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AN APPLICATION OF GRAPHICAL ANALYSIS TO SEMIDESERT SOILS

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ABSTRACT. Geometric models often contain elements of complex geographic processes. The shape of the curve selected to model a specific process can be crucial in determining interpretations. To illustrate this relationship we use the graphical analysis of chaos theory as a geometric model and apply it to the complex process of semidesert-soil production. This application suggests strategies for choosing when human intervention to improve soil production might be appropriate.

ASYMMETRIC curves are useful as models of various real-world settings. All too often, however, the detail of the exact curve shape is of little concern in constructing these models, independent of what they might represent. In this article, we use graphical analysis (Feigenbaum 1980) to demonstrate that exact curve shape is important and that small variations in curve shape can make big differences in the geometric dynamics and the consequent interpretation of a geometric model. The case study is the application of graphical analysis to the study of semidesert soils.

Curve shape can determine how matter and energy spread within a system. The different surface shapes of the tiger and the leopard may determine, for example, that the tiger will have stripes and that the leopard will have spots. When reactions cause diffusion rates of pigmentation in animal coats to vary, different spatial patterns, such as stripes or spots, emerge as standing waves of translation in the underlying surface (Xu, Vest, and Murray 1983; Murray 1988). Seiches, which are standing waves of oscillation rather than of translation, might form stripes on lakes and bays (Mosetti 1982). The position and periods of seiches depend on lake depth and coastline shape. Coastline shape might even cause reactions that influence the diffusion pattern of nearby urban traffic. The position of standing waves of all sorts is controlled by boundary shape as well as by the spread of matter or energy. In sum, shape matters.

Graphical analysis, a tool from mathematical chaos theory, offers a way to understand how small geometric changes can force large geometric differences. Such analysis rests on an ordering of events, in which the output of one stage serves as the input for the next stage. Because graphical analysis cleverly uses the line $y = x$ as a surrogate axis, it produces an easy-to-

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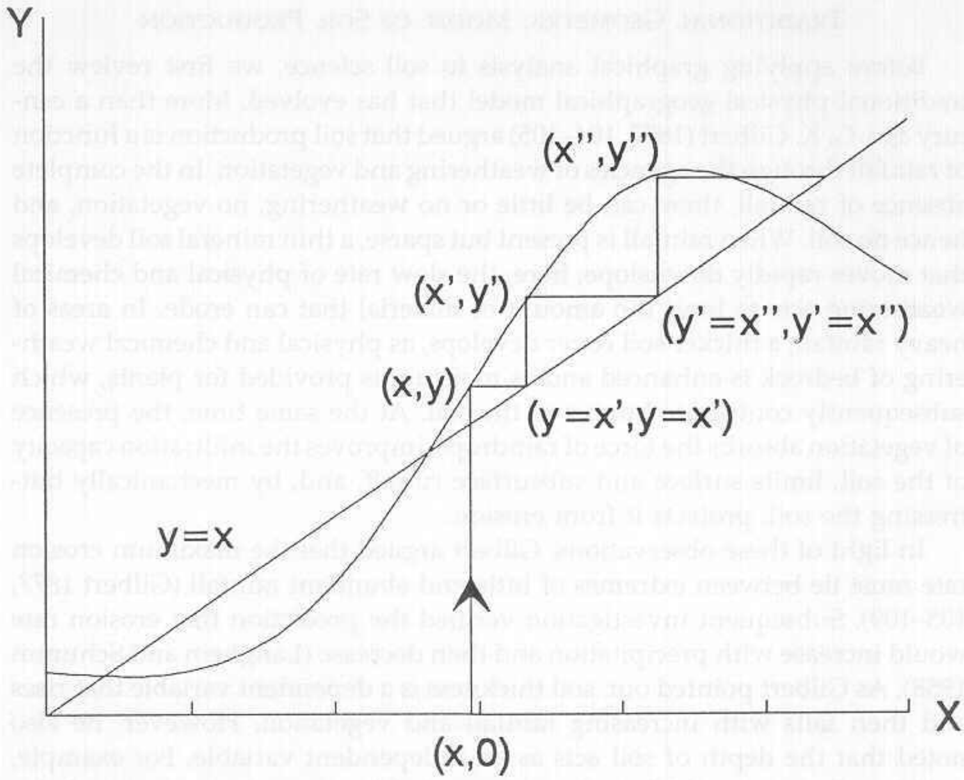

FIG. 1—Simple graphical analysis, using the line $y = x$, in which the output of one stage becomes the input of the next; the input of x leads to an output of y, which is then used as input that produces output y', and so forth.

understand picture of how geometric feedback operates in a system (Feigenbaum 1980; Hofstadter 1981).

