Working Paper

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Ross School of Business Working Paper
Working Paper No. 1300
May 2015

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May 28, 2015

In this paper, we study a conditional upgrade strategy that has recently become very common in the travel industry. After a consumer makes a reservation for a product (e.g., a hotel room), she is asked whether she would like to upgrade her product to a higher-quality (more expensive) one at a discounted price. The upgrade, however, is not fulfilled immediately. The firm fulfills upgrades at check-in if higher-quality products are still available, and the upgrade fee is only charged to the consumer if she gets upgraded. Consumers decide which product type to book and whether to accept an upgrade offer or not based on the anticipated upgrade probability. We model the consumers' decisions using a Poisson-arrival game framework with incomplete information and prove the existence of Bayesian Nash equilibrium. To further study the firm's optimal upgrade pricing strategy and develop managerial insights, we also analyze a fluid model which is the asymptotic version of the stochastic model. Our numerical studies validate that our theoretical results derived from the fluid model carry through to the stochastic model.

Our analysis identifies multiple benefits of conditional upgrades. First, the firm is able to capture more demand by offering conditional upgrades, i.e., the consumers who value original product types lower than the original prices but value higher-quality products higher than the discounted price with upgrades. Second, conditional upgrades enable the firm to improve its market segmentation by inducing more consumers to purchase higher-quality products. Third, conditional upgrades give the firm more flexibility in better matching fixed capacities to stochastic demands. For a firm that is a price taker, offering conditional upgrades is effective in compensating for the firm's lack of ability in setting its prices optimally, and can sometimes generate even higher revenues than being able to optimize product prices. For a firm that has the ability to optimize product prices, conditional upgrades can generate higher revenues than dynamic pricing.

Key words: conditional upgrades; strategic consumers; travel industry; revenue management; Bayesian Nash equilibrium; asymptotic analysis

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1. Introduction

Like many other industries, a big challenge faced by the travel industry is the mismatch between demand and supply across different types of products. In the travel industry (e.g., hotels, airlines, car rental companies, cruise lines), consumers usually make reservations in advance and the products are perishable in the sense that they do not generate value for the firm after the end of the booking period. The capacity for each type of product is fixed, but due to the stochastic demand across different product types over time, firms frequently find capacity of some product types under-utilized while capacity of other product types in shortage at the end of the booking period. Ideally, firms should be able to eliminate the demand-supply mismatch by having enough flexibility in pricing their products. However, in reality, different industries face different constraints in achieving flexibility to set prices.

In the hotel industry, a lot of firms lack the ability to adjust prices dynamically. Due to consumer resistance, dynamic pricing (i.e., adjusting prices for the same product over time during the booking period) is not as common as in the airline industry. Some hotels do not use dynamic pricing at all but only use variable pricing (i.e., setting different nightly rates for the same room based on expected demand but keeping the rate for a room offered on a particular night fixed during the booking period) as their primary pricing strategy. Others use dynamic pricing for their "best available rates" but have had a hard time convincing their consumers, especially corporate travel buyers. For example, hotel chains would like to change prices dynamically and give large travel accounts a negotiated discount off the dynamic best available price. However, according to the survey by Business Travel News conducted on 221 travel buyers, more than two-thirds said that they did not use dynamic pricing in their hotel program (Baker 2010). Instead, most travel buyers negotiate a fixed corporate rate which does not change dynamically. 16% of travel buyers used dynamic pricing only with select hotel chains, 9% used dynamic pricing only in low-volume markets, and only 6% reported that their use of dynamic pricing is standard.

Even with variable pricing, hotels still face constraints on setting room rates optimally. In competitive industries such as travel, firms usually have several direct competitors, hence have less flexibility to adjust product prices as they like. Since consumers can compare prices for similar products very easily on the Internet where online travel agencies such as Orbitz and Expedia have provided such services, most firms providing similar products set similar prices for at least some of their products. For example, the following three hotels all reside in Ann Arbor, Michigan: Hilton Garden Inn, Residence Inn by Marriott, Sheraton. These are all upscale mid-priced hotels, and are located within 1 mile from each other. Thus, they are direct competitors in the local market. As a result, all three hotels use exactly the same (variable rather than dynamic) pricing strategy for standard rooms (with either one king-size bed or two queen-size beds). For example, the price in September and October 2013 was $169 for weekdays and $139 for Friday/Saturday nights.

While hotels have struggled with widespread acceptance of dynamic pricing and some are price takers in the market, many hotels have recently adopted a new type of conditional upgrade policy. This new strategy works in the following way. After a consumer makes a reservation, she is offered an upgrade option which she decides whether to accept or not. If she accepts the upgrade offer, then she will be notified whether she gets upgraded or not during check-in. By accepting the upgrade offer, the consumer agrees that she will pay the associated upgrade fee if her upgrade is fulfilled by the hotel later. The hotel fulfills upgrades if there are higher-quality products still available by the check-in date. Many of the hotels use Nor1, a leading technological company, to offer the upgrades and decide the price of the upgrades.[1] These new upgrades are different from the upgrades historically offered by hotels where elite travelers may be upgraded for free at check-in as part of their consumer loyalty program benefits. First, these are paid upgrades instead of free upgrades. Second, they are conditional upgrades because a consumer does not know whether she will be upgraded and pay the upgrade fee when she accepts an upgrade offer; the upgrades are fulfilled conditional on the availability of higher-quality products by the check-in date. Third, they are offered to not only elite members but also regular consumers. Fourth, instead of being offered at check-in, the upgrades we consider are offered in advance, usually right after the original booking.

However, offering conditional upgrades may result in some consumers, who would purchase higher-quality products when the firm does not offer conditional upgrades, deliberately booking less expensive products as they hope to get upgraded and pay less than the original price of higher-quality products they actually prefer. Thus, conditional upgrades have the potential to cannibalize the higher-quality product sales. When using the conditional upgrade strategy, it is important for the firm to carefully account for such consumer behaviors in setting upgrade prices optimally. In this paper, we study how firms can properly manage the trade-off between the conditional upgrade strategy's potential benefits and potential threats such as cannibalization. More specifically, the research questions we investigate are: 1) what is the optimal conditional upgrade pricing strategy for the firm when consumers may deliberately choose lower-quality products with upgrades? 2) When and why are conditional upgrades profitable/non-profitable for the firm? 3) How profitable is the conditional upgrade strategy compared to other types of upgrade strategy as well as being able to set product prices optimally, in particular, can it replace product price optimization and dynamic pricing?

To answer these questions, we study a model where consumers select which product type to book and whether or not to accept an upgrade offer based on the anticipation of future upgrade probability. Our model analyzes the upgrade policy as currently implemented by hotels and Nor1, where upgrade prices are static over time. Our analysis indicates that conditional upgrades significantly improve revenues of the firm by "demand expansion", "price correction", and "risk management". The conditional upgrades are "real options" that consumers purchase from the firm to be exercised with an upgrade fee if the higher-quality products are still available by the end of the booking

[1] Besides hotels, Nor1 is also expanding its business to airlines, cruise lines, car rentals.

period. We find that this type of options expands the firm's demand by capturing the consumers who are not willing to pay the full price of higher-quality products but still value higher-quality products significantly more than regular products. If the firm does not have pricing flexibility due to competition or other industry constraints, conditional upgrades can be an instrument to correct the firm's original price for higher-quality products and reoptimize the firm's demand segmentation to improve demand-supply matching. Our numerical studies show that by properly using conditional upgrades, the firm can capture at least the revenue potential from being able to optimize the higher-quality product price. Interestingly, we also identify situations where conditional upgrades can generate even higher revenues than the case where the firm can set both product prices optimally but do not offer upgrades. This implies that conditional upgrades can compensate for the firm's lack of ability to set the optimal product prices by managing prices and capacities in a more flexible way. Moreover, offering conditional upgrades generate higher revenues than offering last-minute upgrades in most cases. Thus, our paper provides an analytical justification for "conditional" upgrades becoming more popular in travel industries. Finally, if the firm does have the ability to set product prices optimally, then our numerical results indicate that the revenue improvements with conditional upgrades are generally larger than the revenue improvements with dynamic pricing. By offering conditional upgrades, the firm allocates the consumers who accept the upgrade offers to different types of products at the end of the booking period. One of our interesting findings is that this ex-post allocation flexibility that the firm gains with conditional upgrades is generally more valuable than the pricing flexibility one has in dynamic pricing. Interestingly, these observations hold true even for the case where the firm sets only a static upgrade price, indicating that the potential of conditional upgrades to "correct" for mispricing of product prices may be even higher when dynamic upgrade prices can be used.

2. Literature Review

Although upgrades are widely used in service industries such as travel, there is limited academic literature that focuses on upgrades in service industries. Most of the literature studies upgrades in the context of airlines where upgrades are offered to preferred travelers as a perk or if the flight's economy cabin is overbooked (see for example Karaesmen and Van Ryzin 2004). Gallego and Stefanescu (2009) is one of a handful of papers that study upgrades in detail. They first study free upgrades by generalizing the traditional network revenue management model (where product prices are fixed and demands for different product types are independent) to explicitly account for upgrades. They also study paid upgrades and find that if a primary capacity provider has complete freedom to select prices, upgrades cannot improve profits. The result found by Gallego and Stefanescu (2009) is based on a fluid model. By considering demand randomness, we find that the firm can strictly improve revenues with conditional upgrades compared to having complete freedom to select product prices. Biyalogorsky et al. (2005) study conditional upgrades where the upgrade fee is charged at the time of upgrade request (i.e., a consumer pays the upgrade fee even if she does not get upgraded at the end) and find that upgrades increase the provider's profits when the probability of selling higher-quality units at full price is sufficiently high. The upgrade strategy studied in Biyalogorsky et al. (2005) is similar to an industry practice where only passengers who hold more expensive "upgradable class" tickets can be upgraded if there is available capacity at the fulfillment time. In our paper, we analyze a more recent upgrade strategy pioneered by Nor1 for the travel industry (i.e., selling conditional upgrades where the fee is paid only if the upgrade is fulfilled). Furthermore, unlike Gallego and Stefanescu (2009) and Biyalogorsky et al. (2005), we model the strategic consumer behavior and analyze conditional upgrades with a Bayesian game. The strategic consumer behavior significantly changes the insights.

There is also a stream of literature studying multi-product inventory management with provider-driven demand substitution. Hsu and Bassok (1999), Bassok et al. (1999) study full downward substitution where a consumer can be served by another product with superior quality. Netessine et al. (2002), Shumsky and Zhang (2009) study single-level upgrades where consumers may be upgraded by at most one product level. Although primarily focusing on inventory management or capacity management, these papers also consider upgrades. The main difference from our paper is that in these papers, the upgrade decision is entirely made by the provider and no additional fee is charged to the consumer, while in our paper, consumers get to decide whether they would like to be upgraded to a higher-quality product if it is still available by the end of the booking period. Moreover, in the above papers, consumers are not strategic when making their product purchasing decisions and do not take the future upgrade possibility into consideration, while we model this strategic behavior of consumers.

A growing literature in operations management studies the interaction between consumers' strategic behavior and firm's decisions (see Netessine and Tang 2009 for a detailed review). For example, a problem that has been extensively studied is the consumers' deliberate waiting to purchase later in anticipation of a price decrease when the firm can change prices over time (Su 2007, Elmaghraby et al. 2008, Gallego et al. 2008, Yin et al. 2009, Levin et al. 2010, Mersereau and Zhang 2012). Aviv and Pazgal (2008), Osadchiy and Vulcano (2010), Correa et al. (2013) model the strategic consumers' purchasing decisions as a game with incomplete information and assume Poisson arrival of consumers to capture the randomness in the number of players in the game. We adopt the same assumption to model the random arrival of consumers over time to book different types of products. While the papers mentioned above consider a single product type and focus on the consumers' decision of "buy-now-or-wait", we model a firm selling multiple substitutable product types and study the consumers' decisions on which type of product to book and whether to accept an upgrade offer or not.

Jerath et al. (2010) study the effect of strategic consumer behavior if competing firms offer last-minute sales through opaque channels versus through direct channels. Fay and Xie (2008) study probabilistic selling where the firm creates a probabilistic product by creating uncertainty about the type of product that a consumer will eventually receive. In opaque and probabilistic selling, the different product types are horizontally differentiated (i.e., differentiated based on a single characteristic other than quality), while with conditional upgrades, the different product types are vertically differentiated (i.e., they can be ordered according to quality). With the conditional upgrade strategy, the provider sells an option to the consumer so that the consumer can obtain a higher-quality product if the capacity is available at the fulfillment time. Due to the quality difference between the product types, consumers pay an exercise fee when the upgrade option is fulfilled, which is different from opaque and probabilistic selling. Our paper is also methodologically different than the above papers in that we model the consumers' booking decisions as a Bayesian game with Poisson arrivals. In our paper, a consumer forms an expectation about the upgrade probability based on her arrival time and the product availability information, and decides which product type to book and whether or not to accept an upgrade offer.

3. Model

We consider a firm that sells two types of perishable products, regular and high-quality (e.g., standard rooms and suites in a hotel). The firm has K_H high-quality products and K_R regular products. The products are consumed at time T and consumers arrive to book the products during the booking period $[0,T]$. The products are perishable in the sense that they have no value to the firm after time T. The high-quality products are sold at price p_H and the regular products are sold at price p_R ($p_H > p_R$). After a consumer books a regular product, the firm may offer an upgrade opportunity so that the consumer can pay an additional fee p to upgrade the product to a high-quality one if high-quality products are still available by the end of the booking period. Although the firm does not guarantee the fulfillment of an upgrade, a consumer only needs to pay the upgrade fee if she actually obtains an upgrade, and she is obliged to pay in this case. The firm offers upgrades to γ proportion of consumers.[2] Another interpretation is that $(1-\gamma)$ proportion of consumers are inattentive (i.e., do not consider the upgrade offer) when making their

[2] In reality, travel firms sell through multiple channels and may offer conditional upgrades in selected channels only. For example, Hilton offers conditional upgrades to consumers who book their rooms in hilton.com while it does not offer conditional upgrades if consumers book through online travel agencies.

purchasing decisions even if the firm offers them conditional upgrades.[3] We assume, consistent with industry practice, that if the firm does not have enough remaining high-quality products to satisfy all consumers that have accepted the upgrade offers, these consumers are rationed randomly, that is, the probability that a consumer gets upgraded does not depend on her booking time. The firm's goal is to optimally choose the upgrade price given product prices so that its revenue from selling two types of products as well as collecting upgrade fees is maximized. As stated before, settings where firms are price takers on product prices but can set upgrade price are common in practice. In Section 7 where we evaluate the revenue performance of conditional upgrades, we will also consider a firm that is not a price taker at all and demonstrate that conditional upgrades also have great value for such a firm.

Consumers arrive to the market following a Poisson process with rate λ. Each consumer is characterized by a pair of valuations (v_R, v_H), where v_R denotes her valuation for regular products and v_H denotes her valuation for high-quality products. A consumer observes her private valuations when arriving to the market. The valuations of consumers are jointly distributed in the two-dimensional support Ω which is a finite subset of $\mathbb{R}_+^2$. The joint probability density function is denoted by $f(v_R, v_H)$.[4] By allowing a joint distribution of consumers' valuations for different product types, we are able to capture not only the consumers' heterogeneity in the willingness to pay but also their heterogeneity in the valuation differential between different product types, which is important in making decisions regarding upgrades. Thus, the way we model consumer valuations is more general than the traditional approach used by the market segmentation literature (e.g., Mussa and Rosen 1978, Moorthy and Png 1992) where consumers' valuations for different product types are proportional. The Poisson arrival rate, consumer valuation distribution, percentage of consumers that are offered upgrades, and product prices and capacities are common information for the firm and the consumers.

Consumers are strategic in the sense that a consumer booking at time t and seeing products are still available anticipate the probability $q(t)$ of actually obtaining an upgrade if she accepts the upgrade offer. Consumers' rational expectations on the upgrade probability $q(t)$ depend on the arrival time because we allow consumers to infer the upgrade probability from the fact that

[3] As studied in the recent economics literature, consumers may pay attention to part of the price, menu of products or offerings. When a firm offers a multi-dimensional product, consumers may take only a subset of these dimensions into consideration. This is exemplified by Spiegler (2006), where a consumer samples one price dimension from each firm selling a product with a complicated pricing scheme (e.g., health insurance plans); Gabaix and Laibson (2006), where some consumers do not observe the price of an add-on before choosing a firm; Armstrong and Chen (2009), who extend the notion of "captive" consumers to those who always consider one dimension of a product but not another (e.g., price but not quality).

[4] Our equilibrium analysis for the stochastic model can be generalized to time-dependent arrival rates and time-dependent consumer valuation distributions.

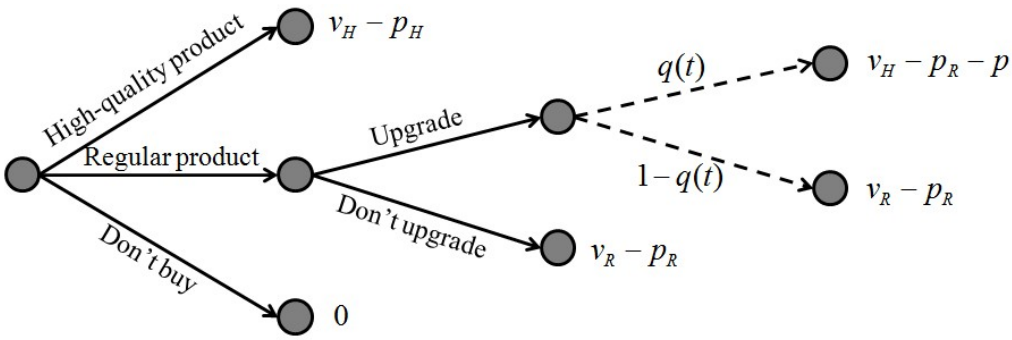


Figure 1 **Consumer decision process**

products have not been fully booked by time t. Figure 1 depicts the consumer decision process and the payoffs from each possible decision. We use "H" to denote booking a high-quality product, use "U" to denote booking a regular product and accepting an upgrade offer, use "R" to denote booking a regular product without upgrade, and use "N" to denote not booking any product. The consumers that are not offered upgrades choose from "H", "R", and "N". Note that if $p \geq p_H - p_R$, nobody accepts the upgrade offer because the total price to pay in order to get a high-quality product through upgrade is at least as large as the original price for high-quality products. This is equivalent to the case where upgrades are not offered.

The firm needs to decide when to stop selling each product type, taking into account the instant booking levels for each product type where upgrades is considered as a unique type. Define $N_H(t)$, $N_U(t)$, $N_R(t)$ as the demand stream booking each product type, respectively. Note that $N_U(t)$ is the arrival process of consumers booking a regular product and accepting the upgrade offer, and $N_R(t)$ is the arrival process of consumers booking a regular product and not accepting the upgrade offer, hence $N_U(t)$ and $N_R(t)$ are mutually exclusive. Due to the decomposition property of Poisson processes, $N_H(t)$, $N_U(t)$, and $N_R(t)$ are independent Poisson processes. We assume the firm cannot "bump" consumers upon check-in (i.e., the firm has to accommodate check-in requests of all reservation holders). The firm stops selling high-quality products when $N_H(t) \geq K_H$ and stops selling regular products when $N_R(t) \geq K_R$, that is, the firm tries to sell as many products as possible. Moreover, the firm stops selling both product types at the same time when $N_H(t) + N_U(t) + N_R(t) \geq K_H + K_R$. Note that this stopping rule allows the firm to accept more bookings for regular products during the booking period than the capacity (because some of the consumers booking regular products with upgrades may later get upgraded and free up some capacity for regular products) while ensuring no bumping of consumers.

A consumer does not observe the firm's instant capacities (also, how many consumers have arrived and the booking decisions they have made) when she makes her booking decision. However, consumers can observe whether a product type is fully booked or still available when making booking decisions. As the firm stops selling some product type, consumers are restricted to fewer choices.

When the high-quality products are unavailable, consumers can only book regular products without upgrades. When the regular products are unavailable, consumers can only book high-quality products. When both types of products are unavailable, consumers cannot book any product. We can see that when at least one product type is unavailable, the consumer decision becomes a simple take-it-or-leave-it decision, so consumers do not anticipate the upgrade probability anymore. Let τ denote the first time when some product type is unavailable ($\tau = T$ if the firm never stops selling any type of product during the booking period), then τ is the (random) stopping time of the consumer booking game that strategic consumers play regarding upgrades.

4. Consumer Booking Equilibrium

Before deriving the firm's optimal conditional upgrade policy, we first need to analyze how strategic consumers make their booking decisions. In this section, we derive and characterize the symmetric pure-strategy equilibrium of the consumer booking game for a given upgrade price p. Upon arrival, a consumer observes her valuations for two product types (v_R, v_H) and arrival time t as well as the availability of product types, and books the product type that maximizes her expected utility. For a consumer that is offered an upgrade, the key to her booking decision is the expected upgrade probability $q(\cdot)$ she anticipates which is a function of her booking time t. Let $a_t(v_R, v_H|q(t))$ denote the consumer's utility-maximizing decision if she arrives at time t, has valuations (v_R, v_H), and anticipates the upgrade probability to be $q(t)$.[5] Similarly, let $a'_t(v_R, v_H)$ denote the utility-maximizing decision of a consumer that is not offered an upgrade.

Now we derive $a_t(v_R, v_H|q(t))$ and $a'_t(v_R, v_H)$. Figure 1 shows the consumers' utilities from booking different product types. The consumer's utility from booking a high-quality product is $v_H - p_H$, the utility from booking a regular product without upgrade is $v_R - p_R$, the expected utility from booking a regular product with upgrade is $q(t)(v_H - p_R - p) + [1 - q(t)](v_R - p_R)$, the utility from not booking any product is zero. Thus, the consumer chooses to book a high-quality product if $v_H - p_H \geq \max\{q(t)(v_H - p_R - p) + [1 - q(t)](v_R - p_R), v_R - p_R, 0\}$; she chooses to book a regular product with upgrade if $q(t)(v_H - p_R - p) + [1 - q(t)](v_R - p_R) \geq \max\{v_H - p_H, v_R - p_R, 0\}$; she chooses to book a regular product without upgrade if $v_R - p_R \geq \max\{v_H - p_H, q(t)(v_H - p_R - p) + [1 - q(t)](v_R - p_R), 0\}$; otherwise, she does not book any product. We can simplify the above decision rule to the following:

- If $p \geq p_H - p_R$,

$$a_t(v_R, v_H|q(t)) = \begin{cases} H \text{ if } v_H - v_R \geq p_H - p_R \text{ and } v_H \geq p_H, \\ R \text{ if } v_H - v_R < p_H - p_R \text{ and } v_R \geq p_R, \\ N \text{ otherwise.} \end{cases}$$

[5] We use $q(\cdot)$ to denote the whole function, and $q(t)$ to denote its value at t.

- If $0 \leq p < p_H - p_R$,

$$a_t(v_R, v_H|q(t)) = \begin{cases} H \text{ if } v_H - v_R \geq \frac{p_H - p_R - q(t)p}{1-q(t)} \text{ and } v_H \geq p_H, \\ U \text{ if } p \leq v_H - v_R < \frac{p_H - p_R - q(t)p}{1-q(t)} \text{ and } q(t)v_H + [1-q(t)]v_R \geq p_R + q(t)p, \\ R \text{ if } v_H - v_R < p \text{ and } v_R \geq p_R, \\ N \text{ otherwise.} \end{cases}$$

The utility-maximizing decision of consumers that are not offered upgrades, $a'_t(v_R, v_H)$, is same as $a_t(v_R, v_H|q(t))$ with $p \geq p_H - p_R$. We next focus on consumers that are offered upgrades and find their equilibrium strategy.

If $0 \leq p < p_H - p_R$, $a_t(v_R, v_H|q(t))$ divides Ω into four subsets. Given $q(\cdot)$, $a_t(v_R, v_H|q(t))$ is uniquely determined for each (v_R, v_H) and each t, and $a_t(v_R, v_H|q(t))$ can be easily computed by plugging $q(t)$ into the equation of $a_t(v_R, v_H|q(t))$. Thus, we use $q(\cdot)$ to define the consumer's strategy in the booking game. The reason for using $q(\cdot)$ as the strategy instead of $a_t(v_R, v_H|q(\cdot))$ is that the corresponding strategy space has fewer dimensions and the computational burden of equilibrium is smaller. The strategy space is then defined as $\mathcal{Q} = \{q(\cdot) : [0,T] \to [0,1]$, such that $q(\cdot)$ is differentiable$\}$. $\mathcal{Q}$ contains all differentiable functions of $t \in [0,T]$ taking values between 0 and 1.

To find the symmetric equilibrium $q^*(\cdot)$, we first fix one consumer (we call this consumer the acting consumer) and calculate the expected upgrade probability for the acting consumer if she books a regular product and accepts an upgrade offer when all other consumers are making their decisions based on $q(\cdot)$. Denote this resulting upgrade probability for the acting consumer as $b(q(\cdot))$, $b(q(\cdot))$ is also a function of t. Then, $q^*(\cdot)$ is the solution to $b(q^*(\cdot)) = q^*(\cdot)$. We can write $b(q(\cdot))$ as $b(q(\cdot)) = g(q(\cdot))/h(q(\cdot))$, where $g(q(\cdot))$ is the unconditional expected probability that a consumer arriving at time t accepts an upgrade offer and gets upgraded at the end of the booking period, and $h(q(\cdot))$ is the probability that both product types are still available by time t. So, $b(q(\cdot))$ is the expected upgrade probability conditioning on the fact that products are still available at time t.

