# Lectures on the Symmetries and Interactions of Particle Physics

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## Lectures discuss:

- Spacetime symmetries of the Standard Model
- Internal (gauge) symmetries of the Standard Model
- Observables and their precision tests of the theory
- Higgs boson theory and its discovery

## Special Relativity

We begin with a statement of our most cherished symmetries. Laws of physics should be invariant under special relativity transformations: rotations (3 of them) and velocity boosts (3 of them). This implies that the length $ds^2$ should be invariant under transformations

$$ds^2 = c^2 dt^2 - dx_i dx_i = c^2 dt'^2 - dx'_i dx'_i \tag{1}$$

where $c$ is the same in all reference frames. Define it to be $c = 1$.

Construct Lorentz four-vector $dx^\mu = (dt, dx^1, dx^2, dx^3)$ where $\mu = 0, 1, 2, 3$. Define metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

Then $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Any "position" 4-vector $x^\mu$ contracted with itself is its length and should be invariant.

$$x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = x'^\mu x'_\mu \tag{3}$$

What are the transformations on $x^\mu$ that leave its length invariant?

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \text{suppressing indices}: \;\; x' = \Lambda x \tag{4}$$

Substituting this into the invariance-requirement equation above gives

$$x^T g x = x'^T g x' = (\Lambda x)^T g (\Lambda x) = x^T (\Lambda^T g \Lambda) x \tag{5}$$

Thus we have to find matrices $\Lambda$ that satisfy

$$g = \Lambda^T g \, \Lambda \tag{6}$$

If $g = E$ where $E = \mathrm{diag}(1, 1, 1, 1)$ is the identity matrix, it would be much more familiar to you. In that case

$$E = \Lambda^T \Lambda \implies \Lambda^T = \Lambda^{-1} \tag{7}$$

This last condition is the definition of special orthogonal matrices, which must have $\det \Lambda = \pm 1$. The set of matrices is then $SO(4)$, $4 \times 4$ orthogonal matrices.

## Definition of a group

$SO(4)$ is a "group", which has a very precise mathematical meaning.

A group $G$ is a collection of elements $g \in G$ endowed with a multiplication operator that satisfies four axioms:

1. **Closure:** For every $g_1, g_2 \in G$, $g_1 g_2 \in G$
2. **Associativity:** For all $g_1, g_2, g_3 \in G$, $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
3. **Identity:** There exists an $e \in G$ such that for all $g \in G$, $eg = ge = g$
4. **Inverse:** For every $g \in G$ there is a $g^{-1} \in G$ such that $gg^{-1} = g^{-1} g = e$

The mathematics of group theory plays a significant role in the description of symmetries, which includes the symmetries of the Standard Model.

Note, there is no requirement that $g_1 g_2 = g_2 g_1$.

If this equality is satisfied it is an "Abelian group" (e.g., $U(1)$, $SO(2)$); otherwise, it is called a "Non-Abelian group" (e.g., $SU(2)$, $SO(3)$, etc.).

## $Z_2$ discrete group

One of the simplest groups of all is the $Z_2$ group. It has two elements $\{1, -1\}$ and group multiplication is normal multiplication.

Multiplication table:

| | 1 | -1 |
|---|---|---|
| 1 | 1 | -1 |
| -1 | -1 | 1 |

This is sometimes called the even/odd group.

Forms a group because the four axioms are respected:
1. Closure: check
2. Associativity: check
3. Identity: check
4. Inverse: check

This is an example of a discrete abelian group.

## Example of a Group, $SO(2)$

These are simply two-dimensional rotations that you are used to. Every group element is parameterized by a rotation angle $\theta$:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{8}$$

Multiply two elements together and we get

$$\begin{aligned} g(\theta_1)g(\theta_2) &= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix} & (9) \\ &= \begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 + \cos\theta_1\sin\theta_2 \\ -\cos\theta_1\sin\theta_2 - \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} = g(\theta_3) & (10) \end{aligned}$$

Thus, closure is satisfied by $g(\theta_1)g(\theta_2) = g(\theta_3)$ where $\theta_3 = \theta_1 + \theta_2$.

Associativity obviously works; the identity element is when $\theta = 0$; and, the inverse of $g(\theta)$ is $g(-\theta)$, which is in $SO(2)$. Thus, all the group axioms are satisfied.

## Back to the Lorentz Group, $SO(3,1)$

However, in our case our metric tensor $g$ is not the identity matrix but rather has a mixed metric of three $-1$ entries and one $+1$ entry. Nevertheless the elements $\Lambda$ that satisfy $g = \Lambda^T g\,\Lambda$ form a group, called $SO(3,1)$.

Here are a few examples of elements in $SO(3,1)$:

$$\Lambda_R = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix}, \quad \text{where } \mathbf{R} \text{ are the } 3\times 3 \text{ rotation matrices } SO(3). \tag{11}$$

$$\Lambda_{B_x} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{velocity boost in the } x \text{ direction} \tag{12}$$

where

$$\cosh\eta = \gamma \;\text{ and }\; \sinh\eta = \beta\gamma, \quad \text{with } \; \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \;\text{ and } \beta = v/c. \tag{13}$$

With a little algebra you can see that $x' = \Lambda_{B_x} x$ is equivalent to what you are used to seeing

$$ct' = \frac{1}{\sqrt{1 - v^2/c^2}} \left( ct - \frac{v}{c} x \right) \tag{14}$$

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}} (x - vt) \tag{15}$$

$$y' = y \tag{16}$$

$$z' = z \tag{17}$$

Summary of this: $SO(3,1)$ is a group and the matrices $\Lambda$ are its elements, and their transformations on Lorentz vectors are rotations and boosts that we are familiar with.

What does all of this have to do with particle physics?

The "Symmetry Invariance Principle"

## Symmetry Invariance Principle

When we say that nature is invariant under some symmetry, it means

- All objects in the theory have well defined transformation properties (i.e., well defined "representation" of the symmetry group) under the symmetry, and
- Every interaction is invariant (i.e., a "singlet") under the symmetry transformations

The "objects" of particle physics are particle fields.

The interactions in particle physics are the operators in the lagrangian.

## Singlet and Triplet representations of $SO(3)$

Representations of groups can be intuitively understood from tensor analysis from rotations, the group $SO(3)$ with elements $R_{ij}$ satisfying the condition $R^T = R^{-1}$ and $\det R = 1$.

Let's start with a vector. If we rotate the vector $v$ we get

$$v \to v' = R\,v, \quad \text{or equivalently} \quad v_i' = R_{ij} v_j. \tag{18}$$

The vector $v$ is definite transformations properties under $SO(3)$ and it has three independent elements $(v_x, v_y, v_z)$ and so it defines a "three-dimensional representation" or "triplet representation" of $SO(3)$. Or, for short, $\mathbf{3}$.

There is always the trivial or "singlet" representation:

$$c \to c' = c \;\; \text{singlet representation.} \tag{19}$$

This is the $\mathbf{1}$ representation, or sometimes called the "scalar representation".

We have just defined rather precisely the $\mathbf{1}$ and $\mathbf{3}$ representations of $SO(3)$ from the scalar and vector. What about tensors? Does a tensor form a separate representation of $SO(3)$. Yes, but it's slightly more complicated!

## Tensor representations of $SO(3)$

Let us look at the tensor formed from two vectors: $T_{ij} = a_i b_j$. This tensor has 9 elements. However, there are subspaces of these 9 elements that have definite and closed transformation properties under $SO(3)$.

The most obvious is the *trace*: $\tau = \mathrm{Tr}(T) = a_i b_i$. Under rotation it is preserved.

$$\tau' \longrightarrow R_{il} a_l R_{ik} b_k = R^T_{ki} R_{il} a_l b_k = \delta_{kl} a_l b_k = a_l b_l = \tau \tag{20}$$

The trace of the tensor therefore is a singlet $\mathbf{1}$ representation of $SO(3)$.

Now let us look at the *anti-symmetric tensor* $A^T = -A$,

$$A_{ij} = a_i b_j - a_j b_i = \begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} \quad \text{(3 independent elements)} \tag{21}$$

The anti-symmetric tensor does not change its character under transformations

$$A'_{ml} = R_{mi} R_{lj} A_{ij} = R_{mi} R_{lj} (-A_{ji}) = -R_{lj} R_{mi} A_{ji} = -R_{li} R_{mj} A_{ij} = -A'_{lm}$$

Thus the anti-symmetric tensor forms a $\mathbf{3}_A$ representation of $SO(3)$.

Let's now take the symmetric tensor $S = S^T$,

$$S_{ij} = a_i b_j + a_j b_i = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} \quad \text{(6 distinct entries)} \tag{22}$$

However, we have already used the trace to form a representation, so we need to "subtract out the trace". What we really need is the traceless symmetric tensor $\hat{S}^T = \hat{S}$ with $\mathrm{Tr}(\hat{S}) = 0$:

$$\hat{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & -S_{11} - S_{22} \end{pmatrix} \quad \text{(5 independent elements)} \tag{23}$$

This traceless symmetric tensor preserves its character under transformations

$$\hat{S}_{ml} = R_{mi} R_{lj} \hat{S}_{ij} = R_{mi} R_{lj} \hat{S}_{ji} = R_{li} R_{mj} \hat{S}_{ij} = \hat{S}_{lm} \tag{24}$$

Thus, the traceless symmetric tensor forms a $\mathbf{5}_S$ representation of $SO(3)$.

So, the 9 elements of the tensor form a reducible rep of $SO(3)$, which can be decomposed into three irreps of dimension $\mathbf{1}$, $\mathbf{3}$ and $\mathbf{5}$. In group theory language we we did was show that

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}_A \oplus (\mathbf{1} \oplus \mathbf{5})_S \tag{25}$$

There are an infinite number of representations of $SO(3)$

All representations can found by taking tensor products of the vectors

$$T_{ij} = a_i b_j \tag{26}$$

$$T_{ijk\ldots} = a_i b_j c_k \ldots \tag{27}$$

There are techniques to do this, and tables exist that classify all representations.