One might consider, for example, a geometric model that is S-shaped and that rises to a peak and then descends (Fig. 1). The first geometric input is the x coordinate of $(x, \emptyset)$; the geometric output derived from this input is y, the second coordinate of (x, y). Set $y = x'$; the value remains the same, but the role is shifted from second to first coordinate. Use x' as the next input. Yet another output, y', results, which in turn is used as the next input, x'', forming yet another output, y'' (Feigenbaum 1980; Gleick 1987; Devaney and Keen 1989). The successive use of an output as the next input represents a kind of geometric feedback. The resultant pattern of directed lines, bouncing back and forth between points on the curve and points on $y = x$, is referred to as the orbit of the original seed value, x, with respect to the given curve. When different seeds are chosen, different orbits emerge; however, all are sequences of points linked continuously in space and ordered in a one-way fashion in time. The line $y = x$ compresses the computations. As the relative position of the curve and the line $y = x$ shifts, so does the pattern of geometric feedback (Feigenbaum 1980).

Traditional Geometric Model of Soil Production

Before applying graphical analysis to soil science, we first review the traditional physical geographical model that has evolved. More than a century ago, G. K. Gilbert (1877, 104–105) argued that soil production is a function of rainfall through the agencies of weathering and vegetation. In the complete absence of rainfall, there can be little or no weathering, no vegetation, and hence no soil. When rainfall is present but sparse, a thin mineral soil develops that moves rapidly downslope; here, the slow rate of physical and chemical weathering acts to limit the amount of material that can erode. In areas of heavy rainfall, a thicker soil cover develops, as physical and chemical weathering of bedrock is enhanced and as moisture is provided for plants, which subsequently contribute humus to the soil. At the same time, the presence of vegetation absorbs the force of raindrops, improves the infiltration capacity of the soil, limits surface and subsurface runoff, and, by mechanically buttressing the soil, protects it from erosion.

In light of these observations, Gilbert argued that the maximum erosion rate must lie between extremes of little and abundant rainfall (Gilbert 1877, 105–109). Subsequent investigation verified the prediction that erosion rate would increase with precipitation and then decrease (Langbein and Schumm 1958). As Gilbert pointed out, soil thickness is a dependent variable that rises and then falls with increasing rainfall and vegetation. However, he also noted that the depth of soil acts as an independent variable. For example, as thin soils become thicker, the soils can hold more water and provide for a denser cover of vegetation that will inhibit erosion by surface flow. The added and more constant soil moisture will also contribute to further weathering by frost and by solution, which causes the soil to thicken. When the soil has become sufficiently thick, it actually serves to protect the bedrock from frost action and eventually from solution. Furthermore, a deep soil acts as a reservoir for rainwater and tends to equalize the flow and hence to decrease the erosive power of streams at the base of soil-covered slopes (Gilbert 1877, 105). Hence thin soils will slowly thicken until the weathering rate matches the erosion rate.

Mindful of this relationship, M. A. Carson and M. J. Kirkby (1972, 105) plotted bedrock weathering rate against soil thickness, instead of erosion rate against rainfall or vegetation. The result was a graph that reflected the feedback relationships between soil depth and weathering rate (Fig. 2). The weathering rate, schematized on the graph as the rate of production of new soil, is a function of soil thickness. The alternative analysis does not capture this feedback; an increased rate of erosion or weathering cannot increase rainfall or vegetation.

