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The Fractal Theory of Central Place Geometry: A Diophantine Analysis of Fractal Generators for Arbitrary Löschian Numbers

We develop material to determine whether or not an arbitrary number is Löschian; the procedure embodied in the theorems achieves the desired result more swiftly than do previous solutions to this problem. The correspondence between a partition of the central place lattice and a quadratic form permits the rapid determination of the lattice coordinates of an arbitrary Löschian number and of the exact shape of a single fractal generator used to form an entire central place hierarchy associated with an arbitrary Löschian number. Central place hierarchies may be generated geometrically using a single shape applied initially to a hexagon and subsequently, scaled appropriately, to resultant polygons. Fractional dimensions of arbitrary central place hierarchies, measuring their "space-filling" characteristics, follow naturally from this general procedure.

The Diophantine equation, $L = x^2 + xy + y^2$, generates the set of Löschian numbers which correspond to the points of a triangular lattice with integral coordinates (Dacey 1964, 1965). To generate Löschian numbers requires mere substitution for x and y; to determine whether or not an arbitrary number, L, is Löschian is more difficult, although the Diophantine equation is solvable (Mordell 1969). The material below displays a technique, in Theorems 1, 2, and 3, that appears computationally more efficient than do earlier methods of determining whether or not L is Löschian. It builds away from this Diophantine equation to find other quadratic expressions, in Theorem 4, that generate single lines of Löschian numbers in the triangular lattice, and uses these equations to cut across the Diophantine equation in order to determine the lattice coordinates of an arbitrary Löschian number. Finally, with the power of Theorem 4 displayed in attacking problems from classical central place theory, its fundamental role in determining the number of sides in, and exact shape of, a fractal generator

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Errors corrected on galley but not changed by Press corresponding to an arbitrary L is examined in detail. It is in this contemporary view of central place geometry, in which a single fractal generator applied to an hexagonal initiator (and, when scaled appropriately, to subsequent teragons [resultant polygons]) produces the entire hierarchy with correct cell-size and net orientation, that Diophantos meets the fractal.

1. A CHARACTERIZATION OF LÖSCHIAN NUMBERS

In 1975 John U. Marshall used number theory in an attempt to determine which positive integers L are Löschian; that is, which positive integers can be represented by the quadratic form $Q(x, y) = x^2 + xy + y^2$. His results involved examining the prime factorization of L and noting that each prime 2 or of the form $6k - 1$ must occur to an even power for L to be Löschian. Marshall did not prove that each such L is Löschian (it is), and in fact his results lead to a simpler algorithm if primes are examined modulo 3 and if certain simple congruence considerations are taken into account (Marshall 1975).

Definition 1. (Birkhoff and MacLane 1960). Two integers x and y are said to be congruent modulo a positive integer m if m is a divisor of $x - y$; i.e., x and y have the same remainder when divided by m. We write $x \equiv y (\text{mod } m)$.

Congruence modulo 3 is a particularly interesting case. For,

$$x \equiv y \equiv 2 (\text{mod } 3) \text{ implies } xy \equiv 1 (\text{mod } 3),$$

and

$$x \equiv y \equiv 1 (\text{mod } 3) \text{ implies } xy \equiv 1 (\text{mod } 3).$$

[Of course, $x \equiv 0(\text{mod } 3)$ implies $xy \equiv 0(\text{mod } 3)$.]

Note then that $x^2 + xy + y^2 \equiv 0$ or $1(\text{mod } 3)$. For,

$$x \equiv 0 (\text{mod } 3) \text{ implies } x^2 + xy + y^2 \equiv y^2 (\text{mod } 3).$$

$$y \equiv 0 (\text{mod } 3) \text{ implies } x^2 + xy + y^2 \equiv x^2 (\text{mod } 3).$$

If neither of these is true,

$$x^2 + xy + y^2 \equiv 0 (\text{mod } 3) \text{ if } x \equiv y (\text{mod } 3)$$

(either $1 + 1 + 1$ or $2 + 2 + 2$) and

$$x^2 + xy + y^2 \equiv 1 (\text{mod } 3) \text{ if } x \not\equiv y (\text{mod } 3)$$

$$(1 + 2 + 1).$$

Thus,

Lemma 1. *L is non-Löschian if* $L \equiv 2(\text{mod } 3)$.

For example, $L = 32759$ is immediately shown to be non-Löschian, without bothering with its prime factorization as $L = 17 \times 41 \times 47$.

Lemma 2. *If 2 occurs to an odd power in the prime factorization of L, then L is non-Löschian.*

Proof. Let $L = 2^n k$, k odd, and suppose $L = x^2 + xy + y^2$. Write $L = x^2 + xy + y^2 = d^2(s^2 + st + t^2)$, s and t having no factors in common. Then $s^2 + st + t^2$ is odd (s, t odd implies $s^2 + st + t^2$ is the sum of 3 odds; one of s, t even implies $s^2 + st + t^2$ is the sum of 2 evens and an odd). Thus n is even, because each time 2 occurs as a factor of d, it occurs twice as a factor of L.

In fact, the same condition will apply to any prime factor of L which is congruent to $2 \pmod 3$. But some number theory must be introduced first.

Definition 2. An integer x is said to be a quadratic residue modulo m if $x \equiv y^2 \pmod m$ for some y (Niven and Zuckerman 1960).

Example 1. Modulo 2, any integer is a quadratic residue ($0^2 = 0$, $1^2 = 1$). Only integers congruent to 0 or to $1 \pmod 3$ are quadratic residues modulo 3, since $0^2 = 0$, $1^2 = 1$, $2^2 = 4 \equiv 1 \pmod 3$. Integers congruent to $0, 1, -1 \pmod 5$ are quadratic residues modulo 5. In general, $(p-1)/2$ of the first $(p-1)$ positive integers are quadratic residues modulo p (Niven and Zuckerman 1960; McCoy 1965).

Definition 3. If p is an odd prime, the Legendre symbol $(a/p) = 1$ if a is a quadratic residue modulo p and $(a/p) = -1$ if a is a nonresidue modulo p. For example, $(1/3) = 1$ and $(2/3) = -1$. Clearly $(a/p) = (b/p)$ if $a \equiv b \pmod p$, $(a^2/p) = 1$ and $(ab/p) = (a/p)(b/p)$. We state without proof two lemmas from elementary number theory (Niven and Zuckerman 1960; McCoy 1965).

Lemma 3. $(-1/p) = (-1)^{(p-1)/2}$.

Lemma 4. *Quadratic Reciprocity Law (Gauss). If p, q are odd primes, then* $(p/q) = (q/p)(-1)^{(p-1)(q-1)/4}$.

With these lemmas, we can establish

Lemma 5. *If p is a prime* > 3, $(-3/p) = 1$ *if and only if* $p \equiv 1 \pmod 3$.

Proof.

$$
\begin{aligned}
(p/3) &= (3/p)(-1)^{(p-1)(3-1)/4} && \text{(Lemma 4)} \\
&= (3/p)(-1)^{(p-1)/2} \\
&= (3/p)(-1/p) && \text{(Lemma 3)} \\
&= (-3/p).
\end{aligned}
$$

But, if $p \equiv 1 \pmod 3$, then $(p/3) = (1/3) = 1$;

and if $p \equiv 2 \pmod 3$, then $(p/3) = (2/3) = -1$.