**Representation of dimension $d$:** Group elements $g \in G$ are mapped to $d \times d$ matrices $M(g)$ that preserve all the group multiplications. I.e., if $g_1 g_2 = g_3$ then $M(g_1)M(g_2) = M(g_3)$.

**Warning**: Often it is said that

"$X$ is a representation $r$ of the symmetry group $G$",

whereas what is really meant is

"$X$ is an object such that when a symmetry transformation of $G$ is applied, it transforms under the $r$ representation"

What does this have to do with the Lorentz group?

The relevance is because

1. $SO(3)$ and $SU(2)$ are closely related, and

2. The representations of $SO(3,1)$ can be classified in terms of representations of $SU(2) \times SU(2)$

Next, I will remind you why point 1 is correct.

In order to show you that point 2 is correct, I will need to tell you about "generators" for group elements and the Lie algebras that they form.

The connection between $SO(3)$ and $SU(2)$

$SU(2)$ is the set of all $2\times 2$ complex matrices of $\det = +1$ that satisfies $A^\dagger A = I$,

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1. \tag{28}$$

There are three independent parameters required to specify an element here of $SU(2)$. Here is one

$$A = \begin{pmatrix} \cos(\theta/2)\exp\{\frac{i}{2}(\psi+\phi)\} & \sin(\theta/2)\exp\{\frac{i}{2}(\psi-\phi)\} \\ -\sin(\theta/2)\exp\{-\frac{i}{2}(\psi-\phi)\} & \cos(\theta/2)\exp\{-\frac{i}{2}(\psi+\phi)\} \end{pmatrix} \tag{29}$$

$$\text{where } \; 0 \le \theta \le \pi, \quad 0 \le \psi \le 4\pi, \quad 0 \le \phi \le 2\pi. \tag{30}$$

The objects that transform under $SU(2)$ elements like this are called "spinors" $\chi_\alpha$, where $\alpha = 1, 2$. They are the analogy of vectors in three dimensional rotation group ($SO(3)$):

$$\chi'_\alpha = A_{\alpha\beta}\,\chi_\beta, \quad \alpha, \beta = 1, 2. \tag{31}$$

## 2-to-1 homomorphic mapping of $SU(2)$ onto $SO(3)$

*Theorem* (Cornwell 1984): "There exists a two-to-one homomorphic mapping of the group $SU(2)$ onto the group $SO(3)$. If $A \in SU(2)$ maps onto $R(A) \in SO(3)$, then $R(A) = R(-A)$ and the mapping may be chosen so that

$$R(A)_{jk} = \frac{1}{2}\mathrm{Tr}\left(\sigma_j A \sigma_k A^{-1}\right), \quad j,k = 1,2,3 \tag{32}$$

$$\text{where} \;\; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{33}$$

are the Pauli spin matrices."

This implies that the representations of $SO(3)$ are also representations of $SU(2)$. In addition there are "spinor" representations of $SU(2)$ that have no analog in $SO(3)$.

The "defining representation" of $SU(2)$ with $2 \times 2$ elements $A_{\alpha\beta}$ is the lowest dimension spinor representation.

## Representations of $SU(2)$

*You already know the $SU(2)$ representations well from quantum mechanics!*

In QM you analyzed spin carefully and found basis functions of the form

$|\ell m\rangle$ where $\ell = 0,\ 1/2,\ 1,\ 3/2,\ 2, \ldots,$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$

Each $\ell$ labels a distinct irreducible representation of $SU(2)$, and the number of $m$'s ($= 2\ell+1$) is the dimensionality of the representation.

The representations of integer $\ell$ are equivalent to $SO(3)$ reps (bosonic reps), and the half-integer representations are the spinor representations (fermionic reps).

Ok, now we understand that $SU(2)$ and $SO(3)$ have lots of different representations of different dimensions.

Before we show how this relates to the Lorentz group and the labels we give to elementary particles, we must introduce two more concepts: generators and Lie algebras. We will do this first through the more intuitive $SO(3)$ and then show them for $SU(2)$ and then for the Lorentz group $SO(3, 1)$.

## Generators of $SO(3)$ group elements

Let's consider the matrix

$$s_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{34}$$

And now compute $e^{i\phi s_x}$:

$$e^{i\phi s_x} = 1 + i\phi s_x + \frac{1}{2!} i^2 \phi^2 s_x^2 + + \frac{1}{3!} i^3 \phi^3 s_x^3 + \cdots = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \tag{35}$$

which is a rotation about the $x$ axis. In this sense $s_x$ is a "generator" of the rotations.

The word "generator" comes from the fact that an infinitesimal rotation about the $x$-axis is $\propto \phi$, and so one can build up or "generate" the full rotation by adding an infinite number of infinitesimal rotations:

$$R_x(\phi) = \lim_{N\to\infty} \left(1 + i\frac{\phi}{N} s_x\right)^N \longrightarrow \; e^{i\phi s_x}. \tag{36}$$

Similarly, rotations about the $y$ and $z$ axis are generated by

$$s_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad s_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

The set of three generators $s_x$, $s_y$ and $s_z$ of $SO(3)$ forms a real Lie algebra.

What is a Lie algebra?

# Reminder: Algebra

(Szekeres 2004) "An algebra consists of a vector space A over a field K together with a law of composition or product of vectors, $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ denoted

$$(A, B) \to AB \in \mathcal{A} \quad (A, B \in \mathcal{A}), \tag{38}$$

which satisfies a pair of distributive laws:

$$A(aB + bC) = aAB + bAC, \quad (aA + bB)C = aAC + bBC \tag{39}$$

for all scalars $a, b \in K$ and vectors $A$, $B$ and $C$."

## Definition of a Lie algebra

**Real Lie algebra:** Real lie algebra $\mathcal{L}$ of dimension $n\,(\geq 1)$ is a real vector space of dimension $n$ equipped with a "Lie product" or "commutator" $[a, b]$ defined for every $a$ and $b$ of $\mathcal{L}$ such that (Cornwell 1984)

1. $[a, b] \in \mathcal{L}$ for all $a, b \in \mathcal{L}$
2. $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$ for all $a, b, c \in \mathcal{L}$ and all real numbers $\alpha$ and $\beta$
3. $[a, b] = -[b, a]$ for all $a, b \in \mathcal{L}$
4. for all $a, b, c \in \mathcal{L}$ the Jacobi identity holds

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

The commutator algebra of the $SO(3)$ generators is

$$[s_x, s_y] = is_z \quad \text{et cyclic} \tag{40}$$

and $s_x$, $s_y$ and $s_z$ are the "basis vectors" of the vector space.

These commutator relationships are very familiar! They are exactly the same as $SU(2)$, whose generators are $\sigma_i/2$:

$$\left[\frac{\sigma_x}{2}, \frac{\sigma_y}{2}\right] = i\frac{\sigma_z}{2} \quad \text{et cyclic} \tag{41}$$

Thus, we see that although the $SU(2)$ group is different than the $SO(3)$ group, the Lie algebras are isomorphic. This is another way in which $SO(3)$ and $SU(2)$ are very similar.

Now, all this will help in understanding why $SO(3,1) \simeq SU(2) \times SU(2)$.

## Generator algebra of $SO(3,1)$

The Lorentz group has three rotations and three boosts, and thus there are six total generators acting on the four-vectors.

The three generators of rotation are

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

The three generators of boosts are

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{43}$$

## The algebra of Lorentz group generators

The generators form an algebra with commutation relations

$$[J_x, J_y] = iJ_z \ \text{ et cyclic;} \qquad [K_x, K_y] = -iJ_z \ \text{ et cyclic;} \tag{44}$$

$$[J_x, K_y] = iK_z \ \text{ et cyclic;} \qquad [J_x, K_x] = 0 \ \text{ et cetera.} \tag{45}$$

The algebra can be recast more simply be redefining

$$A = \frac{1}{2}(J + iK) \ \text{ and } \ B = \frac{1}{2}(J - iK) \tag{46}$$

which leads to

$$[A_x, A_y] = iA_z \text{ et cyclic;} \quad [B_x, B_y] = iB_z \text{ et cyclic;} \tag{47}$$

$$\text{and } \ [A_i, B_j] = 0, \ \text{ for all } i, j = x, y, z. \tag{48}$$

This appears to be the same algebra as that of two independent $SU(2)$ algebras.

This is why $SO(3,1)$ representations can be classified as $SU(2) \times SU(2)$ representations.

This is also why $SU(2)$ spin shows up everywhere in quantum mechanics. It's because of Lorentz symmetry!

$SU(2) \times SU(2)$ Representations of the Lorentz Group

| Lorentz Rep | Total Spin | Elementary Particle Quantum Field |
|---|---|---|
| $(0,0)$ | $0$ | "scalars" (Higgs boson) |
| $\left(\frac{1}{2},0\right)^\sharp$ | $\frac{1}{2}$ | "left spinor" (leptons, quarks, neutrinos)* |
| $\left(0,\frac{1}{2}\right)^\sharp$ | $\frac{1}{2}$ | "right spinor" (leptons, quarks, neutrinos)* |
| $\left(\frac{1}{2},\frac{1}{2}\right)$ | $1$ | "vector gauge field" ($\gamma$, $Z$, $W^\pm$, gluons)† |
| $\left(1,\frac{1}{2}\right)$ | $\frac{3}{2}$ | "Rarita-Schwinger field" (no SM particle) |
| $(1,1)$ | $2$ | "spin-2 field" (graviton) |

♯ The Lorentz group has an operation that allows a spinor $\xi$ that transforms under $SU(2)_L$ to transform as a spinor under $SU(2)_R$: $\chi = i\sigma_2\xi^*$. Related by parity transformation $K \to -K$ (i.e., $A \to B$ generators.)

