Quantum Corrections To Gravity: Polishing The Window Into The Black Hole Microstates.

by

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To my wife.

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Abstract

A thorough understanding of quantum gravity is one of the greatest challenges of modern theoretical physics, given the incompatibility of general relativity and quantum mechanics. In order to address this challenge many physicists compute quantum corrections to classical gravitational backgrounds as means towards a full quantum description of gravitational phenomena.

In this work we focus on developing efficient techniques to compute such quantum corrections. The standard techniques in the literature can be quite involved since they include contributions from unphysical field components that decouple and do not affect the final result. We propose a novel streamlined method in which the quantum corrections at one loop are computed exclusively from physical states.

A key element of our method is the identification of states called boundary modes. These states are pure gauge configurations with non-normalizable gauge parameters, a subtlety that renders them physical albeit pure gauge. Boundary modes are a central element of our method due to their non-trivial nature and since we choose to work exclusively with physical states.

We analyze the characteristics of these boundary modes in detail and use the proposed method to compute logarithmic corrections to extremal four dimensional black holes with $\mathcal{N} \geq 2$ supersymmetries, as well as logarithmic corrections to supergravity in $\mathrm{AdS}_2 \times S^2$.

We then use our new method to compute the one loop divergence of $\mathcal{N} = 8$ supergravity in AdS_4. We show that the divergence is topological in nature and is due to the presence of

boundary modes in the supergravity theory.

Chapter 1

Introduction.

One of the greatest challenges of modern theoretical physics is finding a microscopic understanding of black hole entropy. In order to achieve that goal one needs a theoretical framework that accurately encompasses quantum mechanics and general relativity. String theory satisfies that requirement, providing a consistent description of quantum gravity; significant progress has been made since its inception and particularly since the advent of the AdS/CFT correspondence [1].

The goal of this work is to develop tools and techniques to obtain concrete results and increase efficiency in the study of quantum corrections to black hole entropy and supergravity. We study a novel quasi on-shell method to compute logarithmic corrections to black hole entropy, applying it first to extremal black holes with $\mathrm{AdS}_2 \times S^2$ near horizon geometry and $\mathcal{N} \geq 2$ supersymmetry, then later to $\mathcal{N} = 8$ supergravity on AdS_4.

In this chapter we discuss some of the highlights of our results, as well as the motivations that inspired us in the first place.

1.1 Black Hole Thermodynamics.

Black hole thermodynamics stands out as an interesting research direction since classical intuition at times contradicts established results in physics. As an example and to motivate our discussion, let us remark that classically a black hole is completely determined by its mass, charge, and angular momentum, irrespective of the various possible configurations of the object that collapsed in the black hole. Put differently, two black holes with the same parameters are completely identical, irrespective of any differences between the original collapsing objects.

In this context a black hole would have only one internal configuration—the degeneracy of states of a black hole would be unity—and given our modern understanding of statistical mechanics the entropy of a black hole would then vanish: $S \sim \ln 1 = 0$.

This conclusion is however problematic. One can devise a thought experiment in which a closed system consisting of a black hole and some surrounding matter is constructed. As matter falls into the event horizon, it carries its associated entropy into the black hole. If black holes had zero entropy, then the total entropy of such closed system would diminish, violating the second law of thermodynamics.

Thus, the classical description of black holes is not enough to address their thermodynamical properties. Motivated by contradictions such as the one above, the study of black hole entropy has been under the attention of theoretical physicists since the early 1970's, when Hawking, Bardeen, and Carter [2] proposed the laws of black hole mechanics. In summary, they are:

- Zeroth Law: The surface gravity κ of a stationary black hole is constant over the horizon surface.

- First Law: Perturbations around a black hole solution satisfy

$$dM = \frac{\kappa}{8\pi} dA_{\mathrm{Hor}} + \Omega dJ + \Phi dQ, \tag{1.1.1}$$

  where M is the mass of the black hole, κ is its surface gravity, A_{Hor} the area of the horizon, Ω and J are angular velocity and momentum, Φ and Q are electric potential and charge.

- Second Law: The area A of the event horizon of a black hole does not decrease with time.
- Third Law: It is impossible to reduce the surface gravity κ of a black hole to zero by a finite amount of processes.

The similarity of these laws to the laws of thermodynamics led Hawking, Bardeen, Carter and Bekenstein to bridge between both fields. This similarity implies a fundamental identification of black hole parameters with thermodynamic quantities. In particular and most relevant to this dissertation, the area of the event horizon is identified with the entropy of the black hole. This identification was made by Bekenstein and Hawking[1] [3]:

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{Hor}}}{4}. \tag{1.1.2}$$

The existence of nonzero entropy for a black hole implies the existence of an associated temperature of the event horizon, given by $\frac{1}{T} = \frac{\partial S}{\partial E}$. This associated temperature is named after Hawking, and given by

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}, \tag{1.1.3}$$

where we see that the surface gravity of the black hole κ plays the role of temperature. Comparison of the laws of thermodynamics with the laws of black hole thermodynamics tells us that this identification is expected, since the TdS term in the first law is identified with the κdA term in the black hole first law. The existence of a nonzero temperature on its own turn implies that black holes emit black body radiation, which contradicts the classical intuition that nothing is emitted by a black hole. The study of black hole thermodynamics is indeed an intriguing research topic.

Hawking [4] addressed the existence of a horizon temperature by considering quantum effects. He argued that whenever particle-antiparticle pairs are created in the vicinity of

[1] We use Planck units in this dissertation, defined as $c = G = \hbar = k_B = 1$. In MKS units this formula reads $S_{\mathrm{BH}} = \frac{k_B c^3}{G\hbar}\frac{A_{\mathrm{Hor}}}{4}$.

a black hole it is possible for one of the pair constituents to escape and be measured as outgoing radiation with temperature $T_{\mathrm{H}} = \frac{\kappa}{2\pi}$. In light of such results, one concludes that the classical description definitely does not explain the thermodynamic properties of black holes, and that the inclusion of quantum effects is a direction worth exploring.

The motivating question is then whether a consistent theory of quantum gravity could correctly predict the degeneracies of black hole microstates, and consequently black hole entropy.

1.2 Quantum Descriptions And The Quantum Entropy Function.

The advent of string theory motivated researchers to use it as a tool to develop quantum descriptions of black hole entropy. Great progress was made in the late 90's when Strominger and Vafa [5] studied a class of five dimensional black holes with four supersymmetries arising from compactification of a string theory around a five dimensional manifold[2]. They showed that black hole entropy could be described microscopically by counting the degeneracies of objects such as fundamental strings and D-branes carrying the same charges of the black hole, and that in the limit of large charges the microscopic counting agrees with the Bekenstein-Hawking formula.

Two important features of the black holes in [5] are extremality and the presence of supersymmetry. A black hole is said to be extremal when it has vanishing Hawking temperature. These solutions are then stable under Hawking radiation, in the sense that the black holes will not evaporate.

A black hole solution that is invariant under a supercharge is called a BPS state. The importance of supersymmetry for our purposes lies in that a representation of the supersymmetry algebra that is BPS has a smaller dimension when compared to a non-BPS rep-

[2]Type II string theory on $K3 \times S^1$ or heterotic string theory on T^5.

resentation, and the difference is finite. As an example, in four dimensions with $\mathcal{N} = 2$ supersymmetries, a BPS representation has four states while a non-BPS representation has sixteen states. This means that one cannot continuously vary the parameters of a BPS solution to violate supersymmetry invariance since this variation would be associated with a discontinuous variation in number of states. The upshot is that BPS solutions are protected against quantum fluctuations.

These two features are important to the computation of Strominger and Vafa since they guarantee that the parameters retain their values even when quantum effects are considered. Having a class of black holes with parameters protected from quantum fluctuations was a key step in the understanding of their degeneracies in terms of stringy objects. The microscopic description of black hole entropy provided by Strominger and Vafa was an important milestone in the development of the AdS/CFT correspondence [1], proposed two years later.

Inspired by AdS/CFT, Ashoke Sen addressed the problem of generalizing Wald's entropy to include quantum effects by proposing the quantum entropy function formalism [6]. The formalism consists of focusing on extremal black holes with horizon geometries of the form $\mathrm{AdS}_2 \times \mathcal{M}$, where $\mathcal{M}$ is a compact manifold, and analyzing the string theoretical partition function around the given black hole geometry. Since the geometry contains an AdS_2 factor by assumption, the partition function is generally divergent due to the infinite volume of AdS but this divergence can be regularized by addition of suitable boundary counter terms. The finite part of the string theory partition function is then identified with the black hole microstate degeneracy[3]. Schematically, we have

$$\begin{aligned} Z_{\mathrm{ren}} &= e^{-W_{\mathrm{ren}}} = d_{\mathrm{microstates}}, \qquad (1.2.1) \\ S_{BH} &= \ln d_{\mathrm{microstates}} = -W_{\mathrm{ren}}. \end{aligned}$$

Here Z_{ren} is the renormalized partition function, W_{ren} is the renormalized part of the effective

[3]One is accustomed to approach this kind of partition function in the saddle point approximation; Sen's statement is that the partition function is exactly the degeneracy of states.

action, $d_{\text{microstates}}$ is the degeneracy of microstates and S_{BH} is the entropy of the black hole in question.

We will work with Sen's quantum entropy function all throughout this dissertation, in the one loop regime to be described next.

1.3 One Loop Determinants And The Heat Kernel.

In this work we will concern ourselves entirely with one loop corrections. The one loop approximation involves the expansion of the Lagrangian $\mathcal{L}(\phi)$ of the system up to quadratic order in fluctuations $\delta\phi$ around some background value $\bar{\phi}$, where ϕ is representative of all possible fields. In this setting the Euclidean path integral (1.2.1) takes a Gaussian form:

$$e^{-W} = \int \mathcal{D}[\delta\phi] e^{-\delta\phi\Lambda(\bar{\phi})\delta\phi} = \frac{1}{\sqrt{\det \Lambda(\bar{\phi})}}. \tag{1.3.1}$$

This is the first set of quantum corrections one computes, with the advantage of easy evaluation of the path integral since it has Gaussian form. In our example (1.3.1), $\Lambda(\bar{\phi})$ is a kinetic operator such as $-\nabla^2 + m^2$ that could a priori depend on the background values $\bar{\phi}$.

The one loop determinant $\det \Lambda$ can be written as a product over the spectrum of Λ,

$$\det \Lambda = \prod_i \lambda_i \Rightarrow W = \frac{1}{2} \ln \det \Lambda = \frac{1}{2} \sum_i \ln \lambda_i. \tag{1.3.2}$$

We regularize the sum over eigenvalues using the heat kernel method. Given the eigenfunctions of Λ, one can write the modified Green's function

$$K(x, x'; s) = \sum_i e^{-s\lambda_i} f_i(x) f_i^*(x'), \tag{1.3.3}$$

which satisfies the heat equation[4]

$$(\partial_s + \Lambda_x)\, K(x,x';s) = 0; \tag{1.3.4}$$

hence the reference to heat. The heat kernel parameter s has units of length squared. One can remove the dependence on the eigenfunctions by taking $x = x'$, integrating over the manifold of interest, and using the normalization of the eigenfunctions $f_i(x)$. The resulting function of s will be referred by us as the heat kernel $D(s)$,

$$D(s) = \int d^N x\ K(x,x;s) = \sum_i e^{-s\lambda_i}. \tag{1.3.5}$$

$D(s)$ is an object of interest for us because the one loop determinant (1.3.2) is then expressed as

$$W = \frac{1}{2}\sum_i \ln\lambda_i = -\frac{1}{2}\int_{\epsilon^2}^{\infty} ds \frac{D(s)}{s}, \tag{1.3.6}$$

where ϵ is a UV regulator with dimensions of length, added by hand since the integral (1.3.6) diverges at small s. This relation is vital to our work since W is the input in Sen's quantum entropy function and since $D(s)$ is an object we can compute. Recalling that λ_i is the spectrum of the kinetic operator Λ, one can compute the one loop effective action W and thus one loop corrections to black hole entropy given knowledge of the spectrum of fields living around the black hole geometry.

[4] The Green's function is subject to the boundary condition $K(x,x';s=0) = \delta(x-x')$, a consequence of the orthonormality of the eigenfunctions $f_i(x)$.

1.4 Logarithmic Corrections to Black Hole Entropy.

We are interested in finding corrections to the Bekenstein-Hawking area formula within our one loop approximation,

$$S = \frac{A_{\rm Hor}}{4} + \alpha \ln A_{\rm Hor} + \beta + \gamma \frac{1}{A_{\rm Hor}} + \cdots \tag{1.4.1}$$

In this large area expansion the coefficients α, β, γ depend on the parameters of the black hole and the matter content; we shall focus on the logarithmic piece. To understand that recall that we have an expression for the effective action W that is computable from the spectrum of fields in a given geometry. With that expression in hand, one computes the trace of the energy momentum tensor,

$$T^{\mu}_{\mu} = \frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}}. \tag{1.4.2}$$

The trace of the energy momentum tensor will have have a divergent piece and a renormalized piece, in analogy with the effective action:

$$T^{\mu}_{\mu,\rm tot} = T^{\mu}_{\mu,\rm div} + T^{\mu}_{\mu,\rm ren}. \tag{1.4.3}$$

Here $T^{\mu}_{\mu,\rm div}, T^{\mu}_{\mu,\rm ren}$ are computed from $W_{\rm div}, W_{\rm ren}$ in analogy with (1.4.2). The renormalized part of the effective action is the input in Sen's quantum entropy function, which is of great interest to us. The heat kernel $D(s)$ can be expanded in a power series of s around $s = 0$ yielding three kinds of contribution,

$$D(s) = \text{singular terms} + D_0 + \mathcal{O}(s). \tag{1.4.4}$$

The constant term D_0 produces a logarithmically divergent contribution to W,

$$W_{\text{div}} \sim \frac{1}{2} D_0 \ln \epsilon^2. \tag{1.4.5}$$

If we momentarily assume classical scale invariance, then $T^{\mu}_{\mu,\text{tot}} = 0$ and so

$$T^{\mu}_{\mu,\text{ren}} = -T^{\mu}_{\mu,\text{div}} = -\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta W_{\text{div}}}{\delta g^{\mu\nu}} = \frac{2}{\text{Vol}} \frac{\partial W_{\text{div}}}{\partial \ln \epsilon^2} = \frac{1}{\text{Vol}} D_0, \tag{1.4.6}$$

where Vol is the volume of the manifold, which for AdS geometries is a priori divergent. In this situation one works with the regularized volume. At this point we can lift the assumption of classical scale invariance, keeping in mind that non scale invariant theories will additionally have a classical non anomalous contribution to the trace of the energy momentum tensor.

The quantum entropy function formalism tells us that $S = -W_{\text{ren}}$ for the class of black holes in which we are interested, and the logarithmic dependence of the entropy is determined by

$$\frac{\partial S}{\partial \ln A_{\text{Hor}}} = -\frac{\partial W_{\text{ren}}}{\partial \ln a^2} = -\frac{1}{2} \int d^N x T^{\mu}_{\mu,\text{ren}} = -\frac{1}{2} D_0. \tag{1.4.7}$$

Here, a is a physical scale, such as the radius of AdS space. The dependence on the physical scale $\ln a$ has the opposite sign from that of the regulator ϵ. The result for the logarithmic correction to the entropy is therefore

$$\delta S = \frac{1}{2} D_0 \ln A_{\text{Hor}}. \tag{1.4.8}$$

The upshot is that among the corrections to the Bekenstein-Hawking entropy formula, the logarithmic contribution deserves special attention since it can be computed in the low energy theory near the black hole horizon. It provides a good test for microscopic descriptions of black holes as well as guidance whenever no such descriptions exist. The evaluation of such corrections was outlined above and is a priori straightforward—one computes quadratic fluctuations around a given background followed by evaluation of the functional determinants.

Inspired by this rationale, the first goal of this dissertation is to improve on the computation of logarithmic corrections to the Bekenstein-Hawking formula by identifying the role of each contribution and streamlining the computations.

1.5 Overview of Results.

We have outlined above the procedure for computing logarithmic corrections to black hole entropy. This is a well known program spearheaded by Sen and collaborators, who have published several papers in the area [6, 7, 8, 9, 10]. Although seemingly straightforward, the computation can become quite involved due to the large collection of fields present and the existence of non trivial couplings of the unphysical components of gauge fields and ghosts.

We use a vector field in flat space as an example to motivate our study. In four dimensions a vector field A^μ has four components, one of which is pure gauge, one of which violates the gauge condition[5], and two physical components. In the old fashioned Gupta-Bleuler quantization one works in the small Hilbert space of physical states, while in BRST quantization one accounts for all components but includes two ghosts in the path integral, the contribution of which cancels the contributions of the unphysical components. Schematically,

$$4 - 2 = 2_{\text{phys}}. \tag{1.5.1}$$

In this context it is a matter of taste whether one takes the approach on the right hand side of (1.5.1), with fewer pieces to be evaluated, or the one on the left hand side of (1.5.1) where symmetries are explicit. However, if the matter content includes many more fields with gauge symmetry, and importantly if all these extra unphysical components couple non trivially to the physical parts, then taking the approach on the right of (1.5.1) seems not

[5] We will refer to field configurations set to zero by a gauge condition as gauge violating since their propagation violates the chosen gauge. At this point we are not specifying any gauge.

only advantageous but also intuitive.

As one progresses to curved spaces one learns that such an on-shell point of view is not perfectly accurate. In fact, the cancellation between unphysical components and ghosts leaves a remainder,

$$4 - 2 = 2_{\text{phys}} + \varepsilon. \tag{1.5.2}$$

This contribution ε is due to states we will call *boundary modes*. Boundary modes are normalizable pure gauge configurations with non normalizable gauge parameters,

$$\begin{aligned} A_\mu = \nabla_\mu \lambda, \ \text{with} \\ \int |A_\mu|^2 dV < \infty, \\ \int |\lambda|^2 dV = \infty. \end{aligned} \tag{1.5.3}$$

These configurations are not canceled by ghosts, since that would require non normalizable ghosts; alternatively, when one is dividing the path integral by the gauge group, boundary modes are not modded out since their gauge parameters are not in any gauge orbit of the gauge group. Hence, boundary modes are physical, albeit formally pure gauge. Their contribution to the path integral is essentially off shell since $F_{\mu\nu} \equiv 0$ for boundary mode configurations.

In this dissertation we propose to study the role and behavior of boundary modes with the objective of streamlining the evaluation of logarithmic corrections to the Bekenstein-Hawking formula.

In chapter 2 we focus on four dimensional extremal black holes with $\text{AdS}_2 \times S^2$ near horizon geometry and $\mathcal{N} \geq 2$ supersymmetries. We separate the matter content into $\mathcal{N} = 2$ supersymmetric multiplets, and compute the on shell spectrum of each field and their heat kernels. We shortcut the diagonalization of mass matrices, obtaining the eigenvalues of each field using group theory.

We then study the spectrum of boundary modes and compute their off shell contribution.

The final result agrees with the literature:

$$\delta S = \frac{1}{12}\left(23 - 11(\mathcal{N}-2) - n_V + n_H\right)\ln A_{\text{Hor}} \tag{1.5.4}$$

where $\mathcal{N}$ is the number of supersymmetries, n_V, n_H are the number of $\mathcal{N} = 2$ vector- and hypermultiplets. The upshot is that one is indeed able to take a (quasi-) on shell approach on the evaluation of quantum corrections to gravity, provided that the boundary mode contributions are taken into account. Another important point is that *all* the off shell information present in the one loop approximation is encoded by boundary modes.

This chapter is based on [11], written in collaboration with Cynthia Keeler and Finn Larsen.

In chapter 3 we further elucidate the role of boundary modes in quantum corrections to supergravity. We compute the entire off-shell spectrum of supergravity in $\text{AdS}_2 \times S^2$, from the point of view of the effective AdS_2 theory after compactification. Then by classifying all Kaluza-Klein towers of states into pure gauge, gauge violating, and physical, we check that the cancellations between unphysical components are not perfect and that they indeed leave boundary modes behind.

We identify boundary modes as harmonic modes of living in AdS_2 and address the incorrect notion sometimes found in the AdS/CFT literature that physical boundary states are at the "end" of the physical Kaluza-Klein towers; in fact they are also not in the "end" of the unphysical towers and there seems to be no simple accurate identification.

This chapter is based on [12], written in collaboration with Finn Larsen.

Supersymmetry is a key piece in our analysis of quantum corrections, partly because of its role in the quantum entropy function formalism, but also because supersymmetry allows us to classify fields nicely into supermultiplets. It is then natural that we extend our study of quantum corrections to black hole entropy to more broadly encompass quantum corrections to supergravity.

Supergravity rose to fame in part because supersymmetry alleviates the grave divergences one finds in quantum gravity. Recently Bern, Davies, and Dennen showed that four dimensional supergravity in flat space with $\mathcal{N} = 5$ supersymmetries is ultraviolet finite up to four loops [13]. It is still not clear what divergences exist for curved space supergravity, if there are any [14, 15, 16, 17, 18, 19, 20]. However, several groups since the 1980's have found that in curved backgrounds—such as AdS—the vacuum amplitude diverges as early as one loop [21, 22, 23, 24, 25, 26].

This discussion on the presence of divergences in supergravity was fueled by the proposition of Duff and Nieuwenhuizen [27] that classically dual fields can have inequivalent quantum descriptions, followed by almost immediate backlash by Siegel [28] claiming exactly the opposite, that classically dual fields have identical quantum descriptions. The (in)equivalence of quantum descriptions fits in the discussion of the paragraph above as departures from classical relations such as this proposed inequivalence are possible sources of divergences.

As an example of a classical duality, we look at a 2-form field $B_{\mu\nu}$ and a scalar ϕ, related to each other in four dimensions by

$$3\nabla_{[\mu}B_{\nu\sigma]} = H_{\mu\nu\sigma} = \epsilon_{\mu\nu\sigma\lambda}\nabla^{\lambda}\phi, \tag{1.5.5}$$

where we represented the field strength of $B_{\mu\nu}$ as $H_{\mu\nu\sigma}$. Quantum inequivalence is the statement that while the 2-form and the scalar are classically dual, their path integrals differ.

Chapter 4 draws inspiration from the role of boundary modes on quantum corrections to gravity to address quantum inequivalence. We compute the one loop divergence of the massless multiplet in $\mathcal{N} = 8$ supergravity on AdS$_4$, finding a non vanishing a anomaly. This non vanishing anomaly is a consequence of quantum inequivalence.

We identify inequivalences such as the one between scalars and 2-forms as being due to boundary modes: for a pure gauge configuration $B_{\mu\nu} = \nabla_{[\mu}A_{\nu]}$, the duality is trivial

$$H_{\mu\nu\sigma} \equiv 0 = \epsilon_{\mu\nu\sigma\lambda}\nabla^{\lambda}\phi, \quad \Rightarrow \phi = \text{const.}, \tag{1.5.6}$$

mapping boundary mode configurations into constant scalar states, which are non normalizable in a non compact geometry. Thus, the classical duality doesn't map the physical 2-form boundary states into any physical scalar state. The conclusion is that quantum inequivalence is not only a real effect, it is intimately connected to boundary modes as well.

An important consequence is that an observed divergence in supergravity is duality frame dependent. By suitable dualization of an arbitrary number of 2-forms into scalar fields or vice-versa, one can change such divergence. In fact, in the frame that arises naturally from compactification of eleven dimensional $\mathcal{N} = 1$ supergravity around S^7, with seven 2-forms and one 3-form, there is no divergence.

We also address the topological nature of the inequivalence between 2-forms and scalars. The resulting divergence, or equivalently the corresponding boundary mode contribution to one loop effects, is proportional to the Euler invariant χ of the manifold. This is one important source of the disagreement regarding quantum inequivalence: if boundary conditions such that $\chi = 0$ are chosen, then no inequivalence is present, and also no divergences are observed in the supergravity theory.

This chapter is based on [29], written in collaboration with Finn Larsen.

In chapter 5 we extend the computation carried for the AdS_4 massless multiplet to all massive multiplets resulting from the Kaluza-Klein compactification of eleven dimensional supergravity around S^7, as well as studying the boundary modes that arise naturally from the eleven dimensional theory.

This chapter is based on work in progress.

Chapter 2

Logarithmic Corrections to $\mathcal{N} \geq 2$ Black Hole Entropy.

2.1 Motivation and Summary.

In this first chapter we present a novel approach in the computation of logarithmic corrections to black hole entropy, to be detailed and extended in the following chapters. The computation of logarithmic corrections from the low energy theory is straightforward in principle [30]: determine the quadratic fluctuations around the black hole background and then compute the resulting functional determinant using standard techniques. However, in practice these steps can be quite laborious. The theories of interest in string theory generally have elaborate matter content that results in many distinct contributions to quantum corrections. Gauge symmetries (including diffeomorphism invariance) further complicate the situation by introducing ghost sectors that can be quite nontrivial. The logarithmic corrections to black holes were developed in many recent works including [31, 32, 33, 34, 35].

The goal of this chapter is to present a simplified computation of logarithmic corrections to black hole entropy. The streamlined procedure we present promotes transparency and makes it realistic to address more complicated settings. In this chapter we limit ourselves to BPS black holes which have $\mathrm{AdS}_2 \times S^2$ near horizon geometry. In this context important aspects of our strategy are these:

- *The Spectrum of Chiral Primaries:* a large number of interactions between different fields generally leads to unwieldy matrices at intermediate stages of the computation. We diagonalize the interactions by first computing the spectrum of chiral primaries. This spectrum encodes all information about the interactions that is needed.

  In order to highlight the origin of these simplifications in symmetry principles we give a self-contained derivation of the black hole spectrum. Our method is indirect but it is efficient and new to this context. Further, our independent computation of the spectrum identifies several details that have previously been overlooked.