As rock is weathered from the parent material, it is moved downslope. If the rate of weathering falls short of the rate of downslope movement, a soil thins; if the weathering rate exceeds the transport rate, the soil thickens. A logical extension of Gilbert's argument suggests that the maximum rate

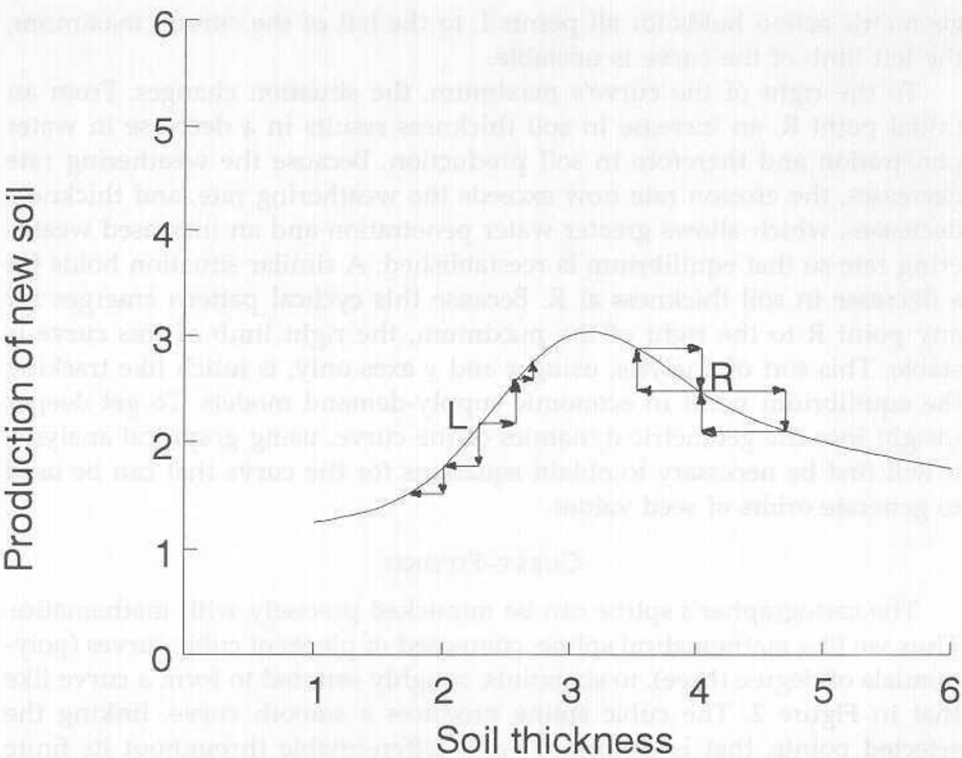

FIG. 2—Feedback relationships between soil depth and weathering rate. Units of measure on the *x* and *y* axes are abstract. *Source:* Carson and Kirkby 1972, 105.

of new soil production occurs at a point between very thin and very thick soils. Conceivably such a local maximum might have been modeled by a simple quadratic equation, graphed as a concave-down parabola (Langbein and Leopold 1968). However, the physical reality, in which the rate of change of soil production rises gradually with increased soil thickness before reaching a maximum, dictated a curve of the sort depicted in Figure 2.

The simple geometrical characteristics of this model reflect the relative stability of the rate of production of new soil as a function of soil thickness, or as feedback. For a point L on the curve located to the left of the maximum, an increase in soil thickness brings about a corresponding increase in soil production, because water penetration is not slowed to an extent sufficient to inhibit weathering. This further increase in soil thickness results in still more new soil, and so on, an action represented on the curve by the staircase approaching the curve's maximum. Similarly, a decrease in soil production, perhaps from erosion or from overgrazing, from L results in a thinner soil that sustains less biological and chemical activity, which leads to still further decrease in soil production and an even thinner soil, and so on, an action represented on the curve by a staircase descending from L. Because this geometric action holds for all points L to the left of the curve's maximum, the left limb of the curve is unstable.

To the right of the curve's maximum, the situation changes. From an initial point R, an increase in soil thickness results in a decrease in water penetration and therefore in soil production. Because the weathering rate decreases, the erosion rate now exceeds the weathering rate, and thickness decreases, which allows greater water penetration and an increased weathering rate so that equilibrium is reestablished. A similar situation holds for a decrease in soil thickness at R. Because this cyclical pattern emerges for any point R to the right of the maximum, the right limb of this curve is stable. This sort of analysis, using x and y axes only, is much like tracking the equilibrium point in economic supply-demand models. To get deeper insight into the geometric dynamics of the curve, using graphical analysis, it will first be necessary to obtain equations for the curve that can be used to generate orbits of seed values.

