# Arithmetic of the Asai L-function for Hilbert modular forms

by

Adam Rubin Kaye

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Doctoral Committee:

Professor Kartik Prasanna, Chair
Professor Stephen M. DeBacker
Professor Jeffrey C. Lagarias
Professor Andrew Snowden
Professor Martin J. Strauss

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# CHAPTER I

## Introduction

In this thesis, we will study the arithmetic of the Asai $L$-function for Hilbert modular forms over a real quadratic field. This is motivated by a $p$-adic analog of the Beilinson conjecture for Hilbert modular surfaces. This $p$-adic analog, which is expected to involve $p$-adic deformations in the weight direction (and is thus not covered by the general $p$-adic Beilinson conjecture of Perrin-Riou [30]), does not seem to be stated anywhere in the literature. The relevant $p$-adic $L$-function should interpolate critical values of the Asai $L$-function for Hilbert modular forms of *non-parallel weight*. Thus we are led to study the arithmetic of the special values of such $L$-functions, building on previous work of Shimura. Our main results include two theorems on rationality of such $L$-values that refine previous work of Shimura. We also make some partial progress towards integrality results.

Let $F$ be a real quadratic field and let $\mathcal{O}_F$ be the ring of integers in $F$. For simplicity, we will also assume throughout that $F$ has narrow class number equal to 1. Let $\mathcal{H}$ denote the complex upper half plane and $f : \mathcal{H}^2 \to \mathbb{C}$ a holomorphic Hilbert modular form of weight $(k_1, k_2)$. We assume that $f$ is a normalized Hecke eigenform.

It turns out there are two natural $L$-functions that one can associate with $f$. One such, the standard $L$-function is defined by

$$L(f,s)=\sum_{\mathfrak{a}} C(\mathfrak{a})N(\mathfrak{a})^{-s},$$

where the sum is over integral ideals $\mathfrak{a}$ of $\mathcal{O}_F$ and the terms $C(\mathfrak{a})$ denote the Fourier coefficients of $f$. This $L$-function is the one that appears in the Birch and Swinnerton-Dyer conjecture for the form $f$.

Another $L$-function that one may associate with $f$ is the so-called *Asai $L$-function*, defined by

$$L^{\mathrm{As}}(f,s)=\zeta(2s-k_1-k_2+2)\sum_{n=1}^{\infty} C(n)n^{-s},$$

where $\zeta(s)$ is the Riemann-Zeta function. Note that the sum is over the positive integers rather than integral ideals of $\mathcal{O}_F$. Asai [1] used Rankin's method to show that this $L$-function admits an analytic continuation and a functional equation. The Asai $L$-function for forms $f$ of weight $(2,2)$ is related to the Hasse-Weil zeta function of Hilbert modular surfaces. Specifically, the $L$-function $L(H^2(X),s)$ for a Hilbert modular surface is a product of Asai $L$-functions corresponding to such forms, and consequently it is the Asai $L$-function which arises in the statement of Beilinson's conjectures for $H^2$ of Hilbert modular surfaces. We will now recall this connection.

Let $X$ be a smooth projective surface over $\mathbb{Q}$. Then for each prime $l$, $H^2_{\text{ét}}(X\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Q}_l)$ is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module and as $l$ varies these form a compatible system of finite-dimensional $l$-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let $L^{(2)}(X,s)$ denote the $L$-function associated with this compatible system. The (conjectural) functional equation satisfied by this $L$-function relates the values $L^{(2)}(X,s)$ with $L^{(2)}(X,3-s)$. Then Beilinson's conjecture (stated in [4]) for the motive $H^2(X)$ concerns the special values $L^{(2)}(X,s)$ for integers $s\geq 2$ or equivalently the leading term in the Taylor expansion at integers $s\leq 1$. These special values are expected to be related to regulators of elements in certain higher Chow groups attached to $X$. Precisely, the higher Chow group (or motivic cohomology) in question that is conjecturally associated with $L^{(2)}(X,j)$ is the group

$$H^3_{\mathcal{M}}(X,\mathbb{Q}(j)) = \mathrm{CH}^j(X,2j-3)\otimes\mathbb{Q}.$$

Beilinson then constructs a regulator map

$$H^3_{\mathcal{M}}(X,\mathbb{Q}(j)) \to H^3_{\mathcal{D}}(X,\mathbb{R}(j)) \tag{1.0.1}$$

where $H^*_{\mathcal{D}}(X,\mathbb{R}(\cdot))$ denotes real Deligne cohomology. The target of the map (1.0.1) admits a natural $\mathbb{Q}$-structure. The conjecture then says that the map (1.0.1) is an isomorphism and the difference between the determinant of this $\mathbb{Q}$-structure and the one induced from (the image of) $H^3_{\mathcal{M}}(X,\mathbb{Q}(j))$ is given by the $L$-value $L^{(2)}(X,j)$, up to certain elementary factors. In fact, one needs to be a bit more careful for two reasons. Firstly, one must work not with $H^3_{\mathcal{M}}(X,\mathbb{Q}(j))$ but rather with the subspace of *integral* elements denoted $H^3_{\mathcal{M}}(X_{\mathbb{Z}},\mathbb{Q}(j))$. Secondly, for the point $j=2$ there is a correction factor involving the Neron-Severi group, that is related to the Tate conjecture for $X$. We will ignore such subtleties for the moment.

We now specialize to the case of a Hilbert modular surface $X$. Since such a surface is not compact, one should consider the intersection cohomology $IH^2$; for the purposes of the introduction we ignore the issue of noncompactness relegating the more precise description to Chapter II. For such an $X$, the representation $H^2_{\text{ét}}(X\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Q}_l)$ decomposes according to the action of the Hecke algebra attached to the group $G_F=\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2$. Let $\mathbb{A}$ denote the adeles over $\mathbb{Q}$. The automorphic representations $\pi$ of $G_F(\mathbb{A})$ that contribute to $H^2_{\text{ét}}(X\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathbb{Q}_l)$ can be classified and the interesting representations are those where the infinite components are discrete series of weight 2, namely those that correspond to Hilbert modular forms $f$ of parallel weight two.

Fixing such a $\pi$ (and $f$), let $W_\ell(\pi)$ denote the $\pi$-isotypic component of $H^2_{\text{ét}}(X \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$. We write $W(\pi)$ for the compatible system of Galois representations $W_\ell(\pi)$. The key point that connects Beilinson's conjecture to the Asai $L$-function is the following result of Brylinski-Labesse.

**Theorem I.1** (Brylinski-Labesse, [9]). *There exists a finite set of primes $Q$ such that*

$$L_Q(W(\pi), s) = L_Q^{\mathrm{As}}(f, s)$$

*where $L_Q$ denotes the $L$-function with Euler factors at primes dividing $Q$ removed.*

As a result, Beilinson's conjecture reduces more or less to a corresponding conjecture relating the $\pi$-isotypic component of $H^3_{\mathcal{M}}(X, \mathbb{Q}(j))$ to special values of the Asai $L$-function $L^{\mathrm{As}}(f, s)$. Let us now briefly recall previous work on Beilinson's conjecture (and $p$-adic versions). conjecture:

(A) The original paper of Beilinson [4] treats the case of the Rankin-Selberg $L$-function $L(f \times g, 2)$ where $f$ and $g$ are classical modular forms of weight two. This may be viewed as a special case of the setting described above by taking $F$ to be the quadratic $\mathbb{Q}$-algebra $\mathbb{Q} \times \mathbb{Q}$ instead of a real quadratic field.

(B) An unpublished preprint of Ramakrishnan [35] discusses the case of a Hilbert modular form of weight $(2, 2)$ at $s = 2$.

(C) The paper [25] of Guido Kings treats the case of Hilbert modular forms of weight $(2, 2)$ at $s \geq 3$.

More recently, there has been interest in formulating and proving $p$-adic versions of Beilinson's conjecture, in which the regulator is replaced by a $p$-adic regulator and the $L$-function by a $p$-adic $L$-function. In the setting of (A) above, this was studied in the paper [6] of Bertolini, Darmon and Rotger. The $p$-adic $L$-function in question is that obtained by interpolating (the algebraic parts of) critical values of the Rankin-Selberg $L$-functions $L(f_\kappa, g, s)$ where $\mathcal{F}$ is a Hida family of forms containing $f$ and $\kappa$ is an arithmetic point of $\mathcal{F}$ corresponding to a classical modular form $f_\kappa$ of weight $\geq 3$. The point is that the original Rankin-Selberg $L$-function $L(f \times g, s)$ has no critical values since the weights of $f$ and $g$ are equal but the $L$-function $L(f_\kappa \times g, s)$ is critical at the points $2 \leq s \leq k-1$ where $k$ is the weight of $f_\kappa$. We note that to define the algebraic part of $L(f_\kappa \times g, j)$, one needs to divide it by an appropriate period. In this case, the period may be taken to be the Petersson inner product $\langle f_\kappa, f_\kappa \rangle$ of the form $f_\kappa$, up to certain elementary factors.

We consider instead the case (B) with a Hilbert modular form $f$ of parallel weight two. Again, the Asai $L$-function $L(f^{\mathrm{As}}, s)$ has no critical points. The idea then is to vary such an $f$ in a Hida family $\mathcal{F}$ of forms whose weights are $(k, 2)$. When we specialize $\mathcal{F}$ to a form $f_\kappa$ of weight $(k, 2)$ with $k > 2$, the Asai $L$-function $L(f_\kappa^{\mathrm{As}}, s)$ is critical in the range $2 \leq s \leq k-1$. Thus one may hope to construct a $p$-adic $L$-function by interpolating the algebraic parts of these $L$-values. However, one runs into a problem since the transcendental period that one must divide by is no longer so easy to define. Indeed the form $f_\kappa$ is not a "product" of forms as in the case $F = \mathbb{Q} \times \mathbb{Q}$ and the period must somehow distinguish one of the two infinite places of the field $F$. It turns out that there are two way to do this and thus two periods that can be defined. We remark that the exact relation between these two periods is not obvious and is an open question that deserves further study.

Motivated by the above discussion, for the rest of the introduction, we use the symbol $f$ to denote a Hilbert modular form (for $F$) of weight $(k, 2)$, and discuss in turn the two periods that one can attach to such an $f$ that capture the transcendental part of critical values of the Asai $L$-function of $f$. The first period is defined by using the Jacquet-Langlands correspondence. The idea here is due to Shimura (in [**?**]). One assumes that there exists a quaternion algebra $B$ over $F$ which is split at one infinite place and ramified at the other infinite place. In other words, we need to assume that the representation $\pi$ is discrete series for at least one finite place. Let $\tau_1, \tau_2$ be the infinite places of $F$ and $B$ be a quaternion algebra that is split at $\tau_1$ and ramified at $\tau_2$. Let $\pi^B$ denote the Jacquet-Langlands transfer of $\pi$ to $B^\times$. Also let $f^B$ be a nonzero vector in $\pi^B$ of weight $(k, 0)$ at infinity. The form $f^B$ may be viewed as a section of an automorphic vector bundle $\mathcal{V}_B$ on the Shimura curve $X_B$ attached to $B^\times$. Now using the fact that $X_B$ and $\mathcal{V}_B$ admit models over $\bar{\mathbb{Q}}$ and even over $F$, the form $f^B$ can be normalized up to a non-zero element in the field $F\mathbb{Q}(f)$ which is the compositum of $F$ with the field $\mathbb{Q}(f)$ generated by the Hecke eigenvalues of $f$. In fact, one can also integrally normalize $f^B$ at least at good primes using suitable integral models of $X_B$ and $\mathcal{V}_B$, as in [24] §1. The first period is then defined to be the Petersson inner product of $f^B$ with respect to a suitable measure.

$$\mathcal{Q}(f, B) := \langle f^B, f^B \rangle.$$

The following theorem is due to Shimura.

**Theorem I.2** (Shimura [43], [44])**.**

$$\frac{L^{\mathrm{As}}(f, k-1)}{\pi^k \langle f^B, f^B \rangle} = \frac{L^{\mathrm{As}}(f, k-1)}{\pi^k \mathcal{Q}(f, B)} \quad \textit{lies in } \bar{\mathbb{Q}}.$$

Our first main result refines this to

**Theorem I.3.** *(See Chapter VI , Cor. VI.18) Suppose that $\mathbb{Q}(f) \subseteq F$ and that every prime ideal $\mathfrak{p}$ of $F$ dividing the discriminant of $B$ is split over $\mathbb{Q}$. Then*

$$\frac{L^{\mathrm{As}}(f, k-1)}{\pi^k \mathcal{Q}(f, B)} \quad \textit{lies in } F_{\mathrm{ab}},$$

*where $F_{\mathrm{ab}}$ denotes the maximal abelian extension of $F$.*

The proof of this result is obtained by carefully analyzing Shimura's proof of algebraicity, which is rather difficult and involves several steps. The brief idea is as follows. One views $B$ as a quadratic space over $F$ equipped with the quadratic form given by the reduced norm. Then $f^B \times f^B : \mathcal{H}^2 \to \mathbb{C}$ can be viewed as an automorphic form for the identity component $\mathrm{G}_B$ of the group $\mathrm{GO}(B)$, since $\mathrm{G}_B$ is isomorphic to $(B^\times \times B^\times)/F^\times$. Shimura defines another automorphic form $\tilde{E}_k$ on $\mathrm{G}_B$ such that

$$\langle f^B \times f^B, \tilde{E}_k \rangle \sim L^{\mathrm{As}}(f^B, k-1)\langle f^B, f^B \rangle.$$

Here $\sim$ denotes equality up to certain well-understood factors. He then shows that $\tilde{E}_k$ is defined over $\bar{\mathbb{Q}}$ which implies $\langle f^B \times f^B, \tilde{E}_k \rangle \sim C\langle f^B, f^B \rangle^2$ with $C \in \bar{\mathbb{Q}}$. The idea then is to refine Shimura's proof by showing that $\tilde{E}_k$ is in fact defined over a smaller field.

The Hermitian symmetric domain for $\mathrm{G}_B$ does not have any cusps, so the rationality properties of $\tilde{E}_k$ cannot be studied through $q$-expansions. Instead the idea is to evaluate $\tilde{E}_k$ at certain CM points and show that the resulting values, suitably normalized, are rational. Let $K$ be a totally imaginary degree 2 extension of $F$ and $w_K \in \mathcal{H}^2$ a CM point corresponding to $K$. One needs to study the value $\tilde{E}_k(w_K)$ divided by an appropriate CM period attached to $K$. Now $K$ itself can be viewed as a quadratic space over $\mathbb{Q}$ of dimension 4. The Shimura variety $\mathrm{GO}(K)$ has a distinguished CM point $z_K$ associated with $K$ and one can define an automorphic form $G_k$ on $\mathrm{GO}(K)$ with the property that

$$\tilde{E}_k(w_K) \sim G_k(z_K).$$

The problem then reduces to showing that the form $G_k$ on $\mathrm{GO}(K)$ is rational. To prove this, one evaluates $G_k$ at CM points $z_L$ on $\mathrm{GO}(K)$ corresponding to imaginary quadratic fields $L$. Finally, one shows that $G_k(z_L)$ is related to special values of Rankin-Selberg type $L$-function of two forms for $\mathrm{GL}_2$ over $\mathbb{Q}$, where the form of large weight is a theta series lift from $L$. The algebraicity (and rationality) of critical values of such $L$-functions is well understood, and can be used as a first step in the above sequence of arguments.

While, in principle, the constructions above only require $K$ to be a quadratic CM extension of $F$, in practice many simplifications occur if we assume that $K$ is biquadratic over $\mathbb{Q}$. For example, in this case the group $\mathrm{GO}(K)$ may be viewed as a the orthogonal group of a quaternion algebra $B_K$ over $\mathbb{Q}$. Moreover various constructions involving reflex fields and CM periods are easier to keep track of if $K$ is biquadratic. To prove the main theorem we therefore first pick a CM field $K$ that is biquadratic, a subfield $L$ of $K$ that is imaginary quadratic and find a dense set of points $z_L$ on $\mathrm{GO}(K)$ such that

$$G_k(z_L)/\pi^k\Omega_{z_L} \in L_{\mathrm{ab}}, \tag{1.0.2}$$

for an appropriate CM period $\Omega_{z_L}$ in $\mathbb{C}^\times/L_{\mathrm{ab}}^\times$ associated with the CM point $z_L$. The condition that every prime $\mathfrak{p}$ of $F$ dividing the discriminant of $B$ is split over $\mathbb{Q}$ is used to ensure that one may in fact pick a biquadratic field $K$ that embeds in $B$.

In order to study $p$-adic $L$-functions, it would be useful to have an integrality result for the Asai $L$-function. In principle, it seems possible to further refine the method described above to obtain such an integrality result but the technical details get complicated. The idea would be to first prove an integrality statement for the values in (1.0.2) and then leverage that to prove integrality of $G_k$ and eventually integrality of $\tilde{E}_k$. In Chapter VII, we show the following partial result. For certain specific points $z_L$ we define an integrally normalized period $\Omega^*_{z_L}$ and a constant $M_L$

such that

**Theorem I.4.**

$$\pi^k G_k(z_L)/(\Omega^*_{z_L})^{2k}$$

*is $p$-integral for all $p \nmid M_L$.*

The proof of this uses a result from [32] on integrality of certain Rankin-Selberg $L$-values. Unfortunately, we are not able to show the theorem above as yet for a dense set of points $z_L$ and so we cannot conclude from this that $G_k$ is $p$-integral.

Now we discuss another approach to studying the critical values of $L^{\mathrm{As}}(f, s)$, which is explained in Chapter VIII. This approach uses the original integral representation of Asai, together with a period that was defined by Harris [16] using rational structures on coherent cohomology. The period $\Omega_f$ so obtained does not make use of the Jacquet-Langlands correspondence to transfer to a quaternion algebra but instead uses directly a construction on the Hilbert modular surface. The main result (which we prove only in the case of forms of full level) is then:

**Theorem I.5.** *Let $f$ be a Hilbert modular form of full level and weight $(k, \ell)$ with $k > \ell$. Then*

$$\frac{L^{\mathrm{As}}(f, k-1)}{\Omega_f} \quad \textit{lies in } F\mathbb{Q}(f)$$

*where $\mathbb{Q}(f)$ denotes the field generated by the Hecke eigenvalues of $f$.*

As mentioned before, it would be interesting to compare the periods $\mathcal{Q}(f, B)$ and $\Omega_f$. Note that both these periods can be normalized up to elements in $F\mathbb{Q}(f)$ and even up to $p$-units for good primes $p$.

## CHAPTER II

# Hilbert Modular Surfaces and their cohomology

### 2.1 Classical Definition

Let $\mathcal{H}$ denote the complex upper half plane. The classical definition of a Hilbert modular surface is given as a complex surface that is a quotient of $\mathcal{H}^2$. Fix a real quadratic field $F$ and denote the two embeddings $F \hookrightarrow \mathbb{R}$ by $\tau_1$, $\tau_2$ respectively. These extend to embeddings $\tau_i : \mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ (for $i = 1, 2$) by applying $\tau_i$ to each entry. We have an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ given by fractional linear transformations:

$$\gamma(z) = \frac{az+b}{cz+d} \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), z \in \mathcal{H}.$$

Combining this with the two embeddings of $\mathrm{SL}_2(F)$ gives the action of $\mathrm{SL}_2(F)$ on $\mathcal{H}^2$:

$$\gamma((z_1, z_2)) = \left( \frac{a^{\tau_1} z_1 + b^{\tau_1}}{c^{\tau_1} z_1 + d^{\tau_1}}, \frac{a^{\tau_2} z_2 + b^{\tau_2}}{c^{\tau_2} z_2 + d^{\tau_2}} \right)$$

$$\text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F), (z_1, z_2) \in \mathcal{H}^2.$$

Let $\mathcal{O}_F$ denote the ring of integers in $F$. Then the image $(\tau_1, \tau_2) : \mathrm{SL}_2(\mathcal{O}_F) \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup so the quotient $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathcal{H}^2$ is a complex surface. This is the simplest example of a Hilbert modular surface. In general we can quotient by any arithmetic subgroup of $\mathrm{SL}_2(F)$ - that is any $\Gamma \subset \mathrm{SL}_2(F)$ such that $\Gamma \cap \mathrm{SL}_2(\mathcal{O}_F)$ has finite index in both $\Gamma$ and $\mathrm{SL}_2(\mathcal{O}_F)$.

**Definition II.1** (Classical, see [46])**.** A *Hilbert modular surface* is the complex surface $\Gamma\backslash\mathcal{H}^2$ where $\Gamma$ is an arithmetic subgroup of $\mathrm{SL}_2(F)$ for some real quadratic field $F$.

*Remark* II.2. We can also allow the case where $F = \mathbb{Q} \oplus \mathbb{Q}$ and $\tau_1, \tau_2$ correspond to projection onto the first, second coordinate respectively and embedding in $\mathbb{R}$. Then $\mathrm{SL}_2(\mathcal{O}_F) = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ and

$$\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathcal{H}^2 \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}.$$

In general we could let $\Gamma = \Gamma_1 \times \Gamma_2$ where each $\Gamma_i$ is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then the 'Hilbert modular surface' $\Gamma\backslash\mathcal{H}^2$ is the product of the modular curves $\Gamma_i\backslash\mathcal{H}$ for $i = 1, 2$. Thus we may think of a product of modular curves as a 'degenerate' Hilbert modular surface.