* The fermions (leptons, quarks, and neutrinos) are often treated as a four-dimensional representation $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$.

† The gauge fields $A^\mu$ take $(1/2,1/2)$ representation form by $\sigma_\mu^{\alpha\dot{\alpha}}A^\mu$, where $\alpha = 1,2$ is spinor index for $SU(2)_L$ and $\dot{\alpha} = 1,2$ is spinor index for $SU(2)_R$. We'll call this $\sigma A$ for short.

## Interactions preserving Lorentz invariance

From mathematical point of view

*particle:* an object that has a well-defined transformation property under Lorentz symmetry.

If nature is to be invariant under the Lorentz symmetry then the only interactions allowed among particles are those that are singlets under both left and right spin groups.

You know how to do this!

There are two facts you learned from early days that help:

1. Tensor product of spin $s_1$ and $s_2$ give spin $|s_1 - s_2|$ and $s_1 + s_2$ results, and
2. If all tensor indices are contracted, the result is a scalar – invariant!

We can manipulate and understand invariants using these facts, and build up all Lorentz invariants of the theory.

## Products of spin – Majorana fermion mass

Just as in QM we realized that

$$|1/2\rangle \otimes |1/2\rangle = |0\rangle \oplus |1\rangle \tag{49}$$

we can do the same thing with particles, with the slight complication that we need to keep track of the left and right $SU(2)$'s separately.

Consider the spinor $f_R = (0, 1/2)$, and let's ask if $f_R \cdot f_R$ interaction is ok:

$$\text{left}: \quad 0 \otimes 0 = 0 \quad \text{contains singlet} \tag{50}$$

$$\text{right}: \quad 1/2 \otimes 1/2 = 0 \oplus 1 \quad \text{contains singlet} \tag{51}$$

so this is an invariant. It is the mass operator: $m f_R \cdot f_R$.

To be more precise, there is a spinor-metric on the contraction which is $i\sigma^2 = \varepsilon$. The mass operator is

$$m\, f_R^T i\sigma^2 f_R \quad \text{(Majorana mass)} \tag{52}$$

If $f_R$ has charge (e.g., electric charge) this term is not invariant, and not allowed. In the Standard Model only right-handed neutrinos qualify for this type of mass:

$$M_R\, \nu_R^T i\sigma^2 \nu_R. \tag{53}$$

## Dirac fermion mass

What if we have $f_L = (1/2, 0)$ and $f_R = (0, 1/2)$.

We learned early that a Lorentz invariant is $\chi_R^T i\sigma^2 f_R$, and we also learned that $i\sigma^2 f_L^*$ transforms like a RH-fermion. Thus, identifying $\chi_R = i\sigma^2 f_L^*$, we have

$$\chi_R^T i\sigma^2 f_R \;=\; (i\sigma^2 f_L^*)^T i\sigma^2 f_R = f_L^\dagger i(\sigma^2)^T i\sigma^2 f_R = f_L^\dagger f_R \tag{54}$$

which used the facts that $(\sigma^2)^T = -\sigma^2$ and $\sigma^2\sigma^2 = 1$.

Likewise $f_R^\dagger f_L$ is an invariant, which is just the conjugate of $f_L^\dagger f_R$.

Therefore, we have identified a new fermion bilinear invariant (i.e., mass term):

$$m_f\,(f_L^\dagger f_R + f_R^\dagger f_L). \tag{55}$$

This is often called Dirac mass.

## Four component spinor representation

We have been talking about $f_L$ and $f_R$, and making mass terms that connect the two. But you have more commonly heard only labels like "electron", "muon", "quarks." We can put the two-component $f_L$ and $f_R$ spinors, Weyl spinors, into a four-component spinor, the Dirac spinor as this:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{56}$$

We can construct the projection operators $P_L\Psi = (\psi_L \; 0)$ and $P_R\Psi = (0 \; \psi_R)$ from

$$P_{L,R} = \frac{1}{2}(1 \mp \gamma^5) \quad \text{where } \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{57}$$

The four dimensional analogy to the $\sigma^\mu$ matrices are the $\gamma^\mu$ matrices, where in the Weyl representation they are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{58}$$

The $\gamma^0$ acting on $\Psi$ interchanges $\psi_L \leftrightarrow \psi_R$ (parity operation).

## Four-component spinor invariants

In this four component notation, we can write the Majorana and Dirac mass terms.

For the Majorana mass, let us define the built-up four-component spinor of $\chi_L$ (which transforms under $(1/2, 0)$ representation) to be

$$\Psi_M = \begin{pmatrix} \chi_L \\ i\sigma^2 \chi_L^* \end{pmatrix} \quad \text{and} \quad \Psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{59}$$

The Majorana and Dirac mass terms are then

$$m\, \Psi_M^T(-i\gamma^0\gamma^2)\Psi_M \quad \text{(Majorana mass)} \tag{60}$$

$$m_D\, \bar{\Psi}_D \Psi_D \quad \text{where } \bar{\Psi}_D = \Psi_D^\dagger \gamma^0 \quad \text{(Dirac mass)} \tag{61}$$

## Vector particle invariants

A vector particle $A_\mu$ (e.g., photon) has many invariants. Easy: just contract all the Lorentz indices.

The invariants up to dimension four are

$$A_\mu A^\mu, \quad \partial_\mu A^\mu, \quad \partial_\mu A^\mu \partial^\nu A_\nu, \quad \partial_\mu A^\nu \partial^\mu A_\nu, \quad A^\mu A^\nu A_\mu A_\nu. \tag{62}$$

If $A_\mu$ is the gauge field of a $U(1)$ invariant theory, such as QED, interactions must be invariant *also* under gauge transformations

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \Lambda \tag{63}$$

The only interaction that is invariant under both Lorentz symmetry and gauge symmetry is

$$F_{\mu\nu}F^{\mu\nu}, \quad \text{where } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{64}$$

This is the well-known kinetic energy term in the QED lagrangian

$$\mathcal{L}^{KE}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{65}$$

## Vector particles interacting with fermions

The vector $A_\mu$ interaction with fermions requires us to consider its $(1/2,1/2)$ representation: $\sigma_\mu A^\mu$. A general interaction

$$(1/2,0)\otimes(1/2,1/2)\otimes(0,1/2) = \text{ contains } (0,0) \text{ contains singlet} \tag{66}$$

If $\chi_R=(0,1/2)$ and $f_L=(1/2,0)$ then we can have interaction

$$\chi_R^T\cdot\sigma_\mu A^\mu\cdot f_L \tag{67}$$

where the first (second) $\cdot$ refers to $SU(2)_R$ ($SU(2)_L$) contraction.

Consider $\chi_R=i\sigma^2 f_L^*$. Interaction becomes

$$(i\sigma^2 f_L^*)^T\cdot\sigma_\mu A^\mu\cdot f_L = -if_L^\dagger(\sigma^2)^T\sigma^2\sigma_\mu A^\mu\cdot f_L = if_L^\dagger\sigma_\mu A^\mu\cdot f_L \tag{68}$$

In four component language we see this interaction as

$$i\bar\Psi\gamma^\mu A_\mu\Psi. \tag{69}$$

Scalar interactions with itself, fermions and vector bosons

The invariant interactions of vector bosons with scalars is also easy. If we assume real scalar $\phi$, we have

$$\partial_\mu \phi \partial^\mu \phi, \quad \phi, \quad \phi^2, \quad \partial^\mu \partial_\mu \phi, \quad \phi^3, \quad \phi^4, \quad \text{etc.} \tag{70}$$

For charged complex scalar $\Phi$ (like the Higgs boson doublet) invariance under Lorentz and charge symmetry allow

$$(\partial_\mu \Phi^*)(\partial^\mu \Phi), \quad \Phi^*\Phi, \quad (\Phi^*\Phi)^2, \quad \text{etc.} \tag{71}$$

Interactions with the vector bosons are

$$A^\mu A_\nu \Phi^* \Phi, \quad A^\mu \Phi^* \, \partial_\mu \Phi, \quad \text{etc.} \tag{72}$$

Interactions with fermions include

$$\phi \nu_R^T i\sigma^2 \nu_R, \quad \Phi \nu_R^T i\sigma^2 \nu_R, \quad \phi f_R^\dagger f_L, \quad \Phi f_R^\dagger f_L, \quad \text{etc.} \tag{73}$$

Lorentz invariance is too general for what is witnessed in nature

Lorentz invariance alone allows us to classify particles and gives strong constraints on what particles are allowed to interact. For example, one cannot have the interactions

$$A^\mu f_R, \quad f_L^\dagger f_L f_R, \quad \Phi f_L A_\mu, \quad \text{etc.} \quad \text{(Lorentz forbidden)} \tag{74}$$

But there are many other interactions forbidden that Lorentz invariance alone does not preclude. These include

$$A_\mu A^\mu, \quad e_L^\dagger \sigma^\mu A_\mu \cdot u_L, \quad \tau_R^T \sigma^2 \tau_R, \quad \mu_L^\dagger \Phi\, t_R, \quad \text{etc.} \tag{75}$$

These are forbidden by "internal" gauge symmetries. The Standard Model particles are charged not only under $SU(2)_L \times SU(2)_R$ Lorentz symmetry, but also under $SU(3)_c \times SU(2)_W \times U(1)_Y$ gauge symmetries.

*Interactions must be invariant under the transformations of every symmetry.*

We discuss next the gauge symmetries of the Standard Model.