Of course, -3 is a quadratic residue modulo 3, since -3 is congruent to $0 \pmod 3$.

Definition 4. If $L = p^n k$, where k and p have no common factors (are relatively prime), then we say p^n exactly divides L and write $p^n \| L$.

REMARK. Suppose $L = x^2 + xy + y^2 = d^2(s^2 + st + t^2)$, s and t relatively prime. Then, if $p^{2n+1} \| L$, p is a divisor of $s^2 + st + t^2$, since p occurs as a divisor of d^2 in even powers.

THEOREM 1. *If* $p^{2n+1} \| L$, $p \equiv 2 (\mathrm{mod}\, 3)$, *then L is non-Löschian.*

Proof. By Lemma 2, we may assume p is odd. Suppose L were Löschian, say $L = x^2 + xy + y^2 = d^2(s^2 + st + t^2)$. Then p is a divisor of $s^2 + st + t^2$; say $s^2 + st + t^2 = kp$. But then $4s^2 + 4st + 4t^2 = 4kp$ or $(2s + t)^2 + 3t^2 \equiv 0 (\mathrm{mod}\, p)$ or $(2s + t)^2 \equiv -3t^2 (\mathrm{mod}\, p)$. Thus -3 is a quadratic residue mod p, which is impossible for $p \equiv 2(\mathrm{mod}\, 3)$. Q.E.D.

Marshall obtained the equation $(2s + t)^2 + 3t^2 = 4kp$ in a slightly different manner, involving the quadratic formula. Moreover, he stated that L was necessarily Löschian if all the prime factors occurring to an odd power were either 3 or congruent to $1(\mathrm{mod}\, 3)$, since -3 is a quadratic residue modulo such p. But the preceding argument only establishes the necessity of the condition of Theorem 1, not its sufficiency. Its sufficiency is established below, using the general theoretical framework in which the necessity was proved.

DEFINITION 5. If $Ax^2 + Bxy + Cy^2$ is a quadratic form, then $D = B^2 - 4AC$ is called its discriminant. (Note the similarity to the discriminant of high-school algebra.) (Birkhoff and MacLane 1960). Thus, the form $Q(x, y) = x^2 + xy + y^2$ has discriminant -3.

THEOREM 2. *If L is odd and square-free, then L is Löschian if and only if* -3 *is a quadratic residue of every prime factor of L.*

Proof. Mordell (1969, pp. 423–24) establishes that quadratic forms $Ax^2 + Bxy + Cy^2$ represent odd square-free integers L if and only if $B^2 - 4AC$ is a quadratic residue of every prime factor of L.

Thus, square-free integers L are Löschian if and only if they contain no factors congruent to $2(\mathrm{mod}\, 3)$. Combining Theorems 1 and 2, we obtain

THEOREM 3. *L is Löschian if and only if every prime factor which is congruent to* $2(\mathrm{mod}\, 3)$ *occurs to an even power in the prime factorization of L.*

Proof. The "only if" portion of the theorem is of course Theorem 1. So we must establish the "if" portion.

a) If L is square-free, this is established by Theorem 2.

b) If $L = d^2$, then $L = d^2 + (d \times 0) + 0^2$.

c) If $L = d^2k$, where k is square-free, then $k = x^2 + xy + y^2$ for some x, y by Theorem 2. Then $L = (dx)^2 + (dx)(dy) + (dy)^2$. Q.E.D.

Of course, Theorem 3 does not establish what the representation of a number L is or whether the representation is unique (material in the next section will do so). What it does is determine an easy method of testing whether a number is Löschian. It is easy to check the criterion if a prime factorization is known. But when L is non-Löschian, it can often be determined more quickly. For if $L \equiv 2(\mathrm{mod}\, 3)$, it is non-Löschian. Further, if $p^n \| L$, then L is non-Löschian if $L/p^n \equiv 2(\mathrm{mod}\, 3)$, since any number congruent to $2(\mathrm{mod}\, 3)$ must have some prime congruent to $2(\mathrm{mod}\, 3)$ occurring in its prime factorization.

EXAMPLE 2. $L = 2691$ is non-Löschian, since $3^2 \| 2691 = 9 \times 299$, and $299 \equiv 2(\mathrm{mod}\, 3)$. Marshall's method required factoring $2691 = 3^2 \times 13 \times 23$.

EXAMPLE 3. $L = 131336$ is not Löschian. Since $2|6, 4|36, 8|336$, but 16 is not a divisor of 1336, $2^3 \| L$. (Recall that N is divisible by 2^k if and only if its last k digits are.)

EXAMPLE 4. $L = 175$ is Löschian, since $L = 5^2 \times 7$, and 5 occurs to an even power in L. But $M = 125 = 5^3$ is not Löschian, nor is $N = 245 = 5 \times 7^2$.

EXAMPLE 5. $L = 85$ is non-Löschian, even though $85 \equiv 1 \pmod 3$, since $L = 5 \times 17$, and both 5 and 17 are congruent to $2 \pmod 3$.

EXAMPLE 6. Representation of a number is not unique. For example, $49 = 7^2 + (7 \times 0) + 0^2 = 5^2 + (5 \times 3) + 3^2$.

Thus, the theorems above give a method quicker than Marshall's in determining whether or not a given number is Löschian. In the worst cases, a complete factorization is necessary, but often only a few factors need be examined. The following facts about products of Löschian numbers follow immediately from Theorem 3.

COROLLARY 1. (Marshall 1975) *If L_1, L_2 are Löschian, so is L_1L_2.*

COROLLARY 2. (Marshall 1975) *If L_1 is Löschian, L_2 non-Löschian, then L_1L_2 is non-Löschian.*

COROLLARY 3. *If L_1 and L_2 are non-Löschian, L_1L_2 is Löschian if and only if whenever $p \equiv 2 \pmod 3$ occurs to an odd power in the prime factorization of L_1 and L_2, then it occurs to an odd power in the prime factorization of the other.* (Marshall observed that the product of two non-Löschian numbers might or might not be Löschian, but did not give the above conditions.)

The centrality of the prime 3 in determining whether or not a number is Löschian is now quite clear. Perhaps this is a consequence of the underlying triangular lattice of points.

2. DETERMINING A REPRESENTATION FOR ARBITRARY L

Suppose that a coordinate system is introduced into the triangular lattice and that the lattice points are coordinatized with reference to these axes. The manner in which coordinatization takes place is such that the positive y-axis is inclined at an angle of 60 degrees with respect to the positive x-axis. When each axis is endowed with the usual Euclidean metric, it follows that the line $x = y$ is inclined at an angle of 30 degrees to the positive x-axis. Figure 1 shows one-sixth of this lattice and shows labels for lattice points lying in the 1/12th-portion of the lattice lying between $x = y$ and the y-axis. The parallel rays emanating from lattice points on the y-axis serve to partition the set of lattice points; note that the scales along the y-axis and the line $x = y$ are different, so that (1, 1) is farther from (0, 0) than is (0, 1) (Figure 1).