## 2.2 Adelic description

Let $G := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2 / F$ denote the Weil restriction of $\mathrm{GL}_2 / F$ to $\mathbb{Q}$ (see [47] for a construction of the Weil restriction). Thus for a $\mathbb{Q}$ algebra $R$, the $R$ points of $G$ are given by $G(R) = \mathrm{GL}_2(F \otimes R)$. Let $\mathfrak{X} = \mathcal{H}^\pm \times \mathcal{H}^\pm$ where $\mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$. Since $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ we have a natural action of $G(\mathbb{R})$ on $\mathfrak{X}$ where each copy of $\mathrm{GL}_2(\mathbb{R})$ acts on the respective copy of $\mathcal{H}^\pm$ by fractional linear transformations. Let

$$L_\infty := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}$$

and $K_\infty = L_\infty \times L_\infty \subset G(\mathbb{R})$. Since $\mathrm{GL}_2(\mathbb{R})/L_\infty \cong \mathcal{H}^\pm$ we have $\mathfrak{X} \cong G(\mathbb{R})/K_\infty$.

Let $\mathbb{A}$ denote the ring of adeles over $\mathbb{Q}$ and $\mathbb{A}_k$ denote the ring of adeles over any number field $k$. In addition we denote the finite and infinite parts of the adeles with subscript $f$ and $\infty$ respectively, so $\mathbb{A}_\infty = \mathbb{R}$ and $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$.

Then

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty = G(\mathbb{Q})\backslash (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)) = G(\mathbb{Q})\backslash (\mathfrak{X} \times G(\mathbb{A}_f))$$

comes with a continuous right $G(\mathbb{A}_f)$ action. For each compact open subgroup $K \subset G(\mathbb{A}_f)$ one can construct the space

$$G(\mathbb{Q})\backslash (\mathfrak{X} \times G(\mathbb{A}_f))/K$$

and these form a directed system as $K$ varies. The theory of canonical models ([38], [11], [12], [27]) defines a Shimura variety $S_K$ over $\mathbb{Q}$, such that

$$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash (\mathfrak{X} \times G(\mathbb{A}_f))/K.$$

Given a compact open subgroup $K$ the strong approximation theorem ([31]) implies that there exist $x_1, \ldots, x_n \in G(\mathbb{A}_f)$ such that

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{n} G(\mathbb{Q}) x_i G(\mathbb{A}_\infty)^+ K$$

and $\{\det(x_i)\}$ is a complete set of representatives for $F^\times \det(G(\mathbb{A}_\infty)^+ K)\backslash \mathbb{A}_F^\times$. Here the $+$ denotes the subgroup of $G(\mathbb{A}_\infty)$ consisting of elements with positive determinant. Then

$$G(\mathbb{A}_\infty)^+/K_\infty = (\mathrm{GL}_2(\mathbb{R})^+ \times \mathrm{GL}_2(\mathbb{R})^+)/(L_\infty \times L_\infty) = \mathcal{H} \times \mathcal{H}.$$

Thus each element of $G(\mathbb{A})$ is represented by a class $[gx_i, z]$ for $g \in G(\mathbb{Q}), i \in \{1, \ldots, n\}, z \in G_\infty^+/K_\infty = \mathcal{H}^2$. Then $[gx_i, z] \cong [x_i, z']$ in $G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K$ if and only if $g \in G(\mathbb{Q}) \cap x_i G_\infty^+ K x_i^{-1}$ and $gz = z'$. Thus the complex points of $S_K$ are naturally isomorphic to

$$\coprod_{i=1}^{n} \Gamma_i \backslash \mathcal{H}^2$$

where

$$\Gamma_i = G(\mathbb{Q}) \cap x_i G(\mathbb{A}_\infty)^+ K x_i^{-1}.$$

Thus the Shimura variety $S_K$ is a generalization of the classical Hilbert modular surfaces. Although $S_K$ is defined over $\mathbb{Q}$, a given component $\Gamma_i \backslash \mathcal{H}^2$ may not be. However there will be a canonical abelian number field $k_{\Gamma_i}$ depending only on $\Gamma_i$ such that the component $\Gamma_i \backslash \mathcal{H}^2$ is defined over $k_{\Gamma_i}$.

## 2.3 Compactification

In general, a Hilbert modular surface $\Gamma \backslash \mathcal{H}^2$ may not be smooth and it will never be compact. The singular points arise from points in $\mathcal{H}^2$ with non-trivial stabilizer in $\Gamma$ so if $\Gamma$ is torsion-free, $\Gamma \backslash \mathcal{H}^2$ will be smooth.

Similar to the case of modular curves we can add finitely many cusps to compactify $\Gamma \backslash \mathcal{H}^2$. Consider the map $i : F \to \mathbb{C}^2$ given by $i : a \mapsto (a^{\tau_1}, a^{\tau_2})$. Then $\mathcal{H}^2 \cup i(F) \cup \{\infty\}$ is compact. The boundary $F \cup \{\infty\}$ corresponds to $\mathbb{P}^1(F)$ and has an action of $\mathrm{GL}_2(F)$ given by

$$\gamma([x, y]) = [ax + by, cx + dy] \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F).$$

$\Gamma \backslash \mathbb{P}^1(F)$ will be finite and these points are the cusps of $\Gamma \backslash \mathcal{H}^2$. The compactification of $S_K$ given by including the cusps is called the Baily-Borel-Satake compactification and we denote it by $\bar{S}_K$.

The singularities at the cusps can be resolved using the methods described by

Hirzebruch ([22]) and Ash, Mumford, Rapoport, Tai ([2]) and this is called the smooth toroidal resolution of $\bar{S}_K$. Take a Hilbert modular surface $\Gamma\backslash\mathcal{H}^2$ and consider the cusp $\infty$. For any $t > 0$ define

$$W_t = \{(z_1, z_2) \in \mathcal{H}^2 : \Im(z_1)\Im(z_2) > t\}.$$

This is a neighborhood of $\infty$ in $\mathcal{H}^2$. The stabilizer of $\infty$ in $\Gamma$ is given by

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \Gamma \right\}.$$

Thus a neighborhood of $\infty$ in $\Gamma\backslash\mathcal{H}^2$ is given by $\Gamma_\infty\backslash W_t$. Let

$$A = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \right\}, N = \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in \Gamma_\infty \right\}.$$

Then $\Gamma_\infty = A \ltimes N$. To construct the smooth toroidal compactification we define an infinite chain of rational curves $\Sigma$ and an $A$-action on $\Sigma$ such that $A\backslash\Sigma$ is a finite polygon of rational curves. The resolution adds $\Sigma$ to $N\backslash W_t$ in such a way that $A\backslash(N\backslash W_t \cup \Sigma)$ is smooth and compact. We denote the Shimura variety of the toroidal compactification by $\tilde{S}_K$ and there is a natural map $\pi_K : \tilde{S}_K \to \bar{S}_K$ such that the pre-image of each cusp is a finite chain of rational curves. Harder, Langlands and Rapoport ([15]) show that $\tilde{S}_K$ and the map $\pi_K$ can be defined over $\mathbb{Q}$. Let $\bar{S}_K^\infty$ denote the cusps in $\bar{S}_K$ and $\tilde{S}_K^\infty = \pi_K^{-1}(\bar{S}_K^\infty) \subset \tilde{S}_K$. It is important to note that the construction of $\tilde{S}_K$ is not canonical. However we will soon see that there is a piece of the cohomology of $\tilde{S}_K$ that is independent of the choices made in the construction of $\tilde{S}_K$.

## 2.4 Cohomology of Hilbert Modular Surfaces, Hecke Operators, and Hilbert Modular Forms

Since $\tilde{S}_K$ is defined over $\mathbb{Q}$ we have an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on

$$W_l := H^2_{\text{ét}}(\tilde{S}_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_l)$$

for each prime $l$. There is a natural map $H^2_c(S_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_l) \to W_l$ and define $IW_l$ to be the image of this map. Furthermore let $W_l^\infty$ be the subgroup of consisting of elements supported in $\tilde{S}_K^\infty$. Harder et al. ([15]) showed that there is a decomposition

$$W_l = IW_l \oplus W_l^\infty.$$

Recall that for each cusp $s$ of $\bar{S}_K$, $\pi_K^{-1}(s)$ is a polygon of rational curves. Let $n(s)$ denote the number of irreducible components of this polygon. Let $\mathbb{Q}_l(1)$ denote the $l$-adic Galois Tate module and $\mathbb{Q}_l(-1)$ be its dual. Then

$$W_l^\infty = \bigoplus_{s \in \bar{S}_K^\infty} (\mathbb{Q}_l(-1))^{\oplus n(s)}.$$

Let $K, L$ be two compact open subgroups of $G(\mathbb{A}_f)$ such that $L \subset K$. Then there is a natural map $R(1) : S_L \to S_K$. In general if there exists $x \in G(\mathbb{A}_f)$ such that $x^{-1}Lx \subset K$ then there is a map $R(x) : S_L \to S_K$ defined by $R(x)(\gamma) = R(1)(\gamma x)$ for $\gamma \in G(\mathbb{A})$.

**Definition II.3.** For any $K$ and $x$ set $L := K \cap xKx^{-1}$. Then there is a diagram

$$\begin{tikzcd} & S_L \arrow[dl, "R(1)"'] \arrow[dr, "R(x)"] & \\ S_K & & S_K \end{tikzcd}$$

which defines a correspondence $T_x : S_K \to S_K$. This is the *Hecke correspondence* $T_x$. The *Hecke algebra* over $k \supset \mathbb{Q}$ is the algebra generated over $k$ by the $T_x$ acting $S_K$ is denoted by $H_k^K$. We will sometimes write simply $H_k$.

Since the Hecke correspondences act on $S_K$ and not $\tilde{S}_K$, there is an induced action of the Hecke algebra on $IW_l$, but not $W_l^\infty$. This allows us to write $IW_l \otimes \bar{\mathbb{Q}}_l$ as a direct sum over irreducible $H_{\bar{\mathbb{Q}}}$ representations. Let $\tau$ be an irreducible $H_{\bar{\mathbb{Q}}}$ representation

$$\tau : H_{\bar{\mathbb{Q}}} \to \mathrm{GL}(W_l(\tau))$$

where $E(\tau)$ is a number field over which $\tau$ is defined and $W_l(\tau)$ is an $E(\tau) \otimes \mathbb{Q}_l$ module. For each $\tau$ there is a multiplicity $m(\tau, K)$ such that if

$$W_l'(\tau) = W_l(\tau)^{\oplus m(\tau,K)}$$

and $\bar{W}_l'$ is the extension of $W_l'$ to $E(\tau) \otimes \bar{\mathbb{Q}}_l$ then

$$IW_l \otimes \bar{\mathbb{Q}}_l \cong \bigoplus_\tau \bar{W}_l'(\tau).$$

The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $S_K$ induces an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $W_l'(\tau)$. Let

$$IH^2_{\text{ét}}(\tilde{S} \times \bar{\mathbb{Q}}, \mathbb{Q}_l) := \lim_K IH^2_{\text{ét}}(\tilde{S}_K \times \bar{\mathbb{Q}}, \mathbb{Q}_l).$$

Let $H^f_{\bar{\mathbb{Q}}_l}$ be the algebra generated by all the $H^K_{\bar{\mathbb{Q}}_l}$'s and decompose $IH^2_{\text{ét}}(\tilde{S} \times \bar{\mathbb{Q}}, \mathbb{Q}_l)$ by the action of $H^f_{\bar{\mathbb{Q}}_l}$. There is a result which says

**Lemma II.4** (see [34] p. 305)**.** *$IH^2_{\acute{e}t}(\tilde{S} \times \bar{\mathbb{Q}}, \mathbb{Q}_l)$ decomposes into a sum of components $\bar{W}_l(\tau)_f$, each occurring once, such that*

$$\bar{W}_l(\tau)_f^K = \bar{W}_l'(\tau)$$

*for all $K$ and for each $\tau$ there is an compact open subgroup $K(\tau)$ such that*

$$\bar{W}_l(\tau)_f^{K(\tau)} \cong \bar{W}_l(\tau).$$

This induces an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $W_l(\tau)$.

Let $\pi = \pi_\infty \times \pi_f$ be an irreducible, admissible representation of $G(\mathbb{A})$ on $V_\pi = V_{\pi_\infty} \otimes V_{\pi_f}$. Then $H_{\mathbb{C}}^K$ acts on $V_{\pi_f}^K$. If $V_{\pi_f}$ is defined over a number field and $V_{\pi_f}^K \neq 0$, this gives us a representation $\tau$. We say such a $\tau$ is associated to $\pi$.

Certain representations $\pi$ come from Hilbert modular forms:

**Definition II.5.** Given integers $k_1, k_2$ and an arithmetic $\Gamma \subset \mathrm{SL}_2(F)$ a holomorphic function $f$ on $\mathcal{H} \times \mathcal{H}$ is a *Hilbert modular form* of weight $(k_1, k_2)$ if it satisfies

$$f(\gamma(z_1, z_2)) = \prod_{i=1,2} (c^{\tau_i} z_i + d^{\tau_i})^{k_i} f(z_1, z_2)$$

for all $(z_1, z_2) \in \mathcal{H}^2, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If $k_1 = k_2 = k$, say $f$ has parallel weight $k$.

**Theorem II.6** (Harder, Langlands, Rapoport, [15])**.** *$W_l(\tau) \neq 0$ if and only if $\tau$ is associated to an irreducible admissible representation $\pi$ of $G(\mathbb{A})$ and $\pi$ is one-dimensional or $\pi$ is cuspidal automorphic of weight 2.*

For a definition of 'cuspidal automorphic of weight 2' see [36] and [13]. We will use the following result

**Theorem II.7** ([13], 3.10)**.** *A cuspidal automorphic weight 2 representation of $G(\mathbb{A})$ corresponds to a classical holomorphic Hecke eigenform on $\mathcal{H} \times \mathcal{H}$ of parallel weight 2.*

# CHAPTER III

# Beilinson's Conjecture

## 3.1 Class Number Formula

Beilinson's conjecture on regulators are an important generalization of the class number formula for number fields. For $K$ a number field, the class number formula states that the Dedekind zeta function of $K$ has a power series expansion $\zeta_K(s) = \frac{c}{s-1} + c_0 + c_1(s-1) + \ldots$ where

$$c = \frac{2^{r_1}(2\pi)^{r_2} h_K \operatorname{reg}_K}{w_k\sqrt{|D_K|}}.$$

Here $r_1$ is the number of real places of $K$, $r_2$ the number of complex places, $h_K$ the class number, $w_k$ the number of roots of unity, $D_K$ the discriminant, and $\operatorname{reg}_K$ is the regulator of $K$. Let us recall the definition of the regulator map.

**Definition III.1.** Regulator map for number fields

Let $K$ be a number field with $r_1$ real embeddings $\tau_1, \ldots, \tau_{r_1}$ and $r_2$ conjugate pairs of complex embeddings $\tau_{r_1+1}, \overline{\tau_{r_1+1}}, \ldots, \tau_{r_1+r_2}, \overline{\tau_{r_1+r_2}}$. Then the group of units in $\mathcal{O}_K$ has rank $m := r_1 + r_2 - 1$. Let

$$Y = \left\{ x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} : \sum_{i=0}^{m} x_i = 0 \right\}.$$

Define $\operatorname{reg} : \mathcal{O}_K^\times \to Y$ by

$$\operatorname{reg} : x \mapsto (\log(|x^{\tau_1}|), \ldots, \log(|x^{\tau_{r_1+r_2}}|)).$$

This map lands in $Y$ because of the product formula and the image $\mathrm{reg}(\mathcal{O}_K^\times)$ is a lattice in $Y$. The regulator of $K$ is defined by

$$\mathrm{reg}_K := \det(\mathrm{reg}(\mathcal{O}_K^\times) \oplus \langle (1,1,\dots,1)\rangle).$$

The regulator is expected to be transcendental for all $K \neq \mathbb{Q}$ and this is significant because every other term in the class number formula is algebraic (except for the well understood powers of $\pi$). This is the idea behind Beilinson's conjecture: find a map of $\mathbb{R}$-vector spaces whose determinant with respect to a suitable $\mathbb{Q}$-structure calculates the transcendental part of the special values of $L$-functions, and that generalize the regulator above.

### 3.2 L-functions

**Definition III.2.** Let $V_\bullet = \{V_l\}$ be a compatible system of finite dimensional $l$-adic $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ representations. The $L$-function of $V_\bullet$ is defined by

$$L(V_\bullet, s) = \prod_p Z_p(V_\bullet, p^{-s})^{-1}$$

where

$$Z_p(V_\bullet, T) = \det(1 - F_p T|_{V_l^{I_p}})$$

for some $l \neq p$. Here $F_p$ denotes the (geometric) Frobenious at $p$ and $I_p$ denotes the inertia subgroup at $p$. This is constructed in [11].

### 3.3 Beilinson's Conjecture for Surfaces

We will now describe Beilinson's conjecture for the case of a surface $X$ defined over $\mathbb{Q}$. Let $\bar{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$. Then the groups $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ form a compatible system of finite dimensional $l$-adic Galois representations. Thus we can construct the $L$-function

$$L^{(2)}(X, s) := L(H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_\bullet), s).$$

Beilinson's conjecture claims that the leading coefficient of this $L$-function at $s = 1$ is given by the volume of a regulator map. Part of the domain of the regulator is a certain higher Chow group on $X$:

**Definition III.3.** Let $X$ be a surface over $\mathbb{Q}$ and $k$ be a number field. The *higher Chow group* $CH^2(X_k, 1)$ is generated by finite formal $k$-rational sums $\sum_i (C_i, f_i)$ where each $C_i$ is a closed irreducible curve on $X_{\bar{k}}$, $f_i$ is a function on $X_{\bar{k}}$ that is invertible at the generic point of $C_i$ such that

$$\sum_i \operatorname{div}(f_i) = 0$$

where both sides are viewed as 0-cycles on $X_{\bar{k}}$. For $g, h$ rational functions on $X_{\bar{k}}$ and $C$ a curve on $X_{\bar{k}}$ define

$$\mathrm{Tame}_C(g, h) = (-1)^{\mathrm{ord}_C(g)\mathrm{ord}_C(h)} \frac{g^{\mathrm{ord}_C(h)}}{h^{\mathrm{ord}_C(g)}}|_C.$$

Quillen ([33]) showed

$$\sum_C \operatorname{div}(\mathrm{Tame}_C(g, h)) = 0$$

as zero-cycle on $X_{\bar{k}}$. Two formal sums of the form $\sum_i (C_i, f_i)$ are said to be equivalent if they differ by a linear combination of elements of the form $\sum_j \operatorname{div}(\mathrm{Tame}_{C_j}(g_j, h_j))$. $CH^2(X_k, 1)$ is the group of such equivalence classes, rational over $k$.

The target of the regulator map comes from the cohomology of the analytification of $X$:

**Definition III.4.** Let $H_B^{\cdot}$ denote Betti cohomology and $\mathbb{R}(j) = (2\pi i)^j \mathbb{R}$. Let

$$Y_{\mathbb{R}} := H^{1,1}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{C}) \cap H_B^2(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(1)).$$

Then $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $Y_{\mathbb{R}}$ via the action on $X \times_{\mathbb{Q}} \mathbb{C}$ so we can let $Y_{\mathbb{R}}^+$ denote the invariants of complex conjugation. The regulator will map into $Y_{\mathbb{R}}^+$.

Let $H^*_{\mathrm{DR}}(X_k)$ denote the DeRahm cohomology of $X_k$ and $F^\bullet$ the algebraically defined Hodge filtration on $H^*_{\mathrm{DR}}$. Then there is an exact sequence

$$0 \to \sum_{\sigma \in \hom(k,\mathbb{C})} (F^2 H^2_{\mathrm{DR}}(X_k) \otimes_{k,\sigma} \mathbb{C})^+ \xrightarrow{p} H^2_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(1))^+ \to Y^+_{\mathbb{R}} \to 0$$

where the map $p$ is defined by taking the isomorphism

$$H^2_{\mathrm{DR}}(X_k) \otimes_{k,\sigma} \mathbb{C} \cong H^2_B(X(\mathbb{C}), \mathbb{C})$$

and projecting by

$$C = \mathbb{R}(1) \oplus \mathbb{R}(2) \to \mathbb{R}(1).$$

$\sum_{\sigma \in \hom(k,\mathbb{C})} (F^2 H^2_{\mathrm{DR}}(X_k) \otimes_{k,\sigma} \mathbb{C})^+$ has a $\mathbb{Q}$-structure given by $\sum_\sigma F^2 H^2_{\mathrm{DR}}(X_k)^+$ and $H^2_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(1))^+$ has a $\mathbb{Q}$-structure given by $H^2_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}(1))^+$. These induce a $\mathbb{Q}$-structure on $Y^+_{\mathbb{R}}$ and allow us to define the volume of a $\mathbb{Q}$-lattice in $Y^+_{\mathbb{R}}$ up to an element of $\mathbb{Q}^\times$.

The vector space $H^{1,1}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ has a pairing $\langle \cdot, \cdot \rangle$ defined by

$$\langle \omega_1, \omega_2 \rangle = \int_{X(\mathbb{C})} \omega_1 \wedge \omega_2.$$

To define $\mathrm{reg} : CH^2(X_k, 1) \to Y^+_{\mathbb{R}}$ it is in fact enough to define how $\mathrm{reg}(\beta)$ behaves under $\langle , \rangle$ for each $\beta \in CH^2(X_k, 1)$ (see [34] p.347).

**Definition III.5.** Let $\beta = \sum_i (C_i, f_i) \in CH^{2,1}(X)$ and $C'_i = C_i \setminus \mathrm{div}(f_i)$. Then

$$\mathrm{reg} : CH^{2,1}(X) \to Y^+_{\mathbb{R}}$$

is uniquely defined by the property

$$\langle \mathrm{reg}(\beta)^\sigma, \omega^\sigma \rangle = \frac{1}{2\pi i} \sum_i \int_{C'^\sigma_i(\mathbb{C})} \log |f^\sigma_j| \cdot \omega^\sigma|_{C_i}$$

for each $\sigma : k \to \mathbb{C}$.