Strong, weak and hypercharge forces

The Standard Model particles also transform as representations of the strong, weak and hypercharge forces, which in group theory language is

$$SU(3)_c \times SU(2)_W \times U(1)_Y \quad \text{(Standard Model gauge groups).} \tag{76}$$

If a particle $\varphi$ transforms as $d$ dimensional representation $R$ of group $G$, then

$$\varphi \to \varphi' = e^{i\theta_k T_k^R} \varphi \tag{77}$$

where $T^R$ are $d \times d$ dimensional generator matrices associated with the representation $R$, and $\theta_k$ are the parameters of the group, analogous to the angle of rotation in $SO(2)$.

Global symmetries mean $\theta_k$ do not depend on spacetime, whereas with local symmetries they do, $\theta_k(x)$.

Gauge symmetries are local internal symmetries.

## Hypercharge gauge symmetry

Hypercharge is a $U(1)$ gauge symmetry, and its generator is the hypercharge operator $Y$, and the parameter we can define as $\alpha$.

Under gauge transformation

$$\psi \to \psi' = e^{i\alpha(x)Y}\psi \tag{78}$$

Let's look at the transformation of the kinetic operator

$$\begin{aligned}
\psi_L^\dagger \sigma_\mu \partial^\mu \cdot \psi_L \;&\to\; (e^{i\alpha Y}\psi_L)^\dagger \sigma_\mu \partial^\mu \cdot e^{i\alpha Y}\psi_L & (79)\\
&=\; \psi_L^\dagger e^{-i\alpha Y}\sigma_\mu e^{i\alpha Y} \cdot (iY\partial^\mu \alpha \psi_L + \partial^\mu \psi_L) & (80)\\
&=\; \psi_L^\dagger \sigma_\mu \cdot \partial^\mu \psi_L + \psi_L^\dagger \sigma_\mu \cdot (iY\partial^\mu \alpha)\psi_L & (81)
\end{aligned}$$

The kinetic term would be invariant if it weren't for $\partial^\mu \alpha \neq 0$ contribution.

Introduce covariant derivative $D^\mu = \partial^\mu - iYA^\mu$ (introducing gauge field $A^\mu$), and one finds

$$\psi_L^\dagger \sigma_\mu D^\mu \cdot \psi_L, \quad \text{is invariant when} \tag{82}$$

$$A_\mu \to A'_\mu = A_\mu + i\partial_\mu \alpha. \tag{83}$$

| Field | $SU(3)$ | $SU(2)_L$ | $T^3$ | $\frac{Y}{2}$ | $Q = T^3 + \frac{Y}{2}$ |
|---|---|---|---|---|---|
| $g_\mu^a$ (gluons) | $\mathbf{8}$ | $\mathbf{1}$ | 0 | 0 | 0 |
| $(W_\mu^\pm, W_\mu^0)$ | $\mathbf{1}$ | $\mathbf{3}$ | $(\pm 1, 0)$ | 0 | $(\pm 1, 0)$ |
| $B_\mu^0$ | $\mathbf{1}$ | $\mathbf{1}$ | 0 | 0 | 0 |
| $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ | $\mathbf{3}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $\frac{1}{6}$ | $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ |
| $u_R$ | $\mathbf{3}$ | $\mathbf{1}$ | 0 | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $d_R$ | $\mathbf{3}$ | $\mathbf{1}$ | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $-\frac{1}{2}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |
| $e_R$ | $\mathbf{1}$ | $\mathbf{1}$ | 0 | $-1$ | $-1$ |
| $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $\frac{1}{2}$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $\Phi^c = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $-\frac{1}{2}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |

Invariants under $SU(3)_c$ and $SU(2)$ are found when tensor products yield a singlet $\mathbf{1}$. Under $SU(3)$ conjugate representations are distinct (i.e., $\bar{Q}_L$ is $\bar{\mathbf{3}}$). Conjugate reps for $SU(2)$ are not distinct.

$$SU(3)_c: \qquad \bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} + \mathbf{8} \tag{84}$$

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6} \tag{85}$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} + \cdots \tag{86}$$

$$SU(2)_W: \qquad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3} \tag{87}$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} + \cdots \tag{88}$$

## The problem with elementary particle masses

The fermions of the Standard Model and some of the gauge bosons have mass.

This is a troublesome statement since gauge invariance appears to allow neither.

First, we illustrate the concepts with a massive $U(1)$ theory – spontaneously broken QED.

The lagrangian of QED is

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{89}$$

where

$$D_\mu = \partial_\mu + ieA_\mu \tag{90}$$

and $Q = -1$ is the charge of the electron. This lagrangian respects the $U(1)$ gauge symmetry

$$\psi \rightarrow e^{-i\alpha(x)}\psi \tag{91}$$

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha(x). \tag{92}$$

Vector gauge boson mass

If we wish to give the photon a mass we may add to the lagrangian the mass term

$$\mathcal{L}_{mass} = \frac{m_A^2}{2} A_\mu A^\mu. \tag{93}$$

However, this term is not gauge invariant since under a transformation $A_\mu A^\mu$ becomes

$$A_\mu A^\mu \to A_\mu A^\mu + \frac{2}{e} A^\mu \partial_\mu \alpha + \frac{1}{e^2} \partial_\mu \alpha \partial^\mu \alpha \tag{94}$$

Simply adding a mass term does not respect the gauge symmetry.

Way to give photon mass while retaining the gauge symmetry: Higgs boson.

Higgs boson is a condensing complex scalar field $\langle \Phi \rangle \neq 0$, charged under the symmetry in question.

## The Higgs mechanism

Let us suppose that the photon in QED has a mass. Invoke the Higgs mechanism in a gauge invariant manner by introducing the field $\Phi$ with charge $q$ to the lagrangian:

$$\mathcal{L} = \mathcal{L}_{QED} + (D_\mu \Phi)^* (D^\mu \Phi) - V(\Phi) \tag{95}$$

where

$$V(\Phi) = \mu^2 |\Phi|^2 + \lambda |\Phi|^4 \tag{96}$$

and it is assumed that $\lambda > 0$ and $\mu^2 < 0$.

With $\mu^2 < 0$ the minimum of this potential is not at $\Phi = 0$ (algebra on next slide).

The field $\Phi$ is then said to "get a vacuum expectation value" or a "vev", which freezes some charge (i.e., the charge of $\Phi$) into the vacuum.

Since $\Phi$ is a complex field we have the freedom to parametrize it as

$$\Phi = \frac{1}{\sqrt{2}}\phi(x)e^{i\xi(x)}, \tag{97}$$

where $\phi(x)$ and $\xi(x)$ are real scalar fields. The scalar potential with this choice simplifies to

$$V(\Phi) \to V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{98}$$

Minimizing the scalar potential one finds

$$\left.\frac{dV}{d\phi}\right|_{\phi=\phi_0} = \mu^2\phi_0 + \lambda\phi_0^3 = 0 \implies \phi_0 = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{99}$$

In the unitary gauge, where $\alpha(x) = \xi(x)/\phi_0$, one finds

$$(D_\mu\Phi)^*(D^\mu\Phi) \to \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}e^2q^2\phi^2 A_\mu A^\mu \tag{100}$$

At the minimum of the potential $\langle\phi\rangle = \phi_0$, so one can expand the field $\phi$ about its vev, $\phi = \phi_0 + h$, and identify the fluctuating degree of freedom $h$ with a propagating real scalar boson.

The Higgs boson mass and self-interactions are obtained by expanding the lagrangian about $\phi_0$. The result is

$$-\mathcal{L}_{Higgs} = \frac{m_h^2}{2}h^2 + \frac{\mu'}{3!}h^3 + \frac{\eta}{4!}h^4 \tag{101}$$

where

$$m_h^2 = 2\lambda\phi_0^2, \quad \mu' = \frac{3m_h^2}{\phi_0}, \quad \eta = 6\lambda = 3\frac{m_h^2}{\phi_0^2}. \tag{102}$$

The mass of the Higgs boson is not dictated by gauge couplings here, but rather by its self-interaction coupling $\lambda$ and the vev.

The complex Higgs boson kinetic terms can be expanded to yield

$$\Delta\mathcal{L} = \frac{1}{2}e^2q^2\phi_0^2 A_\mu A^\mu + e^2q^2hA_\mu A^\mu + \frac{1}{2}e^2q^2h^2A_\mu A^\mu. \tag{103}$$

The first term is the mass of the photon, $m_A^2 = e^2q^2\phi_0^2$. A massive vector boson has a longitudinal degree of freedom, in addition to its two transverse degrees of freedom, which accounts for the degree of freedom lost by virtue of "gauging away" $\xi(x)$.

The second and third terms of eq. 103 set the strength of interaction of a single Higgs boson and two Higgs bosons to a pair of photons:

$$hA_\mu A_\nu \text{ Feynman rule} \ : \ i2e^2q^2\phi_0 g_{\mu\nu} = i2\frac{m_A^2}{\phi_0} \tag{104}$$

$$hhA_\mu A_\nu \text{ Feynman rule} \ : \ i2e^2q^2 g_{\mu\nu} = i2\frac{m_A^2}{\phi_0^2} \tag{105}$$

after appropriate symmetry factors are included.

The general principles that we have learned here
1/ Higgs mechanism gives gauge bosons mass in gauge-invariant way
2/ Higgs boson couples with strength of particle's mass

Chiral fermion masses

In quantum field theory a four-component fermion can be written in its chiral basis as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{106}$$

where $\psi_{L,R}$ are two-component chiral projection fermions. A mass term in quantum field theory is equivalent to an interaction between the $\psi_L$ and $\psi_R$ components

$$m\bar{\psi}\psi = m\psi_L^\dagger \psi_R + m\psi_R^\dagger \psi_L. \tag{107}$$

In vectorlike QED, the $\psi_L$ and $\psi_R$ components have the same charge and a mass term can simply be written down. However, let us now suppose that in our toy $U(1)$ model, there exists a set of chiral fermions where the $P_L\psi = \psi_L$ chiral projection carries a different gauge charge than the $P_R\psi = \psi_R$ chiral projection. In that case, we cannot write down a simple mass term without explicitly breaking the gauge symmetry.