Curve-Fitting

The cartographer's spline can be mimicked precisely with mathematics. Thus we fit a mathematical spline, composed of pieces of cubic curves (polynomials of degree three), to six points, roughly selected to form a curve like that in Figure 2. The cubic spline produces a smooth curve, linking the selected points, that is continuous and differentiable throughout its finite length, over a domain bounded between the x-values of 1 and 6 (Fig. 3a).

When the set of six points is changed, raising the maximum by one unit and also the three succeeding points, a completely different curve results (Fig. 3b). Even though the first two distinguished points are the same in figures 3a and 3b, the pieces of cubic curves that fit between them are different, to ensure the smoothness of the entire bounded curve—the interpolation is a global one. It requires inverting a matrix of entries derived from the initial set of data points, which can be achieved quickly by using any of a number of commercially available spreadsheets as long as the set of data points is relatively small. Further variation in the distinguished data points produces still other curves from the infinite set of curves with the desired stability characteristics for the production of new soil (Figs. 3c and 3d). The details of spline interpolation, cubic and otherwise, are available in elementary linear-algebra textbooks. What is important here is that all curves in Figure 3 have the same general shape as does the real-world curve of Figure 2.

Graphical Analysis of Semidesert Soil Production

We next consider how graphical analysis might permit deeper geometric distinctions to be made and interpret them in the context of soil science, specifically, an analysis of semidesert soils. These soils are typically quite thin and highly vulnerable to changes in biomass; they are soils in a desert with some rainfall, some vegetation, and hence some soil thickness, which

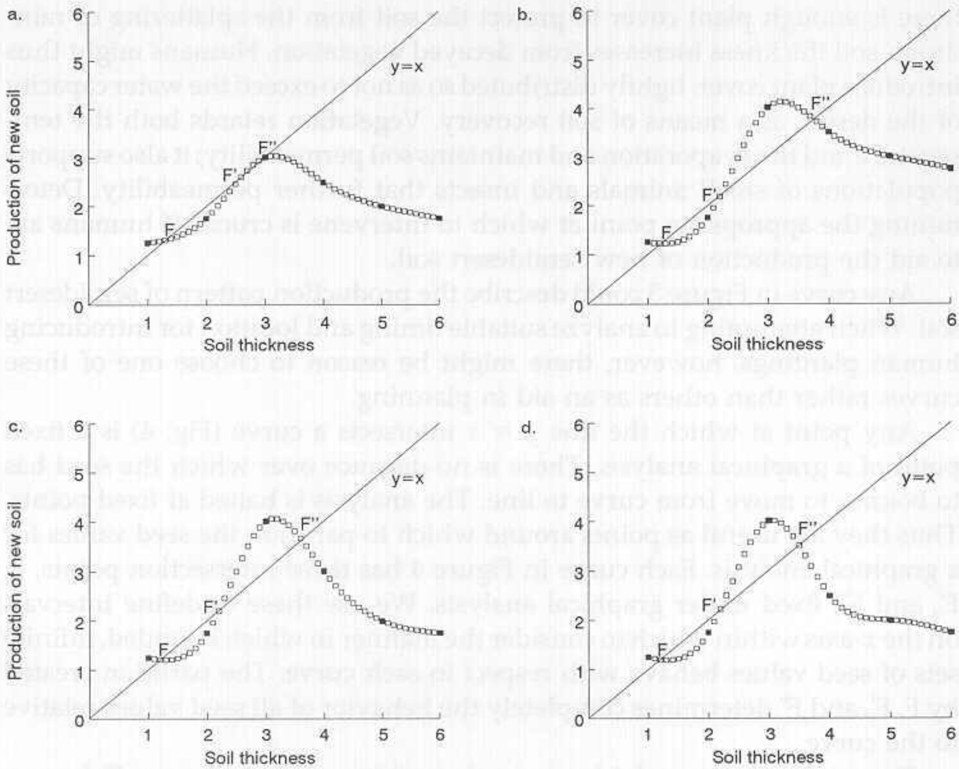

FIG. 3—Curve fitting by cubic spline interpolation. Four similar curves, a, b, c, d, displayed as discrete curves.

reflect equilibrium between the erosion rate and the soil-production rate. Increase or decrease in biomass is associated with soil thickness, so that the organic semidesert soil system feeds on itself. In its natural cycle, feedback is crucial in maintaining and promoting the health of the soil.