Let $X_{\mathcal{O}_k}$ be a proper model for $X$ over $\mathcal{O}_k$. Then Beilinson ([4]) defines a subspace $CH^2(X_{\mathcal{O}_k}, 1) \otimes \mathbb{Q}$ in $CH^2(X_k, 1) \otimes \mathbb{Q}$. Beilinson's conjecture concerns the image of this integral part of the higher Chow group under the regulator map.

The last piece needed to define the regulator is the analogue of the vector $(1, 1, \ldots, 1)$ in the determinant. This is played by the Neron-Severi group $NS(X)$ and $\mathrm{reg}\,|_{NS(X)}$ is defined as the cycle class map.

**Conjecture III.6** (Beilinson's Conjecture, [4])**.** *Let $X_{\mathcal{O}_K}$ be a proper model of $X$, so that we can define the integral part of the Chow group: $CH^2(X_{\mathcal{O}_K}, 1) \otimes \mathbb{Q}$. Then* $\mathrm{reg}(CH^2(X_{\mathcal{O}_K}, 1) \otimes \mathbb{Q} \oplus NS(X_k) \otimes \mathbb{Q})$ *is a $\mathbb{Q}$-lattice of $Y_{\mathbb{R}}^{+}$ and its volume is equal to the leading coefficient of $L^{(2)}(X, s)$ at $s = 1$ up to a non-zero rational number.*

### 3.4 Beilinson's Example

When Beilinson originally made his conjecture, he gave some evidence for it in the case when $X$ is a product of modular curves. In this case he was able to construct elements in $CH^2(X_{\mathcal{O}_K}, 1)$ and explicitly compute their image under the regulator map. These are called Beilinson-Flach elements:

**Definition III.7.** Let $M$ be a modular curve and $X = M \times M$. For each modular unit $u$ on $M$ we can define a *Beilinson-Flach element* $\Delta_u$. Since $u$ is a modular unit it's divisor on $M$ can be written as $\sum_i c_i - c_i'$ where each $c_i, c_i'$ is a cusp on $M$. Let $\alpha_i$ be a modular unit with divisor $c_i - c_i'$ for each $i$. Then the formal sum

$$D_i = (\{c_i\} \times M, \alpha_i) + (M \times \{c_i'\}, \alpha_i)$$

has divisor $(c_i, c_i) - (c_i', c_i')$. Let $\Delta$ denote the diagonally embedded copy of $M$ in $X$. Then the formal sum

$$\Delta_u = (\Delta, u) - \sum_i D_i$$

has divisor 0 and therefore represents an element of $CH^2(X_k,1)$. This is the Beilinson-Flach element associated to $u$. See [4] p.2064 for details and the argument that $\Delta_u \in CH^2(X_{\mathcal{O}_k},1)$.

The groups $H^2(X)$ (both Betti and étale) decompose as

$$H^2(X) = (H^2(M) \otimes H^0(M)) \oplus (H^1(M) \otimes H^1(M)) \oplus (H^0(M) \otimes H^1(M))$$

and we will focus on the $H^1(M) \otimes H^1(M)$ part. $H^1(M)$ decomposes into irreducible components corresponding to weight 2 eigenforms. Let $f$ and $g$ be two such eigenforms. Then the component of $L^{(2)}(M \times M, s)$ corresponding to $(f,g)$ is in fact $L(f \otimes g, s)$, the Rankin-Selberg convolution of $f$ and $g$ (a full explanation of this is in [4] Ch.2 Sec. 6). Let $\omega_g$ be the form $2\pi i g(z)dz$ and

$$\eta_f^{\mathrm{ah}} := \frac{\bar{f}^*(z)d\bar{z}}{\langle f^*, f^* \rangle}$$

where $f^*$ is the form given by applying complex conjugation to the coefficients of $f$. Then the component of $Y_{\mathbb{R}}^+$ corresponding to $(f,g)$ is given by $\omega_g \otimes \eta_f^{\mathrm{ah}}$. Beilinson's result is:

**Theorem III.8** (Beilinson)**.** *Let $f, g$ be weight 2 normalised newforms of levels $N_f$, $N_g$ and nebentypus characters $\chi_f$, $\chi_g$ with $\chi := \chi_f^{-1}\chi_g^{-1}$ non-trivial. Also let $N :=$ lcm$(N_f, N_g)$. Then there is a modular unit $u_\chi$ attached to the character $\chi$. We have*

$$\frac{L(f \otimes g, 2)}{\langle f^*, f^* \rangle} = 16\pi^3 N^{-2} \tau(\chi^{-1}) reg\langle \Delta_{u_\chi}, \omega_g \otimes \eta_f^{ah} \rangle.$$

*Here $\tau(\chi^{-1})$ denotes the Gauss sum associated to $\chi^{-1}$.*

Note that although our statement here concerns the value of $L(f \otimes g, s)$ at $s = 2$ and the statement of Beilinson's conjecture concerns the value at $s = 1$, the functional equation relates the values $s \leftrightarrow 3 - s$.

### 3.5 Ramakrishnan's results for Hilbert modular surfaces

Ramakrishnan ([34]) was able to prove a result similar to Beilinson's for the case of $X$ a Hilbert modular surface. The significant part of this proof lies in finding elements of $CH^2(\tilde{S}_K, 1)$ because the Beilinson-Flach elements do not exist on a Hilbert modular surface.

**Definition III.9.** Given an open compact subgroup $K \subset G(\mathbb{A}_f)$ there is a modular curve $M$ that can be diagonally embeded in $\bar{S}_K$ by through the diagonal embedding $\bar{\mathcal{H}}^{\pm} \to \bar{\mathcal{H}}^{\pm} \times \bar{\mathcal{H}}^{\pm}$. The *Hirzebruch-Zagier cycle* $\bar{C}_{g,K}$ on the Hilbert modular surface $\bar{S}_K$ is the image of this diagonal embedding under the Hecke operator $T_g$.

**Definition III.10.** A formal sum

$$\sum_i (C_i, f_i)$$

is called $K$-*admisible* if each $C_i$ is a Hirzebruch-Zagier cycle $\bar{C}_{g,K}$ on $\bar{S}_K$, each $f_i$ is a modular unit on $C_i$ and

$$\sum_i \mathrm{div}(f_i) = 0$$

as a 0-cycle on $\bar{S}_K \times \bar{\mathbb{Q}}$.

Any such sum will also be rational over some number field. For a given Hirzebruch-Zagier cycle $\bar{C}_{g,K}$, there is a cycle $\tilde{C}_{g,K} \in \tilde{S}_K$ mapping to it under the natural map $\bar{S}_K \to \tilde{S}_K$. However if we lift the modular units $f_i$ to $\tilde{S}_K$ they may not satisfy $\sum_i \mathrm{div}(f_i) = 0$ on $\tilde{S}_K$. Ramakrishnan ([34], p.352) proves

**Lemma III.11.** *Given any $K$-admissible $k$-rational formal sum $\beta = \sum_i (C_i, f_i)$ there exists a finite formal sum $\sum_j (E_j, g_j)$ such that each $E_j$ is a component of a resolution of a cusp in $\tilde{S}_K$, each $g_j$ is a $\mathcal{O}_k$-integral function on $E_j$, and*

$$\sum_i div(f_i) + \sum_j div(g_j) = 0$$

*as a* 0*-cycle on* $\tilde{S}_K \times k$*. The formal sum* $\sum_j (E_j, g_j)$ *is defined over* $\mathcal{O}_k$ *and we call*

$$\tilde{\beta} := \sum_i (C_i, f_i) + \sum_j (E_j, g_j)$$

*a lift of* $\beta$*.*

A lift of a $K$-admissible sum $\beta$ will not be unique, but any two lifts will differ by an integrally defined sum supported on the cusps of $\tilde{S}_K$.

**Definition III.12.** Let $\mathfrak{R}'_K(k)$ be the $\mathbb{Q}$-subspace of $CH^{2,1}(\tilde{S}_K \times k) \otimes \mathbb{Q}$ generated by lifts of $K$-admissible $k$-rational sums $\sum_i (C_i, f_i)$.

If $M$ is an open subgroup of $K$ then there is a natural map $p_{K,M} : \tilde{S}_M \to \tilde{S}_K$. This induces a map $p_{K,M*}$ on the corresponding Chow groups.

**Definition III.13.** Define $\mathfrak{R}_K(k)$ to be the $\mathbb{Q}$-subspace of $CH^{2,1}(\tilde{S}_K \times k) \otimes \mathbb{Q}$ generated by $\{p_{K,M*}(\mathfrak{R}'_K(k)) | M \text{ open in } K\}$

Ramakrishnan then shows that $\mathfrak{R}_K(k)$ is integral.

**Theorem III.14** (Ramakrishnan, [34] Thm. 12.19)**.** *For every character* $\omega$ *of* $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$*, we have* $\mathrm{reg}^{\omega}(\mathfrak{R}_K(\omega) \oplus NS(\tilde{S}_K, \omega))$ *is a* $\mathbb{Q}(\omega)$*-structure of* $Y_{\mathbb{R}}(\omega)^+$*. Furthermore, it's volume is equal (up to a nonzero element of* $\mathbb{Q}(\omega)$*) to the leading coefficient at* $s = 1$ *of* $L^{(2)}(\tilde{S}_K(\omega), s)$*).*

# CHAPTER IV

# $p$-adic Beilinson Conjecture

## 4.1 $p$-adic Rankin-Selberg

Let $f$ be a weight $k$ modular form and $g$ a weight 2 modular form with Nebentypus characters (see [28] for a definition) $\chi_f$, $\chi_g$ and levels $N_f$, $N_g$ respectively. Let $N := \mathrm{lcm}(N_f, N_g)$ and replace $\chi_f$ and $\chi_g$ by the corresponding characters mod $N$. Suppose also that $\chi := \chi_f^{-1}\chi_g^{-1}$ is non-trivial and primitive. Let

$$\sigma_{m,\chi}(n) := \sum_{d|n} \chi(d) d^m$$

and define the Eisenstein series

$$E_{m,\chi}(z) := 2^{-1}L(\chi, 1-m) + \sum_{n=1}^{\infty} \sigma_{m-1,\chi}(n) q^n.$$

In addition let

$$C(k,j) := \frac{(-1)^{k-1-j}2^{k-1}(2\pi)^{2j-1}(iN)^{k-2j}\tau(\chi^{-1})}{((j-1)!)^2}$$

where $\tau(\chi^{-1})$ denotes the Gauss sum attached to $\chi^{-1}$.

**Proposition IV.1** (Shimura, see [39] or [5]). *Suppose $k \geq 3$ and $(k+1)/2 \leq j \leq k-1$. Then*

$$L(f \otimes g, j) = C(k,j)\langle f^*(z), \delta_{2j-k}^{k-1-j} E_{2j-k,\chi}(z) g(z)\rangle.$$

*where $f^*$ is the form given by applying complex conjugation to the coefficients of $f$, $\delta_{2j-k}^{k-1-j}$ is the Shimura-Maass derivative operator, and $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product.*

We need $k \geq 3$ because the values $j \in [(k+1)/2, k-1]$ are exactly the critical values for $L(f \otimes g, j)$ that are right of center, and this set will be empty with $k < 3$. So when $k = 2$ there are no critical values. Hida was able to $p$-adically interpolate the critical values to obtain a $p$-adic $L$-function that is defined at $k = 2, j = 2$.

Since $\langle f^*, \delta_{2j-k}^{k-1-j} E_{2j-k,\chi} g \rangle / \langle f^*, f^* \rangle$ is algebraic, it makes sense to define

$$L^{\mathrm{alg}}(f \otimes g, j) := C(k,j)^{-1} \frac{L(f \otimes g, j)}{\langle f^*, f^* \rangle} = \frac{\langle f^*, \delta_{2j-k}^{k-1-j} E_{2j-k,\chi} g \rangle}{\langle f^*, f^* \rangle} \in \bar{\mathbb{Q}}$$

for $k \geq 3, (k+1)/2 \leq j \leq k-1$.

Let $\mathbf{f}$ be a Hida family of ordinary $p$-adic modular forms with tame level $N$ defined on a non-empty open subset $U_{\mathbf{f}}$ of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$ that is contained in a single residue class modulo $p-1$. Then for each $k \in U_{\mathbf{f}} \cap \mathbb{Z}_{\geq 2}$ and $(k+1)/2 \leq j \leq k-1$ we define

$$L_p(\mathbf{f}, g)(k, j) := \frac{\mathcal{E}(f_k, g, j)}{\mathcal{E}(f_k)\mathcal{E}^*(f_k)} L^{\mathrm{alg}}(f_k \otimes g, j)$$

where $\mathcal{E}(f_k, g, j)$, $\mathcal{E}(f_k)$, and $\mathcal{E}^*(f_k)$ are certain Euler factors (see [6] for a definition). Hida ([21]) shows that this extends to a function on all of $U_{\mathbf{f}} \times \mathbb{Z}_p$. Thus we can specialize $L_p(\mathbf{f}, g)(k, j)$ to $k = 2, j = 2$ even though the Rankin-Selberg $L$-function has no critical values when $f$ and $g$ both have weight 2.

## 4.2 $p$-adic Beilinson for a product of modular curves

Perrin-Riou ([30]) has stated a general conjecture which is a $p$-adic analog of Beilinson's conjecture, but it involves $p$-adic interpolation by varying the cyclotomic variable of an $L$-function. Some cases of this have been proven (see [3] and [8]). In [6], Bertolini, Darmon, and Rotger proved a $p$-adic analogue of Beilinson's result which lies outside of Perrin-Riou's conjecture because it involves $p$-adic interpolation across varying weights as well. Let $K_p$ be an extension of $\mathbb{Q}_p$. The $p$-adic regulator may be viewed as a map

$$\mathrm{reg}_p : CH^2(X,1) \to (\mathrm{Fil}^1(H^2_{\mathrm{dR}}(X/K_p)))^\vee$$

(as described in [29], [14], and [7]).

Let $Y$ denote the open modular curve $Y_1(N)$ and $X$ denote the complete modular curve $X_1(N)$. Let $\mathcal{E} \to Y$ denote the universal elliptic curve over $Y$. Define $\mathcal{L} := R^1\pi_*(\mathcal{E} \to Y)$ and $\mathcal{L}_r := \mathrm{Sym}^r\mathcal{L}$. $\mathcal{L}_r$ is a coherent sheaf over $X$ and comes equipped with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r \to \mathcal{L}_r \otimes \Omega^1_X(\log \text{cusps})$$

where $\Omega^1_X(\log \text{cusps})$ is the sheaf of regular differentials on $Y$ with log poles at the cusps. Let $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)$ be the de Rahm cohomology of $\mathcal{L}_r$. Let $f$ and $g$ be two modular forms as in the previous section. Let $f$,$g$ be cuspidal eigenforms as in the previous section. We also assume that $f$ and $g$ are both ordinary at $p$ and have level $N$ where $p \nmid N$. This implies ([6], p. 7) that there is a canonical unit root subspace $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\mathrm{ur}} \subset H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)$ and a canonical projection onto the unit root subspace. Let $\eta_f^{\mathrm{ah}}$ be the element of $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)$ represented by the form

$$\frac{\bar{f}^*(z)d\bar{z}}{\langle f^*, f^*\rangle}$$

where $f^*$ is the form given by applying complex conjugation to the Fourier coefficients of $f$. Let $\eta_f^{\mathrm{ur}}$ be the projection of $\eta_f^{\mathrm{ah}}$ to the unit root subspace. Let $\omega_g$ be the element of $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$ represented by $2\pi i g(z)dz$. Then $\omega_g \otimes \eta_f^{\mathrm{ur}} \in \mathrm{Fil}^1(H^2_{\mathrm{dR}}(X/\mathbb{C}_p))$ and an analogue of Beilinson's conjecture is:

**Theorem IV.2** (Bertolini, Darmon, Rotger [6])**.** *Let $f$, $g$ be weight 2 modular forms as in the previous section and suppose there is a Hida family $\mathbf{f}$ that specializes to $f$ at 2. Then*

$$L_p(\mathbf{f}, g)(2,2) = \frac{\mathcal{E}(f,g,2)}{\mathcal{E}(f)\mathcal{E}^*(f)} reg_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}).$$

### 4.3 $p$-adic Asai $L$-function

We now return to the case where the surface $X$ is a Hilbert modular surface $\tilde{S}_K$. Recall that $W_l = H^2_{\text{ét}}(\tilde{S}_K \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_l)$ decomposes as $W_l = IW_l \oplus W_l^\infty$ and the action of the Hecke algebra $H_{\bar{\mathbb{Q}}}$ on $IW_l$ gives a decomposition

$$IW_l \otimes \bar{\mathbb{Q}}_l = \bigoplus_\tau \bar{W}_l'(\tau)$$

over irreducible $H_{\bar{\mathbb{Q}}}$ representations. Beilinson's conjecture concerns the special value of the $L$-function

$$L^{(2)}(\tilde{S}_K, s) = L(W_\bullet, s)$$

at $s = 1$. Thus we are led to consider the $L$-function $L(W_\bullet(\tau), s)$ for an irreducible $H_{\bar{\mathbb{Q}}}$ representation $\tau$. The more interesting case is when $\tau$ is cuspidal automorphic of weight 2 and therefore corresponds to a parallel weight 2 Hilbert modular form.

Let $\mathfrak{d}$ denote the different of $F$ and $\omega \in F$ be a generator for the inverse different of $F$: $(\omega) = \mathfrak{d}^{-1}$. For simplicity let us take $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ and assume $\omega >> 0$. Let $f : \mathcal{H}^2 \to \mathbb{C}$ be a Hilbert modular form of weight $(2,2)$ for $\Gamma$. It can be shown ([46], [1]) that $f$ has a Fourier expansion of the form

$$f(z) = \sum_{\mu \in \mathcal{O}_F, \mu >> 0} C(\mu) e^{2\pi i \operatorname{Tr}(\mu\omega z)}.$$

**Definition IV.3.** The *Asai $L$-function* of the weight $(k,l)$ Hilbert modular form $f$ with Fourier expansion

$$f(z) = \sum_{\mu \in \mathcal{O}_F, \mu >> 0} C(\mu) e^{2\pi i \operatorname{Tr}(\mu\omega z)}$$

is defined to be

$$L^{\mathrm{As}}(f,s) := \zeta(2s-k-l+2)\sum_{n=1}^{\infty} C(n)n^{-s}.$$

This is different from the standard $L$-function associated to $f$, which would take into account all the Fourier coefficients. Instead the Asai $L$-function only uses the Fourier coefficients indexed by $\mathbb{Z}$.

**Theorem IV.4** (Brylinski-Labesse)**.** *Given $\tau$ cuspidal automorphic with associated Hilbert modular form $f$, there exists a finite set of primes $Q$ such that*

$$L_Q(W_\bullet(\tau),s) = L_Q^{As}(f,s).$$

*The subscript $Q$ means the Euler factors at primes $p|Q$ are omitted.*

Thus to give a $p$-adic analogue of Ramakrishnan's result requires a $p$-adic Asai $L$-function. As for the case of the Rankin-Selberg convolution $L(f\otimes g,s)$, the Asai $L$-function has no critical values at parallel weight $(2,2)$. However if $f$ is a Hilbert modular form with weights $(k,2)$ with $k\geq 3$, $L^{\mathrm{As}}(f,j)$ will be a critical value for $(k+1)/2 \leq j \leq k-1$. Thus we are led to study the special values $L^{\mathrm{As}}(f,k-1)$ with $f$ of weight $(k,2)$. If these values can be $p$-adically interpolated then specializing to $k=2$ will give a $p$-adic analogue of the special value appearing in Beilinson's conjecture. Completing this is an open question that deserves further study.

# CHAPTER V

# Algebraicity of the Asai $L$-function

## 5.1 Summary of Shimura's proof

We begin by giving a rough outline of Shimura's proof of the algebraicity of the Asai $L$-function which is done in three papers ([42], [43], [44]). In this section we will us $\sim$ to denote when two quantities are related by a known factor which can be explicitly calculated and is an algebraic multiple of a power of $\pi$. We begin by listing the the essential objects of the proof (these will be described more precisely later):

1. $f$ - a Hilbert modular form over a real quadratic field $F$ of weight $(k, l)$ with $k > l$.
2. $B$ - a quaternion algebra over $F$ that is split at exactly 1 infinite place.
3. $f^B$ - the Jacquet-Langlands transfer of $f$ from $\mathrm{GL}_2(F)$ to $B^\times$. We will assume that such a transfer exists.
4. $E_k$ - an automorphic form on $\mathrm{GO}(B)$ where the quadratic form on $B$ is the norm form.
5. $K$ - a CM field of degree 4 over $\mathbb{Q}$ with totally real subfield $F$.
6. $w_K$ - a CM point on the Shimura variety for $\mathrm{GO}(B)$ that corresponds to $K$.
7. $\mathcal{D}(K, s)$ - a series that computes the value of $E_k$ at $w_K$.

8. $G_k$ - an automorphic form on $\mathrm{GO}(K)$ where the quadratic form on $K$ is a twist of the norm form.

9. $z_K$ - a CM point on the Shimura variety for $\mathrm{GO}(K)$ that corresponds to $K$.

10. $L$ - an imaginary quadratic field.

11. $z_L$ - a CM point on the Shimura variety for $\mathrm{GO}(K)$ that corresponds to $L$.

12. $\theta_L$ - a modular form on $\mathrm{GL}_2(\mathbb{Q})$ that is a theta series from $L$.

13. $\Omega$ - a modular form on $\mathrm{GL}_2(\mathbb{Q})$ that will be described later.