The resolution to this conundrum of masses for chiral fermions resides in the Higgs sector. If the Higgs boson has just the right charge, it can be utilized to give mass to the chiral fermions. For example, if the charges are $Q[\psi_L] = 1$, $Q[\psi_R] = 1 - q$ and $Q[\Phi] = q$ we can form the gauge invariant combination

$$\mathcal{L}_f = y_\psi \, \psi_L^\dagger \Phi \psi_R + c.c. \tag{108}$$

where $y_f$ is a dimensionless Yukawa coupling. Now expand the Higgs boson about its vev, $\phi = (\phi_0 + h)/\sqrt{2}$, and we find

$$\mathcal{L}_f = m_\psi \, \psi_L^\dagger \psi_R + \left(\frac{m_\psi}{\phi_0}\right) h \psi_L^\dagger \psi_R + c.c. \tag{109}$$

where $m_\psi = y_\psi \phi_0/\sqrt{2}$ is mass of fermion!

This same Yukawa interaction gives rise to an Higgs-fermion interaction:

$$h\bar{\psi}\psi \text{ (Feynman rule) } : \; i\frac{m_\psi}{\phi_0}. \tag{110}$$

Just as was the case with the gauge bosons, the generation of fermion masses by the Higgs boson leads to an interaction of the physical Higgs bosons with the fermion proportional to the fermion mass.

The bosonic electroweak lagrangian is an $SU(2)_L \times U(1)_Y$ gauge invariant theory

$$\mathcal{L}_{bos} = |D_\mu \Phi|^2 - \mu^2 |\Phi|^2 - \lambda |\Phi|^4 - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} \tag{111}$$

where $\Phi$ is an electroweak doublet with Standard Model charges of $(\mathbf{2}, 1/2)$ under $SU(2)_L \times U(1)_Y$ ($Y = +1/2$). In our normalization electric charge is $Q = T^3 + \frac{Y}{2}$, and the doublet field $\Phi$ can be written as two complex scalar component fields $\phi^+$ and $\phi^0$:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{112}$$

The covariant derivative and field strength tensors are

$$D_\mu \Phi = \left( \partial_\mu + ig \frac{\tau^a}{2} W^a_\mu + ig' \frac{Y}{2} B_\mu \right) \Phi \tag{113}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{114}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g f^{abc} W^b_\mu W^c_\nu \tag{115}$$

The minimum of the potential does not occur at $\Phi = 0$ if $\mu^2 < 0$. Instead, one finds that the minimum occurs at a non-zero value of $\Phi$ – its vacuum expectation value (vev).

With gauge transformation the vev of $\Phi$ is

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{where} \quad v \equiv \sqrt{\frac{-\mu^2}{\lambda}}. \tag{116}$$

This vev carries hypercharge and weak charge into the vacuum, and what is left unbroken is electric charge.

Our symmetry breaking pattern is then simply $SU(2)_L \times U(1)_Y \to U(1)_{EM}$. The original group, $SU(2)_L \times U(1)_Y$, has a total of four generators and $U(1)_{EM}$ has one generator. Thus, three generators are 'broken'.

Goldstone's theorem tells us that for every broken generator of a symmetry there must correspond a massless field. These three massless Goldstone bosons we can call $\phi_{1,2,3}$. We now can rewrite the full Higgs field $\Phi$ as

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ h + i\phi_3 \end{pmatrix} \tag{117}$$

The fourth degree of freedom of $\Phi$ is the Standard Model Higgs boson $h$. It is a propagating degree of freedom. The other three states $\phi_{1,2,3}$ can all be absorbed as longitudinal components of three massive vector gauge bosons $Z, W^{\pm}$.

The four electroweak gauge boson mass eigenstates are

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^{(1)} \mp iW_\mu^{(2)}\right), \quad M_W^2 = \frac{1}{4}g^2v^2 \tag{118}$$

$$Z_\mu = \frac{gW_\mu^{(3)} - g'A_\mu}{\sqrt{g^2+g'^2}}, \qquad M_Z^2 = \frac{1}{4}(g'^2+g^2)v^2 \tag{119}$$

$$A_\mu = \frac{gB_\mu + g'W^{(3)}}{\sqrt{g^2+g'^2}}, \qquad M_A^2 = 0 \tag{120}$$

It is convenient to define $\tan\theta_W = g'/g$. By measuring interactions of the gauge bosons with fermions it has been determined experimentally that $g = 0.65$ and $g' = 0.35$, and therefore $\sin^2\theta_W = 0.23$.

**Important Exercise:** From eqs. 111, 113 and 117 derive the mass eigenvector and eigenvalue equations for $A_\mu$, $Z_\mu$ and $W_\mu^\pm$ given above.

## Higgs boson interactions to bosons

After performing the redefinitions of the fields above, the kinetic terms for the $W_\mu^\pm, Z_\mu, A_\mu$ will all be canonical. Expanding the Higgs field about the vacuum, the contributions to the lagrangian involving Higgs boson interaction terms are

$$\mathcal{L}_{h\,int} = \left[m_W^2 W_\mu^+ W^{-,\mu} + \frac{m_Z^2}{2} Z_\mu Z^\mu\right]\left(1+\frac{h}{v}\right)^2 \quad (121)$$

$$-\frac{m_h^2}{2}h^2 - \frac{\xi}{3!}h^3 - \frac{\eta}{4!}h^4 \quad (122)$$

where

$$m_W^2 = \frac{1}{4}g^2 v^2, \;\; m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2 \;\; \Longrightarrow \;\; \frac{m_W^2}{m_Z^2} = 1-\sin^2\theta_W \quad (123)$$

$$m_h^2 = 2\lambda v^2, \;\; \xi = \frac{3m_h^2}{v}, \;\; \eta = 6\lambda = \frac{3m_h^2}{v^2}. \quad (124)$$

From our knowledge of the gauge couplings, the value of the vev $v$ can be determined from the masses of the gauge bosons: $v \simeq 246\,\mathrm{GeV}$.

The Feynman rules for Higgs boson interactions among the bosons are

$$hhh \; : \; -\frac{i3m_h^2}{v} \tag{125}$$

$$hhhh \; : \; -i\frac{3m_h^2}{v^2} \tag{126}$$

$$hW_\mu^+W_\nu^- \; : \; i2\frac{m_W^2}{v}g^{\mu\nu} \tag{127}$$

$$hZ_\mu Z_\nu \; : \; i2\frac{m_Z^2}{v}g_{\mu\nu} \tag{128}$$

$$hhW_\mu^+W_\nu^- \; : \; i2\frac{m_W^2}{v^2}g_{\mu\nu} \tag{129}$$

$$hhZ_\mu Z_\nu \; : \; i2\frac{m_Z^2}{v^2}g_{\mu\nu} \tag{130}$$

## Fermion masses and their couplings to the Higgs boson

Fermion masses are also generated in the Standard Model through the Higgs boson vev, which in turn induces an interaction between the physical Higgs boson and the fermions. Let us start by looking at $b$ quark interactions. The relevant lagrangian for couplings with the Higgs boson is

$$\Delta\mathcal{L} = y_b Q_L^\dagger \Phi b_R + c.c. \text{ where } Q_L^\dagger = (t_L^\dagger \; b_L^\dagger) \tag{131}$$

where $y_b$ is the Yukawa coupling. The Higgs boson, after a suitable gauge transformation, can be written simply as

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} \tag{132}$$

and the interaction lagrangian can be expanded to

$$\begin{aligned} \Delta\mathcal{L} &= y_b Q_L^\dagger \Phi b_R + c.c. = \frac{y_b}{\sqrt{2}} (t_L^\dagger \; b_L^\dagger) \begin{pmatrix} 0 \\ v+h \end{pmatrix} b_R + h.c. & (133) \\ &= m_b (b_R^\dagger b_L + b_L^\dagger b_R) \left(1 + \frac{h}{v}\right) = m_b \bar{b} b \left(1 + \frac{h}{v}\right) & (134) \end{aligned}$$

where $m_b = y_b v/\sqrt{2}$ is the mass of the $b$ quark.

The quantum numbers are good for $b_L^\dagger b_R$ mass term.

Under $SU(2)$ the interaction $Q_L^\dagger \Phi b_R$ is invariant because $\mathbf{2} \times \mathbf{2} \times \mathbf{1} \in \mathbf{1}$ contains a singlet.

And under $U(1)_Y$ hypercharge the interaction is invariant because $Y_{Q_L^\dagger} + Y_\Phi + Y_{b_R} = -\frac{1}{6} + \frac{1}{2} - \frac{1}{3}$ sums to zero.