- *Simplified Functional Determinants:* we reduce the field content of the 4D theory to a set of fields on AdS_2 and its boundary. The only functional determinants we need are those for massless scalars and fermions in AdS_2. The additional data that is special to each field we consider is encoded in a discrete sum over masses. This organization of the computation represents a simplification because it does not require measures and contours for continuous complex eigenvalues. We also do not need explicit wave functions.

- *Gauge-fixing and Ghosts:* we compute quantum corrections by summing over contributions from physical fields only. The unphysical sector comprising pure gauge modes, longitudinal modes, and ghosts ultimately cancel in the physical quantities of interest. We use an on-shell method where these quantities are not needed in intermediate stages of the computation.

- *Boundary Modes:* gauge symmetries (including supersymmetry and diffeomorphism invariance) give rise to physical modes that localize on the boundary. We determine the quantum numbers of these modes by analyzing the action of the relevant symmetry. Their contribution is then computed by treating them as 2D fields on the S^2.

The physical modes that contribute to the one-loop functional determinant are the 4D bulk modes, the 2D boundary modes, and the 0D zero-modes. Adding the contributions together our final result for logarithmic corrections to extremal black hole entropy in theories with $\mathcal{N} \geq 2$ SUSY becomes

$$S = \frac{1}{4}A_H + \frac{1}{12}\left[23 - 11(\mathcal{N}-2) - n_V + n_H\right]\log A_H. \tag{2.1.1}$$

This final result agrees perfectly with those reported by A. Sen and collaborators [36, 37, 7]. Some important special cases of the formula:

- The $\mathcal{N} = 4$ theory. Such theories have $n_V = n_H + 1$ because one $\mathcal{N} = 2$ vector is part of the $\mathcal{N} = 4$ supergravity multiplet while each $\mathcal{N} = 4$ matter multiplet is composed of one $\mathcal{N} = 2$ vector and one $\mathcal{N} = 2$ hyper. In this case the logarithmic correction vanishes independently of the number of $\mathcal{N} = 4$ matter multiplets.

- The $\mathcal{N} = 6$ theory: $n_V = 7$ and $n_H = 4$ so that the logarithmic correction is $\delta S = -2\log A_H$.

- The $\mathcal{N} = 8$ theory: $n_V = 15$ and $n_H = 10$ so $\delta S = -4\log A_H$.

We evaluate the functional determinants using heat kernel techniques. In 4D the leading term in the heat kernel is a double pole. These double poles cancel in each $\mathcal{N} = 2$ multiplet by itself. This corresponds to vanishing cosmological constant in 4D and is due to the degeneracy of bosons and fermions in the on-shell SUSY multiplets.

The simple pole in the heat kernel receives contributions from the 2D boundary modes that are non-trivial since there is not the same number of bosonic and fermionic symmetries. It also receives a contribution from mixing between the bulk modes. It is a consistency check on our computations that the sum of these terms vanish for any theory with at least $\mathcal{N} = 4$ [22]. For the more general theories we consider the coefficient of the pole in the heat kernel is non-trivial. This part of our result can be interpreted as the renormalization of the gravitational coupling constant.

The logarithmic corrections to black hole entropy are encoded in the constant term of the heat kernel so contributions from both bulk modes and boundary modes must be computed with sufficient precision that the constant is determined. Additionally, there are contributions from zero-modes.

The indirect methods we pursue in this chapter stress the origin of particle spectra in symmetry but at times they leave room for suspicion. In the next chapter we will present the explicit mode expansions that underpin the physical spectrum.

This chapter is organized as follows. In section 2.2 we determine the spectrum of chiral primaries using an indirect argument that exploits symmetries. We resolve a discrepancy with results reported in the literature. In section 2.3 we review the simple heat kernels we need. We provide a self-contained presentation in order to highlight the complete absence of advanced techniques. In section 2.4 we apply the heat kernels to the physical spectrum determined in section 2.2. We thus compute the contribution to the heat kernel from all bulk modes. In section 2.5 we discuss gauge symmetries and use them to determine the spectrum of boundary modes. This yields an additional contribution to the heat kernel. In section 2.6 we briefly review the correction to the heat kernel due to zero-modes on the boundary. Finally, in section 2.6.2 we add the various contibutions to the heat kernel and we discuss the relation to trace anomalies. This gives the logarithmic correction to black hole entropy (2.1.1).

2.2 Classical Modes.

The *spectrum* of the black hole is the set of quantum numbers for fluctuations around the black hole background. In this section we use symmetry principles to determine the BPS part of the spectrum.

We consider a 4D theory with (at least) $\mathcal{N} = 2$ SUSY. We further focus on the near horizon region of black holes that preserve at least some of the supersymmetry. This geometry always takes the form $\mathrm{AdS}_2 \times S^2$. The attractor mechanism ensures that gravity and the graviphoton are the only fields turned on in the near horizon geometry of the black hole [38].

Fields in the $\mathrm{AdS}_2 \times S^2$ background are classified by the quantum numbers of the $SL(2) \times SU(2)$ isometries. We are particularly interested in the lowest weight representations which we denote by (h, j). Here h is the lowest eigenvalue of the L_0 generator of $SL(2)$ and j refers to the $SU(2)$ representation. The (h, j) representation thus has degeneracy $(2j+1)$ from its $SU(2)$ representation and also an infinite tower of states with L_0 values $h, h+1, h+2, \ldots$. The BPS spectrum of the black hole is a list of the (h, j) that are realized by fluctuations in the background.

The massless field content of a general theory with $\mathcal{N} \geq 2$ SUSY can be decomposed into a set of $\mathcal{N} = 2$ multiplets:

- A supergravity multiplet.
- $\mathcal{N} - 2$ (massive) gravitino multiplets (because two of the $\mathcal{N}$ gravitinos are in the $\mathcal{N} = 2$ supergravity multiplet).
- n_V vector multiplets.
- n_H hyper multiplets.

2.2.1 Determination of BPS Spectra.

It is useful to organize the particle content of $\mathcal{N} = 2$ multiplets according to their *helicity content*. Suppose that the maximum helicity state in a given $\mathcal{N} = 2$ multiplet is λ. Upon action with one of the two SUSY generators we then find two states with helicity $\lambda - \frac{1}{2}$ and, upon action with both of them, we find a single state with helicity $\lambda - 1$. This universal structure gives the helicity content of each $\mathcal{N} = 2$ multiplet. This helicity content is presented

Supergravity multiplet:	$\lambda = \ \pm 2, \ \pm \frac{3}{2} \times 2, \ \pm 1.$
Gravitino multiplet:	$\lambda = \ \pm \frac{3}{2}, \ \pm 1 \times 2, \ \pm \frac{1}{2}.$
Vector multiplet:	$\lambda = \ \pm 1, \ \pm \frac{1}{2} \times 2, \ 0 \times 2.$
Hyper multiplet:	$\lambda = \ \pm \frac{1}{2} \times 2, \ 0 \times 4.$

Table 2.1: Helicity content of $\mathcal{N} = 2$ multiplets.

in Table 2.1. The notation $\times 2$ indicates a multiplicity of 2. In the first three kinds of multiplets we included the CPT conjugate states with negative helicity as one must in field theory realizations. The hypermultiplet was automatically CPT invariant but we double its field content anyway. With this convention the hypermultiplet is a "full" hyper with 4 real scalars and two Weyl spinors.

The field equations for quadratic fluctuations are linear. Moreover, we can introduce global flavor symmetries unique to each type of $\mathcal{N} = 2$ supermultiplet and this ensures that there is no mixing between different types of $\mathcal{N} = 2$ supermultiplets. We can therefore consider the supergravity multiplet, the (massive) gravitino multiplets, the vector multiplets, and the hyper multiplets independently.

The expansion of four-dimensional fields in partial waves on S^2 gives an effective 2D theory on AdS_2. The $SU(2)$ representations that appear are determined by the general rules that govern Kaluza-Klein reduction on homogeneous spaces [39]. In the case of the coset $S^2 = SU(2)/U(1)$ the quantum number under $U(1)$ can be identified with the helicity λ and the $SU(2)$ representations that appear in the reduction are precisely those where λ appears in the decomposition of $SU(2)$ with respect to $U(1)$. Thus the allowed angular momentum quantum numbers for a helicity mode λ are $j = |\lambda|, |\lambda| + 1, \ldots$. Starting from the helicity content of the fields in Table 2.1 we can therefore present the $SU(2)$ content in terms of towers. This is in Table 2.2. The BPS spectrum of the black hole amounts to the specification of the value of the AdS_2 energy h for each of these $SU(2)$ multiplets. These energies depend on couplings between the fields. The simplification captured by the enumeration in Table 2.2 is that these couplings respect the partial wave expansion: only

Supergravity multiplet:	$j =$	$(k+2)\times 2,\ (k+\frac{3}{2})\times 4,\ (k+1)\times 2.$
Gravitino multiplet:	$j =$	$(k+\frac{3}{2})\times 2,\ (k+1)\times 4,\ (k+\frac{1}{2})\times 2.$
Vector multiplet:	$j =$	$(k+1)\times 2,\ (k+\frac{1}{2})\times 4,\ k\times 2.$
Hyper multiplet:	$j =$	$(k+\frac{1}{2})\times 4,\ k\times 4.$

Table 2.2: $SU(2)$ content of $\mathcal{N}=2$ multiplets. $k = 0, 1, \ldots$

fields with the same j can mix.

The actual value of the AdS_2 energy h is determined by supersymmetry as follows. The $\mathrm{AdS}_2 \times S^2$ geometry preserves the supergroup $SU(2|1,1)$. This supergroup has 8 SUSY charges, the same as the number in $\mathcal{N}=2$ SUSY in four dimensions. These generators can be represented in terms of two component spinors $\mathcal{Q}^A$ $(A = 1,2)$ and their conjugates. The corresponding charges all have quantum numbers $h = 1/2$ and $j = 1/2$. They transform as doublets of the global $SU(2)$ symmetry acting on the $A = 1,2$ index. We will suppress reference to this global $SU(2)$ in the following in order to avoid confusion with the $SU(2)$ rotation group. Since SUSY is preserved by the background, fluctuating fields must organize themselves into supermultiplets after the mixing is taken into account. Starting from a lowest weight state (h,j) a supermultiplet is obtained by acting with the supercharges that function as creation operators.

The fields we consider will all be in chiral multiplets of the form

$$(k,k), 2(k+\frac{1}{2}, k-\frac{1}{2}), (k+1, k-1), \tag{2.2.1}$$

with the possible values of $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$. In the special case where $k = \frac{1}{2}$ the $SU(2)$ quantum number $j = -\frac{1}{2}$ of the final term in (2.2.1) should be interpreted as an empty representation.

The chiral multiplets (2.2.1) are *short* multiplets. They are special in two (related) ways: the lowest weight state has $h = j$ and also the supercharges always act in a manner that *lowers* the spin. A generic long representation would have four active supercharges so that the span of spins in a single multiplet would be two. Such representations are therefore too large for our purpose.

There is a unique way to organize the fields with the $SU(2)$ content in Table 2.2 into chiral multiplets of the form (2.2.1). This gives the list of fields This is the complete spectrum of

Supergravity multiplet:	$2[(k+2,k+2),\ 2(k+\frac{5}{2},\ k+\frac{3}{2}),\ (k+3,k+1)]$.
Gravitino multiplet:	$2[(k+\frac{3}{2},k+\frac{3}{2}),2(k+2,k+1),(k+\frac{5}{2},k+\frac{1}{2})]$.
Vector multiplet:	$2[(k+1,k+1),2(k+\frac{3}{2},k+\frac{1}{2}),(k+2,k)]$.
Hyper multiplet:	$2[(k+\frac{1}{2},k+\frac{1}{2}),2(k+1,k),(k+\frac{3}{2},k-\frac{1}{2})]$.

Table 2.3: $\mathcal{N}=2$ black hole spectrum. $k=0,1\ldots$

the black hole. In particular the spectrum is determined entirely by symmetries.

2.2.2 Explicit Computations.

The determination of the on-shell spectrum using symmetry constraints illuminates its group theory origin. However, the indirect nature of the method may leave some conceptual unease. It is therefore worthwhile to consider an alternative, the explicit diagonalization of the action expanded to quadratic order. This approach was carried out over a decade ago for the case of pure $\mathcal{N}=2$ SUGRA [40] and for the maximally supersymmetric theory with $\mathcal{N}=8$ SUSY [41, 42]. Combination of the final tables in these references yields towers of multiplets that can be compared with our results in Table 2.3 that apply to the slightly more general case where $\mathcal{N}=2$ SUGRA is coupled to $\mathcal{N}-2$ (massive) gravitini multiplets, n_V vector multiplets, and n_H hypermultiplets. The results in the references agree precisely with Table 2.3 *with one exception*: all previous works report an additional chiral multiplet. In our notation the additional states that were reported correspond to the extension of one of the two supergravity multiplet towers in Table 2.3 to include the mode $k=-1$. Thus the primary states reported in [40, 41, 42], but absent from our analysis are

$$(1,1),2(\frac{3}{2},\frac{1}{2}),(2,0). \tag{2.2.2}$$

It is instructive to find the origin of this discrepancy.

As a starting point for this specific purpose it is sufficient to consider 4D Einstein gravity coupled to a $U(1)$ gauge field

$$\mathcal{L}_4 = \frac{1}{16\pi G}\left[\mathcal{R}^{(4)} - \frac{1}{4}F_{IJ}F^{IJ}\right]. \tag{2.2.3}$$

We use 4D indices $I, J, \ldots$, AdS$_2$ indices $\mu, \nu, \ldots$, and S^2 indices $\alpha, \beta, \ldots$. One solution to this theory is the AdS$_2 \times S^2$ geometry supported by the magnetic monopole $F_{\alpha\beta} = 2\epsilon_{\alpha\beta}$. With this normalization the AdS$_2$ and S^2 radii are both "1". The Freund-Rubin reduction on S^2 is realized by the 4D geometry:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + X d\Omega_2^2, \tag{2.2.4}$$

where $g_{\mu\nu}$ and X are arbitrary functions of the 2D coordinates x^μ, $\mu = 1, 2$. The effective 2D Lagrangian becomes

$$\mathcal{L}_2 = \frac{1}{4G}\left[X\mathcal{R}^{(2)} + 2 - \frac{2}{X} + \frac{(\nabla X)^2}{2X}\right]. \tag{2.2.5}$$

The equations of motion are obtained upon variation of $\mathcal{L}_2$ by the scalar X

$$\mathcal{R}^{(2)} + \frac{2}{X^2} + \frac{(\nabla X)^2}{2X^2} - \frac{1}{X}\nabla^2 X = 0, \tag{2.2.6}$$

and by the metric $g_{\mu\nu}$

$$\begin{aligned} X(\mathcal{R}^{(2)}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}^{(2)}) + \frac{1}{2X}[\nabla_\mu X\nabla_\nu X - \frac{1}{2}g_{\mu\nu}(\nabla X)^2] \\ -g_{\mu\nu}(1 - \frac{1}{X}) + g_{\mu\nu}\nabla^2 X - \nabla_\mu\nabla_\nu X = 0. \end{aligned} \tag{2.2.7}$$

Recall that the Riemann tensor has just a single component in 2D so after contractions $\mathcal{R}^{(2)}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\mathcal{R}^{(2)}$ identically for any 2D geometry, not just for symmetric geometries. The first term in (2.2.7) therefore vanishes identically. We write this term temporarily because it reminds us that (2.2.7) is the Einstein equation while (2.2.6) is the equation of motion for the 2D matter field X. As a check on (2.2.6) and (2.2.7) note that the AdS_2 geometry satisfies these equations with $\mathcal{R}^{(2)} = -2$ and $X = 1$. This corresponds to AdS_2 and S^2 radii equal to "1".

The Einstein equation (2.2.7) decomposes into the trace

$$\nabla^2 X = 2(1 - \frac{1}{X}), \tag{2.2.8}$$

and (upon use of (2.2.8)) the traceless equation

$$(\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla^2)\sqrt{X} = 0. \tag{2.2.9}$$

Taking (2.2.8) in isolation we find that small variations δX around the background $X = 1$ satisfy a Klein-Gordon equation with $m^2 = 2$. In AdS_2 scalar excitations with this mass have conformal weight $h = 2$. The excitations described by the Freund-Rubin compactification (2.2.4) are spherically symmetric (j=0) so this mode would have quantum numbers $(h, j) = (2, 0)$. Comparison with 2.2.2 shows that this is exactly the mode that the explicit analyses recognize as physical but our indirect analysis does not. We will show that the discrepancy is due to the constraints expressed by (2.2.9).

For perspective on the discrepancy recall the elementary counting of degrees of freedom. Perturbative 2D gravity is described by the symmetric tensor $\delta g_{\mu\nu} = h_{\mu\nu}$ with 3 components. Diffeomorphisms $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ impose equivalences that render two components of $h_{\mu\nu}$ redundant. The equations of motion resulting from variations of those two components further impose two constraints so the net number of degrees of freedom in pure 2D gravity is -1. This awkward counting is special to 2D where it is indeed well known for theories such as dilaton gravity (see eg [43]). It implies that the *combination* of 2D gravity (described by $h_{\mu\nu}$) *and* a scalar field (in the present context the 2D scalar field X) will have no degrees of freedom.

There are several known exceptions to this simple type of counting: there may be important quantum effects (captured by a class of matrix models) or there may be classical degrees of freedom in less than 2D. In the present context there are indeed 1D boundary states but they should not be confused with bulk degrees of freedom which is where we differ from previously reported results.

To make the general discussion on the counting of degrees of freedom more explicit we fix the gauge $g_{zz} = 1, g_{zt} = 0$ and so consider the 2D geometry in the form

$$ds^2 = -e^{2\rho} dt^2 + dz^2, \tag{2.2.10}$$

where $\rho = \rho(t, z)$ is an arbitrary function. In this gauge we can represent the background AdS_2 as either just the Poincaré patch (with $e^{2\rho_0} = e^{2z}$) or global AdS_2 (with $e^{2\rho_0} = \cosh^2 z$) or as an AdS_2 black hole (with $e^{2\rho_0} = \sinh^2 z$). For any of these backgrounds the zz and zt-components of (2.2.9) give

$$\begin{aligned} (\partial_z^2 - 1)\delta X &= 0, \\ \partial_z (e^{-\rho_0} \partial_t \delta X) &= 0, \end{aligned} \tag{2.2.11}$$

after linearization. The first equation was simplified using (2.2.8). These equations are *constraints* on fluctuations δX. If δX were a propagating field, we would be able to specify δX and its time derivative $\partial_t \delta X$ for all z at an initial time and then use the equations of motion to find δX at later times. The constraints (2.2.11) show that this is impossible: once we have given δX and $\partial_t \delta X$ for large z, initial conditions are specified for all z. Thus δX is in fact a boundary degree of freedom.

We have not yet analyzed the equation of motion (2.2.6) which relates the curvature $\mathcal{R}^{(2)}$

to the scalar field X. The Ricci curvature of (2.2.10) is

$$\mathcal{R}^{(2)} = -2e^{-\rho}\partial_z^2 e^{\rho}, \tag{2.2.12}$$

so (2.2.6) can be recast as

$$2\delta X = \nabla^2 \delta X = -\frac{1}{2}\delta\mathcal{R}^{(2)} = e^{-\rho_0}(\partial_z^2 - 1)\delta\rho. \tag{2.2.13}$$

This demonstrates that perturbations $\delta\rho$ with $\partial_z^2\delta\rho = 1$ are independent degrees of freedom.

In summary, in this subsection we analyzed the spherically symmetric sector of gravity comprising the 2D metric $h_{\mu\nu}$ and the scalar field X encoding the size of the S^2. We find that after taking gauge fixing and constraints into account the bulk theory has no physical states but two boundary degrees of freedom remain.

2.2.3 Boundary Modes.

Table 2.3 enumerates all *bulk* modes of the black holes. In addition to these modes there are *boundary* modes. The boundary modes are closely associated with gauge symmetries of the theory. Each component of a gauge symmetry allows the removal of one component field. Additionally, the equation of motion for the component thus removed ceases to be dynamical: it becomes a *constraint*. As discussed in the previous subsection, constraints limit the dynamics of the theory by restricting the independent initial data. In the context of AdS_2 each constraint gives rise to one boundary mode.

We first consider the supergravity multiplet. The perturbation h_{IJ} of the 4D metric has 10 components. Diffeomorphisms $\delta_\xi h_{IJ} = \partial_I\xi_J + \partial_J\xi_I$ are generated by the vector field ξ_I with 4 components. Thus the graviton has 6 components subject to 4 constraints. This yields a net of 2 physical degrees of freedom in bulk, as it should. But in addition the boundary data on the 4 constraints give rise to 4 boundary degrees of freedom. These boundary degrees of freedom have the quantum numbers of the diffeomorphism generator ξ_I. In particular, they have helicity content $\lambda = \pm 1, 0, 0$.

A chiral gravitino Ψ_I has 6 components after the Rarita-Schwinger constraint $\gamma^I \Psi_I = 0$ is taken into account. After gauge fixing of local supersymmetry (generated by a chiral spinor with two components) it has 4 components subject to two constraints. This yields a net of two physical degrees of freedom but also two boundary degrees of freedom.

Finally, the graviphoton $\mathcal{A}_I$ has four components. The $U(1)$ gauge symmetry removes one component so three components remain which are subject to one constraint. This gives two physical components for the graviton in bulk but also a single boundary degree of freedom.

Proceeding similarly for the (massive) gravitino multiplet and the vector multiplet, the helicity content of all physical boundary modes can be found on Table 2.4.

The hyper multiplet is not mentioned since it does not have any gauge degrees of freedom and therefore no boundary states. The helicity content in Table 2.4 in turn determines the $SU(2)$ content of the boundary modes, reported in Table 2.5.

Supergravity multiplet:	$\lambda = \pm 1, \pm \frac{1}{2} \times 2, 0 \times 3.$
Gravitino multiplet:	$\lambda = \pm \frac{1}{2}, 0 \times 2.$
Vector multiplet:	$\lambda = 0.$

Table 2.4: Boundary mode helicity content.

Supergravity multiplet:	$j = (k+1) \times 2,\ (k + \frac{1}{2}) \times 4,\ k \times 3.$
Gravitino multiplet:	$j = (k + \frac{1}{2}) \times 2,\ k \times 2.$
Vector multiplet:	$j = k.$

Table 2.5: Boundary Mode $SU(2)$ content. $k = 0, 1, \ldots$

Our discussion of boundary states here focuses on gauge invariance. As such it is based on the off-shell (un-physical) components of the various fields. The on-shell supersymmetry realized by the fields we consider does not extend a simple way to these off-shell degrees of freedom. In the absence of further data it is therefore not possible to compute the conformal weights of these fields from superconformal invariance alone.

Two of the three $j = 0$ fields in the supergravity multiplet are the $\delta\rho$ and δX discussed explicitly in the previous subsection. Similar computations for the remaining fields -detailed in the next chapter- determine the full spectrum of boundary states. In section 2.5 we determine spectrum of boundary states by exploiting symmetries.

2.3 The Heat Kernel Expansion: Elementary Examples.

This section reviews the basics of the heat kernel method [30, 44, 45]. We introduce notation and also give elementary evaluations of the key examples that later will be generalized.

2.3.1 Functional Determinants and the Heat Kernel.

In this subsection we briefly review the role of the heat kernel method in the evaluation of functional determinants presented in the Introduction, and direct it to our intended applications. One loop quantum corrections are encoded in Euclidean path integrals taking a Gaussian form which we present schematically as

$$e^{-W} = \int \mathcal{D}\phi \; e^{-\phi\Lambda\phi} = \frac{1}{\sqrt{\det\Lambda}}. \tag{2.3.1}$$

The kinetic operator generally includes a mass term $\Lambda = -\Delta + m^2$. We suppress the indices on ϕ that enumerate components of the field such as those that incorporate Lorentz structure.

After UV regulation the effective action W becomes

$$W = \frac{1}{2}\ln\det\Lambda = \frac{1}{2}\sum_i \ln\lambda_i = -\frac{1}{2}\int_{\epsilon^2}^{\infty} ds \frac{D(s)}{s}, \tag{2.3.2}$$

where $\{\lambda_i\}$ are the eigenvalues of Λ and the heat kernel

$$D(s) = \mathrm{Tr}\ e^{-s\Lambda} = \sum_i e^{-s\lambda_i}. \tag{2.3.3}$$

We use a notation where the eigenvalues λ_i are assumed discrete even though in practice they may be continuous. Also, in cases where the fields are fermionic the determinant in (2.3.1) should be in the numerator instead and then the contribution to the effective action (2.3.2) will enter with the opposite sign.

The heat kernel terminology arises because it is often useful to express $D(s)$ as

$$D(s) = \int d^D x\ K(x, x; s), \tag{2.3.4}$$

where the Green's function satisfies the heat equation

$$(\partial_s + \Lambda_x)\, K(x, x'; s) = 0, \tag{2.3.5}$$

with the boundary condition $K(x, x'; s) = \delta(x - x')$ at $s = 0$. The Green's function can be expanded on a complete basis as

$$K(x, x'; s) = \sum_i e^{-s\lambda_i} f_i(x) f_i^*(x'), \tag{2.3.6}$$

where $\{f_i\}$ are the normalized eigenfunctions of Λ with eigenvalues $\{\lambda_i\}$. Inserting this expansion in (2.3.4) and using the normalization condition we do indeed recover (2.3.3).