In this section $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product on $B^\times$ or $B^\times \times B^\times$. The argument begins by noting that automorphic forms on $\mathrm{GO}(B)$ can be restricted to automorphic forms on $B^\times \times B^\times$ and showing that the Asai $L$-function of a HMF $f$ can be computed as $\langle f^B \times f^B, \tilde{E}_k|_{B^\times \times B^\times} \rangle$ where $\tilde{E}_k$ is a modified version of $E_k$. Since this integral representation is an inner product of automorphic forms on a Shimura variety it gives the algebraicity of $L^{\mathrm{As}}(f, s)/\langle f^B \times f^B \rangle$ if $f^B$ and $E_k$ are algebraic. $f^B$ is chosen to be algebraic, but one must show that $E_k$ is algebraic. The relevant Shimura curves do not have cusps, so algebraicity is determined by examining values at CM points and showing these are algebraic up to appropriate CM periods.

The Shimura variety for $\mathrm{GO}(B)$ has CM points $w_K$ corresponding to certain CM fields $K$ and the value of $E_k$ at such a CM point is computed by the series $\mathcal{D}(K, s)$:

$$E(w_K) \sim \mathcal{D}(K, s_0)$$

for some integer $s_0$. Viewing $K$ as a quadratic space we can take the group $\mathrm{GO}(K)$ and the corresponding Shimura variety will have a CM point $z_K$ corresponding to $K$. The automorphic form $G_k$ on $\mathrm{GO}(K)$ is constructed so that

$$G_k(z_K) \sim \mathcal{D}(K, s_0).$$

The Shimura variety for $\mathrm{GO}(K)$ will also have CM points $z_L$ corresponding to imaginary quadratic fields $L$ and

$$G_k(z_L) \sim L(\theta_L \otimes \Omega, s_1)$$

for some integer $s_1$. The algebraicity of $L^{\mathrm{As}}(f, s)$ is then proven by bootstrapping up from the algebraicity of $L(\theta_L \otimes \Omega, s_1)$ (up to a CM period $\Omega_{\mathrm{CM},L}$) as follows:

1. Given $K$ we construct the automorphic form $G_k$ on $\mathrm{GO}(K)$ and show that for infinitely many $z_L$ the value $G_k(z_L)/\Omega_{\mathrm{CM},L} \sim L(\theta_L \otimes \Omega, s_1)/\Omega_{\mathrm{CM},L}$ is algebraic. Thus $G_k$ is algebraic.

2. This implies that

$$G_k(z_K)/\Omega_{\mathrm{CM},K} \sim \mathcal{D}(K, s_0)/\Omega_{\mathrm{CM},K} \sim E_k(w_K)/\Omega_{CM,K}$$

   is algebraic for a CM period $\Omega_{\mathrm{CM},K}$ depending on $K$. Doing this for infinitely many $w_K$ shows that $E_k$ is algebraic.

3. The algebraicity of $E_k$ implies the algebraicity of $\tilde{E}_k$. Thus $\langle f^B \times f^B, \tilde{E}_k \rangle$ is an algebraic multiple of $\langle f^B, f^B \rangle^2$. Since

$$\langle f^B, f^B \rangle L^{\mathrm{As}}(f, s_2) \sim \langle f^B \times f^B, \tilde{E}_k \rangle$$

   we get an algebraicity result for

$$L^{\mathrm{As}}(f, s_2)/\langle f^B, f^B \rangle$$

   .

## 5.2 Automorphic forms on a quaternion algebra

This section defines automorphic forms on general quaternion algebras as in [44]. Let $F$ be a totally real number field with infinite places $\tau_1, \ldots, \tau_n$ and $B$ a quaternion algebra over $F$ that is split at $\tau_1, \ldots, \tau_r$ and ramified at the other infinite places. Fix an isomorphism

$$B \otimes_{\mathbb{Q}} \mathbb{R} \to \mathrm{M}_2(\mathbb{R})^r \times \mathbb{H}^{n-r}$$

that extends the map $F \to \mathbb{R}^n$ defined by $a \mapsto (a^{\tau_1}, \ldots, a^{\tau_n})$. Let $G$ be the $\mathbb{Q}$-rational algebraic group whose $A$-points for any $\mathbb{Q}$-algebra $A$ are $(B \otimes_{\mathbb{Q}} A)^{\times}$. Then we define the adelization $G_{\mathbb{A}} \cong (B \otimes_{\mathbb{Q}} \mathbb{A})^{\times}$ and the infinite part $G_\infty \cong \mathrm{GL}_2(\mathbb{R})^r \times (\mathbb{H}^{\times})^{n-r}$. Let $N$ denote the norm map $N : B_{\mathbb{A}} \to F_{\mathbb{A}}$. Let $G_{\infty+}$ denote the identity component of $G_\infty$ and $G_{\mathbb{A}+}$ (resp. $G_{\mathbb{Q}+}$) denote the subgroup of $G_{\mathbb{A}}$ (resp. $G_{\mathbb{Q}}$) consisting of elements with infinite part in $G_{\infty+}$. For $\gamma \in G_{\mathbb{A}}$ we denote its infinite part by $\gamma_\infty$ and for $1 \leq i \leq n$ we denote the infinite part at $\tau_i$ by $\gamma^{(i)}$. This group $G_{\infty+}$ acts on $\mathcal{H}^r$ by

$$\gamma(z_1, \ldots, z_r) = \left( \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_r z_r + b_r}{c_r z_r + d_r} \right)$$

for

$$\gamma^{(i)} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, (z_1, \ldots, z_r) \in \mathcal{H}^r.$$

For such a $\gamma$, $(z_1, \ldots, z_r)$ and $k = \sum_{i=1}^r k_i \tau_i$, $k_i \in \mathbb{Z}$ define

$$j(\gamma, z)^k := \prod_{i=1}^r (c_i z_i + d_i)^{k_i} |N(\gamma^{(i)})|^{-k_i/2}.$$

For a $\mathbb{C}$ valued function $f$ on $\mathcal{H}^r$ define

$$(f|_k \gamma)(z) = j(\gamma, z)^{-k} f(\gamma(z)).$$

Fix a maximal order $\mathcal{O}_B$ of $B$ and define

$$\Gamma_N = \{\gamma \in \mathcal{O}_B : N(\gamma) = 1, \gamma - 1 \in N\mathcal{O}_B\}.$$

**Definition V.1.** A *congruence subgroup of* $G_{\mathbb{Q}+}$ *is a subgroup* $\Gamma \subset G_{\mathbb{Q}+}$ *such that* $\Gamma_N \subset \Gamma$ *and* $[\Gamma \mathcal{O}_F^{\times} : \Gamma_N \mathcal{O}_F^{\times}] < \infty$ *for some* $N$.

Let $\mathcal{A}_k(\Gamma)$ denote the set of meromorphic functions $f : \mathcal{H}^r \to \mathbb{C}$ (also meromorphic at the cusps) such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$. Let $\mathcal{A}_k$ denote the union of $\mathcal{A}_k(\Gamma)$ over all congruence subgroups $\Gamma$. Similarly define $\mathcal{M}_k(\Gamma)$, $\mathcal{M}_k$ as holomorphic functions satisfying $f|_k\gamma = f$. Furthermore define $\mathcal{S}_k(\Gamma)$ (resp. $\mathcal{S}_k$) to be the subset of $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{M}_k$) consisting of functions which vanish at the cusps.

Let $W_0$ be an open compact subgroup of $G(\mathbb{A}_f)$ and $W = W_0 G_{\infty+}$.

**Definition V.2.** Let $\mathcal{M}_k(W)$ be the set of all $\mathbb{C}$-valued functions $\mathbf{f}$ on $G_\mathbb{A}$ such that

$$\mathbf{f}(\alpha\gamma w) = \mathbf{f}(\gamma) \text{ for all } \alpha \in G_\mathbb{Q}, \gamma \in G_\mathbb{A}, w \in W, w_\infty = 1$$

and for every $\gamma \in G_\mathbb{A}$ with $\gamma_\infty = 1$ there is an element $g_\gamma \in \mathcal{M}_k$ such that

$$\mathbf{f}(\gamma y) = (g_\gamma|_k y)(\mathbf{i})$$

for all $y \in G_{\infty+}$. Here $\mathbf{i} = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathcal{H}^r$.

**Lemma V.3** (see [44] p.574)**.** *For any such $W$ there exists an integer $h$ and elements $x_1, \ldots, x_h \in G_\mathbb{A}$ each with $x_{i\infty} = 1$ such that*

$$F_\mathbb{A}^\times = \bigsqcup_{i=1}^{h} N(x_i) F^\times N(W).$$

*Let $\Gamma_i = x_i W x_i^{-1} \cap G_\mathbb{Q}$. Then we have a bijection*

$$\mathcal{M}_k(W) \cong \prod_{i=1}^{h} \mathcal{M}_k(\Gamma_i)$$

*given by $\mathbf{f} \mapsto (f_1, \ldots, f_g)$ where*

$$\mathbf{f}(\alpha N(x_i)^{-1} x_i w) = (f_i|_k w)(\mathbf{i}).$$

*In fact $G_\mathbb{Q} \backslash G_\mathbb{A} / W \cong \bigsqcup_{i=1}^{h} \Gamma_i \backslash \mathcal{H}^r$.*

Suppose for each place $\nu$ of $F$, $W_\nu = \mathcal{O}_{B\nu}^\times$. Then $N(W) \subset \mathcal{O}_{F\nu}^\times$ for each finite place $\nu$ and $N(W) \subset F_{\nu+}^\times$ for each infinite place $\nu$. Thus $h$ divides the narrow class number of $F$. From now on we assume that $W_\nu = \mathcal{O}_{B\nu}^\times$ for each place $\nu$ and $F$ has narrow class number 1. Thus our lemma implies $G_\mathbb{Q}\backslash G_\mathbb{A}/W$ has only one component $\Gamma\backslash\mathcal{H}^r$ in this case ($\Gamma = W \cap G_\mathbb{Q}$).

### 5.3 The Asai $L$-function on a quaternion algebra

In this section we describe a variation of the Asai $L$-function which applies to automorphic forms over quaternion algebras $B$ that are not $\mathrm{GL}_2$.

Let $F$ be a real quadratic field with infinite places $\tau_1, \tau_2$ and $B$ a quaternion algebra over $F$ that is unramified at $\tau_1$ and ramified at $\tau_2$. Suppose that $F$ has a unit of norm $-1$ and take $\nu \in F$ to be a totally positive generator of the inverse different of $F$. Also assume that $F$ has odd discriminant and narrow class number 1. Since in this case $r = 1$, $k$ has the form $k = k_1\tau_1$ with $k_1 \in \mathbb{Z}$. Let $f^B$ be in $\mathcal{S}_k$. Further assume that $f^B$ is an eigenform with eigenvalue $\chi(\mathfrak{a})$ for the Hecke operator $T(\mathfrak{a})$ (here $\mathfrak{a}$ runs over fractional ideals in $E$, but $\chi()$ is understood to be 0 if $\mathfrak{a}$ is not integral). We define the modified Asai $L$-function of $f$

$$L^{*\mathrm{As}}(f^B, s) = \sum_{0<b\in\mathbb{Q}} \chi(b\nu)(b\nu)^{-s}.$$

This $L$-function satisfies a functional equation with center $3/2$. The critical points right of center are $2, \ldots, k/2$ when $k$ is even, $k > 2$. Our main result concerns the rightmost critical point $s = k/2$. In the case when $B = \mathrm{GL}_2(F)$ this differs from our previous definition only by the $\zeta$-function factor and the power of $\nu$.

**Theorem V.4** (Shimura, [44] Thm. 3.1). *Let $f^B$ be an eigenform with eigenvalues $\chi(\mathfrak{a})$ and weight $(k, 0)$. Then*

$$J(f)(z) := \sum_{0\ll b\in\mathcal{O}_F} \chi(b\nu)(b\nu)^{(k/2-1)\tau_1} e^{2\pi i\,\mathrm{Tr}(b\nu z)}$$

*is a weight $(k, 2)$ Hilbert modular form for $F$.*

In the case where $f^B$ has weight $k$, $J(f)$ will have weight $(k,2)$ and

$$L^{\mathrm{As}}(J(f),s) = \nu^{s\tau_1}\zeta(2s-k)L^{*\,\mathrm{As}}(f^B, s+1-k/2). \tag{5.3.1}$$

**Definition V.5.** When $f$ is a Hilbert modular form for $F$ that is in the image of the map $J$ we let $f^B$ be an eigenform with $J(f^B) = f$ and say $f^B$ is the *Jacquet-Langlands transfer of $f$ to $B$.*

This transfer was studied by Jacquet and Langlands in [26] sec. 14 and 16.

## 5.4 Automorphic Forms on an Orthogonal Group

Let $E$ be $\mathbb{Q}$ or a real quadratic number field. Let $\tau_1$ denote the unique infinite place if $E = \mathbb{Q}$ and $\tau_1, \tau_2$ denote the two infinite places if $E$ is real quadratic. Let $V$ be a 4 dimensional vector space over $E$ with a quadratic form $S$ of signature $(2,2)$ at $\tau_1$. If $E$ is real quadratic we also require $S$ to be positive definite at $\tau_2$.

Let $V_i = V \otimes_{\tau_i, E} \mathbb{R}$ and $S_i$ denote the extension of $S$ to $V_i$. Consider the quadratic spaces $\mathrm{M}_2(\mathbb{R})$ and $\mathbb{H}$ (the Hamilton quaternions) over $\mathbb{R}$ with the standard quadratic form

$$S(x,y) := xy^\iota + yx^\iota.$$

Here $\iota$ denotes the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ on } \mathrm{M}_2(\mathbb{R})$$

$$(a+bi+cj+dk)^\iota = a - bi - cj - dk \text{ on } \mathbb{H}.$$

There exists isomorphisms of quadratic spaces $V_1 \cong \mathrm{M}_2(\mathbb{R})$ and $V_2 \cong \mathbb{H}$.

Let

$$\mathfrak{N} = \{x \in V_1 \otimes \mathbb{C} : S_1(x,x) = 0, S_1(\bar{x},x) < 0\}.$$

Under our isomorphism $V_1 \cong \mathrm{M}_2(\mathbb{R})$ we can view $\mathfrak{N}$ as a subset of $\mathrm{M}_2(\mathbb{C})$. Suppose

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{N}.$$

Then the condition $S_1(x, x) = 0$ implies $\det(x) = 0$ and the condition $S_1(\bar{x}, x) < 0$ implies $\Re(\bar{a}d - \bar{b}c) < 0$. Thus $c \neq 0$ and we can write

$$x = \begin{pmatrix} a' & a'd' \\ 1 & d' \end{pmatrix}$$

where

$$0 > \Re(\bar{a}'d' - \bar{a}'\bar{d}') = \Re(\bar{a}'(d' - \bar{d}'))$$

$$\Re(2i\bar{a}'\Im(d')) = 2\Im(a')\Im(d').$$

Thus $\mathfrak{N}$ has two components corresponding to $a' \in \mathcal{H}, d' \in \mathcal{H}^-$ or $a' \in \mathcal{H}^-, d' \in \mathcal{H}$ and we can define an isomorphism $\mathbb{C} \times \mathcal{H}^2 \to \mathfrak{N}$ onto the $a' \in \mathcal{H}, d' \in \mathcal{H}^-$ component of $\mathfrak{N}$ by

$$\zeta \times (z_1, z_2) \mapsto \zeta \cdot \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}.$$

Let $p : \mathcal{H}^2 \to \mathfrak{N}$ denote the map

$$p : (z_1, z_2) \mapsto \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}$$

so that each element in the $a' \in \mathcal{H}, d' \in \mathcal{H}^-$ component of $\mathfrak{N}$ is a unique scalar multiple of a unique element in the image of $p$.

Let

$$G(V) = \{\alpha \in \mathrm{GL}(V, E) : S(\alpha x, \alpha y) = \nu(\alpha) S(x, y), \nu(\alpha) \in E_{\gg 0}\}.$$

Then $G(V)$ acts on $\mathrm{M}_2(\mathbb{C})$ through our isomorphism $V_1 \otimes \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$. Let $G_+(V)$ be the subgroup of $G(V)$ that preserves the components of $\mathfrak{N}$. For any $\alpha \in G_+(V)$

and $z \in \mathcal{H}^2$ there is a unique $\mu(\alpha, z) \in \mathbb{C}$, $\alpha z \in \mathcal{H}^2$ such that

$$\alpha p(z) = \mu(\alpha, z) \cdot p(\alpha z).$$

This defines an action of $G_+(V)$ on $\mathcal{H}^2$ and an automorphy factor $\mu : G_+(V) \times \mathcal{H}^2 \to \mathbb{C}$.

For $k \in \mathbb{Z}$, $\mathfrak{f}$ a function on $\mathcal{H}^2$ and $\alpha \in G_+(V)$ we define

$$(\mathfrak{f}||_k\alpha)(z) = \mu(\alpha, z)^{-k}|N(\alpha)|^{k\tau_1/2}\mathfrak{f}(\alpha z).$$

**Definition V.6.** Then we say a function $\mathfrak{f}$ on $\mathcal{H}^2$ is a *weight $k$ automorphic form* for a congruence subgroup $\Gamma \subset G_+(V)$ if $\mathfrak{f}||_k\alpha = \mathfrak{f}$ for all $\alpha \in \Gamma$.

We now present two examples of these automorphic forms. In the case where $E = \mathbb{Q}$ we pick a congruence subgroup $X$ in $V$ and a modular form $\Omega$ with $q$-expansion coefficients $\omega(a)$ for $a \in \mathbb{Z}$ and define a $\mathbb{C}$-valued function $G_k$ on $\mathcal{H}^2 \times \mathbb{C}$ by

$$G_k(z, s; X) := \sum_{x \in X} \omega(-S[x])S(x, p(z))^{-k}|S(x, p(z))|^{-2s}.$$

In the case where $E = F$ a real quadratic field we denote the extension of $S$ to $V_1$ by $S_1$ and for $x \in F$ we denote by $x_1$ it's image in $V_1$. We can then take a congruence subgroup $X$ of $B$ and a modular form $\Omega$ (with $q$-expansion coefficients $\omega$) and define

$$E_k(z, s; X) := \sum_{x \in X} \omega(-\operatorname{Tr}_{F/\mathbb{Q}}(S[x]))S_1(x_1, p(z))^{-k}|S_1(x_1, p(z))|^{-2s}.$$

Although Shimura studied these automorphic forms in general, for $E_k$ we will only be interested in the case where the modular form $\Omega$ is constant and equal to 1. In this case $\omega(a) = 1$ for $a = 0$ and otherwise $\omega(a) = 0$ so we can also write

$$E_k(z,s;X) := \sum_{x \in X \mid \mathrm{Tr}_{F/\mathbb{Q}}(-S[x])=0} S_1(x_1,p(z))^{-k}|S_1(x_1,p(z))|^{-2s}.$$

### 5.5 Integral representation

Let $w \in \mathcal{H}$ and denote by $d\mu(w)$ the measure $y^{-2}dxdy$ on $\mathcal{H}$ where $w = x+iy$. For $f^B, g^B$ automorphic forms of level $k$ with respect to the congruence subgroup $\Gamma \subset B_+^\times$, let $\langle \cdot,\cdot \rangle$ denote the Petersson inner product

$$\langle f^B, g^B \rangle = \mu(\Gamma \backslash \mathcal{H})^{-1} \int_{\Gamma\backslash\mathcal{H}} \overline{f^B(w)} g^B(w) \Im(w)^k d\mu(w).$$

Let $\epsilon \in F$ be a totally positive unite such that $\nu = \epsilon/\sqrt{D}$. Also choose $\delta \in B$ such that $N(\delta) = -\epsilon$. Let

$$\mathcal{O}_F^\vee := \{a \in F : \mathrm{Tr}_{F/\mathbb{Q}}(ab) \in \mathbb{Z} \text{ for all } b \in \mathcal{O}_F\}.$$

Further define

$$A(w,w',s) := \pi^{-s}\Gamma(k+s)\Im(w)^s\Im(w')^s,$$

$$X_0 := \{b \in B : N(b) \in \mathcal{O}_F^\vee\},$$

and

$$\tilde{E}_k(w,w',s) := \pi^{-|k|}E_k((w,\delta\bar{w}'),s;X_0)j(\delta,\bar{w}')^{-k}|j(\delta,w')|^{-2s}.$$

Shimura proves

**Theorem V.7** ([44], sec. 6)**.**

$$\int_{\Gamma\backslash\mathcal{H}} \overline{f(w)} A(w,w',s)\tilde{E}_k(w,w',s)\Im(w)^k d\mu(w)$$

$$= (-2i)^k(4\pi)^{1-s-k}\epsilon^{k+s-1}\Gamma(k+s-1)L^{*\,\mathrm{As}}(f^B,k/2-s)\overline{f^B(w')}.$$

Setting $s = 0$ and taking the Petersson inner product against $f^B(w')$ we get

$$\int_{(\Gamma\backslash\mathcal{H})^2} \overline{f^B(w)} f^B(w') A(w, w', 0) \tilde{E}_k(w, w', 0) \Im(w)^k \Im(w')^k d\mu(w) d\mu(w')$$

$$= (-2i)^k (4\pi)^{1-k} \epsilon^{k-1} \Gamma(k-1) L^{*\,\mathrm{As}}(f^B, k/2) \langle f^B, f^B \rangle \mu(\Gamma\backslash\mathcal{H}).$$

**Proposition V.8.**

$$\int_{(\Gamma\backslash\mathcal{H})^2} \overline{f^B(w)} f^B(w') A(w, w', 0) \tilde{E}_k(w, w', 0) \Im(w)^k \Im(w')^k d\mu(w) d\mu(w')$$

$$= C \langle f^B, f^B \rangle^2 \mu(\Gamma\backslash\mathcal{H})^2$$

*with $C$ algebraic.*

*Proof.* This will be shown in Section 5.10 as a consequence of the algebraicity of $\tilde{E}_k(w, w', 0)$. □

We also note that $\mu(\Gamma\backslash\mathcal{H})$ is a rational multiple of $\pi$, so the proposition and theorem together imply

**Theorem V.9** (Shimura [44] Thm. 3.3)**.**

$$\frac{L^{*\,\mathrm{As}}(f^B, k/2)}{\pi^k \langle f^B, f^B \rangle} \in \bar{\mathbb{Q}}.$$

### 5.6 CM Points on the orthogonal Shimura variety

Let $V$ be a quadratic space over $E$ where $E$ is a real quadratic field or $\mathbb{Q}$. Let $Y$ be an $E$-algebra which is a direct sum of CM fields over $E$ and copies of $E$ and let $\rho$ denote the unique positive involution of $Y$. Let $J_Y$ denote the set of homomorphisms $Y \to \mathbb{C}$. Suppose we have an embedding $h : Y \to \mathrm{End}(V, E)$ that maps the identity to the identity and satisfies

$$S(h(a)x, y) = S(x, h(a^\rho)y).$$

If we set

$$Y^u := \{y \in Y : yy^\rho = 1\}$$

then $h$ maps $Y^u$ into $G(S)$. We also assume $h(Y^u)$ is contained in $G_+(S)$.

