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Rotationally invariant integrals of arbitrary dimensions

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Abstract: In this note integrals over spherical volumes with rotationally invariant densities are computed. Exploiting the rotational invariance, and using identities in the integration over Gaussian functions, the general n-dimensional integral is solved up to a one-dimensional integral over the radial coordinate. The volume of an n-sphere with unit radius is computed analytically in terms of the $\Gamma(z)$ special function, and its scaling properties that depend on the number of dimensions are discussed. The geometric properties of n-cubes with volumes equal to that of their corresponding n-spheres are also derived. In particular, one finds that the length of the side of such an n-cube asymptotes to zero as n increases, whereas the longest straight line that can fit within the cube asymptotes to the constant value $\sqrt{2e\pi} \simeq 4.13$. Finally, integrals over power-law form factors are computed for finite and infinite radial extent.

1 Preliminaries

Our interest is to consider integrals over a volume in n-dimensional space that is rotationally invariant and has a density that is rotationally invariant. The first requirement implies that we integrate over the volume of S_n. S_n is defined[1] to be

$$S_n = \{\vec{r} \in R^n \,|\, r \leq 1\}. \tag{1}$$

where $\vec{r} = (x_1, x_2, \ldots, x_n)$ and $r^2 = x_1^2 + x_2^2 + \ldots + x_n^2$. For integrals over a finite radius one rescales the coordinates such that radius of the sphere integrated over is $r = 1$. The second requirement is that the mass density $\rho(\vec{r})$ that is assumed throughout the volume, i.e., the integrand of the integral, must be rotationally invariant and therefore can only depend on the distance from the origin: $\rho(\vec{r}) \to \rho(r)$. Therefore, the types of integrals under consideration are

$$I_n[\rho(r)] = \int_{S_n} \rho(r) dx_1 dx_2 \cdots dx_n. \tag{2}$$

[1]We use the notation S_n rather than S^n to distinguish from the common definition by topologists that S^n is unit sphere in R^{n+1} such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. The definition S_n used here is equivalent to the unit ball B^n definition by topologists (Munkres 2000).

2 Volumes of S_2 and S_3

If one is not clever, solving integrals of the eq. (2) type is very hard as the number of dimensions increases. Let's illustrate this with the simple example of computing the volume of S_2:

$$V_2 = I_2[1] = \int_{S_2} dx_1 dx_2 \tag{3}$$

where V_n in general denotes the volume of S_n. S_2 is just a filled in circle with $r = 1$ and π is its volume[2].

Solving for V_2 using cartesian coordinate systems is painful, as the derivation below suggests:

$$\begin{aligned} V_2 &= \int_{-1}^{+1} \left(\int_{-\sqrt{1-x_1^2}}^{+\sqrt{1-x_1^2}} dx_2 \right) dx_1 \\ &= \int_{-1}^{+1} \left(2\sqrt{1-x_1^2} \right) dx_1 \\ &= \left[x_1\sqrt{1-x_1^2} + \sin^{-1} x_1 \right]_{-1}^{+1} \\ &= \left(0 + \frac{\pi}{2} \right) - \left(0 - \frac{\pi}{2} \right) = \pi \end{aligned} \tag{4}$$

This approach is unnecessarily difficult, and often prohibitively difficult, as the number of dimensions increases or more complex functions $\rho(r)$ are introduced.

An obviously easier approach is to take advantage of the symmetries involved and use polar coordinates, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{5}$$

where r is of course the radius and θ is the coordinate angle that sweeps over the entire surface. Its range is 0 to 2π. The differential volume element in polar coordinates must be computed. The answer is

$$dx_1 dx_2 = r dr d\theta \tag{6}$$

Our integral then becomes

$$V_2 = \int_{S_2} dx_1 dx_2 \longrightarrow \int_{S_2} r dr d\theta \tag{7}$$

This integral is now separable, meaning the radial part can be integrated separately and independently from the angular sweep variable. The result is trivial to solve:

$$V_2 = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 r dr \right) = 2\pi \cdot \frac{1}{2} = \pi. \tag{8}$$

[2]Note I am using "volume" V_n technically here as the integral over S_n, whereas we would normally and more colloquially say that the filled in circle of S_2 is an "area". The "volume" of S^1 is 2, whereas nontechnically we would say it is a "length." . We shall define the technical term "area" A_n later.

Let's now compute the volume of S_3. To make the computation simpler we again exploit the symmetry and define a coordinate r which is fixed length radius from the origin, and angular coordinates ϕ and θ that sweep over the entire surface of the sphere at fixed r. The standard spherical coordinates used for this purpose are

$$\begin{aligned} x_1 &= r\cos\phi\sin\theta \\ x_2 &= r\sin\phi\sin\theta \\ x_3 &= r\cos\theta \end{aligned} \tag{9}$$

Angular coverage requires $0 \le \phi < 2\pi$ and $0 \le \theta < \pi$.