Now we derive $g(q(\cdot))$ and $h(q(\cdot))$. With a slight abuse of notation, we use $N_H(t|q(\cdot))$, $N_U(t|q(\cdot))$, $N_R(t|q(\cdot))$ to denote the arrival processes of other consumers (as seen by the acting consumer) booking each product type given that the strategy they are using is $q(\cdot)$. Let $\tau(q(\cdot))$ denote the stopping time of the consumer booking game (i.e., the time when the firm stops selling at least one product type) if the acting consumer chooses to book a regular product and accept an upgrade offer and all other consumers make their booking decisions based on $q(\cdot)$. Then, we have

$$g(q(\cdot)) = \mathop{\mathbb{E}}_{N_H(t|q(\cdot)), N_U(t|q(\cdot)), N_R(t|q(\cdot))} \left\{ \min\left\{ \frac{[K_H - N_H(\tau(q(\cdot))|q(\cdot))]^+}{N_U(\tau(q(\cdot))|q(\cdot)) + 1}, 1 \right\} \cdot \mathbb{1}\{t \leq \tau(q(\cdot))\} \right\}$$

where the "+1" term represents the acting consumer, and

$$h(q(\cdot)) = \mathbb{P}(N_H(t|q(\cdot)) < K_H, N_R(t|q(\cdot)) < K_R, N_H(t|q(\cdot)) + N_U(t|q(\cdot)) + N_R(t|q(\cdot)) < K_H + K_R).$$

Note that $g(q(\cdot))$ and $h(q(\cdot))$ are both functions of t. To completely characterize $g(q(\cdot))$ and $h(q(\cdot))$, it remains to characterize $N_H(t|q(\cdot))$, $N_U(t|q(\cdot))$, $N_R(t|q(\cdot))$ as well as $\tau(q(\cdot))$.

LEMMA 1. *(Myerson 1998: Environmental equivalence property of games with Poisson arrivals[6]) From the perspective of any one player, the arrival process of other players is also a Poisson process with the same rate as the total arrival rate.*

Lemma 1 implies that $N_H(t|q(\cdot))$, $N_U(t|q(\cdot))$, $N_R(t|q(\cdot))$ are indeed Poisson processes. Moreover, they have the same distributions as the overall arrival processes. Given $q(\cdot)$, the probabilities of any other consumer that is offered an upgrade booking each type of product are as follows:

$$\xi_H^\gamma(t|q(\cdot)) = \iint_\Omega \mathbb{1}\{a_t(v_R, v_H|q(\cdot)) = H\} f(v_R, v_H)\, \mathrm{d}v_R\, \mathrm{d}v_H,$$
$$\xi_U^\gamma(t|q(\cdot)) = \iint_\Omega \mathbb{1}\{a_t(v_R, v_H|q(\cdot)) = U\} f(v_R, v_H)\, \mathrm{d}v_R\, \mathrm{d}v_H,$$
$$\xi_R^\gamma(t|q(\cdot)) = \iint_\Omega \mathbb{1}\{a_t(v_R, v_H|q(\cdot)) = R\} f(v_R, v_H)\, \mathrm{d}v_R\, \mathrm{d}v_H.$$

The probabilities of any other consumer that is not offered an upgrade booking each type of product are as follows:

$$\xi'_H(t) = \iint_\Omega \mathbb{1}\{a'_t(v_R, v_H) = H\} f(v_R, v_H)\, \mathrm{d}v_R\, \mathrm{d}v_H,$$
$$\xi'_R(t) = \iint_\Omega \mathbb{1}\{a'_t(v_R, v_H) = R\} f(v_R, v_H)\, \mathrm{d}v_R\, \mathrm{d}v_H.$$

Thus, the arrival rates of $N_H(t|q(\cdot))$, $N_U(t|q(\cdot))$, $N_R(t|q(\cdot))$ are $\lambda_H(t|q(\cdot)) = \lambda\gamma\xi_H^\gamma(t|q(\cdot)) + \lambda(1-\gamma)\xi'_H(t)$, $\lambda_U(t|q(\cdot)) = \lambda\gamma\xi_U^\gamma(t|q(\cdot))$, $\lambda_R(t|q(\cdot)) = \lambda\gamma\xi_R^\gamma(t|q(\cdot)) + \lambda(1-\gamma)\xi'_R(t)$, respectively.

Next, we derive the stopping time $\tau(q(\cdot))$. Define the following auxiliary stopping times:

- $\tau_H(q(\cdot)) = \inf\{t \geq 0 : N_H(t|q(\cdot)) \geq K_H\}$.
- $\tau_R(q(\cdot)) = \inf\{t \geq 0 : N_R(t|q(\cdot)) \geq K_R\}$.
- $\tau_T(q(\cdot)) = \inf\{t \geq 0 : N_H(t|q(\cdot)) + N_U(t|q(\cdot)) + 1 + N_R(t|q(\cdot)) \geq K_H + K_R\}$.

$\tau_H(q(\cdot))$ is the time when high-quality products are fully booked, $\tau_R(q(\cdot))$ is the time when regular products are fully booked, $\tau_T(q(\cdot))$ is the time when the total demand reaches the firm's total

[6] Myerson (1998) first proved the environmental equivalence property of games with Poisson arrivals. Myerson (1998) provides a proof for the case of discrete player type set, but it is easily generalized to the case of continuous player type set (in our problem, the player type set is continuous because we assume a continuous valuation support). We refer the readers that are interested in theories of Poisson games to Myerson (1998), Myerson (2000) and Milchtaich (2004).

capacity so both product types are fully booked simultaneously. Then, the stopping time of the consumer booking game is $\tau(q(\cdot)) = \min\{\hat{\tau}(q(\cdot)), T\}$, where

$$\hat{\tau}(q(\cdot)) = \min\{\tau_H(q(\cdot)), \tau_R(q(\cdot)), \tau_T(q(\cdot))\} = \begin{cases} \tau_H(q(\cdot)) \text{ if } \tau_H(q(\cdot)) \le \tau_T(q(\cdot)), \\ \tau_R(q(\cdot)) \text{ if } \tau_R(q(\cdot)) \le \tau_T(q(\cdot)), \\ \tau_T(q(\cdot)) \text{ if } \tau_H(q(\cdot)) > \tau_T(q(\cdot)) \text{ and } \tau_R(q(\cdot)) > \tau_T(q(\cdot)). \end{cases}$$

$\hat{\tau}(\cdot)$ can be interpreted as the stopping time when $T \to \infty$. Note that the second equality in the above equation follows from the fact that $\tau_H(q(\cdot)) \le \tau_T(q(\cdot))$ implies $\tau_H(q(\cdot)) < \tau_R(q(\cdot))$ and that $\tau_R(q(\cdot)) \le \tau_T(q(\cdot))$ implies $\tau_R(q(\cdot)) < \tau_H(q(\cdot))$.

THEOREM 1. *There exists a symmetric pure-strategy equilibrium $q^*(\cdot)$ of the consumer booking game. $q^*(\cdot)$ is increasing in the arrival time of the consumer. Moreover, by equipping $\mathcal{Q}$ with the uniform norm $\|q(\cdot)\|_\infty = \sup_{0 \le t \le T} |q(t)|$, there exists a constant $\bar{\alpha}$ such that for any $q_1(\cdot), q_2(\cdot) \in \mathcal{Q}$, we have $\|b(q_1(\cdot)) - b(q_2(\cdot))\|_\infty \le \bar{\alpha}\|q_1(\cdot) - q_2(\cdot)\|_\infty$. Thus, if $\bar{\alpha} < 1$, $b(q(\cdot))$ is a contraction mapping and the equilibrium is unique.*

Theorem 1 states that the consumer booking game indeed has a symmetric pure-strategy equilibrium $q^*(\cdot)$ which is the solution to $b(q^*(\cdot)) = q^*(\cdot)$. $q^*(\cdot)$ is an increasing function because a consumer that arrives later and still finds both product types are available will have better knowledge that demand has realized to be weak, and hence form a higher probability of getting upgraded. Theorem 1 also gives a sufficient condition for $q^*(\cdot)$ to be unique.[7] However, due to the complicated structure of our consumer booking game with Poisson arrivals, it is not possible to derive the closed-form equilibrium or further analyze the firm's optimal upgrade pricing policy analytically (the firm's revenue function is given in Appendix B). We are able to derive some interesting results about the value of conditional upgrades in the stochastic model by focusing on special case valuation functions for consumers (which we do in Section 7.1). However, to study conditional upgrades in greater depth and develop more managerial and policy insights, we are going to first analyze a fluid model which is the asymptotic version of our stochastic model (i.e., scale up the capacities and demand rates by n and let $n \to \infty$). One may consider our fluid model as a deterministic approximation of the stochastic model where the consumer booking game is essentially with perfect information. However, as verified by our numerical examples in Sections 5.3 and 6, our fluid model is very accurate in approximating the stochastic model and the results and insights derived from the fluid model also hold in the stochastic model. In Section 7, we study a special case of the stochastic model analytically as well as the general stochastic model numerically, and derive additional insights.

[7] The formula of $\bar{\alpha}$ is complicated and is given in the proof of Theorem 1 in Appendix A. Our numerical studies indicate that $\bar{\alpha} < 1$ is satisfied when the product prices are far apart enough from each other and the capacity-demand ratio is moderately large. Note that $\bar{\alpha} < 1$ is a sufficient but not necessary condition for the equilibrium to be unique. In our extensive numerical studies with bivariate uniform and bivariate normal valuation distributions, we do not observe multiple equilibria to arise. In fact, as the capacities and demand rates increase proportionally to infinity, the equilibrium is provably unique (Theorem 3).

5. Fluid Model

In this section, we derive and analyze the fluid model. In Section 5.1, we derive the asymptotic consumer booking equilibrium by scaling up the problem size by n and letting $n \to \infty$. In the problem instance scaled by n, the consumer arrival rate is $n\lambda(t)$ and the firm's capacities are nK_H and nK_R. For other variables, we add a subscript of n to specify the problem size. Based on Section 5.1, in Section 5.2, we study the firm's optimal upgrade pricing strategy. In Section 5.3, we evaluate the performance of the fluid model.

5.1. Consumer Booking Equilibrium

The following theorem characterizes the equilibrium upgrade probability in the asymptotic scenario of the consumer booking game. As $n \to \infty$, $q^*(\cdot)$ converges to a constant q_f, where the subscript of f denotes the fluid model (we also use s to denote the stochastic model).

THEOREM 2. *(i) As $n \to \infty$, for any $q(\cdot) \in \mathcal{Q}$, the auxiliary stopping times converge to*

$$\tau_H^\infty(q(\cdot)) = \inf\left\{t \geq 0 : \int_0^t \lambda_H(s|q(\cdot))\,\mathrm{d}s \geq K_H\right\},$$
$$\tau_R^\infty(q(\cdot)) = \inf\left\{t \geq 0 : \int_0^t \lambda_R(s|q(\cdot))\,\mathrm{d}s \geq K_R\right\},$$
$$\tau_T^\infty(q(\cdot)) = \inf\left\{t \geq 0 : \int_0^t [\lambda_H(s|q(\cdot)) + \lambda_U(s|q(\cdot)) + \lambda_R(s|q(\cdot))]\,\mathrm{d}s \geq K_R + K_H\right\},$$

a.s., respectively. The stopping time of the consumer booking game converges to $\tau^\infty(q(\cdot)) = \min\{\hat{\tau}^\infty(q(\cdot)), T\}$ a.s., where

$$\hat{\tau}^\infty(q(\cdot)) = \begin{cases} \tau_H^\infty(q(\cdot)) \text{ if } \tau_H^\infty(q(\cdot)) \leq \tau_T^\infty(q(\cdot)), \\ \tau_R^\infty(q(\cdot)) \text{ if } \tau_R^\infty(q(\cdot)) \leq \tau_T^\infty(q(\cdot)), \\ \tau_T^\infty(q(\cdot)) \text{ if } \tau_H^\infty(q(\cdot)) > \tau_T^\infty(q(\cdot)) \text{ and } \tau_R^\infty(q(\cdot)) > \tau_T^\infty(q(\cdot)). \end{cases}$$

(ii) As $n \to \infty$, the equilibrium upgrade probability $q^{n*}(\cdot)$ converges pointwise to q_f which is the (time-independent) solution of the following equation:

$$q_f = \min\left\{\frac{\left[K_H - \int_0^{\tau^\infty(q_f)} \lambda_H(t|q_f)\,\mathrm{d}t\right]^+}{\int_0^{\tau^\infty(q_f)} \lambda_U(t|q_f)\,\mathrm{d}t}, 1\right\}. \tag{1}$$

Our primary goal of studying the fluid model is to derive closed-form solutions which will provide us sharp insights about how consumers make upgrading decisions and how the firm's optimal upgrade price depends on problem parameters. To be able to obtain closed-form solutions, we will assume that the consumers' valuations for two types of products are jointly uniformly distributed in the two-dimensional support $\Omega = \{(v_R, v_H) : 0 \leq v_R \leq v_H \leq u\}$, that is, for consumers that value high-quality products at v_H, their valuations for regular products are uniformly distributed over $[0, v_H]$. u is the upper bound of consumer valuations ($u > p_H$). Thus, the valuation support Ω is now an upper triangular subset of $\mathbb{R}_+^2$, and the joint probability density is $f(v_R, v_H) = 2/u^2$. Our analysis can be easily generalized if we move Ω within $\mathbb{R}_+^2$ to allow for different upper and lower bounds of consumer valuations. Moreover, we have numerically tested our results when consumers' valuations follow a bivariate normal distribution, and we find that all results in the paper (for both the fluid model and the stochastic model) carry through to the case with bivariate normal distribution.

Now we calculate q_f by solving (1). We first need to derive the demand segmentation in the fluid model for a given q (i.e., $\lambda_H(q)$, $\lambda_U(q)$, $\lambda_R(q)$). Figure 2 plots all five possible demand segmentations of consumers that are offered upgrades. Throughout the paper, we use the superscript "a" through "e" consistent with Figure 2 to specify which case we are referring to. Case a also gives the demand segmentation of consumers that are not offered upgrades. In each ease, the proportions of consumers booking each product type, $\xi_H(q)$, $\xi_U(q)$, $\xi_R(q)$, can be calculated as the ratio between the area of each region where the consumer decision is to book the corresponding product type and the area of the entire valuation support Ω. The results are shown below. The overall demand rates are $\lambda_H(q) = \lambda\gamma\xi_H^i(q) + \lambda(1-\gamma)\xi_H^a$, $\lambda_U(q) = \lambda\gamma\xi_U^i(q)$, $\lambda_R(q) = \lambda\gamma\xi_R^i(q) + \lambda(1-\gamma)\xi_R^a$ in Case i.

Case a If $p \geq p_H - p_R$ (i.e., the firm does not offer upgrades), the consumer segmentation is

$$\xi_H^a = \frac{1}{u^2}(u - p_H + 2p_R)(u - p_H), \qquad \xi_U^a = 0, \qquad \xi_R^a = \frac{1}{u^2}(p_H - p_R)(2u - p_H - p_R).$$

Case b If $p < p_H - p_R$ (i.e., the firm offers upgrades) and $q = 1$, because upgrades are guaranteed to be fulfilled, nobody books a high-quality product directly. The consumer segmentation in this case is

$$\xi_H^b = 0, \qquad \xi_U^b = \frac{1}{u^2}(p_R + u - p)(u - p_R - p), \qquad \xi_R^b = \frac{1}{u^2}\left[-p^2 + 2(u - p_R)p\right].$$

Case c If $p < p_H - p_R$ and $q < 1$ and $(p_H - p_R - qp)/(1-q) \geq u$, since $q < 1$, by booking a regular product and accepting an upgrade offer instead of booking a high-quality product directly, a consumer risks not being upgraded and ending up consuming a regular product. Recall that a consumer books a high-quality product directly if $v_H - v_R \geq (p_H - p_R - qp)/(1-q)$ and $v_H \geq p_H$, where $(p_H - p_R - qp)/(1-q)$ is the minimum valuation differential required to induce one to book a high-quality product directly. If $(p_H - p_R - qp)/(1-q) \geq u$, all consumers that are interested in high-quality products will choose to get them through upgrades. The consumer segmentation in this case is

$$\xi_H^c = 0, \qquad \xi_U^c(q) = \frac{1}{u^2}\left[-\frac{p_R^2}{q} + (u-p)^2\right], \qquad \xi_R^c = \frac{1}{u^2}\left[-p^2 + 2(u - p_R)p\right].$$

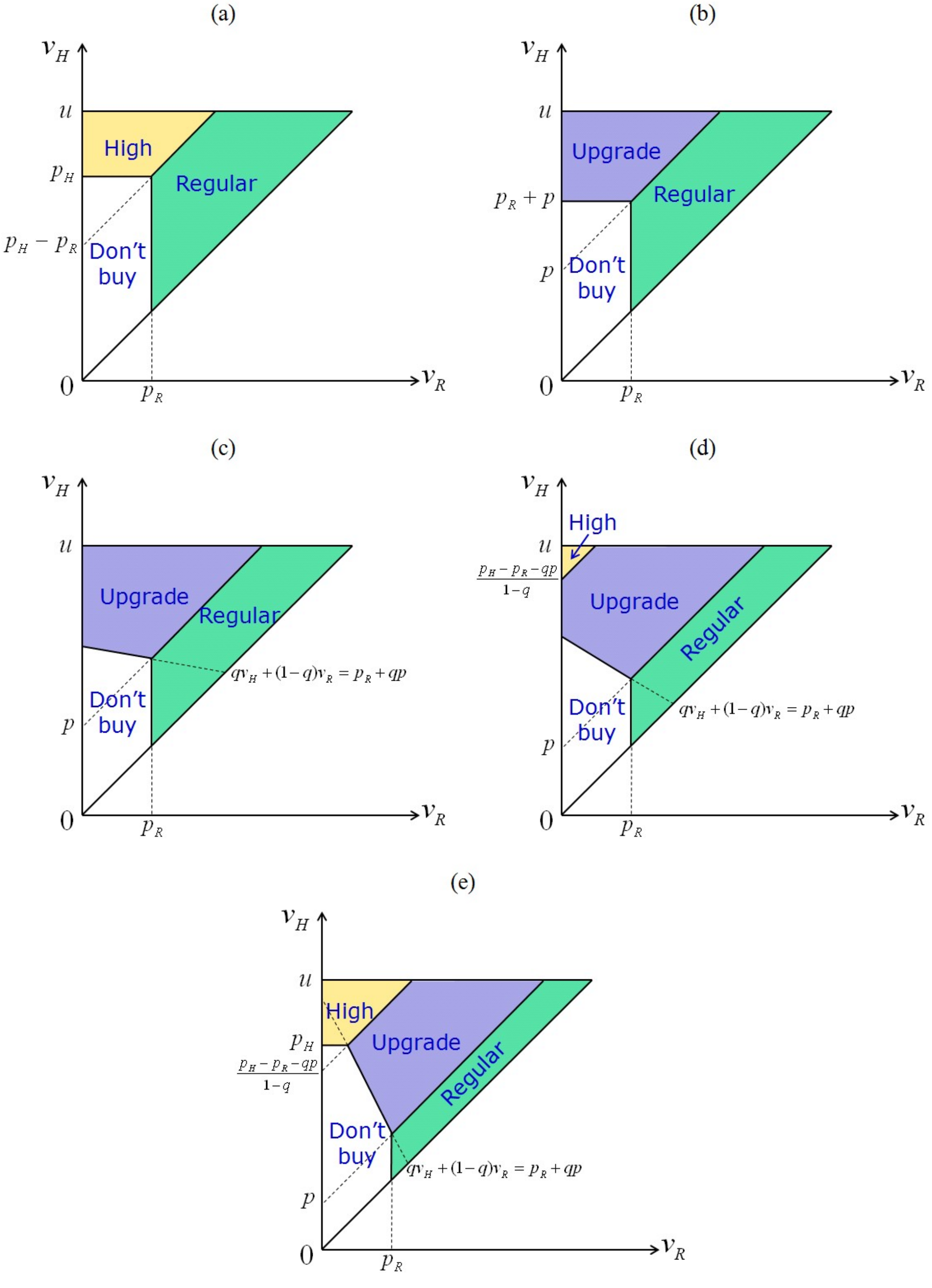


Figure 2 **Demand segmentation given the upgrade price p and the upgrade probability q: (a) no upgrades offered, or $p \geq p_H - p_R$; (b) $p < p_H - p_R$ and $q = 1$; (c) $p < p_H - p_R$ and $q < 1$ and $(p_H - p_R - qp)/(1-q) \geq u$; (d) $p < p_H - p_R$ and $q < 1$ and $p_H \leq (p_H - p_R - qp)/(1-q) < u$; (e) $p < p_H - p_R$ and $q < 1$ and $(p_H - p_R - qp)/(1-q) < p_H$.**

Case d If $p < p_H - p_R$ and $q < 1$ and $p_H \le (p_H - p_R - qp)/(1-q) < u$, since $(p_H - p_R - qp)/(1-q) < u$, the consumers with high enough valuations for high-quality products combined with low enough valuations for regular products will book high-quality products directly. Thus, in this case, high-quality products are sold in both channels (i.e., directly and through upgrades). Further, depending on whether $(p_H - p_R - qp)/(1-q) \ge p_H$ or not, $\xi_H(q)$ and $\xi_U(q)$ take different functional forms. If $(p_H - p_R - qp)/(1-q) \ge p_H$, the consumer segmentation is

$$\begin{aligned}
\xi_H^d(q) &= \frac{1}{u^2}\left(u - \frac{p_H - p_R - qp}{1-q}\right)^2, \\
\xi_U^d(q) &= \frac{1}{u^2}\left[-\left(\frac{p_H - p_R - qp}{1-q}\right)^2 + 2u\left(\frac{p_H - p_R - qp}{1-q}\right) - \frac{p_R^2}{q} + p^2 - 2up\right], \\
\xi_R^d &= \frac{1}{u^2}\left[-p^2 + 2(u - p_R)p\right].
\end{aligned}$$

Otherwise we are in Case e.

Case e If $p < p_H - p_R$ and $q < 1$ and $(p_H - p_R - qp)/(1-q) < p_H$, the consumer segmentation is

$$\begin{aligned}
\xi_H^e(q) &= \frac{1}{u^2}\left[u + p_H - \frac{2(p_H - p_R - qp)}{1-q}\right](u - p_H), \\
\xi_U^e(q) &= \frac{1}{u^2} \cdot \frac{p_H - p_R - p}{1-q} \cdot (2u - p_H - p_R - p), \\
\xi_R^e &= \frac{1}{u^2}\left[-p^2 + 2(u - p_R)p\right].
\end{aligned}$$

We assume $K_H \ge \lambda_H^a T$ and $K_R \ge \lambda_R^a T$, that is, the firm's expected demand when upgrades are not offered does not exceed its capacity for either product type at the prices p_H and p_R. This assumption is reasonable since the utilization rates in travel industries are generally not high (according to Statista[8], the average occupancy rate of the U.S. hotel lodging industry from 2000 to 2013 is only 60%). We would like to note that when the firm offers upgrades, it is still possible under this assumption that the firm's total capacity is fully booked before the end of the booking period, because offering upgrades can generate more demand than the case without upgrades. Thus, our analysis allows for any utilization level with upgrades. Moreover, our numerical analysis indicates that all findings in this paper continue to hold even if the above assumption is not satisfied.

THEOREM 3. *Define*

$$\begin{aligned}
\bar{p} &= u - \sqrt{\frac{1}{\gamma}\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right] + (u - p_H + p_R)^2}, \\
\underline{p} &= p_H - p_R - \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)}{\gamma(u - p_H + p_R)},
\end{aligned}$$

[8] http://www.statista.com/statistics/200161/us-annual-accomodation-and-lodging-occupancy-rate.

$$\underline{p}' = -\frac{1}{p_R}\left[\frac{K_H}{\lambda T}u^2 - u^2 + 2(p_H - p_R)u - p_H^2 + p_H p_R + p_R^2\right] + \frac{1}{\gamma p_R}\sqrt{\gamma\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right]}$$
$$\cdot\sqrt{\left[1 - \gamma - \frac{(1-\gamma)K_H + K_R}{\lambda T}\right]u^2 + 2\gamma(p_H - p_R)u - \gamma p_H^2 - 2(1-\gamma)p_H p_R + \gamma p_R^2}.$$

(i) If $K_H \geq (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$, $q_f = 1$ *for all* $0 \leq p < p_H - p_R$.

(ii) If $K_H < (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$, *the equilibrium upgrade probability is uniquely given by the following:*

- *If* $\underline{p}^+ \geq \underline{p}'^+$ *(where* $x^+ = \max\{x, 0\}$*),*

$$q_f = \begin{cases} 1 & \text{for } \bar{p} \leq p < p_H - p_R, \\ \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H) + \gamma(u - p_H + p_R)^2}{\gamma(u-p)^2} & \text{for } \underline{p}^+ \leq p < \bar{p}, \\ \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)}{\gamma(p_H - p_R - p)^2 + \frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)} & \text{for } 0 \leq p < \underline{p}^+; \end{cases}$$

- *If* $\underline{p}^+ < \underline{p}'^+$,

$$q_f = \begin{cases} 1 & \text{for } \bar{p} \leq p < p_H - p_R, \\ \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H) + \gamma(u - p_H + p_R)^2}{\gamma(u-p)^2} & \text{for } \underline{p}'^+ \leq p < \bar{p}, \\ \frac{2\gamma\frac{K_H}{K_H + K_R}p_R^2}{-\beta - \sqrt{\beta^2 - 4\gamma^2\frac{K_H}{K_H + K_R}p_R^2(u-p)^2}} & \text{for } 0 \leq p < \underline{p}'^+, \end{cases}$$

where $\beta = (u - p_H + 2p_R)(u - p_H) - \gamma(u - p_H + p_R)^2 + \frac{K_H}{K_H + K_R}[2\gamma p_R p - u^2 + (1-\gamma)(2p_H p_R - p_R^2)]$.

(iii) q_f *is increasing in* p.