In Figure 2, the coordinates of each lattice point in Figure 1 are used as x and y values in the Diophantine equation $x^2 + xy + y^2$ to generate the Löschian numbers corresponding to each lattice point. These generated values are displayed in Figure 2 below the distinguished points assigned coordinates in Figure 1.

The quadratic expressions below the parallel rays in Figure 2 are the ones that generate exactly the set of Löschian values along the corresponding rays (proven in Theorem 4, below). For example, the quadratic expression $3x^2 + 6x + 4$ generates the second row (above $x = y$) of Löschian numbers: if $x = 0$, the value of the quadratic expression is 4; if $x = 1$, the corresponding value is 13; if $x = 2$, the corresponding value is 28; and so forth. Clearly, these are identical to the values generated by the Diophantine equation $x^2 + xy + y^2$ along that line. Because we now have two distinct quadratic equations generating the same values, this new equation may be used to cut across the Diophantine equation in order to establish what the representation of a number L is and whether or not that representation is unique (as shown below).

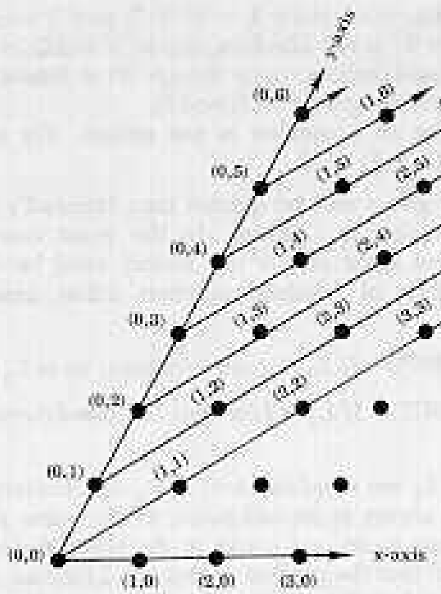

FIG. 1. Coordinatization of a Triangular Lattice. The y-axis is inclined at an angle of 60 degrees to the x-axis, and the line $y = x$ is inclined at an angle of 30 degrees to the y-axis.

THEOREM 4. *The Löschian numbers, L, lying ~~among~~ along any single one of the parallel rays in Figures 2 and 3 are generated by substituting non-negative integral values into a quadratic expression of the form* $3x^2 + 3bx + b^2$, *where* $b = 0, 1, \ldots,$ [illegible] *counts the number of units in the y-direction the ray is translated from* $x = y$.

Proof. Suppose $b = k$; that is, suppose that an arbitrary horizontal ray is translated k units from the line $y = x$. Let the ordered pair (x_k, y_k) denote the coordinates of a lattice point on this ray.

Using the Diophantine equation to generate the Löschian number, L_k, associated with (x_k, y_k), it follows that $L_k = x_k^2 + x_k y_k + y_k^2$. The point (x_k, y_k) lies on the line parallel to the y-axis denoted by $x = x_k$ (Figure 3). Label the intersection point of $x = x_k$ with $y = x$ as the ordered pair (x_0, y_0). Thus, $x_k = x_0$. Further, $x_0 = y_0$ since (x_0, y_0) lies on $y = x$. Finally, $y_k = y_0 + k = x_0 + k$, since the spacing between parallel rays is one unit. (This also follows from the spacing between lines parallel to the y-axis; for then $y_k = kx_k$.) Therefore, $L_k = x_0^2 + x_0(x_0 + k) + (x_0 + k)^2 = 3x_0^2 + 3kx_0 + k^2$. Since the choice for k was arbitrary, the result follows: $L = 3x^2 + 3bx + b^2$.

The following example shows how to use Theorem 4 to determine the b-value corresponding to an arbitrary Löschian number. Suppose that it has been determined that the number 397 is Löschian. Express 397 as $3x^2 + 3bx + b^2$, for some b. To do so, examine values of b from 0 to $[(397)^{1/2}] = 19$. When $397 = 3x^2 + 3bx + b^2$, it follows that any admissible value of b must be such that $397 - b^2$ is divisible by 3. Thus, Table 1 shows candidate values for b in the first column and eliminates from consideration, in the second column, those for which $(397 - b^2)$ is not divisible by 3. Of those not eliminated, only b-values that produce a quadratic equation such that $D^{1/2} = ((3b)^2 - 12b^2)^{1/2}$ as an integer are possible. These are shown in the third column of Table 1. In this case $b = 1$ produces the only possible solution. Thus, $L = 397$ lies along the parallel line

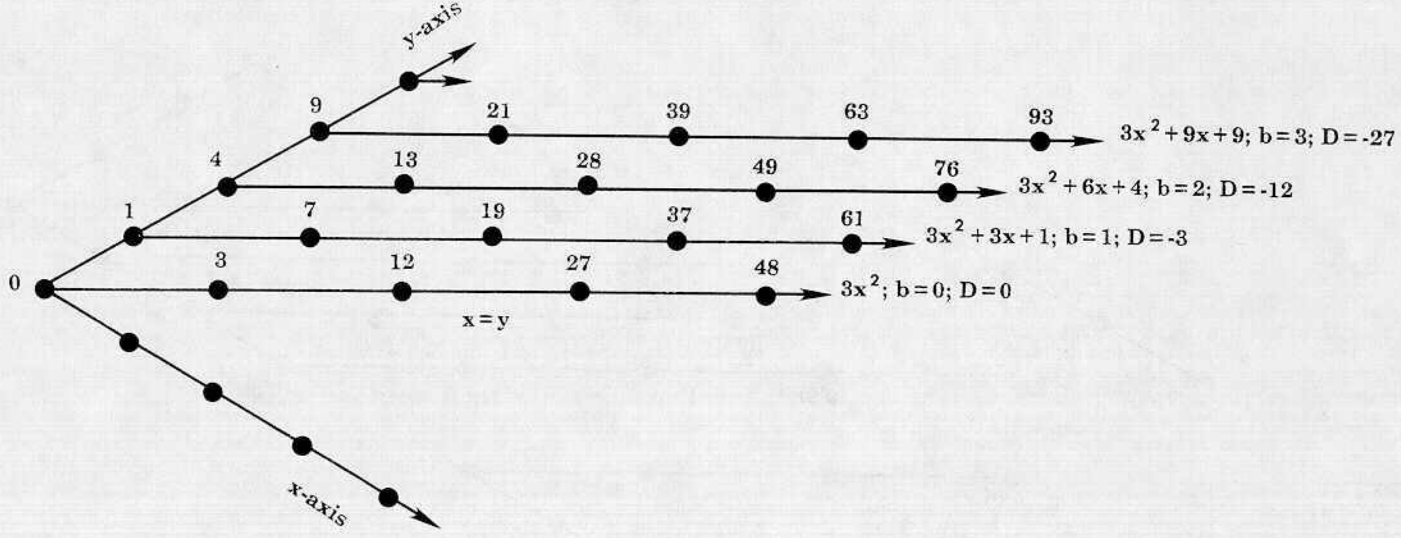

FIG. 2. Löschian Numbers Recorded below Lattice Points from Figure 1. Material associated with quadratic forms generating each line of values is recorded to the right of the line.