The volume element is

$$dx_1 dx_2 dx_3 = r^2 dr \sin\theta d\theta d\phi \tag{10}$$

and the volume integral over S_3 becomes

$$V_3 = \int_{S_3} dx_1 dx_2 dx_3 = \left(\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\right)\left(\int_0^1 r^2 dr\right) \tag{11}$$

$$= (2\cdot 2\pi)\cdot\frac{1}{3} = \frac{4}{3}\pi, \tag{12}$$

which is the well-known result of the volume of a sphere of unit radius in three dimensions.

3 Isolating Radial Integral

The key to the integrals over S_2 and over S_3, and indeed over S_n in general, is to utilize a coordinate system that has the radial distance from the origin as one coordinate, which we call r. The function $\rho(r)$ depends only on that coordinate. The remaining coordinates and how they are defined is not so important. The only requirement is that they angularly sweep over the entire surface at fixed radius r. We found θ variable in the S_2 case and ϕ and θ in the S_3 case, but we could have chosen different variables and it would not have materially affected the difficulty of the problem, or changed the symmetry compatibility of the coordinates to the problem.

For this reason it is helpful to define an generalized differential solid angle $d\Omega_{n-1}$, which is defined to be a differential area on the surface of S_n. When integrated it gives the total area of surface of S_n:

$$A_n = \int_{\partial S_n} d\Omega_{n-1} \tag{13}$$

where ∂S_n denotes the full area of the surface of S_n. In the case of S_2 and S_3 we found

$$S_2: \qquad d\Omega_1 = d\theta \qquad A_1 = 2\pi \tag{14}$$

$$S_3: \qquad d\Omega_2 = \sin\theta d\theta d\phi \qquad A_2 = 4\pi \tag{15}$$

In general, any integral of the kind given in eq. 2 can be rewritten as the area of S_n multiplied by radial integral:

$$\begin{aligned} I_n[\rho(r)] &= \int_{S_n} \rho(r) dx_1 dx_2 \cdots dx_n \\ &= \left(\int_{\partial S_n} d\Omega_{n-1} \right) \int_0^1 \rho(r) r^{n-1} dr \\ &= A_n \, R_n[\rho(r)] \end{aligned} \tag{16}$$

where,

$$R_n[\rho(r)] = \int_0^1 \rho(r) r^{n-1} dr \tag{17}$$

If we find a way to tabulate A_n values, then the difficult n-dimensional integrals $I_n[\rho(r)]$ become the much easier to solve one-dimensional integrals $R_n[\rho(r)]$. However, determining A_n can be difficult for high dimensional spaces or fractional n, as is sometimes needed in quantum field theory, for example.

4 Solid Angle Integrals over S_n

One path to determine A_n for arbitrary real value of n is through manipulations of a multi-dimensional gaussian integral. Let us define

$$\begin{aligned} G_n &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(x_1^2 + x_2^2 + \ldots + x_n^2)} dx_1 dx_2 \cdots dx_n \\ &= \int_{-\infty}^{+\infty} e^{-x_1^2} dx_1 \int_{-\infty}^{+\infty} e^{-x_2^2} dx_2 \cdots \int_{-\infty}^{+\infty} e^{-x_n^2} dx_n = (G_1)^n. \end{aligned} \tag{18}$$

where

$$G_1 = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}, \quad \text{therefore} \quad G_n = \pi^{\frac{n}{2}}. \tag{19}$$

However, G_n can be written also in generalize spherical coordinates since the integrand is spherically symmetric and the integral is over a sphere with radius at infinity.

$$G_n = \left(\int_{\partial S_n} d\Omega_{n-1} \right) \int_0^\infty e^{-r^2} r^{n-1} dr = A_n \int_0^\infty e^{-r^2} r^{n-1} dr. \tag{20}$$

The radial integral has a form similar to the Gamma function, defined as (Arfken 2001)

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}, \quad Re(z) > 0. \tag{21}$$

The two most important properties of the Γ function that we will have occasion to use later are

$$z\Gamma(z) = \Gamma(z+1) \quad \text{and} \quad \Gamma(z+1) = z!, \tag{22}$$

where $z!$ is the normal factorial function $z! = 1 \cdot 2 \cdot 3 \cdots z$ when z is an integer, yet it is still defined when z is non-integer by the $\Gamma(z+1)$ function.

