FRACTALS TAKE A CENTRAL PLACE

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ABSTRACT. The geometry of central place theory is shown to be a (small) proper subset of the geometry of fractal curves: curves of fractional dimension which have only recently been displayed in a graphically provocative manner as computer-generated images (Mandelbrot 1977; 1983). The exact procedure for making this correspondence between a theory from economics and geography with one from pure mathematics is displayed in the text. It lends itself to replication by hand or by machine. As is usual with alignments of this sort, a wide variety of related projects follows naturally; some of these are indicated at appropriate points in the text.

Introduction

"Fractal" curves are curves that "fill" a fractional part of a space; such curves have long been a source of pathological counterexample in mathematics, and more recently, have been applied to fundamental problems in a variety of disciplines. Elementary calculus exploits the use of the absolute value function, $y = |x|$, to disprove that continuity implies differentiability; this V-shaped, continuous, absolute value function has only one point, at the "corner," at which the function is non-differentiable. Karl Weierstrass, a nineteenth century mathematician, sought a continuous curve that was nowhere differentiable and found one using a sequence of alternations of the absolute value function of the following sort; transform the letter "V," replacing each segment by two suitably placed copies of the letter "N," and repeat this procedure through n steps (Hahn 1956, pp. 1962–63). The limit as n approaches infinity yields a curve that is continuous but is composed only of corner points and so is nowhere differentiable. Koch superimposed the endpoints of Weierstrass's curve, producing a continuous curve that is nowhere differentiable surrounding a bounded zone of the plane (Mandelbrot 1983, p. 41). Filling a bounded zone, using Weierstrass's procedure with lines, led to the notion of "space-filling" curves. Peano created curves formed from one-dimensional line segments, which, when twisted and transformed infinitely, filled a two-dimensional space (Hahn 1956, pp. 1965–66), suggesting integration, the analytic companion of differentiation.

The use of space-filling curves to disprove topological conjectures continues to the present (Steen and Seebach 1970, pp. 137–38), and by the early twentieth century, mathematicians Fatou and Julia focused on a systematic theoretical organization of these sets extending beyond the calculus (Sullivan, 1982). Today, mathematicians specializing in topological dynamics create theorems about these sets and about sets that fill only part of a space; that the broader mathematical community sees the constructive potential for this activity is suggested by the series of four invited addresses on dynamical systems given in the "Colloquim" Lectures to the American Mathematical Society (Sullivan, 1982).

Moreover, prior to the presence of high-speed computing machinery, we could visualize these complicated curves only in the mathematical twilight of our minds. Thus the emergence of Benoit Mandelbrot's works (Mandelbrot 1977; 1983), displaying an elegant array of computer-generated "fractal" curves, suggests various applications for these curves in situations where shift in scale is fundamental. Mandelbrot's computer-generated three-dimensional landscapes are reminiscent of Erwin Raisz's block diagrams (Raisz 1948, pp. 120–121); his description of Minkowski's "sausage" for smoothing curves is similar to John Nystuen's use of epsilon discs to identify the domain of the boundary dweller (Nystuen 1967); and his concern for "How long is the coast of Britain?" echoes the persistence of the geographic scale problem (Mandelbrot 1983, pp. 264–65, 32, 25). Cartographers Waldo Tobler and Harold Moellering observed the potential for fractals to contribute to their research in shape theory and the transformation of shape; Michael Goodchild noted Mandelbrot's "model for the Pareto distributions observed for certain geographic areas" in an article dealing with various aspects of the location-allocation problem; and John Nystuen saw the significance of applying Mandelbrot's notion of "self-similarity" to the design of urban facilities dependent on dendritic networks for entry and exit (cf. Mandelbrot, 1977; Tobler, 1984; Moelle-

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ring, 1978; Goodchild 1979, p. 247; Nystuen, 1978; 1984).