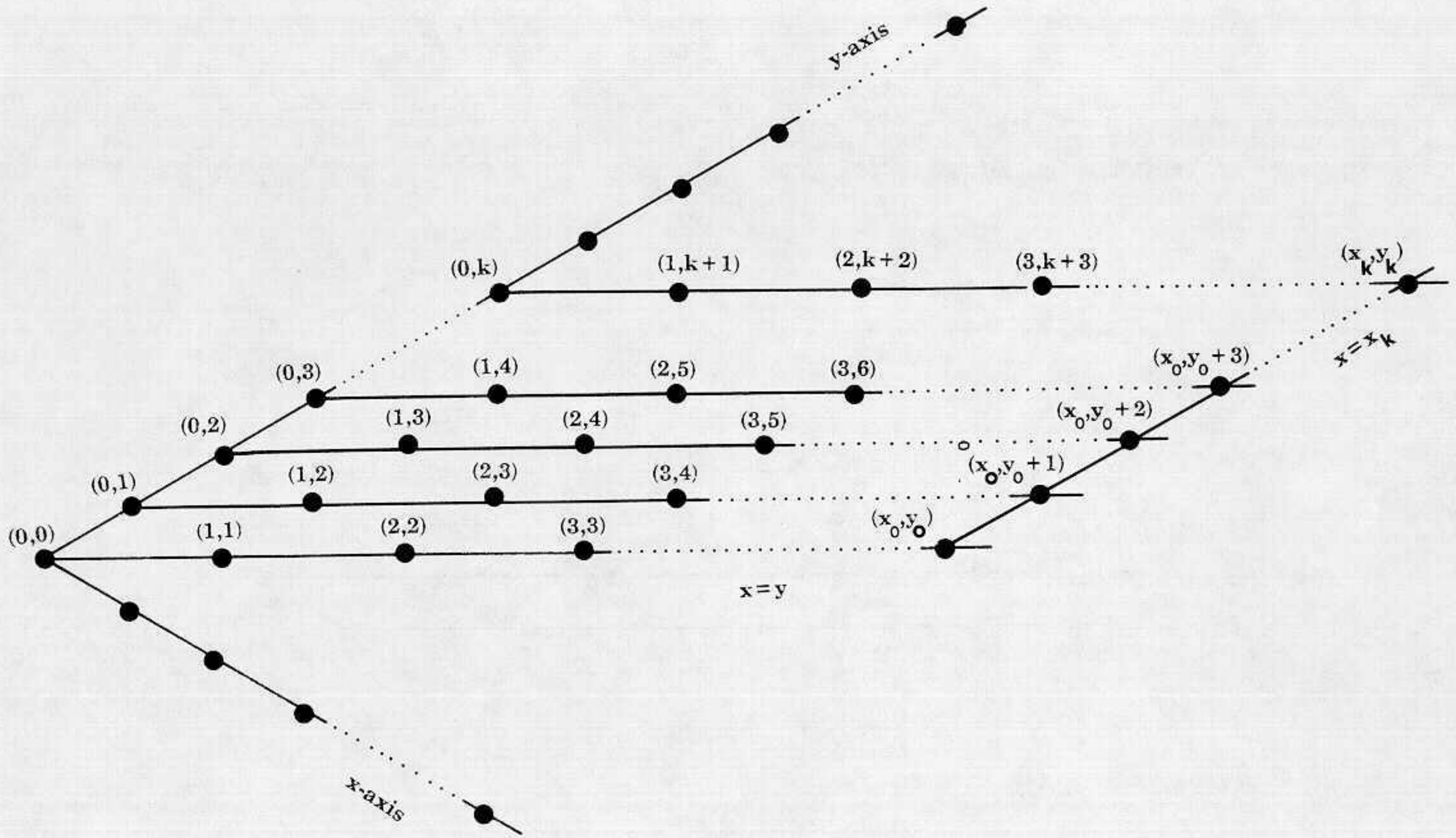

FIG. 3. Generalized Coordinatization of a Triangular Lattice in the 30-degree Wedge between the Line $y = x$ and the y-Axis (k rows are labeled); to accompany the proof of Theorem 4.

TABLE 1
Algorithm to Determine Lattice Position and Coordinates Applied to $L = 397$

b	Is $(397 - b^2)/3$ an integer?	Is $D^{1/2}$ an integer?
0	No	
1	Yes	$(4761)^{1/2} = 69$
2	Yes	$(4752)^{1/2}$; No
3	No	
4	Yes	$(4716)^{1/2}$; No
5	Yes	$(4689)^{1/2}$; No
6	Yes	$(4656)^{1/2}$; No
7	Yes	$(4617)^{1/2}$; No
8	Yes	$(4572)^{1/2}$; No
9	No	
10	Yes	$(4464)^{1/2}$; No
11	Yes	$(4401)^{1/2}$; No
12	No	
13	Yes	$(4257)^{1/2}$; No
14	Yes	$(4176)^{1/2}$; No
15	No	
16	Yes	$(3996)^{1/2}$; No
17	Yes	$(3897)^{1/2}$; No
18	No	
19	Yes	$(3681)^{1/2}$; No

displaced one unit in the y-direction from $x = y$. By Theorem 4, the equation $3x^2 + 3x + 1$ generates the entire set of Löschian numbers along this line.

To find the coordinates of the lattice point that gives rise to the Löschian number $L = 397$, solve simultaneously the system

$$397 = 3x^2 + 3x + 1$$

$$397 = x^2 + xy + y^2.$$

Thus, $x = 11$ and $y = 12$. There are no other lattice points that give rise to the Löschian number 397 because there are no other integral solutions in the third column of Table 1.

Consider instead the number $L = 49$. Table 2 shows candidate values for b in the first column and eliminates from consideration, in the second column, those for which $(49 - b^2)$ is not divisible by 3. Of those, two b-values, $b = 2$ and $b = 7$, produce integral values for $D^{1/2}$ in the third column. Thus, $L = 49$ lies along the parallel line displaced two units from $x = y$ as well as along the parallel line displaced 7 units from $x = y$. By Theorem 4, the equations $3x^2 + 6x + 4$ and $3x^2 + 21x + 49$ generate Löschian numbers long these lines.

TABLE 2
Algorithm to Determine Lattice Position and Coordinates Applied to $L = 49$

b	Is $(49 - b^2)/3$ an integer?	Is $D^{1/2}$ an integer?
0	No	
1	Yes	$(585)^{1/2}$; No
2	Yes	$(576)^{1/2} = 24$
3	No	
4	Yes	$(540)^{1/2}$; No
5	Yes	$(512)^{1/2}$; No
6	No	
7	Yes	$(441)^{1/2} = 21$

To find the coordinates of the lattice points that give rise to these Löschian numbers, solve simultaneously the systems of equations

$$\left.\begin{aligned} 49 &= 3x^2 + 6x + 4 \\ 49 &= x^2 + xy + y^2 \end{aligned}\right\} \text{ yielding } x = 3,\ y = 5$$

and

$$\left.\begin{aligned} 49 &= 3x^2 + 21x + 49 \\ 49 &= x^2 + xy + y^2 \end{aligned}\right\} \text{ yielding } x = 0,\ y = 7\,.$$

There are no other distinct lattice points (up to symmetric pairs such as $(7, 0)$, $(-7, 0)$) that give rise to the Löschian number 49 because there are no other integral solutions in the third column of Table 2.