Thus, the interaction is invariant under all gauge groups, and we have found a suitable way to give mass to the bottom quark.

| Field | $SU(3)$ | $SU(2)_L$ | $T^3$ | $\frac{Y}{2}$ | $Q = T^3 + \frac{Y}{2}$ |
|---|---|---|---|---|---|
| $g_\mu^a$ (gluons) | $\mathbf{8}$ | $\mathbf{1}$ | 0 | 0 | 0 |
| $(W_\mu^\pm, W_\mu^0)$ | $\mathbf{1}$ | $\mathbf{3}$ | $(\pm 1, 0)$ | 0 | $(\pm 1, 0)$ |
| $B_\mu^0$ | $\mathbf{1}$ | $\mathbf{1}$ | 0 | 0 | 0 |
| $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ | $\mathbf{3}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $\frac{1}{6}$ | $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ |
| $u_R$ | $\mathbf{3}$ | $\mathbf{1}$ | 0 | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $d_R$ | $\mathbf{3}$ | $\mathbf{1}$ | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $-\frac{1}{2}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |
| $e_R$ | $\mathbf{1}$ | $\mathbf{1}$ | 0 | $-1$ | $-1$ |
| $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $\frac{1}{2}$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $\Phi^c = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$ | $\mathbf{1}$ | $\mathbf{2}$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $-\frac{1}{2}$ | $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ |

How does this work for giving mass to the top quark? Obviously, $Q_L^\dagger \Phi t_R$ is not invariant. However, we have the freedom to create the conjugate representation of $\Phi$ which still transforms as a $\mathbf{2}$ under $SU(2)$ but switches sign under hypercharge: $\Phi^c = i\sigma^2\Phi^*$. This implies that $Y_{\Phi^c} = -\frac{1}{2}$ and

$$\Phi^c = \frac{1}{\sqrt{2}}\begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{135}$$

when restricted to just the real physical Higgs field expansion about the vev. Therefore, it becomes clear that $y_t Q_L^\dagger \Phi^c t_R + c.c.$ is now invariant since the $SU(2)$ invariance remains $\mathbf{2}\times\mathbf{2}\times\mathbf{1}\in\mathbf{1}$ and $U(1)_Y$ invariance follows from $Y_{Q_L^\dagger} + Y_{\Phi^c} + Y_{t_R} = -\frac{1}{6} - \frac{1}{2} + \frac{2}{3} = 0$. Similar to the $b$ quark one obtains an expression for the mass and Higgs boson interaction:

$$\begin{aligned} \Delta\mathcal{L} &= y_t Q_L^\dagger \Phi^c t_R + c.c. = \frac{y_t}{\sqrt{2}}(t_L^\dagger\; b_L^\dagger)\begin{pmatrix} v+h \\ 0 \end{pmatrix} t_R + c.c. &(136)\\ &= m_t(t_R^\dagger t_L + t_L^\dagger t_R)\left(1+\frac{h}{v}\right) = m_t\,\bar{t}t\left(1+\frac{h}{v}\right) &(137)\end{aligned}$$

where $m_t = y_t v/\sqrt{2}$ is the mass of the $t$ quark.

The mass of the charged leptons follows in the same manner, $y_e E_L^\dagger \Phi e_R + c.c.$, and interactions with the Higgs boson result.

In all cased the Feynman diagram for Higgs boson interactions with the fermions at leading order is

$$h\bar{f}f \; : \; i\frac{m_f}{v}. \tag{138}$$

We see from this discussion several important points.

- Higgs boson can give mass to all massive vector bosons ($W^{\pm}$, $Z$)
- Higgs boson can give mass to all fermions
- Higgs boson phenomenology (production, decays) entirely determined once its mass is known

## Standard Model Lagrangian

We now put the elements together to write the SM lagrangian:

$$\begin{aligned}\mathcal{L}_{SM} &= |D_\mu \Phi|^2 - \mu^2|\Phi|^2 - \lambda|\Phi|^4 \\ &-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu} - \frac{1}{4}G^{k,\mu\nu}G^k_{\mu\nu} \\ &-if^\dagger_{Li}\gamma^\mu D_\mu f_{Li} - if^\dagger_R\gamma^\mu D_\mu f_{Ri} \\ &-y^d_{ij}Q^\dagger_{Li}\Phi d_{Rj} - y^u_{ij}Q^\dagger_{Li}\Phi^c u_{Rj} - y^e_{ij}E^\dagger_{Li}\Phi e_{Rj} + h.c.\end{aligned} \tag{139}$$

where

$$D_\mu = \partial_\mu + ig\frac{\tau^a}{2}W^a_\mu + ig'\frac{Y}{2}B_\mu + ig_s\lambda^k G^k_\mu \tag{140}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{141}$$

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - gf^{abc}W^b_\mu W^c_\nu \tag{142}$$

$$G^k_{\mu\nu} = \partial_\mu G^k_\nu - \partial_\nu G^k_\mu - g_s f'^{klm}G^l_\mu G^m_\nu \tag{143}$$

Lagrangian is Lorentz invariant and gauge invariant and allows mass for all particles.

Observables in the Standard Model

The input parameters of the Standard Model are

$$g,\ g',\ g_s,\ y_t,\ y_b,\ y_e, \ldots \tag{144}$$

From these input parameters, and knowledge of the elementary particle identities, we can calculate all the observables, which include

$$\sigma(e^+e^- \to \mu^+\mu^-),\ \sigma(p\bar{p} \to W^+Z),\ \cdots \tag{145}$$

$$M_Z,\ M_W,\ m_t,\ m_e,\ \cdots \tag{146}$$

$$\Gamma(\mu \to e\bar{\nu}_e\nu_\mu),\ \Gamma(Z \to \tau\bar{\tau}),\ \Gamma(h \to \gamma\gamma),\ \cdots \tag{147}$$

$$A_{FB}(Z \to b\bar{b}),\ A_{LR}(Z \to e^+e^-) \tag{148}$$

There are an infinite number of observables that can be defined.

## Calculating Observables

To learn how to compute observables is the subject of other lectures.

We shall just give the answers to the calculation of the observables in terms of the lagrangian parameters.

To do that, it is convenient to define three derived parameters, $e$, $s_W$ and $c_W$, where

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}, \quad s_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad c_W = \frac{g}{\sqrt{g^2 + g'^2}} \tag{149}$$

These are defined merely for convenience, and not strictly necessary. Note also that

$$g = \frac{e}{s_W}, \quad g' = \frac{e}{c_W}, \quad \text{and } c_W^2 = 1 - s_W^2 \tag{150}$$

## Leading-order results for some key observables

$$\hat{M}_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2, \qquad \hat{M}_W^2 = \frac{1}{4}g^2 v^2, \quad m_f = y_f v \tag{151}$$

$$\left.\frac{i4\pi\hat{\alpha}}{q^2}\right|_{q^2\to 0} = \left.\frac{e^2}{q^2}\right|_{q^2\to 0} \quad (\gamma^* \to e^+e^- \text{ Compton scattering}) \tag{152}$$

$$\frac{\hat{G}_F}{\sqrt{2}} = \frac{g^2}{8\hat{M}_W^2}\left(= \frac{1}{2v^2}\right) \quad (\mu \text{ decay}) \tag{153}$$

$$\hat{\Gamma}(Z \to f\bar{f}) = \frac{N_c}{48\pi}\frac{g^2}{c_W^2}M_Z(g_V^2 + g_A^2), \text{ where } g_V = T_f^{(3)} - 2Q_f s_W^2, \quad g_A = T_f^{(3)},$$
$$\text{and } T_f^{(3)} = \pm\frac{1}{2} \text{ for up/down fermions} \tag{154}$$

$$\hat{A}_{LR}^e = \frac{\Gamma(Z \to e_L^+e_L^-) - \Gamma(Z \to e_R^+e_R^-)}{\Gamma(Z \to e_L^+e_L^-) + \Gamma(Z \to e_R^+e_R^-)} \equiv \frac{(1/2 - \hat{s}_{\text{eff}}^2)^2 - (\hat{s}_{\text{eff}}^2)^2}{(1/2 - \hat{s}_{\text{eff}}^2)^2 - (\hat{s}_{\text{eff}}^2)^2} \tag{155}$$

$\hat{s}_{\text{eff}}^2$ is often called $\sin^2\theta_\ell^{\text{eff}}$, and at leading order it is $\hat{s}^2 = s_W^2$!

Six key observables and their measured values

Let's simplify and exchange $\{g, g', v\}$ for $\{e, s, v\}$ and calculate

$$\hat{\alpha} = \frac{e^2}{4\pi} = 1/137.03599911(46) \tag{156}$$

$$\hat{G}_F = \frac{1}{\sqrt{2}v^2} = 1.16637(1) \times 10^{-5}\,\mathrm{GeV}^{-2} \tag{157}$$

$$\hat{M}_Z = \frac{ev}{2sc} = 91.1875 \pm 0.0021\,\mathrm{GeV} \tag{158}$$

$$\hat{M}_W = \frac{ev}{2s} = 80.385 \pm 0.015\,\mathrm{GeV} \tag{159}$$

$$\hat{s}^2_{\mathrm{eff}} = s^2_W = 0.23098 \pm 0.00027 \text{ (SLAC } A_{LR} \text{ only)} \tag{160}$$

$$\hat{\Gamma}_{l^+l^-} = \frac{v}{96\pi}\frac{e^3}{s^3c^3}\left[\left(-\frac{1}{2} + 2s^2_W\right)^2 + \frac{1}{4}\right] = 83.989 \pm 0.100\,\mathrm{MeV} \tag{161}$$

Left column is name of observable, center column is calculation in terms of lagrangian parameters, and right column is the experimental measurement.

Over-constrained system = ability to test the theory

In our analysis we have six observables

$$\{\hat{\alpha}, \quad \hat{G}_F, \quad \hat{M}_Z, \quad \hat{M}_W, \quad \hat{s}^2_{\mathrm{eff}}, \quad \hat{\Gamma}_{l^+l^-}\} \tag{162}$$

that are written in terms of just three parameters $\{e, s, v\}$.

Test by solving for $\{e, s, v\}$ using the first three very well measured observables:

$$e = \sqrt{4\pi\hat{\alpha}} = 0.30282 \tag{163}$$

$$v = \left(\frac{1}{\sqrt{2}\hat{G}_F}\right)^{1/2} = 246.22\,\mathrm{GeV} \tag{164}$$

$$s = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{M}_Z^2}}\right)^{1/2} = 0.46062 \tag{165}$$

Calculation of $M_W$ prediction in terms of $\hat{\alpha}$, $\hat{G}_F$ and $\hat{M}_Z$

Having expressed $e$, $v$ and $s$ in terms of observables $\hat{\alpha}$, $\hat{G}_F$ and $\hat{M}_Z$, we can now compute the remaining observables $M_W$, $\hat{s}^2_{eff}$ and $\hat{\Gamma}_{\ell\ell}$ in terms of $\hat{\alpha}$, $\hat{G}_F$ and $\hat{M}_Z$.