This paper exhibits procedure to generate fractal sets and then uses it to show that the entire geometry of central place theory is but a small subset of the theory of fractal geometry; for, one style of fractal iteration sequence alone, using various (but related) "generators," will produce all possible central place nets. Consequently, the alignment of central place theory with fractal geometry does not merely produce one, two, or three (i.e., K = 3,4,7) cases of central place nets; rather, because fractal iteration sequences deal with infinite processes, they yield all cases. Nor does this alignment present mere technique for verification of a geometry of central places; the geometry of the central place model is well-known and has been clearly, and comprehensively, discussed in Michael Dacey's masterful 1965 article (Dacey 1965). The material presented below does show exact procedure for the merging of two separate theories – one from pure mathematics and one from economics and geography. In doing so, it suggests, in general, the power of one to enrich the other through the lodging of one discipline in the house of the other, and in particular, a theory in its own right derived from associating mathematically-specialized central place concepts with mathematically-broader fractal concepts. Recently, R. H. Atkin has demonstrated the richness of this sort of approach in (among other things) his analysis of the internal dynamics of urban structure using material from combinatorial and algebraic topology (Atkin, 1974; 1981). In a parallel vein, the broader conceptual base offered here could present means for assessing the dynamic structure of shared space between cities by evaluating the changing dimensions of an entire urban landscape across the continuum of fractional values that reflects the infinity of variation in real-world constraints.

Procedure for generating simple fractal sets

As with Weierstrass's creation of a nowhere differentiable curve, the strategy that underlies the physical development of fractal curves involves replacing, successively, the edges of a given regular polygon with a pre-determined pattern. To represent this replacement, the notation of Figure 1 will prove convenient; in that Figure, the shape above each arrow indicates that each edge of the closed curve on the left is to be replaced by the pattern

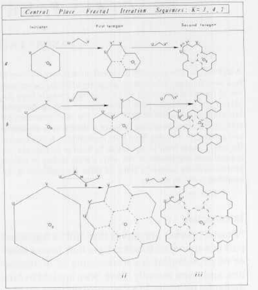

Figure 1.

above the arrow, generating the closed curve on the right. Thus in Figure 1.a, side $\overline{UV}$ of the hexagon in Figure 1.a.i is replaced (outside the hexagonal boundary) by a 'bent' or 'broken' shape with the included angle equal to 120°. This same shape replaces the side adjacent to $\overline{UV}$ at V (within the hexagonal boundary), and so forth around the hexagon, until U is reached from the left-hand side. Since the application of this shape alternated back and forth between 'inside' and 'outside' the hexagon, the area in Figure 1.a.ii is the same as that in Figure 1.a.i. Similarly, this pattern could be applied, at a scale made to match the length of a side, to Figure 1.a.ii creating Figure 1.a.iii; in this case the broken shape $\overline{UV'}$ is used to replace each side of Figure 1.a.ii. Iteration of this procedure produces increasingly complicated curves; the shape at the left end of this sequence is the 'initiator,' the pattern that is applied to the initiator is the 'generator,' and the shapes that appear at various stages in the iteration sequence are 'teragons' (Mandelbrot 1983, pp. 50, 48).

The 'broken' character of the teragons derives from the application of a 'broken' generator; indeed, Mandelbrot comments that "I coined fractal from the Latin adjective *fractus*. The corresponding Latin verb *frangere* means 'to break' to create irregular fragments" (Mandelbrot 1983, p. 4). The curve formed as the limit of an infinite iteration sequence is a "fractal" curve if and only if its dimension, D, does not coincide with one of the standard Euclidean dimensions 0, 1,2, or 3. Mandelbrot's formula for determining the dimension D is $D = (\log N) / (\log k)$ where N represents the number of segments of equal length into which the generator is broken (e.g., in Figure 1.a, $N = 2$), and k is derived from the concept of "self-similarity," discussed below (Mandelbrot 1983, p. 44).