The geometric characterization of the central place lattice as a set of integral lattice points lying along a set of lines parallel to $x = y$ permitted the algebraic determination, in Theorem 4, of a quadratic form, $3x^2 + 3bx + b^2$, that generated exactly the set of lattice points along any single line. When this Theorem was applied to the Diophantine equation $L = x^2 + xy + y^2$ it was a simple matter to determine the lattice coordinates of an arbitrary Löschian number and to assess whether or not these coordinates were unique.

3. FRACTAL GENERATION OF CENTRAL PLACE HIERARCHIES

Central place hierarchies may be generated geometrically using a single shape applied initially to a hexagon and subsequently, scaled appropriately, to resultant polygons. S. Arlinghaus has shown this previously (1985); to motivate the reader, a brief description of how the $K = 4$ hierarchy is generated is repeated here. Beyond that, Theorem 4 is used to show how the actual shape of a fractal generator, that will generate correctly an entire central place hierarchy with hexagonal cells of the correct size at each level and with correct orientation of the layers of hexagonal nets, may be determined for any Löschian number.

EXAMPLE. When the generator shaped like a half-hexagon is applied to each side of an initial hexagon (Figure 4a), alternately inside and outside that hexagon, and when the generated material is highlighted and the initial sides removed, a second polygonal figure (a first "teragon") emerges (Figure 4b). When the hexagons suggested by the boundary are completed in the natural way, these hexagonal cells are of the correct size, so that when they are superimposed on the initial hexagon (aligning O, U, and V), adjacent levels of the $K = 4$ central place hierarchy are generated (the orientation of the nets is also correct). When this procedure is repeated (Figure 4c), successive levels of the $K = 4$ central place hierarchy are also correctly generated (S. Arlinghaus 1985). When this iteration is carried out ad infinitum, Mandelbrot's formula for calculating fractional dimension yields a value of 1.5849625 for this "space-filling" process (Mandelbrot 1977, 1983).

Arlinghaus's 1985 paper explains how to generate central place hierarchies for $K = 3$, $K = 4$, $K = 7$, $K = 12$, $K = 13$, and $K = 19$; when the correct generator is selected, it yields, through simple application to an initial hexagon, and to subsequent teragons, the entire hierarchy associated with each Löschian number—with cells of the correct size relative to one another at each level of the

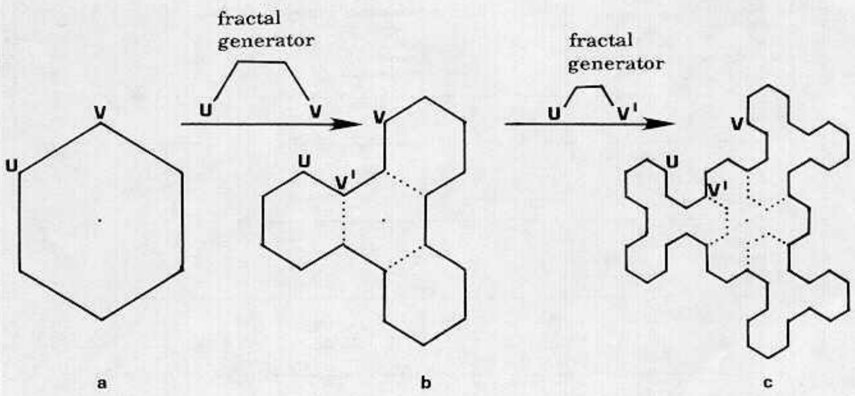

FIG. 4. Fractal Generator for the $K = 4$ Hierarchy: A. Initiator; B. First Teragon; and C. Second Teragon. The dotted lines within the teragons suggest self-similarity characteristics. When the teragons are stacked, at point 0, onto the initiator (through geometric translation), two layers in the $K = 4$ central place hierarchy are evident, as should be the iterative procedure to generate the entire hierarchy.

hierarchy, and with the correct orientation of superimposed hexagonal nets. This earlier paper contains a conjecture that K-values associated with lattice points lying along parallel lines in the underlying triangular central place lattice have generator shapes clearly related to each other. Further, it suggests that sets of three parallel lines within that lattice serve as a base for suggesting generator types for all other triples of parallel lines (S. Arlinghaus 1985). A line joining lattice points which all generate $K = 3$ type hierarchies (with respect to orientation of stacked nets of hexagonal cells) is type T_3; one of type $K = 4$ is type T_4; and, one of type $K = 7$ is of type T_7.

The point of such a conjecture is to investigate whether there is a general procedure which, when one is given an arbitrary Löschian number, will guarantee a correct selection of a single generator to be applied to an hexagonal initiator, yielding a central place hierarchy of stacked nets of correct cell sizes and orientation. Theorem 4, and the values of the discriminant for the associated quadratic forms, permit this.

4. THE APPLICATION OF THEOREM 4 TO CENTRAL PLACE FRACTALS

The Number of Sides in a Set of Fractal Generators

Figure 5 shows that the number of sides in a fractal generator increases directly with the size of the Löschian number. Examination of the column in Figure 5, "Number, n, of sides in a generator," reveals a numerical pattern, proceeding downward from the column top, of the form $(2, 3, 3)$, $(6, 7, 7)$, $(10, 11, 11)$, $(14, 15, 15)$, $(18, 19, 19), \ldots, (n, n + 1, n + 1)$. The first entry in each ordered triple lies along a line of Figure 5 in which L (and b) are congruent to $0 (\mathrm{mod}\, 3)$.

To find the number of sides, n, for a single generator that produces the entire central place hierarchy suited to L, with L congruent to $0 (\mathrm{mod}\, 3)$ along the y-axis, use the following procedure. Divide the b-value by 3; denote the quotient as $j = b/3$. (Dividing by 3 counts triples and moves the calculation to a T_3-line each time.) Then, $n = 2 + 4j$, as demonstrated in Table 3. (The appearance of the "2" in the first summand reflects the presence of a two-sided generator in the case $b = 0$; the "4" in the second summand comes from the fact that every fourth horizontal line is of the same T-type.)

Line number by b-value (Fig. 2): D congruent to:	Löschian number, L	Number, n, of sides in a fractal generator	generator shape
b = 0; D = 0	3	2	
b = 1; D = -3 D congruent to 1(mod 4)	7	2	
b = 2; D = -12 D congruent to 0(mod 4)	4	3	
b = 3; D = -27	9	6	
b = 4; D = -48 D congruent to 0(mod 4)	16	7	
b = 5; D = -75 D congruent to 1(mod 4)	25	7	
b = 6; D = -108	36	10	
b = 7; D = -147 D congruent to 1(mod 4)	49	11	
b = 8; D = -192 D congruent to 0(mod 4)	64	11	
b = 9; D = -243	81	14	
b = 10; D = -300 D congruent to 0(mod 4)	100	18	
b = 11; D = -363 D congruent to 1(mod 4)	121	18	
b = 12; D = -432	144	18	

FIG. 5: Generator Shapes for Selected Löschian Numbers, Primarily for Those along the y-Axis

TABLE 3
Numerical Determination of Generator Size for Selected Values

n	b	$j = b/3$	$n = 2 + 4 \times j$
2	0	0	$2 = 2 + 4 \times 0$
6	3	1	$6 = 2 + 4 \times 1$
10	6	2	$10 = 2 + 4 \times 2$
14	9	3	$14 = 2 + 4 \times 3$
18	12	4	$18 = 2 + 4 \times 4$
...	...	...	...