Examples:

$$\hat{M}_W(\text{prediction}) = \frac{ev}{2s} = \left( \frac{4\pi\hat{\alpha}/\sqrt{2}\hat{G}_F}{1 - \sqrt{1 - \frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{M}_Z^2}}} \right)^{1/2} = 80.939\,\text{GeV} \tag{166}$$

$$\hat{s}^2_{eff}(\text{prediction}) = s^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{M}_Z^2}} = 0.21215 \tag{167}$$

Tree-level predictions for $\hat{M}_W$, $\hat{s}^2_{\text{eff}}$ and $\Gamma_\ell$

Using the calculations of $\{e, s, v\}$ we find the following predictions for $\hat{M}_W$, $\hat{s}^2_{\text{eff}}$ and $\hat{\Gamma}_\ell$, their corresponding experimental measurement, and the number of $\sigma$ the prediction is off from the measurement

| Observable | Prediction | Measurement | Error |
|---|---|---|---|
| $\hat{M}_W$ | 80.939 GeV | $80.385 \pm 0.015$ GeV | $37\sigma$ |
| $\hat{s}^2_{\text{eff}}$ | 0.21215 | $0.23098 \pm 0.00027$ | $70\sigma$ |
| $\hat{\Gamma}_\ell$ | 84.841 MeV | $83.989 \pm 0.100$ MeV | $8.5\sigma$ |

Even if we put $\hat{\alpha} \to \alpha(M_Z) = 1/129.0$, we still find problems

| Observable | Prediction | Measurement | Error |
|---|---|---|---|
| $\hat{M}_W$ | 79.972 GeV | $80.385 \pm 0.015$ GeV | $27\sigma$ |
| $\hat{s}^2_{\text{eff}}$ | 0.23085 | $0.23098 \pm 0.00027$ | $0.5\sigma$ |
| $\hat{\Gamma}_\ell$ | 83.427 MeV | $83.989 \pm 0.100$ MeV | $5.6\sigma$ |

Does this mean the Standard Model is a very bad theory?

No, the problem is that we have pretended that the theory computation was perfect, whereas they were only tree-level results.

There are "radiative corrections" to all the observables: quantum loops that are higher-order in the couplings.

The measurements are so good that we need these higher-order calculations to test the theories compatibility.

The next slide shows a set of observables and their compatibilities with the Standard Model after doing state-of-the-art calculations. From LEP Electroweak Working Group.


This figure is from the LEP Electroweak Working Group.

## Discovery of the Higgs Boson

It was on announced on July 4, 2012 that a boson of mass 126 GeV was discovered. Overtime it is looking more and more like the Standard Model Higgs boson.

Discovery takes place mainly from two channels:

$$\sigma(gg \to H) \times Br(H \to \gamma\gamma) \ \text{ and,} \tag{168}$$

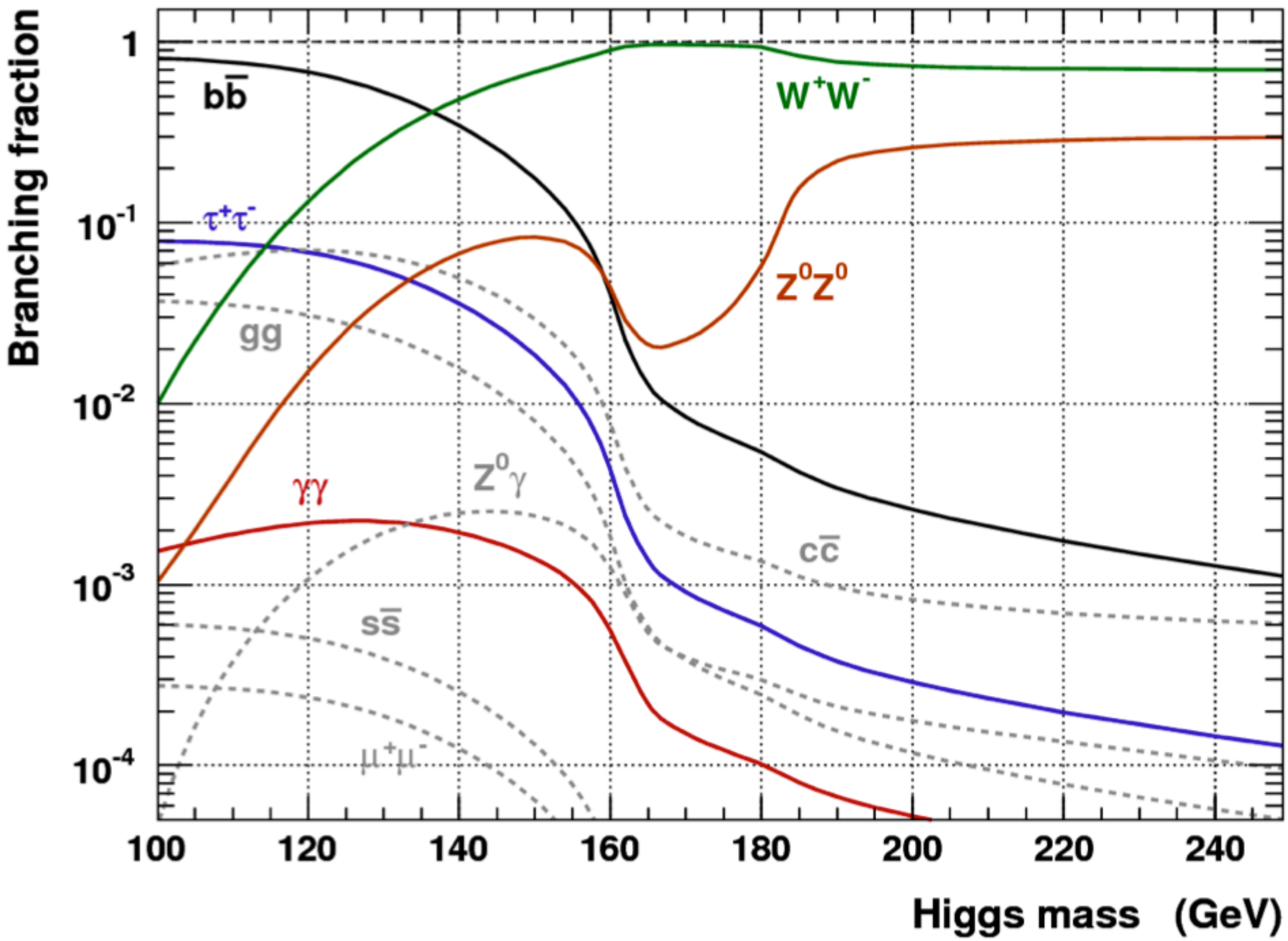
$$\sigma(gg \to H) \times Br(H \to ZZ^* \to \ell^+\ell^-\ell'^+\ell'^-) \tag{169}$$

The $gg \to H$ cross-section comes about from a loop of top quarks

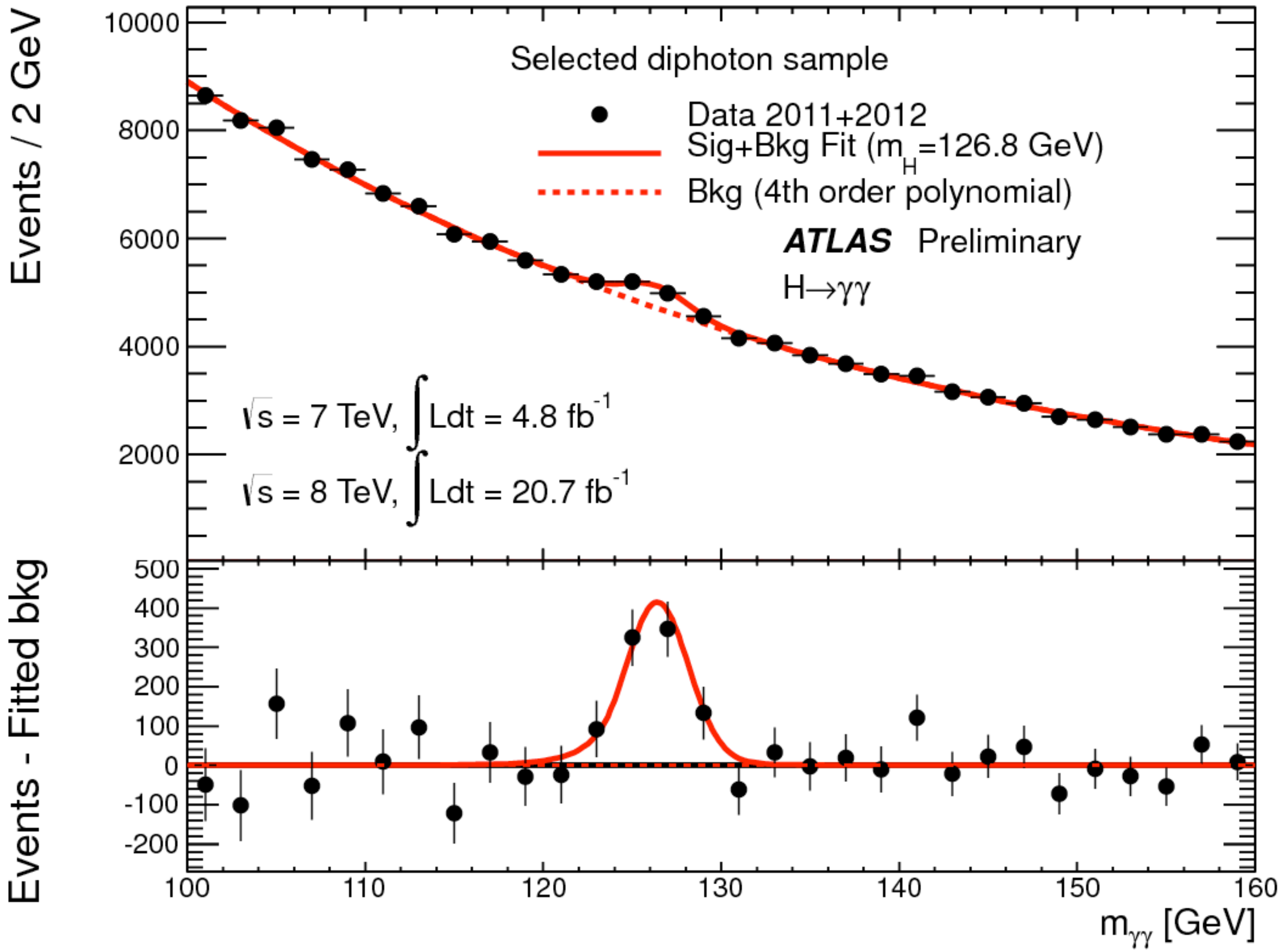
$$gg \to (t\bar{t})_{loop} \to H. \tag{170}$$

