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SMALLEST TOURNAMENTS WITH GIVEN ABELIAN AUTOMORPHISM GROUP

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DEDICATED TO PROF. FRANK HARARY ON HIS COMPLETING 70 YEARS

Given an abelian group A of odd order, let $t(A)$ be the smallest number of vertices possible in a (round-robin) tournament with automorphism group A. $t(A)$ is determined for all finite abelian groups A of odd order, and it is shown that $t(A) = d(A)$, the smallest number of vertices in a digraph with automorphism group A.

In 1938, Roberto Frucht [4] proved that any abstract group can be realized as the automorphism group of a graph. Two natural questions that arise from this fact are:

1. What is the smallest number of vertices possible in such a graph?
2. Does the theorem remain true for other graph-like structures?

Frank Harary has had much to do, either directly or indirectly, with the answer to the first question. He and Palmer [5] found the smallest graph whose group is cyclic of order 3. After Sabidussi [8] made an early attempt at the problem for cyclic groups, Meriwether solved it [6], but his results remained unpublished. Frank suggested to me that I try the problem for abelian groups, and the suggestion bore fruit [1]. He also suggested to Czerniakiewicz that she try to find the smallest multigraph with given cyclic group, and again the results were positive [3]. More recently, Harary and I [2] determined the smallest digraphs with given abelian automorphism group. In fact, this last problem was the most straightforward to approach. One of the earliest approaches to the second problem was by Moon, who showed [7] that every group of odd order was the automorphism group of some tournament (it is impossible for a group of even order to be the group of a tournament).

It is the purpose of this paper to find the smallest tournaments with given abelian automorphism group of odd order. It seems quite likely from reading Moon's paper [7] that the results of this paper were within his grasp, had he turned his attention to them. But the proofs will be fairly easy now, given the presence of the Arlinghaus-Harary results on digraphs [2].

DEFINITION 1. A (round-robin) tournament T consists of a set of nodes $V(T)$ such that each two distinct nodes are joined by exactly one directed arc between them. (Thus a tournament with p nodes has $p(p-1)/2$ arcs.) If an arc is oriented from x to y, we say "x defeats y" or $x \to y$.

DEFINITION 2. An automorphism of a tournament is a permutation g of the nodes which preserves orientation. That is $xg \to yg$ if and only if $x \to y$.

It is thus clear that no tournament can have automorphism group of even order, since such a group would have an element of order two, which would reverse the orientation of some arc of the tounament.

DEFINITION 3. Let A be a finite abelian group. Let $d(A)$ be the smallest number of vertices possible in a digraph with group A. If $|A|$ is odd, let $t(A)$ be the smallest number of vertices possible in a tournament with group A.

Let C_n denote the cyclic group of order n. Now let A be written as $\prod_{i=1}^{n} C_{p_i^{a_i}}$, the p_i not necessarily distinct. Let $s_A = \sum_{i=1}^{n} p_i^{a_i}$. In Arlinghaus and Harary [2], it was shown that any permutation group isomorphic to A moved at least s_A letters. Indeed, unless 2 was one of the primes, such a permutation group was either generated by n distinct $p_i^{a_i}$ cycles, one for each i, or moved more than s_A Letters. Applying this to the case of tournaments, we have established

LEMMA 1. *If A is an abelian group of odd order, $A \cong \prod_{i=1}^{n} C_{p_i^{a_i}}$, then $t(A) \geq s_A$, with equality possible only if A is generated by n $p_i^{a_i}$- cycles, one for each i.*

Thus the permutation group structure of a tournament with s_A vertices, should it exist, is uniquely determined (even though the tournament itself may not be unique). But the cyclic tournaments of the next definition (Moon [7]) are the basis for the construction of just such a tournament.

DEFINITION 4. Let $\mathcal{T}_n$ be the tournament consisting of n vertices $v_1, \ldots, v_n$, with $v_i \to v_j$ if and only if $0 < (j-i) \bmod n \leq (n-1)/2$. $\mathcal{T}_3$ and $\mathcal{T}_5$ are exhibited in Figure 1.