To find the number of sides, $n + 1$, for a single generator that produces the entire central place hierarchy suited to L, when L is congruent to $1(\text{mod}\,3)$ along the y-axis, use the following procedure. Apply the procedure of the preceding paragraph to the greatest Löschian number along the y-axis that is less than L and is congruent to $0(\text{mod}\,3)$. This produces a value, n. The number of sides in the generator suited to producing the hierarchy for L congruent to $1(\text{mod}\,3)$ is then $n + 1$. To understand how to draw this fractal generator of $(n + 1)$ sides, consider whether $j = b/3$ is odd or even.

1. Suppose j is even.

a. When the discriminant D, of the quadratic expression that generates the parallel line of lattice points including L, is congruent to $1(\text{mod}\,4)$, the corresponding fractal generator is asymmetric with respect to the perpendicular bisector of the initiator side.

b. When D is congruent to $0(\text{mod}\,4)$, the fractal generator is bilaterally symmetric with respect to the perpendicular bisector of the initiator side.

2. Suppose j is odd.

a. When D is congruent to $1(\text{mod}\,4)$, the fractal generator is bilaterally symmetric with respect to the perpendicular bisector of the initiator side.

b. When D is congruent to $0(\text{mod}\,4)$, the fractal generator is asymmetric with respect to the perpendicular bisector of the initiator side.

The following examples demonstrate this idea.

1. If $L = 36$, so that L is congruent to $0(\text{mod}\,3)$, the corresponding quadratic expression is $3x^2 + 18x + 36$; since $18 = 3b$, it follows that $b = 6$. Therefore, $j = 6/3 = 2$ so that $n = 2 + 4j = 10$. Thus, the central place fractal generator corresponding to $L = 36$ has 10 sides.

2. If $L = 64$, so that L is congruent to $1(\text{mod}\,3)$, the corresponding quadratic expression is $3x^2 + 24x + 64$. The greatest Löschian number along the y-axis that is less than $64 = 8^2$, and is a multiple of 3, is the number $36 = 6^2$, so that $b = 6$. Thus (using Table 3), the number of sides in a fractal generator for the $L = 64$ hierarchy is $10 + 1 = 11$. Further, because $j = 2$ is even and because the associated value of D is congruent to $0(\text{mod}\,4)$, the generator is bilaterally symmetric with respect to the initiator side.

Thus, it is possible to ascertain the number of sides for a central place fractal generator, for any value of L along the y-axis in Figure 2, using only numerical properties of the number L and associated algebraic forms. This procedure for determining quadratic expressions corresponding to arbitrary values of L is based on viewing L as being both of the form $x^2 + xy + y^2$ and $3x^2 + 3bx + b^2$.

The Shape of a Set of Fractal Generators

The previous subsection displayed empirical evidence in Figure 5 to suggest a general method for determining the number of sides in a fractal generator used to generate a central place hierarchy. Figure 5 may also be used to suggest the following construction in order to provide a systematic strategy for choosing the generator shape of the initial (y-axis) entry in each horizontal line of Figure 2.

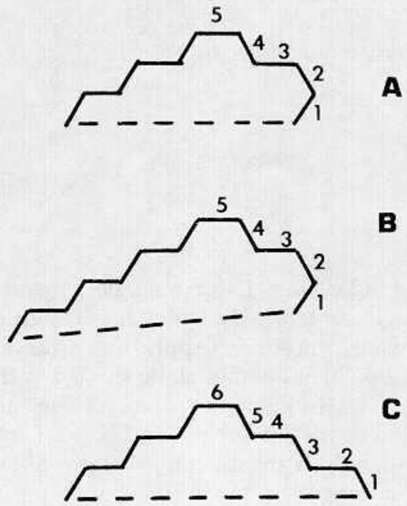

FIG. 6. Fractal Generators: A. $L = 36$; B. $L = 49$; and C. $L = 64$. Dashed lines show an edge of the hexagonal initiator; solid lines show the edge of the fractal generator. The top of the $L = 36$ generator is $[n/2] = [10/2] = 5$ edges from the initiator base; the top of the $L = 49$ generator is $[n/2] = [11/2] = 5$ edges from the initiator base; and the top of the $L = 64$ generator is $[(n/2) + 1] = [(11/2) + 1] = 6$ edges from the initiator base.

Construction 1

Given a Löschian number $L > 1$ (otherwise, the generator is a degenerate form), along the y-axis in Figure 2.

1. Suppose L is congruent to $0(\text{mod}\, 3)$ (Figure 6a).

Use a lattice of regular unit hexagons. Using the procedure of the previous section, determine n, the number of sides in a generator. Then, beginning at the right-hand side of an edge of a possible initiator, trace along three sides of a single unit hexagon (numbered 1, 2, 3 in Figure 6a), and continue moving toward the left, upward along steps in the hexagonal lattice, until about half the sides, $[n/2]$, have been exhausted. (The symbol, $[n/2]$, represents the greatest integer less than or equal to $n/2$.) Then, descend along hexagonal steps until the left-hand end-point of the initiator edge has been reached. Figure 6a displays this idea for $L = 36$, and $n = 10$ (values are from Figure 5).

2. Suppose L is congruent to $1(\text{mod}\, 3)$, and suppose that

a) j is even and that D is congruent to $1(\text{mod}\, 4)$, or, that j is odd and that D is congruent to $0(\text{mod}\, 4)$ (Figure 6b).

The procedure for shaping the generator is identical to that in (1) above, except that the last step on the left will touch the initiator edge. Thus, the final segment of the generator will be incident with a segment of an initiator edge, as shown in Figure 6b. Clearly, it is not symmetric with respect to the perpendicular bisector of the initiator edge. Here, $L = 49$, $n = 11$.

b) j is even and that D is congruent to $0(\text{mod}\, 4)$, or, that j is odd and that D is congruent to $1(\text{mod}\, 4)$ (Figure 6c).

The procedure for shaping the generator involves calculating the number, n, of sides required and arranging them symmetrically, with respect to the perpendicular bisector of an initiator edge, as in Figure 6c, which displays the case $L = 64$, $n = 11$. In this case the top step has the value $[(n/2) + 1]$.