In Figure 1.a, application of the generator to the initiator produces the first teragon which contains three copies of the initiator at a reduced scale (suggested by the dashed lines inside the first teragon). Application of the generator to the first teragon transforms it into the second teragon which contains three images of the first teragon (again suggested by dashed lines within the second teragon). Repeated application of the generator produces teragons with increasingly lacy edges, but in all cases the (n+1)st-teragon contains three copies, at a reduced scale, of the nth-teragon. This numerical invariant that measures shape but permits scale to shift, will be called the invariant of self-similarity. In the case of Figure 1, the smallest Euclidean dimension in which all teragons can be embedded is 2; thus, the invariant three, of this example, will be represented as k^2, where the exponent corresponds to this minimum embedding criterion. (Had three appeared as an invariant of self-similarity in a sequence of three-dimensional (one-dimensional) teragons, we should write $k^3 = 3$ ($k = 3$)). Thus $k^2 = 3$ in Figure 1.a, so that $k = \sqrt{3}$, and since $N = 2$, $D = (\log 2)/(\log \sqrt{3}) = 1.2618595$. Thus the curve that results from the infinite sequence associated with Figure 1.a is a "fractal" curve of fractional dimension 1.2618595.

The plausibility of assigning a "fractional" dimension to a curve comes from viewing it as a highly contorted one-dimensional Euclidean line that appears to "fill" more Euclidean space than does a single straight line, but less than does a Euclidean plane region. Mandelbrot's formula for D produces larger values for D as larger amounts of space are "filled," reflecting this plausibility from a notational standpoint (Polya, 1954). The notion of fractional dimension goes beyond standard Euclidean dimension since it includes it as proper subset; Mandelbrot observes that "In fact, having recognized the inadequacies of standard dimension, numerous scholars . . . had already been groping towards broken, anomalous, continuous dimensions of all kind. These approaches had remained mutually unrelated [and] . . . few definitions of dimension were used more than once" (Mandelbrot 1983, p. 16). Indeed, a traditional difficulty in identifying a basic set of spatial assumptions from which to classify spatial phenomena (as noted for example by Nystuen (Nystuen 1968)), involves problems associated with the placement of objects into more than one class as a result of changes in the scale of observation. Mandelbrot's continuum of fractional dimensions offers the potential to resolve this dichotomy in spatial classification.

Fractal generation of central place nets

The invariant of self-similarity, k^2, produced in the iteration sequence of Figure 1.a appears to serve the same function as the K-value of central place theory. (Dacey noted a relationship of this sort between central place theory, repetition theory, and iterative processes of various kinds; regrettably, Mandelbrot's work was not available to him in 1965 (Dacey 1965, p. 115)). The remainder of this section determines the relation of the fractal k-value to the central place K-value and exhibits fractal iteration sequences that generate the standard central place nets associated with $K = 3$, $K = 4$, and $K = 7$, as well as those derived from other points of a triangular lattice, selected to expose the reader to generator selection technique appropriate to obtaining nets for higher K-values associated with an arbitrary lattice point.

Standard central place hierarchies

The generator in the iteration sequence in Figure 1.a transforms a hexagonal initiator (Figure 1.a.i) into a first teragon composed of three hexagons (Figure 1.a.ii) which is again transformed (by the same generator) into the shape in Figure 1.a.iii which contains three copies of the first teragon. The following stacking procedure of the initiator and teragons yields the $K = 3$ central place hierarchy (Figure 1.a):

a) stack the initiator on the first teragon so that O_0 is superimposed on O_1 as a geometric translation in the direction of the arrow;
b) stack the first teragon on the second teragon so that O_1 is superimposed on O_2 as a geometric translation in the direction of the arrow;

c) continue this sequence indefinitely, as considerations of scale demand.

This generator produces teragons with cells of exactly the right size to use to form a central place net K = 3; the stacking procedure used to align the teragons to form the entire central place net is not as straightforward as it is in the cases that follow. Figure 1.b shows a fractal iteration sequence whose invariant of self-similarity is $k^2 = 4$. When initiator and teragons are stacked, with centers $O_0, O_1, O_2, \ldots$ superimposed in the obvious way (as a geometric translation in the direction of the arrow), a K = 4 central place net emerges. Application of the generator shown in Figure 1.c, to the hexagonal initiator, produces a fractal iteration sequence with an invariant of self-similarity $k^2 = 7$; the teragons provide hexagonal cells of diameter suited to forming a K = 7 central place hierarchy. Again, the entire central place net appears easily when the teragons are stacked in such a way that centers $O_0, O_1, O_2, \ldots$, and vertices U, V, V', . . . line up in the natural way. (Figure 1.c.ii is tilted to fit the teragons together neatly; otherwise, stacking follows the arrows, as above). In all cases of Figure 1, unit hexagons form the basis of the second teragons in order to ease visual comparisons of the most complicated forms in that Figure.