As an example, in flat space with D Euclidean dimensions the eigenfunctions of the kinetic operator are plane waves e^{ikx} and the eigenvalues are $k^2 + m^2$. Expression (2.3.6) becomes a Gaussian integral which upon integration gives the Green's function

$$K_{\mathrm{flat}}(x, x'; s) = \left(\frac{1}{4\pi s}\right)^{\frac{D}{2}} e^{-\frac{1}{4s}(x-x')^2 - m^2 s}. \tag{2.3.7}$$

Inserting this expression in (2.3.4) we find the heat kernel for a massless scalar field

$$D_{\text{flat}}(s) = \left(\frac{1}{4\pi s}\right)^{\frac{D}{2}} \text{Vol}. \tag{2.3.8}$$

This expression gives the leading asymptotic behavior for small s (small distance) in any geometry. A standard approach to curved space examples is to correct the flat space result (2.3.8) perturbatively (see *eg.* [44]). This gives an expansion in small s with coefficients that are scalars formed from the curvature. For example, for a minimally coupled scalar field

$$K^s(s) = \left(\frac{1}{4\pi s}\right)^{\frac{D}{2}} \left[1 + \frac{s}{6}\mathcal{R} + \frac{s^2}{360}(5\mathcal{R}^2 - 2R_{IJ}R^{IJ} + 2R_{IJKL}R^{IJKL}) + \ldots\right]. \tag{2.3.9}$$

Similar expansions apply to other fields.

In our computations we will actually *not* employ the heat equation (2.3.5) and, related to that, we will avoid the explicit eigenfunctions. Instead we will compute $D(s)$ directly from (2.3.3) by explicit summation over eigenvalues. In the homogeneous spaces we focus on the corresponding heat kernel density is then given by

$$K(s) = \frac{1}{\text{Vol}} D(s). \tag{2.3.10}$$

For a sphere S^2 with radius a the volume is simply $\text{Vol}_S = 4\pi a^2$. For AdS_2 the volume diverges but it can be regulated near the boundary

$$\text{Vol}_A = 2\pi a^2 \int_0^{\rho_{\max}} d\rho \sinh\rho = 2\pi a^2(\cosh\rho_{\max} - 1). \tag{2.3.11}$$

In the context of AdS/CFT it is often appropriate to remove the $\cosh\rho_{\max}$ by adding terms that are intrinsic and local on the boundary. This gives $\text{Vol}_A = (-2\pi a^2)$ for the renormalized volume of AdS_2. We do not use this value since a positive volume makes it easier to track signs for fermions and bosons. The dependence of the actual (regulated) volume (2.3.11) on a cut-off will anyway cancel in physical results so we can effectively take $\mathrm{Vol}_A = +2\pi a^2$ when an explicit volume is needed.

Although our strategy is to compute $D(s)$ using the sum (2.3.3) we will quote results in terms of $K(s)$ using the relation (2.3.10). This practice will facilitate comparison with the literature.

2.3.2 The Scalar Field on S^2.

The heat kernel on the two-sphere S^2 is of special importance to us since it will serve as the building block for all our computations.

The determination of this heat kernel is particularly simple because the eigenvalue problem of the Laplacian on S^2 has been studied by all physics students since their first course in quantum mechanics. The possible eigenvalues of $-\nabla^2$ are $l(l+1)$ with each value of the orbital angular momentum $l = 0, 1, \ldots$ appearing with degeneracy $2l+1$ corresponding to the possible azimuthal quantum numbers $m = -l, \ldots, l$. The corresponding eigenfunctions are the spherical harmonics Y_{lm}. These basic facts immediately give the heat kernel (density) for a minimally coupled scalar field on S^2:

$$K_S^s(s) = \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} e^{-sk(k+1)}(2k+1). \tag{2.3.12}$$

We can expand for small s using the Euler-MacLaurin formula in the form simplified for functions with $f^{(n)}(\infty) = 0$:

$$\begin{aligned}\sum_{k=0}^{\infty} f(k) &= \int_0^{\infty} dk f(k) + \frac{1}{2}(f(0) + f(\infty)) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(f^{(2n-1)}(\infty) - f^{(2n-1)}(0)\right) \\ &= \int_0^{\infty} dk f(k) + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \frac{1}{720}f'''(0) + \ldots\end{aligned} \tag{2.3.13}$$

The sum (2.3.12) then gives

$$K_S^s(s) = \frac{1}{4\pi a^2}\left[\int_0^\infty dk\ e^{-sk(k+1)}(2k+1) + \frac{1}{2} - \frac{1}{12}(2-s) + \frac{1}{720}(-12s) + \mathcal{O}(s^2)\right] \tag{2.3.14}$$
$$= \frac{1}{4\pi a^2 s}\left(1 + \frac{1}{3}s + \frac{1}{15}s^2 + \dots\right).$$

2.3.3 The Fermion Field on S^2.

Relativistic fermions on S^2 transform in the $2j+1$ dimensional representations of the rotation group with half-integral values $j = \frac{1}{2}, \frac{3}{2}, \dots$. The square of the Dirac operator is a scalar so it commutes with the angular momentum operator. Indeed, these operators are essentially the same (see *eg.* [46]):

$$-D_F^2 = \vec{J}^2 + \frac{1}{4}. \tag{2.3.15}$$

The eigenvalues needed for the heat kernel are thus $j(j+1) + \frac{1}{4} = (j+\frac{1}{2})^2$. Introducing the integer $k = j - \frac{1}{2} = 0, 1, \dots$ we write the analogue of (2.3.12) for one fermionic degree of freedom:

$$K_S^f(s) = \frac{1}{4\pi a^2}\sum_{k=0}^{\infty} e^{-s(k+1)^2}(2k+2) = \frac{1}{2\pi a^2}\sum_{k=0}^{\infty} e^{-sk^2} k. \tag{2.3.16}$$

We evaluate this expression using the Euler-MacLaurin formula (2.3.13):

$$K_S^f(s) = \frac{1}{4\pi a^2}\left[\int_0^\infty dk\ e^{-sk^2} 2k + \left(-\frac{1}{12}\cdot 2 + \frac{1}{720}(-12s) + \mathcal{O}(s^2)\right)\right] \tag{2.3.17}$$
$$= \frac{1}{4\pi a^2 s}\left(1 - \frac{1}{6}s - \frac{1}{60}s^2 + \dots\right).$$

We employ the convention that the heat kernel for the spinor on the sphere has the same sign as a scalar. Fermion statistics will of course ultimately change the sign of the contribution to the one loop determinant but we will take this into account manually when needed.

2.3.4 Scalars and Fermions on AdS$_2$.

The expansion of the heat kernel in curvature invariants has the structure (2.3.9) for all fields. The only local distinction between S^2 and AdS$_2$ is the sign of the curvature. Further, by dimensional analysis each power of curvature is accompanied by one power of the expansion parameter s. Thus we can find the heat kernels on AdS$_2$ from the S^2 results by changing the sign of s. The overall sign of the heat kernel is such that the asymptotics (2.3.7) apply for small s.

Applying the $s \to -s$ rule to the scalar on S^2 (2.3.14) we find

$$K_A^s(s) = \frac{1}{4\pi a^2 s}\left(1 - \frac{1}{3}s + \frac{1}{15}s^2 + \dots\right), \tag{2.3.18}$$

for the massless scalar on AdS$_2$. The fermion on S^2 (2.3.17) similarly gives

$$K_A^f(s) = -\frac{1}{4\pi a^2 s}\left(1 + \frac{1}{6}s - \frac{1}{60}s^2 + \dots\right), \tag{2.3.19}$$

for each fermionic degree of freedom on AdS$_2$. We take fermion statistics into account through the overall sign in (2.3.19).

The $s \to -s$ rule relates the local terms in the heat kernels on S^2 and AdS$_2$ but there are no correspondingly simple continuations of individual eigenvalues and eigenfunctions [47]. For example, the scalar spectrum on S^2 is $\lambda_S = l(l+1)$ with $l = 0, 1, \dots$. The scalar spectrum AdS$_2$ similarly includes a discrete branch for which $\lambda_A = -m^2 = -h(h-1)$ with $h = 1, 2, \dots$. These highest weight type modes are important as they correspond to massive on-shell particles (in Lorentzian signature). However, the quantum fluctuations on AdS$_2$ are encoded in an unrelated continuous branch with $\lambda_A = p^2 + \frac{1}{4}$ with $p \in R$. These are strictly off-shell modes which correspond to conformal weights $h = \frac{1}{2} + ip$ and "mass" $m^2 \le -\frac{1}{4}$ below the Breitenlohner-Freedman bound (for $p \neq 0$).

The expression (2.3.3) for a heat kernel as a "sum" over eigenvalues in the case of AdS$_2$

becomes an integral. For a scalar field [48, 49],

$$K_A^s(s) = \frac{1}{2\pi a^2}\int_0^\infty e^{-(p^2+\frac{1}{4})s} p \tanh \pi p \; dp = \frac{1}{4\pi a^2 s}\left(1 - \frac{1}{3}s + \frac{1}{15}s^2 + \ldots\right). \tag{2.3.20}$$

The Plancherel measure $\mu(p) = p \tanh \pi p$ arises as the eigenvalue space dual of the real space measure $\sqrt{-g} = \sinh \rho$ on AdS$_2$. This agrees with (2.3.18) as it should. The leading term for small s agrees with the flat space result (2.3.7) both in magnitude and in sign even though this is not manifest in the prefactor of (2.3.20) (related to AdS$_2$ volume (2.3.11)).

2.3.5 AdS$_2 \times S^2$.

For minimally coupled fields the kinetic operator on the product space is a sum of kinetic operators on the factors. In this situation the eigenfunctions on the full space are products of eigenfunctions on each factor space and so the eigenvalues on the product space are equal to the sum of eigenvalues on each factor. The full Green's function (2.3.7) therefore becomes a product of contributions from each factor and this result descends to the heat kernel.

The heat kernel of a minimally coupled boson on AdS$_2 \times S^2$ is thus

$$K_4^s(s) = K_S^s(s)K_A^s(s) = \frac{1}{16\pi^2 a^4 s^2}\left(1 + \frac{1}{45}s^2 + \ldots\right), \tag{2.3.21}$$

where the individual factors were copied from (2.3.14) and (2.3.18). Similarly the heat kernel of a minimally coupled Dirac fermion on AdS$_2 \times S^2$ becomes

$$K_4^f(s) = 4K_S^f(s)K_A^f(s) = -\frac{1}{4\pi^2 a^4 s^2}\left(1 - \frac{11}{180}s^2 + \ldots\right), \tag{2.3.22}$$

where the individual factors were taken from (2.3.17) and (2.3.19). The overall factor of 4 counts the number of fermionic degrees of freedom. In our conventions the overall minus sign came from AdS$_2$ (2.3.19) but not from the S^2 (2.3.17). This correctly accounts for statistics on AdS$_2 \times S^2$.

An important benchmark in the following section will be the heat kernel of a full hypermultiplet with *no* couplings taken into account. This is the heat kernel of four scalars and one Dirac fermion (with four fermionic degrees of freedom), all minimally coupled:

$$K_4^{\rm min}(s) = 4K_4^s(s) + K_4^f(s) = \frac{1}{4\pi^2 a^4 s^2} \cdot \frac{1}{12} s^2. \tag{2.3.23}$$

In this case the divergences cancel to two leading orders, both of order s^{-2} and of order s^{-1}. Thus quantum corrections do not induce a cosmological constant, nor a renormalization of the Newton constant. The leading nontrivial term in the heat kernel is constant, corresponding to a marginal operator in the action. This order is responsible for the logarithmic corrections to black hole entropy in which we are interested.

2.4 Quantum Corrections to $\mathcal{N} = 2$ Multiplets.

The supergravity fields propagating in the AdS$_2 \times S^2$ background interact with each other, in addition to the interaction with the background. This modifies their heat kernels from the canonical values such as those given in (2.3.21) and (2.3.22). In this section we combine the quantum numbers computed in section 2.2 with the elementary methods from section 3 to determine the quantum corrections with interactions taken into account.

2.4.1 The Hypermultiplet.

The classical spectrum in Table 2.3 gives the eigenvalues of scalars in the hypermultiplet as four towers with $(h, j) = (k+1, k)$ with $k = 0, 1, \ldots$. From the AdS$_2$ point of view these are on-shell particles with mass level $m^2 = h(h-1) = k(k+1)$ and degeneracy $2k+1$ from an $SU(2)$ quantum number.

The AdS$_2$ heat kernels presented in (2.3.18) and (2.3.19) are for massless particles ($h = 1$) with unit degeneracy but AdS$_2$ mass and degeneracy due to $SU(2)$ spin j present a minimal modification

$$K_A(h, j; s) = K_A(h = 1, j = 0; s)\ e^{-h(h-1)s}(2j + 1). \tag{2.4.1}$$

The heat kernel for the four towers with $(h, j) = (k + 1, k)$ therefore becomes

$$\begin{aligned} K_4^{H,b}(s) &= 4 \cdot K_A^s(s) \cdot \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} e^{-sk(k+1)}(2k + 1) \\ &= 4K_A^s(s) \cdot K_S^s(s) \\ &= \frac{1}{4\pi^2 a^4 s^2} \left(1 + \frac{1}{45}s^2 + \dots\right). \end{aligned} \tag{2.4.2}$$

The sum over particles in AdS$_2$ reduced to (2.3.12) which was evaluated already in (2.3.14) where it was interpreted as the heat kernel in S^2.

Although in this section we take an AdS$_2$ perspective, the final result (2.4.2) agrees with (2.3.21) for four massless scalars in AdS$_2 \times S^2$. This is expected because the scalar fields in hypermultiplets interact only minimally with the background. The absence of scalar couplings in turn is well known from the fact that the attractor mechanism in the AdS$_2 \times S^2$ background applies to scalars in vector multiplets but not to those in hypermultiplets [38].

The fermions in a hypermultiplet are more complicated because couplings to the graviphoton background introduce effective masses. For a fermion the dictionary between conformal weight and spacetime mass is $m^2 = h(h-1) + \frac{1}{4} = (h - \frac{1}{2})^2$ with the shift of $\frac{1}{4}$ the $SL(2)$ analogue of the $SU(2)$ shift in (2.3.15). The AdS$_2$ heat kernel for the two towers of hypermultiplet fermions in Table 2.3 then gives

$$
\begin{aligned}
K_4^{H,f}(s) &= K_A^f(s)\cdot\frac{1}{4\pi a^2}\sum_{k=0}^{\infty}\left(e^{-sk^2}(2k+2)+e^{-s(k+1)^2}2k\right) \\
&= K_A^f(s)\cdot\frac{1}{2\pi a^2}\left(\sum_{k=0}^{\infty}e^{-s(k+1)^2}(2k+2)+1\right) \\
&= K_A^f(s)\cdot\frac{1}{2\pi a^2 s}\left(1-\frac{1}{6}s-\frac{1}{60}s^2+\ldots+s\right) \\
&= -\frac{1}{4\pi^2 a^4 s^2}\left(1-\frac{11}{180}s^2+\ldots+s(1+\frac{1}{6}s)+\ldots\right).
\end{aligned}
\tag{2.4.3}
$$

The second line was obtained by a simple shift of indices and the third line used the summation formula (2.3.16)-(2.3.17). In the final line we used the AdS_2 heat kernel (2.3.19). We refrained from collecting all terms in the final result in order to stress that the first set of terms are the "kinematical" (not due to interactions) contributions present even for non-interacting fermions (as in (2.3.22)) while the second set of terms can be attributed to the interactions between the fermions.

The heat kernel for the full hypermultiplet is obtained by the addition of contributions from bosons (2.4.1) and fermions (2.4.3):

$$
K_4^H(s) = \frac{1}{4\pi^2 a^4 s^2}\left(\frac{1}{12}s^2-(s+\frac{1}{6}s^2)+\ldots\right) = \frac{1}{4\pi^2 a^4}\left(-\frac{1}{s}-\frac{1}{12}+\ldots\right). \tag{2.4.4}
$$

In the first form we recognize the first term as the canonical (non-interacting) result (2.3.23) and so the second one can be attributed to the interactions. In the context of logarithmic corrections to the area law we focus on the *constant* term in (2.4.4). It is amusing that the role of the interactions for this term is precisely to *change the sign* of the quantum corrections. Such an effect could conceivably go unnoticed in some circumstances. Our result agrees (of course) with that reported by A. Sen [7].

2.4.2 The Vectormultiplet.

For the $\mathcal{N} = 2$ vector multiplet it is well-known that the bosonic degrees of freedom are sensitive to the interactions: the attractor mechanism determines the horizon values of the scalar fields in terms of the charges of the vector fields. Thus the excitations of the scalar fields in vector multiplets acquire a mass in AdS$_2$. This should be contrasted with the scalar fields in hypermultiplets which remain freely specifiable in the near horizon region as they are moduli.

The effect of interactions on the heat kernel for the bosons in the vector multiplet are captured again by the spectrum in Table 2.3 which we take into account using (2.4.1). This gives

$$\begin{aligned} K_4^{V,b}(s) &= 2 \cdot K_A^s(s) \cdot \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} \left(e^{-sk(k+1)}(2k+3) + e^{-s(k+1)(k+2)}(2k+1) \right) \\ &= 2 \cdot K_A^s(s) \cdot \frac{1}{2\pi a^2} \left(\sum_{k=0}^{\infty} e^{-sk(k+1)}(2k+1) + \frac{1}{2} \right) \\ &= 2 \cdot K_A^s(s) \cdot \frac{1}{2\pi a^2 s} \left(1 + \frac{1}{3}s + \frac{1}{15}s^2 + \ldots + \frac{1}{2}s \right) \\ &= \frac{1}{4\pi^2 a^4 s^2} \left(1 + \frac{1}{45}s^2 + \ldots + \frac{1}{2}s(1 - \frac{1}{3}s) + \ldots \right). \end{aligned} \tag{2.4.5}$$

The second line was obtained by a simple shift of summation indices and the third line used the evaluation of (2.3.12) given in (2.3.14). The heat kernel for a scalar in AdS$_2$ was given in (2.3.18).

According to Table 2.3 the four *fermionic* degrees of freedom are, in contrast to the bosons, minimally coupled. The contribution of the fermions to the heat kernel is therefore captured by the AdS$_2 \times S^2$ result (2.3.22)

$$K_4^{V,f}(s) = -\frac{1}{4\pi^2 a^4 s^2} \left(1 - \frac{11}{180}s^2 + \ldots \right). \tag{2.4.6}$$

Adding 2.4.5 and (2.4.6) we find the result for the $\mathcal{N}=2$ vector multiplet

$$K_4^V(s) = \frac{1}{4\pi^2 a^4 s^2}\left(\frac{1}{12}s^2 + \frac{1}{2}s(1-\frac{1}{3}s)\right) = \frac{1}{4\pi^2 a^4}\left(\frac{1}{2s} - \frac{1}{12}\right). \tag{2.4.7}$$

Again the "$\frac{1}{12}s^2$" is the benchmark contribution that one gets from four fermions and four bosons in the $\mathrm{AdS}_2 \times S^2$ background before interactions are taken into account. The "$\frac{1}{2}s(1-\frac{1}{3}s)$" can thus be attributed to the couplings between the bosons in the vector multiplet, the same interactions that give rise to the attractor mechanism for $\mathcal{N}=2$ black holes. The effect of interactions on the constant term in the heat kernel is to flip its sign.

2.4.3 The Gravitino Multiplet.

Combining the spectrum of the fermions in Table 2.3 with the rule (2.4.1) we find the heat kernel

$$\begin{aligned} K_4^{3/2,f}(s) &= 2\cdot K_A^f(s)\cdot\frac{1}{4\pi a^2}\sum_{k=0}^{\infty}\left(e^{-s(k+1)^2}(2k+4) + e^{-s(k+2)^2}(2k+2)\right) \\ &= 2\cdot K_A^f(s)\cdot\frac{1}{2\pi a^2}\sum_{k=0}^{\infty} e^{-s(k+1)^2}(2k+2) \\ &= 2\cdot K_A^f(s)\cdot\frac{1}{2\pi a^2 s}\left(1 - \frac{1}{6}s - \frac{1}{60}s^2 + \ldots\right) \\ &= -\frac{1}{4\pi^2 a^4 s^2}\left(1 - \frac{11}{180}s^2 + \ldots\right). \end{aligned} \tag{2.4.8}$$

The summation is the same as for the minimal fermion (2.3.17). There are contributions from interactions in intermediate steps but they ultimately cancel each other.

The quantum numbers of the bosons in Table 2.3 are shifted relative to free bosons. The effect of this shift is to remove the leading term in the sum over modes on the sphere, which is easily taken into account:

$$
\begin{aligned}
K_4^{3/2,b}(s) &= K_A^s(s) \cdot \frac{1}{4\pi a^2} \sum_{k=1}^{\infty} e^{-sk(k+1)}(2k+1) \\
&= K_A^s(s) \cdot \frac{1}{4\pi a^2} \left(\sum_{k=0}^{\infty} e^{-sk(k+1)}(2k+1) - 1 \right) \\
&= K_A^s(s) \cdot \frac{1}{4\pi a^2 s} \left(1 + \frac{1}{3}s + \frac{1}{15}s^2 + \ldots - s + \ldots \right) \\
&= \frac{1}{4\pi^2 a^4 s^2} \left(1 + \frac{1}{45}s^2 + \ldots - s(1 - \frac{1}{3})s + \ldots \right).
\end{aligned} \tag{2.4.9}
$$

The "$-s(1-\frac{1}{3}s)$" can be attributed to the couplings between components of a vector field relative to those of scalar degrees of freedom.

Adding (2.4.8) and (2.4.9) we find the heat kernel for a complete $\mathcal{N}=2$ multiplet for a massive gravitino:

$$
K_4^{3/2} = \frac{1}{4\pi^2 a^4 s^2} \left(\frac{1}{12}s^2 - s(1 - \frac{1}{3}s) \right) = \frac{1}{4\pi^2 a^4} \left(-\frac{1}{s} + \frac{5}{12} \right). \tag{2.4.10}
$$

2.4.4 The Graviton Multiplet.

The quantum numbers $(h,j) = (k+\frac{5}{2}, k+\frac{3}{2})$ from Table 2.3 give the contribution from the four fermion degrees of freedom as

$$
\begin{aligned}
K_4^{\text{grav},f}(s) &= 4 \cdot K_A^f(s) \cdot \frac{1}{4\pi a^2} \sum_{k=0}^{\infty} e^{-s(k+2)^2}(2k+4) \\
&= 4 \cdot K_A^f(s) \cdot \frac{1}{4\pi a^2} \left(\sum_{k=0}^{\infty} e^{-s(k+1)^2}(2k+2) - 2e^{-s} \right) \\
&= 4 \cdot K_A^f(s) \cdot \frac{1}{4\pi a^2 s} \left(1 - \frac{1}{6}s - \frac{1}{60}s^2 - 2se^{-s} \right) \\
&= -\frac{1}{4\pi^2 a^4 s^2} \left(1 - \frac{11}{180}s^2 + \ldots - 2s(1 - \frac{5}{6}s) + \ldots \right).
\end{aligned} \tag{2.4.11}
$$

As in previous cases the "$-2s(1-\frac{5}{6}s)$" can be attributed to the couplings between components of a gravitino field relative to those of a free fermion.

Finally, inserting the quantum numbers in Table 2.3 for bosons in the supergravity multiplet into (2.4.1) we find

$$\begin{aligned}
K_4^{\text{grav},b}(s) &= K_A^s(s)\cdot\frac{1}{4\pi a^2}\sum_{k=0}^{\infty}\left(e^{-s(k+2)(k+1)}(2k+5)+e^{-s(k+3)(k+2)}(2k+3)\right) \\
&= K_A^s(s)\cdot\frac{1}{4\pi a^2}\left(2\sum_{k=0}^{\infty}e^{-s(k+2)(k+1)}(2k+3)-e^{-2s}\right) \\
&= K_A^s(s)\cdot\frac{1}{2\pi a^2}\left(\sum_{k=0}^{\infty}e^{-s(k+1)}(2k+1)-1-\frac{1}{2}e^{-2s}\right) \\
&= K_A^s(s)\cdot\frac{1}{2\pi a^2 s}\left(1+\frac{1}{3}s+\frac{1}{15}s^2+\ldots-\frac{3}{2}s+s^2+\ldots\right) \\
&= \frac{1}{4\pi^2a^4s^2}\left(1+\frac{1}{45}s^2+\ldots-\frac{3}{2}s+\frac{3}{2}s^2+\ldots\right)
\end{aligned} \tag{2.4.12}$$

Adding (2.4.11) and (2.4.12) the complete result for the heat kernel of the $\mathcal{N}=2$ gravity multiplet becomes

$$\begin{aligned}
K_4^{\text{grav}}(s) &= \frac{1}{4\pi^2a^4s^2}\left(\frac{1}{12}s^2-\frac{3}{2}s(1-\frac{1}{3}s)+2s(1+\frac{1}{6}s)\right) \\
&= \frac{1}{4\pi^2a^4s^2}\left(\frac{1}{12}s^2+(\frac{1}{2}s-\frac{1}{6}s^2)\right)=\frac{1}{4\pi^2a^4}\left(\frac{1}{2s}-\frac{1}{12}\right).
\end{aligned} \tag{2.4.13}$$

2.4.5 Summary.

In summary, we have computed the contributions to heat kernels of the $\mathcal{N}\geq 2$ theory from physical non-zero modes. The result is

$$K_{\text{nzm}}=\frac{1}{4\pi^2a^4}\left((\frac{1}{2s}-\frac{1}{12})+(\mathcal{N}-2)(-\frac{1}{s}+\frac{5}{12})+n_V(\frac{1}{2s}-\frac{1}{12})+n_H(-\frac{1}{s}-\frac{1}{12})\right) \tag{2.4.14}$$

The notation "nzm" is a reminder that at this point interactions have been taken into account but the focus was on non-zero modes. Corrections due to zero-modes will be considered in the next two sections.