Then since the closure of $h(Y^u)$ is compact it has a fixed point $z$ in $\mathfrak{Z}$ and we refer to points of this type as CM points.

The CM point gives us a representation $\psi : Y \to \mathbb{C}$ which is the unique one satisfying $\psi(a) = \mu(h(a), w)$ for $a \in Y^u$.

We now recall Proposition 5.4 from [42]:

**Proposition V.10.** *There exists an element $v \in V$ such that $V = h(Y)v$ and for any such choice of $v$ there is a unique element $\delta \in Y$ such that*

$$S(h(a)v, h(b)v) = \mathrm{Tr}_{Y/E}(\delta a b^\rho)$$

*for all $a, b \in Y$. This $\delta$ satisfies $\delta^\rho = \delta$; $\delta^\psi < 0$; $\delta^\alpha > 0$ for all $\alpha \in J_Y$ not equal to $\psi$ or $\psi\rho$.*

## 5.7 Evaluating $E_k$ at CM Points

Using the notation from the previous section, take $w_K$ a CM point associated to $Y$ of the form $Y = K \oplus F \oplus F$ where $K$ is a CM field of degree 2 over $F$. Define $W \subset V$ by $W := h(F \oplus F)v$ (this is the $v$ described in proposition V.10 satisfying $h(Y)v = V$). Thus $V$ is spanned by $W$ and $h(K)v$ so each element of $V$ can be written as $h(a)v + x$ with $a \in K, x \in W$. But then

$$S(h(a)v + x, h(b)v + y) = \mathrm{Tr}_{K/E}(\delta_0 a b^\rho) + S'(x, y)$$

for $a, b \in K$ and $x, y \in W$ where $S'$ denotes the restriction of $S$ to $W$ and $\delta_0$ is the projection of the $\delta$ from V.10 to $K$. Let $u := p(w_K)$ which Shimura shows ([42], page 328, 5.10) satisfies $S_1(W,u)=0$. Thus $S_1(h(b)v_1+y_1,u)=S(v_1,h(b^\rho)u)=b^{\rho\psi}S(v_1,u)$. Setting $d:=S(v_1,u)$ we see that

$$S_1(h(b)v_1+y_1,p(w))^{-k}|S(h(b)v_1+y_1,p(w))|^{-2s}$$

$$=d^{-k}|d|^{-2s}b^{-k\rho\psi}|b^\psi|^{-2s}.$$

Thus we can write

$$E_k(w_K,s;X_0)=\sum_{h(a)v+y\in X_0:\mathrm{Tr}_{K/\mathbb{Q}}(-\delta_0 aa^\rho)+\mathrm{Tr}_{E/\mathbb{Q}}(-S'[y])=0} d^{-k}|d|^{-s2}a^{-k\rho\psi}|a^\psi|^{-2s}.$$

Now suppose that $X_0$ decomposes as $h(X_1)v+X_2$ for congruence subsets $X_1\subset K$, $X_2\subset W$. We can define a modular form

$$\Omega(z):=\sum_{y\in X_2}\mathbf{e}_{\mathbb{Q}}(\mathrm{Tr}_{E/\mathbb{Q}}(S'[y])z).$$

For $\Omega$ to be well-defined we need to verify that $S'[y]$ is totally positive, but $S$ is positive definite at $\tau_2$ so $S'$ must be positive there and $S_1$ is negative definite on $h(X_1)$ by the proposition so $S_1$ must be positive definite on $X_2$. This makes $\Omega(z)$ a weight 2 modular form for $\mathrm{GL}_2(\mathbb{Q})$. Writing the $q$-expansion of $\Omega(z)=\sum_{a\in\mathbb{Z}}\omega(a)\mathbf{e}_{\mathbb{Q}}(az)$ we have

$$\omega(a)=\#\{y\in X_2:\mathrm{Tr}_{E/\mathbb{Q}}(S'[y])=a\}.$$

Therefore

$$E_k(w_K,0;X_0)=d^{-k}\sum_{a\in X_1}\omega(\mathrm{Tr}_{K/\mathbb{Q}}(-\delta_0 aa^\rho))a^{-k\rho\psi}.$$

We now define the functions $\mathcal{D}(s)$ which will correspond to the values of $E_k$ at CM points.

**Definition V.11.** Let $K/\mathbb{Q}$ be a degree 4 CM field, $X$ a congruence subset $X\subset K$, and take a modular form $G(z)=\sum_{n\in\mathbb{Z}}g(n)\mathbf{e}_{\mathbb{Q}}(nz)$. Choose an embedding $\phi:K\to$

$\mathbb{C}$ and an element $\eta$ in the totally real subfield of $K$ satisfying $\eta^\phi > 0$ and $\eta$ negative at all places other than $\phi$ and $\rho\phi$. Then

$$\mathcal{D}_k(s) := \sum_{b \in X} g(\mathrm{Tr}_{K/\mathbb{Q}}(\eta b b^\rho)) b^{-k\rho\phi} |b^\phi|^{-2s}.$$

Choosing $K$ as the field corresponding to a CM point, $X_1$ for the congruence subset $X$, $\Omega(z)$ for $G(z)$, $\psi$ for $\phi$, and $-\delta_0$ for $\eta$ we see that

$$d^{-k}\mathcal{D}_k(0) = d^{-k} \sum_{a \in X_1} \omega'(\mathrm{Tr}_{K/\mathbb{Q}}(-\epsilon a a^\rho)) a^{-k\rho\psi} = E_k(w_K, 0; X_0). \tag{5.7.1}$$

Note that Proposition 9.1 in [42] shows $\mathcal{D}_k(s)$ converges absolutely at $s = 0$ for $k \geq 6$.

## 5.8 Relating $\mathcal{D}(s)$ and $G_k$

The next step is to relate the values $\mathcal{D}_k(0)$ to the values of various $G_k$'s at CM points. Recall that our construction of $G_k$ began with $V$ (a 4 dimensional $\mathbb{Q}$ vector space) with quadratic form $S$ of signature $(2, 2)$. In the previous section we had a degree 4 CM field $K$ and the definition of $\mathcal{D}_k$ involved summing over elements in $X_1$, a congruence subset of $K$. We construct $G_k$ by taking the modular form to be $H(z)$ and the quadratic space to be $K$ viewed as a $\mathbb{Q}$-vector space with quadratic form given by:

$$S(a, b) = -\mathrm{Tr}_{K/\mathbb{Q}}(\delta_0 a b^\rho),$$

where $\delta_0 = \delta_0^\rho \in K$ and there is a place $\psi : K \to \mathbb{C}$ such that $\delta_0^\psi$ is negative and $\delta_0$ is positive at every place of $K$ other than $\psi$ and $\psi\rho$. So we can take $\delta_0$ and $\psi$ as in the previous section.

The Shimura variety for $\mathrm{GO}(K)$ has an obvious CM point $z_K$ given by the regular representation of $K$, that is

$$h : K \mapsto \mathrm{End}_{\mathbb{Q}}(K), h(k) : a \mapsto ak \text{ for all } a \in K.$$

Applying Proposition V.10 it is clear that we can take $v = 1$ and $\delta = -\epsilon$. Thus implies that $a^{\psi\rho} = S(a, p(w))$ Thus we can write

$$\mathcal{D}_k(0) = \sum_{a \in X_1} \omega'(-S[a]) S(a, p(z_K))^{-k}.$$

But this is the value of $G_k$ at the CM point $z_K$ so

$$G_k(z_K, 0; X_1) = \mathcal{D}_k(0). \tag{5.8.1}$$

## 5.9 Evaluating $G_k$ at CM Points

Take $G_k, X_1$ as in the previous section. Consider a CM point $z_L$ corresponding to a map $h : Y = L \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{End}(K, \mathbb{Q})$ where $L$ is an imaginary quadratic field. We now proceed to evaluate $G_k(z_L, s; X_1)$ as we did with $E_k$.

Proposition V.10 implies there is an $v \in K$ such that $h(Y)v = K$ and setting $M := h(\mathbb{Q} \oplus \mathbb{Q})v \subset K$ we have

$$S(h(a)v + x, h(b)v + y) = \mathrm{Tr}_{L/\mathbb{Q}}(-\delta_1 a b^\rho) + S'(x, y)$$

for $a, b \in L$ and $x, y \in M$ where $S'$ is the restriction of $S$ to $M$ and $\delta_1$ is the projection of $\delta$ from Prop V.10. Letting $u := p(z_L)$ we have $S(M, u) = 0$ and $S(h(b)v + y, u) = b^{\rho\phi} S(v, u)$ for some character $\phi$ of $L$. So we can set $e := S(v, u)$ and get

$$G_k(z_L, s; X_1) = \sum_{h(b)v + x \in X_1} \omega'(\mathrm{Tr}_{L/\mathbb{Q}}(-\delta_0 a a^\rho) - S'[x]) e^{-k} |e|^{-2s} a^{-k\rho\phi} |a^\psi|^{-2s}.$$

Just as before we suppose $X_1$ decomposes as $h(X_3)v + X_4$ for congruence subsets $X_3 \subset L$ and $X_4 \subset M$. $S'$ is totally positive so it makes sense to define a modular form

$$H(z) := \sum_{x \in X_4} \mathbf{e}_{\mathbb{Q}}(S'[x] z)$$

which has weight one. We can then look at the $q$-expansion

$$\Omega'(z) = H(z)\Omega(z) = \sum_{a\in\mathbb{Z}} \omega'(a)\mathbf{e}_{\mathbb{Q}}(az)$$

where

$$\omega'(a) = \sum_{x\in X_4} \omega(a - S'[x]).$$

Since $\Omega(z)$ was a weight 2 modular form and $H(z)$ is weight 1, $\Omega'(z)$ is a weight 3 modular form. Therefore

$$G_k(z_L, 0; X_1) = e^{-k} \sum_{b\in X_3} \omega''(-2\delta_1 bb^\rho) b^{-k\rho\phi}.$$

Define

$$\theta_k(z) = \sum_{b\in X_3} b^{k\phi}\mathbf{e}_{\mathbb{Q}}(-2\delta_1 bb^\rho z)$$

which is a modular form of weight $k+1$. Then

$$D(\Omega', \theta_k, k) = \sum_{a\in\mathbb{Z}} \omega'(a) \left( \sum_{b\in X_3 : -2\delta_1 bb^\rho = a} b^{k\phi} \right) a^{-k}$$

$$= \sum_{b\in X_3} \omega'(-2\delta_1 bb^\rho) b^{k\phi} (-2\delta_1 bb^\rho)^{-k} = (-2\delta_1)^{-k} e^k G_k(z_L, 0; X_1).$$

Hence

(5.9.1) $$G_k(z_L, 0; X_1) = (-2\delta_1)^k e^{-k} D(\Omega', \theta_k, k).$$

## 5.10 Proof of proposition V.8

Given a CM field $M$ and two a characters $\eta_1, \eta_2 : M \to \mathbb{C}$, Shimura ([42], Thm. 1.1) a period $p_M(\eta_1, \eta_2) \in \mathbb{C}^\times/\bar{\mathbb{Q}}^\times$ (we will explicitly define a refinement of it in the next chapter). Let $V$ be a quadratic space over a totally real field $E$ and let $w$ be a CM point for $\mathrm{GO}(V)$ coming from a map $h : Y \to \mathrm{End}_E(V)$ with $Y = M$ or $Y = M \oplus E \oplus E$. V.10 associates an embedding $\eta : M \to \mathbb{C}$ with $w$. Define

$$\Omega_w := p_M(\eta, \eta)^2.$$

Similarly, if $z$ is a CM point for $B_+^\times$ corresponding to an embedding $K \hookrightarrow B$ we have an infinite place $\phi : K\mathbb{C}$ defined by $\phi(a) = j(a, z)$ and define the period $\Omega_z = p_K(\phi, \phi)$.

Let $\mathcal{U}_k$ be the union over all congruence subgroups $\Gamma \subset G_+(V)$ of all weight $k$ automorphic forms for $\Gamma$.

**Definition V.12.** $\mathcal{U}_k(\bar{\mathbb{Q}})$ is the set of $\mathfrak{f} \in \mathcal{U}_k$ such that for every CM point $w$ as above where $\mathfrak{f}$ is holomorphic we have

$$\mathfrak{f}(w)/\Omega_w^k \in \bar{\mathbb{Q}}.$$

**Lemma V.13** (Shimura, [42] Thm. 6.5)**.** *For every $k$ and every point $w' \in \mathcal{H}^2$ there exists a $T \in \mathcal{U}_k(\bar{\mathbb{Q}})$ such that $T$ is holomorphic at $w'$ and $T(w') \neq 0$.*

Such a $T$ is called a rational uniformizer.

*Proof of Prop. V.8.* Returning to the setting of the previous section, we had shown

$$G_k(z_L, 0; X_1) = (-2\delta_1)^k e^{-k} D(\Omega', \theta_k, k).$$

Proposition 9.5 of [42] shows $D(\Omega', \theta_k, k)/\pi^k \Omega_{z_L}^k \in \bar{\mathbb{Q}}$. Let $T$ be a weight $k$ rational uniformizer for the point $z_K$ of section 5.8. Then the function $G_k(z, 0; X_1)/T(z)$ takes values in $\bar{\mathbb{Q}}$ at every point $z_L$ as in section 5.9. By [42] p.357 these $z_L$ are dense in $\mathcal{H}^2$. The function $G_k(z, 0; X_1)/T(z)$ is an automorphic function (a weight 0 automorphic form) on $\mathcal{H}^2$. Such a function is algebraic on a dense set of CM points then it is algebraic at every CM point. Thus $G_k/\pi^k \in \mathcal{U}_k$.

This, along with equations 5.7.1 and 5.8.1, implies that $E_k(w_K, 0; X_0)/\pi^k \Omega_{z_K} \in \bar{\mathbb{Q}}$. In the next chapter we will prove $\Omega_{z_K} = \Omega_{w_K}$ and therefore $\tilde{E}_k(w_K, 0)/\Omega_{w_K} \in \bar{\mathbb{Q}}$. There is an infinite set of pairs of CM points $w, w' \in \mathcal{H}$ for $B_+^\times$ each with CM field $K$, such that the point $w_K = (w, w')$ is a CM point corresponding to $K$ and satisfies

$\Omega_{w_K} = \Omega_w \Omega_{w'}$ (this will be proved in the next chapter). The points $w_K$ of this form are dense in $\mathcal{H}^2$ and this implies $\tilde{E}_k(w, w', 0)$ can be written as a linear combination

$$A(w, w', 0)\tilde{E}_k(w, w', 0) = \sum_{i,j} c_{ij} f_i(w) f_j(w')$$

with $c_{ij} \in \bar{\mathbb{Q}}$ and $f_i, f_j$ normalized eigenforms for $B_+^\times$ (see [44] lemma 4.1). Set $f_1 = f^B$ so $\langle f^B, f_i \rangle = 0$ for all $i \neq 1$.

Then

$$\int_{(\Gamma\backslash\mathcal{H})^2} \overline{f^B(w)} f^B(w') A(w, w', 0)\tilde{E}_k(w, w', 0) \Im(w)^k \Im(w')^k d\mu(w) d\mu(w')$$

$$= c_{1,1} \langle f, f \rangle^2 \mu(\Gamma\backslash\mathcal{H})^2,$$

which completes the proof.

□

### 5.11 An Example

Since the strategy above is rather involved, we now work out an example similar to (but not the same) as the case considered in the previous sections. We will now compute an explicit example of the value of $G_k$ at a CM point as described in the previous section. We take $V = \mathbb{Q}^4$ (with standard basis $v_1, v_2, v_3, v_4$) and $S = \mathrm{diag}(s_1, s_2, s_3, -1)$ where $s_i > 0$ for $i = 1, 2$ and $s_3 < 0$. Let $L = \mathbb{Q}(\zeta)$ the imaginary quadratic field where $\zeta^2 = -s_3$. Set $Y = L \oplus \mathbb{Q} \oplus \mathbb{Q}$ and define $h : Y \to \mathrm{End}_{\mathbb{Q}}(V)$ by

$$h(a + b\zeta, c_1, c_2) = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & a & b \\ & & \zeta^2 b & a \end{pmatrix}.$$

This corresponds to a CM point $w$. We identify $V_{\mathbb{R}}$ with $\mathrm{M}_2(\mathbb{R})$ by

$$v_1 \mapsto \sqrt{\frac{s_1}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v_2 \mapsto \sqrt{\frac{s_2}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$v_3 \mapsto \sqrt{\frac{-s_3}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v_4 \mapsto \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can calculate

$$u = p(w) = \sqrt{\frac{2}{-s_3}}v_3 + \sqrt{2}iv_4 \mapsto \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

One choice for the $v$ in proposition V.10 is $v = v_1 + v_2 + v_3$.

Thus $S(v, u) = \sqrt{\frac{2}{-s_3}}S(v_3, v_3) = -\sqrt{-2s_3}$. We can also calculate $M = h(\mathbb{Q} \oplus \mathbb{Q})v = \mathbb{Q}v_1 + \mathbb{Q}v_2$ and $h(L)v = \mathbb{Q}v_3 + \mathbb{Q}v_4$. In particular $h(a + b\zeta)v = av_3 + b\zeta^2 v_4$. We can also calculate that for $a + b\zeta, c + d\zeta \in L$

$$S(h(a + b\zeta)v, h(c + d\zeta)v) = \mathrm{Tr}_{L/\mathbb{Q}}\left(\frac{s_3}{2}(a + b\zeta)(c - d\zeta)\right)$$

so if we let $\epsilon$ be as in the previous section $\epsilon = -s_3/2$. Thus $(-2\epsilon)^k e^{-k} = (-1)^k s_3^{k/2} 2^{-k/2}$ and we are left with

$$G_k(w, 0; X_1) = \frac{(-1)^k s_3^{k/2}}{2^{k/2}} L(\Omega'' \otimes \theta_k, k).$$

Both $\Omega''$ and $\theta_k$ depend on the congruence subgroup $X_1$. Let us first assume that $X_1 = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4$. This splits up as $h(X_3)v + X_4$ where $X_4 = \mathbb{Z}v_1 + \mathbb{Z}v_2$ and $h(X_3)v = \mathbb{Z}v_3 + \mathbb{Z}v_4$. Thus $a + b\zeta \in L$ is in $X_3$ if and only if $a \in \mathbb{Z}$ and $b\zeta^2 \in \mathbb{Z}$. But

$$a \in \mathbb{Z} \text{ and } b\zeta^2 \in \mathbb{Z} \iff a + b\zeta \in \frac{1}{\zeta}\mathbb{Z}[\zeta]$$

so $X_3 = (1/\zeta)\mathbb{Z}[\zeta]$. Thus

$$\theta_k(z) = \sum_{b \in X_3} b^k \mathbf{e}_{\mathbb{Q}}(s_3 b b^\rho z) = \sum_{b \in \mathbb{Z}[\zeta]} \zeta^{-k} b^k \mathbf{e}_{\mathbb{Q}}(b b^\rho z).$$

If $\mathbb{Z}[\zeta]$ has class number 1 then this is $\zeta^{-k}$ times the theta series associated to a Hecke character.

## CHAPTER VI

# Rationality of the Asai $L$-function I

### 6.1 Rational version of Shimura's Period

Let $K$ be a CM-field with totally real subfield $F$ and $\Phi$ a CM-type of $K$. Define $J_K$ be the set of embeddings $K \to \mathbb{C}$ and $I_K$ the free $\mathbb{Z}$-module on $J_K$. Furthermore define

$$I_K^0 = \left\{ \sum_{\tau \in J_K} c_\tau \tau \in I_K : c_\tau + c_{\tau\rho} \text{ is independent of } \tau \right\}.$$

Let $[F : \mathbb{Q}] = n$ (although we will actually only use the case $n \leq 2$). There exists some $a \in K$ such that $\Im(a^\phi) > 0$ for every $\phi \in \Phi$. If $\Phi = \sum \tau_i$ then $w := (a^{\tau_1}, \ldots, a^{\tau_n})$ defines a point in $\mathcal{H}^n$. Let $f, g$ be Hilbert modular forms on any congruence subgroup of weights

$$(k_1, \ldots, k_{r-1}, k_r + 1, k_{r+1}, \ldots, k_n) \text{ and } (k_1, \ldots, k_n)$$

respectively such that $f(w)g(w) \neq 0$ and both $f$ and $g$ are defined over $\mathbb{Q}_{ab}$.