The key to using fractal geometry to obtain central place nets rests in choosing the correct generator to apply to the hexagonal initiator. Once the shape of such a generator has been determined (not always an easy task), it remains to construct the generator. Variations in detail of the procedure used to construct the generator for K = 7 (Figure 1.c) will yield positions for precise generator placement for any K-value. To produce the K = 7 central place net, it is required that $\angle UAB = \angle ABV = 120°$, that $\overline{AU} = \overline{AB} = \overline{BV}$, and that M is the midpoint of both UV and AB. Use of the Law of Cosines in $\triangle AUM$ gives $(\overline{UM})^2 = (\overline{AU})^2 + (\overline{AM})^2 - 2 \cdot \overline{AU} \cdot \overline{AM} \cdot \cos 120°$, or, when $\overline{UM}$ = 2.3 cm (for example) and $\overline{AM} = (\overline{AU})/2$, it follows that $(2.3)^2 = (7\ \overline{AU})/4$. Thus, $\overline{AU}$ = 1.7386366 cm. Because the value of 2.3 used for $\overline{UM}$ would vary with enlargement or reduction of Figure 1, calculate the size of $\angle AUM$, an angular measure which remains fixed under such geometric transformation. To do so, use the Law of Sines in $\triangle AUM$: $(\sin 120°)/(2.3) = (\sin(\angle AUM))/(1.7386366)$, so that $\angle AUM = 19.106605°$. With a good estimate of $\angle AUM$, we have enough information to characterize the generator for Figure 1.c and to construct the corresponding central place hierarchy, from a fractal iteration sequence, to any level of detail.

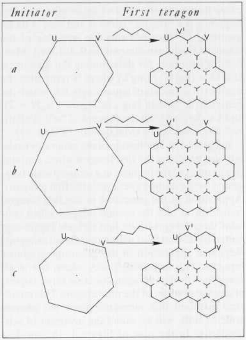

Figure 2.

The teragons of Figure 1 show only the first two stages in fractal iteration sequences; passing abstractly to the limit as the number of stages approaches infinity yields the dimension of each central place landscape and consequently, a measure of how completely that net "fills" the space that contains it. For K = 3, D = 1.2618595 as shown above. When K = 4, the generator is broken into three pieces, so that N = 3 and the invariant of self-similarity is four so that $k^2 = 4$. Consequently, D = (log N)/(log k) = (log 3)/(log 2) = 1.5849625, where the exponent of k reflects the minimum Euclidean dimension in which all teragons may be embedded. Similarly, when K = 7, D = (log 3)/(log $\sqrt{7}$) = 1.1291501. This suggests that independent of the number of hexagonal boundary lines introduced into the central place landscape, there is always "unfilled" space to use as trade areas, and that values of D closer to 2 represent greater penetration of the net into the space it partitions. The constructions in Figure 1 suggest, from mathematically inductive evidence, that pla-

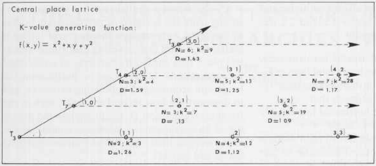

Figure 3.

ne central place K-values are identical to the fractal invariant of self-similarity, k^2, in which the exponent represents an upper bound for the dimension of any fractal curve in the plane. Thus $K = k^2$ is the relation that serves to formulate central place nets from fractal iteration sequences in the plane (this paper deals with existence criteria, uniqueness is not addressed although it appears that such investigation might prove fruitful).