2.5 Boundary States.

As we have stressed, the spectrum in Table 2.3 enumerates physical modes only. In particular, gauge conditions have been imposed that fix the gauge symmetry. These conditions remove all unphysical states except that, for each continuous gauge symmetry, a single physical boundary mode remains. We discussed the mechanism for this in some detail in section 2.2.

The physical boundary states contribute to the quantum corrections to black holes just like all other physical states. In this section we compute their contributions to the heat kernel.

2.5.1 Localization on the Boundary.

A 4D gauge symmetry reduces to a tower of 2D gauge symmetries in AdS_2. Each entry in the tower gives rise to a single mode on the boundary of AdS_2. These towers were presented as a list in Table 2.5.

The contribution from each entire tower will amount to a field on the S^2 that is localized on AdS_2. We need to find the spectrum of these fields on S^2. This can be accomplished by considering the structure of gauge transformations. This introduces gauge dependence at intermediate stages but our final result is gauge invariant.

In the following we consider the boundary modes for each $\mathcal{N} = 2$ multiplet in turn.

The Vector Multiplet

Modes that are pure gauge from the 4D point of view take the form of a gauge variation

$$\delta \mathcal{A}_I = \nabla_I \Lambda, \tag{2.5.1}$$

where Λ is the $U(1)$ gauge parameter. Among these modes those that preserve the Lorentz gauge condition

$$\nabla_I \mathcal{A}^I = 0, \tag{2.5.2}$$

are

$$-\nabla^I \delta \mathcal{A}_I = -\nabla^2 \Lambda = 0, \tag{2.5.3}$$

just like a massless scalar from the 4D point of view. From the 2D point of view there is a tower of fields in AdS_2 with masses given by

$$m^2 = k(k+1), \tag{2.5.4}$$

with $k = 0, 1, \ldots$. Each field is pure gauge so its contribution to physical processes cancels with the corresponding unphysical mode. This cancellation is imperfect and leaves the AdS_2 zero-mode $\nabla_A^2 \Lambda = 0$. We interpret this mode as a physical mode on the AdS_2 boundary. As we recombine all 2D fields $k = 0, 1, \ldots$ we find a physical scalar field on S^2. The quantum corrections due to these physical states are computed by the scalar determinant on the sphere (2.3.14) and gives

$$K^V_{\mathrm{bndy}} = \frac{1}{2\pi a^2} \cdot \frac{1}{4\pi a^2 s}\left(1 + \frac{1}{3}s\right) = \frac{1}{4\pi^2 a^4}\left(\frac{1}{2s} + \frac{1}{6}\right). \tag{2.5.5}$$

The overall factor is the volume of AdS_2. The sign is the one appropriate for a physical boson. The simple pole in the parameter s is mild for a 4D field but entirely standard for a 2D field.

The Gravitino Multiplet

The gauge symmetry of a gravitino is the SUSY variation

$$\delta \Psi_I = \nabla_I \epsilon. \tag{2.5.6}$$

The SUSY transformation that preserves the Lorentz gauge condition on the gravitino

$$\gamma^I \delta\Psi_I = 0, \tag{2.5.7}$$

satisfies the Weyl's equation

$$\gamma^I \nabla_I \epsilon = 0. \tag{2.5.8}$$

The physical boundary state that remains is therefore a Weyl fermion on S^2. Our previous computation of the heat kernel for a single fermionic degree of freedom (2.3.17) then gives

$$K^{(3/2)}_{\rm bndy} = -\frac{1}{2\pi a^2}\cdot\frac{1}{4\pi a^2 s}\cdot 2\left(1-\frac{1}{6}s\right) = \frac{1}{4\pi^2 a^4}\left(-\frac{1}{s}+\frac{1}{6}\right). \tag{2.5.9}$$

An explicit factor of two counted the two components of the Weyl fermion. The overall minus sign is appropriate for a physical fermion.

The gravitino supermultiplet also includes two vector multiplets. Each realizes a standard $U(1)$ gauge symmetry and gives rise to a boundary mode that contributes (2.5.5) to the heat kernel. The total boundary contribution to the gravitino supermultiplet therefore becomes

$$K^{(3/2)}_{\rm bndy} = \frac{1}{4\pi^2 a^4}\cdot\frac{1}{2}. \tag{2.5.10}$$

There is no pole in s because the boundary states in this multiplet fill out a super multiplet with equal number of fermions and bosons on the boundary.

The Graviton Multiplet

The gauge symmetries of gravity are the 4D diffeomorphisms ξ^I acting on gravitational perturbations as

$$\delta h_{IJ} = \nabla_I \xi_J + \nabla_J \xi_I. \tag{2.5.11}$$

The coordinate transformations that preserve the Lorentz (harmonic) gauge condition

$$\nabla^I h_{\{IJ\}} = \nabla^I (h_{IJ} + h_{JI} - g_{IJ} h^K_K) = 0, \tag{2.5.12}$$

satisfy

$$(g_{IJ}\nabla^2 + R_{IJ})\xi^J = 0. \tag{2.5.13}$$

The Ricci curvature is $R_{\mu\nu} = -g_{\mu\nu}$ on AdS_2 and $R_{\alpha\beta} = +g_{\alpha\beta}$ on the S^2.

The diffeomorphisms ξ^α generate vector modes on S^2 so the angular momentum of the corresponding boundary modes is restricted to $k = 1, 2, \ldots$. The Ricci curvature gives a contribution $\Delta m^2 = -1$ to the effective mass and the dualization to a scalar field gives an identical contribution. The spectrum of the two scalar boundary modes with $\nabla^2_A \xi^\alpha = 0$ therefore becomes

$$m^2_S = k(k+1) - 2, \tag{2.5.14}$$

with $k = 1, \ldots$. The mass-shift $\Delta m^2 = -2$ is such that the leading AdS_2 boundary mode is massless also on the S^2.

The pure gauge modes generated by ξ^μ decompose into an AdS_2 scalar $\nabla_\mu \xi^\mu$, an AdS_2 vector $\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$, and an AdS_2 traceless tensor. The AdS_2 scalar mixes with the pure gauge mode from the graviphoton such that all three of these are independent even though ξ^μ has only two components. The AdS_2 zero-modes of the scalar and the traceless tensor both give rise to physical boundary states with the spectrum (2.5.4) of a standard scalar field on S^2. However, the AdS_2 vector has zero modes that generates a tower of boundary modes with the shifted effective mass

$$m^2_S = k(k+1) + 2. \tag{2.5.15}$$

These three towers all have $k = 0, \ldots$. The leading terms with vanishing angular momentum $j = k = 0$ are essentially the boundary states denoted δX and $\delta\rho$ in section 2.2 except that here the gauge is different and the graviphoton is taken into account.

The sum of contributions from all five bosonic boundary modes yields

$$K^{\rm grav,b}_{\rm bndy} = 2\cdot\frac{1}{4\pi^2a^4}\cdot\frac{1}{2}\left(\frac{1}{s}-\frac{2}{3}\right)e^{2s} + 2\cdot\frac{1}{4\pi^2a^4}\cdot\frac{1}{2}\left(\frac{1}{s}+\frac{1}{3}\right) + \frac{1}{4\pi^2a^4}\cdot\frac{1}{2}\left(\frac{1}{s}+\frac{1}{3}\right)e^{-2s} \tag{2.5.16}$$

$$= \frac{1}{4\pi^2a^4}\cdot\frac{5}{2}(\frac{1}{s}+\frac{1}{3}).$$

Despite the various shifts of masses and angular momentum quantum numbers this is identical to the heat kernel of five free scalars on the S^2.

The $\mathcal{N}=2$ supersymmetry acts on the two gravitini in the graviton multiplet as

$$\delta\Psi^A_I = (\delta^A_B\nabla_I - \frac{1}{4}\hat{F}\epsilon_{AB}\gamma_I)\epsilon^B, \tag{2.5.17}$$

where the background graviphoton field strength $\hat{F}=\frac{1}{2}F_{JK}\gamma^{JK}=\epsilon_{\alpha\beta}\gamma^{\alpha\beta}$. This differs from a generic gravitino (2.5.6) by the dependence on the graviphoton background. It is because of this dependence that $\mathcal{N}=2$ SUSY is preserved. The field strength contributions to (2.5.17) are such that the AdS_2 ground state energy $(-\nabla_\mu\nabla^\mu)$ of the two fermions adds to $\Delta m^2=-1$. This gives a shift in the effective fermion mass on S^2 such that

$$m^2 = (k+1)^2-1. \tag{2.5.18}$$

The first term is the standard effective mass (2.3.15) on S^2, sometimes written as $j(j+1)+\frac{1}{4}=(j+\frac{1}{2})^2$ with j taking half integer values. The tower $j=\frac{1}{2},\frac{3}{2},\ldots$ is parametrized here by $k=0,1,\ldots$. The mass-shift $\Delta m^2=-1$ is such that the leading AdS_2 boundary mode is massless also on the S^2.

The heat kernel for a single standard fermion on S^2 was given in (2.3.17). Four fermionic boundary degrees of freedom with effective mass (2.5.18) then give

$$K_{bndy}^{\text{grav,f}} = -4 \cdot \frac{1}{4\pi^2 a^4} \cdot \frac{1}{2} \left(\frac{1}{s} - \frac{1}{6} \right) e^s = - \frac{1}{4\pi^2 a^4} \left(\frac{2}{s} + \frac{5}{3} \right) . \tag{2.5.19}$$

Adding the bosonic contribution (2.5.16) we have

$$K_{\text{bndy}}^{\text{grav}} = \frac{1}{4\pi^2 a^4} \left(\frac{1}{2s} - \frac{5}{6} \right) , \tag{2.5.20}$$

for the complete contribution of boundary states to the heat kernel of the $\mathcal{N} = 2$ supergravity multiplet.

Summary

In summary, the contribution to the heat kernel of the $\mathcal{N} = 2$ theory from boundary modes is

$$K_{\text{bndy}} = \frac{1}{4\pi^2 a^4} \left((\frac{1}{2s} - \frac{5}{6}) + (\mathcal{N} - 2) \cdot \frac{1}{2} + n_V (\frac{1}{2s} + \frac{1}{6}) \right) . \tag{2.5.21}$$

We can add this to the bulk contribution (2.4.14) and find

$$K_{\text{phys}} = \frac{1}{4\pi^2 a^4} \left((\frac{1}{s} - \frac{11}{12}) + (\mathcal{N} - 2) \cdot (-\frac{1}{s} + \frac{11}{12}) + n_V (\frac{1}{s} + \frac{1}{12}) + n_H (-\frac{1}{s} - \frac{1}{12}) \right) . \tag{2.5.22}$$

As a nontrivial consistency check on (2.5.22) note that the coefficient of $1/s$ is the same for each type of $\mathcal{N} = 2$ multiplet, except that the sign alternates as the spin of the SUSY multiplet changes. This is precisely the property needed to ensure that these terms cancel in any theory with $\mathcal{N} = 4$ SUSY, as they should.

Another interesting special case is the pure $\mathcal{N} = 3$ theory which is scale invariant at this level [50]. The $\mathcal{N} = 3$ matter multiplets have $n_H = n_V = 1$ so an arbitrary number of those can be added without violating scale invariance.

2.6 Zero Modes.

The boundary states are zero modes from the AdS_2 point of view but they are generally non-trivial on the S^2. The true 4D zero-modes are the boundary states that are also zero modes on the S^2. These zero mode contributions require special considerations.

The zero mode content of each multiplet can be read off from the spectrum of boundary states. The vector multiplet has one bosonic zero-mode from gauge symmetry: the $k = 0$ entry in (2.5.4). The gravitino multiplet has two bosonic zero-modes, both from gauge symmetry. The gravity multiplet also has two bosonic zero-modes: the $k = 1$ entry in (2.5.14). These both have angular momentum $j = 1$. Finally, the gravity multiplet also has four fermionic zero-modes, the $k = 0$ entry in (2.5.18).

For the zero-modes we cannot use the Euclidean path integral (2.3.1) (repeated here for easy reference)

$$e^{-W} = \int \mathcal{D}\phi \; e^{-\phi\Lambda\phi} = \frac{1}{\sqrt{\det\Lambda}}, \tag{2.6.1}$$

since they correspond to vanishing eigenvalues of the matrix Λ. However, each zero-mode is just a field in zero dimensions so in this sector the path integral reduces to an ordinary integral. The scale dependence of N_0 zero-modes with scaling dimension Δ is

$$e^{-W} = \int \mathcal{D}\phi_0 = \mathrm{Vol}[\phi_0] \sim \epsilon^{-N_0\Delta}. \tag{2.6.2}$$

In contexts where (2.6.1) applies it is understood that the dependence on physical parameters is encoded in *ratios* of integrals of this general form. The scale dependence due to a single zero-mode is similarly computed from ratios of integrals (2.6.2) computed at different scales.

The naïve inclusion of N_0 zero-modes in the heat kernel (2.3.3):

$$D(s) = \sum_i e^{-s\lambda_i} = \sum_{\lambda_i \neq 0} e^{-s\lambda_i} + N_0, \tag{2.6.3}$$

corresponds to a term $W = N_0 \ln \epsilon$ in the effective action according to (2.3.2). Thus the correct zero-mode contribution $W \sim \Delta N_0 \ln \epsilon$ from (2.6.2) is larger than the naïve result by a factor of the scaling dimension Δ. After generalization to multiple fields with either bosonic or fermionic statistics we have

$$K_{\rm zm} = \frac{1}{8\pi^2 a^4} \sum_{i \in B} N_{0,i}(\Delta_i - 1) - \frac{1}{8\pi^2 a^4} \sum_{i \in F} N_{0,i}(2\Delta_i - 1), \tag{2.6.4}$$

for the correction to the heat kernel $K(s)$ due to zero-modes. Each fermionic zero-mode counts with double weight because of the leading spin degeneracy in (2.3.16).

Vector fields have dimension $\Delta_1 = 1$ so they were already taken correctly into account in the naïve heat kernel. Since the zero-modes in the vector and (massive) gravitino multiplet are all due to vector fields these multiplets do not get corrected. It is only the supergravity-multiplet that is corrected due to zero-modes.

Disregarding the vector, the bosonic zero-modes in the gravity multiplet are just $k = 1$ in (2.5.14). Each of these two states have angular momentum $j = 1$ so there are $N_0^b = 2 \cdot (2j+1) = 6$ bosonic zero-modes in the path integral. These fields have scaling dimension $\Delta_2 = 2$. Similarly, (2.5.18) gives $N_0^f = 4$ fermionic zero-modes in the path integral. They have scaling dimension $\Delta_{3/2} = \frac{3}{2}$. The zero-mode contribution to a general $\mathcal{N} = 2$ theory simply becomes

$$K_{zm} = \frac{1}{8\pi^2 a^4} \cdot (6 \cdot (2-1) - 4 \cdot (3-1)) = \frac{1}{4\pi^2 a^4} \cdot (3 - 4) = \frac{1}{4\pi^2 a^4} (-1) \,. \tag{2.6.5}$$

2.6.1 Summary.

The sum of contributions to the heat kernel from non-zero modes (2.4.14), boundary modes (2.5.21), and zero modes (2.6.5) is

$$K_{\rm tot} = \frac{1}{4\pi^2 a^4} \left((\frac{1}{s} - \frac{23}{12}) + (\mathcal{N} - 2)(-\frac{1}{s} + \frac{11}{12}) + n_V(\frac{1}{s} + \frac{1}{12}) + n_H(-\frac{1}{s} - \frac{1}{12}) \right) . \tag{2.6.6}$$

This is the main result of our computations.

2.6.2 Logarithmic Corrections to Black Hole Entropy.

Recall the result (1.4.8) derived in the Introduction,

$$\delta S = \frac{1}{2} D_0 \ln A_H = 4\pi^2 a^4 K_0 \ln A_H, \tag{2.6.7}$$

where we extended (1.4.8) by writing it in terms of K_0, the constant term in the heat kernel density we evaluated (2.3.10).

The relation (1.4.7) between the trace anomaly and the logarithmic correction to the entropy is interesting and quite general. It is corrected only by the treatment of zero-modes. Our formula (2.6.4) for the zero-mode contribution to the heat kernel was constructed precisely so that the entropy formula (2.6.7) would be maintained for this contribution as well.

The constant term in the heat kernel expansion K_0 is easily read off from the total heat kernel (2.6.6). The relation (2.6.7) then gives the logarithmic correction so the entropy

$$\delta S = \frac{1}{12} \left(23 - 11(\mathcal{N} - 2) - n_V + n_H\right) \ln A_H. \tag{2.6.8}$$

This is the final result advertised in the motivation of this chapter as (2.1.1).

Chapter 3

Quantum Corrections to Supergravity on $\mathbf{AdS}_2 \times S^2$.

3.1 Motivation and Summary.

In this chapter we develop the on-shell method in the context of supergravity on $\mathrm{AdS}_2 \times S^2$. We focus on supergravity since the couplings organize physical states efficiently according to quantum numbers such as the conformal dimension. However, unphysical modes are often unwieldy since auxiliary fields and ghosts involved in the off-shell theory also couple non-minimally. These complications seem excessive for determinants of quadratic fluctuations so it may be advantageous to work in the small Hilbert space that focusses entirely on the physical modes. The resulting on-shell strategy is the same as the one presented in chapter 2; it is simpler but it must address global aspects that remain after gauge fixing of local symmetries. Specifically, there will be boundary modes in AdS.

Our results for quantum corrections (reported in chapter 2) are not new as they were previously reported in [7, 51] but we present explicit details that develop concepts and resolve issues in the literature.

An important motivation for developing quantum corrections in AdS_2 and specifically the role of boundary modes is that they play a central role also in other settings. Some recent discussions are:

- Boundary states are standard in AdS_3 partition functions [52, 53] and they presumably play a similar role in higher dimensional AdS spaces [45, 54].

- Quantum corrections in AdS_2 geometry appear for Wilson loops in AdS_5 [55]. Subtleties remain in this context [56, 57, 58, 59].

- $\mathrm{AdS}_2 \times S^2$ is conformally equivalent to Minkowski space so these modes may also be related to the physical boundary modes that play a role in scattering amplitudes [60, 61, 62] and to those that appear in the context of holography in Minkowski space [63].

- Our set-up is an explicit realization of $\mathrm{AdS}_2/\mathrm{CFT}_1$ holography. Many open questions remain in this context [64, 65, 6, 66].

In our computation we organize the field content on AdS_2 into towers of partial waves due to the reduction on the S^2. We analyze this 2D spectrum with gauge fixing terms included in the equations of motion but not imposed as constraints. In our presentation we explicitly identify some towers as unphysical (they violate the gauge condition) and others as pure gauge (the action of diff $\times$ gauge on the background), with the remaining fields constituting the physical bulk spectrum. Equivalently, we match both the unphysical and gauge towers with ghosts and determine the "small" departure from perfect cancellation. In either construction, the bulk spectrum is thus augmented by physical modes that are formally pure gauge albeit with non-normalizable gauge function. These are the boundary modes.

In our construction each local symmetry in 4D gives rise to a tower of boundary modes in AdS_2. We interpret such a tower as a single field on S^2. There is exactly one such boundary field on S^2 for each symmetry. It may appear that we have lost a dimension:

the boundary of $\mathrm{AdS}_2 \times S^2$ has one dimension, in addition to the S^2 dimensions. Indeed, at an intermediate stage there is one mode for each boundary momentum on AdS_2 but we reinterpret the resulting sum as the volume of AdS_2. It is in this sense that we find exactly one mode on S^2 for each 4D symmetry.

We express quantum corrections to the geometry as heat kernel sums over the spectrum. In the "large" Hilbert space these are traces over the full spectrum with unphysical modes cancelled by ghosts with "wrong" statistics. These sums can be reorganized as traces over the physical spectrum in the "small" Hilbert space where boundary states are included and all modes appear with a positive sign. The boundary fields include components that are zero-modes on $\mathrm{AdS}_2 \times S^2$ and such modes require special treatment [36]. The complete partition function thus comprises modes in 4D (bulk), 2D (boundary), and 0D (zero-modes).

The main idea of our computation can be illustrated clearly by considering a standard (minimally coupled) vector field A_I in $\mathrm{AdS}_2 \times S^2$. The partial wave expansion on S^2 gives four towers of 2D fields: two physical (spatially transverse), one unphysical (violating the gauge condition), and one longitudinal (pure gauge). In the old-fashioned Gupta-Bleuler formalism the unphysical and the longitudinal towers "cancel" (due to a Ward identity) and in BRST formalism both towers are cancelled by ghosts. Either way, for each partial wave the mode that is formally pure gauge can be arranged to require a non-normalizable gauge function on AdS_2 and this gives rise to a single physical longitudinal mode that survives as an AdS_2 boundary mode.

Standard AdS/CFT lore sometimes suggests that physical boundary states are at the "end" of the physical towers but we find this rule to be misleading. Indeed, since boundary states arise formally as states that are pure gauge it may be more appropriate to interpret them as the "end" of the *unphysical* towers. However, ultimately it turns out that couplings between boundary modes render such shortcuts unreliable. One aspect of this is that modes generated by symmetries generally do not continue smoothly from general partial wave component l to the "small" values $l = 0, 1$.

As we have indicated, boundary states can be interpreted as modes that are formally "pure gauge". An alternative perspective ties them to harmonic modes on AdS_2 which play a special role when fields of higher spin are dualized to scalars. We find that the dual of gravity includes an interesting harmonic scalar satisfying a higher order equation of motion with solutions for both $m^2 = 0$ and $m^2 = 2$. It is the latter that gives rise to physical boundary modes for gravity. This twist on the harmonic condition may be significant in other settings.

The detailed considerations are instructive but they are unfortunately somewhat cumbersome even in the simple example of $\mathrm{AdS}_2 \times S^2$. That is a byproduct of analyzing $\mathcal{N} = 2$ supergravity off-shell without introducing a full-fledged off-shell formalism. Several asymmetries give rise to a non-Hermitan action for off-shell fields which manifests itself by awkward degenerate eigenvectors. For example, diffeomorphisms act on gauge fields but gauge transformations do not act on the metric. The pay-off for addressing these practical complications is considerable conceptual clarity.

This chapter is organized as follows. In section 3.2 we present the details of a minimally coupled vector field on $\mathrm{AdS}_2 \times S^2$. We reduce from 4D to 2D, diagonalize the off-shell 2D equations in Lorentz gauge, and discuss the physical spectrum. We specify the boundary modes as pure gauge modes with non-normalizable gauge function and also as harmonic modes. In section 3.3 we compute the heat kernel of the vector field as a sum over all physical states in bulk and on the boundary. We compare with the standard off-shell computation. In section 3.4 we discuss the analogous aspects of the bosonic fields in the $\mathcal{N} = 2$ supergravity multiplet. We also address additional features: degenerate eigenvalues and modes, the harmonic condition on the scalar dual to a tensor field, residual 2D diffeomorphism invariance, and the role of (Conformal) Killing Vectors. In section 3.5, we discuss the heat kernels of the bosonic fields with special emphasis on the cancellation of off-shell modes and the contribution of physical boundary states. In section 3.6 we turn to the gravitini in the $\mathcal{N} = 2$ supergravity multiplet. We again diagonalize the equations of motion entirely without any gauge condition imposed and only then discuss supersymmetry and the constraints inherent in the Rarita-Schwinger equation. Finally, in section 3.7 we compute the heat kernel for the gravitini and assemble the full result for supergravity on $\mathrm{AdS}_2 \times S^2$.

3.2 Vector Fields in $\mathbf{AdS}_2 \times S^2$.

In this section we analyze a vector field in $\mathrm{AdS}_2 \times S^2$ from the AdS_2 point of view. We determine the full set of modes in 4D Lorentz gauge and identify the physical subset with special attention paid to the boundary modes.

3.2.1 2D Effective Theory.

Our starting point is a 4D vector field a_I on $\mathrm{AdS}_2 \times S^2$ with standard Maxwell action

$$\mathcal{L}_{\mathrm{Maxwell}} = -\frac{1}{4} F_{IJ} F^{IJ}. \tag{3.2.1}$$

In order to extract the physical content of the theory we impose Lorentz gauge

$$\nabla_I a^I = 0. \tag{3.2.2}$$

In the quantum theory this is implemented by modifying the Maxwell action (3.2.1) to

$$\mathcal{L}_{\mathrm{Lorentz}} = -\frac{1}{4} F_{IJ} F^{IJ} - \frac{1}{2\xi} \left(\nabla_I a^I\right)^2. \tag{3.2.3}$$

In the following we take Feynman gauge $\xi = 1$ and freely integrate by parts without keeping boundary terms. The action then simplifies to

$$\mathcal{L}_{\mathrm{Feynman}} = \frac{1}{2} a^J \nabla^I (\nabla_I a_J - \nabla_J a_I) + \frac{1}{2} a^J \nabla_J \nabla_I a^I = \frac{1}{2} a^I (g_{IJ} \nabla^2 - R_{IJ}) a^J. \tag{3.2.4}$$

We want to represent this theory as an effective theory in 2D by reduction on S^2. In so doing the capital latin indices $I, J, \ldots$ in the 4D total space divide into the indices $\mu, \nu, \ldots$ on AdS_2 and the indices $\alpha, \beta, \ldots$ that refer to S^2. The reduction to 2D on S^2 is realized by a partial wave expansion in spherical harmonics:

$$
\begin{aligned}
a_\mu &= b_\mu^{(lm)}(x) Y_{lm}(y), \\
a_\alpha &= b^{(lm)}(x) \epsilon_{\alpha\beta} \nabla^\beta Y_{lm}(y) + \tilde{b}^{(lm)}(x) \nabla_\alpha Y_{lm}(y).
\end{aligned}
\tag{3.2.5}
$$

A sum over angular momentum quantum numbers l, m is implied. The allowed angular momenta for the 2D gauge fields $b_\mu^{(lm)}$ are $l = 0, 1, \ldots$ but the 2D scalar fields $b^{(lm)}(x), \tilde{b}^{(lm)}(x)$ are defined only for $l = 1, 2, \ldots$ since these fields multiply spherical harmonics with derivatives acting on them.