Figure 7 shows the first teragons that result from applying each of these generators to an hexagonal initiator. When the small hexagons are filled in, in the natural manner within the boundary suggested by the first teragon, and when this

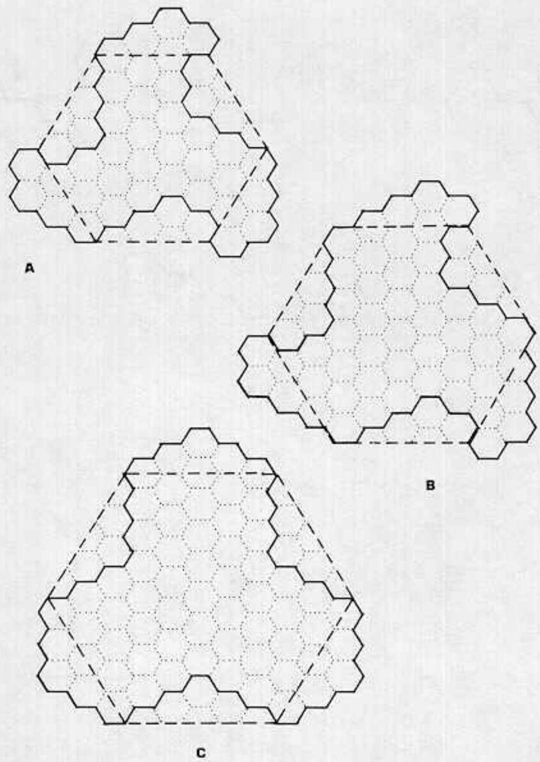

FIG. 7. First Teragons: A. $L = 36$; B. $L = 49$; and C. $L = 64$. Dashed lines show the hexagonal initiator; solid lines show the first teragon boundary; dotted lines show hexagonal cells to verify count.

filled-in teragon is superimposed on the hexagonal initiator, two adjacent levels with hexagonal nets of correct cell size and net orientation emerge for each Löschian number. Nets of successively finer cell size may be generated through iteration of the process.

Within Construction 1, central place nets derived from Löschian numbers congruent to $0 (\mathrm{mod}\, 3)$, or from Löschian numbers congruent to $1 (\mathrm{mod}\, 3)$ whose quadratic generating expression has discriminant congruent to $0 (\mathrm{mod}\, 4)$ for j even and congruent to $1 (\mathrm{mod}\, 4)$ for j odd, are such that the generator does not cross its corresponding initiator side, although it is anchored to it at endpoints. Those with Löschian numbers congruent to $1 (\mathrm{mod}\, 3)$ whose quadratic generating expression has discriminant congruent to $1 (\mathrm{mod}\, 4)$ for j even and congruent to $0 (\mathrm{mod}\, 4)$ for j odd, are such that the corresponding fractal generator crosses, or is incident with some portion of, its corresponding initiator side at positions other than the endpoints. This is abstractly parallel to the use of the discriminant of a quadratic equation to measure the number of crossings of the x-axis made by the parabola representing that quadratic equation.

5. THE NUMBER OF SIDES IN, AND THE SHAPE OF, A FRACTAL GENERATOR FOR ARBITRARY L

The previous material permits the determination of fractal generators for Löschian numbers L lying on the y-axis in Figure 2. Given a Löschian number L' such that L' does not lie on the y-axis in Figure 2, number-theoretic material exists to determine, swiftly, the quadratic equation generating the lattice points of the line

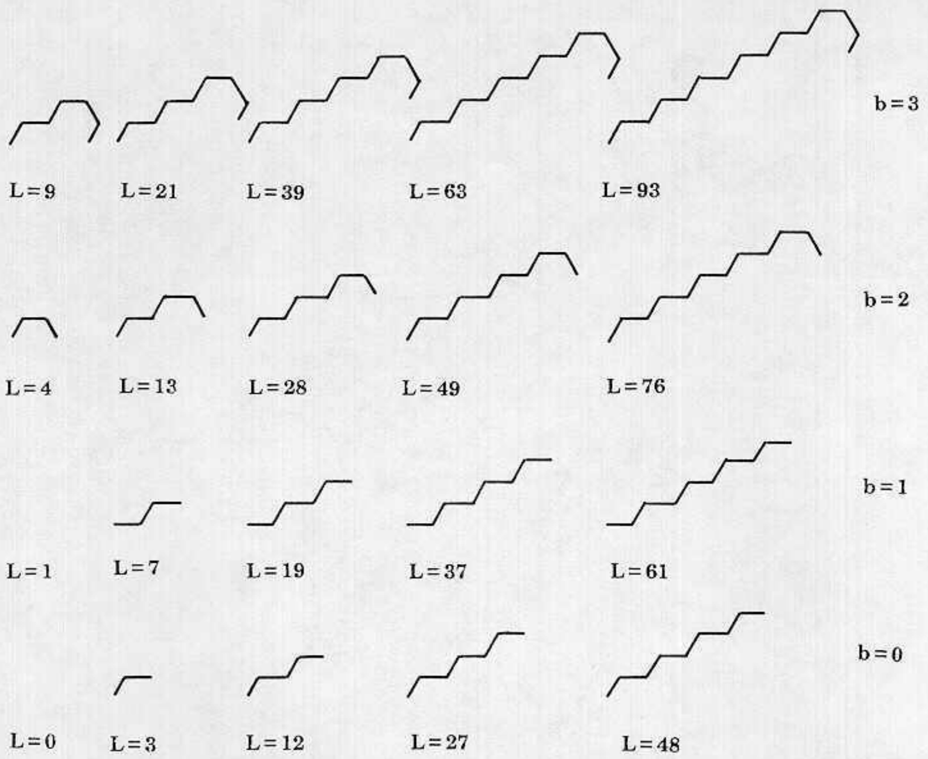

FIG. 8. Central Place Fractal Generator Shapes, L Arbitrary.

containing L', parallel to $y = x$ (section 2). The number of lattice-point positions separating L' from the y-axis entry along the appropriate line parallel to $y = x$ will determine completely the generator for the central place hierarchy associated with L', given that the number of sides in, and the shape of, this leading entry is known (section 4). Figure 8 shows generators for all of the positions labeled in Figure 2.

The pattern here is straightforward; add two edges, to form a step at the left of the generator, for each position that L' is removed from the leading entry on the y-axis along a line parallel to $y = x$. The basic generator structure remains the same across any line parallel to $y = x$, representing number-theoretic stability; the increase in number of sides corresponds to the idea of geometric translation of shape across a line parallel to $y = x$. Figure 9 shows generators applied to hexagonal initiators (producing first teragons), associated with the first three Löschian numbers lying along the line corresponding to $b = 3$; recursive use of each of these generators, scaled to fit teragon sides, would yield complete central place hierarchies for $L = 9$, $L = 21$, $L = 39$.

Thus, a single fractal generator that will produce an entire central place hierarchy when applied originally to an hexagonal initiator, and recursively to teragons, may be completely determined for an arbitrary Löschian number. What remains is to determine the position of the initiator relative to the lattice of unit hexagons.

6. THE POSITION OF THE INITIATOR

The partition of the triangular lattice into sets of lines parallel to $y = x$ also yields information concerning the position of the initiator relative to an underlying lattice of unit hexagons centered on the triangular lattice. Löschian numbers

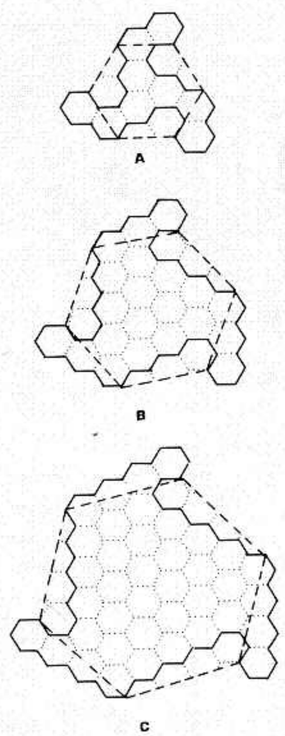

FIG. 9. First Teragons: A. $L = 9$; B. $L = 21$; and C. $L = 39$. Dashed lines show the hexagonal initiator; solid lines show the first teragon boundary; dotted lines show hexagonal cells to verify count.

congruent to 0(mod 3) produce hexagonal initiators and teragons centered at intersection points of the lattice of unit hexagons. Löschian numbers congruent to 1(mod 3) produce hexagonal initiators and teragons centered at the center of a unit hexagon. This fact, used together with the generator-shaping procedure permits rapid placement of central place initiators and generators on graph paper.