Theorem 3 gives the equilibrium upgrade probability q_f for any upgrade price p set by the firm. If the firm's capacity for high-quality products is very large (i.e., $K_H \geq (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$), consumers accepting upgrade offers are guaranteed to get upgraded. In equilibrium, being aware of the very high chance to get upgraded, all consumers who are interested in high-quality products and offered upgrades choose to book regular products and accept upgrade offers. If the firm's capacity for high-quality products is not very large (i.e., $K_H < (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$), the equilibrium upgrade probability q_f increases with the upgrade price p. This is because fewer consumers accept upgrade offers when the upgrade price is higher. As the upgrade price p decreases from $p_H - p_R$ to 0, as shown by the proof of Theorem 3 (in Appendix A), the market segmentation takes the form in Cases b, c, d, e in sequence.[9] Case d or e occurs only if the upgrade price is low (i.e., $0 \leq p < \underline{p}^+$), meaning the

[9] $\bar{p}$ is the threshold between Case b and Case c, $\underline{p}$ is the threshold between Case c and Case d when $\tau^\infty(q_f) \geq T$, $\underline{p}'$ is the threshold between Case c and Case d when $\tau^\infty(q_f) < T$. If $\underline{p}^+ \geq \underline{p}'^+$, when Case c switches to Case d, we have $\tau^\infty(q_f) \geq T$; and vice versa. q_f takes the same form in Cases d and e.

equilibrium upgrade probability is small enough. Thus, the consumers with high enough valuations for high-quality products and low enough valuations for low-quality products will book high-quality products directly even if they are offered upgrades. In Case b or c, the upgrade probability is large enough so that consumers would like to obtain high-quality products through upgrades if they are given the offers.

5.2. Optimal Upgrade Pricing

In this section, based on the equilibrium consumer booking decision characterized in Section 5.1, we study the firm's optimal conditional upgrade pricing strategy. The firm's goal is to maximize its revenue from selling both types of products and charging upgrade fees. Recall that $p \geq p_H - p_R$ corresponds to the case without upgrades. In this case, the firm's revenue is $\Pi_{N,f} = p_R\lambda_R^a \min\{K_R/\lambda_R^a, T\} + p_H\lambda_H^a \min\{K_H/\lambda_H^a, T\}$, where the subscript of N denotes no upgrades. If the firm offers upgrades with $p < p_H - p_R$, its revenue is

$$\begin{aligned}
\Pi_f(p) = {} & p_R[\lambda_U(q_f) + \lambda_R]\tau^\infty(q_f) + p\lambda_U(q_f)\tau^\infty(q_f)q_f + p_H\lambda_H(q_f)\tau^\infty(q_f) \\
& + \mathbb{1}\{\tau^\infty(q_f) = \tau_R^\infty(q_f)\}p_H \min\left\{\lambda_H^a[T - \tau^\infty(q_f)], K_H - [\lambda_H(q_f) + \lambda_U(q_f)]\tau^\infty(q_f)\right\} \\
& + \mathbb{1}\{\tau^\infty(q_f) = \tau_H^\infty(q_f)\}p_R \min\left\{\lambda_R^a[T - \tau^\infty(q_f)], K_R - [\lambda_U(q_f) + \lambda_R]\tau^\infty(q_f)\right\}.
\end{aligned}$$

The first line of $\Pi_f(p)$ is the revenue collected before the consumer booking game stops. The first term is the revenue from selling regular products (including the revenue from consumers accepting upgrade offers), the second term is the revenue from collecting upgrade fees, the third term is the revenue from selling high-quality products. The second line of $\Pi_f(p)$ is the revenue from selling high-quality products after regular products are fully booked, where $\lambda_H^a[T - \tau^\infty(q_f)]$ is the demand and $K_H - [\lambda_H(q_f) + \lambda_U(q_f)]\tau^\infty(q_f)$ is the remaining capacity for high-quality products. The third line of $\Pi_f(p)$ is the revenue from selling regular products after high-quality products are fully booked. Since $\Pi_f(p) = \Pi_{N,f}$ at $p = p_H - p_R$, we limit ourselves to $0 \leq p \leq p_H - p_R$ in studying $\Pi_f(p)$ in the remainder of the paper. When the optimal upgrade price is achieved at $p_f^* = p_H - p_R$, we know that it is optimal for the firm not to offer upgrades.

THEOREM 4. *The optimal upgrade price is* $p_f^* = \min\left\{\max\left\{(p_{foc}^b)^+, \bar{p}\right\}, p_H - p_R\right\}$, *where*

$$p_{foc}^b = \frac{2u - \sqrt{u^2 + 9p_R^2}}{3}.$$

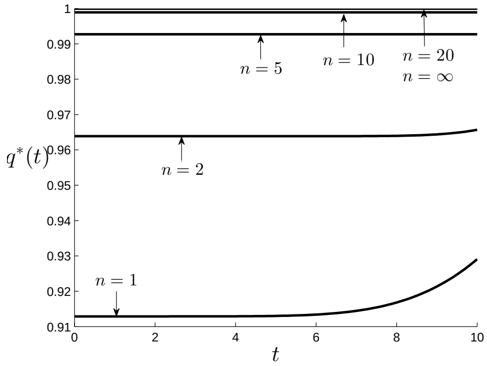
Moreover, the optimal pricing induces $q_f = 1$.

Theorem 4 characterizes the optimal upgrade price. The optimal upgrade price results in an equilibrium consumer segmentation in Case b (p_{foc}^b is the optimal price in Case b, Case b occurs for $\bar{p} \leq p \leq p_H - p_R$) where the upgrade probability is equal to one. Recall that Theorem 3 states q_f

is increasing in p (or always equal to one if the high-quality product capacity is very large). Thus, Theorem 4 states that the firm should choose an upgrade price that is high enough. If an upgrade price results in some consumers being rationed for upgrades, that means too many consumers are willing to pay for the upgrades and the current upgrade price is too low. The firm should increase the upgrade price to extract more surplus from consumers while still being able to sell out high-quality products after fulfilling upgrades. Thus, under the optimal upgrade pricing policy, strategic consumers who are offered upgrades purchase high-quality products through upgrades instead of booking directly. Note that because of the deterministic feature, our fluid model captures an ideal situation where the firm and consumers have perfect knowledge about the demand stream for each product type. In the stochastic model, because of the demand randomness, the equilibrium upgrade probability may not be exactly equal to one under the optimal upgrade price, so consumers with very high valuations for high-quality products and very low valuations for regular products may choose to book high-quality products directly even if upgrades are offered at the optimal price. However, consistent with the insight we developed from the fluid model, in the stochastic model, the firm should generally charge a high enough upgrade price that results in a high upgrade probability for consumers (Tables 1 and 2 in the next subsection provide a set of examples).

5.3. Performance Evaluation of Fluid Model

We now evaluate how well the fluid model approximates the stochastic model for relatively small values of n (we know that as $n \to \infty$, the fluid model converges to the stochastic model). In Figure 3, we provide an illustrative example for the comparison between the consumer purchasing equilibria in the stochastic model for different values of n and the consumer booking equilibrium in the fluid model. For example, in Figure 3, we see that the upgrade probability in the fluid model is 1. We also see that when $n = 5$, in the stochastic model, the average upgrade probability is 0.9927. We note that in this example, $n = 5$ corresponds to a relatively small hotel with 60 rooms ($n(K_H + K_R) = 60$). Furthermore, in the example in Figure 3, we see that when $n = 5$, the percentage of consumers that would make a different decision in the stochastic model (with respect to which type of product to book) than in the fluid model is only 0.73%. In Tables 1 and 2, we examine the gap between the consumer booking equilibria in the stochastic model and in the fluid model with more examples. Table 1 provides examples with different product prices, Table 2 provides examples with different product capacities. We can see that the equilibrium upgrade probability in the stochastic model is closer to one when the product price differential is larger, or when the high-quality product capacity is large, both indicating a smaller probability that the firm runs out of high-quality products. Overall, we observe that the equilibrium upgrade probability is increasing in the product price differential, and increasing in the high-quality product capacity.



n	$\mathbb{E}_t[q^*(t)]$	Demand segmentation			ΔDemand
		High	Upgrade	Regular	
1	0.9152	9.00%	25.20%	30.19%	2.03%
2	0.9640	9.00%	25.51%	30.20%	1.72%
5	0.9927	9.00%	26.49%	29.75%	0.73%
10	0.9989	9.00%	27.02%	29.47%	0.20%
20	1	9.00%	27.21%	29.37%	0.02%
∞	1	9.00%	27.22%	29.36%	–

Figure 3 A numerical example on the asymptotic convergence of consumer booking equilibrium under the optimal upgrade price. ($\lambda=1$, $T=10$, $K_H=5$, $K_R=7$, $p_H=160$, $p_R=70$, $\gamma=0.5$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$; "ΔDemand" is defined as the expected percentage of consumers that would make a different booking decision in the stochastic model than predicted by the fluid model)

	$p_H=130$		$p_H=140$		$p_H=150$		$p_H=160$	
	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand
$p_R=60$	0.9879	1.34%	0.9953	0.63%	0.9985	0.24%	0.9996	0.08%
$p_R=70$	0.9770	2.25%	0.9898	1.21%	0.9962	0.55%	0.9989	0.20%
$p_R=80$	0.9615	3.42%	0.9809	2.05%	0.9919	1.06%	0.9972	0.45%
$p_R=90$	0.8680	23.35%	0.9685	3.11%	0.9849	1.79%	0.9939	0.88%

Table 1 Numerical examples on the gap between the consumer booking equilibria (under the optimal upgrade price) in the stochastic model and in the fluid model with different product prices: the time-average equilibrium upgrade probability ($\mathbb{E}_t[q^*(t)]$) and the expected percentage of consumers that would make a different booking decision in the stochastic model than predicted by the fluid model (ΔDemand). ($\lambda=1$, $T=100$, $K_H=50$, $K_R=70$, $\gamma=0.5$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

	$K_H=40$		$K_H=50$		$K_H=60$		$K_H=70$	
	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand	$\mathbb{E}_t[q^*(t)]$	ΔDemand
$K_R=60$	0.9487	3.05%	0.9919	1.06%	0.9997	0.05%	1.0000	0.00%
$K_R=70$	0.9487	3.06%	0.9919	1.06%	0.9997	0.05%	1.0000	0.00%
$K_R=80$	0.9487	3.06%	0.9919	1.06%	0.9997	0.05%	1.0000	0.00%
$K_R=90$	0.9487	3.06%	0.9919	1.06%	0.9997	0.05%	1.0000	0.00%

Table 2 Numerical examples on the gap between the consumer booking equilibria (under the optimal upgrade price) in the stochastic model and in the fluid model with different product capacities: the time-average equilibrium upgrade probability ($\mathbb{E}_t[q^*(t)]$) and the expected percentage of consumers that would make a different booking decision in the stochastic model than predicted by the fluid model (ΔDemand). ($\lambda=1$, $T=100$, $p_H=150$, $p_R=80$, $\gamma=0.5$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

Table 3 provides an illustrative example for the asymptotic convergence of the firm's optimal upgrade price and revenue. The derivation of the stochastic revenue function, $\Pi_s(p)$, is given in Appendix B. By comparing the stochastic revenues using the optimal upgrade price derived from the fluid model and using the optimal upgrade price for the stochastic model, we can evaluate the performance of the fluid model. From Table 3, we clearly see that by using the optimal upgrade price derived from the fluid model, the firm's revenue deviates by an almost negligible amount from the optimal revenue in the stochastic model even for very small problem sizes (less than or equal to 0.1% even for $n = 1$). The optimal upgrade price itself may have some error especially when the problem size is small, but our numerical studies indicate that the revenue function in the stochastic model is quite flat in the region around the optimal upgrade price, hence the deviation of the optimal revenue is significantly smaller than the deviation of the optimal upgrade price. In Tables 4 and 5, we examine the deviation of optimal upgrade price and optimal revenue in the stochastic model caused by the fluid solution with more examples. Table 4 provides examples with different product prices, Table 5 provides examples with different product capacities. We can see that similar to the observation from analyzing the consumer booking equilibrium, the optimal upgrade price and revenue deviations caused by the fluid solution are smaller when the product price differential is larger, or when the high-quality product capacity is larger, both indicating a smaller probability that the firm runs out of high-quality products. Overall we observe that the pricing heuristic derived from the fluid model performs very well in terms of giving the firm close-to-optimal revenues in the stochastic model. Thus, by studying the fluid model, we can develop managerial insights that will carry through to the stochastic model and provide an excellent heuristic for the stochastic problem.

n	Fluid solution		Stochastic solution		$\Delta p^* = \|p_f^* - p_s^*\|$	$\Delta\Pi^* = \frac{\Pi_s(p_s^*) - \Pi_s(p_f^*)}{\Pi_s(p_s^*)}$
	p_f^*	$\Pi_s(p_f^*)$	p_s^*	$\Pi_s(p_s^*)$		
1	36.7	620.7	40.3	621.1	3.6	0.07%
2	36.7	1265.4	40.4	1266.4	3.7	0.08%
5	36.7	3192.2	38.4	3192.8	1.7	0.02%
10	36.7	6396.9	37.1	6397.0	0.5	0.00%
20	36.7	12798.0	36.7	12798.0	0.0	0.00%

Table 3 **A numerical examples on the asymptotic convergence of optimal upgrade price and revenue. (**$\lambda = 1$**,** $T = 10$**,** $K_H = 5$**,** $K_R = 7$**,** $p_H = 160$**,** $p_R = 70$**,** $\gamma = 0.5$**,** v_R **and** v_H **are jointly uniformly distributed over** $\Omega = \{(v_R, v_H) : 0 \le v_R \le v_H \le 200\}$**)**

6. Analysis of Optimal Upgrade Pricing

Now that we have obtained the optimal upgrade pricing strategy, we explore it further and develop managerial and policy insights for firms. We are first interested in when the conditional upgrade policy increases firms' revenues and when it can actually decrease revenues. We identify some benefits of conditional upgrades and show that by optimally deciding when to offer upgrades and at which price to offer upgrades, the firm benefits from offering conditional upgrades to more strategic

	$p_H=130$		$p_H=140$		$p_H=150$		$p_H=160$	
	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$
$p_R=60$	3.3	0.07%	1.6	0.01%	0.6	0.00%	0.2	0.00%
$p_R=70$	5.3	0.18%	2.8	0.05%	1.3	0.01%	0.5	0.00%
$p_R=80$	7.4	0.35%	4.5	0.14%	2.3	0.04%	1.0	0.01%
$p_R=90$	11.5	0.62%	6.3	0.28%	3.7	0.10%	1.8	0.02%

Table 4 Numerical examples on the gap between the firm's optimal upgrade prices as well as revenues in the stochastic model and in the fluid model with different product prices: the price error ($\Delta p^*=|p_f^*-p_s^*|$) and the revenue error ($\Delta\Pi^*=\frac{\Pi_s(p_s^*)-\Pi_s(p_f^*)}{\Pi_s(p_s^*)}$). ($\lambda=1$, $T=100$, $K_H=50$, $K_R=70$, $\gamma=0.5$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

	$K_H=40$		$K_H=50$		$K_H=60$		$K_H=70$	
	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$	Δp^*	$\Delta\Pi^*$
$K_R=60$	6.4	0.28%	2.3	0.04%	0.1	0.00%	0.0	0.00%
$K_R=70$	6.4	0.28%	2.3	0.04%	0.1	0.00%	0.0	0.00%
$K_R=80$	6.4	0.28%	2.3	0.04%	0.1	0.00%	0.0	0.00%
$K_R=90$	6.4	0.28%	2.3	0.04%	0.1	0.00%	0.0	0.00%

Table 5 Numerical examples on the gap between the firm's optimal upgrade prices as well as revenues in the stochastic model and in the fluid model with different product capacities: the price error ($\Delta p^*=|p_f^*-p_s^*|$) and the revenue error ($\Delta\Pi^*=\frac{\Pi_s(p_s^*)-\Pi_s(p_f^*)}{\Pi_s(p_s^*)}$). ($\lambda=1$, $T=100$, $p_H=150$, $p_R=80$, $\gamma=0.5$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

consumers. Then, we characterize when it is optimal to offer conditional upgrades for free. Finally, we demonstrate the importance of accounting for strategic consumer behavior with conditional upgrades by evaluating the cost of ignoring strategic consumer behavior.

6.1. When to Offer Upgrades?

The following result states when offering conditional upgrades at the optimal price increases or decreases the firm's revenue. For the conditional upgrade policy to be beneficial (i.e., $p_f^*<p_H-p_R$), the product price differential should be large enough. When the product price differential is small, it is optimal not to offer upgrades (or alternatively set the upgrade price at $p_f^*=p_H-p_R$).

THEOREM 5. *Offering conditional upgrades increases the revenue if*

$$p_H>\frac{2u+3p_R-\sqrt{u^2+9p_R^2}}{3}$$

and decreases the revenue otherwise.

The fundamental trade-off regarding whether the firm should offer upgrades is as follows. If the firm offers upgrades, some consumers, who book high-quality products when the firm does not offer upgrades, will now book regular products and accept upgrade offers instead, and hence the firm's revenue from direct sales of high-quality products decreases. This is the cannibalization effect of conditional upgrades. On the other hand, some consumers who book regular products when the firm does not offer upgrades will now accept upgrade offers; also, some consumers who do not book any product when the firm does not offer upgrades will now purchase regular products and accept upgrade offers (these consumers' valuations for regular (high-quality) products are lower than p_R (p_H), but their valuations for high-quality products are higher than or equal to $p_R + p$). These two types of consumers bring additional revenues to the firm. This is the demand improvement effect of conditional upgrades. One important factor that determines which of these two effects is stronger is the product price differential. If the price differential is small and the firm offers upgrades, the cannibalization effect is significant, as a lot of consumers will book high-quality products if the firm does not offer upgrades, and these consumers will switch to upgrades under the optimal upgrade price (Theorem 4). Moreover, since the high-quality product price is already close to the regular product price, there will not be many consumers who originally book regular products or don't book any products and now switch to upgrades, hence the demand improvement effect is not significant. Therefore, the firm's revenue is hurt if upgrades are offered in this case.

Thus, the firm benefits from offering conditional upgrades if the product price differential is large enough. This finding has important implications for the companies in travel industries regarding whether and when they should use the conditional upgrade strategy. Travel managers tend to believe that upgrades should only be offered between similar product types, as they feel that they may be giving consumers too much benefit by offering them the opportunity to get a product that is much better than the originally booked type. However, this common wisdom does not take into account the consumers' strategic behavior that they may deliberately book a lower-quality product than desired in anticipation of getting upgraded later. Our analysis suggests that as a response to such strategic consumer behavior, the firm should be able to extract more revenues by offering upgrades between product types that are priced not so closely, but also charging sufficiently large amounts for the upgrades.

We provide the following example for the stochastic model where as the product price differential becomes smaller, offering upgrades switches from increasing the firm's revenue to decreasing the firm's revenue: $\lambda = 1$, $T = 100$, $K_H = 70$, $K_R = 50$, $p_R = 80$, $\gamma = 0.5$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \leq v_R \leq v_H \leq 200\}$. For this example, Theorem 5 would predict that offering upgrades benefits the firm when $p_H \geq 110$ and hurts the firm when $p_H \leq 109$. From the numerical analysis for the stochastic model, we find that offering upgrades benefits the firm when $p_H \geq 111$ and hurts the firm when $p_H \leq 110$, which is very close to the result indicated by the fluid heuristic.

From our analysis above, we have seen two benefits of conditional upgrades. First, the optimal conditional upgrade strategy can lead to demand expansion. Second, offering upgrades can shift some consumers from regular products to high-quality products. We use the following example (in the stochastic model) to illustrate these two benefits of conditional upgrades: $\lambda = 1$, $T = 100$, $K_H = 70$, $K_R = 50$, $p_H = 150$, $p_R = 80$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \le v_R \le v_H \le 200\}$. For this example, if the firm does not offer upgrades, 26.25% of consumers book high-quality products and 29.75% of consumers book regular products. If the firm offers upgrades to half of the consumers (i.e., $\gamma = 0.5$), 13.13% of consumers book high-quality products directly, 27.41% of consumers book regular products and accept the upgrade offers, and 23.09% of consumers book regular products without upgrades. Compared to the case without upgrades where the total demand is 56%, the firm increases the total demand to 63.63% by offering upgrades to half of the consumers (i.e., demand expansion effect). Moreover, offering upgrades decreases the demand for regular products from 29.75% to 23.09% and increases the demand for high-quality products from 26.25% to 40.54% including the consumers who accept the upgrade offers (i.e., demand segmentation reoptimization effect). We will identify more benefits of conditional upgrades in later sections.

THEOREM 6. *The optimal upgrade price and the optimal revenue are increasing in* γ.

How does the firm's revenue change with the proportion of strategic consumers it offers conditional upgrades to? Theorem 6 states that the firm's revenue becomes higher when it offers conditional upgrades to more strategic consumers. Note that Theorem 6 incorporates the possibility that it is optimal not to offer conditional upgrades, as the optimal upgrade price and revenue would be constant in γ in this case. For a firm that sells conditional upgrades at the optimal upgrade price, the presence of strategic consumers is actually not a bad thing. Although strategic consumers create the cannibalization effect of conditional upgrades, they also allow the firm to benefit from demand expansion and demand segmentation reoptimization. By appropriately choosing the upgrade price, the firm can compensate the revenue loss due to cannibalization by the revenue gains due to the benefits of conditional upgrades and earn a higher revenue overall. Figure 4 plots the firm's optimal revenue in the stochastic model as a function of the proportion of strategic consumers it offers conditional upgrades to, which is an increasing function. Therefore, given that the upgrade price is properly chosen, the firm benefits from offering conditional upgrades to as many consumers as possible even if consumers are strategic.

6.2. Free Upgrades

Next, we consider the extreme case where it is optimal for the firm to offer conditional upgrades for free. As we mentioned in the beginning, the recent trend is that firms in the travel industry are offering fewer free upgrades and introducing paid upgrades. The following theorem states that the optimal upgrade price is zero when the regular products are very expensive (i.e., $p_R \ge u/\sqrt{3}$) and the firm has such an overabundant high-quality product capacity (i.e., $K_H \ge (\lambda T/u^2)[(u -$

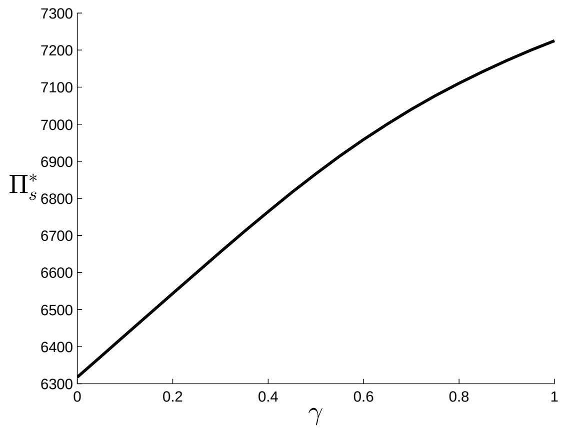


Figure 4 Firm's optimal revenue in the stochastic model as a function of the percentage of consumers offered upgrades. ($\lambda = 1$, $T = 100$, $K_H = 50$, $K_R = 70$, $p_H = 150$, $p_R = 80$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \le v_R \le v_H \le 200\}$)

$p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)])$ that it could satisfy all demand for both product types in expectation using only the high-quality product capacity when the upgrade price is zero. Clearly, this is a very restrictive condition and is not very likely to be satisfied in reality. Thus, our analysis indicates that the conditional upgrades should generally be fulfilled with fees, which is consistent with the industry trend.

THEOREM 7. *$p_f^* = 0$ if and only if $p_R \ge u/\sqrt{3}$ and $K_H \ge (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$.*

The trade-off that the firm is managing when giving free upgrades is as follows. When upgrades are free, the firm will get a number of consumers, who would not have booked any product at a higher upgrade price, to book regular products and accept upgrade offers. In the mean time, the firm will earn less revenue from consumers that would have accepted upgrade offers anyway at a higher upgrade price. As the regular product price p_R becomes higher, we can clearly see from Figure 2b that the number of the first type of consumers discussed above becomes larger, and the firm also earns more additional revenue from each of these consumers (at $p = 0$, the firm earns p_R from each consumer). However, the number of the second type of consumers discussed above becomes smaller. Therefore, if the regular product price is high enough (i.e., $p_R \ge u/\sqrt{3}$), the revenue improvement due to the first type of consumers will dominate the revenue loss due to the second type of consumers. Moreover, as Theorem 4 states, the optimal upgrade price results in the upgrade probability equal to one. Thus, for $p = 0$ to be optimal, we need the high-quality product capacity to be larger than or equal to the expected demand for high-quality products and upgrades, which results in $K_H \ge (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$.

We provide the following example for the stochastic model where the optimal policy is to offer free upgrades when the regular product price p_R is high enough: $\lambda = 1$, $T = 100$, $K_H = 70$, $K_R = 50$, $p_H = 150$, $\gamma = 0.5$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \leq v_R \leq v_H \leq 200\}$. For this example, Theorem 7 would predict that $p_f^* = 0$ when $p_R \geq 116$. We find the exact same result for the stochastic model ($p_s^* = 0$ if and only if $p_R \geq 116$).

6.3. Cost of Ignoring Strategic Consumer Behavior

Finally, we investigate how important it is for the firm to take strategic consumer behavior into consideration when offering conditional upgrades. We measure the importance of accounting for strategic consumer behavior by the revenue loss (in the stochastic model) if the firm mistakenly assumes consumers are myopic while they are in fact strategic. Myopic consumers do not consider future utilities from possibly getting upgrades and make their booking decisions in a two stages. A myopic consumer first chooses among booking a high-quality product or booking a regular product (ignoring the upgrade opportunity) or booking no product. In the first stage, she books a high-quality product if $v_H - p_H \geq \max\{v_R - p_R, 0\}$, books a regular product if $v_R - p_R \geq \max\{v_H - p_H, 0\}$, and does not book any product otherwise. If a myopic consumer books a regular product, then upon receiving an upgrade offer, she accepts the offer if her utility from getting upgraded dominates her utility from consuming the regular product. In the second stage, she accepts the upgrade offer if $v_H - p_R - p \geq v_R - p_R$, or equivalently, $v_H - v_R \geq p$. Table 6 gives the revenue loss results if the firm mistakenly assumes strategic consumers are myopic. As the results indicate, the cost of ignoring strategic consumer behavior is non-negligible and can be very significant in some cases (exceeding 10%). Across all 16 examples given in Table 6, the average revenue loss is 6.79%. According to recent data from Sageworks which is a financial information company, the net profit margin of U.S. hotel industry is 5% in 2013 and the five-year average margin is −1% (Biery 2014). Given the low net profit margin in the hotel industry, the cost of ignoring strategic consumer behavior is significant.