Inserting the expansions (3.2.5) into the 4D Lagrangian (3.2.4) we find the 2D effective action on AdS_2

$$
\begin{aligned}
\mathcal{L}_{2D} =& \frac{1}{2} l(l+1) b^{(lm)} \left[\nabla_A^2 - l(l+1)\right] b^{(lm)} + \frac{1}{2} l(l+1) \tilde{b}^{(lm)} \left[\nabla_A^2 - l(l+1)\right] \tilde{b}^{(lm)} \\
&+ \frac{1}{2} b^{(lm)\mu} \left[\nabla_A^2 + 1 - l(l+1)\right] b_\mu^{(lm)}.
\end{aligned}
\tag{3.2.6}
$$

The 2D Laplacian on AdS_2 is denoted $\nabla_A^2 = \nabla^\mu \nabla_\mu$. We still imply a sum over fields $l = 0, 1, \ldots$. This rule correctly takes into account that the $l = 0$ mode is missing for $b^{(lm)}$ and $\tilde{b}^{(lm)}$ but it is not missing for $b_\mu^{(lm)}$. Curvature terms from commutation of derivatives were evaluated using the block diagonal Ricci tensor with $R_{\mu\nu} = -g_{\mu\nu}$ and $R_{\alpha\beta} = +g_{\alpha\beta}$ of $\mathrm{AdS}_2 \times S^2$ with unit radii.

The gauge variation of the Lorentz gauge condition (3.2.2) is

$$
\nabla^I \delta A_I = \nabla^I \nabla_I \Lambda = (\nabla_A^2 - l(l+1))\Lambda.
\tag{3.2.7}
$$

We will variously interpret this as the equation of motion for the pure gauge mode or as the ghost action

$$\mathcal{L}_{\rm ghost} = \tilde{c}^{(lm)}(\nabla_A^2 - l(l+1))c^{(lm)}. \tag{3.2.8}$$

The ghost spectrum $m^2 = l(l+1)$ with $l = 0, 1, \dots$ is identical to two scalar fields except for anti-commuting statistics.

3.2.2 Dualizing 2D Vectors.

The Hodge decomposition of a 1-form into an exact form, a co-exact form, and a harmonic form can be presented in components as

$$b_\mu^{(lm)} = b_{\mu\perp}^{(lm)} + b_{\mu\|}^{(lm)} + b_{\mu 0}^{(lm)}, \tag{3.2.9}$$

where $b_{\mu\perp}^{(lm)}$ is transverse

$$\nabla^\mu b_{\mu\perp}^{(lm)} = 0, \tag{3.2.10}$$

and $b_{\mu\|}^{(lm)}$ is longitudinal

$$\epsilon^{\mu\nu}\nabla_\nu b_{\mu\|}^{(lm)} = 0, \tag{3.2.11}$$

while $b_{\mu 0}^{(lm)}$ satisfies both of the above. In order to avoid over counting of modes we insist that

$$\nabla^\mu b_{\mu\|}^{(lm)} \neq 0 \quad , \ \epsilon^{\mu\nu}\nabla_\nu b_{\mu\perp}^{(lm)} \neq 0. \tag{3.2.12}$$

This is because the modes satisfying both of (3.2.10) and (3.2.11) are the harmonic modes denoted $b_{\mu 0}^{(lm)}$. The harmonic component of the vector field satisfies

$$(\nabla_A^2 + 1) b_{\mu 0}^{(lm)} = 0. \tag{3.2.13}$$

We dualize the irreducible components of the 2D vector $b_\mu^{(lm)}$ to scalars as $b_{\mu\perp}^{(lm)} =$

$\epsilon_{\mu\nu}\nabla^\nu b_\perp^{(lm)}$ and $b_{\mu\|}^{(lm)} = \nabla_\mu b_\|^{(lm)}$. This gives the expansion

$$b_\mu^{(lm)} = \epsilon_{\mu\nu}\nabla^\nu b_\perp^{(lm)} + \nabla_\mu b_\|^{(lm)} + \nabla_\mu b_0^{(lm)}, \tag{3.2.14}$$

For definiteness the harmonic mode was presented as a longitudinal mode $b_{\mu 0}^{(lm)} = \nabla_\mu b_0^{(lm)}$ with $b_0^{(lm)}$ harmonic

$$\nabla_A^2 b_0^{(lm)} = 0, \tag{3.2.15}$$

but we might as well have dualized it to a transverse mode. In our convention the scalar components $b_\|^{(lm)}$ and $b_\perp^{(lm)}$ cannot be harmonic on AdS_2.

3.2.3 The Spectrum.

The complete field content of the 4D vector field from a 2D point of view is:

- **Modes on S^2:** $\tilde{b}^{(lm)}, b^{(lm)}$ with $l = 1, 2, \ldots$
- **Modes on $\mathbf{AdS}_2$:** $b_{\mu\perp}^{(lm)} = \epsilon_{\mu\nu}\nabla^\nu b_\perp^{(lm)}$ and $b_{\mu\|}^{(lm)} = \nabla_\mu b_\|^{(lm)}$ with $l = 0, 1, \ldots$
- **Ghosts:** $\tilde{c}^{(lm)}, c^{(lm)}$ with $l = 0, 1, \ldots$
- **Harmonic modes:** $b_{\mu 0}^{(lm)} = \nabla_\mu b_0^{(lm)}$ with $l = 0, 1, \ldots$

In the fully dualized theory there is almost symmetry between AdS_2 and S^2 after appropriate interpretations. One departure from perfect symmetry is the "subtraction" of the leading $l = 0$ entry from the scalars $b^{(lm)}, \tilde{b}^{(lm)}$ which represent the vector on S^2 that only has range $l = 1, 2, \ldots$. This contrasts with the scalars $b_\|^{(lm)}, b_\perp^{(lm)}$ from the AdS_2 vector. These have the full range $l = 0, 1, \ldots$ and also "add" the harmonic fields $b_0^{(lm)}$.

Each 2D field is a scalar field on AdS_2 with mass given by $m^2 = l(l+1)$. At the level of counting, the modes on AdS_2 cancel exactly with the ghosts. The net physical spectrum is therefore essentially just the modes on S^2 forming two towers with $l = 1, 2, \ldots$. These correspond to the partial wave expansions of two physical modes with helicity $\lambda = \pm 1$ that we expect from a 4D vector field.

It is instructive to go beyond counting and construct physical modes explicitly. We first assume $l \geq 1$ and consider the gauge condition (3.2.2). It amounts to

$$\nabla^\mu b^{(lm)}_{\mu\|} - l(l+1)\tilde{b}^{(lm)} = \nabla^2_A b^{(lm)}_{\|} - l(l+1)\tilde{b}^{(lm)} = 0, \tag{3.2.16}$$

in terms of 2D modes. Only one linear combination of the modes $b^{(lm)}_{\|}$, $\tilde{b}^{(lm)}$ satisfies the gauge condition. On-shell the equations of motion impose $\nabla^2_A b^{(lm)}_{\|} = l(l+1) b^{(lm)}_{\|}$ so the physical modes are those that satisfy $\tilde{b}^{(lm)} = b^{(lm)}_{\|}$.

We next consider the 4D gauge symmetry $a_I \to a_I + \nabla_I \Lambda$. Expanding the gauge function Λ in spherical harmonics

$$\Lambda = \lambda^{(lm)}(x) Y_{lm}(y), \tag{3.2.17}$$

this amounts to the 2D transformations

$$\begin{aligned} \tilde{b}^{(lm)} &\to \tilde{b}^{(lm)} + \lambda^{(lm)}, \\ b^{(lm)}_{\mu\|} &\to b^{(lm)}_{\mu\|} + \nabla_\mu \lambda^{(lm)}. \end{aligned} \tag{3.2.18}$$

The field configurations identified after (3.2.16) as satisfying the gauge condition on-shell have $\tilde{b}^{(lm)} = b^{(lm)}_{\|}$ with $b^{(lm)}_{\mu\|} = \nabla_\mu b^{(lm)}_{\|}$. Therefore these are precisely those that are gauge equivalent to the vacuum. Such pure gauge configurations decouple from processes involving states that do satisfy the gauge condition.

The modes $b^{(lm)}$ and $b^{(lm)}_{\mu\perp} = \epsilon_{\mu\nu} \nabla^\nu b^{(lm)}_{\perp}$ do not enter the gauge conditions (3.2.16) at all, nor are they acted on by the gauge transformations (3.2.18). These therefore form two towers of physical modes. Since we assumed $l \geq 1$ from the outset the range of these towers is $l = 1, 2, \cdots$ as expected.

The lowest spherical harmonic $l = 0$ requires special consideration. Indeed, the scalar fields $b^{(00)}, \tilde{b}^{(00)}$ from the S^2 components of the vector field are non-existent because partial waves on S^2 have $l \geq 1$. Further, for $l = 0$ the on-shell condition on the scalars $b_{\|}^{(00)}, b_{\perp}^{(00)}$ due to the AdS$_2$ components of the vector field reduces to the harmonic condition on AdS$_2$ and in (3.2.12) we specifically exempt harmonic modes. Thus there are no modes at $l = 0$ before even considering the gauge condition and the possibility of pure gauge modes.

In summary, the more detailed discussion identifies the physical modes as the towers $b^{(lm)}$, $b_{\perp}^{(lm)}$ with $l = 1, 2, \ldots$. Importantly, these are not simply the modes $b^{(lm)}$, $\tilde{b}^{(lm)}$ that were defined with range $l = 1, 2, \ldots$ from the outset. Indeed, the mode $b_{\perp}^{(lm)}$ was defined for $l = 0, 1, \ldots$ but the harmonic condition removed the $l = 0$ entry.

3.2.4 Boundary Modes.

The discussion of the spectrum so far deferred consideration of the harmonic modes $b_0^{(lm)}$ introduced in (3.2.14). These give rise to boundary modes. Several comments are in order:

- There is exactly one harmonic mode for each partial wave (lm): the AdS$_2$ vector $b_{\mu}^{(lm)}$ is dualized to two scalar components $b_{\perp}^{(lm)}$ and $b_{\|}^{(lm)}$ but the harmonic mode $b_0^{(lm)}$ is "shared" between these fields as it is both longitudinal and transverse.

- The tower of harmonic modes begins at $l = 0$ just like all other components of the AdS$_2$ vector.

- The harmonic condition implies that these modes are zero-modes on AdS$_2$. The tower of harmonic modes — one for each (lm) — identifies the configuration space of harmonic modes as a field on S^2. The equation of motion of this field identifies the leading $l = 0$ mode as physical.

- The scalar Laplacian $(-\nabla_A^2)$ in Euclidean AdS$_2$ has eigenvalues $c_2 = \frac{1}{4} + s^2$ with s real for fields in the principal continuous representations of $SL(2)$. These representations are AdS$_2$ analogues of plane waves in flat space. The harmonic mode has $c_2 = 0$ and belongs to a principal discrete representation with no flat space analogue.

- The harmonic modes are formally pure gauge since they are longitudinal. However, they are physical because the gauge function that generates them is non-normalizable. For us the term harmonic mode is synonymous with the term boundary mode because AdS/CFT lore posits that pure gauge degrees of freedom localize on the boundary.

The harmonic modes were constructed explicitly some time ago [49]. In our discussion we write the Euclidean AdS_2 BH metric in complex form as

$$ds_2^2 = a^2(d\eta^2 + \sinh^2\eta d\theta^2) = a^2\frac{4}{(1-|z|^2)^2}dzd\bar{z}, \tag{3.2.19}$$

where θ has period 2π and $z = \tanh\frac{\eta}{2}e^{i\theta}$. The conformal factor in the second expression diverges as the AdS_2 boundary $|z| = 1$ is approached but this does not affect the harmonic condition which is conformally invariant. We can therefore choose a standard complete set of harmonic modes such as [1]

$$u_n = \frac{1}{\sqrt{2\pi n}}z^n, \qquad n = 1, 2\ldots, \tag{3.2.20}$$

and their complex conjugates. These modes cannot appear as components of a scalar field on AdS_2 since the normalization condition

$$\int \sqrt{g}d^2z\ |u_n|^2 = \int \frac{2a^2d^2z}{(1-|z|^2)^2}\ |u_n|^2 \to \infty, \tag{3.2.21}$$

diverges at the boundary due to the conformal factor. However, derivatives of the modes (3.2.20) are subject to a conformally invariant normalization condition so they are legitimate components of a vector field. The modes (3.2.20) are normalized so

$$\int \sqrt{g}d^2z\ |\nabla_z u_n|^2 = 1, \tag{3.2.22}$$

[1] We omit the constant on AdS_2 (corresponding to $n = 0$) since only derivatives of the basis parametrize vector fields.

in standard conventions where $d^2z = 2dxdy$. Vector fields formed from gradients of harmonic modes are therefore physical even though they are formally pure gauge. We interpret them as boundary modes.

3.2.5 BRST Quantization.

Our old-fashioned discussion of physical modes extends immediately to the more streamlined BRST quantization. For completeness we briefly outline this generalization.

The physical fields $b_\perp^{(lm)}, b^{(lm)}$ are BRST invariant. Other BRST invariant field configurations are those that have no anti-ghosts $\tilde{c}^{(lm)} = 0$ and also satisfy $\tilde{b}^{(lm)} = b_\parallel^{(lm)}$.

The ghost states $c^{(lm)}$ are BRST exact since they are BRST transforms of pure gauge fields. The gauge fields with $\tilde{b}^{(lm)} = b_\parallel^{(lm)}$ are also BRST exact since they are BRST transforms of anti-ghosts $\tilde{c}^{(lm)}$.

This accounting leaves just the physical fields $b_\perp^{(lm)}, b^{(lm)}$ with $l = 1, 2, \ldots$.

The spherically symmetric fields $l = 0$ must be considered separately. The antighost fails to be BRST invariant and the ghost is the BRST transform of a pure gauge function. The remaining two fields $b_\perp^{(00)}, b_\parallel^{(00)}$ are not independent on-shell and can be formally presented as the BRST transform of the anti-ghost $\tilde{c}^{(00)}$, albeit with a non-normalizable field configuration.

In summary, the BRST cohomology agrees with the physical states discussed above in a more elementary formalism. As before, it can be parametrized in terms of the physical fields $b_\perp^{(lm)}, b^{(lm)}$ with $l = 1, 2, \ldots$ and the harmonic fields $b_0^{(lm)}$ with $l = 0$.

3.3 Logarithmic Quantum Corrections: The Vector Field.

In this section we compute functional determinants with the heat kernel method [45, 30, 44]. We first review the elementary heat kernels that we need, including the basic contribution from boundary modes. We then compare the on-shell and off-shell computations of the heat kernel for a vector field.

3.3.1 Elementary Heat Kernels.

The basic heat kernel for a massless scalar on the sphere S^2 is

$$K_S^s = \frac{1}{4\pi a^2}\sum_{k=0}^{\infty} e^{-k(k+1)s}(2k+1) = \frac{1}{4\pi a^2 s}(1+\frac{1}{3}s+\frac{1}{15}s^2+\ldots). \tag{3.3.1}$$

Each component of a vector field on S^2 has the same spectrum as a scalar field on S^2 but the $k=0$ mode is absent from the partial wave expansion. Therefore the heat kernel for a vector on S^2 is

$$K_S^v = \frac{1}{4\pi a^2}\sum_{k=1}^{\infty} e^{-k(k+1)s}(2k+1) = K_S^s - \frac{1}{4\pi a^2} = \frac{1}{4\pi a^2 s}(1-\frac{2}{3}s+\frac{1}{15}s^2+\ldots). \tag{3.3.2}$$

We also need the scalar heat kernel on AdS$_2$. The representation of a heat kernel as an expansion around flat space shows that the local terms are determined from K_S^s by flipping the sign of terms that are odd in the curvature so:

$$K_A^s = \frac{1}{4\pi a^2 s}(1-\frac{1}{3}s+\frac{1}{15}s^2+\ldots). \tag{3.3.3}$$

Although this rule of thumb applies for local terms, there is no similar continuation of eigenvalues and eigenfunctions [49, 47, 48]. The heat kernels above refer to 2D fields on AdS$_2$ and S^2. We assemble these 2D heat kernels into heat kernels for 4D fields on AdS$_2\times S^2$ by summing over towers of the form

$$K_4^s = K_A^s \cdot \frac{1}{4\pi a^2}\sum_j e^{-m_j^2 s}(2j+1), \tag{3.3.4}$$

where each value of angular momentum j on S^2 has a specific value of the effective AdS$_2$ mass $m_j^2 = h_j(h_j-1)$. For example, dimensional reduction of a massless 4D scalar field on S^2 gives a tower of 2D fields with the AdS$_2$ Casimir $h_j(h_j-1)$ identical to the S^2 Casimir $j(j+1)$. In this case the spectrum is $(h,j)=(k+1,k)$ with $k=0,\ldots$ so $h_j=j+1$ and the sum in (3.3.4) reduces to the sum in (3.3.1). We therefore find

$$K_4^s = K_A^s K_S^s = \frac{1}{16\pi^2 a^4 s^2}(1 + \frac{1}{45}s^2 + \ldots). \tag{3.3.5}$$

The physical components arising from reduction of a 4D vector field is restricted to helicities ± 1 but otherwise identical to two 4D scalar fields. The conformal weights for a single tower of this type is therefore again $(h, j) = (k+1, k)$ but with $k = 1, \ldots$ because the angular momentum $j = 0$ on the S^2 is prohibited. The sum over S^2 quantum numbers reduces to (3.3.2) and so we find

$$K_4' = \frac{1}{16\pi^2 a^4 s^2}(1-\frac{1}{3}s+\frac{1}{15}s^2+\ldots)(1-\frac{2}{3}s+\frac{1}{15}s^2+\ldots) = \frac{1}{16\pi^2 a^4 s^2}(1-s+\frac{16}{45}s^2+\ldots), \tag{3.3.6}$$

for a 4D scalar with partial wave $j = 0$ missing.

3.3.2 Counting Boundary Modes.

The harmonic modes are zero-modes from the AdS_2 point of view. Their heat kernel is given by a sum over a complete set of modes that takes the schematic form

$$K(x, x'; s) = \sum_i f_i(x) f_i^*(x'). \tag{3.3.7}$$

We presented all harmonic modes in (3.2.20). At equal points the sum over all harmonic modes for the vector field in the geometry (3.2.19) gives

$$\begin{aligned} K(x, x; s) &= \sum_{n=1}^{\infty} |\nabla u_n|^2 + \text{c.c.} = 2\sum_{n=1}^{\infty} g^{z\bar{z}} \partial_z u_n \partial_{\bar{z}} u_n^* \\ &= \sum_{n=1}^{\infty} \frac{(1-r^2)^2}{a^2} \frac{1}{2\pi n} n^2 r^{2(n-1)} = \frac{1}{2\pi a^2}. \end{aligned} \tag{3.3.8}$$

The expression is independent of the position r, as expected in a homogeneous space.

Homogeneity of AdS_2 allows us to write alternatively

$$K(x,x;s) = \frac{1}{\mathrm{Vol}}\int \sqrt{g}d^2z \sum_i |f_i(x)|^2 = \frac{1}{\mathrm{Vol}_c}N_c, \tag{3.3.9}$$

where Vol_c is the regulated AdS_2 volume and N_c is the regulated number of harmonic modes. Thus the equal point heat kernel can be interpreted as the *density* of harmonic modes in AdS_2.

We interpret the finite density (3.3.8) as the contribution to the heat kernel from a single massless boundary mode rather than a field on the 1D boundary of AdS_2.

3.3.3 Heat Kernel for a 4D Vector Field: The Off-shell Method.

We can arrive at the heat kernel for a 4D vector field by adding contributions from all four components of the vector field and then cancel two unphysical components by introducing ghosts. This is the strategy that is most commonly used.

In this off-shell method the two towers originating from vector components along S^2 are treated identically. They were denoted $b^{(lm)}, \tilde{b}^{(lm)}$ in the explicit mode expansion (3.2.5). From the AdS_2 point these are towers of scalars fields with the leading partial wave $j = 0$ missing so their heat kernel is given by (3.3.6).

In the off-shell method the two towers of scalars originating from vector components along AdS_2 are also treated identically. They were denoted $b_{\parallel}^{(lm)}, b_{\perp}^{(lm)}$ in the explicit mode expansion. The direct computation of the heat kernel on AdS_2 requires consideration of a complete set of vector modes on AdS_2 and subsequent summation over the S^2 tower. The appropriate modes were identified in [49]. For the present purpose recall that heat kernels can be represented as a local expansion. We can therefore take a short-cut and simply invert the sign of the linear term in (3.3.6), corresponding to the interchange $A \leftrightarrow S$. This gives

$$2\tilde{K}_4' = \frac{1}{8\pi^2a^4s^2}(1 + s + \frac{16}{45}s^2 + \ldots). \tag{3.3.10}$$

The final contribution to the off-shell computation are the two ghosts (3.2.8) which are standard scalars with heat kernel given in (3.3.5) except for an overall sign due to statistics. The net result for the 4D vector field then becomes

$$K_4^v = 2K_4' + 2\tilde{K}_4' - 2K_4^s = \frac{1}{8\pi^2 a^4 s^2}(1 + \frac{31}{45}s^2 + \ldots). \tag{3.3.11}$$

3.3.4 Heat Kernel for a 4D Vector Field: The On-shell Method.

The on-shell computation focusses on the physical components of the 4D vector field. These are two towers of scalar fields on AdS_2 with angular momentum on the S^2 $l = 1, 2, \ldots$. In our explicit mode expansions these two towers of physical modes are $b^{(lm)}, b_\perp^{(lm)}$ with $l = 1, 2, \ldots$. They each contribute to the heat kernel with K_4' given in (3.3.6).

In the on-shell computation the only additional contribution is a single tower of boundary modes on AdS_2 with partial wave expansion $l = 0, 1, \ldots$. There is one such mode for each of the AdS_2 pairs $b_\perp^{(lm)}, b_\parallel^{(lm)}$ $l = 0, 1, \ldots$ or, equivalently, one for each gauge function $\lambda^{(lm)}$ $l = 0, 1, \ldots$. For each entry in the tower the AdS_2 part contributes with a factor of the regulated AdS_2 volume with normalization (3.3.8). The sum (3.3.4) over the S^2 tower of boundary modes thus contributes a simple scalar field on S^2 (3.3.1).

In the on-shell computation the heat kernel for the 4D vector field becomes

$$\begin{aligned} K_4^v &= 2K_4' + \frac{1}{2\pi a^2}K_S^s \\ &= \frac{1}{8\pi^2 a^4 s^2}(1 - s + \frac{16}{45}s^2) + \frac{1}{8\pi^2 a^4}(\frac{1}{s} + \frac{1}{3}) \\ &= \frac{1}{8\pi^2 a^4 s^2}(1 + \frac{31}{45}s^2). \end{aligned} \tag{3.3.12}$$

This agrees with the off-shell result (3.3.11).

The off-shell and the on-shell computations are related by a simple rearrangement.

$$
\begin{aligned}
K_4^v &= 2K_4' + 2\tilde{K}_4' - 2K_4^s = 2K_4' + 2(K_A^s + \frac{1}{4\pi a^2})K_S^s - 2K_A^s K_S^s \\
&= 2K_4' + \frac{1}{2\pi a^2}K_S^s.
\end{aligned}
\tag{3.3.13}
$$

The key is that the subtraction of the $l = 0$ mode for a vector on S^2 included in (3.3.6) amounts to an addition of the boundary mode in AdS_2 that is implicitly included in (3.3.10).

Some mild virtual aspects remain in on-shell method. The heat kernel (3.3.3) of a bulk field in AdS_2 implicitly sums over the continuum of off-shell modes of plane-wave type. Similarly, the boundary mode has fixed wave function on AdS_2 but the sum over the tower of S^2 partial waves probes the configuration space off-shell. The simplification of the on-shell computation is that we do not need to determine the explicit spectrum of the gauge violating modes, longitudinal modes, and the corresponding ghosts. It is known from the outset that these contributions must cancel so we may as well not compute them in the first place. Instead, we include just the boundary modes which appear with positive sign, as expected from physical modes.

3.4 Supergravity in $\mathrm{AdS}_2 \times S^2$ - Bosonic Sector.

In this section we analyze the bosonic sector of $\mathcal{N} = 2$ supergravity in $\mathrm{AdS}_2 \times S^2$. The matter content is a tensor field h_{IJ} coupled to a vector field a_I. We derive the linearized equations of motion from the AdS_2 point of view, then diagonalize them explicitly and find the full spectrum and all eigenvectors. Finally, we write the modes in a basis where their gauge transformations are manifest. This classifies the modes as gauge violating, pure gauge, or physical.

3.4.1 4D Theory.