**Proposition VI.1** (Shimura, [45] Thm. 7.10 and Thm. 9.6)**.** *Let $K_\Phi$ be the reflex field of $K$ at $\Phi$. Then every Hilbert modular function defined over $\mathbb{Q}_{ab}$ takes values in $K_{\Phi,ab}$ at $w$. If $[K : \mathbb{Q}] = 2$ then $K_{\text{ab}} = \Phi, ab$ and if $[K : \mathbb{Q}] = 4$ and $K$ is cyclic Galois or biquadratic then $K_{\Phi,ab} \subset K_{ab}$.*

Restricting to the cases where $K_{\Phi,ab} \subset K_{ab}$, the period $p_K(\tau_r, \Phi)$ is defined by

$$p_K(\tau_r, \Phi) = f(w)/g(w) \mod K_{ab}^\times.$$

This is independent of the choice of $f$ and $g$ because if $f_0, g_0$ were other choices, then $fg_0/gf_0$ is a Hilbert modular function and therefore $f(w)/g(w) \equiv f_0(w)/g_0(w)$ mod $K_{ab}^{\times}$.

Suppose another choice of $a$ is made. Let $a_0$ define the point $w_0 \in \mathcal{H}^n$. Suppose $a_0 = Aa + B$ for $A, B \in \mathcal{O}_F$. Then by looking at the $q$-expansion of $f(Az)$ we see that $f(Az)$ is defined over $\mathbb{Q}_{ab}$. Let $\gamma = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$. The $q$-expansion further shows that $f(\gamma(Az)) = \omega f(Az)$ where $\omega$ is a root of unity. Thus $f(\gamma(Az))$ is $\mathbb{Q}_{ab}$-rational and $f(\gamma(Aw)) = f(w_0)$. Since $f(\gamma(Az))/f(z)$ is a $\mathbb{Q}_{ab}$-rational automorphic function, $f(w_0)/f(w) = f(\gamma(Aw))/f(w) \in K_{ab}^{\times}$. The same holds for $g$ and so any choice of $a$ in $\mathcal{O}_F a + \mathcal{O}_F$ gives the same period. But any two choices $a_1$, $a_2$ will lie in $\mathcal{O}_F a + \mathcal{O}_F$ for some $a$ and thus our period is well-defined.

We now extend $p_K$ to a bilinear map $p_K : I_K \times I_K \to \mathbb{C}^{\times}/K_{ab}^{\times}$. First, if $\tau \notin \Phi$, then $\tau \in \Phi\rho$ and we define $p_K(\tau, \Phi) = p_k(\tau, \Phi\rho)^{-1}$. Now $p_k(\tau, \Phi)$ makes sense for any $\tau \in J_K$ and $\Phi$ a CM type. The CM types of $K$ generate $I_K^0$ and in the case $[K : \mathbb{Q}] = 2$, $I_K^0$ is free module generated by the CM types. In the case $[K : \mathbb{Q}] = 4$ let $J_K = \{\tau_1, \tau_2, \tau_1\rho, \tau_2\rho\}$ and

$$\Phi_1 = \tau_1 + \tau_2,\ \Phi_2 = \tau_1 + \tau_2\rho$$

$$\Phi_3 = \tau_1\rho + \tau_2,\ \Phi_4 = \tau_1\rho + \tau_2\rho$$

be the CM types of $K$. Then

$$I_K^0 \cong \mathbb{Z}[\Phi_1, \ldots, \Phi_4]/(\Phi_1 + \Phi_4 - \Phi_2 - \Phi_3).$$

For any $\tau \in J_K$ we have

$$p_K(\tau, \Phi_1)p_K(\tau, \Phi_4) = p_K(\tau, \Phi_2)p_K(\tau, \Phi_3) = 1$$

and so

$$\frac{p_K(\tau, \Phi_1)p_K(\tau, \Phi_4)}{p_K(\tau, \Phi_2)p_K(\tau, \Phi_3)} = 1.$$

Thus in both the cases $[K : \mathbb{Q}] = 2$ or $4$ we can extend the definition of $p_K$ linearly to $J_K \times I_K^0$. We further define

$$p_K(\tau, \tau') := \sqrt{p_K(\tau, \tau' - \tau'\rho)}$$

which extends $p_K$ to $J_K \times I_K$. We then extend $p_K$ linearly in the first variable to $I_K \times I_K$. This now satisfies

$$p_K(\alpha, \beta) = p_K(\alpha\gamma, \beta\gamma) \text{ for } \gamma \text{ an automorphism of } K, \text{ and}$$

$$p_K(\alpha, \beta\rho) = p_K(\alpha\rho, \beta) = p_K(\alpha, \beta)^{-1}.$$

**Proposition VI.2.** *Let $L$ be an imaginary quadratic field and $K$ a biquadratic CM field over $L$. Let $\psi_1$ be an infinite place of $L$ and $\psi, \phi$ the places of $K$ that restrict to $\psi_1$ on $L$. Then*

$$p_L(\psi_1, \psi_1) = p_K(\psi, \psi + \phi)$$

*up to an element in $K_{ab}^{\times}$.*

*Proof.* Let $F$ be the totally real subfield of $K$ and $f$, $g$ be Hilbert modular forms on $\mathrm{SL}_2(F)$ of weights $(k+1, l)$, $(k, l)$ respectively. Let $\alpha \in L$ such that $\alpha^{\psi_1} \in \mathcal{H}$. Then $\alpha^{\psi} = \alpha^{\phi} \in \mathcal{H}$. Therefore $f(\alpha^{\psi}, \alpha^{\phi})/g(\alpha^{\psi}, \alpha^{\phi}) = p_K(\psi, \psi + \phi)$ by definition.

Let $\bar{f}$ and $\bar{g}$ denote the restriction of $f$ and $g$ respectively to the diagonally embedded $\mathcal{H} \in \mathcal{H}^2$. Then $\bar{f}$ and $\bar{g}$ are modular forms on $\mathrm{SL}_2(\mathbb{Q})$ of weights $k + l + 1$ and $k + l$ respectively. In addition

$$\bar{f}(\alpha^{\psi_1}) = f(\alpha^{\psi_1}, \alpha^{\psi_1}) = f(\alpha^{\psi}, \alpha^{\phi})$$

and similarly for $g$. Thus

$$p_L(\psi_1, \psi_1) = \bar{f}(\alpha^{\psi_1})/\bar{g}(\alpha^{\psi_1}) = p_K(\psi, \psi + \phi).$$

$\square$

### 6.2 Rankin-Selberg Convolution and the Rational Period

Let $L$ be an imaginary quadratic field and set $p(L) = p_L(\tau_1, \tau_1) = p_L(\rho\tau_1, \rho\tau_1)$. Here $p_L$ is our new version of Shimura's period which is an element of $\mathbb{C}^\times/L_{ab}^\times$. We will show that, for certain modular forms $f$ and $g$, the special values of the Rankin-Selberg convolution $L$-function $L(f, g, s)$ is an $L_{ab}$-rational multiple of $p(L)$.

The modular form $f$ that we consider is determined by a congruence subset $X$ of $L$, an integer $k$, a constant $c < 0$, $c \in \mathbb{Q}$ and an infinite place $\psi$ of $L$. Recall that $e_{\mathbb{Q}}(x) = \exp(2\pi i x)$. $f$ is then defined by

$$f(z) := \sum_{a \in X} a^{k\psi} e_{\mathbb{Q}}(caa^\rho z).$$

$f$ is a weight $k+1$ modular form. Let $g$ be any modular form of weight $l < k$ and $l \equiv k \mod 2$. Let $f$ and $g$ have $q$-expansions

$$f(z) = \sum a(n) e_{\mathbb{Q}}(nz),\; g(z) = \sum b(n) e_{\mathbb{Q}}(nz).$$

We consider the Dirichlet series

$$D(f, g, s) = \sum a(n) b(n) n^{-s}.$$

**Theorem VI.3** (Shimura, [41] Thm. 5.6)**.** *Let $q$ be a weight $2k$ modular form defined over $L_{ab}$ and $w \in L$ such that $w^\psi \in \mathcal{H}$ and $q(w^\psi) \neq 0$. Then*

$$\frac{D(f, g, k)}{\pi^k q(w^\psi)} \in L_{ab}.$$

From the definition of $p(L)$ we can see that $q(w^{\psi}) = p(L)^{2k}$ so

$$\frac{D(f,g,k)}{\pi^k p(L)^{2k}} \in L_{ab}. \tag{6.2.1}$$

## 6.3 Rationality of $G_k$ at CM points

Write $G_k(z)$ for the function $G_k(z, 0; X_1)$ which is a weight $k$ automorphic form for the orthogonal group $\mathrm{GO}(K)$ of $K$ viewed as a quadratic space over $\mathbb{Q}$. Recall that $K$ is a biquadratic CM field with totally real subfield $F$ and the quadratic form on $K$ is defined by

$$S(a,b) = -\operatorname{Tr}_{K/\mathbb{Q}}(\delta_0 a b^{\rho})$$

where $\delta_0 \in F$ has $N_{F/\mathbb{Q}}(\delta_0) < 0$. The resulting Shimura variety has a specific CM point $z_K$ and a dense set of CM points $z_L$ corresponding to embeddings $h : L \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{End}(K, \mathbb{Q})$. The CM point $z_K$ is described in section 5.8 and we will describe the points $z_L$ later in this section.

Given a CM point $z_L$ we have an embedding $\psi : L \to \mathbb{C}$ defined by

$$\phi(a) = \mu(h^u(a), z_L) \text{ for all } a \in L^u.$$

Define the period $\Omega_{z_L}$ to be $p_L(\phi,\phi)^2$. Similarly for the CM point $z_K$ we have an embedding $\psi : K \to \mathbb{C}$ defined by

$$\psi(a) = \mu(h^u(a), z_K) \text{ for all } a \in K^u.$$

and we take the period $\Omega_{z_K}$ to be $p_K(\psi,\psi)^2$.

**Lemma VI.4.** *For each $z_L$, $G_k(z_L)/\pi^k \Omega_{z_L}^k \in L_{ab}$.*

*Proof.* We have explicitly calculated

$$G_k(z_L) = cD(\Omega', \theta_k, k)$$

where $c \in \mathbb{Q}_{ab}$, $\Omega'$ is a modular form with rational coefficients and $\theta_k$ is a weight $k+1$ theta series defined by

$$\theta_k(z) = \sum_{a \in X_3} a^{k\phi} e_{\mathbb{Q}}(-2\delta_1 a a^{\rho} z)$$

for some congruence subset $X_3$ of $L$. Our result 6.2.1 implies

$$\frac{D(\Omega', \theta_k, k)}{\pi^k \Omega_{z_L}^k} \in L_{ab}.$$

□

We know restrict to the case when $K$ is biquadratic. Recall that the quadratic form on $K$ is given by

$$S(a,b) = -\operatorname{Tr}_K(\delta_0 a b^{\rho})$$

for some $\delta_0 = \delta_0^{\rho} \in K$ such that $\delta_0^{\psi}$ is negative and $\delta_0^{\phi}$ is positive if $\phi \neq \psi, \psi\rho$.

**Lemma VI.5.** *There is a quaternion algebra $B_K$ over $\mathbb{Q}$ such that the quadratic space associated to $B_K$ is isomorphic to a scalar multiple of the quadratic space associated to $K$. Here the quadratic form on $B$ is given by*

$$S_{B_K}(a,b) = ab^{\rho} + ba^{\rho}.$$

*Proof.* Let $F = \mathbb{Q}(\sqrt{d})$ (the totally real subspace of $K$) and $K = F(\sqrt{-c})$ where $d, c \in \mathbb{Q}$ and $d, c > 0$. Let $\delta_0 = a + b\sqrt{d}$. Then in the basis $\{1, \sqrt{d}, \sqrt{-c}, \sqrt{-cd}\}$ for $K$, $S$ is given by

$$S = \begin{pmatrix} -4a & -4bd & 0 & 0 \\ -4bd & -4ad & 0 & 0 \\ 0 & 0 & -4ca & -4cbd \\ 0 & 0 & -4cbd & -4cad \end{pmatrix}.$$

When $a \neq 0$ this can be diagonalized by taking the basis $\{1, \sqrt{d} - \frac{bd}{a}, \sqrt{-c}, \sqrt{-c}(\sqrt{d} - \frac{bd}{a})\}$ to get

$$S = \mathrm{diag}(-4a, -4ad(1 - \frac{b^2}{a^2}d), -4ac, -4acd(1 - \frac{b^2}{a^2}d)).$$

Let $c_0 = -d(1 - \frac{b^2}{a^2}d)$ and take the quaternion algebra $B_K = \mathbb{Q}(i,j)$ where $i^2 = c_0$, $j^2 = -c$. Then

$$S_{B_K} = \mathrm{diag}(1, -c_0, c, -cc_0) = -4a \cdot S.$$

The case with $a = 0$ is similar. $\square$

There is a natural map $B_K^\times \times B_K^\times \to \mathrm{GO}(B_K)$ where the element $(\beta, \gamma) \in B_K^\times \times B_K^\times$ acts by $\alpha \mapsto \beta\alpha\gamma^\iota$. Given $\gamma = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$ and $z, z' \in \mathcal{H}$ we have

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} z & -zz' \\ 1 & -z' \end{pmatrix} = (pz+q) \begin{pmatrix} \frac{mz+n}{pz+q} & -\frac{mz+n}{pz+q}z' \\ 1 & -z' \end{pmatrix}$$

and

$$\begin{pmatrix} z & -zz' \\ 1 & -z' \end{pmatrix} \begin{pmatrix} q & -n \\ -p & m \end{pmatrix} = (pz'+q) \begin{pmatrix} z & -z\frac{mz'+n}{pz'+q} \\ 1 & \frac{mz'+bn}{pz'+q} \end{pmatrix}.$$

Thus the map $B_K^\times \times B_K^\times \to \mathrm{GO}(B_K)$ satisfies

$$(\beta, \gamma)(z, z') = (\beta z, \gamma z')$$

and

$$\mu((\beta, \gamma), (z, z')) = j(\beta, z) j(\gamma, z').$$

We now describe a dense set $\{z_L\}$ of CM points for $\mathrm{GO}(B_K)$. Here we make an identification $B_K \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$. Let $L = \mathbb{Q}(\sqrt{-c})$ which is an imaginary quadratic subfield of $K$. Then there is an embedding $i : L \hookrightarrow B_K$ given by

$$a + b\sqrt{-c} \mapsto a + bj \text{ for } a, b \in \mathbb{Q}.$$

This gives us a CM point $w \in \mathcal{H}$ for the Shimura curve associated to $B_K^\times$ by taking the fixed point of $i(L^\times) \subset B^\times$. Such a CM point gives us an embedding $\eta : L \to \mathbb{C}$ defined by

$$i(a)\begin{pmatrix} z \\ 1 \end{pmatrix} = a^\eta \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

This then defines the period $\Omega_z$:

$$\Omega_z := p_L(\eta, \eta).$$

Consider the point $(z, z) \in \mathcal{H}^2$.

**Lemma VI.6.** *$(z, z)$ is a CM point for $\mathrm{GO}(B_K)$ and its associated period satisfies*

$$\Omega_{(z,z)} = \Omega_z^2.$$

*Proof.* The embedding $i$ allows us to write $B = i(L) \oplus i(L)\alpha$ with $i(a)\alpha = \alpha i(a^\rho)$ for all $a \in L$. Define an homomorphism $h : L \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{End}(B_K, \mathbb{Q})$ by

$$h(1, -1, 1) : i(a) + i(b)\alpha \mapsto -i(a^\rho) + i(b)\alpha$$

$$h(1, 1, -1) : i(a) + i(b)\alpha \mapsto i(a)^\rho + i(b)\alpha$$

$$h(c, 1, 1) : i(a) + i(b)\alpha \mapsto i(a) + i(c)i(b)\alpha$$

for $a, b \in L$, $c \in L^u$. For $c \in L^u$ we have

$$h^u(c)^2 : \gamma \mapsto i(c)\gamma i(c^\rho).$$

Thus $(z, z)$ is the fixed point of $h^u(L)$ and

$$\mu(h^u(c), (z, z)) = \eta(c) \text{ for } c \in L^u.$$

By definition $\Omega_{(z,z)} = p_L(\eta, 2\eta) = \Omega_z^2$. $\square$

**Corollary VI.7.** *Let $z, z' \in \mathcal{H}$ such that $z$ is a CM point coming from an embedding $L \to B_K$ and $z'$ is a CM point from a conjugate embedding $L \to B_K$. Then $(z, z')$ is a CM point for* $\mathrm{GO}(B)$ *and*

$$\Omega_{(z,z')} = \Omega_z \Omega_{z'}.$$

*Proof.* Let $i : L \to B_K$ be the embedding corresponding to $z$ and $j : L \to B_K$ the embedding corresponding to $z'$ wher $j$ is defined by $j(a) = \beta i(a)\beta^{-1}$. Then we can take $h' : L \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{End}(B_K, \mathbb{Q})$ to be $h'((a,b,c))(\gamma) = h((a,b,c))(\gamma\beta)\beta^{-1}$ where $(a,b,c) \in L \oplus \mathbb{Q} \oplus \mathbb{Q}$ and $h$ is as defined in the previous proof. This $h'$ will have $(z, z')$ as a fixed point and the same argument shows that

$$\Omega(z, z') = \Omega_z \Omega_{z'}.$$

$\square$

Let $K_1 = \mathbb{Q}(\sqrt{-c}), K_2 = \mathbb{Q}(\sqrt{-cd})$ be the two imaginary quadratic subfields of $K$.

**Lemma VI.8.** *There exist embeddings $\mathbb{Q}(\sqrt{-c}) \hookrightarrow B_K$, $\mathbb{Q}(\sqrt{-c}) \hookrightarrow B_K$ corresponding to CM points $z_1, z_2 \in \mathcal{H}$ such that the CM point $z_K$ for* $\mathrm{GO}(B_K)$ *coming from the regular representation of $K$ satisfies $z_K = (z_1, z_2)$.*

*Proof.* Choose an isomorphism $B_K \otimes \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$ by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} 0 & \sqrt{c_0} \\ \sqrt{c_0} & 0 \end{pmatrix},$$

$$j \mapsto \begin{pmatrix} 0 & -\sqrt{c} \\ \sqrt{c} & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} \sqrt{cc_0} & 0 \\ 0 & -\sqrt{cc_0} \end{pmatrix}.$$

Recall that the action of $\mathrm{GO}(B)$ on $\mathcal{H}^2$ is defined via the embedding $\mathcal{H}^2 \to B_K \otimes \mathbb{C}$ given by $(z, z') \mapsto \begin{pmatrix} z & -zz' \\ 1 & -z' \end{pmatrix}$. Then it can be explicitly calculated that the action of an element $a \in \mathbb{Q}(\sqrt{-c})$ acts by a rotation on the coordinate $z'$. This defines an embedding $i : \mathbb{Q}(\sqrt{-c}) \to B_K$ with a fixed point $z_2 \in \mathcal{H}$ such that for $a \in \mathbb{Q}(\sqrt{-c})$ the image of $a$ in $\mathrm{GO}(B_K)$ is given by action on the right by $i(a)$. Each element of $\mathbb{Q}(\sqrt{-cd})$ maps into the connected component of $GO(B_K)$ and commutes with each element of $\mathbb{Q}(\sqrt{-c})$ so the action of $\mathbb{Q}(\sqrt{-cd})$ must be given by rotations in the $z$ coordinate ($\mathbb{Q}(\sqrt{-cd})$ must act trivially on $z$ or $z'$ and if it didn't act trivially on $z'$ then mapping the connected component of $\mathrm{GO}(B_K)$ to $B_K^\times \times_{\mathbb{Q}} B_K^\times$ and projecting onto the second coordinate would induce an isomorphism $\mathbb{Q}(\sqrt{-cd}) \cong \mathbb{Q}(\sqrt{-c})$). This gives an embedding $\mathbb{Q}(\sqrt{-cd}) \in B_K$ with fixed point $z_1 \in \mathcal{H}$ and $z_K = (z_1, z_2)$. $\square$

**Lemma VI.9.**

$$\Omega_{z_K} = \Omega_{z_1}\Omega_{z_2}.$$

*Proof.* Let $\psi_1, \psi_2$ be the infinite places of $K_1, K_2$ given by restricting the place $\psi$ of $K$. Let $\phi$ be the other place of $K$ that restricts to $\psi_1$. Then $\phi \neq \psi\rho$ and $\phi\rho$ restricts to $\psi_2$. Then Proposition VI.2 shows

$$p_{K_1}(\psi_1, \psi_1) = p_K(\psi, \psi + \phi), \text{ and } p_{K_2}(\psi_2, \psi_2) = p_K(\psi, \psi + \phi\rho).$$

Thus

$$p_{K_1}(\psi_1, \psi_1)p_{K_2}(\psi_2, \psi_2) = p_K(\psi, \psi + \phi)p_K(\psi, \psi + \phi\rho)$$

$$= p_K(\psi, \psi)^2 p_K(\psi, \phi)p_K(\psi, \phi\rho) = p_K(\psi, \psi)^2.$$

$\square$

**Lemma VI.10.** *Let $z_0 \in \mathcal{H}$ be a CM point for $B_K^\times$. There is a weight $k$ automorphic form $T$ on $B_K^\times$ that is non-zero at $z_0$ and satisfies $T(z)/\Omega_z^k \in M_{ab}$ for each CM point $z$ where $M$ is the imaginary quadratic field associated to $z$.*

*Proof.* From the definition of $\Omega_z$ as a ratio of values of usual modular forms, we see that $\Omega_z$ is a period of CM elliptic curve defined over $M_{\text{ab}}$, with complex multiplication by $M$. The result then follows from observing that the Shimura curve associated with $B_K^\times$ is also a solution to a moduli problem (abelian surfaces with multiplication by $B_K$) and for a CM point $z$ on this curve attached to an imaginary quadratic field $M$, the corresponding abelian surface is isogenous to a product of two elliptic curves over $M_{\text{ab}}$ with CM by $M$. $\square$

**Lemma VI.11.** *There is a weight $k$ automorphic form $T$ on $\mathrm{GO}(B_K)$ which is non-zero at $z_K$ and satisfies $T(z_K)/\Omega_{z_K}^k \in K_{ab}$. Furthermore, for each $z_L$, $T(z_L)/\Omega_{z_L}^k \in K_{ab}$.*

*Proof.* Let $z_1, z_2 \in \mathcal{H}$ such that $z_K(z_1, z_2)$. Let $T_1, T_2$ be weight $k$ forms on $B_K^\times$ that do not vanish at $z_1, z_2$ respectively (as given in the previous lemma). Then $T := T_1 \times T_2$ has the desired properties by Corollary VI.7 and Lemma VI.9. $\square$

**Proposition VI.12.**

$$G_k(z_K)/\pi^k\Omega_{z_K}^k \in K_{ab}.$$

*Proof.* $G_k/\pi^k T$ is an automorphic function on $\mathrm{GO}(K)$. Since both $T(z_L)/\Omega_{z_L}^k$ and $G_k(z_L)/\pi^k\Omega_{z_L}^k$ are in $K_{ab}$ for every $z_L$ we know that $G_k(z_L)/\pi^k T(z_L) \in K_{ab}$ for every $z_L$ such that $T(z_L) \neq 0$. Since these $z_L$ are dense in $\mathcal{H}^2$ we can conclude that $G_k/\pi^k T$ is defined over $K_{ab}$ which implies $G_k(z_K)/\pi^k T(z_K) \in K_{ab}$. Then

$$\frac{G_k(z_K)}{\pi^k\Omega_{z_K}^k} = \frac{G_k(z_K)}{\pi^k T(z_K)}\frac{T(z_K)}{\Omega_{z_K}^k} \in K_{ab}.$$

$\square$

## 6.4 Rationality of $E_k$

Let $F$ be a real quadratic field and $B$ a quaternion algebra over $F$ which is split at one infinite place.