	$p_H = 90$	$p_H = 100$	$p_H = 110$	$p_H = 120$
$p_R = 30$	10.80%	8.45%	6.39%	4.52%
$p_R = 40$	10.27%	8.16%	6.17%	4.40%
$p_R = 50$	9.44%	7.57%	5.55%	3.83%
$p_R = 60$	8.09%	6.96%	4.85%	3.19%

Table 6 Percentage revenue loss in the stochastic model if the firm prices conditional upgrades assuming consumers are myopic while consumers are strategic. ($\lambda = 1$, $T = 100$, $K_H = 70$, $K_R = 50$, $\gamma = 1$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \leq v_R \leq v_H \leq 200\}$)

7. Revenue Performance of Conditional Upgrades

In this section, we evaluate the conditional upgrade strategy's revenue performance. We will first consider a firm that is a price taker on product prices but can set upgrade price, as we have assumed so far. An interesting question is how much of the revenue potential does the conditional upgrade strategy capture compared to setting product prices optimally? In Section 7.1, we compare the conditional upgrade strategy to product price optimization. Our interesting finding is that conditional upgrades as a lever can compensate for the firm's lack of ability to optimize product prices and even generate higher revenues than product price optimization. In Section 7.2, we compare conditional upgrades to an alternative way of offering upgrades, in which case the firm offers upgrades at the end of the booking period and can decide the upgrade price based on demand realizations during the booking period. We find that the value of offering conditional upgrades in advance and collecting consumers' upgrading decisions in advance is in general greater than the value of pricing flexibility for upgrades. Moreover, we will also consider a firm that is not a price taker. As dynamic pricing would be another strategy that is naturally considered by such a firm, in Section 7.3, we compare the revenue performance of conditional upgrades to the revenue performance of dynamic pricing. Surprisingly, offering conditional upgrades outperforms using dynamic pricing.

7.1. Conditional Upgrades vs Product Price Optimization

Following from our previous analysis for the fluid model, Corollary 1 states that when offering conditional upgrades is profitable, offering conditional upgrades to all consumers (which is the optimal strategy to offer upgrades, as shown in Theorem 6) enables the firm to capture all of the revenue potential from optimally setting the price for high-quality products. Recall that as Theorem 4 indicates, when it is optimal to offer upgrades (i.e., when $p_f^* < p_H - p_R$), consumers choose to obtain high-quality products through upgrades, and the equilibrium outcome is equivalent to the firm selling regular products at price p_R and high-quality products at price $p_R + p_f^*$. Thus, the high-quality product price is replaced by $p_R + p_f^*$ which results in a higher revenue (note that p_f^* does not depend on p_H). In this case, $p_R + p_f^*$ is also the optimal high-quality product price for a firm that is a price taker on only regular products. When it is optimal not to offer upgrades (i.e., when $p_f^* = p_H - p_R$), however, the firm may increase revenue by increasing p_H. Thus, the upgrade price can "correct" the price for high-quality products when it is sub-optimally high. This is consistent with our finding in Section 6.1 that offering conditional upgrades can alter consumer segmentation and shift more consumers to high-quality products. By offering upgrades, the firm can offer a lower price for the high-quality products that is somewhat disguised.

COROLLARY 1. *Consider two scenarios: 1) the firm is a price taker on both product prices but offers conditional upgrades, 2) the firm is a price taker on the regular product price but the firm can optimize the high-quality product price, and no conditional upgrades are offered. With* $\gamma = 1$*, when it is optimal to offer upgrades in the first scenario, these two scenarios result in the same revenue.*

Next, we explore what happens with stochastic demand. We first establish an analytical result for a special case of the stochastic model. Assume consumers have homogeneous valuations, v_H for high-quality products and v_R for regular products ($v_H \geq p_H$, and $v_R \geq p_R$). Theorem 8(i) states that with stochastic demand, optimal upgrade pricing results in higher revenues than optimal high-quality product pricing ($\Pi_{N,s}(p^*_{H,s})$ is the revenue when the firm is a price taker on only regular products and can set the high-quality product price optimally). When demand is stochastic and no upgrades are allowed, if the realized demand exceeds product capacity for either type, the firm cannot capture this excess demand. However, with upgrades, during the booking period, the firm does not allocate the consumers who accept upgrade offers to specific product types; after demand is fully realized, the firm then gets to allocate more of these consumers to the product type that has weaker demand. Thus, the firm is able to better match its capacity to demand and improve capacity utilization. Moreover, Theorem 8(ii) states that optimal upgrade pricing even results in higher revenues than optimal pricing for both product types if the original regular product price is not too far away from optimal ($\Pi^*_{N,s}$ is the revenue when the firm is not a price taker and can set both product prices optimally).

THEOREM 8. *Consider the stochastic model with homogeneous consumer valuations.*
(i) When $p_H > p^*_{H,s}$*, we have* $\Pi^*_s \geq \Pi_{N,s}(p^*_{H,s})$*; moreover,* $\Pi^*_s > \Pi_{N,s}(p^*_{H,s})$ *if* $v_H - p^*_{H,s} \geq v_R - p_R$*.*
(ii) When $p_H > p^*_{H,s}$ *and* p_R *is close enough to* $p^*_{R,s}$*, we have* $\Pi^*_s > \Pi^*_{N,s}$ *if* $v_H - p^*_{H,s} \geq v_R - p^*_{R,s}$*.*

Next, we examine our original stochastic model (with heterogeneous consumer valuations) numerically. In Table 7, we compare the firm's revenue when it is a price taker, $\Pi_{N,s}$, to 1) the revenue when the firm offers upgrades at the optimal price (taking the product prices as given), Π^*_s, and 2) the revenue when the firm is a price taker on only regular products and can set the high-quality product price optimally, $\Pi_{N,s}(p^*_{H,s})$. We see that optimal upgrade pricing results in strictly higher revenues than optimal high-quality product pricing. For example, suppose the firm is a price taker selling regular products at price 90 and high-quality product at price 130. Suppose now that the firm achieves flexibility to set price optimally for high-quality products. Optimizing p_H results in only a 0.13% improvement in revenue. However, if the firm keeps p_H at 130, p_R at 90, and offers conditional upgrades, it increases revenue by 1.30%. In all of the examples in Table 7, the firm is able to obtain higher revenues by offering conditional upgrades than by being able to optimize the high-quality product price. Thus, Table 7 clearly shows that the conditional upgrade strategy is a very valuable form of flexibility for the firm, and in fact may be at least as valuable as the flexibility to set price for one product type optimally.

	$p_H=130$		$p_H=140$		$p_H=150$		$p_H=160$	
	$\Delta\Pi_s^*$	$\Delta\Pi_{N,s}(p_{H,s}^*)$	$\Delta\Pi_s^*$	$\Delta\Pi_{N,s}(p_{H,s}^*)$	$\Delta\Pi_s^*$	$\Delta\Pi_{N,s}(p_{H,s}^*)$	$\Delta\Pi_s^*$	$\Delta\Pi_{N,s}(p_{H,s}^*)$
$p_R=60$	4.57%	4.12%	10.44%	9.96%	18.65%	18.13%	29.75%	29.19%
$p_R=70$	2.93%	2.36%	8.36%	7.76%	16.28%	15.64%	27.27%	26.57%
$p_R=80$	1.67%	0.98%	6.58%	5.85%	14.36%	13.59%	25.43%	24.58%
$p_R=90$	1.30%	0.13%	4.96%	4.09%	12.67%	11.75%	23.97%	22.95%

Table 7 Percentage revenue improvements in the stochastic model from $\Pi_{N,s}$ (i.e., the revenue from not offering upgrades and using the given product prices) by 1) optimal upgrade pricing ($\Delta\Pi_s^*=\frac{\Pi_s^*-\Pi_{N,s}}{\Pi_{N,s}}$), and 2) optimal pricing of high-quality products ($\Delta\Pi_{N,s}(p_{H,s}^*)=\frac{\Pi_{N,s}(p_{H,s}^*)-\Pi_{N,s}}{\Pi_{N,s}}$). ($\lambda=1$, $T=100$, $K_H=50$, $K_R=70$, $\gamma=1$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

In Table 8, we go one step further and compare the firm's revenue when it is a price taker, $\Pi_{N,s}$, to 1) the revenue when the firm offers upgrades at the optimal price (taking the product prices as given), Π_s^*, and 2) the revenue when the firm is not a price taker and can set both product prices optimally, $\Pi_{N,s}^*$. We observe that the flexibility of conditional upgrades in better allocating capacity to stochastic demand allows the firm to earn higher revenues than optimizing both product prices when the regular product price that the firm is forced to offer is not too far away from optimal. For example, if the firm is forced to offer high-quality products at price 130 and regular products at price 100, optimizing p_H and p_R (the optimal product prices are $p_{H,s}^*=129.1$ and $p_{R,s}^*=92.7$) results in only a 0.36% improvement in revenue. However, if the firm keeps p_H at 130, p_R at 100, and offers conditional upgrades, it increases revenue by 2.92%. In Table 8, $\Pi_s^*>\Pi_{N,s}^*$ for at least $90\le p_R\le 100$. Thus, the conditional upgrade strategy is effective in capturing the revenue potential from being able to optimize product prices. Additionally, the benefit of conditional upgrades in matching fixed capacities to stochastic demands is more significant when the capacity-demand mismatch without upgrades is more severe. We can see this from the examples given in Table 8. The optimal product prices in this case are $p_{H,s}^*=129.1$ and $p_{R,s}^*=92.7$. As we move p_H and p_R away from optimal so that the capacity-demand mismatch becomes more severe, the revenue improvement of conditional upgrades increases.

7.2. Conditional Upgrades vs Last-Minute Upgrades

Now we consider another type of upgrades that the firm offers to consumers at the last minute and compare it to conditional upgrades that are offered in advance. In this case, the firm offers upgrades at the end of the booking period (e.g., at check-in), and chooses the upgrade price after demand realizations during the booking period. During the booking period, strategic consumers

	$p_H = 130$		$p_H = 140$		$p_H = 150$		$p_H = 160$	
	$\Delta\Pi^*_s$	$\Delta\Pi^*_{N,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{N,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{N,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{N,s}$
$p_R = 70$	2.93%	5.29%	8.36%	10.84%	16.28%	18.95%	27.27%	30.19%
$p_R = 80$	1.67%	1.81%	6.58%	6.72%	14.36%	14.52%	25.43%	25.61%
$p_R = 90$	1.30%	0.16%	4.96%	4.13%	12.67%	11.78%	23.97%	22.99%
$p_R = 100$	2.92%	0.36%	3.49%	2.73%	11.07%	10.25%	22.71%	21.80%

Table 8 Percentage revenue improvements in the stochastic model from $\Pi_{N,s}$ (i.e., the revenue from not offering upgrades and using the given product prices) by 1) optimal upgrade pricing ($\Delta\Pi^*_s = \frac{\Pi^*_s - \Pi_{N,s}}{\Pi_{N,s}}$), and 2) optimal pricing of both product types ($\Delta\Pi^*_{N,s} = \frac{\Pi^*_{N,s} - \Pi_{N,s}}{\Pi_{N,s}}$). ($\lambda = 1$, $T = 100$, $K_H = 50$, $K_R = 70$, $\gamma = 1$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \le v_R \le v_H \le 200\}$)

choose among booking a high-quality product or booking a regular product or booking no product based on the anticipated upgrade probabilities and prices. We use a similar (stochastic) model to analyze last-minute upgrades; the model and analysis are described in Appendix C.

In Table 9, we compare the firm's revenue when it is a price taker, $\Pi_{N,s}$, to 1) the optimal revenue when the firm offers conditional upgrades, Π^*_s, and 2) the optimal revenue when the firm offers last-minute upgrades, $\Pi^*_{LM,s}$. As Table 9 shows, conditional upgrades result in higher revenues than last-minute upgrades in all cases. Across all examples given in Table 9, on average, conditional upgrades improve the revenue by 13.08%, whereas last-minute upgrades improve the revenue by only 2.36% (offering last-minute upgrades may even decrease the firm's revenue in some cases). Although last-minute upgrades give the firm more pricing flexibility (i.e., the firm can dynamically determine the upgrade price based on demand realizations during the booking period), conditional upgrades give the firm other advantages that appear to be more valuable. First, the firm has better flexibility in managing capacities with conditional upgrades. By offering upgrades in advance and letting consumers reveal their upgrading decisions in advance, the firm is able to better control the time to stop selling each product type and improve its capacity utilizations. With last-minute upgrades, the firm loses the ability to observe consumers' upgrading decisions in advance, and hence cannot improve capacity utilizations as effectively. Second, with conditional upgrades, by committing to the upgrade price up front, the firm can induce more consumers, who would not purchase any product without upgrades being offered, to purchase from the firm. With last-minute upgrades, however, the demand expansion effect is weakened. Across all examples given in Table 9, on average, conditional upgrades generate 13.61% more demand than last-minute upgrades. Additionally, with conditional upgrades, the firm can overbook regular products without having to "bump" consumers during check-in, because by observing consumers' upgrading decisions in advance, the firm can overbook regular products as long as it knows that enough consumers (who have accepted upgrade offers) can be switched to high-quality products. However, if upgrades are offered at check-in and the firm overbooks regular products, it has the risk of having to bump some consumers. In this case, the firm chooses the upgrade price at the end of the booking period based on its belief about the probability of consumers (who have booked regular products) accepting the upgrade offer. It may occur that not enough consumers are actually willing to pay for the upgrades at the price chosen by the firm, so the firm will incur penalty costs from bumping consumers. Note that in the examples given in Table 9, the penalty cost per consumer, c, is equal to zero. So, we are comparing the conditional upgrade revenue to an upper bound of the last-minute upgrade revenue.

	$p_H = 130$		$p_H = 140$		$p_H = 150$		$p_H = 160$	
	$\Delta\Pi_s^*$	$\Delta\Pi_{LM,s}^*$	$\Delta\Pi_s^*$	$\Delta\Pi_{LM,s}^*$	$\Delta\Pi_s^*$	$\Delta\Pi_{LM,s}^*$	$\Delta\Pi_s^*$	$\Delta\Pi_{LM,s}^*$
$p_R = 60$	4.57%	0.05%	10.44%	4.03%	18.65%	5.03%	29.75%	14.86%
$p_R = 70$	2.93%	−0.03%	8.36%	0.05%	16.28%	0.68%	27.27%	5.98%
$p_R = 80$	1.67%	0.23%	6.58%	−0.06%	14.36%	0.09%	25.43%	5.69%
$p_R = 90$	1.30%	0.83%	4.96%	0.21%	12.67%	−0.01%	23.97%	0.19%

Table 9 Percentage revenue improvements in the stochastic model from $\Pi_{N,s}$ (i.e., the revenue from not offering upgrades and using the given product prices) by 1) offering conditional upgrades ($\Delta\Pi_s^* = \frac{\Pi_s^* - \Pi_{N,s}}{\Pi_{N,s}}$), and 2) offering last-minute upgrades ($\Delta\Pi_{LM,s}^* = \frac{\Pi_{LM,s}^* - \Pi_{N,s}}{\Pi_{N,s}}$). ($\lambda = 1$, $T = 100$, $K_H = 50$, $K_R = 70$, $\gamma = 1$, $c = 0$, v_R and v_H are jointly uniformly distributed over $\Omega = \{(v_R, v_H) : 0 \leq v_R \leq v_H \leq 200\}$)

7.3. Conditional Upgrades vs Dynamic Pricing

As we have seen, the flexibility of conditional upgrades in better allocating capacity to demand allows a product-price-taking firm to achieve higher revenues than being able to optimize product prices and offering last-minute upgrades in many cases. Now, suppose the firm is not a price taker at all and can set both product prices optimally. In Table 10, we compare the firm's revenue from optimal product pricing, $\Pi_{N,s}^*$, to 1) the optimal revenue from the conditional upgrade strategy (using the optimal static product prices), Π_s^*, and 2) the optimal revenue from dynamic pricing, $\Pi_{D,s}^*$. We use the classic multiproduct dynamic pricing model in Gallego and van Ryzin (1997) to compute the expected revenue from optimal dynamic pricing.[10] Interestingly, we find that conditional upgrades generate more revenues than dynamic pricing in all examples in Table 10. The firm gains different types of flexibility from conditional upgrades and dynamic pricing. By using dynamic pricing, the firm can adjust the allocation of consumers to different product types by changing product prices during the booking period. However, the firm does not have the flexibility to change product assignments after purchase. With conditional upgrades, the firm's product assignments of consumers who have accepted upgrade offers are made after demand is fully realized. As Table 10 shows, the ex-post allocation flexibility created by conditional upgrades has more revenue potential

[10] Note that in Gallego and van Ryzin (1997), consumers do not postpone their purchases due to the anticipated future price drops. Thus, the dynamic pricing revenue we are comparing the upgrade revenue to is an upper bound on dynamic pricing revenues (Levin et al. 2010).

than the pricing flexibility created by dynamic pricing. Therefore, for a firm that is not a price taker and has the ability to set optimal static product prices, the conditional upgrade strategy can serve as a substitute to dynamic pricing and in fact generate higher revenues.

	$K_H=20$		$K_H=30$		$K_H=40$		$K_H=50$	
	$\Delta\Pi^*_s$	$\Delta\Pi^*_{D,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{D,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{D,s}$	$\Delta\Pi^*_s$	$\Delta\Pi^*_{D,s}$
$K_R=50$	6.42%	5.19%	4.02%	2.43%	3.07%	2.21%	1.95%	1.75%
$K_R=60$	6.42%	5.20%	4.02%	2.43%	3.07%	2.21%	1.95%	1.75%
$K_R=70$	6.42%	5.20%	4.02%	2.43%	3.07%	2.21%	1.95%	1.75%
$K_R=80$	6.42%	5.20%	4.02%	2.43%	3.07%	2.21%	1.95%	1.75%

Table 10 Percentage revenue improvements in the stochastic model from $\Pi^*_{N,s}$ (i.e., the revenue from optimal product pricing without upgrades) by 1) optimal upgrade pricing given the optimal product prices ($\Delta\Pi^*_s = \frac{\Pi^*_s-\Pi^*_{N,s}}{\Pi^*_{N,s}}$), and 2) optimal dynamic pricing ($\Delta\Pi^*_{D,s} = \frac{\Pi^*_{D,s}-\Pi^*_{N,s}}{\Pi^*_{N,s}}$). ($\lambda=1$, $T=100$, $\gamma=1$, v_R and v_H are jointly uniformly distributed over $\Omega=\{(v_R,v_H):0\le v_R\le v_H\le 200\}$)

Even if the firm is a monopoly in the local market and can freely determine its product prices, implementing variable pricing (i.e., charging different prices for the same product consumed at different times) or dynamic pricing (i.e., changing the price over time for the same product consumed at the same time) may still create consumer dissatisfaction. Recall that many hotels are having a hard time to convince business consumers to accept dynamic pricing. While variable pricing has become more acceptable over time in travel-related industries, most firms still have constraints on how much they can freely adjust prices based on demand. For example, if demand is very low on a given day, optimal pricing for that particular day may result in the hotel setting severely discounted prices for its rooms. But many hotels are reluctant to do that as they believe offering rooms below certain price levels may undercut their image and damage their brand. Compared to changing the product prices, changing the upgrade price may be a more benign strategy. The hotel would not suffer from reputational effects as consumers would usually consider upgrades as a benefit offered to them. Thus, overall we conclude that the conditional upgrade strategy is a very good alternative to unconstrained variable/dynamic pricing.

8. Conclusion

In this paper, we study the conditional upgrade policy that has become popular especially in the travel industry. We model the consumers' strategic behavior of anticipating the upgrade probability when making booking decisions and derive the firm's optimal upgrade price incorporating the strategic consumer behavior. We find that offering conditional upgrades improves the firm's revenue so long as the product price differential is not too small. Thus, our paper provides conditions on when firms will benefit from conditional upgrades. We also find that unlike the "markdown pricing" settings, the existence of more strategic consumers benefits the firm when the firm offers conditional upgrades.

Moreover, we derive managerial insights about why the conditional upgrade strategy is effective in generating more revenues. First, conditional upgrades expand the firm's demand as some consumers, who wouldn't buy any of the products without the upgrade option, start purchasing when conditional upgrades are introduced. Second, the optimal upgrade pricing strategy can work as a product price correction mechanism and reoptimize the firm's demand segmentation. With conditional upgrades, more consumers become willing to purchase high-quality products (including purchasing through upgrades). This is especially helpful when the firm's high-quality product demand is weak. By properly offering conditional upgrades at the optimal upgrade price, the firm can capture at least the revenue potential from optimizing the high-quality product price. Third, the conditional upgrade strategy is one novel way of risk management. The extra flexibility created by the upgrade channel allows the firm to better allocate its capacities across product types to stochastic demands and improve utilization. We have seen that the conditional upgrade strategy not only can compensate for the firm's lack of ability in setting its product prices optimally, but it can also result in even higher revenues than optimized product prices. If the firm already has the ability of setting static product prices optimally, we have observed that offering conditional upgrades can generate higher revenues than using dynamic pricing. We have also seen that conditional upgrades generally outperform last-minute upgrades. Finally, we have derived a simple fluid model which can be very effective in estimating optimal upgrade prices for the underlying stochastic model even when overall capacity is very low.

References

Armstrong, M., Y. Chen. 2009. Inattentive consumers and product quality. *J. European Economic Association* **7**(2-3) 411–422.

Aviv, Y., A. Pazgal. 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing Service Oper. Management* **10**(3) 339–359.

Baker, M. 2010. Dynamic hotel pricing making limited inroads among buyers. *Business Travel News* (September 23) URL: http://www.businesstravelnews.com/Hotel-News/BTN-Research–Dynamic-Hotel-Pricing-Making-Limited-Inroads-Among-Buyers/?a=btn.

Bassok, Y., R. Anupindi, R. Akella. 1999. Single-period multiproduct inventory models with substitution. *Oper. Res.* **47**(4) 632–642.

Biery, M.E. 2014. How U.S. hotels are faring. *Forbes* (January 5) URL: http://www.forbes.com/sites/sageworks/2014/01/05/how-u-s-hotels-are-faring/.

Biyalogorsky, E., E. Gerstner, D. Weiss, J. Xie. 2005. The economics of service upgrades. *J. Service Res.* **7**(3) 234–244.

Caldentey, R., G. Vulcano. 2007. Online auction and list price revenue management. *Management Sci.* **53**(5) 795–813.

Correa, J.R., R. Montoya, C. Thraves. 2013. Contingent preannounced pricing policies with strategic consumers. Working paper, University of Chile, Chile.

Elmaghraby, W., A. Gülcü, P. Keskinocak. 2008. Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *Manufacturing Service Oper. Management* **10**(1) 126–148.

Fay, S., J. Xie. 2008. Probabilistic goods: A creative way of selling products and services. *Marketing Sci.* **27**(4) 674–690.

Gabaix, X., D. Laibson. 2006. Shrouded attributes, consumer myopia, and information suppression in competitive markets. *Quarterly J. Economics* **121**(2) 505–540.

Gallego, G., R. Phillips, Ö. Şahin. 2008. Strategic management of distressed inventory. *Production and Oper. Management* **17**(4) 402–415.

Gallego, G., C. Stefanescu. 2009. Upgrades, upsells and pricing in revenue management. Working paper, Columbia University, New York, NY.

Gallego, G., G. van Ryzin. 1997. A multiproduct dynamic pricing problem and its applications to network yield management. *Oper. Res.* **45**(1) 24–41.

Hsu, A., Y. Bassok. 1999. Random yield and random demand in a production system with downward substitution. *Oper. Res.* **47**(2) 277–290.

Jerath, K., S. Netessine, S.K. Veeraraghavan. 2010. Revenue management with strategic customers: Last-minute selling and opaque selling. *Management Sci.* **56**(3) 430–448.

Karaesmen, I., G. Van Ryzin. 2004. Overbooking with substitutable inventory classes. *Oper. Res.* **52**(1) 83–104.

Levin, Y., J. McGill, M. Nediak. 2010. Optimal dynamic pricing of perishable items by a monopolist facing strategic consumers. *Production and Oper. Management* **19**(1) 40–60.

Mersereau, A.J., D. Zhang. 2012. Markdown pricing with unknown fraction of strategic customers. *Manufacturing Service Oper. Management* **14**(3) 355–370.

Milchtaich, I. 2004. Random-player games. *Games and Economic Behavior* **47**(2) 353–388.

Moorthy, K.S., I.P.L. Png. 1992. Market segmentation, cannibalization, and the timing of product introductions. *Management Sci.* **38**(3) 345–359.

Mussa, M., S. Rosen. 1978. Monopoly and product quality. *J. Economic Theory* **18**(2) 301–317.

Myerson, R.B. 1998. Population uncertainty and poisson games. *International J. Game Theory* **27**(3) 375–392.

Myerson, R.B. 2000. Large poisson games. *J. Economic Theory* **94**(1) 7–45.

Netessine, S., G. Dobson, R.A. Shumsky. 2002. Flexible service capacity: Optimal investment and the impact of demand correlation. *Oper. Res.* **50**(2) 375–388.

Netessine, S., C.S. Tang. 2009. *Consumer-driven demand and operations management models*. Springer, New York, NY.

Osadchiy, N., G. Vulcano. 2010. Selling with binding reservations in the presence of strategic consumers. *Management Sci.* **56**(12) 2173–2190.