The 4D action for the gravity-graviphoton system is just standard Einstein-Maxwell

$$\mathcal{L}_{\rm EM} = \frac{1}{2}\left[R - \frac{1}{4}F_{IJ}F^{IJ}\right]. \tag{3.4.1}$$

The physical content of the theory can be extracted by imposing Lorentz gauge

$$\begin{aligned}\nabla^I h_{IJ} - \frac{1}{2}\nabla_J h^I{}_I &= 0,\\ \nabla^I a_I &= 0,\end{aligned} \tag{3.4.2}$$

on the perturbations $\delta g_{IJ} = h_{IJ}$, $\delta A_I = a_I$. We once again implement this in the quantum theory by adding gauge fixing terms to the action and taking Feynman gauge $\xi = 1$. The gauge fixed action is

$$\mathcal{L}_{\rm Feynman} = \frac{1}{2}\left[R - \frac{1}{4}F^2 - \frac{1}{2}\left(\nabla^I h_{IJ} - \frac{1}{2}\nabla_J h^I{}_I\right)^2 - \frac{1}{2}(\nabla^I a_I)^2\right]. \tag{3.4.3}$$

We consider the magnetic $\mathrm{AdS}_2 \times S^2$ background. In our units the background reads

$$F_{\alpha\beta} = 2a\epsilon_{\alpha\beta}, R_{\mu\nu} = -a^{-2}g_{\mu\nu}, R_{\alpha\beta} = a^{-2}g_{\alpha\beta}. \tag{3.4.4}$$

We take the scale $a = 1$ in this section but restore it later.

When analyzing the spectator vector field in $\mathrm{AdS}_2 \times S^2$ we diagonalized the 4D action before reducing it on S^2. In the present context it is simpler to take the linearized equations of motion in 4D, reduce them on S^2, and only then diagonalize. We therefore first consider the gauge fixed Maxwell's equations in 4D:

$$\nabla^I F_{IJ} + \nabla_J \nabla^I a_I = 0. \tag{3.4.5}$$

Perturbing around the background (3.4.4) and keeping only linear terms yields

$$-2\nabla^{\alpha}h_{\mu}{}^{\beta}\epsilon_{\alpha\beta} + (\nabla_A^2 + \nabla_S^2 + 1)a_{\mu} = 0. \tag{3.4.6}$$

$$-2\nabla_{\mu}h^{\mu\alpha}\epsilon_{\alpha\beta} + \nabla^{\alpha}(h^{\mu}_{\mu} - h^{\gamma}_{\gamma})\epsilon_{\alpha\beta} + (\nabla_A^2 + \nabla_S^2 - 1)a_{\alpha} = 0. \tag{3.4.7}$$

An analogous computation for Einstein's equations yields

$$-\frac{1}{2}(\nabla^2 - 2)h_{\alpha\beta} + \frac{1}{4}g_{\alpha\beta}\big[(\nabla^2 + 2)h^{\gamma}_{\gamma} + (\nabla^2 - 2)h^{\rho}_{\rho}\big] = g_{\alpha\beta}\epsilon^{\gamma\delta}\nabla_{\gamma}a_{\delta}, \tag{3.4.8}$$

$$-\frac{1}{2}(\nabla^2 + 2)h_{\mu\nu} + \frac{1}{4}g_{\mu\nu}\big[(\nabla^2 - 2)h^{\gamma}_{\gamma} + (\nabla^2 + 2)h^{\rho}_{\rho}\big] = -g_{\mu\nu}\epsilon^{\alpha\beta}\nabla_{\alpha}a_{\beta}, \tag{3.4.9}$$

$$\frac{1}{2}(\nabla^2 - 2)h_{\mu\alpha} = \epsilon_{\alpha\beta}\left(\nabla_{\mu}a^{\beta} - \nabla^{\beta}a_{\mu}\right). \tag{3.4.10}$$

The graviphoton equations of motion (3.4.6) -(3.4.7) are more complicated than those for a spectator vector field because here we allow the metric to fluctuate as well. Similarly, the vector field terms in (3.4.8)-(3.4.10) constitute nontrivial kinetic mixing.

3.4.2 2D Effective Theory.

We want to represent the 4D equations of motion (3.4.6)-(3.4.10) as towers of 2D equations. The physics of the 2D theory is determined by Kaluza-Klein reduction in homogeneous spaces

[39]. As in (3.2.5) we expand the 4D fields in partial waves:

$$
\begin{aligned}
h_{\{\mu\nu\}}(x,y) &= H^{(lm)}_{\{\mu\nu\}}(x)Y_{(lm)}(y), \\
h_\rho{}^\rho(x,y) &= H^{(lm)}{}_\rho{}^\rho(x)Y_{(lm)}(y), \\
h_{\mu\alpha}(x,y) &= \tilde{B}^{(lm)}_\mu(x)\nabla_\alpha Y_{(lm)}(y) + B^{(lm)}_\mu(x)\epsilon_{\alpha\beta}\nabla^\beta Y_{(lm)}(y), \\
h_{\{\alpha\beta\}}(x,y) &= \phi^{(lm)}(x)\nabla_{\{\alpha}\nabla_{\beta\}}Y_{(lm)}(y) + \tilde{\phi}^{(lm)}(x)\nabla_{\{\alpha}\epsilon_{\beta\}\gamma}\nabla^\gamma Y_{(lm)}(y), \\
h_\alpha{}^\alpha(x,y) &= \pi^{(lm)}(x)Y_{(lm)}(y), \\
a_\mu(x,y) &= b^{(lm)}_\mu(x)Y_{(lm)}(y), \\
a_\alpha(x,y) &= \tilde{b}^{(lm)}(x)\nabla_\alpha Y_{(lm)}(y) + b^{(lm)}(x)\epsilon_{\alpha\beta}\nabla^\beta Y_{(lm)}(y).
\end{aligned}
\tag{3.4.11}
$$

Sum over angular momentum quantum numbers (lm) is implied. Curly brackets around indices indicate that we remove the 2D trace: $h_{\{\alpha\beta\}} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h_\gamma{}^\gamma$, and analogously for $h_{\{\mu\nu\}}$, [23]. We also expand the generators of diffeomorphisms and gauge transformations in spherical harmonics,

$$
\begin{aligned}
\xi_\mu(x,y) &= \xi^{(lm)}_\mu(x)Y_{(lm)}(y), \\
\xi_\alpha(x,y) &= \zeta^{(lm)}(x)\nabla_\alpha Y_{(lm)}(y) + \xi^{(lm)}(x)\epsilon_{\alpha\beta}\nabla^\beta Y_{(lm)}(y), \\
\Lambda(x,y) &= \lambda^{(lm)}(x)Y_{(lm)}(y).
\end{aligned}
\tag{3.4.12}
$$

The allowed range for the angular momentum quantum number of each mode can be read off from the expressions (3.4.11) and (3.4.12). The modes with a single (double) derivative acting on the spherical harmonic functions are missing the first (the first two) modes. Table 3.1 below summarizes the allowed range of l for all 2D modes defined in (3.4.11) and (3.4.12). Inserting the partial wave expansion (3.4.11) into the Maxwell equations (3.4.6)-(3.4.7) we find

$$
\Big((\nabla_A^2 - l(l+1)+1)b^{(lm)}_\mu(x) - 2l(l+1)B^{(lm)}_\mu(x)\Big)Y_{(lm)}(y) = 0, \tag{3.4.13}
$$

2D Field or Gauge Parameter	Range
$H^{(lm)}_{\{\mu\nu\}}$, $H^{(lm)}{}_{\rho}{}^{\rho}$, $\pi^{(lm)}$, $b^{(lm)}_{\mu}$; $\xi^{(lm)}_{\mu}$, $\lambda^{(lm)}$	$l = 0, 1...$
$\tilde{B}^{(lm)}_{\mu}$, $B^{(lm)}_{\mu}$, $\tilde{b}^{(lm)}$, $b^{(lm)}$; $\zeta^{(lm)}$, $\xi^{(lm)}$	$l = 1, 2...$
$\phi^{(lm)}$, $\tilde{\phi}^{(lm)}$	$l = 2, 3...$

Table 3.1: Allowed range of l for all 2D modes.

$$\begin{aligned} &\left((\nabla_A^2 - l(l+1))\tilde{b}^{(lm)}(x) - 2\nabla^{\mu} B^{(lm)}_{\mu}(x)\right)\nabla_{\alpha} Y_{(lm)}(y) + \\ &\left((\nabla_A^2 - l(l{+}1))b^{(lm)}(x) + 2\nabla^{\mu}\tilde{B}^{(lm)}_{\mu}(x) + \pi^{(lm)}(x) - H^{(lm)}{}_{\rho}{}^{\rho}(x)\right)\epsilon_{\alpha\beta}\nabla^{\beta} Y_{(lm)}(y) = 0. \end{aligned} \tag{3.4.14}$$

The dependence on the S^2 coordinates can be integrated out by contracting (3.4.13) and (3.4.14) with the appropriate spherical harmonic functions and using their orthonormality conditions. The result is one equation that is a vector from the AdS$_2$ point of view and two equations that are scalars.

Dimensional reduction of the Einstein equations (3.4.8)-(3.4.10) proceeds similarly. For brevity we just present a summary of all 2D effective equations of motion.

2D Equations of Motion - Summary

The equations defined for $l = 0, 1...$ are

$$(\nabla_A^2 - l(l+1) + 1)b^{(lm)}_{\mu} - 2l(l+1)B^{(lm)}_{\mu} = 0, \tag{3.4.15}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) - 2)\pi^{(lm)} - 2l(l+1)b^{(lm)} = 0, \tag{3.4.16}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) + 2)H^{(lm)}_{\{\mu\nu\}} = 0, \tag{3.4.17}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) - 2)H^{(lm)}{}_{\rho}{}^{\rho} - 2\pi^{(lm)} + 2l(l+1)b^{(lm)} = 0. \tag{3.4.18}$$

The equations defined for $l = 1, 2...$ are

$$(\nabla_A^2 - l(l+1))\tilde{b}^{(lm)} - 2\nabla^{\mu} B^{(lm)}_{\mu} = 0, \tag{3.4.19}$$

$$(\nabla_A^2 - l(l+1))b^{(lm)} + 2\nabla^\mu \tilde{B}_\mu^{(lm)} + \pi^{(lm)} - H^{(lm)}{}_\rho{}^\rho = 0, \tag{3.4.20}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) - 1)\tilde{B}_\mu^{(lm)} + \nabla_\mu b^{(lm)} = 0, \tag{3.4.21}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) - 1)B_\mu^{(lm)} - \nabla_\mu \tilde{b}^{(lm)} + b_\mu^{(lm)} = 0. \tag{3.4.22}$$

The equations defined for $l = 2, 3...$ are

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) + 2)\phi^{(lm)} = 0, \tag{3.4.23}$$

$$-\frac{1}{2}(\nabla_A^2 - l(l+1) + 2)\tilde{\phi}^{(lm)} = 0. \tag{3.4.24}$$

The complete set of equations has $10 + 4 = 14$ components as expected for gravity coupled to a gauge field. They are organized into 6 scalar equations, 3 vector equations (with two components each), and one equation that is a symmetric traceless tensor (with two components).

3.4.3 Spectrum.

To compute the 2D spectrum we must diagonalize the system of 2D equations of motion presented above. To disentangle the equations we dualize each of the 2D vectors $B_\mu^{(lm)}, \tilde{B}_\mu^{(lm)}, b_\mu^{(lm)}$ into two scalars and one harmonic mode, as in (3.2.14). A new feature is that we also need to dualize the symmetric traceless tensor $H_{\{\mu\nu\}}^{(lm)}$ to scalars [67]. We write

$$H_{\{\mu\nu\}}^{(lm)} = \nabla_{\{\mu}\nabla_{\nu\}} H_+^{(lm)} + \nabla_{\{\mu}\epsilon_{\nu\}\rho}\nabla^\rho H_\times^{(lm)} + \nabla_{\{\mu}\nabla_{\nu\}} H_0^{(lm)}. \tag{3.4.25}$$

The configuration space of scalars $H_+^{(lm)}, H_\times^{(lm)}$ could generate all possible $H_{\{\mu\nu\}}^{(lm)}$. Indeed, to avoid that some $H_{\{\mu\nu\}}^{(lm)}$ are counted twice we require:

$$\begin{aligned} \nabla_A^2(\nabla_A^2 - 2)H_+^{(lm)} &\neq 0, \\ \nabla_A^2(\nabla_A^2 - 2)H_\times^{(lm)} &\neq 0. \end{aligned} \tag{3.4.26}$$

For those configurations that could have been represented in either H_+ or $H_\times$ form we introduced the harmonic mode $H_0^{(lm)}$, written to be definite in its H_+ form. The harmonic mode satisfies

$$\nabla_A^2(\nabla_A^2 - 2)H_0^{(lm)} = 0. \tag{3.4.27}$$

To verify these claims it is useful to first compute

$$\nabla^\mu H_{\{\mu\nu\}}^{(lm)} = \frac{1}{2}\nabla_\nu(\nabla_A^2 - 2)(H_+^{(lm)} + H_0^{(lm)}) + \epsilon_{\nu\mu}\nabla^\mu(\nabla_A^2 - 2)H_\times^{(lm)}, \tag{3.4.28}$$

in AdS_2 and then use this identity to find $H_+^{(lm)}, H_\times^{(lm)}$ in terms of $H_{\{\mu\nu\}}^{(lm)}$. The resulting expressions involve the inverse of the operator $\nabla_A^2(\nabla_A^2 - 2)$ which is invertible on the appropriate subspaces due to (3.4.26).

After dualization of all fields to scalars the equations of motion (3.4.15)-(3.4.24) can be recast as 14 Klein-Gordon equations coupled by a 14×14 block diagonal mass matrix. We find that 5 components of the mass matrix are diagonal in our basis. The remaining blocks in the equations of motion are the 2×2 block,

$$(\nabla_A^2 - l(l+1))\begin{pmatrix} B_\perp^{(lm)} \\ b_\perp^{(lm)} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2l(l+1) & 0 \end{pmatrix}\begin{pmatrix} B_\perp^{(lm)} \\ b_\perp^{(lm)} \end{pmatrix}, \tag{3.4.29}$$

the 3×3 block,

$$(\nabla_A^2 - l(l+1))\begin{pmatrix} B_\parallel^{(lm)} \\ b_\parallel^{(lm)} \\ \tilde{b}^{(lm)} \end{pmatrix} = \begin{pmatrix} 2 & 2 & -2 \\ 2l(l+1) & 0 & 0 \\ 4+2l(l+1) & 4 & -4 \end{pmatrix}\begin{pmatrix} B_\parallel^{(lm)} \\ b_\parallel^{(lm)} \\ \tilde{b}^{(lm)} \end{pmatrix}, \tag{3.4.30}$$

and the 4×4 block

$$(\nabla_A^2 - l(l+1))\begin{pmatrix} H^{(lm)} \\ \tilde{B}_\parallel^{(lm)} \\ \pi^{(lm)} \\ b^{(lm)} \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4 & 4l(l+1) \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4l(l+1) \\ 1 & -2(2+l(l+1)) & -1 & -4 \end{pmatrix}\begin{pmatrix} H^{(lm)} \\ \tilde{B}_\parallel^{(lm)} \\ \pi^{(lm)} \\ b^{(lm)} \end{pmatrix}. \tag{3.4.31}$$

The final 4×4 block is the most complicated with eigenvectors

$$\begin{aligned} V_0 &= 2l(l+1)\tilde{B}_\parallel^{(lm)} + \pi^{(lm)}, \\ V_1 &= {H^{(lm)}}_\rho^{\ \rho} - 2(2+l(l+1))\tilde{B}_\parallel^{(lm)} - \frac{l-1}{l+1}\pi^{(lm)} - 2(l+1)b^{(lm)}, \\ V_2 &= -l{H^{(lm)}}_\rho^{\ \rho} + 2(2+l(l+1))l\tilde{B}_\parallel^{(lm)} + (l+2)\pi^{(lm)} - 2l^2 b^{(lm)}, \\ V_3 &= -{H^{(lm)}}_\rho^{\ \rho} + 2(2+l(l+1))\tilde{B}_\parallel^{(lm)} + 4b^{(lm)}. \end{aligned} \tag{3.4.32}$$

Our result for the spectrum and the corresponding modes is reported in Table 3.2. Comments:

- The eigenvectors V_n with $n = 0, 1, 2, 3$ were defined in (3.4.32).

Mode	Mass	Range
$H_{+}^{(lm)}$	$m^2 = l(l+1) + 2$	$l = 0, 1...$
$H_{\times}^{(lm)}$	$m^2 = l(l+1) + 2$	$l = 0, 1...$
$V_0 = 2l(l+1)\tilde{B}_{\parallel}^{(lm)} + \pi^{(lm)}$ #	$m^2 = l(l+1) + 2$	$l = 0, 1...$
$\tilde{B}_{\perp}^{(lm)}$	$m^2 = l(l+1) + 2$	$l = 1, 2...$
$b_{\perp}^{(lm)} - lB_{\perp}^{(lm)}$	$m^2 = l(l-1)$	$l = 0, 1...$
V_1	$m^2 = l(l-1)$	$l = 1, 2...$
V_2	$m^2 = (l+1)(l+2)$	$l = 0, 1...$
$b_{\perp}^{(lm)} + (l+1)B_{\perp}^{(lm)}$	$m^2 = (l+1)(l+2)$	$l = 1, 2...$
$\tilde{b}^{(lm)} - b_{\parallel}^{(lm)} - 2B_{\parallel}^{(lm)}$ #	$m^2 = l(l+1)$	$l = 0, 1, ...$
$b_{\parallel}^{(lm)} + l(l+1)B_{\parallel}^{(lm)}$ ‡	$m^2 = l(l+1)$	$l = 1, 2...$
$B_{\parallel}^{(lm)} + b_{\parallel}^{(lm)} - \tilde{b}^{(lm)}$	$m^2 = l(l+1) - 2$	$l = 1, 2...$
$\phi^{(lm)}$	$m^2 = l(l+1) - 2$	$l = 2, 3...$
$\tilde{\phi}^{(lm)}$	$m^2 = l(l+1) - 2$	$l = 2, 3...$
V_3 †	$m^2 = l(l+1) - 2$	$l = 1, 2...$

Table 3.2: Spectrum of $\mathcal{N} = 2$ supergravity in $\mathrm{AdS}_2 \times S^2$.

- We express our results for the eigenvalues as scalar masses defined in the usual way

$$(-\nabla_A^2 + m^2)X = 0. \tag{3.4.33}$$

- We do not indicate the harmonic modes explicitly. In the present context they can be absorbed in $\parallel$ components and $+$ components.

- The mode labeled with † does not apply for $l = 1$ and the two modes labelled with # similarly do not apply at $l = 0$. We inspect these special cases later.

- The entry labeled with ‡ is not a true eigenvector. Instead it is a generalized eigenvector associated with a repeated eigenvalue. We discuss the details of this issue in Appendix A.

3.4.4 Gauge Violating, Longitudinal, and Physical States.

At this point we have diagonalized the gauge fixed equations of motion but we did not yet analyse gauge symmetry. To do so we first write the gauge conditions (3.4.2) in components

$$\nabla^\mu h_{\{\mu\nu\}} + \nabla^\alpha h_{\alpha\nu} - \frac{1}{2}\nabla_\nu h^\alpha_\alpha = 0, \tag{3.4.34}$$

$$\nabla^\alpha h_{\{\alpha\beta\}} + \nabla^\mu h_{\mu\beta} - \frac{1}{2}\nabla_\beta h^\mu_\mu = 0, \tag{3.4.35}$$

$$\nabla^\mu a_\mu + \nabla^\alpha a_\alpha = 0, \tag{3.4.36}$$

and then insert the partial wave expansion (3.4.11) to find the 2D version of the gauge conditions

$$\nabla^\mu H^{(lm)}_{\{\mu\nu\}} - l(l+1)\tilde{B}^{(lm)}_\nu - \frac{1}{2}\nabla_\nu \pi^{(lm)} = 0, \tag{3.4.37}$$

$$\nabla^\mu b^{(lm)}_\mu - l(l+1)\tilde{b} = 0. \tag{3.4.38}$$

$$l(l+1)\left[\nabla^\mu \tilde{B}^{(lm)}_\mu + \frac{1}{2}(2 - l(l+1))\phi^{(lm)} - \frac{1}{2}H^{(lm)\ \rho}_{\quad\ \rho}\right] = 0, \tag{3.4.39}$$

$$l(l+1)\left[\nabla^\mu B^{(lm)}_\mu + \frac{1}{2}(2 - l(l+1))\tilde{\phi}^{(lm)}\right] = 0. \tag{3.4.40}$$

The factors of $l(l+1)$ in front of (3.4.39) and (3.4.40) are due to the integration over the S^2 coordinates. We retained them to stress that these equations apply only for $l \geq 1$. The field components that are only defined at $l \geq 1$ similarly appear with a prefactor $l(l+1)$ so that the $l = 0$ component is not needed; and the fields $\phi^{(lm)}$, $\tilde{\phi}^{(lm)}$ that are defined only for $l \geq 2$ both have a prefactor that vanishes at $l = 0, 1$.

Our next step is to dualize the 2D vectors and the 2D tensor using (3.2.14) and (3.4.25).

The gauge conditions defined for $l = 0, 1, ...$ become

$$\nabla_\nu \left[\frac{1}{2}(\nabla_A^2 - 2) H_+^{(lm)} - l(l+1)\tilde{B}_\parallel^{(lm)} - \frac{1}{2}\pi^{(lm)} \right] + \epsilon_{\nu\mu}\nabla^\mu \left[\frac{1}{2}(\nabla_A^2 - 2) H_\times^{(lm)} - l(l+1)\tilde{B}_\perp^{(lm)} \right] = 0, \tag{3.4.41}$$

$$\nabla_A^2 b_\parallel^{(lm)} - l(l+1)\tilde{b} = 0, \tag{3.4.42}$$

and those defined for $l = 1, 2, ...$ become

$$\nabla_A^2 \tilde{B}_\parallel^{(lm)} + \frac{1}{2}(2 - l(l+1))\phi^{(lm)} - \frac{1}{2} H^{(lm)}{}_\rho{}^\rho = 0, \tag{3.4.43}$$

$$\nabla_A^2 B_\parallel^{(lm)} + \frac{1}{2}(2 - l(l+1))\tilde{\phi}^{(lm)} = 0, \tag{3.4.44}$$

We can project (3.4.41) and obtain two linearly independent scalar equations by applying ∇^ν or $\epsilon^{\nu\mu}\nabla_\mu$ and then inverting the resulting overall Laplacian ∇_A^2. Our results below will indeed justify the inversion except for the special case $l = 0$ which we reconsider later. With this exception we can therefore simply require both square brackets in (3.4.41) to vanish.

Our final step is to eliminate the kinetic operators ∇_A^2 from (3.4.41) - (3.4.44) by using the equations of motion. This gives the on-shell gauge conditions:

$$\frac{1}{2} l(l+1) H_+^{(lm)} - l(l+1)\tilde{B}_\parallel^{(lm)} - \frac{1}{2}\pi^{(lm)} = 0, \tag{3.4.45}$$

$$\frac{1}{2} H_\times^{(lm)} - \tilde{B}_\perp^{(lm)} = 0, \tag{3.4.46}$$

$$\tilde{b}^{(lm)} - b_\parallel^{(lm)} - 2B_\parallel^{(lm)} = 0, \tag{3.4.47}$$

$$(2 + l(l+1))\tilde{B}_\parallel^{(lm)} + 2b^{(lm)} + \frac{1}{2}(2 - l(l+1))\phi^{(lm)} - \frac{1}{2} H^{(lm)}{}_\rho{}^\rho = 0, \tag{3.4.48}$$

$$(2 + l(l+1))B_\parallel^{(lm)} - 2\tilde{b}^{(lm)} + 2b_\parallel^{(lm)} + \frac{1}{2}(2 - l(l+1))\tilde{\phi}^{(lm)} = 0. \tag{3.4.49}$$

As mentioned above, these equations apply only for $l \geq 1$ and we return to $l = 0$ later.

The modes presented in subsection 3.4.3 were identified only by their eigenvalues so we can freely choose a new basis by taking linear combinations of modes with the same mass. The gauge conditions (3.4.45)-(3.4.49) specify particular linear combinations that are set to zero by the gauge conditions. We collect these gauge violating modes in Table 3.3. Our next

Gauge Violating Modes	Mass
$\frac{1}{2}l(l+1)H_+^{(lm)} - l(l+1)\tilde{B}_\parallel^{(lm)} - \frac{1}{2}\pi^{(lm)}$	$m^2 = l(l+1)+2$
$\frac{1}{2}H_\times^{(lm)} - \tilde{B}_\perp^{(lm)}$	$m^2 = l(l+1)+2$
$\tilde{b}^{(lm)} - b_\parallel^{(lm)} - 2B_\parallel^{(lm)}$	$m^2 = l(l+1)$
$(2+l(l+1))\tilde{B}_\parallel^{(lm)} + 2b^{(lm)} + \frac{1}{2}(2-l(l+1))\phi^{(lm)} - \frac{1}{2}H^{(lm)}{}_\rho{}^\rho$	$m^2 = l(l+1)-2$
$(2+l(l+1))B_\parallel^{(lm)} - 2\tilde{b}^{(lm)} + 2b_\parallel^{(lm)} + \frac{1}{2}(2-l(l+1))\tilde{\phi}^{(lm)}$	$m^2 = l(l+1)-2$

Table 3.3: Gauge violating modes.

step is to take equivalences under gauge and diffeomorphism transformations into account. The variations of the 4D fields are:

$$\begin{aligned} \delta a_I &= \nabla_I \Lambda' + \xi^J F_{JI} + \nabla_I(\xi^J A_J), \\ \delta h_{IJ} &= \nabla_I \xi_J + \nabla_J \xi_I. \end{aligned} \tag{3.4.50}$$

The gauge field varies under diffeomorphisms but the metric fluctuations do not vary under gauge transformations. It is therefore advantageous to remove field components in a specific order: first exploit diffeomorphisms and then gauge transformations. In particular, we have not yet specified a gauge for the background gauge fields A_J although the field strength is of course specified in (3.4.4). We take this into account by redefining diffeomorphisms to include a compensating gauge transformation that removes the A_J dependence. We implement this by henceforth taking $\Lambda' = \Lambda - \xi^J A_J$ in (3.4.50).

