle method. For complete explanation see Ezekiel, "Methods of Correlation Analysis", second edition, Appendix 1, p.464.

Variable	Regression Coefficient	Equation IV	Check
X_1	+1.901487	845.768750	+1,608.218283
X_2	-0.158599	17,130.231250	-2,716.837546
X_3	-0.250680	13,461.137500	-3,374.437948
X_4	+0.016024	250,317.875000	+4,011.093629
sums		-471.957500	-471.9636

(cont.)

Equation Y	Computation of R^2	Means	Computation of a
-0.694875	-1.321296	2.046875	+3.892106
-26.089125	+4.137709	17.753125	-2.815628
-31.818750	+7.976324	18.193750	-4.560809
-471.957500	-7.562647	169.687500	+2.719072
+9.760750	+3.230090		-.765259

Check on solution of normal equations. Computation of correlation coefficient. Computation of constant, "a", in prediction equation. (See Ezikiel, "Methods of Correlation Analysis", second edition, Appendix 1, p.467)

$$R^2 = 3.230090/9.760750$$
$$= .330926$$

$$R = .575$$

$$a = 1.991250-(-.765259) = 2.756509$$

Prediction equation:

$$(z+2.00)' = 2.756509 = 1.901487X_1 - 0.158599X_2 - 0.250680X_3 + 0.016024X_4$$

$$Y' = z' = 0.756509 + 1.901487X_1 - 0.158599X_2 - 0.250680X_3 + 0.016024X_4$$