Shumsky, R.A., F. Zhang. 2009. Dynamic capacity management with substitution. *Oper. Res.* **57**(3) 671–684.

Spiegler, R. 2006. Competition over agents with boundedly rational expectations. *Theoretical Economics* **1**(2) 207–231.

Su, X. 2007. Intertemporal pricing with strategic customer behavior. *Management Sci.* **53**(5) 726–741.

Yin, R., Y. Aviv, A. Pazgal, C.S. Tang. 2009. Optimal markdown pricing: Implications of inventory display formats in the presence of strategic customers. *Management Sci.* **55**(8) 1391–1408.

Appendix A: Proofs of Theorems and Lemmas

LEMMA A1. *For any* $a_1, a_2, a_3, b_1, b_2, b_3 \in [-1,1]$, *we have*

(i) $|a_1a_2 - b_1b_2| \le |a_1 - b_1| + |a_2 - b_2|$.

(ii) $|a_1a_2a_3 - b_1b_2b_3| \le |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3|$.

Proof of Lemma A1 (i) $|a_1a_2 - b_1b_2| = |a_2(a_1 - b_1) + b_1(a_2 - b_2)| \le |a_1 - b_1| + |a_2 - b_2|$.

(ii) $|a_1a_2a_3 - b_1b_2b_3| = |a_2a_3(a_1 - b_1) + a_3b_1(a_2 - b_2) + b_1b_2(a_3 - b_3)| \le |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3|$. □

LEMMA A2. *Let* N_{λ_1} *and* N_{λ_2} *be two Poisson variables with means* λ_1 *and* λ_2, *respectively.* $n \ge 0$ *is an integer. Then, for every* n, *there exist* $\alpha_c(n), \alpha_p(n) \in (0,1]$ *such that*

(i) $\left|\mathbb{P}(N_{\lambda_1} \le n) - \mathbb{P}(N_{\lambda_2} \le n)\right| \le \alpha_c(n)\,|\lambda_1 - \lambda_2|$.

(ii) $\left|\mathbb{P}(N_{\lambda_1} = n) - \mathbb{P}(N_{\lambda_2} = n)\right| \le \alpha_p(n)\,|\lambda_1 - \lambda_2|$.

Moreover, $\alpha_c(n)$ *is decreasing in* n.

Proof of Lemma A2 (i) The case of $n \ge 1$ is proved by Caldentey and Vulcano (2007) (Lemma A3 in online appendix). In particular, $\alpha_c(n) = \mathbb{P}(N_n = n)$. When $n = 0$, $\left|\mathbb{P}(N_{\lambda_1} \le 0) - \mathbb{P}(N_{\lambda_2} \le 0)\right| = \left|e^{-\lambda_1} - e^{-\lambda_2}\right| \le \sup_{\lambda>0}\{e^{-\lambda}\}\,|\lambda_1 - \lambda_2| = |\lambda_1 - \lambda_2|$, hence $\alpha_c(0) = 1$. It is easy to see that $\alpha_c(n) \in (0,1]$. $\alpha_c(n)$ is decreasing in n because $\alpha_c(n+1)/\alpha_c(n) = (1 + 1/n)^n e^{-1} < 1$.

(ii) When $n = 0$, $\alpha_p(n) = \alpha_c(n) = 1$. When $n \ge 1$,

$$\left|\mathbb{P}(N_{\lambda_1} = n) - \mathbb{P}(N_{\lambda_2} = n)\right| = \left|\frac{e^{-\lambda_1}\lambda_1^n}{n!} - \frac{e^{-\lambda_2}\lambda_2^n}{n!}\right| \le \frac{|\lambda_1 - \lambda_2|}{n!}\sup_{\lambda>0}\left\{\left|\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda}\right|\right\}.$$

We have $\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda} = e^{-\lambda}\lambda^{n-1}(n - \lambda)$. Thus, $\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda} > 0$ for $0 < \lambda < n$ and $\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda} < 0$ for $\lambda > n$. Moreover, $\frac{\mathrm{d}^2(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda^2} = e^{-\lambda}\lambda^{n-2}[\lambda^2 - 2n\lambda + n(n-1)]$. Solving $\frac{\mathrm{d}^2(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda^2} = 0$ yields $\lambda = n - \sqrt{n}$ and $\lambda = n + \sqrt{n}$.

Since $\left.\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda}\right|_{\lambda=0} = 0$ and it follows from L'Hospital's Rule that $\lim_{\lambda\to\infty} \frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda} = 0$, we know that $\sup_{\lambda>0}\left\{\left|\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda}\right|\right\}$ is attained at either $\lambda = n-\sqrt{n}$ or $\lambda = n+\sqrt{n}$. Thus,

$$\sup_{\lambda>0}\left\{\left|\frac{\mathrm{d}(e^{-\lambda}\lambda^n)}{\mathrm{d}\lambda}\right|\right\} = \max\left\{e^{-(n-\sqrt{n})}(n-\sqrt{n})^{n-1}\sqrt{n}, e^{-(n+\sqrt{n})}(n+\sqrt{n})^{n-1}\sqrt{n}\right\},$$

and hence

$$\alpha_p(n) = \max\left\{\frac{e^{-(n-\sqrt{n})}(n-\sqrt{n})^{n-1}\sqrt{n}}{n!}, \frac{e^{-(n+\sqrt{n})}(n+\sqrt{n})^{n-1}\sqrt{n}}{n!}\right\} = \frac{\max\left\{\mathbb{P}(N_{n-\sqrt{n}} = n), \mathbb{P}(N_{n+\sqrt{n}} = n)\right\}}{\sqrt{n}}.$$

It is easy to see that $\alpha_p(n) \in (0,1]$. □

LEMMA A3. *For any $a, b \in [-1,1]$ and integer $n \geq 0$, we have $|a^n - b^n| \leq n|a-b|$.*

Proof of Lemma A3 $|a^n - b^n| = |a-b| \cdot \left|\sum_{i=0}^{n-1} a^i b^{n-1-i}\right| \leq |a-b| \cdot \sum_{i=0}^{n-1} |a^i b^{n-1-i}| \leq n|a-b|$. □

Proof of Theorem 1 In order to show the existence of $q^*(\cdot)$, we need to prove that the mapping $b(q(\cdot))$ from $\mathcal{Q}$ to $\mathcal{Q}$ has the fixed-point property. By the Schauder-Tychonoff Fixed-Point Theorem, we need to prove: 1) $\mathcal{Q}$ is convex and compact, 2) $b(q(\cdot))$ is continuous. Convexity of $\mathcal{Q}$ is easy to verify. To prove compactness, by the Arzela-Ascoli Theorem, we need to prove that $\mathcal{Q}$ is closed, bounded, and equicontinuous. Closedness and boundedness of $\mathcal{Q}$ are easy to verify. To prove equicontinuity, first pick a $q(\cdot)$ from $\mathcal{Q}$. For any $t_1, t_2 \in [0,T]$, we have $|q(t_1) - q(t_2)| \leq \sup_{0\leq t\leq T}\{|q'(t)|\}\,|t_1 - t_2|$. Next, let $\bar{q}' = \sup_{q(\cdot)\in\mathcal{Q}} \sup_{0\leq t\leq T}\{|q'(t)|\}$. Note that $\bar{q}'$ is finite because each $q(\cdot)$ is bounded. Then, for any $\epsilon > 0$, there exists $\delta = \epsilon/\bar{q}'$ such that if $|t_1 - t_2| < \delta$, then for all $q(\cdot) \in \mathcal{Q}$, $|q(t_1) - q(t_2)| \leq \bar{q}'|t_1 - t_2| < \epsilon$. Thus, we have proved equicontinuity of $\mathcal{Q}$.

Next, we prove that $b(q(\cdot))$ is a continuous mapping. In order to obtain a sufficient condition for the uniqueness of $q^*(\cdot)$, we will prove a stronger result that $b(q(\cdot))$ is Lipschitz continuous, that is, there exists a constant $\bar{\alpha} \geq 0$ such that for any $q_1(\cdot), q_2(\cdot) \in \mathcal{Q}$, $\left\|b(q_1(\cdot)) - b(q_2(\cdot))\right\|_\infty \leq \bar{\alpha}\left\|q_1(\cdot) - q_2(\cdot)\right\|_\infty$. For a given arrival time t, we start by bounding $\left|b(q_1(\cdot)) - b(q_2(\cdot))\right|$ from above as follows:

$$\left|b(q_1(\cdot)) - b(q_2(\cdot))\right| = \frac{\left|g(q_1(\cdot))h(q_2(\cdot)) - g(q_2(\cdot))h(q_1(\cdot))\right|}{h(q_1(\cdot))h(q_2(\cdot))} \leq \frac{\left|g(q_1(\cdot)) - g(q_2(\cdot))\right| + \left|h(q_1(\cdot)) - h(q_2(\cdot))\right|}{h(q_1(\cdot))h(q_2(\cdot))}, \quad \text{(A1)}$$

where the inequality follows from Lemma A1(i).

We analyze (A1) part by part. We first bound the denominator of (A1) from below as follows:

$$\begin{aligned} h(q_1(\cdot)) &\geq \mathbb{P}(N_H(T|q_1(\cdot)) < K_H, N_R(T|q_1(\cdot)) < K_R, N_H(T|q_1(\cdot)) + N_U(T|q_1(\cdot)) + N_R(T|q_1(\cdot)) < K_H + K_R) \\ &\geq \mathbb{P}(N_\lambda(T) < K_H, N_\lambda(T) < K_R, N_\lambda(T) < K_H + K_R) \\ &= \mathbb{P}\left(N_\lambda(T) < \min\{K_H, K_R\}\right) \stackrel{\text{def}}{=\!=} \alpha_h, \end{aligned}$$

where $N_\lambda(t)$ denotes the Poisson process with rate λ. The above bound is also valid for $h(q_2(\cdot))$, hence

$$h(q_1(\cdot))h(q_2(\cdot)) \geq \alpha_h^2. \quad \text{(A2)}$$

Now, consider the numerator of (A1). To bound $|h(q_1(\cdot)) - h(q_2(\cdot))|$ from above, we can write $h(q(\cdot))$ as

$$h(q(\cdot)) = \sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1} \mathbb{P}(N_H(t|q(\cdot)) = i_H)\,\mathbb{P}(N_R(t|q(\cdot)) = i_R)\,\mathbb{P}(N_U(t|q(\cdot)) < K_H + K_R - i_H - i_R).$$

Then, we have

$$\begin{aligned}
&\big|h(q_1(\cdot))-h(q_2(\cdot))\big|\\
\le& \sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\big|\,\mathbb{P}(N_H(t|q_1(\cdot))=i_H)\,\mathbb{P}(N_U(t|q_1(\cdot))<K_H+K_R-i_H-i_R)\\
&\qquad\qquad -\mathbb{P}(N_H(t|q_2(\cdot))=i_H)\,\mathbb{P}(N_U(t|q_2(\cdot))<K_H+K_R-i_H-i_R)\big|\\
\le& \sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\big|\,\mathbb{P}(N_H(t|q_1(\cdot))=i_H)-\mathbb{P}(N_H(t|q_2(\cdot))=i_H)\big|\\
&+\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\big|\,\mathbb{P}(N_U(t|q_1(\cdot))<K_H+K_R-i_H-i_R)-\mathbb{P}(N_U(t|q_2(\cdot))<K_H+K_R-i_H-i_R)\big|,
\end{aligned}$$

where the first step follows from the fact that $N_R(t|q(\cdot))$ does not depend on $q(\cdot)$, and the second step follows from Lemma A1(i). Define $\xi_H(t|q(\cdot))=\gamma\xi_H^\gamma(t|q(\cdot))+(1-\gamma)\xi_H'(t|q(\cdot))$, $\xi_U(t|q(\cdot))=\gamma\xi_U^\gamma(t|q(\cdot))$, $\xi_R(t|q(\cdot))=\gamma\xi_R^\gamma(t|q(\cdot))+(1-\gamma)\xi_R'(t|q(\cdot))$ as the proportions of total demand rate λ for $N_H(t|q(\cdot))$, $N_U(t|q(\cdot))$, $N_R(t|q(\cdot))$, respectively. We can bound $\big|\,\mathbb{P}(N_H(t|q_1(\cdot))=i_H)-\mathbb{P}(N_H(t|q_2(\cdot))=i_H)\big|$ as follows. Using Lemma A2(ii) yields

$$\begin{aligned}
\big|\,\mathbb{P}(N_H(t|q_1(\cdot))=i_H)-\mathbb{P}(N_H(t|q_2(\cdot))=i_H)\big| &\le \alpha_p(i_H)\left|\int_0^t\lambda[\xi_H(s|q_1(\cdot))-\xi_H(s|q_2(\cdot))]\,\mathrm{d}s\right|\\
&\le \alpha_p(i_H)\lambda\int_0^t\big|\xi_H(s|q_1(\cdot))-\xi_H(s|q_2(\cdot))\big|\,\mathrm{d}s\\
&\le \alpha_p(i_H)\lambda T\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty.
\end{aligned}$$

Similarly, We can use Lemma A2(i) to obtain $\big|\,\mathbb{P}(N_U(t|q_1(\cdot))<K_H+K_R-i_H-i_R)-\mathbb{P}(N_U(t|q_2(\cdot))<K_H+K_R-i_H-i_R)\big|\le\alpha_c(K_H+K_R-1-i_H-i_R)\lambda T\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty$. Combining these two inequalities leads to $\big|h(q_1(\cdot))-h(q_2(\cdot))\big|\le\alpha_{H1}(t)\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_{U1}(t)\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty$, where

$$\begin{aligned}
\alpha_{H1}(t) &= \lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\alpha_p(i_H),\\
\alpha_{U1}(t) &= \lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\alpha_c(K_H+K_R-1-i_H-i_R).
\end{aligned}$$

We further bound $\alpha_{H1}(t)$ and $\alpha_{U1}(t)$ as follows:

$$\alpha_{H1}(t)=\lambda T\,\mathbb{P}(N_R(t|q_1(t))<K_R)\sum_{i_H=0}^{K_H-1}\alpha_p(i_H)\le\lambda T\sum_{i_H=0}^{K_H-1}\alpha_p(i_H)\stackrel{\text{def}}{=\!=}\alpha_{H1}.$$

Similarly,

$$\begin{aligned}
\alpha_{U1}(t) &\le \lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\mathbb{P}(N_R(t|q_1(\cdot))=i_R)\alpha_c(K_H-i_H)\\
&= \lambda T\,\mathbb{P}(N_R(t|q_1(\cdot))<K_R)\sum_{i_H=0}^{K_H-1}\alpha_c(K_H-i_H)\\
&\le \lambda T\sum_{i_H=1}^{K_H}\alpha_c(i_H)\stackrel{\text{def}}{=\!=}\alpha_{U1},
\end{aligned}$$

where the first inequality follows from Lemma A2(i) that $\alpha_c(n)$ is decreasing in n. Thus, we have obtained that

$$\left|h(q_1(\cdot))-h(q_2(\cdot))\right| \le \alpha_{H1}\left\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\right\|_\infty+\alpha_{U1}\left\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\right\|_\infty. \tag{A3}$$

Next, we bound $|g(q_1(\cdot))-g(q_2(\cdot))|$ from above. If $\tau_H(q(\cdot))\le\tau_T(q(\cdot))$ and $\tau_H(q(\cdot))\le T$, then $g(q(\cdot))=0$. Thus, we can write $g(q(\cdot))$ as $g(q(\cdot))=g_1(q(\cdot))+g_2(q(\cdot))+g_3(q(\cdot))$, where

$$\begin{aligned}
g_1(q(\cdot)) &= \mathbb{P}(\tau_R(q(\cdot))\le\tau_T(q(\cdot)),\tau_R(q(\cdot))\le T)g(q(\cdot)|\tau_R(q(\cdot))\le\tau_T(q(\cdot)),\tau_R(q(\cdot))\le T),\\
g_2(q(\cdot)) &= \mathbb{P}(\tau_H(q(\cdot))>\tau_T(q(\cdot)),\tau_R(q(\cdot))>\tau_T(q(\cdot)),\tau_T(q(\cdot))\le T)\\
&\quad \cdot g(q(\cdot)|\tau_H(q(\cdot))>\tau_T(q(\cdot)),\tau_R(q(\cdot))>\tau_T(q(\cdot)),\tau_T(q(\cdot))\le T),\\
g_3(q(\cdot)) &= \mathbb{P}(\tau_H(q(\cdot))>T,\tau_R(q(\cdot))>T,\tau_T(q(\cdot))>T)g(q(\cdot)|\tau_H(q(\cdot))>T,\tau_R(q(\cdot))>T,\tau_T(q(\cdot))>T).
\end{aligned}$$

Now we consider each term of $g(q(\cdot))$. Define $m_R(t|q(\cdot))=\int_0^T\lambda\xi_R(t|q(\cdot))\,\mathrm{d}t$ as the mean value function of $N_R(t|q(\cdot))$. Define $f_{\tau_R(q(\cdot))}(t)$ as the probability density function of $\tau_R(q(\cdot))$, and $f_{\tau_T(q(\cdot))}(t)$ as the probability density function of $\tau_T(q(\cdot))$. We have

$$f_{\tau_R(q(\cdot))}(t)=\frac{e^{-m_R(t|q(\cdot))}[m_R(t|q(\cdot))]^{K_R-1}\lambda\xi_R(t|q(\cdot))}{(K_R-1)!}=\mathbb{P}(N_R(t|q(\cdot))=K_R-1)\lambda\xi_R(t|q(\cdot)),$$

and similarly,

$$f_{\tau_T(q(\cdot))}(t)=\mathbb{P}(N_H(t|q(\cdot))+N_U(t|q(\cdot))+N_R(t|q(\cdot))=K_H+K_R-2)\lambda\left[\xi_H(t|q(\cdot))+\xi_U(t|q(\cdot))+\xi_R(t|q(\cdot))\right].$$

$g_1(q(\cdot))$ can be written as

$$\begin{aligned}
g_1(q(\cdot)) &= \int_t^T f_{\tau_R(q(\cdot))}(s)\,\mathbb{P}(N_H(s|q(\cdot))+N_U(s|q(\cdot))\le K_H-1)\,\mathrm{d}s\\
&= \int_t^T \mathbb{P}(N_R(s|q(\cdot))=K_R-1)\lambda\xi_R(s|q(\cdot))\,\mathbb{P}(N_H(s|q(\cdot))+N_U(s|q(\cdot))\le K_H-1)\,\mathrm{d}s.
\end{aligned}$$

$g_2(q(\cdot))$ can be written as

$$\begin{aligned}
g_2(q(\cdot)) &= \int_t^T f_{\tau_T(q(\cdot))}(s)\cdot\left[\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\frac{K_H-i_H}{K_H+K_R-i_H-i_R}\right.\\
&\qquad \left.\cdot\,\mathbb{P}(N_H(s|q(\cdot))=i_H,N_R(s|q(\cdot))=i_R,N_U(s|q(\cdot))=K_H+K_R-1-i_H-i_R|\tau_T(q(\cdot))=s)\right]\mathrm{d}s\\
&= \int_t^T \mathbb{P}(N_H(s|q(\cdot))+N_U(s|q(\cdot))+N_R(s|q(\cdot))=K_H+K_R-2)\lambda\left[\xi_H(s|q(\cdot))+\xi_U(s|q(\cdot))+\xi_R(s|q(\cdot))\right]\\
&\qquad\cdot\left\{\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\frac{K_H-i_H}{K_H+K_R-i_H-i_R}\binom{K_H+K_R-1}{i_H}\binom{K_H+K_R-1-i_H}{i_R}\right.\\
&\qquad\quad\left.\cdot\left\{\mathop{\mathbb{E}}_{0\le r\le s}[\xi_H(r|q(\cdot))]\right\}^{i_H}\left\{\mathop{\mathbb{E}}_{0\le r\le s}[\xi_R(r|q(\cdot))]\right\}^{i_R}\left\{\mathop{\mathbb{E}}_{0\le r\le s}[\xi_U(r|q(\cdot))]\right\}^{K_H+K_R-1-i_H-i_R}\right\}\mathrm{d}s.
\end{aligned}$$

$g_3(q(\cdot))$ can be written as

$$\begin{aligned}
g_3(q(\cdot)) = \sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\sum_{i_U=0}^{K_H+K_R-2-i_H-i_R} &\mathbb{P}(N_H(T|q(\cdot))=i_H)\,\mathbb{P}(N_R(T|q(\cdot))=i_R)\\
&\cdot\mathbb{P}(N_U(T|q(\cdot))=i_U)\min\left\{\frac{K_H-i_H}{i_U+1},1\right\}.
\end{aligned}$$

Next, notice that

$$\big|g(q_1(\cdot))-g(q_2(\cdot))\big|\le\big|g_1(q_1(\cdot))-g_1(q_2(\cdot))\big|+\big|g_2(q_1(\cdot))-g_2(q_2(\cdot))\big|+\big|g_3(q_1(\cdot))-g_3(q_2(\cdot))\big|. \tag{A4}$$

By using the same approach that is used to bound $\big|h(q_1(\cdot))-h(q_2(\cdot))\big|$, we can bound each term in the right-hand side (RHS) of (A4) from above. Bounding the first term in the RHS of (A4) results in

$$\big|g_1(q_1(\cdot))-g_1(q_2(\cdot))\big|\le\alpha_{H2}\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_{U2}\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty, \tag{A5}$$

where $\alpha_{H2}=\alpha_{U2}=(\lambda T)^2\alpha_c(K_H-1)$. Bounding the second term in the RHS of (A4) results in

$$\big|g_2(q_1(\cdot))-g_2(q_2(\cdot))\big|\le\alpha_{H3}\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_{U3}\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty, \tag{A6}$$

where

$$\alpha_{H3}=\lambda T\left\{\lambda T\alpha_p(K_H+K_R-2)+1+\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\binom{K_H+K_R-1}{i_H}\binom{K_H+K_R-1-i_H}{i_R}\frac{i_H(K_H-i_H)}{K_H+K_R-i_H-i_R}\right\},$$

$$\begin{aligned}\alpha_{U3}=\lambda T\Bigg\{&\lambda T\alpha_p(K_H+K_R-2)+1\\&+\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\binom{K_H+K_R-1}{i_H}\binom{K_H+K_R-1-i_H}{i_R}\frac{(K_H+K_R-1-i_H-i_R)(K_H-i_H)}{K_H+K_R-i_H-i_R}\Bigg\}.\end{aligned}$$

Lemma A3 is used in deriving α_{H3} and α_{U3}. Bounding the third term in the RHS of (A4) results in $\big|g_3(q_1(\cdot))-g_3(q_2(\cdot))\big|\le\alpha_{H4}(t)\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_{U4}(t)\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty$, where

$$\begin{aligned}\alpha_{H4}(t)&=\lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\sum_{i_U=0}^{K_H+K_R-2-i_H-i_R}\mathbb{P}(N_R(T|q_1(\cdot))=i_R)\alpha_p(i_H)\min\left\{\frac{K_H-i_H}{i_U+1},1\right\},\\ \alpha_{U4}(t)&=\lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_R=0}^{K_R-1}\sum_{i_U=0}^{K_H+K_R-2-i_H-i_R}\mathbb{P}(N_R(T|q_1(\cdot))=i_R)\alpha_p(i_U)\min\left\{\frac{K_H-i_H}{i_U+1},1\right\}.\end{aligned}$$

We then have

$$\begin{aligned}\alpha_{H4}(t)&\le\lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_U=0}^{K_H+K_R-2-i_H}\alpha_p(i_H)\min\left\{\frac{K_H-i_H}{i_U+1},1\right\}\overset{\text{def}}{=\!=}\alpha_{H4},\\ \alpha_{U4}(t)&\le\lambda T\sum_{i_H=0}^{K_H-1}\sum_{i_U=0}^{K_H+K_R-2-i_H}\alpha_p(i_U)\min\left\{\frac{K_H-i_H}{i_U+1},1\right\}\overset{\text{def}}{=\!=}\alpha_{U4}.\end{aligned}$$

Thus, we have obtained that

$$\big|g_3(q_1(\cdot))-g_3(q_2(\cdot))\big|\le\alpha_{H4}\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_{U4}\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty, \tag{A7}$$

Thus, by plugging (A5), (A6), (A7) into (A4) and then plugging (A2), (A3), (A4) into (A1), we obtain

$$\big|b(q_1(\cdot))-b(q_2(\cdot))\big|\le\alpha_H\big\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\big\|_\infty+\alpha_U\big\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\big\|_\infty, \tag{A8}$$

where

$$\alpha_H=\frac{\alpha_{H1}+\alpha_{H2}+\alpha_{H3}+\alpha_{H4}}{\alpha_h^2},\qquad\alpha_U=\frac{\alpha_{U1}+\alpha_{U2}+\alpha_{U3}+\alpha_{U4}}{\alpha_h^2}.$$

Note that α_H and α_U do not depend on t.