**Lemma VI.13.** *Let $K$ be a biquadratic CM field over $F$ and with an embedding $K \hookrightarrow B$. There is a dense set $W$ of CM points $(w, w') \in \mathcal{H}^2$ corresponding to embeddings $h : K \oplus F \oplus F \to \mathrm{End}(B, F)$ and they satisfy $\Omega_{(w,w')} = \Omega_w \Omega_{w'}$.*

*Proof.* The proof is the same as that of Lemma VI.7. □

For a field $K_{ab} \subset R \subset \mathbb{C}$ let $\mathcal{U}_k(\Gamma, R)$ be the space of weight $k$ automorphic forms $f$ for $\Gamma \subset B^\times$ such that $f(w)/\Omega_w^k \in R$ for every CM point $w \in W$. We assume that $k$ is even.

**Lemma VI.14.** $\mathcal{U}_k(\Gamma, R) = \mathcal{U}_k(\Gamma, K_{ab}) \otimes_{K_{ab}} R$.

*Proof.* We give a sketch of the argument. It suffices to show that a form $f$ on $B^\times$ of weight $(k, 0)$ is $K_{\mathrm{ab}}$-rational if and only if for all CM points $w$ as above (attached to $K$), we have $f(w)/\Omega_w^k \in K_{\mathrm{ab}}$. The proof is similar to that of Lemma VI.10 but somewhat more involved. Let $X_B$ denote the Shimura variety attached to $B^\times$. In this case, $X_B$ is not itself a moduli space in a natural way. However, it is closely related to a Shimura variety that is a moduli space. Namely, let $V = B$, considered as a $K$-vector space via the embedding $K \hookrightarrow B$. In fact, $V$ may be naturally equipped with a $K$-hermitian form; let $G = \mathrm{GU}_K(V)$ denote the unitary similitude group and $X$ the Shimura variety attached to $G$. Since $G = (B^\times \times K^\times)/F^\times$, there is a natural map $B^\times \to G$ which induces an embedding of Shimura varieties $X_B \to X$, that is defined over $K$. Thus $f$ can be considered as a form on $X$ by extension by zero. Now, $X$ is a moduli space of abelian varieties $A$ with multiplication by $K$. Note that the embedding $K \to \mathrm{End}_{\mathbb{Q}}(A)$ gives an action of $K$ on $H^1(A)$ and one requires that the induced action of

$$K \otimes \mathbb{C} = \mathbb{C} \times \mathbb{C} \tag{6.4.1}$$

on $H^1(A, \mathbb{C})$ is of the form $(\eta \oplus \eta\rho, \psi \oplus \psi)$, where the two factors in the decomposition (6.4.1) correspond to the embeddings $\eta, \psi : K \to \mathbb{C}$. This follows from the fact that the unitary group $\mathrm{U}(V)$ at infinity is given by

$$\mathrm{U}(V)(\mathbb{R}) = \mathrm{U}(1,1) \times \mathrm{U}(2,0).$$

Now, a CM point on $X$ attached to $K$ then corresponds to a product of abelian surfaces $A_1 \times A_2$ where $A_1$ and $A_2$ have CM by $K$ and have CM types

$$\eta + \psi, \quad \eta\rho + \psi$$

respectively. Since the reflex field of such a CM point is $K$ itself (as $K$ is biquadratic), and such CM poiints are dense in $X$, we find that $f$ is rational if and only if its values at all such points are $K_{\mathrm{ab}}$ rational after dividing by an appropriate power of a period of a suitable differential form on $A_1 \times A_2$. This period is

$$p_K(\eta, \eta + \psi) \cdot p_K(\eta\rho, \eta\rho + \psi) = p_K(\eta, \eta + \psi) \cdot p_K(\eta, \eta + \psi\rho) = p_K(\eta, \eta)^2 = \Omega_w^2,$$

which completes the proof. $\square$

**Lemma VI.15.** *Let $R \supset K_{ab}$. Let $f$ be a function on $\mathcal{H}^2$ such that*

$$f(\alpha w, \beta w') = j(\alpha, w)^k j(\beta, w')^k f(w, w')$$

*for all $\alpha, \beta \in \Gamma$ a congruence subgroup of $B^\times$ and $f(w, w')/\Omega_w \Omega_{w'} \in R$ for all $(w, w') \in W$. Furthermore let $h_i \in \mathcal{U}_k(R)$ for $i = 1, \ldots, t$ form a basis of $\mathcal{U}_k(R)$. Then there are unique forms $g_i \in \mathcal{U}_k(R)$ such that*

$$f(w, w') = \sum_{i=1}^{t} g_i(w) h_i(w') \text{ for all } w, w' \in \mathcal{H}.$$

*Proof.* Same as Lemma 4.1 of [44]. □

Equations 5.7.1 and 5.8.1 show that

$$E_k(w_K, 0; X) = cG_k(z_K, 0; X_1)$$

with $c \in K$. The point $w_K$ was constructed so that the embeddings $K \to \mathbb{C}$ associated with $z_K$ and $w_K$ are the same. Thus

$$\Omega_{w_K} = \Omega_{z_K}.$$

Therefore

$$\frac{E_k(w_K, 0; X)}{\pi^k \Omega_{w_K}^k} \in K_{ab}.$$

Choose a $\Gamma \subset B^\times$ such that $f^B \in \mathcal{U}_k(\Gamma, K_{ab})$ and

$$E_k(\alpha w, \beta w', 0; X) = j(\alpha, w)^k j(\beta, w')^k E_k(w, w', 0; X)$$

for all $\alpha, \beta \in \Gamma$. Let $f_1, \ldots, f_t$ be an orthogonal basis for $\mathcal{U}_k(\Gamma, K_{ab})$ with $f_1 = f^B$ and let $\delta \in B$. Then the lemma allows us to write

$$\pi^{-k} E_k(w, w', 0; X_0) = \sum_{i=1}^{t} f_i(w) g_i(w')$$

with each $g_i \in \mathcal{U}_k(\Gamma, K_{ab})$.

Recall our definition:

$$\tilde{E}_k(w, w', 0) := \pi^{-k} j(\delta, \bar{w}')^{-k} E_k(w, \delta\bar{w}', 0; X_0)$$

with $\delta \in B$ such that $N(\delta) = -\delta_0$. Let $M \subset \mathbb{C}$ be a real extension of $F$ that splits $B$. Then we can choose the isomorphism $B_1 \cong \mathrm{M}_2(\mathbb{R})$ such that each element of $B$ has coordinates in $\mathrm{M}_2(M)$. If $K \subset B$ is a quadratic extension of $F$ then $B$ splits over $K$. Thus for $w \in \mathcal{H}$ is a CM point corresponding to $K$ we can choose the isomorphism $B_1 \cong \mathrm{M}_2(\mathbb{R})$ having coordinates in $\mathrm{M}_2(K)$ and then $\overline{j(\delta, w')} = j(\delta, \bar{w}') \in K$.

Define $h_i(w') = j(\delta, w')^{-k}\overline{g_i(\delta\bar{w}')}$ for each $1 \leq i \leq t$. Then $g_i \in \mathcal{U}_k(\Gamma, K_{ab})$ implies $h_i \in \mathcal{U}_k(\Gamma, K_{ab})$. Thus $\langle f^B, h_1\rangle / \langle f^B, f^B\rangle \in K_{ab}\mathbb{Q}(f)$ where $\mathbb{Q}(f)$ is the field generated by the eigenvalues of $f^B$. Furthermore,

$$\langle f^B \times f^B, \tilde{E}_k\rangle = \langle f^B, f^B\rangle^2 \frac{\langle f^B, h_1\rangle}{\langle f^B, f^B\rangle}.$$

This gives us the rationality result

$$\frac{\langle f^B \times f^B, \tilde{E}_k\rangle}{\langle f^B, f^B\rangle^2} \in K_{ab}\mathbb{Q}(f). \tag{6.4.2}$$

But this holds for any biquadratic $K \subset B$ over $F$. A biquadratic CM field $K/F$ embeds in $B$ if and only if $K$ is not split at each finite place where $B$ is ramified. For every finite place $v$ of $F$ with $v$ over $p$ in $\mathbb{Q}$ such $B$ is ramified at $v$ we assume $v/p$ is split. There are finitely many places where $B$ is ramified and thus infinitely many biquadratic CM fields $K/F$ that embed in $B$. We now use a result of Shimura to further restrict the field to $F_{ab}$. For $(a)$ an ideal of $F$ (note we have already assumed $F$ has class number 1), and $M$ a number field over $F$ let $C(M,(a))$ denote the class field of $M$ corresponding to $(a)$ and all the infinite places. Let $\mathfrak{d}(M/F)$ denote the different of $M$ relative to $F$.

**Theorem VI.16** (Shimura [37] Lemma 1.4)**.** *Let $L$ and $M$ be finite extensions of $F$ and $(a)$ an ideal in $F$. Let $L_0$ be the Galois closure of $L$ over $F$ and assume*

$$\mathfrak{d}(M/F) \textit{ is prime to } (a)\mathfrak{d}(L_0/F) \tag{6.4.3}$$

$$M \textit{ and } C(L_0,(a)) \textit{ are linearly disjoint over } F \tag{6.4.4}$$

*Then $C(F,(a)) = C(M,(a)) \cap C(L,(a))$.*

We now further assume that $\mathbb{Q}(f^B) \subseteq F$. Fix $K_1$ to be a biquadratic CM field over $F$ such that equation 6.4.2 holds. Then in fact $\frac{\langle f^B \times f^B, \tilde{E}_k\rangle}{\langle f^B, f^B\rangle^2}$ lies in $C(K_1,(a))$ for some $a \in F$. We now show that there exists a biguadratic CM field $K_2$ over $F$ such that 6.4.3 and 6.4.4 hold with $L = K_1$ and $M = K_2$ and 6.4.2 holds for $K = K_2$. As described, the conditions on $K_2$ for 6.4.2 to hold are purely local and requiring 6.4.3 also adds only local conditions on $K_2$. Now as long as $K_2$ is not contained in $C(K_1, (a))$ 6.4.4 will be satisfied and thus we have infinitely many choices of $K_2$ such that $C(F, (a)) = C(K_1, (a)) \cap C(K_2, (a))$.

For any such $K_2$ let $E = C(K_1, (a)) \cap K_{2,ab}$. Then $\frac{\langle f^B \times f^B, \tilde{E}_k \rangle}{\langle f^B, f^B \rangle^2} \in E$. Suppose we also chose $K_2$ to be split at all the primes dividing $a$. We will show this implies $E \subset C(K_2, (a))$. For each local unit $u \equiv 1 \mod {}^*(a)$ of $K_2$, $u$ is actually given by a local unit in $F$ at some prime dividing $a$. This implies $u$ is a norm from $E$ because it is already a norm from $C(K_1, (a))$. Thus $E \subset C(K_2, (a))$ and therefore

$$C(K_1, (a)) \cap K_{2,ab} \subset C(K_1, (a)) \cap C(K_2, (a)) = C(F, (a)) \subset F_{ab}.$$

Combining this with V.7 we have

**Theorem VI.17.** *For* $\mathbb{Q}(f^B) \subset F$,

$$\frac{L^{*\,\mathrm{As}}(f^B, k/2)}{\pi^k \langle f^B, f^B \rangle} \in F_{ab}.$$

Using 5.3.1 we get the corollary

**Corollary VI.18.** *For* $\mathbb{Q}(f^B) \subset F$,

$$\frac{L^{\mathrm{As}}(J(f^B), k-1)}{\pi^{2k-2} \langle f^B, f^B \rangle} \in F_{ab}.$$

## CHAPTER VII

# Integrality

In this chapter we take a first step toward an integrality result for the Asai $L$-function. We will show that for certain CM points $w$ there is a period $\Omega$ associated with the CM field of $w$ such that the automorphic form $G_k$ evaluated at $w$ is a $p$-integral multiple of $\Omega^{2k}/\pi^{k+1}$.

Let $F = \mathbb{Q}(\sqrt{A})$, $L = \mathbb{Q}(\sqrt{-B})$, $K = FL$ with $A, B \in \mathbb{Z}$, $A, B > 0$ and $(A, B) = 1$. Assume also that $A \equiv B \equiv 1 \mod 4$ so that

$$\mathcal{O}_F = \langle 1, \frac{1+\sqrt{A}}{2} \rangle, \mathcal{O}_L = \langle 1, \sqrt{-B} \rangle, \mathcal{O}_K = \mathcal{O}_F \mathcal{O}_L.$$

(The last equality holds because $F$ and $L$ have relatively prime discriminants.) We make $K$ into a quadratic space of signature $(2, 2)$ with associated bilinear form

$$S(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\sqrt{A} x \bar{y}).$$

We will consider an example of the automorphic form $G_k$ (defined in Section 5.4) where $K$ is the quadratic space, $X = \mathcal{O}_K$, $k$ is odd, and $\Omega$ is a weight 2 modular form. That is

$$G_k(z, s) = \sum_{x \in \mathcal{O}_K} \omega(-S[x]) S(x, p(z))^{-k} |S(x, p(z))|^{-2s}.$$

Let $\alpha = \frac{1+\sqrt{A}}{2} \in K$, $\beta = \sqrt{-B} \in K$ so $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\alpha\beta$. In the basis $\{1, \alpha, \beta, \alpha\beta\}$, the matrix of $S$ is given by

$$S = \begin{pmatrix} 0 & 2A & 0 & 0 \\ 2A & 2A & 0 & 0 \\ 0 & 0 & 0 & 2AB \\ 0 & 0 & 2AB & 2AB \end{pmatrix}$$

This can be diagonalized by changing to the basis $\{\alpha, \alpha\beta, \beta - \alpha\beta, 1 - \alpha\}$ where $S$ becomes

$$S = \mathrm{diag}(2A, 2AB, -2AB, -2A).$$

Fix this as the basis for $\mathcal{O}_K$. We can now construct Shimura's example of a CM point coming from $Y = L \oplus \mathbb{Q} \oplus \mathbb{Q}$ by defining

$$h : L \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathrm{End}(K, \mathbb{Q}), h : (x + \sqrt{-B}y, c, d) \mapsto \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & -By & x \end{pmatrix}.$$

Applying proposition V.10, we can take the vector $v = (1, 1, 0, 1)$ which satisfies $h(L \oplus \mathbb{Q} \oplus \mathbb{Q})v = K$. This choice of $v$ gives us the embedding $L \hookrightarrow K$

$$1 \mapsto h((1, 0, 0))v = 1 - \alpha, \sqrt{-B} \mapsto h((\sqrt{-B}, 0, 0))v = \beta - \alpha\beta.$$

We set $M := h(\mathbb{Q} \oplus \mathbb{Q})v = \mathbb{Q}\alpha + \mathbb{Q}\alpha\beta$ as before and see that for $a, b \in L$ and $x, y \in M$

$$S(h(a)v + x, h(b)v + y) = \mathrm{Tr}_{L/\mathbb{Q}}(\delta a a^\rho) + S'(x, y)$$

with $\delta = -A$.

Next we compute the fixed point $w$ of $h(Y^u)$. It is easy to see that $h(Y^u)$ acts on the point $(0, 0, \sqrt{-1/B}, 1) \in K \otimes \mathbb{C}$ by scalar multiplication, so the point $u := p(w)$

is in the line generated by $(0, 0, \sqrt{-1/B}, 1)$. To properly normalize, we need to look at the map $p$ which comes from the isomorphism $K \otimes \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$. We can choose this isomorphism to be defined by

$$\alpha \mapsto \sqrt{A}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha\beta \mapsto \sqrt{AB}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\beta - \alpha\beta \mapsto \sqrt{AB}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 1 - \alpha \mapsto \sqrt{A}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under this isomorphism we see that $(0, 0, \sqrt{-1/B}, 1)$ maps to

$$\begin{pmatrix} \sqrt{-A} & \sqrt{A} \\ \sqrt{A} & -\sqrt{-A} \end{pmatrix} = \sqrt{A}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

Thus the proper normalization is $u = (0, 0, \sqrt{-1/AB}, \sqrt{1/A})$. This allows us to compute

$$e := S(v, u) = -2\sqrt{A}.$$

The ring of integers in $K$ satisfies $\mathcal{O}_K = h(\mathcal{O}_L)v + X_2$ with $X_2 = \mathbb{Z}\alpha + \mathbb{Z}\alpha\beta \subset M$. Thus our previous evaluation of $G_k$ at CM points (Equation 5.9.1) shows that

$$G_k(w, 0; \mathcal{O}_K) = A^{k/2} D(\Omega H, \theta_k, k) \tag{7.0.1}$$

where

$$H(z) := \sum_{x \in X_2} e_{\mathbb{Q}}(S'[x]z)$$

and

$$\theta_k(z) := \sum_{b \in \mathcal{O}_L} b^{k\psi} e_{\mathbb{Q}}(2Abb^\rho z).$$

The modular form $H(z)$ can also be written as

$$H(z) = \sum_{b \in \mathcal{O}_L} e_{\mathbb{Q}}(2Abb^\rho z).$$

We now follow Prasanna ([32]) to show that the value $D(\Omega H, \theta_k, k)$ is integral. Let $-d$ be the discriminant of $L$, so $\Omega H$ has level $NCd$ with $N, C, d$ pairwise coprime. Define

$$E := E_{k-3,NCd}(z, s, \eta_L) = \sum_{(m,n)} \eta_L(n)(mNCdz + n)^{-k+3}|mNCdz + n|^{-2s}.$$

where $\eta_K$ is the quadratic character associated to $L$ and the sum is over all $(m, n) \in \mathbb{Z}^2$ excluding $(m, n) = (0, 0)$. Then

$$D(\Omega H, \theta_k, k) = \frac{c}{2}\pi^{k+1}\langle \Omega H E, \theta_k^\rho \rangle L_{NCd}(1, \eta_L)^{-1}$$

where $L_{NCd}$ is the $L$-function with Euler factors at $NCd$ removed, $\theta_k^\rho$ is given by applying complex conjugation to the coefficients of $\theta_k$ and

$$c = \frac{4^k NCd}{3(k-1)!} \prod_{p|NCd} \frac{p+1}{p}.$$

However our $\theta_k$ has real coefficients so $\theta_k = \theta_k^\rho$. Furthermore $L(1, \eta_L)/\pi$ is rational with numerator $2h(L)$ where $h(L)$ is the class number of $K$ and

$$L_{NCd}(1, \eta_L) = \prod_{p|NCd} \frac{p - \eta_L(p)}{p} L(1, \eta_L).$$

Thus

$$D(\Omega H, \theta_k, k) = c_0 \pi^k \langle \Omega H E, \theta_k \rangle$$

with $c_0$ algebraic and $p$-integral for all $p > k$, $p \nmid h(L) \prod_{q|NCd} q(q-1)(q+1)$.

Let $\chi_1, \ldots, \chi_{h(L)}$ be the distinct characters of the ideal class group of $L$ which are trivial on principal ideals. Let $\lambda$ be a Hecke character of $L$ that satisfies $\lambda((b)) = b^{k\psi}$ for each $b \in \mathcal{O}_L$. Let $\lambda_i$ be the twist of $\lambda$ by $\chi_i$. Then assuming $\pm 1$ are the only roots of unity in $L$, we have

$$\theta_k(z) = \frac{2}{h(L)} \sum_{i=1}^{h(L)} \theta_{\lambda_i}(2Az).$$

Thus

$$D(\Omega H, \theta_k, k) = \pi^k \sum_{i=1}^{h(L)} c_i \langle \Omega H E, \theta_{\lambda_i}(2Az)\rangle \tag{7.0.2}$$

with each $c_i$ algebraic and $p$-integral for all $p > k$, $p \nmid h(K) \prod_{q|2NAd} q(q-1)(q+1)$.