Defining $t = r^2$ enables us to recast the radial integral result of eq. (20) in terms of the Gamma function

$$\begin{aligned} \int_0^\infty e^{-r^2} r^{n-1} dr &= \frac{1}{2} \int_0^\infty e^{-t} t^{\frac{n}{2}-1} dt \\ &= \frac{1}{2} \Gamma\left(\frac{n}{2}\right) \end{aligned} \tag{23}$$

Combining eqs. (19), (20) and (23) one can solve for A_n:

$$A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{24}$$

This is the area of the surface of S_n, which is equivalent to the integral over the solid angle factor $d\Omega_{n-1}$. We are now in position to use this result to turn $I_n[\rho(r)]$ from an n-dimensional integral to a much easier one dimensional integral over r.

5 Summary of Result

Let us now go back to the original integral of eq. (2). In summary, we have learned we can now rewrite this integral as

$$\begin{aligned} I_n[\rho(r)] &= \int_{S_n} \rho(r) dx_1 dx_2 \cdots dx_n = A_n R_n[\rho(r)] \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \rho(r) r^{n-1} dr \end{aligned} \tag{25}$$

which is now relatively easy to compute, since it only involves a one-dimensional integral over the radius.

6 Example: Areas and Volumes of S_n

Recall that to compute the volume of an S_n sphere, we must compute $I_n[1]$:

$$\begin{aligned} V_n &= I_n[1] = \int_{S_n} \rho(r) dx_1 dx_2 \cdots dx_n \qquad (26) \\ &= \left(\int_{\partial S_n} d\Omega_{n-1}\right) \int_0^1 r^{n-1} dr \\ &= A_n\, R_n[1] = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{n} \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \end{aligned}$$

where the Gamma function identity $z\Gamma(z) = \Gamma(z+1)$ has been used. The volume V_n is defined on S_n, which has radius of unity. The relation between V_n and A_n is simply $V_n = \frac{1}{n} A_n$. If we assume the radius is a the volume becomes $V_n(a) = a^n V_n$.

Let us compute the area and volume of the S_n for the first 10 integers. We also compute ℓ_n, which is the length of a side of an n-cube that has the same volume as the S_n, and is defined to be $\ell_n \equiv (V_n)^{1/n}$. The values $\pm \ell_n/2$ are the places where the cube intersects each x_i axis, and it is these values we put in the table.

Using Stirling's series expansion (Bartle 1964)

$$\ln n! = n \ln n - n + \ln \sqrt{2\pi n} + \mathcal{O}(n^{-1}) \tag{27}$$

we can compute ℓ_n analytically at large n,

$$\frac{\ell_n}{2} = \sqrt{\frac{e\pi}{2n}} = \frac{2.06637}{\sqrt{n}} \quad \text{(large } n\text{)} \tag{28}$$

where $e = 2.71828\cdots$ is the normal e^1 exponential constant. The value of $\ell_n/2$ tends to 0 as $n \to \infty$, which suggests that the faces of the cubes pass infinitesimally closely to the origin as $n \to \infty$, which is somewhat anti-intuitive of a cube with a unit volume.

Note, for $n > 1$ the cube intersects the x_i axis at a position inside the radius of the sphere. This is expected since these are the closest points of the cube's surface to the origin, and the cube has yet further points away, such as at 2^n vertices of the cube, which are always at a distance further than 1 to compensate and render the overall volume equal that of S_n. The distance from the origin to the cube's vertices is the "n-cube hypotenuse" h_n. This value is

$$h_n^2 = \left(\frac{\ell_n}{2}\right)^2 + \left(\frac{\ell_n}{2}\right)^2 + \cdots + \left(\frac{\ell_n}{2}\right)^2 = n\left(\frac{\ell_n}{2}\right)^2 \quad \longrightarrow \quad h_n = \sqrt{n}\,\frac{\ell_n}{2} \tag{29}$$

At large n the value of h_n asymptotes to

$$h_\infty = \sqrt{\frac{e\pi}{2}} = 2.06637\cdots \quad (n \to \infty \text{ limit}). \tag{30}$$

In other words, the unit volume n-cube does extend past the S_n's unit radius in places, but it never extends beyond a distance of ~ 2.07, where the vertices reside. In Fig. 1 the values of $\ell_n/2$ and h_n are plotted as a function of n.