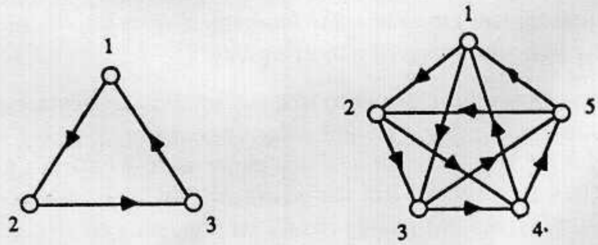

Figure 1

DEFINITION 5. Let T_1 and T_2 be two tournaments with vertex sets V_1 and V_2. Define a new tournament $T_1 \ast T_2$ with vertex set $V_1 \cup V_2$ as follows:

(a) If $x, y \in T_1$, then $x \to y$ in $T_1 \ast T_2$ if and only if $x \to y$ in T_1.

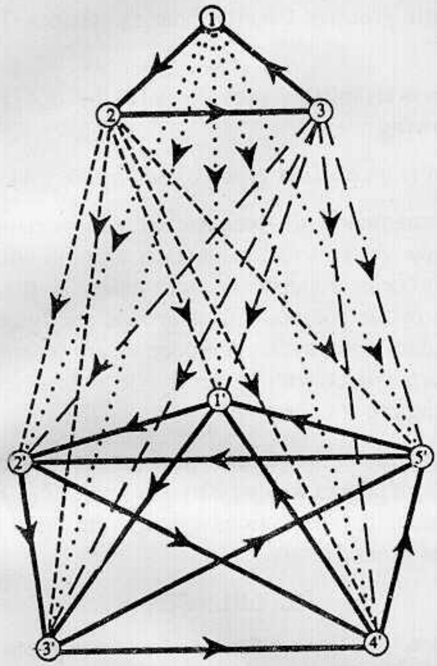

Figure 2

(b) If $x, y \in T_2$, then $x \to y$ in $T_1 \ast T_2$ if and only if $x \to y$ *in* T_2.
(c) If $x \in T_1$, $y \in T_2$, then $x \to y$ *in* $T_1 \ast T_2$.
$\mathcal{T}_3 \ast \mathcal{T}_5$ is illustrated in Figure 2.
Moon [7] established the following

LEMMA 2. *T_n has automorphism group C_n if n is of odd order.*

LEMMA 3. *If T_1 has automorphism group G, and T_2 has automorphism group H, then $T_1 \ast T_2$ has automorphism group $G \times H$.*

Note that, by Lemma 3, $T_1 \ast T_2$ and $T_2 \ast T_1$ have isomorphic automorphism groups, even though they are different tournaments. Equipped with these results, we now establish

THEOREM *If A is an abelian group of odd order,* $A \cong \prod_{i=1}^{n} C_{p_i^{a_i}}$, *then* $t(A) = S_A$.

Proof: By Lemma 1, it is only necessary to find a tournament T with s_A vertices and automorphism group A. Define $T_1 = \mathcal{T}_{p_1^{a_1}}$. For $2 \leq i \leq n$, define $T_i = T_{i-1} * \mathcal{T}_{p_i^{a_i}}$. (If the same cyclic group C_n occurs as a factor of A more than once, use different copies of $\mathcal{T}_n$ each time.) Let $T = T_n$. By Lemmas 2 and 3, T has automorphism group A. Clearly T has s_A vertices. Thus the theorem is established.

Finally, since it was established previously [2] that $d(A) = s_A$ for any abelian A, we have the following

COROLLARY. *If A is an abelian group of odd order,* $t(A) = d(A) = s_A$.

Thus, when a tournament with given abelian group exists, the smallest such tournament has as few vertices as the smallest digraph with the same group. Not only that, the process is similar in both cases: in the case of digraphs, digraphs with given cyclic components as groups are "glued" together using unions (or complements if the cyclic components are isomorphic); in the case of tournaments, tournaments with given cyclic components as groups are "glued" together using the $*$ operation.

What may be nicest about the result is that it combines in a straightforward fashion techniques from papers written 23 years apart [7 (1964), 2 (1987)], but both dealing with fundamental questions arising from Frucht and kept alive by graphical universalist Frank Harary.

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