**Proposition VII.1.** *Let $\theta$ be an integral newform of level $d$, $g$ an integral form of level $Nd$ and $C$ an integer with $(C, d) = 1$. Then $\pi^{2k+1}\langle g, \theta(Cz)\rangle/\Omega^{2k}$ is $p$-integral for all $p \nmid M$ where $M := \prod_{q|NC} q(q+1)$ and $\Omega$ is the period defined in Section 2.3.3 of [32].*

*Proof.* In [32] this is proven (p.942) in the case where $C = 1$. The same argument applies here, except that there it is shown in equation 22 that $\langle \theta_i(d'z), \theta\rangle/\langle \theta, \theta\rangle$ is $p$ integral for $p \nmid M$ and we need this result for

$$\langle \theta_i(d'z), \theta(Cz)\rangle/\langle \theta, \theta\rangle$$

where $T$ is a subset of primes dividing $NC$, $P$ is the product of primes dividing $NC$ that are not in $T$, $d'|P$, $T_i \subset T$, $\theta_i$ is $\theta$ with the Euler factor $(1 - \alpha_q q^{-s})^{-1}$ removed for $q \in T_i$ and $(1 - \beta_q q^{-s})^{-1}$ removed for $q \in T \setminus T_i$.

Let $p$ be a prime in $T_i$ and $\theta_i^{(p)}$ be the form given by putting the Euler factor $(1 - \alpha_p p^{-s})^{-1}$ back in. Then $\theta_i(z) = \theta_i^{(p)}(z) - \alpha_p \theta_i^{(p)}(pz)$. Shimura ([39], Lemma 3) shows that

$$\begin{aligned} \langle \theta(pd'z), \theta(Cz)\rangle &= p^{-k-1} \frac{a(p)}{1 + \alpha_p \beta_p p^{-k-1}} \langle \theta(dz), \theta(Cz)\rangle \\ &= p^{-k-1} \frac{a(p)}{1 + p^{-1}} \langle \theta(d'z), \theta(Cz)\rangle \end{aligned}$$

when $(p, d') = (p, C) = 1$. Thus

$$\begin{aligned} \langle \theta_i(d'z), \theta(Cz)\rangle &= \left(1 - \frac{\alpha_p p^{-k-1} a(p)}{1 + p^{-1}}\right) \langle \theta_i^{(p)}(d'z), \theta(Cz)\rangle \\ &= \frac{1 + p^{-1} - \alpha_p^2 p^{-k-1} - \alpha_p \beta_p p^{-k-1}}{1 + p^{-1}} \langle \theta_i^{(p)}(d'z), \theta(Cz)\rangle \end{aligned}$$

$$= \frac{1-\alpha_p^2 p^{-k-1}}{1+p^{-1}} \langle \theta_i^{(p)}(d'z), \theta(Cz)\rangle$$

$$= \frac{p^k - \alpha_p^2}{p^k(p+1)} \langle \theta_i^{(p)}(d'z), \theta(Cz)\rangle.$$

If $p|C$, $(p, d') = 1$ then

$$\langle \theta(pd'z), \theta(Cz)\rangle = \frac{1}{p^{k+1}} \langle \theta(d'z), \theta((C/p)z)\rangle.$$

So in this case

$$\langle \theta_i(d'z), \theta(Cz)\rangle = \langle \theta_i^{(p)}(d'z), \theta(Cz)\rangle - \frac{\alpha_p}{p^{k+1}} \langle \theta_i^{(p)}(d'z), \theta((C/p)z)\rangle.$$

Similar results hold for primes $p \in T \setminus T_i$ with $\alpha_p$ replaced with $\beta_p$. Thus we can iterate this step for each prime in $T$ to write $\langle \theta_i(d'z), \theta(Cz)\rangle$ as a linear combination

$$\langle \theta_i(d'z), \theta(Cz)\rangle = \sum_{c'|C} b_{c'} \langle \theta(d'z), \theta(c'z)\rangle$$

with each coefficient $b_{c'}$ being $p$-integral for all $p \nmid M$.

Shimura's lemma can be used again to compute

$$\langle \theta(d'z), \theta(c'z)\rangle = \prod_{p||d'c'} p^{-k-1} \frac{a(p)}{1+\alpha_p\beta_p p^{-k-1}} \prod_{p^2|d'c'} \frac{1}{p^{k+1}} \langle \theta, \theta\rangle$$

$$= \prod_{p||d'c'} \frac{a(p)}{p^k(p+1)} \prod_{p^2|d'c'} \frac{1}{p^{k+1}} \langle \theta, \theta\rangle.$$

Therefore $\langle \theta_i(d'z), \theta(Cz)\rangle$ is a $p$-integral multiple of $\langle \theta, \theta\rangle$ for $p \nmid M$ which is the necessary generalization of equation 22 of [32].

□

Combining this with equations 7.0.1 and 7.0.2 we have

**Theorem VII.2.**

$$\frac{\pi^{k+1} G_k(w, 0; \mathcal{O}_K)}{\Omega^{2k}}$$

*is $p$-integral for all $p \nmid h(K) \prod_{q|2NAd} q(q-1)(q+1)$.*

# CHAPTER VIII

# Rationality of the Asai $L$-function II

## 8.1 Asai's Integral Representation

In this chapter we prove a different rationality result for the Asai $L$-function. Work of Harris allows us to define a period $\Omega_f$ associated with the Hilbert modular form $f$ of weight $(k_1, k_2)$ and we show $L^{\mathrm{As}}(f, k_1 - 1)/\Omega_f \in F\mathbb{Q}(f)$ under certain conditions. This gives a smaller field of rationality than our earlier result, but there is no obvious interpretation of $\Omega_f$ as a Petersson inner product.

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with ring of integers $\mathcal{O}_F$. Let $\mathfrak{d}$ denote the different of $F$ and $\omega \in F$ be a generator for the inverse different of $F$: $(\omega) = \mathfrak{d}^{-1}$. Take $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ and let $f : (\mathcal{H}^{\pm})^2 \to \mathbb{C}$ be a Hilbert modular form of weight $(k, l)$ for $\Gamma$ with $k > l$. For each $e = (e_1, e_2)$ with $e_i = \pm 1$ let $\mathcal{H}_e$ denote the component of $\mathcal{H}^{\pm}$ containing $(e_1 i, e_2 i)$. For $a \in F$, let $\operatorname{sgn} a$ denote the pair $\operatorname{sgn} a = (e_1, e_2)$ where $e_i = 1$ if $a^{\tau_i} > 0$ and $e_i = -1$ if $a^{\tau_i} < 0$. In addition let $\mathcal{O}_{F,e} = \{a \in F : \operatorname{sgn} a = e\}$. We also assume that $F$ has odd discriminant and that $F$ has a unit of norm $-1$. Let $\epsilon$ be a unit such that $\epsilon/\sqrt{D}$ is totally positive.

Then $f$ has a Fourier expansion of the form

$$f(z) = \sum_{\mu \in \mathcal{O}_F, \mu >> 0} C(\mu) e^{2\pi i \operatorname{Tr}(\mu \epsilon z/\sqrt{D})}$$

on the $\mathcal{H}^2$ component (see [1]). We want to study the Asai $L$-function

$$L^{\mathrm{As}}(f,s) = \zeta(2s-k-l+2)\sum_{n=1}^{\infty} C(n)n^{-s}.$$

Let $d : \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ be defined by $d : w \mapsto (\epsilon^{-\tau_1}w, \epsilon^{-\tau_2}\bar{w})$. We define $g(w) : \mathcal{H} \to \mathbb{C}$ to be the composition $g = f \circ d$. Also define the completed $L$-function

$$G^*(s) = D^{s/2}(4\pi)^{-s}\Gamma(s)\Gamma(s-l+1)L^{\mathrm{As}}(f,s).$$

**Definition VIII.1.** Let $R$ be a set of representatives for $\mathrm{SL}_2(\mathbb{Z})$ modulo upper triangular matrices and $\lambda$ an even integer $\geq 0$. Define the Eisenstein series $E^*_\lambda$ to be

$$E^*_\lambda(w,s) := \sum_{\gamma \in R} (cw+d)^{-\lambda}|cw+d|^{-2s}.$$

This is independent of the choice of $R$. Furthermore define

$$\tilde{E}_\lambda(w,s) := 2\zeta(2s+\lambda)E^*_\lambda(w,s) = \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} (cw+d)^{-\lambda}|cw+d|^{-2s}.$$

Note that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $g(w)$ satisfies

$$g(\gamma w) = |cw+d|^{2k}(cw+d)^{l-k}g(w).$$

Therefore $g(w)E_{k-l}(w, s+1-k)y^{s-1}dxdy$ is an $\mathrm{SL}_2(\mathbb{Z})$ invariant form on $\mathcal{H}$ (where $w = x+iy$).

**Definition VIII.2.**

$$J(s) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(w)E_{k-l}(w,s+1-k)y^{s-1}.$$

**Proposition VIII.3.**

$$G^*(s) = J(s-k+1).$$

*Proof.* We first note that $g(w)$ has a Fourier expansion

$$g(w) = \sum_{\mu} C(\mu) e^{2\pi i(\mu^{\tau_1} w - \mu^{\tau_2} \bar{w})/\sqrt{D}}.$$

Let $w = x + iy$ with $x, y \in \mathbb{R}$. If $\mu \in \mathbb{Z}$ then $\mu^{\tau_1} w - \mu^{\tau_2} \bar{w} = 2i\mu y$ and if $\mu \notin \mathbb{Z}$ then $\mu^{\tau_1} w - \mu^{\tau_2} \bar{w} = Aiy + C\sqrt{D}x$ for some nonzero integral $C$. Therefore

$$\int_{x=0}^{1} g(w) dx = \sum_{n=1}^{\infty} C(n) e^{-4\pi n y}.$$

Thus

$$\int_{y=0}^{\infty} y^{s-1} \int_{x=0}^{1} g(w) dx dy = \int_{0}^{\infty} \sum_{n=1}^{\infty} C(n) y^{s-1} e^{-4\pi n y/\sqrt{D}} dy$$

$$= \sum_{n=1}^{\infty} C(n) D^{s/2} (4\pi n)^{-s} \int_{0}^{\infty} u^{s-1} e^{-u} du = D^{s/2} (4\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} C(n) n^{-s}.$$

Therefore

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(w) y^{s-1} E^*_{k-l}(z, s - k_1 + 1) dx dy = \int_0^{\infty} \int_0^1 y^{s-1} g(z) dx dy$$

$$= D^{s/2} (4\pi)^{-s} \Gamma(s) \sum_{n} C(n) n^{-s}.$$

Replacing $E^*$ with $E$ gives

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(w) y^{s-1} E_{k-l}(w, s - k + 1) =$$

$$2D^{s/2} (4\pi)^{-s} \Gamma(s) \zeta(2s - k - l + 2) \sum_{n} C(n) n^{-s}.$$

Thus if we multiply the integral by $\Gamma(s - l + 1)$ we get

$$J(s - k + 1) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(w) y^{s-1} \Gamma(s - l + 1) E_{k-l}(z, s - k + 1)$$

$$D^{s/2} (4\pi)^{-s} \Gamma(s) \Gamma(s - l + 1) L^{\mathrm{As}}(f, s) = G^*(s).$$

$\square$

Asai was able to use this integral representation of $L^{\mathrm{As}}(f,s)$ to prove analytic continuation and a functional equation. This method follows the steps of Rankin and Selberg: if $f_1$ and $f_2$ are modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weights $k$ and $l$ respectively, then the same argument with $g(w) := f_1(w)\overline{f_2^*(w)}$ gives an integral representation of $L(f_1 \otimes f_2, s)$. Here $f^*$ denotes the modular form given by applying complex conjugation to the coefficients of $f$ Explicitly, let $f_1(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and $f_2 = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$ and define

$$D(f,g,s) := \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$

**Theorem VIII.4** ([39], Thm. 2)**.** *Let $f_1$ be a cusp form of weight $k$, $f_2$ a form of weight $l$, and both of level* 1*. Assume also that $k > l$. Then*

$$D(f_1, f_2, k-1) = \frac{4^{k-1}\pi^k}{3\Gamma(k-1)} \langle f^*, gE^*_{k-l}(z,0)\rangle.$$

### 8.2 Rationality with Harris' Period

Shimura used the integral representation of the Rankin-Selberg convolution $L$-function to get the following rationality result:

**Theorem VIII.5** ([39], Thm. 3)**.** *Let $f_1, f_2$ be as in the previous theorem and further assume that $f_1$ is a normalized newform. Let $\mathbb{Q}_{f_1}$ and $\mathbb{Q}_{f_2}$ be the number fields generated by the coefficients of $f_1$ and $f_2$ respectively. Then*

$$\frac{D(f_1, f_2, k-1)}{\pi^k \langle f_1, f_1\rangle} \in \mathbb{Q}_{f_1}\mathbb{Q}_{f_2}.$$

Let $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}_2(F)$, and $\mathfrak{g} = Lie(G(\mathbb{R}))$. We identify $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})^2$ via the two infinite places of $F$ and let $K_\infty^+ \subset G(\mathbb{R})$ correspond to the copy of $SO(2)^2$ in $\mathrm{GL}_2(\mathbb{R})^2$. Then under the standard action of $G(\mathbb{R})^+$ on $\mathcal{H}^2$ the stabilizer of $(i,i)$ is $Z_G(\mathbb{R})K_\infty^+$. There is a decomposition

$$\mathfrak{g}_{\mathbb{C}} = Lie(Z_G)_{\mathbb{C}} \oplus \mathfrak{k}_{\infty,\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

under the action of $\mathrm{Ad}(K_\infty^+)$ where $\mathfrak{p}^+$ naturally maps to the holomorphic tangent space to $\mathcal{H}^2$ at $(i,i)$ and $\mathfrak{p}^-$ maps to the anti-holomorphic tangent space. $\mathfrak{p}^+$ decomposes as

$$\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$$

compatibly with the decomposition $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})^2$ and similarly for $\mathfrak{p}^-$. Then a Hilbert modular form $f$ viewed as an automorphic form on $G(\mathbb{Q})\backslash G(\mathbb{A})$ will satisfy $R(\mathfrak{p}^-)f = 0$ where $R(\cdot)$ is the right regular action of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\underline{k} = (k_1, k_2)$ be a pair of integers both even. For $K$ an open compact subgroup of $G(\mathbb{A}_f)$ Harris constructs an $F$-rational line bundle $\mathcal{E}_{\underline{k},0}$ on the Hilbert modular variety $S_K$ whose global sections are canonically isomorphic to Hilbert modular forms of weight $\underline{k}$ ([16] p.159). For a toroidal compactification $\tilde{S}_K$ of $S_K$ there are two extensions of $\mathcal{E}_{\underline{k},0}$ to $\tilde{S}_K$: the subcanonical and canonical extensions which are denoted by $\mathcal{E}_{\underline{k},0}^{\mathrm{sub}}$ and $\mathcal{E}_{\underline{k},0}^{\mathrm{can}}$ respectively and defined in [17], [18], [19]. Harris further defines in [17] a cohomology theory $\bar{H}^*(\mathcal{E}_{\underline{k},0})$ which can be realized as

$$\bar{H}^q(\mathcal{E}_{\underline{k},0}) = \lim_{\rightarrow} Im[H^q(\tilde{S}_K, \mathcal{E}_{\underline{k},0}^{\mathrm{sub}}) \to H^q(\tilde{S}_K, \mathcal{E}_{\underline{k},0}^{\mathrm{can}})]$$

where the limit is taken over all $K$.

Let $I$ be a subset of the infinite places of $F$. Then define $\underline{k}(I)$ to be the pair of integers that agrees with $\underline{k}$ at the places outside of $I$ and replaces $k_i$ with $2 - k_i$ at the places in $I$. If $f$ is a Hilbert modular form, define $f^{\mathcal{J}(I)}$ to be the function obtained by precomposing $f$ with complex conjugation at the places in $I$.

Let $(\pi, H_\pi)$ be a irreducible cuspidal automorphic representation of $G(\mathbb{A})$ generated by the Hilbert modular form $f$ of weight $\underline{k}$. For $I$ as before let $H_\pi^I$ be the subset of $H_\pi$ consisting of functions $\phi \in H_\pi$ satisfying $R(\mathfrak{p}_i^+)\phi = 0$ for $i \in I$ and $R(\mathfrak{p}_i^-)\phi = 0$ for $i$ not in $I$. Then we have

**Lemma VIII.6.** *[16] 1.4.3 Let $|I|$ be the cardinality of $I$. There is a natural embedding*

$$H_\pi^I \hookrightarrow \bar{H}^{|I|}(\mathcal{E}_{\underline{k},0})$$

*of $G(\mathbb{A}^f)$-modules. Furthermore the image of $H_\pi^I$ is an $F\mathbb{Q}(\pi)$-rational subspace of $\bar{H}^{|I|}(\mathcal{E}_{\underline{k},0})$ where $\mathbb{Q}(\pi)$ is the field of definition of $\pi$.*

Define $\mathcal{J}(I)$ to be the element $(\mathcal{J}(I)_1, \mathcal{J}(I)_2) \in G(\mathbb{R})$ such that $\mathcal{J}(I)_i = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if $i \in I$ and $\mathcal{J}(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ otherwise. For $\phi$ a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ let $\phi^{\mathcal{J}(I)}(g) = \phi(g\mathcal{J}(I))$. Then $H_\pi^I = \{\phi^{\mathcal{J}(I)} : \phi \in H_\pi^\emptyset\}$.

Let $f$ be a Hilbert modular form for $F$ of weight $\underline{k} = (k_1, k_2)$ with $k_1 > k_2$, $k_1 \equiv k_2 \equiv 0 \mod 2$. Furthermore assume $f$ is an eigenform in the space of the representation $\pi$ and that $f$ is defined over $F\mathbb{Q}(\pi)$. Choose $I = \{\sigma_2\}$. Then $f^{\mathcal{J}(I)}$ may be viewed as an element in $H^1(\mathcal{E}_{(\underline{k}(I),0)})$ where $\underline{k}(I) = (k_1, 2 - k_2)$.

There are two $F\mathbb{Q}(\pi)$-rational structures on $H_\pi^I$. The first is induced by the rational embedding $H_\pi^I \hookrightarrow \bar{H}^{|I|}(\mathcal{E}_{\underline{k},0})$ and denoted by $H_\pi^I(F\mathbb{Q}(\pi))$. The second is defined applying the operator $\mathcal{J}(I)$ to $H_\pi^\emptyset(F\mathbb{Q}(\pi))$ and is denoted ${}^L H_\pi^I(F\mathbb{Q}(\pi))$. Harris shows that these two rational structures are related by a constant:

**Lemma VIII.7** ([16] Lemma 1.4.5)**.** *There is a number $\nu^I(\pi) \in \mathbb{C}^\times$, defined up to multiplication by an element of $F\mathbb{Q}(\pi)^\times$ such that*

$$\nu^I(\pi) \cdot H_\pi^I(F\mathbb{Q}(\pi)) = {}^L H_\pi^I(F\mathbb{Q}(\pi)).$$

Define the period $\Omega_f$ to be $\nu^I(\pi)$ so that $f^{\mathcal{J}(I)}/\Omega_f$ is an $F\mathbb{Q}(\pi)$-rational element of $H^1(\mathcal{E}_{\underline{k}(I),0})$. Asai had shown that $L^{\mathrm{As}}(f, s)$ may be calculated by restricting $f^{\mathcal{J}(I)}$ to the embedded modular curve and integrating against an Eisenstein series. That is the embedding $d : \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ defined in 8.1 induces an embedding $M_L \to S_K$ where $M_L$ is the modular curve corresponding to the open compact subgroup

$$L = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} K \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cap \mathrm{SL}_2(\mathbb{A}_f) \subset \mathrm{SL}_2(\mathbb{A}_f).$$

The line bundle $\mathcal{E}_{\underline{k}(I),0)}$ on the Hilbert modular surface $S_K$ restricts to the line bundle $\mathcal{E}_{(2+k_1-k_2,0)}$ on the modular curve $M_L$. On $M_L$ there is a pairing

$$\bar{H}^1(\mathcal{E}_{(2+k_1-k_2,0)}) \otimes \bar{H}^0(\mathcal{E}_{(k_1-k_2,0)}) \to \mathbb{C}.$$

This pairing is actually a Tate twist of the Serre duality pairing and is therefore rational over $\mathbb{Q}$ (see [16], p. 165). So when we pair the rationally defined Eisenstein series $E_\lambda(w, s)$ with the restriction of $f^{\mathcal{J}(I)}/\Omega_f$ the result is $F\mathbb{Q}(\pi)$-rational. Thus we tautologically get rationality of $L^{\mathrm{As}}(f, k_1 - 1)/\Omega_f$:

**Theorem VIII.8.** *Let $f$ be a eigenform of weights $k_1, k_2$ with $k_1 \equiv k_2 \equiv 0 \mod 2$ and $k_1 > k_2$. Let $\mathbb{Q}(f)$ be the field of definition of $f$. Then*

$$L^{\mathrm{As}}(f, k_1 - 1)/\Omega_f \in F\mathbb{Q}(f).$$

The relation between the period $\Omega_f$ and the Petersson inner product $\langle f^B, f^B \rangle$ used in the previous chapters deserves further study. The ratio $\langle f^B, f^B \rangle/\Omega_f$ lies in $\bar{\mathbb{Q}}$ but it would be interesting to study its rationality (and integrality) properties.

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