It remains to bound $\left\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\right\|_\infty$ and $\left\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\right\|_\infty$ from above. Define $\bar{v}_H = \sup\{v_H : (v_R,v_H)\in\Omega\}$, $\underline{v}_H = \inf\{v_H : (v_R,v_H)\in\Omega\}$, $\bar{v}_R = \sup\{v_R : (v_R,v_H)\in\Omega\}$, $\underline{v}_R = \inf\{v_R : (v_R,v_H)\in\Omega\}$. Fix t, and without loss of generality, assume $q_1(t) < q_2(t)$. First, consider $\left\|\xi_H(t|q_1(\cdot))-\xi_H(t|q_2(\cdot))\right\|_\infty$. We have

$$\xi_H(t|q(t)) = \gamma\int_{p_H}^{\bar{v}_H}\left[\int_{\underline{v}_R}^{\max\left\{v_H-\frac{p_H-p_R-q(t)p}{1-q(t)},\underline{v}_R\right\}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H + (1-\gamma)\xi_H'(t|q(t)).$$

Since $\max\left\{v_H-\frac{p_H-p_R-q(t)p}{1-q(t)},\underline{v}_R\right\}$ is decreasing in $q(t)$, we have

$$\begin{aligned}
&\left|\xi_H(t|q_1(t))-\xi_H(t|q_2(t))\right| \\
&= \gamma\int_{p_H}^{\bar{v}_H}\left[\int_{\max\left\{v_H-\frac{p_H-p_R-q_2(t)p}{1-q_2(t)},\underline{v}_R\right\}}^{\max\left\{v_H-\frac{p_H-p_R-q_1(t)p}{1-q_1(t)},\underline{v}_R\right\}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H \\
&\le \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\int_{p_H}^{\bar{v}_H}\left[\max\left\{v_H-\frac{p_H-p_R-q_1(t)p}{1-q_1(t)},\underline{v}_R\right\}-\max\left\{v_H-\frac{p_H-p_R-q_2(t)p}{1-q_2(t)},\underline{v}_R\right\}\right]\mathrm{d}v_H.
\end{aligned}$$

Define

$$\bar{q}_H = \sup\left\{0<q<1 : v_H-\frac{p_H-p_R-qp}{1-q}\ge\underline{v}_R\right\} = \begin{cases}\frac{v_H-p_H+p_R-\underline{v}_R}{v_H-\underline{v}_R-p} & \text{if } v_H > p_H-p_R+\underline{v}_R,\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that $\bar{q}_H < 1$, and $v_H-\frac{p_H-p_R-q(t)p}{1-q(t)}\ge\underline{v}_R$ if and only if $q(t)\le\bar{q}_H$. Then,

$$\begin{aligned}
&\max\left\{v_H-\frac{p_H-p_R-q_1(t)p}{1-q_1(t)},\underline{v}_R\right\}-\max\left\{v_H-\frac{p_H-p_R-q_2(t)p}{1-q_2(t)},\underline{v}_R\right\}\\
&=\left[v_H-\frac{p_H-p_R-\min\{q_1(t),\bar{q}_H\}p}{1-\min\{q_1(t),\bar{q}_H\}}\right]-\left[v_H-\frac{p_H-p_R-\min\{q_2(t),\bar{q}_H\}p}{1-\min\{q_2(t),\bar{q}_H(t)\}}\right]\\
&=\frac{(p_H-p_R-p)\left[\min\{q_2(t),\bar{q}_H\}-\min\{q_1(t),\bar{q}_H\}\right]}{(1-\min\{q_1(t),\bar{q}_H\})\,(1-\min\{q_2(t),\bar{q}_H\})}\\
&\le\frac{p_H-p_R-p}{(1-\bar{q}_H)^2}\left[q_2(t)-q_1(t)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\xi_H(t|q_1(t))-\xi_H(t|q_2(t))\right| &\le \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\,(p_H-p_R-p)\left[q_2(t)-q_1(t)\right]\int_{p_H}^{\bar{v}_H}\frac{1}{(1-\bar{q}_H)^2}\,\mathrm{d}v_H\\
&= \alpha_H'\left[q_2(t)-q_1(t)\right],
\end{aligned}$$

where

$$\alpha_H' = \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\,(p_H-p_R-p)\left[(\underline{v}_R-p_R)^+ + \frac{(\bar{v}_H-\underline{v}_R-p)^3-(p_H-\min\{\underline{v}_R,p_R\}-p)^3}{3(p_H-p_R-p)^2}\right].$$

Note that α_H' is finite because Ω is finite. Then, we have

$$\left|\xi_H(t|q_1(t))-\xi_H(t|q_2(t))\right| \le \alpha_H'\left\|q_1(\cdot)-q_2(\cdot)\right\|_\infty. \tag{A9}$$

Second, consider $\left\|\xi_U(t|q_1(\cdot))-\xi_U(t|q_2(\cdot))\right\|_\infty$. We have

$$\xi_U(t|q(t)) = \gamma\int_{p_H}^{\bar{v}_H}\left[\int_{\max\left\{v_H-\frac{p_H-p_R-q(t)p}{1-q(t)},\underline{v}_R\right\}}^{v_H-p} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H + \gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q(t)p-q(t)v_H}{1-q(t)}}^{v_H-p} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H.$$

Note that we write the lower bound of the second integration as p_R+p instead of $\underline{v}_H$ because for $v_H<p_R+p$, we have $v_H-p<\frac{p_R+q(t)p-q(t)v_H}{1-q(t)}$ for all $q(t)$. Since $\max\left\{v_H-\frac{p_H-p_R-q(t)p}{1-q(t)},\underline{v}_R\right\}$ is decreasing in $q(t)$ and $\frac{p_R+q(t)p-q(t)v_H}{1-q(t)}$ is decreasing in $q(t)$ for $v_H\geq p_R+p$, we have

$$\begin{aligned}
\left|\xi_U(t|q_1(t))-\xi_U(t|q_2(t))\right| &= \gamma\int_{p_H}^{\bar{v}_H}\left[\int_{\max\left\{v_H-\frac{p_H-p_R-q_2(t)p}{1-q_2(t)},\underline{v}_R\right\}}^{\max\left\{v_H-\frac{p_H-p_R-q_1(t)p}{1-q_1(t)},\underline{v}_R\right\}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H \\
&\quad+\gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q_2(t)p-q_2(t)v_H}{1-q_2(t)}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H \\
&\leq \alpha'_H\left[q_2(t)-q_1(t)\right]+\gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q_2(t)p-q_2(t)v_H}{1-q_2(t)}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H.
\end{aligned}$$

We can write the last integration equivalently as

$$\gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q_2(t)p-q_2(t)v_H}{1-q_2(t)}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H=\gamma\int_{\underline{v}_R}^{p_R}\left[\int_{v_R+p+\frac{p_R-v_R}{q_2(t)}}^{v_R+p+\frac{p_R-v_R}{q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_H\right]\mathrm{d}v_R.$$

Now we bound this integration from above case by case. Define $\bar{q}_U$ as the solution to $\frac{p_R+\bar{q}_Up-\bar{q}_Uv_H}{1-\bar{q}_U}=\underline{v}_R$, so $\bar{q}_U=\frac{p_R-\underline{v}_R}{p_H-\underline{v}_R-p}$. At $q(t)=\bar{q}_U$, $\frac{p_R+q(t)p-q(t)v_H}{1-q(t)}=v_R$ becomes the negatively-sloped diagonal of the rectangle $\{(v_R,v_H):\underline{v}_R\leq v_R\leq p_R, p_R+p\leq v_H\leq p_H\}$.

- If $q_1(t)<q_2(t)\leq\bar{q}_U$,

$$\begin{aligned}
\gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q_2(t)p-q_2(t)v_H}{1-q_2(t)}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f_t(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H &\leq \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\int_{p_R+p}^{p_H}\frac{(v_H-p_R-p)\left[q_2(t)-q_1(t)\right]}{\left[1-q_1(t)\right]\left[1-q_2(t)\right]}\mathrm{d}v_H \\
&\leq \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\frac{q_2(t)-q_1(t)}{\left(1-\bar{q}_U\right)^2}\int_{p_R+p}^{p_H}(v_H-p_R-p)\,\mathrm{d}v_H \\
&= \alpha'_U\left[q_2(t)-q_1(t)\right],
\end{aligned}$$

where

$$\alpha'_U=\gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\frac{\left(p_H-\underline{v}_R-p\right)^2}{2}.$$

- If $\bar{q}_U\leq q_1(t)<q_2(t)$,

$$\begin{aligned}
\gamma\int_{\underline{v}_R}^{p_R}\left[\int_{v_R+p+\frac{p_R-v_R}{q_2(t)}}^{v_R+p+\frac{p_R-v_R}{q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_H\right]\mathrm{d}v_R &\leq \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\int_{\underline{v}_R}^{p_R}\frac{(p_R-v_R)\left[q_2(t)-q_1(t)\right]}{q_1(t)q_2(t)}\mathrm{d}v_R \\
&\leq \gamma\sup_{(v_R,v_H)\in\Omega}\{f(v_R,v_H)\}\frac{q_2(t)-q_1(t)}{\bar{q}_U^2}\int_{\underline{v}_R}^{p_R}(p_R-v_R)\,\mathrm{d}v_R \\
&= \alpha'_U\left[q_2(t)-q_1(t)\right].
\end{aligned}$$

- If $q_1(t)<\bar{q}_U<q_2(t)$,

$$\begin{aligned}
&\gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+q_2(t)p-q_2(t)v_H}{1-q_2(t)}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H \\
&= \gamma\int_{p_R+p}^{p_H}\left[\int_{\frac{p_R+\bar{q}_Up-\bar{q}_Uv_H}{1-\bar{q}_U}}^{\frac{p_R+q_1(t)p-q_1(t)v_H}{1-q_1(t)}} f(v_R,v_H)\,\mathrm{d}v_R\right]\mathrm{d}v_H+\gamma\int_{\underline{v}_R}^{p_R}\left[\int_{v_R+p+\frac{p_R-v_R}{q_2(t)}}^{v_R+p+\frac{p_R-v_R}{\bar{q}_U}} f(v_R,v_H)\,\mathrm{d}v_H\right]\mathrm{d}v_R \\
&\leq \alpha'_U\left[\bar{q}_U-q_1(t)\right]+\alpha'_U\left[q_2(t)-\bar{q}_U\right] \\
&= \alpha'_U\left[q_2(t)-q_1(t)\right].
\end{aligned}$$

Thus, we obtain that $\left|\xi_U(t|q_1(t)) - \xi_U(t|q_2(t))\right| \le \alpha'_U\,[q_2(t) - q_1(t)]$, where α'_U is finite. Then, we have

$$\left|\xi_U(t|q_1(t)) - \xi_U(t|q_2(t))\right| \le \alpha'_U \left\|q_1(\cdot) - q_2(\cdot)\right\|_\infty, \tag{A10}$$

Finally, by plugging (A9) and (A10) into (A8), we obtain that $\left|b(q_1(\cdot)) - b(q_2(\cdot))\right| \le \bar{\alpha}\left\|q_1(\cdot) - q_2(\cdot)\right\|_\infty$ where $\bar{\alpha} = \alpha_H\alpha'_H + \alpha_U\alpha'_U$ and $\bar{\alpha}$ does not depend on t. Then, we have $\left\|b(q_1(\cdot)) - b(q_2(\cdot))\right\|_\infty \le \bar{\alpha}\left\|q_1(\cdot) - q_2(\cdot)\right\|_\infty$. Therefore, we have proved the Lipschitz continuity of $b(q(\cdot))$, and hence the existence of $q^*(\cdot)$. If $\bar{\alpha} < 1$, $b(q(\cdot))$ is a contraction mapping from $\mathcal{Q}$ to $\mathcal{Q}$, hence $q^*(\cdot)$ is unique. $q^*(\cdot)$ is increasing in t because of the following. For every sample path, $g(q(\cdot))$ is constant in t for $t \le \tau(q(\cdot))$ and is equal to zero for $t > \tau(q(\cdot))$. It is easy to see that $h(q(\cdot))$ is decreasing in t, hence after taking the average of $g(q(\cdot))$ for each sample path, we know that $b(q(\cdot))$ is increasing in t. Therefore, the solution to $b(q(\cdot)) = q(\cdot)$ must be increasing in t. The whole proof is complete. □

Proof of Theorem 2 (i) For any $q(\cdot)$ and any t, as $n \to \infty$, by the Strong Law of Large Numbers, we have

$$\begin{aligned}
\frac{N_H^n(t|q(\cdot))}{n} &\to \int_0^t \lambda_H(s|q(\cdot))\,\mathrm{d}s \qquad \text{a.s.},\\
\frac{N_U^n(t|q(\cdot))}{n} &\to \int_0^t \lambda_U(s|q(\cdot))\,\mathrm{d}s \qquad \text{a.s.},\\
\frac{N_R^n(t|q(\cdot))}{n} &\to \int_0^t \lambda_R(s|q(\cdot))\,\mathrm{d}s \qquad \text{a.s.}.
\end{aligned}$$

Moreover, as $n \to \infty$, we have

$$\begin{aligned}
\tau_T^n(q(\cdot)) &= \inf\left\{t \ge 0 : N_H^n(t|q(\cdot)) + N_U^n(t|q(\cdot)) + 1 + N_R^n(t|q(\cdot)) \ge nK_H + nK_R\right\}\\
&= \inf\left\{t \ge 0 : \frac{N_H^n(t|q(\cdot))}{n} + \frac{N_U^n(t|q(\cdot))}{n} + \frac{1}{n} + \frac{N_R^n(t|q(\cdot))}{n} \ge K_H + K_R\right\}\\
&\to \inf\left\{t \ge 0 : \int_0^t [\lambda_H(s|q(\cdot)) + \lambda_U(s|q(\cdot)) + \lambda_R(s|q(\cdot))]\,\mathrm{d}s \ge K_H + K_R\right\} \qquad \text{a.s.}.
\end{aligned}$$

The convergence of $\tau_H^n(q(\cdot))$ and $\tau_R^n(q(\cdot))$ follows from the same approach, then the convergence of $\tau^n(q(\cdot))$ is obtained.

(ii) To derive $q^{\infty*}(\cdot)$, we need to first derive $g^\infty(q(\cdot))$ and $h^\infty(q(\cdot))$ and then derive $b^\infty(q(\cdot))$. First,

$$\begin{aligned}
g^\infty(q(\cdot)) &= \lim_{n\to\infty} \mathop{\mathbb{E}}_{N_H^n(t|q(\cdot)),N_U^n(t|q(\cdot)),N_R^n(t|q(\cdot))} \left\{\min\left\{\frac{[nK_H - N_H^n(\tau^n(q(\cdot))|q(\cdot))]^+}{N_U^n(\tau^n(q(\cdot))|q(\cdot)) + 1}, 1\right\} \cdot \mathbb{1}\left\{t \le \tau^n(q(\cdot))\right\}\right\}\\
&= \lim_{n\to\infty} \mathop{\mathbb{E}}_{N_H^n(t|q(\cdot)),N_U^n(t|q(\cdot)),N_R^n(t|q(\cdot))} \left\{\min\left\{\frac{[nK_H - N_H^n(\tau^\infty(q(\cdot))|q(\cdot))]^+}{N_U^n(\tau^\infty(q(\cdot))|q(\cdot)) + 1}, 1\right\}\right\} \cdot \mathbb{1}\left\{t \le \tau^\infty(q(\cdot))\right\}\\
&= \lim_{n\to\infty} \mathop{\mathbb{E}}_{N_H^n(t|q(\cdot)),N_U^n(t|q(\cdot)),N_R^n(t|q(\cdot))} \left\{\min\left\{\frac{\left[K_H - \frac{N_H^n(\tau^\infty(q(\cdot))|q(\cdot))}{n}\right]^+}{\frac{N_U^n(\tau^\infty(q(\cdot))|q(\cdot))}{n} + \frac{1}{n}}, 1\right\}\right\} \cdot \mathbb{1}\left\{t \le \tau^\infty(q(\cdot))\right\}\\
&= \min\left\{\frac{\left[K_H - \int_0^{\tau^\infty(q(\cdot))} \lambda_H(t|q(\cdot))\,\mathrm{d}t\right]^+}{\int_0^{\tau^\infty(q(\cdot))} \lambda_U(t|q(\cdot))\,\mathrm{d}t}, 1\right\} \cdot \mathbb{1}\left\{t \le \tau^\infty(q(\cdot))\right\}.
\end{aligned}$$

Second,

$$\begin{aligned}
h^\infty(q(\cdot)) &= \lim_{n\to\infty} \mathbb{P}\left(\frac{N_H^n(t|q(\cdot))}{n} < K_H, \frac{N_R^n(t|q(\cdot))}{n} < K_R, \frac{N_H^n(t|q(\cdot))}{n} + \frac{N_U^n(t|q(\cdot))}{n} + \frac{N_R^n(t|q(\cdot))}{n} < K_H + K_R\right)\\
&= \mathbb{P}\left(\int_0^t \lambda_H(s|q(\cdot))\,\mathrm{d}s < K_H, \int_0^t \lambda_R(s|q(\cdot))\,\mathrm{d}s < K_R, \int_0^t [\lambda_H(s|q(\cdot)) + \lambda_U(s|q(\cdot)) + \lambda_R(s|q(\cdot))]\,\mathrm{d}s < K_H + K_R\right)\\
&= \mathbb{1}\left\{t \le \tau^\infty(q(\cdot))\right\}.
\end{aligned}$$

For $t \le \tau^\infty(q(\cdot))$, $b^\infty(q(\cdot))$ is constant in t, hence so is $q^{\infty *}(\cdot)$. Note that for $t > \tau^\infty(q(\cdot))$, $b^\infty(q(\cdot))$ is not defined. Since the upgrade probability is irrelevant in this case, without loss of generality, we let $b^\infty(q(\cdot))$ take the same value as $t \le \tau^\infty(q(\cdot))$ to preserve the differentiability of $b^\infty(q(\cdot))$. It then follows that $q^{\infty *}(\cdot)$ is given by Part (ii) of the theorem. □

Proof of Theorem 3 With uniform valuation distribution, the formula of $\hat{\tau}^\infty(q)$ in Theorem 2 reduces to

$$\hat{\tau}^\infty(q) = \begin{cases} \frac{K_H}{\lambda_H(q)} & \text{if } \frac{K_H}{\lambda_H(q)} \le \frac{K_H+K_R}{\lambda_H(q)+\lambda_U(q)+\lambda_R}, \\ \frac{K_R}{\lambda_R} & \text{if } \frac{K_R}{\lambda_R} \le \frac{K_H+K_R}{\lambda_H(q)+\lambda_U(q)+\lambda_R}, \\ \frac{K_H+K_R}{\lambda_H(q)+\lambda_U(q)+\lambda_R} & \text{if } \frac{K_H}{\lambda_H(q)} > \frac{K_H+K_R}{\lambda_H(q)+\lambda_U(q)+\lambda_R} \text{ and } \frac{K_R}{\lambda_R} > \frac{K_H+K_R}{\lambda_H(q)+\lambda_U(q)+\lambda_R}. \end{cases} \tag{A11}$$

(i) $q_f = 1$ corresponds to Case b, where $q_f = 1$ requires $K_H \ge (\lambda_H^b + \lambda_U^b)\tau^\infty(1)$. We will show that $K_H \ge (\lambda_H^b + \lambda_U^b)\tau^\infty(1)$ is equivalent to $K_H \ge (\lambda_H^b + \lambda_U^b)T$. First, if $\hat{\tau}^\infty(1) = K_H/\lambda_H^b$, since $K_H/\lambda_H^b \ge K_H/\lambda_H^a \ge T$, we have $\tau^\infty(1) = \min\{K_H/\lambda_H^b, T\} = T$. Second, if $\hat{\tau}^\infty(1) = K_R/\lambda_R^b$, since $K_R/\lambda_R^b \ge K_R/\lambda_R^a \ge T$, we have $\tau^\infty(1) = \min\{K_R/\lambda_R^b, T\} = T$. Third, if $\hat{\tau}^\infty(1) = (K_H + K_R)/(\lambda_H^b + \lambda_U^b + \lambda_R^b)$, suppose $\hat{\tau}^\infty(1) < T$, then it is easy to see that $K_H \ge (\lambda_H^b + \lambda_U^b)\tau^\infty(1)$ is equivalent to $K_R/\lambda_R^b \le (K_H + K_R)/(\lambda_H^b + \lambda_U^b + \lambda_R^b)$ which contradicts the second condition in (A11) for $\hat{\tau}^\infty(1) = (K_H + K_R)/(\lambda_H^b + \lambda_U^b + \lambda_R^b)$ to occur. Thus, we also have $\tau^\infty(1) = T$ in this case. Overall, $K_H \ge (\lambda_H^b + \lambda_U^b)\tau^\infty(1)$ is equivalent to $K_H \ge (\lambda_H^b + \lambda_U^b)T$. Since $\lambda_H^b + \lambda_U^b$ is decreasing in p, if $K_H \ge (\lambda_H^b + \lambda_U^b)T$ at $p = 0$, that is, if $K_H \ge (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$, then $q_f = 1$ for all $0 \le p < p_H - p_R$.

(ii) Next, consider the case of $K_H < (\lambda T/u^2)[(1-\gamma)(u - p_H + 2p_R)(u - p_H) + \gamma(u^2 - p_R^2)]$. We first characterize when $q_f = 1$. In Case b, solving $K_H = (\lambda_H^b + \lambda_U^b)T$ yields $p = \bar{p}$. Since $(\lambda_H^b + \lambda_U^b)T$ is decreasing in p, $q_f = 1$ for $p \ge \bar{p}$.

Now we derive q_f for $0 \le p < \bar{p}$. We first derive q_f for the case of $\hat{\tau}^\infty(q_f) \ge T$ and then incorporate the case of $\hat{\tau}^\infty(q_f) < T$. When $\hat{\tau}^\infty(q_f) \ge T$, Cases c, d, e may occur in sequence as p decreases. Since $(p_H - p_R - q_f p)/(1 - q_f) = \infty > u$ at $p = \bar{p}$, we are in Case c where (1) becomes $q = [K_H - \lambda_H^c T]/[\lambda_U^c(q)T]$. Solving (1) yields

$$q_f = \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H) + \gamma(u - p_H + p_R)^2}{\gamma(u-p)^2} \tag{A12}$$

which is increasing in p. Then, solving the condition for Case c $(p_H - p_R - q_f p)/(1 - q_f) \ge u$ yields $p \ge \underline{p}$, hence Case c occurs where q_f is given by (A12) for $\underline{p}^+ \le p < \bar{p}$ (note that $\underline{p}$ can be negative). For $0 \le p < \underline{p}^+$, Cases d and e may occur. In Case d, (1) becomes $q = [K_H - \lambda_H^d(q)T]/[\lambda_U^d(q)T]$. Solving (1) yields

$$q_f = \frac{\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)}{\gamma(p_H - p_R - p)^2 + \frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)} \tag{A13}$$

which is increasing in p. Then, solving the condition for Case d $(p_H - p_R - q_f p)/(1 - q_f) \ge p_H$ yields

$$p \ge p_H - p_R - \frac{1}{\gamma p_R}\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right] \stackrel{\text{def}}{=\!=} \tilde{p}.$$

So Case d occurs for $\tilde{p}^+ \le p < \underline{p}^+$ and Case e occurs for $0 \le p < \tilde{p}^+$. In Case e, (1) becomes $q = [K_H - \lambda_H^e(q)T]/[\lambda_U^e(q)T]$. Solving (1) also yields (A13). Thus, q_f is given by (A13) for $0 \le p < \underline{p}^+$.

Now, we incorporate the case of $\hat{\tau}^\infty(q_f) < T$. Note that when $\hat{\tau}^\infty(q_f) < T$, we must have $\hat{\tau}^\infty(q_f) = (K_H + K_R)/[\lambda_H(q_f) + \lambda_U(q_f) + \lambda_R]$ which is the last case in (A11), because $\hat{\tau}^\infty(q_f) = K_H/\lambda_H(q_f)$ implies $q_f = 0$ and $\hat{\tau}^\infty(q_f) = K_R/\lambda_R$ implies $q_f = 1$. To analyze the case of $\hat{\tau}^\infty(q_f) < T$, we first show that it may only occur for small enough p. In Cases c, d, e, $\hat{\tau}^{\infty}(q_f) < T$ if and only if $K_H + K_R < [\lambda_H(q_f) + \lambda_U(q_f) + \lambda_R]T$. We will show that when $\hat{\tau}^{\infty}(q_f) \geq T$, $K_H + K_R \geq [\lambda_H(q_f) + \lambda_U(q_f) + \lambda_R]T$ for large p. Using the above derived q_f, in Case c when $\hat{\tau}^{\infty}(q_f) \geq T$, we have

$$\begin{aligned}
\frac{\mathrm{d}[\lambda_H^c + \lambda_U^c(q_f) + \lambda_R^c]}{\mathrm{d}p} &= \frac{2\gamma\lambda p_R}{u^2}\left[\frac{\gamma p_R(u-p)}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H) + \gamma(u-p_H+p_R)^2} - 1\right] \\
&\leq \frac{2\gamma\lambda p_R}{u^2}\left[\frac{\gamma p_R(u-\underline{p})}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H) + \gamma(u-p_H+p_R)^2} - 1\right] \\
&< \frac{2\gamma\lambda p_R}{u^2}\left[\frac{\gamma (u-p_H+p_R)(u-\underline{p})}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H) + \gamma(u-p_H+p_R)^2} - 1\right] \\
&= 0,
\end{aligned}$$

hence $\lambda_H^c + \lambda_U^c(q_f) + \lambda_R^c$ is decreasing in p. Denote $\lambda_N(q) = \lambda - \lambda_H(q) - \lambda_U(q) - \lambda_R$ as the arrival rate of consumers who do not book any product. In Case d, we have

$$\begin{aligned}
\lambda_N^d(q_f) &= \frac{\gamma\lambda p_R^2}{u^2}\left\{\frac{p^2 - 2\left\{p_H - p_R - \frac{1}{\gamma p_R}\left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right]\right\}p}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)}\right. \\
&\qquad \left. + \frac{\frac{1}{\gamma}\left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right] + (p_H-p_R)^2}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)}\right\},
\end{aligned}$$

which is a parabola whose axis of symmetry is $\tilde{p}$. Thus, $\lambda_N^d(q_f)$ is increasing in p, hence $\lambda_H^d(q_f) + \lambda_U^d(q_f) + \lambda_R^d$ is decreasing in p. In Case e, we have

$$\lambda_N^e(q_f) = \frac{\lambda}{\gamma u^2}\left[-\frac{K_H}{\lambda T}u^2 + (u-p_H+2p_R)(u-p_H) + 2\gamma p_H p_R - \gamma p_R^2\right]$$

which is a constant, hence so is $\lambda_H^e(q_f) + \lambda_U^e(q_f) + \lambda_R^e$. Thus, combining the above analysis for Cases c, d, e, we know that $\lambda_H(q_f) + \lambda_U(q_f) + \lambda_R$ is decreasing in p for $0 \leq p < \bar{p}$. This means that $\hat{\tau}^{\infty}(q_f) < T$ may only occur for small enough p.