The $H \to \gamma\gamma$ decay comes about from loop of $W$ bosons and top quarks:

$$H \to \ (W^+W^-)_{loop} + (t\bar{t})_{loop} \ \to \gamma\gamma. \tag{171}$$




ATLAS (Peters 2013)

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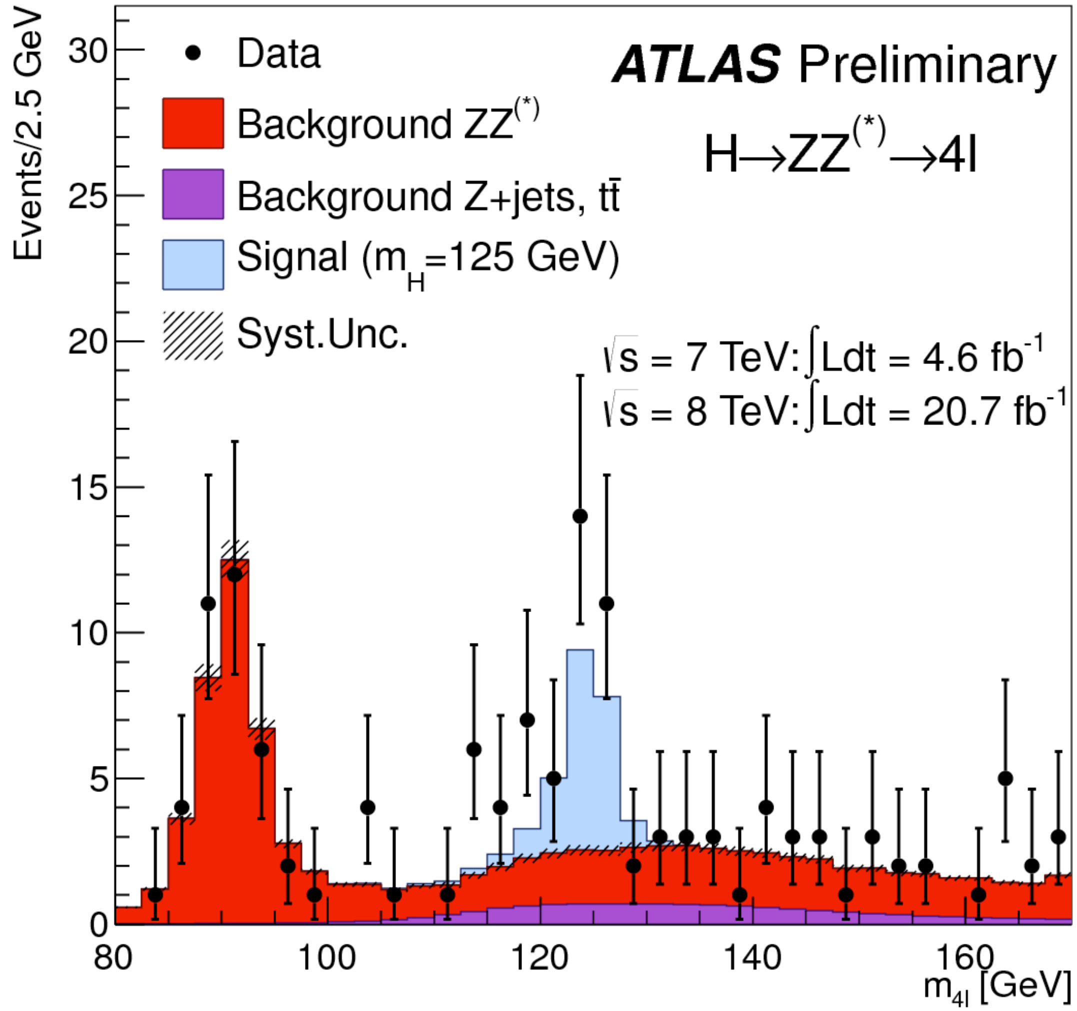


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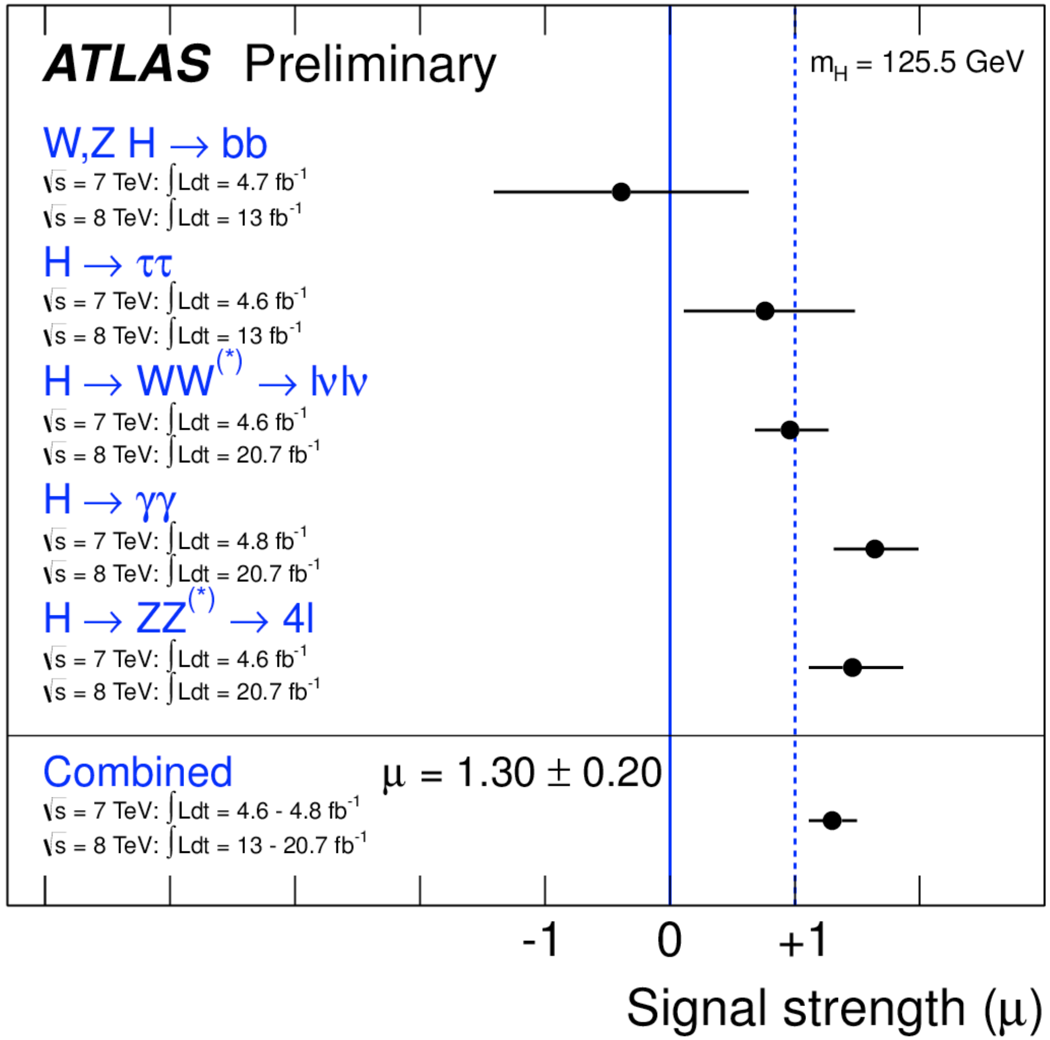


ATLAS (Peters 2013)

| Higgs Boson Decay | $\mu$ ($m_H$=125.5 GeV) |
|---|---|
| $VH \to Vbb$ | $-0.4 \pm 1.0$ |
| $H \to \tau\tau$ | $0.8 \pm 0.7$ |
| $H \to WW^{(*)}$ | $1.0 \pm 0.3$ |
| $H \to \gamma\gamma$ | $1.6 \pm 0.3$ |
| $H \to ZZ^{(*)}$ | $1.5 \pm 0.4$ |
| Combined | $1.30 \pm 0.20$ |

The discovered scalar is acting like a Higgs boson to within about 30% or so.

Many new physics ideas suggest that corrections on the order of 10% or less are to be expected. This could take LHC many years to get to that level, and may require a future $e^+e^-$ linear collider.

## Conclusion

Our discussion of the Standard Model has been focused on understanding

- Particle identities with respect to spacetime symmetries (Lorentz Group)

- Particle identities with respect to internal symmetries (gauge symmetries)

- Particle interactions

- Observables in which particles manifest their behavior and properties

- Higgs boson solution to the elementary particle mass problem

In other lectures you will learn more about how particles interact among the generations (flavor physics) and how they violate CP symmetry, and other special features derivable from the Standard Model lagrangian you have learned here.

And finally, you will learn that the Standard Model is incomplete!