Higher order central place nets

To gain added insight into methods for generator selection, Figure 2 displays the beginnings of fractal iteration sequences required to produce central place hierarchies associated with lattice points (2,2), (3,2), and (3,1); here again, the most complicated teragon boundaries contain the unit hexagons. All exhibit only the first step in the fractal iteration sequence; the hard part in creating these is to find the generator. In Figure 2.a, a generator of four sides, applied alternately within and outside the initiator, yields a first teragon containing twelve copies of the initiator. Thus $k^2 = 12$ and, if this sequence were carried out indefinitely, the dimension of the limiting position would be $D = (\log 4)/(\log \sqrt{12}) = 1.1157718$. In Figure 2.b a five-sided generator, which crosses the path of UV when applied to the initiator produces a first teragon with nineteen images of the initiator. The limiting position for this sequence would have dimension $D = (\log 5)/(\log \sqrt{19}) = 1.0932051$. Finally, Figure 2.c shows a five-sided generator, applied outside UV and then rotated about V to lie within the initiator, that leads to an ultimate partition of the plane with dimension $D = (\log 5) / (\log \sqrt{13}) = 1.2549471$. In these three cases, the associated central place hierarchies ($K = 12$, $K = 19$, and $K = 13$, or f(2,2), f(3,2), and f(3,1) derived from the generating function $f(x,y) = x^2 + xy + y^2$ for lattice points (x,y) (Dacey 1965, p. 113)), emerge by superimposing initiator and teragon centers in the natural way.

Classification of generators

The generator in Figure 2.a is composed of two copies of the generator in Figure 1.a; it may be shown using algebraic technique that fractal generators for central place nets associated with the lattice point (n,n) (n an integer) are composed of $|n|$ copies of the generator in Figure 1.a. This observation suggests grouping all such lattice points into one class, called T_3 from the use of the $K = 3$ generator. These points all fall on one horizontal line in Figure 3, and this collinearity condition suggests looking at the other horizontal lines to see if they determine sets of lattice points associated with other generator types. Indeed, it appears that higher central place K-value fractal sequences sort naturally into one of three mutually exclusive generator types grouped by lattice points on these horizontal lines:

a) Type T_3: the images of the initiator inside the first (and subsequent) teragon. Figueres 1.a and 2.a exhibit this sort of configuration.
b) Type T_4: the characteristic of T_3 does not hold, and in addition, the generator does not cross UV or any other initiator or teragon side. Figures 1.b and 2.b show this style of net.
c) Type T_7: the characteristic of T_3 does not hold, and in addition, the generator

does cross UV and all other initiator and teragon sides. Figures 1.c and 2.c demonstrate this quality.

Evidence from additional fractal constructions, with results coded along the horizontal lines in Figure 3 that show the relationships among these types in a triangular lattice, suggests that

a) along any single horizontal line exactly one of T_3, T_4, or T_7 holds;
b) as one moves from left to right along a horizontal line, the dimension of the central place fractal decreases;
c) there is a cyclic pattern in the order of T_3, T_7, T_4;
d) values of N, the number of sides in the generator, increase by an increment of 2, as one moves from left to right along a single horizontal line.

Conclusions

This paper presents explicit technique for generating central place nets from fractal iteration sequences. It begins with the lowest level central place nets (Figure 1) and then moves to more complicated nets associated with higher K-values (Figure 2). The general strategy of Figure 2 extends to any point of a triangular lattice with integer coordinates; the procedure for generating higher K-value central place nets from fractal iteration sequences rests in choosing a fractal generator that forces K to emerge from k^2. The separation of such generators into three mutually exclusive, but exhaustive, classes shows that the set of central place nets is a subset of the set of fractal iteration sequences. It is a proper subset as it relies only on one basic initiator (the hexagon) and on three basic generator types (T_3, T_4, and T_7). (The geographic and economic implications of nets formed from other initiators and/or generators is an open issue).

Moreover, as the calculations and figures above exhibit, the precise alignment of central place nets with overlays of fractal teragons along the seam K $= k^2$, also issues a challenge. That challenge, resulting from the merging of two disparate bodies of literature, is to explore the power of this geometric alignment in uncovering distortions to central place nets caused by barriers, to understand the implications of this merger for partitioning the landscape according to underlying networks, and to demonstrate how related concepts, such as the fractional dimension D, might explain the potential of an areally spread market to communicate, along teragon links, with point sources of central goods and services.

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