Next, we show that $\hat{\tau}^{\infty}(q_f) < T$ never occurs in Case d or e and may only occur in Case c. We will prove that $K_H + K_R \geq [\lambda - \lambda_N(q_f)]T$ at $p = 0$ in Cases d and e. In Case d at $p = 0$, we have

$$K_H + K_R - [\lambda - \lambda_N(q_f)]T = K_H + K_R - \lambda T + \lambda T(\frac{p_R}{u})^2\left[1 + \frac{(p_H - p_R)^2}{\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)}\right]. \tag{A14}$$

The derivative of (A14) with respect to K_H is

$$1 - \frac{p_R^2(p_H-p_R)^2}{\left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right]^2}$$

which is negative for $\lambda_H^a T \leq K_H < (\lambda T/u^2)[u^2 - 2(p_H - p_R)u + p_H^2 - p_H p_R - p_R^2]$ and positive for $K_H > (\lambda T/u^2)[u^2 - 2(p_H - p_R)u + p_H^2 - p_H p_R - p_R^2]$. Thus, by taking $K_H = (\lambda T/u^2)[u^2 - 2(p_H - p_R)u + p_H^2 - p_H p_R - p_R^2]$, we obtain

$$(A14) \geq K_R - \frac{\lambda T}{u^2}(p_H - p_R)(2u - p_H - p_R) = K_R - \lambda_R^a T \geq 0,$$

hence $K_H + K_R \geq [\lambda - \lambda_N^d(q_f)]T$ at $p = 0$ in Case d. In Case e, note that $\lambda_N^d(q_f) = \lambda_N^e(q_f)$ at $p = \tilde{p}$. Since $\lambda_N^e(q_f)$ stays constant in p and $\lambda_N^d(q_f)$ is increasing in p, $\lambda_N^e(q_f)$ at $p = 0$ is larger than $\lambda_N^d(q_f)$ at $p = 0$. Thus, our analysis for Case d implies that $K_H + K_R > [\lambda - \lambda_N^e(q_f)]T$ at $p = 0$ in Case e as well.

So far, we have known that $\hat{\tau}^\infty(q_f) < T$ may only occur in Case c for small enough p. Now we derive q_f in this case. (1) becomes

$$q = \frac{K_H - \lambda_H^c \frac{K_H + K_R}{\lambda_H^c + \lambda_U^c(q) + \lambda_R^c}}{\lambda_U^c(q) \frac{K_H + K_R}{\lambda_H^c + \lambda_U^c(q) + \lambda_R^c}},$$

and can be simplified to

$$\gamma(u-p)^2 q^2 + \beta q + \gamma k p_R^2 = 0, \tag{A15}$$

where $k = K_H/(K_H + K_R)$. (A15) is a quadratic equation. Now we show that the smaller root $q = [-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}]/[2\gamma(u-p)^2]$ is infeasible because it yields $(p_H - p_R - qp)/(1-q) < u$ which contradicts the condition for Case c to occur. With $q = [-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}]/[2\gamma(u-p)^2]$, we can simplify $(p_H - p_R - qp)/(1-q) < u$ to $\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2} > -\beta - 2\gamma(u-p)(u - p_H + p_R)$. If $-\beta - 2\gamma(u-p)(u-p_H+p_R) < 0$, $\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2} > -\beta - 2\gamma(u-p)(u-p_H+p_R)$ is trivially satisfied. If $-\beta - 2\gamma(u-p)(u-p_H+p_R) \geq 0$, by taking square on both sides of $\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2} > -\beta - 2\gamma(u-p)(u-p_H+p_R)$ and rearranging terms, we obtain $-\beta(u-p_H+p_R) - \gamma(u-p_H+p_R)^2(u-p) - \gamma k p_R^2 (u-p) \geq 0$. Since $-\beta \geq 2\gamma(u-p)(u-p_H+p_R)$, we have $-\beta(u-p_H+p_R) - \gamma(u-p_H+p_R)^2(u-p) - \gamma k p_R^2(u-p) \geq \gamma(u-p)[(u-p_H+p_R)^2 - k p_R^2] > 0$. So, $q = [-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}]/[2\gamma(u-p)^2]$ always leads to $(p_H - p_R - qp)/(1-q) < u$. Thus, in Case c when $\hat{\tau}^\infty(q_f) < T$, the equilibrium q_f is given by the larger root of (A15):

$$q_f = \frac{-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}}{2\gamma(u-p)^2} = \frac{2\gamma k p_R^2}{-\beta - \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}}. \tag{A16}$$

To show that q_f is increasing in p, we need to show that $-\beta - \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}$ is decreasing in p. The derivative of $-\beta - \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}$ with respect to p is

$$\frac{-2\gamma k p_R}{\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}} \left[\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2} + \beta + 2\gamma p_R (u-p) \right]. \tag{A17}$$

If $\beta + 2\gamma p_R(u-p) \geq 0$, (A17)$\leq 0$ trivially. If $\beta + 2\gamma p_R(u-p) < 0$, we have

$$\begin{aligned}
(A17) &\leq \frac{-2\gamma k p_R}{\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}} \left[\sqrt{\beta^2 - 4\gamma^2 p_R^2 (u-p)^2} + \beta + 2\gamma p_R (u-p) \right] \\
&= \frac{-2\gamma k p_R}{\sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}} \sqrt{-\beta - 2\gamma p_R (u-p)} \left[\sqrt{-\beta + 2\gamma p_R(u-p)} - \sqrt{-\beta - 2\gamma p_R (u-p)} \right] \\
&\leq 0.
\end{aligned}$$

Thus, q_f is increasing in p in Case c when $\hat{\tau}^\infty(q_f) < T$.

Finally, we characterize the threshold p in Case c where $\hat{\tau}^\infty(q_f) \geq T$ switches to $\hat{\tau}^\infty(q_f) < T$. When $\hat{\tau}^\infty(q_f) \geq T$, $K_H + K_R = [\lambda_H^c + \lambda_U^c(q_f) + \lambda_R^c]T$ can be simplified to

$$\begin{aligned}
&\gamma p_R^2 p^2 + 2\gamma p_R \left[\frac{K_H}{\lambda T} u^2 - u^2 + 2(p_H - p_R)u - p_H^2 + p_H p_R + p_R^2 \right] p \\
&+ \left[\frac{K_H + K_R}{\lambda T} u^2 - u^2 + 2(1-\gamma) p_H p_R + \gamma p_R^2 \right] \left[\frac{K_H}{\lambda T} u^2 - (u - p_H + 2p_R)(u - p_H) \right] + \gamma p_R^2 (p_H - p_R)^2 = 0.
\end{aligned} \tag{A18}$$

(A18) is a quadratic equation whose smaller root is always negative and larger root (which may also be negative) is $\underline{p}'$. Therefore, by combining all our analysis above, we conclude the following. If $\underline{p}^+ \geq \underline{p}'^+$, we always have $\hat{\tau}^\infty(q_f) \geq T$ for $0 \leq p < \bar{p}$; Case c occurs for $\underline{p}^+ \leq p < \bar{p}$ where q_f is given by (A12), and Case d or e occurs for $0 \le p < \underline{p}^+$ where q_f is given by (A13). If $\underline{p}^+ < \underline{p}'^+$, Case c always occurs for $0 \le p < \bar{p}$ and q_f is given by (A12) for $\underline{p}'^+ \le p < \bar{p}$ and (A16) for $0 \le p < \underline{p}'^+$. Finally, all previous analysis indicates that q_f is the unique solution to (1).

(iii) In the proof of Part (ii), we have shown that q_f is increasing in p in all cases. □

Proof of Theorem 4 If $K_H \ge (\lambda T/u^2)[(u-p_H+2p_R)(u-p_H)+\gamma(p_H-p_R)(2u-p_H+p_R)]$, we have $q_f=1$ for all $0 \le p \le p_H - p_R$, and as found in the proof of Theorem 3(i), $\tau^\infty(1)=T$. Thus, the revenue function is $\Pi_f(p) = p_R\gamma\lambda(\xi_R^b+\xi_U^b)T + p\gamma\lambda\xi_U^b T + (1-\gamma)\Pi_{N,f} = (\gamma\lambda T/u^2)\left[p^3-2up^2+(u^2-3p_R^2)p+u^2p_R-p_R^3\right]+(1-\gamma)\Pi_{N,f}$. The first-order condition is $3p^2-4up+u^2-3p_R^2=0$. The larger root of this quadratic equation is $(2u+\sqrt{u^2+9p_R^2})/3$ which is larger than u; the smaller root is p_{foc}^b. Thus, $\Pi_f(p)$ is increasing in p for $p<p_{foc}^b$ and decreasing in p for $p>p_{foc}^b$; the optimal upgrade price is $\min\left\{(p_{foc}^b)^+, p_H-p_R\right\}$.

Next, consider the case of $K_H < (\lambda T/u^2)[(u-p_H+2p_R)(u-p_H)+\gamma(p_H-p_R)(2u-p_H+p_R)]$. For $\bar{p} \le p \le p_H - p_R$, we have $q_f=1$, hence our proof above indicates that the local optimum is $\min\left\{\max\left\{(p_{foc}^b)^+, \bar{p}\right\}, p_H-p_R\right\}$. For $0 \le p < \bar{p}$, we have $0<q_f<1$ and

$$\Pi_f(p) = \left\{p_R[\lambda_U^i(q_f)+\lambda_R^i] + p\lambda_U^i(q_f)q_f + p_H\lambda_H^i(q_f)\right\}\min\left\{\frac{K_H+K_R}{\lambda_H^i(q_f)+\lambda_U^i(q_f)+\lambda_R^i}, T\right\},$$

where $i=c,d,e$ as we may be in Case c, d, or e. We will show that $\Pi_f(p)$ is increasing in p for $0 \le p < \bar{p}$, thus the global optimal upgrade price is also $p^* = \min\left\{\max\left\{(p_{foc}^b)^+, \bar{p}\right\}, p_H-p_R\right\}$.

First, consider Case c. If $(K_H+K_R)/[\lambda_H^c+\lambda_U^c(q_f)+\lambda_R^c] \ge T$, the revenue function becomes

$$\begin{aligned}\Pi_f(p) &= p_R\gamma\lambda[\xi_U^c(q_f)+\xi_R^c]T + p[K_H-(1-\gamma)\lambda\xi_H^a T] + (1-\gamma)\Pi_{N,f} \\ &= \frac{\frac{\gamma\lambda T}{u^2}}{\frac{\tilde{K}_H}{\gamma\lambda T}u^2+p_R^2}\left\{-p_R^3p^2+\left[(\frac{\tilde{K}_H}{\gamma\lambda T})^2u^4-\frac{\tilde{K}_H}{\gamma\lambda T}p_R^2u^2+2p_R^3u-2p_R^4\right]p+\frac{\tilde{K}_H}{\gamma\lambda T}p_Ru^4\right\}+(1-\gamma)\Pi_{N,f},\end{aligned}$$

where $\tilde{K}_H = K_H-(1-\gamma)\lambda\xi_H^a T$. So, $\Pi_f(p)$ is concave in p. Solving the first-order condition

$$\frac{\mathrm{d}\Pi_f}{\mathrm{d}p} = \frac{\frac{\gamma\lambda T}{u^2}}{\frac{\tilde{K}_H}{\gamma\lambda T}u^2+p_R^2}\left[-2p_R^3p+(\frac{\tilde{K}_H}{\gamma\lambda T})^2u^4-\frac{\tilde{K}_H}{\gamma\lambda T}p_R^2u^2+2p_R^3u-2p_R^4\right]=0$$

yields

$$p = \frac{1}{2}(\frac{\tilde{K}_H}{\gamma\lambda T})^2\frac{u^4}{p_R^3} - \frac{1}{2}\frac{\tilde{K}_H}{\gamma\lambda T}\frac{u^2}{p_R} + u - p_R \stackrel{\text{def}}{=\!=} p_{foc}^c.$$

$\Pi_f(p)$ is increasing in p for $p<p_{foc}^c$ and decreasing in p for $p>p_{foc}^c$. Next, we show $p_{foc}^c>\bar{p}$ so that $\Pi_f(p)$ is always increasing in p in Case c. $p_{foc}^c>\bar{p}$ is equivalent to

$$\sqrt{\frac{\tilde{K}_H}{\gamma\lambda T}u^2+p_R^2} > -\frac{1}{2}(\frac{\tilde{K}_H}{\gamma\lambda T})^2\frac{u^4}{p_R^3}+\frac{1}{2}\frac{\tilde{K}_H}{\gamma\lambda T}\frac{u^2}{p_R}+p_R. \tag{A19}$$

If $\tilde{K}_H/(\gamma\lambda T) > 2(p_R/u)^2$, the RHS of (A19) is negative so (A19) holds. If $\tilde{K}_H/(\gamma\lambda T) \le 2(p_R/u)^2$, after taking square on both sides, (A19) can be simplified to $\tilde{K}_H/(\gamma\lambda T) < 3(p_R/u)^2$. Thus, (A19) always holds.

If $(K_H+K_R)/[\lambda_H^c+\lambda_U^c(q_f)+\lambda_R^c] < T$ in Case c, the revenue function becomes

$$\begin{aligned}\Pi_f(p) &= p_R\gamma\lambda[\xi_U^c(q_f)+\xi_R^c]\tau^\infty(q_f) + p\gamma\lambda_U^c(q_f)\tau^\infty(q_f)q_f + p_H(1-\gamma)\lambda\xi_H^a\tau^\infty(q_f) + p_R(1-\gamma)\lambda\xi_R^a\tau^\infty(q_f) \\ &= p_R(K_H+K_R) + pK_H + (1-\gamma)(p_H-p_R-p)\lambda\xi_H^a\tau^\infty(q_f),\end{aligned}$$

where the last equality follows from using $\tau^{\infty}(q_f) = (K_H + K_R)/\{\gamma\lambda[\xi_U^c(q_f) + \xi_R^c] + (1-\gamma)\lambda(\xi_H^a + \xi_R^a)\}$ and $q_f = [K_H - (1-\gamma)\lambda\xi_H^a\tau^{\infty}(q_f)]/[\gamma\lambda\xi_U^c(q_f)\tau^{\infty}(q_f)]$. The derivative of $\Pi_f(p)$ is

$$\frac{\mathrm{d}\Pi_f}{\mathrm{d}p} = K_H - (1-\gamma)\lambda\xi_H^a\tau^{\infty}(q_f) - \frac{(1-\gamma)\gamma(p_H - p_R - p)\lambda\xi_H^a\tau^{\infty}(q_f)\frac{\mathrm{d}[\xi_U^c(q_f)+\xi_R^c]}{\mathrm{d}p}}{\gamma[\xi_U^c(q_f) + \xi_R^c] + (1-\gamma)(\xi_H^a + \xi_R^a)}.$$

Now we show that $\xi_U^c(q_f) + \xi_R^c$ is decreasing in p and hence

$$\frac{\mathrm{d}\Pi_f}{\mathrm{d}p} \geq K_H - (1-\gamma)\lambda\xi_H^a\tau^{\infty}(q_f) \geq K_H - \lambda\xi_H^a T \geq 0.$$

Using the q_f in (A16), we obtain

$$\xi_U^c(q_f) + \xi_R^c = \frac{1}{u^2}\left[u^2 - 2p_R p + \frac{\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2 (u-p)^2}}{2\gamma k}\right],$$

where $k = K_H/(K_H + K_R)$. Then,

$$\frac{\mathrm{d}[\xi_U^c(q_f) + \xi_R^c]}{\mathrm{d}p} = \frac{p_R}{u^2}\left[\frac{\beta + 2\gamma p_R(u-p)}{\sqrt{\beta^2 - 4\gamma^2 k p_R^2(u-p)^2}} - 1\right],$$

and $\mathrm{d}[\xi_U^c(q_f) + \xi_R^c]/\,\mathrm{d}p \leq 0$ can be simplified to $-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2(u-p)^2} \geq 2\gamma p_R(u-p)$. Note that the feasibility condition for Case c, $(p_H - p_R - q_f p)/(1-q_f) \geq u$, can be simplified to $-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2(u-p)^2} \geq 2\gamma(u-p)(u - p_H + p_R)$. Since $2\gamma(u-p)(u-p_H+p_R) > 2\gamma p_R(u-p)$, $-\beta + \sqrt{\beta^2 - 4\gamma^2 k p_R^2(u-p)^2} \geq 2\gamma p_R(u-p)$ is true in Case c. Thus, $\Pi_f(p)$ is increasing in p.

We have shown that $\Pi_f(p)$ is increasing in Case c. Second, consider Case d. As proved in Theorem 3(ii), we must have $(K_H + K_R)/[\lambda_H^i(q_f) + \lambda_U^i(q_f) + \lambda_R^i] \geq T$ for $i = d, e$. The revenue function in Case d is

$$\begin{aligned}
\Pi_f(p) &= p_R\gamma\lambda[\xi_U^d(q_f) + \xi_R^d]T + p[K_H - \gamma\lambda\xi_H^d(q_f)T - (1-\gamma)\lambda\xi_H^a T] + p_H\gamma\lambda\xi_H^d(q_f)T + (1-\gamma)\Pi_{N,f} \\
&= \gamma\lambda T p_R + (\tilde{K}_H - \frac{2\gamma\lambda T p_R^2}{u^2})p + \frac{\gamma\lambda T}{u^2}\left\{\frac{\left[-(u - p_H + p_R)p + (1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 - (p_H - p_R)u - p_R^2\right]^2}{p_H - p_R - p}\right. \\
&\quad \left. - \frac{p_R^3\left[p^2 - 2(p_H - p_R)p - (1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H - p_R)u + p_R^2\right]}{-(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H - p_R)u - p_H^2 + 2p_H p_R}\right\} + (1-\gamma)\Pi_{N,f}.
\end{aligned}$$

Taking derivatives yields

$$\begin{aligned}
\frac{\mathrm{d}\Pi_f}{\mathrm{d}p} &= \tilde{K}_H - \frac{2\gamma\lambda T p_R^2}{u^2} + \frac{\gamma\lambda T}{u^2}\left\{-(u - p_H + p_R)^2 + \frac{\left[(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 - 2(p_H - p_R)u + p_H^2 - 2p_H p_R\right]^2}{(p_H - p_R - p)^2}\right. \\
&\quad \left. + \frac{2p_R^3(p_H - p_R - p)}{-(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H - p_R)u - p_H^2 + 2p_H p_R}\right\}, \\
\frac{\mathrm{d}^2\Pi_f}{\mathrm{d}p^2} &= \frac{2\gamma\lambda T}{u^2}\left\{\frac{\left[(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 - 2(p_H - p_R)u + p_H^2 - 2p_H p_R\right]^2}{(p_H - p_R - p)^3} - \frac{p_R^3}{-(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H - p_R)u - p_H^2 + 2p_H p_R}\right\}.
\end{aligned}$$

Since the second-order derivative is increasing in p and is equal to zero at $p = \tilde{p}$, $\mathrm{d}^2\Pi_f/\,\mathrm{d}p^2 \geq 0$ and $\Pi_f(p)$ is convex in p in Case d. Moreover, we have

$$\left.\frac{\mathrm{d}\Pi_f}{\mathrm{d}p}\right|_{p=\tilde{p}} = \frac{\gamma\lambda T}{u^2}\left[-(1 - \frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H - p_R)u - p_H^2 + 2p_H p_R\right] = K_H - \lambda_H^a T \geq 0.$$

Thus, $\Pi_f(p)$ is increasing in p in Case d.

Third, consider Case e. The revenue function in Case e is

$$\begin{aligned}
\Pi_f(p) &= p_R\gamma\lambda[\xi_U^e(q_f)+\xi_R^e]T + p[K_H - \gamma\lambda\xi_H^e(q_f)T - (1-\gamma)\lambda\xi_H^a T] + p_H\gamma\lambda\xi_H^e(q_f)T + (1-\gamma)\Pi_{N,f} \\
&= \frac{\gamma\lambda T}{u^2}\left\{\left[-(1-\frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H-p_R)u - p_H^2 + 2p_Hp_R\right]p\right. \\
&\qquad \left. + p_H(u^2-p_H^2) - (2u-2p_H-p_R)\left[-(1-\frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H-p_R)u + p_R^2\right]\right\} + (1-\gamma)\Pi_{N,f}.
\end{aligned}$$

Since

$$\frac{\mathrm{d}\Pi_f}{\mathrm{d}p} = \frac{\gamma\lambda T}{u^2}\left[-(1-\frac{\tilde{K}_H}{\gamma\lambda T})u^2 + 2(p_H-p_R)u - p_H^2 + 2p_Hp_R\right] \geq 0,$$

$\Pi_f(p)$ is increasing in p in Case e.

Therefore, we conclude that if $K_H < (\lambda T/u^2)[(1-\gamma)(u-p_H+2p_R)(u-p_H)+\gamma(u^2-p_R^2)]$, the optimal upgrade price is $p_f^* = \min\left\{\max\left\{(p_{foc}^b)^+, \bar{p}\right\}, p_H - p_R\right\}$ which induces $q_f = 1$. Finally, note that $\bar{p} \leq 0$ if $K_H \geq (\lambda T/u^2)[(1-\gamma)(u-p_H+2p_R)(u-p_H)+\gamma(u^2-p_R^2)]$, so we can always write the optimal upgrade price as $p_f^* = \min\left\{\max\left\{(p_{foc}^b)^+, \bar{p}\right\}, p_H - p_R\right\}$. □

Proof of Theorem 5 Offering upgrades increases the revenue if $p_f^* < p_H - p_R$ and decreases the revenue if $p_f^* = p_H - p_R$. $p_f^* = p_H - p_R$ if and only if $\max\left\{p_{foc}^b, \bar{p}\right\} \geq p_H - p_R$. Since $K_H \geq \lambda_H^a T$, we have $\bar{p} \leq p_H - p_R$. Thus, $p_f^* = p_H - p_R$ if and only if $p_{foc}^b \geq p_H - p_R$ or equivalently, $p_H \leq \left(2u + 3p_R - \sqrt{u^2 + 9p_R^2}\right)/3$. □

Proof of Theorem 6 First, consider the monotonicity of p_f^* in γ. Since p_{foc}^b and $p_H - p_R$ are independent of γ, we only need to show that $\bar{p}$ is increasing in γ. This is true because $K_H/(\lambda T)\cdot u^2 - (u-p_H+2p_R)(u-p_H) = u^2/(\lambda T)\cdot(K_H - \lambda_H^a T) \geq 0$.