Semidesert soil and vegetation are destroyed naturally by various combinations of the removal of small nutrient particles by wind and water; the hardening of clayey soil by the filling of pores from whatever source, including raindrop splatter; and alkalinization of the soil, resulting from the rapid evaporation of the water that leaves mineral deposits to seal the soil. Together, these factors change the pH of the soil so that plants cannot grow successfully. The result is a patchwork quilt of semidesert soils, some patches of which support biomass and others of which do not, like leopard spots on the sandy skin of the desert.

Desertification along desert edges can come from human abuse of this already fragile system. The biological potential is subjected to a downward spiral from which the soil cannot recover; collapse of the soil, as a system supporting biological activity, becomes imminent. The situation can be reversed, if the destruction of this potential has not progressed too far. When there is enough plant cover to protect the soil from the splattering of raindrops, soil thickness increases from decayed vegetation. Humans might thus introduce plant cover, lightly distributed so as not to exceed the water capacity of the desert, as a means of soil recovery. Vegetation retards both the temperature and the evaporation and maintains soil permeability; it also supports populations of small animals and insects that further permeability. Determining the appropriate point at which to intervene is crucial if humans are to aid the production of new semidesert soil.

Any curve in Figure 3 could describe the production pattern of semidesert soil. When attempting to analyze suitable timing and location for introducing human plantings, however, there might be reason to choose one of these curves rather than others as an aid in planning.

Any point at which the line $y = x$ intersects a curve (Fig. 4) is a fixed point of a graphical analysis. There is no distance over which the seed has to bounce to move from curve to line. The analysis is halted at fixed points. Thus they are useful as points around which to partition the seed values for a graphical analysis. Each curve in Figure 4 has three intersection points, F, F′, and F″, fixed under graphical analysis. We use these to define intervals on the x-axis within which to consider the manner in which bounded, infinite sets of seed values behave with respect to each curve. The partition created by F, F′, and F″ determines completely the behavior of all seed values relative to the curve.

Figures 4a, 4b, 4c, and 4d suggest that orbits are pushed from F′ for any seed value on either side of F′ (between F and F″). Thus the relative locations of the fixed points, together with general orbit dynamics associated with sets of seed values, might be used to make good choices from otherwise apparently equivalent models or curves.

In the specific case of the semidesert soils, four models (Figs. 4a, 4b, 4c, and 4d) indicate that human planting is appropriate at a soil thickness immediately to the left of F′ to enlarge the size of the interval between F′ and F″. Such enlargement is important; within this interval an increase in the x-coordinate soil thickness results in an increase in the y-coordinate production of new soil. Orbits to the right of F′ are staircases that exhibit various geometric dynamics as they approach F″. Orbits to the left of F′ are descending graphical staircases; a decrease in soil thickness results in lowering the production of new soil, because the soil is stripped of its own capacity to build itself. Thus intervention slightly to the left of F′ would slide this fixed point to the left, which reduces the unwanted dynamic in the interval to the left of F′ and increases the desirable dynamic to the right of F′. Shape and size both become factors.

The structure of the orbits associated with specific curves suggests the extent to which control can be retained over the geometric dynamics of such an insertion of plants. The most geometric control appears in figures 4a and 4b where the general pattern and direction of feedback are clear. Least control