7. FRACTALAND—NEW DIMENSIONS IN CENTRAL PLACE GEOMETRY

Fractional Dimensions of Arbitrary Central Place Hierarchies

When any one of the types of generators described above is applied successively, first to the sides of a suitable hexagonal initiator, and then (appropriately scaled) to the sides of subsequent teragons, an entire central place hierarchy emerges as overlays of hexagonal nets oriented with respect to one another, and with relative scaling of cell size, suited to the positions and sizes dictated by the Löschian number. Previous work has shown that different central place hierarchies "fill" different amounts of space. The general procedure described above for determining generators for arbitrary Löschian numbers permits the rapid determination of the fractional dimension, F, of an arbitrary central place hierarchy (with infinite iteration in generator application) using Mandelbrot's formula:

$$F = (\log n)/\left(\log(L)^{1/2}\right). \qquad (1)$$

In (1), n is the number of sides in a generator and L is the Löschian number, as above (Mandelbrot 1983; S. Arlinghaus 1985). Table 4 shows fractional dimensions for the Löschian numbers displayed in Figure 2; it is organized according to the set of parallel lines of Figure 2, characterized by the b-values of those parallel lines. The initial (nontrivial) values for n were taken from Figure 5, and subsequent

TABLE 4
Fractional Dimensions of Central Place Hierarchies

b-value; Löschian Number L; Number, n, of sides in a generator
F, fractional dimension, $F = (\log n)/(\log(L)^{1/2})$

$b = 3$:	$L = 9$	$L = 21$	$L = 39$	$L = 63$	$L = 93$.......
	$n = 6$	$n = 8$	$n = 10$	$n = 12$	$n = 14$.......
	$F = 1.631$	$F = 1.367$	$F = 1.257$	$F = 1.200$	$F = 1.165$ ...
$b = 2$:	$L = 4$	$L = 13$	$L = 28$	$L = 49$	$L = 76$.......
	$n = 3$	$n = 5$	$n = 7$	$n = 9$	$n = 11$.......
	$F = 1.585$	$F = 1.255$	$F = 1.168$	$F = 1.129$	$F = 1.107$ ...
$b = 1$:	$L = 1$	$L = 7$	$L = 19$	$L = 37$	$L = 61$.......
		$n = 3$	$n = 5$	$n = 7$	$n = 9$ ....
		$F = 1.129$	$F = 1.093$	$F = 1.078$	$F = 1.069$ ...
$b = 0$:	$L = 0$	$L = 3$	$L = 12$	$L = 27$	$L = 48$.......
		$n = 2$	$n = 4$	$n = 6$	$n = 8$ ....
		$F = 1.262$	$F = 1.116$	$F = 1.087$	$F = 1.074$ ...

values for n, along any single horizontal line, were determined from Figure 8 by adding 2 to the previous value in the horizontal line. The values for F decrease from left to right along each line, reflecting the wider spacing between rival centers, on which hexagonal nets were based. A closer look at Table 4 brings to light some interesting numerical facts: the central place nets associated with $L = 49$ and $L = 7$ have the same fractional dimension, and therefore "fill" the same amount of space, although the pattern of net orientation is not the same, since the former is a T_4-type hierarchy, while the latter is a T_7-type hierarchy. This is obvious algebraically, since

$$(\log 7)/\left(\log(3)^{1/2}\right) = (2(\log 7))/\left(2\left(\log(3)^{1/2}\right)\right)$$
$$= \left(\log 7^2\right)/\left(\log\left(3^{1/2}\right)^2\right) = (\log 49)/\left(\log(9)^{1/2}\right), \quad (2)$$

but it is not obvious geometrically. The implications of this, when applied for example to transportation planning, might suggest the algebraic equivalence of two routing patterns (positioned along net-lines), in penetrating the underlying population, leaving open the geometric freedom to choose one or the other routing pattern based on outside factors such as degree of route curviness or efficiency of administration of route repairs (Arlinghaus and Nystuen 1985, 1986). The appropriateness of employing fractal geometry in urban planning is documented in the work of Batty (1985) and Batty and Longley (1985).

Further, within the set of T_3-type hierarchies, the higher the *b*-value, the larger the value for F; this reflects the increasing complexity in pattern arising from using generators with larger numbers of edges. A number of questions remain open.

1. Within a triple of consecutive *b*-values, such as $b = 0$, $b = 1$, $b = 2$, the relative rates of decrease in F across lines parallel to $y = x$ differ according to T-type hierarchy. What sort of formula might characterize these differences, and what are the implications of this formula for the geographic and economic components of spatial planning associated with each T-type hierarchy?

2. What, if they exist, are the lower bounds for the sequence of F-values across each line parallel to $y = x$? What, if it exists, is the greatest lower bound for the set of sequences associated with the set of all *b*-values, $0 \pmod 3$ [or $1 \pmod 3$ when D is congruent to $0 \pmod 4$, or $1 \pmod 3$ when D is congruent to $1 \pmod 4$ for each of j even and j odd]?

3. Reading sequences of column entries within, say, T_3-type hierarchies, such as 1.262, 1.367, ..., what is the upper bound of this sequence? Again, one can ask a set of related questions, involving least upper bounds and the like, with an eye

toward using elements of the theory of measure and integration with central place hierarchies.

4. With bounding criteria that reflect the capability of nets to fill space, extend the transportation planning idea referred to above (or other suitable ideas) across the entire spectrum (an infinite set) of possible hierarchies.

8. DIRECTIONS FOR FURTHER RESEARCH

Given an arbitrary Löschian number, L, the discussion above provides a systematic strategy for characterizing a single generator that, when applied recursively to an hexagonal initiator and higher level teragons, generates the entire geometry of the central place hierarchy associated with L. It does so by presenting methods to count the number of sides in the generator, and to shape the arrangement of these sides to yield appropriate results. Information concerning initiator position is a corollary to these procedures permitting complete, and relatively easy, determination of generator shape. Once a generator has been determined, then Mandelbrot's formula for fractional dimension applies easily to any central place hierarchy.

These procedures, however, are all "existence" procedures. The issue of uniqueness may be dealt with at two levels. First, any Löschian number that can be derived from multiple lattice points might have more than one fractal generator emerge using the procedure above. Existing results, used in conjunction with Construction 1 and associated material, completely solve this issue. Second, there might be other constructions that yield different schema for producing central-place fractal generators. This notion is mainly of theoretical interest and is unsolved, as are other open questions surrounding patterns in sequences of fractional dimension. For the purpose of empirical application, however, the theoretical questions surrounding the creation of an arbitrary central place hierarchy using fractals, and of determining its fractional dimension, are fully resolved.

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