Second, consider the monotonicity of $\Pi_f(p_f^*)$ in γ. When $p_f^* = p_{foc}^b$, using the revenue function in Case b, the Envelope Theorem yields

$$\frac{\mathrm{d}\Pi_f(p_f^*)}{\mathrm{d}\gamma} = \frac{\lambda T}{u^2}\left[3(p_{foc}^b)^2 - 4up_{foc}^b + u^2 - 3p_R^2\right] - \Pi_{N,f} > 0,$$

because $p_f^* = p_{foc}^b$ (so $p_f^* \neq p_H - p_R$) implies $\Pi_f(p_{foc}^b) > \Pi_{N,f}$ which is equivalent to $(\lambda T/u^2)[3(p_{foc}^b)^2 - 4up_{foc}^b + u^2 - 3p_R^2] > \Pi_{N,f}$. Next, when $p_f^* = \bar{p}$, $\mathrm{d}\Pi_f(p_f^*)/\mathrm{d}\gamma \geq 0$ can be simplified to

$$\begin{aligned}
&2\gamma^2\sqrt{\frac{1}{\gamma}\left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right] + (u-p_H+p_R)^2} \\
&\quad \cdot(u-p_H+p_R)(u^2 - 2p_Hu + 2p_Ru + p_H^2 - 2p_Hp_R - 2p_R^2) \geq a_1\gamma^2 + b_1\gamma + c_1,
\end{aligned} \tag{A20}$$

where

$$\begin{aligned}
a_1 &= 2(u-p_H+p_R)^2(u^2 - 2p_Hu + 2p_Ru + p_H^2 - 2p_Hp_R - 2p_R^2), \\
b_1 &= \left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right](u^2 - 2p_Hu + 2p_Ru + p_H^2 - 2p_Hp_R - 2p_R^2), \\
c_1 &= -\left[\frac{K_H}{\lambda T}u^2 - (u-p_H+2p_R)(u-p_H)\right]^2.
\end{aligned}$$

We show (A20) indeed holds as follows. If $u^2 - 2p_Hu + 2p_Ru + p_H^2 - 2p_Hp_R - 2p_R^2 \leq 0$, we have $a_1 \leq 0$, $b_1 \leq 0$, $c_1 \leq 0$, and hence the RHS of (A20) ≤ 0. Since the left-hand side (LHS) of (A20) ≥ 0, (A20) holds. If

$u^2 - 2p_H u + 2p_R u + p_H^2 - 2p_H p_R - 2p_R^2 > 0$, RHS can be both positive and negative. If RHS≤ 0, again (A20) holds. If RHS> 0, by taking square on both sides and rearranging terms, (A20) is equivalent to

$$\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right]^2 (a_2\gamma^2 + b_2\gamma + c_2) \geq 0, \tag{A21}$$

where

$$\begin{aligned}
a_2 &= 3(u^2 - 2p_H u + 2p_R u + p_H^2 - 2p_H p_R + 2p_R^2)(u^2 - 2p_H u + 2p_R u + p_H^2 - 2p_H p_R - 2p_R^2), \\
b_2 &= 2\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right](u^2 - 2p_H u + 2p_R u + p_H^2 - 2p_H p_R - 2p_R^2), \\
c_2 &= -\left[\frac{K_H}{\lambda T}u^2 - (u - p_H + 2p_R)(u - p_H)\right]^2.
\end{aligned}$$

It is easy to see that $a_2 > a_1$, $b_2 \geq b_1$, $c_2 = c_1$. Thus, RHS> 0 implies that (A21) is satisfied. We have proved that $\Pi_f(p_f^*)$ is increasing in γ when $p_f^* = \bar{p}$. Finally, when $p_f^* = p_H - p_R$ or $p_f^* = 0$, $\Pi_f(p_f^*)$ is constant in γ. Therefore, we conclude that $\Pi_f(p_f^*)$ is increasing in γ overall. □

Proof of Theorem 7 If $K_H < (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$, we have $\bar{p} > 0$, hence $p_f^* > 0$. If $K_H \geq (\lambda T/u^2)[(u - p_H + 2p_R)(u - p_H) + \gamma(p_H - p_R)(2u - p_H + p_R)]$, we have $\bar{p} \leq 0$, hence $p_f^* = 0$ if and only if $p_{foc}^b \leq 0$ which is simplified to $u \leq \sqrt{3}p_R$. □

Proof of Theorem 8 We only prove Part (i) of the theorem; Part (ii) follows from Part (i). First, consider the case of $v_H - p_{H,s}^* \geq v_R - p_R$, so that when the firm does not offer upgrades but can optimize the high-quality product price, consumers book high-quality products. Suppose the firm offers upgrades at price $p = p_{H,s}^* - p_R$. Since $v_H - p_{H,s}^* \geq v_R - p_R$, consumers' utility from booking regular products and accepting upgrade offers, $q^*(t)[v_H - p_R - (p_{H,s}^* - p_R)] + [1 - q^*(t)](v_R - p_R)$, dominates their utility from booking regular products without upgrades, $v_R - p_R$. If $v_H - p_H < v_R - p_R$, consumers book regular products and accept upgrade offers. In this case, we have

$$\begin{aligned}
\Pi_s^* &\geq \Pi_s(p_{H,s}^* - p_R) \\
&= \sum_{i=0}^{\infty} \mathbb{P}(N(T) = i)\left\{[p_R + (p_{H,s}^* - p_R)]\min\{i, K_H\} + p_R \min\{(i - K_H)^+, K_R\}\right\} \\
&> \sum_{i=0}^{\infty} \mathbb{P}(N(T) = i) p_{H,s}^* \min\{i, K_H\} \\
&= \Pi_{N,s}(p_{H,s}^*).
\end{aligned}$$

If $v_H - p_H \geq v_R - p_R$, consumers may book high-quality products, or book regular products and accept upgrade offers. Since $q^*(t)$ is increasing in t (Theorem 1), there exists a threshold arrival time $\bar{t}$ such that consumers arriving before time $\bar{t}$ book high-quality products, and consumers arriving after time $\bar{t}$ book regular products and accept upgrade offers. In this case, we have

$$\begin{aligned}
\Pi_s^* &\geq \Pi_s(p_{H,s}^* - p_R) \\
&= \sum_{i_H=0}^{K_H-1} \mathbb{P}(N(\bar{t}) = i_H)\left\{p_H i_H + \sum_{i_U=0}^{\infty} \mathbb{P}(N(T - \bar{t}) = i_U)\left[p_{H,s}^* \min\{i_U, K_H - i_H\} + p_R \min\{i_U - (K_H - i_H), K_R\}\right]\right\} \\
&\quad + \sum_{i_H=K_H}^{\infty} \mathbb{P}(N(\bar{t}) = i_H) p_H K_H
\end{aligned}$$

$$
\begin{aligned}
&> \sum_{i_H=0}^{K_H-1} \mathbb{P}(N(\bar{t})=i_H)\left\{p^*_{H,s}i_H+\sum_{i_U=0}^{\infty}\mathbb{P}(N(T-\bar{t})=i_U)\left[p^*_{H,s}\min\{i_U,K_H-i_H\}+p_R\min\{i_U-(K_H-i_H),K_R\}\right]\right\}\\
&\quad+\sum_{i_H=K_H}^{\infty}\mathbb{P}(N(\bar{t})=i_H)p^*_{H,s}K_H\\
&> \sum_{i_H=0}^{K_H-1}\left[\mathbb{P}(N(\bar{t})=i_H)p^*_{H,s}i_H+\sum_{i_U=0}^{\infty}\mathbb{P}(N(T-\bar{t})=i_U)p^*_{H,s}\min\{i_U,K_H-i_H\}\right]+\sum_{i_H=K_H}^{\infty}\mathbb{P}(N(\bar{t})=i_H)p^*_{H,s}K_H\\
&= \sum_{i_H=0}^{K_H-1}\sum_{i_U=0}^{\infty}\mathbb{P}(N(\bar{t})=i_H)\,\mathbb{P}(N(T-\bar{t})=i_U)p^*_{H,s}\min\{i_H+i_U,K_H\}+\sum_{i_H=K_H}^{\infty}\mathbb{P}(N(\bar{t})=i_H)p^*_{H,s}K_H\\
&= \sum_{i_H=0}^{\infty}\sum_{i_U=0}^{\infty}\mathbb{P}(N(\bar{t})=i_H)\,\mathbb{P}(N(T-\bar{t})=i_U)p^*_{H,s}\min\{i_H+i_U,K_H\}\\
&= \sum_{i=0}^{\infty}\mathbb{P}(N(T)=i)p^*_{H,s}\min\{i,K_H\}\\
&= \Pi_{N,s}(p^*_{H,s}).
\end{aligned}
$$

Thus, if $v_H-p^*_{H,s}\geq v_R-p_R$, we have $\Pi^*_s>\Pi_{N,s}(p^*_{H,s})$.

Second, consider the case of $v_H-p^*_{H,s}<v_R-p_R$, so that when the firm does not offer upgrades but can optimize the high-quality product price, consumers book regular products. Suppose the firm offers upgrades at price $p=p^*_{H,s}-p_R$. Since $v_R-p_R>v_H-p^*_{H,s}$, consumers' utility from booking regular products without upgrades, v_R-p_R, dominates their utility from booking high-quality products, v_H-p_H, as well as their utility from booking regular products and accepting upgrade offers, $q^*(t)[v_H-p_R-(p^*_{H,s}-p_R)]+[1-q^*(t)](v_R-p_R)$. Then, we have $\Pi^*_s\geq\Pi_s(p^*_{H,s}-p_R)=\Pi_{N,s}(p^*_{H,s})$. $\square$

Appendix B: Revenue Function in the Stochastic Model

In this section, we derive the stochastic revenue function. To differentiate the demand processes from the ones used in Section 4 where we derive the consumer booking equilibrium (which are the number of other consumers as seen by the acting consumer), we use $N^*_i(t)$, $i=H,U,R$, instead of the previous $N_i(t|q^*(\cdot))$ to denote the demand processes for the firm when the consumer booking equilibrium is $q^*(\cdot)$. Since now we are analyzing from the firm's perspective, the environmental equivalence property does not apply, hence the "+1" term in the stopping times does not exist. Again, to represent this difference, we use τ^*_H, τ^*_R, τ^*_T, $\hat{\tau}^*$, and τ^* to denote the stopping times. Moreover, denote $N'_H(t)$ as the demand process for high-quality products after regular products are fully booked ($N'_H(t)$ is a Poisson process with rate λ^a_H). Similarly, denote $N'_R(t)$ as the demand process for regular products after high-quality products are fully booked ($N'_R(t)$ is a Poisson process with rate λ^a_R). The expected revenue in the stochastic model, $\Pi_s(p)$, is as follows:

$$
\begin{aligned}
\Pi_s(p) = \mathop{\mathbb{E}}_{N^*_H(t),N^*_U(t),N^*_R(t)}\Big\{&p_R\left[N^*_R(\tau^*)+N^*_U(\tau^*)\right]+p\min\left\{N^*_U(\tau^*),K_H-N^*_H(\tau^*)\right\}+p_HN^*_H(\tau^*)\\
&+\mathbb{1}\{\tau^*=\tau^*_R\}p_H\mathop{\mathbb{E}}_{N'_H(T-\tau^*)}\left[\min\left\{N'_H(T-\tau^*),K_H-N^*_H(\tau^*)-N^*_U(\tau^*)\right\}\right]\\
&+\mathbb{1}\{\tau^*=\tau^*_H\}p_R\mathop{\mathbb{E}}_{N'_R(T-\tau^*)}\left[\min\left\{N'_R(T-\tau^*),K_R-N^*_U(\tau^*)-N^*_R(\tau^*)\right\}\right]\Big\}.
\end{aligned}
$$

Now we further expand the above revenue function. $\Pi_s(p)$ can be written as $\Pi_s(p) = \Pi_{s1}(p) + \Pi_{s2}(p) + \Pi_{s3}(p) + \Pi_{s4}(p)$, where

$$\begin{aligned}
\Pi_{s1}(p) &= \mathbb{P}(\tau_H^* \le \tau_T^*, \tau_H^* \le T)\Pi_s(p|\tau_H^* \le \tau_T^*, \tau_H^* \le T),\\
\Pi_{s2}(p) &= \mathbb{P}(\tau_R^* \le \tau_T^*, \tau_R^* \le T)\Pi_s(p|\tau_R^* \le \tau_T^*, \tau_R^* \le T),\\
\Pi_{s3}(p) &= \mathbb{P}(\tau_H^* > \tau_T^*, \tau_R^* > \tau_T^*, \tau_T^* \le T)\Pi_s(p|\tau_H^* > \tau_T^*, \tau_R^* > \tau_T^*, \tau_T^* \le T),\\
\Pi_{s4}(p) &= \mathbb{P}(\tau_H^* > T, \tau_R^* > T, \tau_T^* > T)\Pi_s(p|\tau_H^* > T, \tau_R^* > T, \tau_T^* > T).
\end{aligned}$$

Each part of $\Pi_s(p)$ is derived as follows:

$$\begin{aligned}
\Pi_{s1}(p) = &\int_0^T f_{\tau_H^*}(t) \sum_{i_R=0}^{K_R-1} \sum_{i_U=0}^{K_R-i_R} \mathbb{P}(N_R^*(t)=i_R)\,\mathbb{P}(N_U^*(t)=i_U)\\
&\cdot\left\{p_R(i_R+i_U) + p_H K_H + p_R \mathop{\mathbb{E}}_{N_R'(T-t)}\left[\min\{N_R'(T-t), K_R - i_R - i_U\}\right]\right\}\mathrm{d}t,
\end{aligned}$$

where $f_{\tau_H^*}(t) = \mathbb{P}(N_H^*(t) = K_H - 1)\lambda\xi_H^*(t)$.

$$\begin{aligned}
\Pi_{s2}(p) = &\int_0^T f_{\tau_R^*}(t) \sum_{i_H=0}^{K_H-1} \sum_{i_U=0}^{K_H-i_H} \mathbb{P}(N_H^*(t)=i_H)\,\mathbb{P}(N_U^*(t)=i_U)\\
&\cdot\left\{p_R(K_R+i_U) + p i_U + p_H i_H + p_H \mathop{\mathbb{E}}_{N_H'(T-t)}\left[\min\{N_H'(T-t), K_H - i_H - i_U\}\right]\right\}\mathrm{d}t,
\end{aligned}$$

where $f_{\tau_R^*}(t) = \mathbb{P}(N_R^*(t) = K_R - 1)\lambda\xi_R^*(t)$.

$$\begin{aligned}
\Pi_{s3}(p) = &\int_0^T f_{\tau_T^*}(t) \sum_{i_H=0}^{K_H-1} \sum_{i_R=0}^{K_R-1} \binom{K_H+K_R}{i_H}\binom{K_H+K_R-i_H}{i_R}\left\{\mathop{\mathbb{E}}_{0\le s\le t}[\xi_H^*(s)]\right\}^{i_H}\left\{\mathop{\mathbb{E}}_{0\le s\le t}[\xi_R^*(s)]\right\}^{i_R}\\
&\cdot\left\{\mathop{\mathbb{E}}_{0\le s\le t}[\xi_U^*(s)]\right\}^{K_H+K_R-i_H-i_R}\left[p_R(K_H+K_R-i_H) + p(K_H-i_H) + p_H i_H\right]\mathrm{d}t,
\end{aligned}$$

where $f_{\tau_T^*}(t) = \mathbb{P}(N_H^*(t) + N_U^*(t) + N_R^*(t) = K_R + K_H - 1)\lambda\left[\xi_H^*(t) + \xi_U^*(t) + \xi_R^*(t)\right]$.

$$\begin{aligned}
\Pi_{s4}(p) = &\sum_{i_H=0}^{K_H-1} \sum_{i_R=0}^{K_R-1} \sum_{i_U=0}^{K_H+K_R-i_H-i_R-1} \mathbb{P}(N_H^*(T)=i_H)\,\mathbb{P}(N_R^*(T)=i_R)\,\mathbb{P}(N_U^*(T)=i_U)\\
&\cdot\left[p_R(i_R+i_U) + p\min\{i_U, K_H - i_H\} + p_H i_H\right].
\end{aligned}$$

Appendix C: Last-Minute Upgrades

In this section, we introduce the (stochastic) model when the firm offers last-minute upgrades, and derive the consumer booking equilibrium and the firm's optimal revenue. To avoid too much repetition, we keep the description of the model elements that are same as the conditional upgrade model to a minimum, and we focus on explaining notations that are new to or different from the conditional upgrade model.

The firm offers upgrades and announces the upgrade price at the end of the booking period (e.g., during check-in) instead of in advance. Consistent with the conditional upgrade model, the firm offers upgrades to γ proportion of consumers (and consumers know whether they will be offered upgrades or not). Also consistent with the conditional upgrade model, the firm can overbook regular products during the booking period. However, if there are more consumers who do not accept the upgrade offers (i.e., they choose to consume the regular products) than the remaining capacity of regular products by the end of the booking period, the firm incurs a penalty cost c per consumer from "bumping" these consumers. At the end of the booking period, the firm chooses the upgrade price $p \leq p_H - p_R$ based on its belief about the probabilities that the consumers who have booked regular products will accept upgrade offers.

During the booking period, consumers choose which product type to book (high-quality or regular) or not to book any product. When making booking decisions, consumers anticipate the optimal upgrade price that is going to be chosen by the firm at the end of the booking period as well as the corresponding upgrade probability on every sample path of consumers' arrival and booking processes. More specifically, consumers' rational expectations take into account the following: 1) the probability that upgrades will be offered at the end of the booking period (because the firm has unsold high-quality products by then), 2) the probability that the consumer will be willing to accept the upgrade offer (because the upgrade price that the firm charges is low enough), 3) the probability that the consumer will get upgraded if more consumers are willing to accept upgrade offers than the remaining capacity of high-quality products (same as in our conditional upgrade model, we assume random rationing in this case).

Let $a_t(v_R, v_H)$ denote the consumer's utility-maximizing decision if she arrives at time t, has valuations (v_R, v_H) and will be offered an upgrade. $a_t(v_R, v_H) = H$ represents booking a high-quality product, $a_t(v_R, v_H) = R$ represents booking a regular product (the consumer later may or may not accept the upgrade offer), $a_t(v_R, v_H) = N$ represents not booking any product. We now use the fixed-point approach to derive the consumer booking equilibrium. Suppose all other consumers except the acting consumer are using strategy $a_t(v_R, v_H)$. For the acting consumer, given that both product types are still available by her arrival time t, her utility from booking a high-quality product is $v_H - p_H$ which does not depend on other consumers' strategies used in the consumer booking game. Let $u_R(a_t(v_R, v_H))$ denote the acting consumer's expected utility from booking a regular product upon arrival. $u_R(a_t(v_R, v_H))$ incorporates the potential utility gained from being upgraded at the end of the booking period. Let $b(a_t(v_R, v_H))$ denote the resulting optimal strategy for the acting consumer. Then,

$$b(a_t(v_R, v_H)) = \begin{cases} H \text{ if } v_H - p_H \geq u_R(a_t(v_R, v_H)) \text{ and } v_H \geq p_H; \\ R \text{ if } v_H - p_H < u_R(a_t(v_R, v_H)) \text{ and } u_R(a_t(v_R, v_H)) \geq 0; \\ N \text{ otherwise.} \end{cases}$$

The equilibrium condition is that for every t and every (v_R, v_H), we must have $b(a_t(v_R, v_H)) = a_t(v_R, v_H)$. The strategy space has three dimensions, namely, the arrival time dimension, and the two valuation dimensions. Note that different from the conditional upgrade model, we cannot reduce the strategy space to only the arrival time dimension by equivalently defining the anticipated upgrade probability as the strategy used by consumers in the booking game, because with last-minute upgrades, consumers' probabilities to actually get upgraded also depend on their valuations. If $v_H - v_R$ is lower than the upgrade price announced at the end of the booking period, the consumer will not accept the upgrade offer, and hence the upgrade probability is zero; another consumer with $v_H - v_R$ higher than the upgrade price will have a higher upgrade probability in expectation.

Given $a_t(v_R, v_H)$, the probabilities of any other consumer that will be offered an upgrade booking each type of product are as follows:

$$\xi_H^{\gamma}(t|a_t(v_R,v_H)) = \iint\limits_{\Omega} \mathbb{1}\{a_t(v_R,v_H)=H\} f(v_R,v_H)\,\mathrm{d}v_R\,\mathrm{d}v_H,$$

$$\xi_R^{\gamma}(t|a_t(v_R,v_H)) = \iint\limits_{\Omega} \mathbb{1}\{a_t(v_R,v_H)=R\} f(v_R,v_H)\,\mathrm{d}v_R\,\mathrm{d}v_H.$$

The probabilities of any other consumer that will not be offered an upgrade booking each type of product are as follows:

$$\xi'_H(t) = \iint\limits_{\Omega} \mathbb{1}\{a'_t(v_R,v_H)=H\} f(v_R,v_H)\,\mathrm{d}v_R\,\mathrm{d}v_H,$$

$$\xi'_R(t) = \iint\limits_{\Omega} \mathbb{1}\{a'_t(v_R,v_H)=R\} f(v_R,v_H)\,\mathrm{d}v_R\,\mathrm{d}v_H,$$

where $a'_t(v_R, v_H)$ denotes the utility-maximizing decision of a consumer that will not be offered an upgrade:

$$a'_t(v_R,v_H) = \begin{cases} H \text{ if } v_H - v_R \geq p_H - p_R \text{ and } v_H \geq p_H, \\ R \text{ if } v_H - v_R < p_H - p_R \text{ and } v_R \geq p_R, \\ N \text{ otherwise.} \end{cases}$$

The arrival processes of other consumers, $N_H(t|a_t(v_R,v_H))$ and $N_R(t|a_t(v_R,v_H))$, are Poisson processes with rates $\lambda_H(t|a_t(v_R,v_H)) = \lambda\gamma\xi_H^{\gamma}(t|a_t(v_R,v_H)) + \lambda(1-\gamma)\xi'_H(t)$ and $\lambda_R(t|a_t(v_R,v_H)) = \lambda\gamma\xi_R^{\gamma}(t|a_t(v_R,v_H)) + \lambda(1-\gamma)\xi'_R(t)$, respectively. The stopping time of the booking game is $\tau(a_t(v_R,v_H)) = \min\{\hat{\tau}(a_t(v_R,v_H)), T\}$, where $\hat{\tau}(a_t(v_R,v_H)) = \min\{\tau_H(a_t(v_R,v_H)), \tau_T(a_t(v_R,v_H))\}$, and $\tau_H(a_t(v_R,v_H)) = \inf\{t \geq 0 : N_H(t|a_t(v_R,v_H)) \geq K_H\}$, $\tau_T(a_t(v_R,v_H)) = \inf\{t \geq 0 : N_H(t|a_t(v_R,v_H)) + N_R(t|a_t(v_R,v_H)) + 1 \geq K_H + K_R\}$.

$u_R(a_t(v_R,v_H))$ is derived as follows:

$$\begin{aligned} u_R(a_t(v_R,v_H)) = &\mathop{\mathbb{E}}_{N_H(t|a_t(v_R,v_H)),N_R(t|a_t(v_R,v_H))|N_H(t|a_t(v_R,v_H))<K_H, N_H(t|a_t(v_R,v_H))+N_R(t|a_t(v_R,v_H))<K_H+K_R} \\ &\Bigg\{\mathbb{1}\{t \leq \tau(a_t(v_R,v_H))\} \cdot \Bigg\{\mathbb{1}\{\tau(a_t(v_R,v_H)) = \tau_H(a_t(v_R,v_H))\} \cdot (v_R - p_R) + \\ &\mathbb{1}\{\tau(a_t(v_R,v_H)) \neq \tau_H(a_t(v_R,v_H))\} \cdot \Big[(1 - q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)) \cdot (v_R - p_R) + \\ &q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1) \cdot (v_H - v_R - p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1))\Big]\Bigg\}\Bigg\}. \end{aligned}$$

$q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ is the probability that the acting consumer accepts the upgrade offer and gets upgraded on any sample path, $p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ is the optimal upgrade price chosen by the firm at the end of the booking period based on demand realizations on any sample path. The "+1" term represents the acting consumer. Note that consistent with the conditional upgrade model, in the above derivation, the expectation taken over each sample path is conditional expectation (i.e., conditional on that up to the acting consumer's arrival time, both product types are still available).

Next, we derive $q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ and $p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$. We have

$$\begin{aligned} &q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1) = \mathbb{1}\{v_H - v_R \geq p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)\} \cdot \\ &\sum_{i=0}^{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))} \mathbb{P}(i \text{ other consumers accept upgrades}) \cdot \min\left\{\frac{[K_H - N_H(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))]^+}{i+1}, 1\right\}. \end{aligned}$$

We still need to derive $\mathbb{P}(i$ other consumers accept upgrades). Let $\eta_t(p|a_t(v_R,v_H))$ denote the probability that a consumer who arrives at time t and books a regular product will accept the upgrade offer with upgrade price p. We have

$$\eta_t(p|a_t(v_R,v_H)) = \frac{\iint_\Omega \mathbb{1}\{a_t(v_R,v_H)=R\}\mathbb{1}\{v_H - v_R \geq p\} f(v_R,v_H)\,\mathrm{d}v_R\,\mathrm{d}v_H}{\xi_R^\gamma(t|a_t(v_R,v_H))}.$$

Further, define $\eta(p|a_t(v_R,v_H)) = \mathbb{E}_t \eta_t(p|a_t(v_R,v_H))$. We assume that the acting consumer anticipates the other consumers' acceptance of the upgrade offers as a binomial distribution with probability $\eta(p|a_t(v_R,v_H))$.[11] Thus, $\mathbb{P}(i$ other consumers accept upgrades$) = \sum_{j=i}^{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))} \binom{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))}{j}\gamma^j(1-\gamma)^{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))-j}\binom{j}{i}[\eta(p|a_t(v_R,v_H))]^i[1-\eta(p|a_t(v_R,v_H))]^{j-i}$.

$p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ is the maximizer of the net revenue earned at check-in, which is the difference between the revenue from collecting upgrade fees and the cost from bumping consumers due to insufficient regular product capacity. When $\tau(a_t(v_R,v_H)) \neq \tau_H(a_t(v_R,v_H))$, let $\Pi_T(p|N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ denote the firm's expected net revenue from selling upgrades at check-in on any sample path. We have

$$\Pi_T(p|N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1) = \sum_{i=0}^{N_R(t|a_t(v_R,v_H))+1} \mathbb{P}(i \text{ other consumers accept upgrades})\cdot$$
$$\Big\{p\cdot\min\{i,[K_H - N_H(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))]^+\}$$
$$-c\cdot\{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))+1-\min\{i,[K_H-N_H(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))]^+\}-K_R\}^+\Big\}.$$

$\mathbb{P}(i$ other consumers accept upgrades) is calculated using the same approach when we derive $q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$. Note that when $\tau(a_t(v_R,v_H)) = \tau_H(a_t(v_R,v_H))$ (so $N_H(\tau(a_t(v_R,v_H))|a_t(v_R,v_H)) = K_H$), $p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ and $\Pi_T(p|N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)$ are irrelevant, because the firm does not earn any revenue from upgrades (also, the firm does not incur penalty cost, because $\tau(a_t(v_R,v_H)) = \tau_H(a_t(v_R,v_H))$ implies that the firm does not overbook regular products). Moreover, in this case, we naturally have $q(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1) = 0$.

We have characterized the consumer book equilibrium. Then, we can calculate the firm's optimal expected revenue, $\Pi^*_{LM,s}$, as follows:

$$\Pi^*_{LM,s} = \mathbb{E}_{N_H^*(t),N_R^*(t)}\Big\{p_R N_R^*(\tau^*) + p_H N_H^*(\tau^*) + \mathbb{1}\{\tau^* \neq \tau_H^*\}\cdot\Pi_T(p^*(N_H^*(t),N_R^*(t))|N_H^*(t),N_R^*(t))$$
$$+\mathbb{1}\{\tau^*=\tau_H^*\}\cdot R\,\mathbb{E}_{N_R'(T-\tau^*)}\Big[\min\{N_R'(T-\tau^*), K_R - N_R^*(\tau^*)\}\Big]\Big\}.$$

[11] An alternative way is to allow the acting consumer to form a heterogeneous binomial belief about the acceptance of the upgrade offer from each of the other consumers. For any other consumer with arrival time $t_j(j=1,2,...,N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H)))$, the probability of accepting the upgrade offer is $\eta_{t_j}(p^*(N_H(t|a_t(v_R,v_H)), N_R(t|a_t(v_R,v_H))+1)|a_t(v_R,v_H))$, where $t_1,t_2,...,t_{N_R(\tau(a_t(v_R,v_H))|a_t(v_R,v_H))}$ denote the arrival times of other consumers who have booked regular products on any sample path. By using this approach, the computational burden of $\mathbb{P}(i$ other consumers accept upgrades) is significantly larger. The approach we take can be considered as an approximation by assuming that consumers have limited computational capability in the booking game. If the problem size is large enough, the equilibrium booking strategy of consumers becomes time-independent, in which case our approach produces the same result as this alternative approach (the examples we give in the paper have large enough problem sizes so that this occurs).

The revenue function can be expanded conditional on τ^* in the same way as the revenue function from conditional upgrades in Appendix B.