In our on-shell approach we already fixed the gauge in (3.4.2) so at this point we can focus on residual symmetries. The gauge variations (3.4.50) that preserve the gauge conditions

(3.4.2) satisfy

$$\nabla_4^2\Lambda + 2\epsilon_{\alpha\beta}\nabla^\beta\xi^\alpha = 0, \tag{3.4.51}$$
$$(g_{IJ}\nabla_4^2 + R_{IJ})\xi^I = 0.$$

Upon expansion in partial waves (3.4.12) we find the 2D equations of motion for the residual symmetries. The 2D diffeomorphisms $\xi_\parallel^{(lm)}, \xi_\perp^{(lm)}$ have mass $m^2 = l(l+1)+2$ and range $l = 0, 1, \ldots$, the S^2 diffeomorphisms $\zeta^{(lm)}, \xi^{(lm)}$ have mass $m^2 = l(l+1)-2$ and range $l = 1, 2, \ldots$ while the gauge symmetry is an eigenvector satisfying

$$(\nabla_A^2 - l(l+1))\lambda^{(lm)} - 2l(l+1)\xi^{(lm)} = 0 \quad , \; l = 0, 1, \ldots. \tag{3.4.52}$$

We need only consider diffeomorphisms and gauge transformations that satisfy their appropriate on-shell condition.

Inserting the partial wave expansions (3.4.11) and (3.4.12) into the 4D symmetry variations (3.4.50) we find variations of all 2D fields. After complete dualization to scalars, the result is presented below in Table 3.4. The five towers of gauge violating modes identified in (3.4.45)-(3.4.49) are all invariant under symmetry variations as they should be. To obtain the longitudinal states we consider our original list of 14 towers and constrain it with the gauge conditions. For example, condition (3.4.46) allows us to work only with $\tilde{B}_\perp^{(lm)}$ and not worry about $H_\times^{(lm)}$ since these fields are proportional after imposing gauge conditions. After constraining the modes in this way we find combinations that are pure gauge. These are presented in Table 3.5. The mode $b_\parallel^{(lm)} + l(l+1)B_\parallel^{(lm)}$ was a generalized eigenvector prior to gauge fixing. However, the state with mass $l(l+1)$ with which it was degenerate was removed by the gauge condition (3.4.47) and thus $b_\parallel^{(lm)} + l(l+1)B_\parallel^{(lm)}$ is now a true eigenvector. Its symmetry variation $\lambda^{(lm)} + l(l+1)\xi^{(lm)}$ is not diagonal but in view of (3.4.52) it is precisely the combination that is on-shell with mass so $m^2 = l(l+1)$.

There is significant ambiguity in the form of the longitudinal modes we identify. We

Mode	Symmetry Variation	Range
$H^{(lm)\ \rho}_{\ \ \ \ \ \ \ \ \rho}$	$2\nabla^2_A \xi^{(lm)}_{\parallel}$	$l = 0, 1...$
$H^{(lm)}_{+}$	$2\xi^{(lm)}_{\parallel}$	$l = 0, 1...$
$H^{(lm)}_{\times}$	$2\xi^{(lm)}_{\perp}$	$l = 0, 1...$
$\tilde{B}^{(lm)}_{\parallel}$	$\xi^{(lm)}_{\parallel} + \zeta^{(lm)}$	$l = 1, 2...$
$\tilde{B}^{(lm)}_{\perp}$	$\xi^{(lm)}_{\perp}$	$l = 1, 2...$
$B^{(lm)}_{\parallel}$	$\xi^{(lm)}$	$l = 1, 2...$
$B^{(lm)}_{\perp}$	0	$l = 1, 2...$
$\phi^{(lm)}$	$2\zeta^{(lm)}$	$l = 2, 3...$
$\tilde{\phi}^{(lm)}$	$2\xi^{(lm)}$	$l = 2, 3...$
$\pi^{(lm)}$	$-2l(l+1)\zeta^{(lm)}$	$l = 0, 1...$
$b^{(lm)}_{\parallel}$	$\lambda^{(lm)}$	$l = 0, 1...$
$b^{(lm)}_{\perp}$	0	$l = 0, 1...$
$b^{(lm)}$	$-2\zeta^{(lm)}$	$l = 1, 2...$
$\tilde{b}^{(lm)}$	$2\xi^{(lm)} + \lambda^{(lm)}$	$l = 1, 2...$

Table 3.4: Variations of all 2D fields.

can freely add modes proportional to the gauge violating modes since those are themselves invariant under on-shell gauge transformations. Similarly (and perhaps more relevant) we can add modes proportional to the gauge invariant physical states identified below.

After removal of five towers of gauge violating modes and five towers of longitudinal modes there remain four towers of fields that satisfy the gauge condition and cannot be represented as pure gauge states. These are the physical states. Simplifying the modes from our 14 original towers using the gauge conditions and then forming gauge invariant combinations we find the physical modes in Table 3.6. The second line is just $-\frac{l+1}{l-1}V_1$, while the third line is $\frac{1}{l+2}V_2$.

Longitudinal Mode	Symmetry variation	Mass
$2l(l+1)\tilde{B}_{\parallel}^{(lm)} + \pi^{(lm)}$	$2l(l+1)\xi_{\parallel}^{(lm)}$	$m^2 = l(l+1)+2$
$\tilde{B}_{\perp}^{(lm)}$	$\xi_{\perp}^{(lm)}$	$m^2 = l(l+1)+2$
$b_{\parallel}^{(lm)} + l(l+1)B_{\parallel}^{(lm)}$	$\lambda^{(lm)} + l(l+1)\xi^{(lm)}$	$m^2 = l(l+1)$
$\tilde{\phi}^{(lm)}$	$2\xi^{(lm)}$	$m^2 = l(l+1)-2$
$\phi^{(lm)}$	$2\zeta^{(lm)}$	$m^2 = l(l+1)-2$

Table 3.5: Logitudinal modes.

Physical Modes.	Mass	Range
$b_{\perp}^{(lm)} - lB_{\perp}^{(lm)}$	$m^2 = l(l-1)$	$l = 2, \ldots$
$\pi^{(lm)} + 2(l+1)b^{(lm)} + (l+1)(l+2)\phi^{(lm)}$	$m^2 = l(l-1)$	$l = 2, \ldots$
$\pi^{(lm)} - 2lb^{(lm)} + l(l-1)\phi^{(lm)}$	$m^2 = (l+1)(l+2)$	$l = 1, \ldots$
$b_{\perp}^{(lm)} + (l+1)B_{\perp}^{(lm)}$	$m^2 = (l+1)(l+2)$	$l = 1, \ldots$

Table 3.6: Physical modes.

3.4.5 l=1 Modes.

Some of our results warrant special comment for small values of l. In this subsection we reconsider $l = 1$ and in the next we consider $l = 0$.

There are several issues for $l = 1$:

- The part of the 4D graviton that is a symmetric traceless tensor on S^2 vanishes identically for $l = 1$. Consequently the modes $\phi^{(lm)}$ and $\tilde{\phi}^{(lm)}$ are only defined for $l \geq 2$. This leaves 12 2D scalar modes at $l = 1$.

- For $l = 1$ the eigenvalue $m^2 = l(l-1)$ of V_1 coincides with $m^2 = l(l+1) - 2$ of V_3. In fact, $V_3 = -V_1$ for $l = 1$ so in this case our set of modes is incomplete in its generic form. We adress this by introducing a generalized eigenvector $V_3' = 4b^{(1m)} + \pi^{(1m)}$ which is acted on as $\nabla_A^2 V_3' = 4V_1$.

- We have dualized all 2D fields fully to 2D scalars. This can lead to overcounting in case of harmonic fields, which we define as those fields where $m^2 = 0$ after dualization of 2D vectors and those where $m^2 = 0$ or $m^2 = 2$ after dualization of 2D symmetric traceless tensors. There are no modes of this type for $l \geq 2$ but they are present for $l = 0, 1$. We must therefore revisit dualization.

We present for convenience the spectrum and the corresponding modes for $l = 1$ in Table 3.7.

Modes	Mass
$V_2 = -H^{(1m)\ \rho}_{\quad\ \rho} + 8\tilde{B}^{(1m)}_{\parallel} + 3\pi^{(1m)} - 2b^{(1m)}$	$m^2 = 6$
$b^{(1m)}_{\perp} + 2B^{(1m)}_{\perp}$	$m^2 = 6$
$H^{(1m)}_{+}$	$m^2 = 4$
$H^{(1m)}_{\times}$	$m^2 = 4$
$V_0 = 4\tilde{B}^{(1m)}_{\parallel} + \pi^{(1m)}$	$m^2 = 4$
$\tilde{B}^{(1m)}_{\perp}$	$m^2 = 4$
$b^{(1m)}_{\parallel} + 2B^{(1m)}_{\parallel} - \tilde{b}^{(1m)}$	$m^2 = 2$
$b^{(1m)}_{\parallel} + 2B^{(1m)}_{\parallel} \quad \ddagger$	$m^2 = 2$
$b^{(1m)}_{\parallel} + B^{(1m)}_{\parallel} - \tilde{b}^{(1m)}$	$m^2 = 0$
$b^{(1m)}_{\perp} - B^{(1m)}_{\perp}$	$m^2 = 0$
$V_1 = H^{(1m)\ \rho}_{\quad\ \rho} - 8\tilde{B}^{(1m)}_{\parallel} - 4b^{(1m)}$	$m^2 = 0$
$V_3' = 4b^{(1m)} + \pi^{(1m)} \quad \ddagger$	$m^2 = 0$

Table 3.7: Spectrum of $l = 1$ modes.

The modes labeled with ‡ are generalized eigenvectors. The $m^2 = 2$ mode is just the $l = 1$ version of the generalized state $b^{(1m)}_{\parallel} + l(l+1)B^{(1m)}_{\parallel}$ already present for $l \geq 2$. $V_3' = 4b^{(1m)} + \pi^{(1m)}$ is the mode particular to $l = 1$ that was discussed above.

As we have stressed we must take care not to overcount the modes with $m^2 = 0$ that arise from dualization of a 2D vector to a 2D scalar. In order to illuminate the issue that may arise we consider the coupled system of $B^{(1m)}_{\mu}$, $b^{(1m)}_{\mu}$, and $\tilde{b}^{(1m)}$ prior to dualization. The equations of motion (3.4.15), (3.4.19), and (3.4.22) can be presented as

$$(\nabla_A^2 + 1)(b_\mu^{(1m)} - B_\mu^{(1m)}) = 2\nabla_\mu \tilde{b}^{(1m)}, \tag{3.4.53}$$

$$(\nabla_A^2 - 5)(b_\mu^{(1m)} + 2B_\mu^{(1m)}) = -4\nabla_\mu \tilde{b}^{(1m)}, \tag{3.4.54}$$

$$(\nabla_A^2 - 2)\tilde{b}^{(1m)} = 2\nabla^\mu B_\mu^{(1m)}. \tag{3.4.55}$$

Upon dualization to 2D scalars the right hand side of (3.4.53) is manifestly longitudinal so for the perpendicular component $(b_\perp^{(1m)} - B_\perp^{(1m)})$ only the left hand side remains. Taking the curvature terms into account we find that this mode is massless, as indicated in the table. However, recall that in (3.2.12) we explicitly defined dualization of a 2D vector such that dual components do not satisfy the harmonic condition. This mode is therefore disallowed except if the longitudinal mode $(b_\parallel^{(1m)} - B_\parallel^{(1m)})$ is massless as well. In that event the two modes are interpreted together as a single harmonic mode. This harmonic mode forces vanishing $\tilde{b}^{(1m)}$ and this in turn decouples the vector mode $(b_\mu^{(1m)} + 2B_\mu^{(1m)})$. We interpret the massless $(b_\perp^{(1m)} - B_\perp^{(1m)})$ as a harmonic mode in this strong sense.

We next consider the gauge conditions at $l = 1$

$$H_+^{(1m)} - 2\tilde{B}_\parallel^{(1m)} - \frac{1}{2}\pi^{(1m)} = 0, \tag{3.4.56}$$

$$\frac{1}{2}H_\times^{(1m)} - \tilde{B}_\perp^{(1m)} = 0, \tag{3.4.57}$$

$$b_\parallel^{(1m)} + 2B_\parallel^{(1m)} - \tilde{b}^{(1m)} = 0. \tag{3.4.58}$$

$$4\tilde{B}_\parallel^{(1m)} + 2b^{(1m)} - \frac{1}{2}H^{(1m)}{}_\rho{}^\rho = 0, \tag{3.4.59}$$

$$\nabla^\mu B_\mu^{(1m)} = 0. \tag{3.4.60}$$

With the exception of (3.4.60), these are the continuations to $l = 1$ of the higher l conditions (3.4.45)-(3.4.49). The derivation of (3.4.60) is different from the one of (3.4.49) only in that the equations of motion were not used to simplify it so we simply revert to (3.4.40).

If we proceed to dualize the gauge condition (3.4.60) we find that $B_{\parallel}^{(1m)}$ is harmonic which we have disallowed. Thus $B_{\parallel}^{(1m)} = 0$ and so the condition (3.4.58) becomes a condition on the massless mode $b_{\parallel}^{(1m)} + B_{\parallel}^{(1m)} - \tilde{b}^{(1m)}$ in addition to the massive mode $b_{\parallel}^{(1m)} + 2B_{\parallel}^{(1m)} - \tilde{b}^{(1m)}$.

On the other hand we may dualize $B_{\mu}^{(1m)}$ to the true harmonic mode that is shared between $B_{\parallel}^{(1m)}$ and $B_{\perp}^{(1m)}$. This mode satisfies the gauge condition since in this sector we have the constraint $(b_{\mu}^{(1m)} + 2B_{\mu}^{(1m)}) = 0$ and so $B_{\mu}^{(1m)}$ has vanishing divergence as well as vanishing curl. In Table 3.8 we have collected the 5 towers of modes that we project out

Gauge Violating Modes	Mass
$H_{+}^{(1m)} - 2\tilde{B}_{\parallel}^{(1m)} - \frac{1}{2}\pi^{(1m)}$	$m^2 = 4$
$H_{\times}^{(1m)} - 2\tilde{B}_{\perp}^{(1m)}$	$m^2 = 4$
$b_{\parallel}^{(1m)} + 2B_{\parallel}^{(1m)} - \tilde{b}^{(1m)}$	$m^2 = 2$
$4\tilde{B}_{\parallel}^{(1m)} + 2b^{(1m)} - \frac{1}{2}H^{(1m)}{}_{\rho}{}^{\rho}$	$m^2 = 0$
$b_{\parallel}^{(1m)} + B_{\parallel}^{(1m)} - \tilde{b}^{(1m)}$	$m^2 = 0$

Table 3.8: Gauge violating modes at $l = 1$.

due to the gauge conditions. They are themselves gauge invariant. Among the remaining 7 towers there are 5 that we can present as pure gauge. The longitudinal modes are given in Table 3.9. The modes in the third and fourth line of Table 3.9 were generalized eigenvectors

Longitudinal Modes	Mass	Symmetry variation
$4\tilde{B}_{\parallel}^{(1m)} + \pi^{(1m)}$	$m^2 = 4$	$4\xi_{\parallel}^{(1m)}$
$4\tilde{B}_{\perp}^{(1m)}$	$m^2 = 4$	$4\xi_{\perp}^{(1m)}$
$b_{\parallel}^{(1m)} + 2B_{\parallel}^{(1m)}$	$m^2 = 2$	$\lambda^{(1m)} + 2\xi^{(1m)}$
$4b^{(1m)} + \pi^{(1m)}$	$m^2 = 0$	$-6\zeta^{(1m)}$
$b_{\perp}^{(1m)} - B_{\perp}^{(1m)}$	$m^2 = 0$	$2\xi^{(1m)}$

Table 3.9: Logitudinal modes at $l = 1$.

before gauge conditions were imposed but they are now true eigenvectors.

The fifth line refers to the harmonic mode that can be presented either perpendicular or longitudinal form. The longitudinal form can obviously be presented as a pure diffeomorphism. However, the parameter ξ is itself harmonic for $l = 1$ so this symmetry can also be recast in perpendicular form. These presentations are entirely equivalent.

The fourth and fifth line in the table both correspond to modes generated by S^2 diffeomorphisms (with a compensating gauge transformation to keep $\lambda^{(1m)} + 2\xi^{(1m)}$ fixed). Neither of these $l = 1$ modes are smooth continuations of the towers that apply for larger values of l. The last one is the mode that is physical if it is harmonic since then it is formally pure gauge but with non-normalizable gauge function.

The two remaning towers of modes satisfy the gauge conditions and they are not pure gauge. The gauge invariant form of these physical towers are the continuations from higher l; they are collected in Table 3.10.

Physical Modes	Mass
$\pi^{(1m)} - 2b^{(1m)}$	$m^2 = 6$
$b_\perp^{(1m)} + 2B_\perp^{(1m)}$	$m^2 = 6$

Table 3.10: Physical modes at $l = 1$.

3.4.6 l=0 Modes.

The $l = 0$ sector is the truncation of gravity and a vector field to the spherically symmetric sector. It is instructive to analyze this sector in detail.

Prior to any dualization the 2D field content is the 2D graviton $H^{(00)}_{\{\mu\nu\}}$, the AdS$_2$ volume mode $H^{(00)}{}_\rho{}^\rho$, the S^2 volume mode $\pi^{(00)}$, and the 2D gauge field $b_\mu^{(00)}$. There is a total of 6 component fields. The three continuous symmetries generated by gauge symmetry $\lambda^{(00)}$ and the AdS$_2$ diffeomorpisms $\xi_\mu^{(00)}$ are each expected to gauge one component field away and require another to vanish due to a constraint. Thus we expect no physical degrees of freedom in the $l = 0$ sector.

We first consider the equations of motion

$$(\nabla_A^2 + 1) b_\mu^{(00)} = 0, \tag{3.4.61}$$

$$(\nabla_A^2 + 2) H_{\{\mu\nu\}}^{(00)} = 0, \tag{3.4.62}$$

$$(\nabla_A^2 - 2) \pi^{(00)} = 0, \tag{3.4.63}$$

$$(\nabla_A^2 - 2) H^{(00)}{}_\rho{}^\rho + 4\pi^{(00)} = 0. \tag{3.4.64}$$

There is no mixing between the gauge field $b_\mu^{(00)}$ and the gravity modes so we can treat them separately.

The gauge field sector is simply 2D QED. Dualizing the scalars as in (3.2.9) the gauge fixed equation of motion (3.4.61) amounts to two harmonic equations for the dualized scalars $b_\parallel^{(00)}$ and $b_\perp^{(00)}$.

$$\nabla_A^2 b_\parallel^{(00)} = \nabla_A^2 b_\perp^{(00)} = 0. \tag{3.4.65}$$

Once again, recall that we define the scalars dual to vector fields requiring that they do not satisfy the harmonic condition (3.2.12). Both these modes therefore vanish on shell. However, since the equations of motion coincide with the harmonic equation, the harmonic mode $b_{\mu 0}^{(00)} = \nabla_\mu b_0^{(00)}$ does in fact satisfy the equations of motion. This is special to the $l = 0$ sector.

We proceed similarly for the gravity modes described by the symmetric traceless tensor $H_{\{\mu\nu\}}^{(00)}$. We must again take extra care when dualizing. According to (3.4.25) we can dualize to two scalars $H_+^{(00)}, H_\times^{(00)}$ which cannot satisfy the generalized harmonic condition

$$\nabla_A^2 (\nabla_A^2 - 2) X = 0, \tag{3.4.66}$$

and one harmonic scalar $H_0^{(00)}$ that must satisfy this equation.

Inserting the expansion (3.4.25) of $H_{\{\mu\nu\}}^{(00)}$ into (3.4.62) we find that the equations of motion for the two dual scalars $H_+^{(00)}$ and $H_\times^{(00)}$ are precisely the generalized harmonic condition. These modes must therefore must vanish on shell.

However, again we find that since the equations of motion coincide with the harmonic equation, the harmonic mode $H_{\{\mu\nu\}}^{(00)} = \nabla_{\{\mu}\nabla_{\nu\}}H_0^{(00)}$ with $H_0^{(00)}$ satisfying (3.4.66) does in fact satisfy the equations of motion.

The remaining two modes are $H^{(00)}{}_\rho{}^\rho$ and $\pi^{(00)}$. These are already scalars so we do not have to worry about any dualization. The equations of motion (3.4.63)-(3.4.64) indicate that these scalars have $m^2 = 2$. Indeed, they are equivalent to a single "weight-two" scalar with $m^2 = 2$ and satisfying

$$(\nabla_A^2 - 2)^2 H^{(00)}{}_\rho{}^\rho = 0. \tag{3.4.67}$$

Either way, both these scalars remain after the gauge fixed equations of motion are imposed.

Summarizing so far, the fields that are on-shell at $l = 0$ are the harmonic scalar $b_0^{(00)}$ dual to the 2D gauge field, the generalized harmonic scalar $H_0^{(00)}$ dual to the traceless symmetric tensor, and the two scalars $H^{(00)}{}_\rho{}^\rho$ and $\pi^{(00)}$ with $m^2 = 2$.

The 4D gauge condition for diffeomorphisms (3.4.37) simplifies at $l = 0$ to

$$\nabla^\mu H_{\{\mu\nu\}}^{(00)} = \frac{1}{2}\nabla_\nu \pi^{(00)}. \tag{3.4.68}$$

We insert (3.4.28) into (3.4.68), giving the condition

$$\nabla^\mu(\nabla_A^2 - 2)H_0^{(00)} = \frac{1}{2}\nabla_\nu \pi^{(00)}, \tag{3.4.69}$$

We can contract with ∇^ν and find $\nabla_A^2\pi^{(00)} = 0$ in view of the generalized harmonic condition on $H_0^{(00)}$. This conflicts with the equation of motion (3.4.63) so we conclude that $\pi^{(00)} = 0$ after the equations of motion and the gauge condition have been imposed. Further, the gauge condition (3.4.69) then projects on to the $m^2 = 2$ component of $H_0^{(00)}$.

The dualization of the on-shell physical fields $H_{\{\mu\nu\}}^{(00)}$ and $b_\mu^{(00)}$ manifestly presents them as pure gauge. The AdS$_2$-volume $H^{(00)}{}_\rho{}^\rho$ mode is also pure gauge with gauge function chosen such that

$$H^{(00)}{}_\rho{}^\rho = 2\nabla_\rho \xi^{\rho(00)}. \tag{3.4.70}$$

Since $H^{(00)}{}_\rho{}^\rho$ has $m^2 = 2$ the harmonic component of $\xi^{\rho(00)}$ can play no role here. We dualize $\xi^{(00)}_\rho = \nabla_\rho \xi^{(00)}_\parallel$ where $\xi^{(00)}_\parallel$ also has $m^2 = 2$ as already found in (3.4.51). We therefore have

$$H^{(00)}{}_\rho{}^\rho = 2\nabla^2_A \xi^{(00)}_\parallel = 4\xi_\parallel. \tag{3.4.71}$$

on-shell. In particular, it is manifest that all normalizable $H^{(00)}{}_\rho{}^\rho$ are generated by normalizable gauge functions.

In summary, the only physical modes at $l = 0$ are the harmonic modes $b^{(00)}_0$, $H^{(00)}_0$. These modes are pure gauge so we find that in this sector gauge symmetries remove all fields (at least formally). This is the expected result.

3.4.7 Boundary Modes.

As we have stressed, special care must be taken when the dualization of vector or tensor fields gives rise to harmonic modes.

An important example of this situation is a 2D vector field that satisfies (3.2.13)

$$(\nabla^2_A + 1) C_\mu = 0, \tag{3.4.72}$$

since then the dual scalar field X satisfies the harmonic equation $\nabla^2_A X = 0$. In this case the gradient and curl versions of dualization are equivalent so only one of these configurations should be counted.

There are three 2D vector fields in our setting. Their equations of motion simplify when we focus on harmonic fields since those are divergence free and so their couplings to gradients of scalars can be consistently ignored. With these simplifications (3.4.21) becomes

$$(\nabla_A^2 - l(l+1)+1)\tilde{B}_\mu^{(lm)} = 2\tilde{B}_\mu, \tag{3.4.73}$$

and (3.4.15), (3.4.22) combine to

$$(\nabla_A^2 - l(l+1)+1)\begin{pmatrix} B_\mu^{(lm)} \\ \\ b_\mu^{(lm)} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ \\ 2l(l+1) & 0 \end{pmatrix}\begin{pmatrix} B_\mu^{(lm)} \\ \\ b_\mu^{(lm)} \end{pmatrix}. \tag{3.4.74}$$

We must in addition consider the 2D tensor $H_{\{\mu\nu\}}^{(lm)}$ with equations of motion (3.4.17).