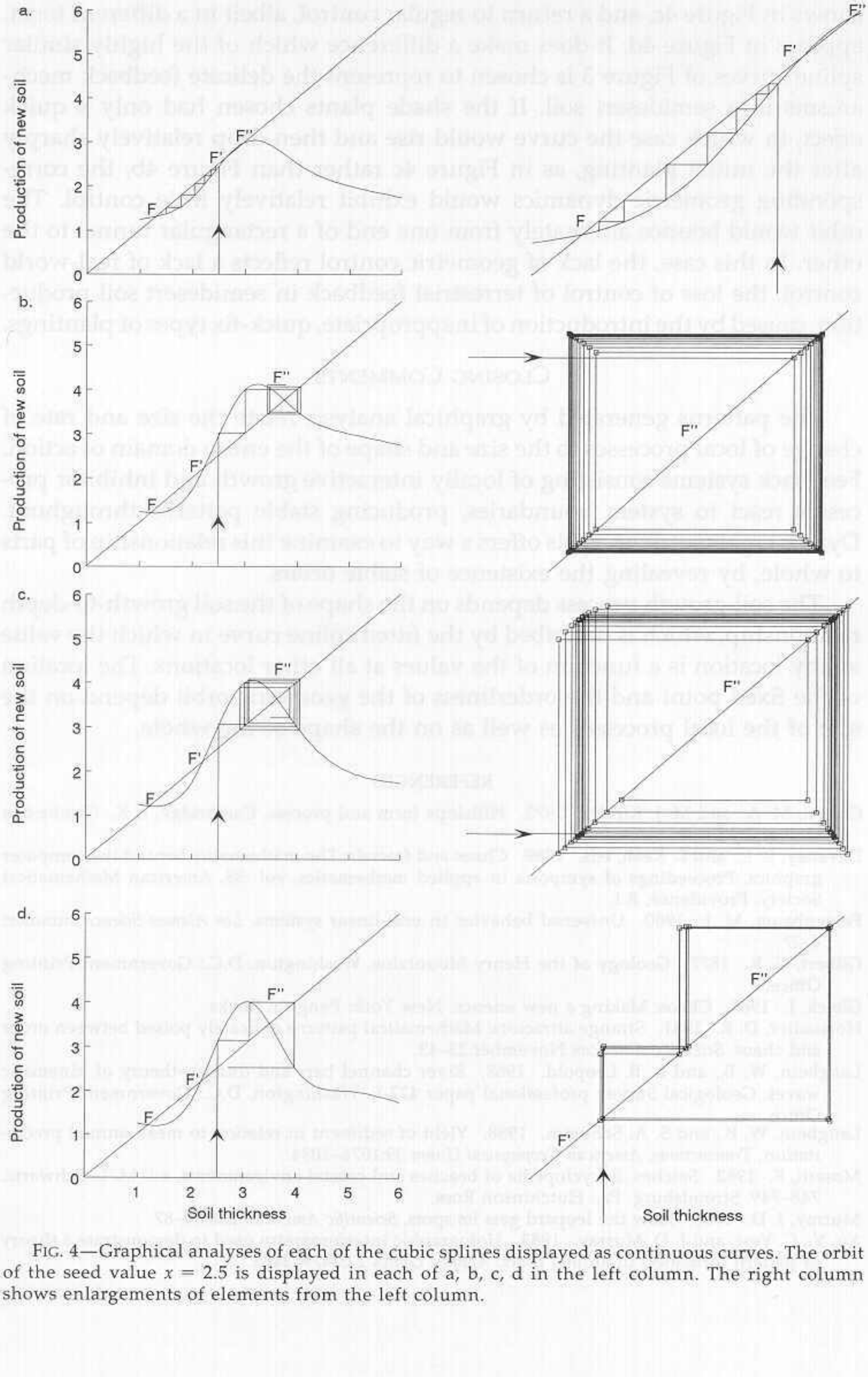

FIG. 4—Graphical analyses of each of the cubic splines displayed as continuous curves. The orbit of the seed value $x = 2.5$ is displayed in each of a, b, c, d in the left column. The right column shows enlargements of elements from the left column.

shows in Figure 4c, and a return to regular control, albeit in a different form, appears in Figure 4d. It does make a difference which of the highly similar spline curves of Figure 3 is chosen to represent the delicate feedback mechanisms in a semidesert soil. If the shade plants chosen had only a quick effect, in which case the curve would rise and then drop relatively sharply after the initial planting, as in Figure 4c rather than Figure 4b, the corresponding geometric dynamics would exhibit relatively little control. The orbit would bounce alternately from one end of a rectangular tunnel to the other. In this case, the lack of geometric control reflects a lack of real-world control: the loss of control of terrestrial feedback in semidesert soil production, caused by the introduction of inappropriate, quick-fix types of plantings.

Closing Comments

The patterns generated by graphical analysis relate the size and rate of change of local processes to the size and shape of the entire domain of action. Feedback systems consisting of locally interactive growth and inhibitor processes react to system boundaries, producing stable patterns throughout. Dynamic geometric analysis offers a way to examine this relationship of parts to whole, by revealing the existence of stable orbits.

The soil-growth process depends on the shape of the soil growth-to-depth relationship, which is described by the fitted spline curve in which the value at any location is a function of the values at all other locations. The location of the fixed point and the orderliness of the geometric orbit depend on the size of the local processes as well as on the shape of the whole.

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