Another interpretation of h_n is that $2h_n$ is the length of the longest straight line that can be fully contained within an n-cube of the same volume as S_n. This line extends from a vertex of the cube at position $\vec{x}$ passing through the origin to the corresponding vertex on the opposite side of the cube at position $-\vec{x}$. When $n = 1$, 2, 3 and ∞ these longest possible straight-line distances are 2, $\sqrt{2\pi} = 2.51$, $(48\pi^2)^{1/6} = 2.79$, and $\sqrt{2e\pi} = 4.13$, respectively. These are always longer lengths than the longest length of a straight line that can fit into the S_n sphere, which is of course 2 for the sphere of unit radius, corresponding to any straight line that passes through the origin.

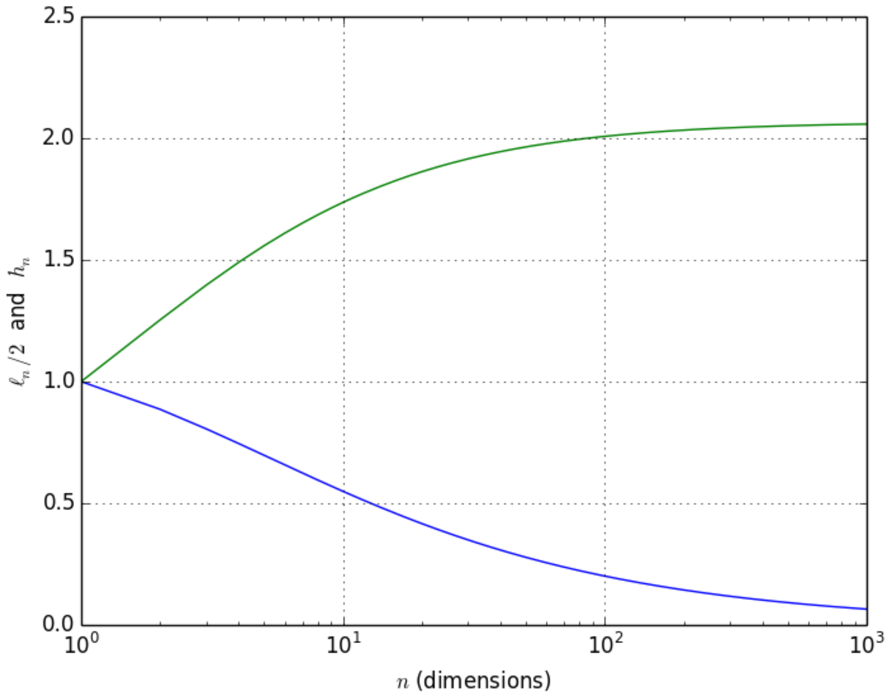

Figure 1: The lower blue curve is a plot vs. number of dimensions n of half the length of an n-cube, $\ell_n/2$, that has the same volume as a unit radius S_n sphere. This length asymptotes to $\ell_\infty \to 0$ as $n \to \infty$, which is indicative of the well-known property of S_n that its volume asymptotes to zero for large dimensions. The upper green curve is a plot of the distance to the furthest point of the surface of the same n-cube to the origin. This furthest point is any one of the 2^n vertices of the cube. Its value asymptotes to $h_\infty = \sqrt{e\pi/2} \simeq 2.066$ as $n \to \infty$. Equivalently, the longest straight line that can fit wholly inside the n-cube has length $2h_n$ and extends from one vertex at position $\vec{x}$ to the opposite vertex at point $-\vec{x}$.

7 Example: Form Factor Integrals

Let us end this discussion by considering another example beyond volume computations:

$$\rho_k(r) = \rho_0 \frac{a^{2k}}{(r^2 + a^2)^k} \tag{31}$$

where a is a smoothing radius parameter and ρ_0 is a constant with units of n-volume density. I will call integrals over eq. (31) “form factor integrals” since $\rho_k(r)$ is reminiscent of form factors in physics, where $\rho \sim \rho_0$ is nearly constant near the center when $r \ll a$ but then falls off rapidly by the power law $\rho \sim 1/r^{2k}$ when $r \gg a$. Let us assume that in this case the radius

n	A_n	V_n	$\ell_n/2$	h_n
1	2	2	1	1
2	2π	π	0.886	1.25
3	4π	$\frac{4}{3}\pi$	0.806	1.40
4	$2\pi^2$	$\frac{1}{2}\pi^2$	0.745	1.49
5	$\frac{8}{3}\pi^2$	$\frac{8}{15}\pi^2$	0.697	1.56
6	π^3	$\frac{1}{6}\pi^3$	0.657	1.61
7	$\frac{16}{15}\pi^3$	$\frac{16}{105}\pi^3$	0.624	1.65
8	$\frac{1}{3}\pi^4$	$\frac{1}{24}\pi^4$	0.596	1.68
9	$\frac{32}{105}\pi^4$	$\frac{32}{945}\pi^4$	0.571	1.71
10	$\frac{1}{12}\pi^5$	$\frac{1}{120}\pi^5$	0.549	1.74
∞	$\to 0$	$\to 0$	$\to 0$	2.07

Table 1: Computations of A_n, V_n and ℓ_n for integer dimensionality n from 1 through 10. V_n is the volume of unit radius S_n, and A_n is the area of the surface of S_n. For spheres of radius a, these are rescaled to $A_n(a) = a^{n-1}A_n$ and $V_n(a) = a^n V_n$. The quantity ℓ_n is the length of a side of an n-dimensional cube that has the same volume as the S_n sphere. It is defined to be $\ell_n \equiv (V_n)^{1/n}$. The cube's intersection points on each x_i axis are $\pm\ell_n/2$, and it is this value that we put in the table. h_n is the furthest point that the cube reaches from the origin, and is the n-dimensional hypotenuse. For large n the values of A_n, V_n and ℓ_n asymptote to 0, whereas h_n asymptotes to $h_\infty = \sqrt{e\pi/2} \simeq 2.066$.

integral is from 0 to r_0. In other words, we wish to compute

$$\begin{aligned} I_n[\rho_k; r_0] &= \int_{S_n(r_0)} \rho_0 \frac{a^{2k}}{(r^2+a^2)^k} dx_1 dx_2 \cdots dx_n \\ &= A_n\, R_n[\rho_k; r_0] \end{aligned} \tag{32}$$

where the notation $S_n(r_0)$ means an n-sphere of radius r_0, and $I_n[\rho; r_0]$ and $R_n[\rho; r_0]$ indicate integrate of r from 0 to r_0. In this notation if r_0 is left unstated, as was the case for the volume integrals over unit radius S_n, its value is assumed to be $r_0 = 1$.

We can rewrite $R_n[\rho_k; r_0]$ as an integral over ξ from the standard range of 0 to 1 by simple transformation of variables $\xi = r/r_0$, leading to

$$I_n[\rho_k; r_0] = \rho_0 r_0^n \left(\frac{a^2}{r_0^2}\right)^k A_n R_n[\tilde{\rho}_k] \tag{33}$$

where

$$R_n[\tilde{\rho}_k] = F(n, k, \xi_0) = \int_0^1 \frac{\xi^{n-1}\, d\xi}{(\xi^2+\xi_0^2)^k} \quad \text{where } \xi_0^2 = a^2/r_0^2. \tag{34}$$

The general solution to $F(n, k, \xi_0)$ with arbitrary arguments involves the evaluation of hypergeometric functions. However, for any particular values of the arguments the integration is often possible to carry out analytically for small integer values of n and k, and usually easy to do numerically for any values of n and k. In the case that $a = 0$ the integral diverges unless $n > 2k$.

It is frequently the case that one wishes to integrate r from 0 to ∞, and in that case and with $a \neq 0$ the integration can be simplified by the substitution $\beta = r/a$, yielding

$$\int_0^\infty \rho_0 \frac{a^{2k}}{(r^2 + a^2)^k} r^{n-1} dr = \rho_0 a^n G(n, k), \quad \text{where} \tag{35}$$

$$G(n, k) = \int_0^\infty \frac{\beta^{n-1}\, d\beta}{(\beta^2 + 1)^k}. \tag{36}$$

Again, it is usually easy to solve this integral numerically for arbitrary values of n and k, and it is often easy to compute analytically for low values of n and k, such as $G(3, 2) = \frac{\pi}{4}$. The integral diverges unless $n < 2k$.

In summary, for integrations of $\rho_k(r)$ of eq. (31) from 0 to finite r_0 in n dimensions one finds

$$I_n[\rho_k; r_0] = \rho_0 r_0^n \left(\frac{a^2}{r_0^2}\right)^k A_n\, F\left(n, k, \frac{a^2}{r_0^2}\right) \tag{37}$$

where the function F is defined in eq. (34) and A_n is defined in eq. (24). For integrations of $\rho_k(r)$ from 0 to ∞ one finds

$$I_n[\rho_k; \infty] = \rho_0 a^n A_n\, G(n, k) \tag{38}$$

where $G(n, k)$ is defined in eq. (36).

References

Arfken, G.B., Webber, H.J. (2001). *Mathematical Methods for Physicists, 5th ed.* New York: Harcourt.

Bartle, R.G. (1964). *The Elements of Real Analysis.* New York: John Wiley & Sons.

Dennery, P., Krzywicki, A. (1996). *Mathematics for Physicists.* Mineola, NY: Dover.

Munkres, J.R. (2000). *Topology, 2nd ed.* Upper Saddle River, NJ: Prentice Hall.