For bulk modes we define mass as the value needed to satisfy the on-shell condition $(-\nabla_A^2 + m^2)X = 0$ with the understanding that eventually we will go off-shell and consider all eigenvalues of the AdS_2 Laplacian $-\nabla_A^2$. This strategy fails for boundary modes since the harmonic equation determines the AdS_2 wave function completely from the outset and so the only option will be to go off-shell on S^2. We will instead record the spectrum of boundary modes as the eigenvalue of the harmonic operator $(\nabla_A^2 + 1)C_\mu = m^2 C_\mu$ for vectors and $(\nabla_A^2 + 2)H_{\{\mu\nu\}}^{(lm)} = m^2 H_{\{\mu\nu\}}^{(lm)}$ for tensors. For boundary modes the "mass" becomes a measure of the distance off-shell along S^2. With this terminology we find the spectrum summarized in Table 3.11. The symmetries of the theory include the tower of 2D diffeomorphisms

Boundary Mode	Mass	Range
$\tilde{B}_\mu^{(lm)}$	$m^2 = l(l+1)+2$	$l = 1, 2\ldots$
$b_\mu^{(lm)} - lB_\mu^{(lm)}$	$m^2 = l(l-1)$	$l = 0, 1\ldots$
$b_\mu^{(lm)} + (l+1)B_\mu^{(lm)}$	$m^2 = (l+1)(l+2)$	$l = 1, 2\ldots$
$H_{\{\mu\nu\}}^{(lm)}$	$m^2 = l(l+1)$	$l = 0, 1\ldots$
$\xi_\mu^{(lm)}, c_\mu^{(lm)}, \tilde{c}_\mu^{(lm)}$	$m^2 = l(l+1)+2$	$l = 0, 1\ldots$

Table 3.11: Boundary mode spectrum.

$\xi_\mu^{(lm)}$. These are 2D vectors so their dualization is also delicate. The residual symmetries remaining after gauge fixing satisfy (3.4.51), which serves as their equation of motion. We have included these modes in our table along with the ghosts $c_\mu^{(lm)}$ and anti-ghosts $\tilde{c}_\mu^{(lm)}$ that satisfy the same equations of motion.

We have not yet specified which modes violate the gauge conditions nor have we determined which modes are pure gauge. In the BRST formalism both of these are anyway cancelled by the ghosts and antighosts. The net effect is that the last line in the table (one tower of modes and two ghost towers) cancels the first line in the table (one tower of modes) *except* for one mode at $l = 0$ that counts with negative sign. The $l = 0$ is the spherical reduction of Einstein-Maxwell which is known to have confusing features in $\mathrm{AdS}_2 \times S^2$. In the present set-up there is -1 mode at $l = 0$ as one expects from an overconstrained system [68].

We can be more explicit about this. When the 2D diffeomorphisms $\xi_\mu^{(lm)}$ are harmonic they can be dualized to a massless scalar that is not normalizable but such that the vector field itself is normalizable and therefore generates a true symmetry. We can use this symmetry to gauge away the metric components $h_{\mu\alpha}$ with mixed indices on AdS_2 and S^2. This justifies a physical on-shell approach that simply omits $\tilde{B}_\mu^{(lm)}$ and $\xi_\mu^{(lm)}$ from the outset and never introduces ghosts.

In AdS_2 the effective mass is related to conformal weight through $m^2 = h(h-1)$. We find that all physical boundary modes have integral conformal weights.

The dualization of the tensor $H_{\{\mu\nu\}}^{(lm)}$ is less familiar. The harmonic tensors introduced in (3.4.25) are formally pure gauge generated by a diffeomorphism that can be dualized to a scalar H_0 that satisfies $\nabla_A^2(\nabla_A^2 - 2)H_0 = 0$. We can interpret such scalar field as two independent scalars with masses $m^2 = 0$ and $m^2 = 2$. The $m^2 = 0$ component corresponds to non-normalizable scalars that generate a normalizable diffeomorphism. These are precisely the boundary modes that were cancelled two paragraphs ago. On the other hand, the $m^2 = 2$ component corresponds to non-normalizable scalar modes that generate non-normalizable diffeomorphisms V_μ. However, these non-normalizable diffeomorphisms in turn generate normalizable tensors $H_{\{\mu\nu\}} = \nabla_\mu V_\nu - \nabla_\nu V_\mu - g_{\mu\nu}\nabla^\lambda V_\lambda$. These are physical fields on AdS_2 even though they are formally pure gauge. As we discuss in Appendix B, the summation over all modes again produces a volume factor but also a multiplicity factor of three. The tensor thus has *three* boundary modes.

The $m^2 = 2$ condition on the scalars H_0 imply that the non-normalizable vector modes V_μ satisfy

$$(\nabla_A^2 - 1)V_\mu = 0. \tag{3.4.75}$$

Interestingly, the definition of Conformal Killing Vectors on AdS_2 imply this equation. However, the CKVs are precisely those that generate a trivial $H^{(lm)}_{\{\mu\nu\}}$ so the non-normalizable vector modes V_μ are the solutions to (3.4.75) that are not CKVs on AdS_2.

We introduced the notion of mass for boundary modes as a measure of off-shellness on S^2. Thus only the $m^2 = 0$ modes are truly on-shell. In the $b^{(lm)}_\mu - lB^{(lm)}_\mu$ tower $l = 0$ is the mode that is formally pure gauge but with non-normalizable gauge function. For $l = 0$ this mode does not mix with gravity and so "gauge" really refers to the gauge field and the problem reduces to the spectator vector field discussed in section 3.2. The $l = 1$ mode in the same tower is also massless and again it is formally pure gauge with non-normalizable gauge function. However, the symmetry is a 2D diffeomorphism accompanied by a compensating gauge transformation such that $b^{(1m)}_\mu + 2B^{(1m)}_\mu$ is fixed. Specifically this mode is the Conformal Killing Vector $\nabla^\alpha Y_{(1m)}$ on S^2 with a compensating gauge transformation so the gauge field a^α is left invariant.

The analogous relation between $l = 0$ tensors $H^{(00)}_{\{\mu\nu\}}$ and 2D diffeomorphisms was discussed above so all the on-shell boundary modes are related to symmetries. These modes were all previously identified in the discussion of the special cases $l = 1$ and $l = 0$. We can interpret the full towers of boundary modes as the off-shell realization of these symmetries. This extrapolation to general partial wave number l is nontrivial because of mixing between modes.

3.5 Quantum Corrections to $\mathrm{AdS}_2 \times S^2$ - Bosonic Sector.

Quantum corrections depend only on the spectrum rather than the explicit modes. We consider in turn the contributions from the physical states, the unphysical states, the boundary modes, and the zero modes. We then add the contributions to find the complete heat kernel.

3.5.1 Physical States.

The bosonic physical spectrum is given below in Table 3.12. In each entry the mass refers

Mass	Multiplicity	Range
$m^2 = l(l-1)$	2	$l = 2, 3 \ldots$
$m^2 = (l+1)(l+2)$	2	$l = 1, 2 \ldots$

Table 3.12: Physical bosonic spectrum.

to the value of m^2 such that $(-\nabla_A^2 + m^2)X = 0$ is the on-shell condition. The bulk result we present agrees with [40, 42, 41] [2]. Quantum corrections necessarily consider modes that are off-shell. For modes with $m^2 = 0$ there is a continuous spectrum off-shell with eigenvalues $\lambda \geq \frac{1}{4}$ for the Euclidean operator $(-\nabla_A^2)$. The contributions from this continuous spectrum on AdS_2 is encoded in the AdS_2 heat kernel (3.3.3). We subsequently sum over the four towers of modes on S^2 using (3.3.4). This gives

$$\begin{aligned} K_4^{\text{bulk},b}(s) &= 2K_A^s(s) \cdot \frac{1}{4\pi a^2} \cdot \left(\sum_{l=2}^{\infty} e^{-sl(l-1)}(2l+1) + \sum_{l=1}^{\infty} e^{-s(l+2)(l+1)}(2l+1) \right) \qquad (3.5.1) \\ &= K_A^s(s) \cdot \frac{1}{\pi a^2} \left(\sum_{l=0}^{\infty} e^{-sl(l+1)}(2l+1) - 1 - \frac{1}{2}e^{-2s} \right) \\ &= \frac{1}{4\pi^2 a^4 s^2} \left(1 - \frac{3}{2}s + \frac{137}{90}s^2 + \ldots \right) \end{aligned}$$

[2] Except that we find the S^2 volume mode $\pi^{(00)}$ to be unphysical. This discrepancy was stressed in chapter 1

3.5.2 Unphysical States.

The full spectrum of modes include some that violate the gauge condition and others that are pure gauge. These two groups of modes coincide precisely. Each has the spectrum in Table 3.13. In our physical quantization scheme we simply omit these modes. They are not

Mass	Multiplicity	Range
$m^2 = l(l+1)+2$	2	$l = 0, 1 \ldots$
$m^2 = l(l+1)$	1	$l = 0, 1 \ldots$
$m^2 = l(l+1)-2$	2	$l = 1, 2 \ldots$

Table 3.13: Unphysical bosonic spectrum.

allowed even virtually so they do not run in loops.

In standard covariant quantization we would instead impose the gauge condition and then argue using Ward identities that the pure gauge modes decouple. The upshot will be that indeed these states give no net contribution to the quantum corrections. This structure is of course expected but our construction provides explicit details.

Similarly, in BRST quantization we allow all the modes and then include b and c-ghosts that contribute with opposite sign. These ghost modes will have exactly the same spectrum because they are essentially the pure gauge modes (and their dual constraints). Again there will be no net contribution to the quantum corrections.

The unphysical modes with $m^2 = 0$ are special and they are worth discussing. They are the harmonic gauge mode $b_0^{(00)}$, the Conformal Killing Vector on S^2 generated by $\zeta^{(1m)}$ and the Killing Vector on S^2 generated by $\xi^{(1m)}$. Each is a harmonic mode $\nabla_A^2 X = 0$ on AdS$_2$. The standard covariant quantization above implicitly realizes each of these harmonic modes in both their gradient and curl form. In the off-shell theory these two forms are not equivalent so the two members of the pair are distinct field configurations. Each is equivalent to a massless scalar and the two contributions cancel just as they do for higher l.

The harmonic modes and the Killing Vector on S^2 ultimately give boundary states and those we treat differently (in the next subsection). One may therefore object as a matter of principle that the harmonic modes should not be included among the unphysical modes. This question is an ambiguity in the quantization scheme that does not have a "correct" resolution since no physical quantity will depend on it.

3.5.3 Boundary Modes.

Each boundary mode receives the constant contribution (3.3.8) from the AdS_2 part. This must be multiplied by the S^2 tower using (3.3.4). The harmonic modes from the two mixed/gravity towers $b_\mu^{(lm)}, B_\mu^{(lm)}$ combine to give

$$\begin{aligned} K_4^{\text{mix bndy},b}(s) &= \frac{1}{2\pi a^2}\cdot\frac{1}{4\pi a^2}\cdot\left(\sum_{l=0}^{\infty} e^{-sl(l-1)}(2l+1) + \sum_{l=1}^{\infty} e^{-s(l+2)(l+1)}(2l+1)\right) \\ &= \frac{1}{8\pi^2 a^4}\left(2\sum_{l=0}^{\infty} e^{-sl(l+1)}(2l+1) + 2 - e^{-2s}\right) \end{aligned} \tag{3.5.2}$$

The harmonic modes from pure gravity reside in the tensors $H_{\{\mu\nu\}}$ (which count with weight three) and in the almost cancelling towers $\tilde{B}_\mu^{(lm)}, \xi_\mu^{(lm)}$. These contributions combine to give

$$\begin{aligned} K_4^{\text{grav bndy},b}(s) &= \frac{1}{2\pi a^2}\cdot\frac{1}{4\pi a^2}\cdot\left(3\sum_{l=0}^{\infty} + \sum_{l=1}^{\infty} e^{-2s} - \sum_{l=0}^{\infty} e^{-2s}\right)(2l+1)e^{-sl(l+1)} \\ &= \frac{1}{8\pi^2 a^4}\left(3\sum_{l=0}^{\infty}(2l+1)e^{-sl(l+1)} - e^{-2s}\right). \end{aligned} \tag{3.5.3}$$

The sum of contributions from all bosonic boundary modes becomes

$$\begin{aligned} K_4^{\text{bndy},b}(s) &= \frac{1}{8\pi^2 a^4}\left(5\sum_{l=0}^{\infty}(2l+1)e^{-sl(l+1)} + 2 - 2e^{-2s}\right) \\ &= \frac{1}{8\pi^2 a^4 s}\cdot 5\left(1 + \frac{1}{3}s + \frac{13}{15}s^2 + \cdots\right). \end{aligned} \tag{3.5.4}$$

Ultimately we only need the first two orders. At that precision the boundary modes are equivalent to five free scalar fields on S^2. The addition of $2 - 2e^{-2s}$ in the exact result introduces corrections at higher order.

3.5.4 Zero Modes.

Zero-modes are on-shell boundary modes. They are

- The pure gauge mode $b_\mu^{(00)}$.
- The modes $b_\mu^{(1m)} - B_\mu^{(1m)}$ (with compensating gauge transformation so $b_\mu^{(1m)} + 2B_\mu^{(1m)}$ is fixed) are due to Killing Vectors on S^2. These are in the $l = 1$ sector so there are $2l + 1 = 3$ modes of this kind.
- The on-shell modes $H_{\mu\nu}^{(00)}$ are generated by 2D diffeomorphisms on AdS_2. The sum over these modes give a multiplicity factor of 3.

The zero-modes require special considerations because they are not damped in the Euclidean path integral. As explained in detail by Sen and collaborators, they can be incorporated by a change of variable to the corresponding symmetry parameter [36, 9, 10]. For gauge symmetry it turns out that the naive treatment is correct but for diffeomorphisms the zero modes were undercounted by a factor of two. Each of our $3 + 3 = 6$ zero modes that are due to gravity already contributed $\frac{1}{8\pi^2 a^4}$ but this should be multiplied by two. This correction contributes

$$K_4^{\mathrm{zm},b} = \frac{1}{8\pi^2 a^4} \cdot 6, \tag{3.5.5}$$

to the heat kernel.

3.5.5 Summary.

Adding contributions from bulk (4D), boundary (2D), and the zero-modes (0D) we find

$$K_4^b(s) = \frac{1}{4\pi^2 a^4 s^2}(1 + s + \frac{241}{45}s^2 + \cdots). \tag{3.5.6}$$

as the total contributions from bosonic modes.

3.6 Supergravity in $\mathrm{AdS}_2 \times S^2$ - Fermionic Sector.

In this section we analyze the two gravitini in $\mathcal{N} = 2$ supergravity in $\mathrm{AdS}_2 \times S^2$. We derive the equations of motion in AdS_2 point of view via a partial wave expansion and diagonalize them. Only then do we fix the gauge and identify longitudinal states.

3.6.1 4D Theory.

The matter content is a pair of Majorana gravitino fields Ψ_{IA}, where $A = 1, 2$ is an R index. The action for the 4D gravitini is

$$\mathcal{L} = -\bar{\Psi}_{AI}\Gamma^{IJK}D_J\Psi_{AK} + \frac{1}{2}\bar{\Psi}_{AI}\left(F^{IJ}_{AB} + \frac{1}{2}\Gamma^{IJKL}F_{AB,KL}\right)\Psi_{BJ}. \tag{3.6.1}$$

We do not bother matching upstairs and downstairs indices when summing over A, B. We work with a magnetic background that couples differently to each of the 4D gravitini, so we incorporate index structure in A, B: $F^{\alpha\beta}_{AB} = 2\epsilon_{AB}\epsilon^{\alpha\beta}$.

The supersymmetry that leaves the Lagrangian (3.6.1) invariant is

$$\delta\Psi_{AI} = \left(\delta_{AB}D_I - \frac{1}{4}\hat{F}_{AB}\gamma_I\right)\theta_B, \tag{3.6.2}$$

for some arbitrary spinor θ_B.

We vary the Lagrangian to obtain the 4D equation of motion,

$$\Gamma^{IJK}D_J\Psi_{AK} - \frac{1}{2}(F^{IJ}_{AB} + \frac{1}{2}\Gamma^{IJKL}F_{AB,KL})\Psi_{JB} = 0. \tag{3.6.3}$$

We split the AdS_2 and S^2-components of the equations of motion, rewrite them in terms of the 2D gamma matrices $\gamma^\mu, \gamma^\alpha$, and use the expression for the background field strength.

Our conventions are summarized in Appendix C. The result is

$$
\begin{aligned}
&\gamma^{\mu\nu} \otimes \gamma^{\alpha} D_{\nu} \Psi_{A\alpha} - \gamma^{\mu\nu} \otimes \gamma^{\alpha} D_{\alpha} \Psi_{A\nu} + \gamma^{\mu} \otimes \gamma^{\alpha\beta} \gamma_S D_{\alpha} \Psi_{A\beta} + i\gamma^{\mu\nu} \otimes \gamma_S \epsilon_{AB} \Psi_{B\nu} = 0, \\
&\quad \gamma^{\mu} \otimes \gamma^{\alpha\beta} \gamma_S D_{\beta} \Psi_{A\mu} - \gamma^{\mu} D_{\mu} \otimes \gamma^{\alpha\beta} \gamma_S \Psi_{A\beta} + \gamma^{\mu\nu} D_{\mu} \otimes \gamma^{\alpha} \Psi_{A\nu} - \epsilon_{AB} \epsilon^{\alpha\beta} \Psi_{B\beta} = 0. \quad (3.6.4)
\end{aligned}
$$

Each term is written explicitly as a tensor product to stress that the gamma matrices in AdS and the sphere are in different Clifford algebras and therefore commute. The matrix γ_S is the sphere analog of Γ_5.

3.6.2 Partial Wave Expansion.

We denote spherical spinors with definite angular momentum quantum number $\eta_{(\sigma lm)}$. The index $\sigma = \pm$ labels the two components of $\eta_{(\sigma lm)}$. A complete set of complex spinors on S^2 is then given by $\eta_{(\sigma lm)}$ and $\gamma_S \eta_{(\sigma lm)}$ [46, 69] satisfying

$$
\begin{gathered}
\gamma^{\alpha} D_{\alpha} \eta_{(\sigma lm)} = i(l+1) \eta_{(\sigma lm)}, \\
l = 0, 1...
\end{gathered} \quad (3.6.5)
$$

We expand the gravitino wavefunction in spinor spherical harmonics according to

$$
\Psi_{A\mu} = \Psi^{(\sigma lm)}_{+A\mu} \otimes \eta_{(\sigma lm)} + \Psi^{(\sigma lm)}_{-A\mu} \otimes \gamma_S \eta_{(\sigma lm)}, \quad (3.6.6)
$$

$$
\begin{aligned}
\Psi_{A\alpha} = \Psi^{(\sigma lm)}_{+A} \otimes D_{(\alpha)} \eta_{(\sigma lm)} + \Psi^{(\sigma lm)}_{-A} \otimes D_{(\alpha)} \gamma_S \eta_{(\sigma lm)} \\
+ \chi^{(\sigma lm)}_{+A} \otimes \gamma_{\alpha} \eta_{(\sigma lm)} + \chi^{(\sigma lm)}_{-A} \otimes \gamma_{\alpha} \gamma_S \eta_{(\sigma lm)}.
\end{aligned} \quad (3.6.7)
$$

We expanded the vector index on the gravitino along the sphere in the basis

$$
D_{(\alpha)} \eta_{(\sigma lm)}, \quad \gamma_{\alpha} \eta_{(\sigma lm)}, \quad (3.6.8)
$$

where

$$D_{(\alpha)} = D_\alpha - \frac{1}{2}\gamma_\alpha\gamma^\beta D_\beta. \tag{3.6.9}$$

The spinors $D_{(\alpha)}\eta_{(\sigma lm)}$ and $\gamma_\alpha\eta_{(\sigma lm)}$ pick out the spin-3/2 part and the spin-1/2 part of the Rarita-Schwinger field on S^2. The spin-3/2 part is not defined for $l = 0$ so the AdS$_2$ field $\Psi_{\pm A}$ is only defined for $l \geq 1$.

Complex conjugation is given by

$$\eta^*_{(\sigma lm)} = i\sigma\gamma_S\eta_{(-\sigma lm)}. \tag{3.6.10}$$

The 4D fields Ψ_{IA} are Majorana and thus (3.6.10) gives the conjugation property

$$(\Psi^{(\sigma lm)}_{\pm\mu A})^* = \mp i\sigma\Psi^{(-\sigma lm)}_{\mp\mu A}. \tag{3.6.11}$$

The components $\Psi_{\pm A}$ and $\chi_{\pm A}$ transform analogously.

3.6.3 Equations of Motion: 2D Theory.

We now insert the spinor harmonic expansion (3.6.6) and (3.6.7) into the 4D equations of motion (3.6.4). We drop the spinor harmonic indices (σlm) to simplify the notation.

We contract the $I = \mu$ equation of motion in (3.6.4) with $\gamma_{\rho\mu}$ then insert the expansion in spinor harmonics.

$$\begin{aligned} 0 = &\left(2D_\mu\chi_{-A} + i(l+1)\Psi_{-\mu A} + \frac{1}{2}((l+1)^2-1)\gamma_\mu\Psi_{+A} + i(l+1)\gamma_\mu\chi_{+A} + i\epsilon_{AB}\Psi_{+\mu B}\right)\otimes\gamma_S\eta \\ &+\left(2D_\mu\chi_{+A} - i(l+1)\Psi_{+\mu A} + \frac{1}{2}((l+1)^2-1)\gamma_\mu\Psi_{-A} - i(l+1)\gamma_\mu\chi_{-A} + i\epsilon_{AB}\Psi_{-\mu B}\right)\otimes\eta. \end{aligned} \tag{3.6.12}$$

There is an obvious redundancy in this equation, since the first line is related to the second through complex conjugation. We multiply (3.6.12) by $(\gamma_S\eta)^\dagger$ and integrate over the sphere coordinates to find

$$0 = 2D_\mu \chi_{-A} + i(l+1)\Psi_{-\mu A} + \frac{1}{2}((l+1)^2 - 1)\gamma_\mu \Psi_{+A} + i(l+1)\gamma_\mu \chi_{+A} + i\epsilon_{AB}\Psi_{+\mu B}. \quad (3.6.13)$$

These are the 2D equations of motion. We could alternatively have multiplied by $\eta^\dagger$ and kept the second line of (3.6.12).

The procedure is repeated for the $I = \alpha$ equations of motion (the second equation in (3.6.4)). The difference is that the sphere dependent part now carries a vector index. We find

$$\begin{aligned} 0 &= (-\gamma^\mu \Psi_{+A\mu} + \gamma^\mu D_\mu \Psi_{+A} + i\epsilon_{AB}\Psi_{+B}) \otimes D^{(\alpha)}\gamma_S \eta \\ &+ (-\gamma^\mu \Psi_{-A\mu} + \gamma^\mu D_\mu \Psi_{-A} + i\epsilon_{AB}\Psi_{-B}) \otimes D^{(\alpha)}\eta \\ &+ \left(-\frac{i}{2}(l+1)\gamma^\mu \Psi_{+A\mu} + \gamma^\mu D_\mu \chi_{+A} + \gamma^{\mu\nu} D_\mu \Psi_{-A\nu} + i\epsilon_{AB}\chi_{+B}\right) \otimes \gamma^\alpha \gamma_S \eta \\ &+ \left(\frac{i}{2}(l+1)\gamma^\mu \Psi_{-A\mu} + \gamma^\mu D_\mu \chi_{-A} + \gamma^{\mu\nu} D_\mu \Psi_{+A\nu} + i\epsilon_{AB}\chi_{-B}\right) \otimes \gamma^\alpha \eta. \end{aligned} \quad (3.6.14)$$

The operators $D^{(\alpha)}$ and γ^α are orthogonal so we can project (3.6.14) and integrate over the sphere degrees of freedom,

$$0 = -\frac{i}{2}(l+1)\gamma^\mu \Psi_{+A\mu} + \gamma^\mu D_\mu \chi_{+A} + \gamma^{\mu\nu} D_\mu \Psi_{-A\nu} + i\epsilon_{AB}\chi_{+B} \quad (3.6.15)$$

$$0 = \frac{1}{2}[(l+1)^2 - 1]\Big[- \gamma^\mu \Psi_{+A\mu} + \gamma^\mu D_\mu \Psi_{+A} + i\epsilon_{AB}\Psi_{+B}\Big] \quad (3.6.16)$$

The prefactor $[(l+1)^2 - 1]$ in (3.6.16) stresses that this equation does not apply for $l = 0$. It is analogous to the overall factors of $l(l+1)$ present in some the bosonic sector equations of motion that were not defined at $l = 0$.

The complete equations of motion in AdS_2 are (3.6.13), (3.6.15), and (3.6.16). We will work for now with $l = 1, 2...$. The $l = 0$ components will be treated separately.

In order to decouple our equations of motion we define the combinations

$$\hat{\Psi}_{\mu A} = \Psi_{+\mu A} - i\Psi_{-\mu A}, \tag{3.6.17}$$
$$\hat{\Psi}_{A} = \Psi_{+A} - i\Psi_{-A},$$
$$\hat{\chi}_{A} = \chi_{+A} - i\chi_{-A},$$

and the conjugate fields

$$\tilde{\Psi}_{\mu A} \equiv \Psi_{+\mu A} + i\Psi_{-\mu A}, \tag{3.6.18}$$

with analogous relations defining $\hat{\Psi}_A$ and $\hat{\chi}_A$.

Complex conjugation in this basis is given by

$$(\hat{\Psi}^{(\sigma l m)}_{\mu A})^* = -\sigma \tilde{\Psi}^{(-\sigma l m)}_{\mu A}. \tag{3.6.19}$$

Where we restored the harmonic indices temporarily. The fields $\tilde{\Psi}_{\mu A}$ are related to $\hat{\Psi}_{\mu A}$ via complex conjugation according to (3.6.19). The fields $\tilde{\Psi}_{\mu A}$ present no new information.

By inspection of the equations of motion we see that the 2D Rarita-Schwinger field $\Psi_{\mu A}$ is dependent on the fields Ψ_A and χ_A. Hence, we use (3.6.13) to express $\Psi_{\mu A}$ in terms of the other modes and simplify the remaining equations (3.6.15) and (3.6.16).

Recall that the index A takes two values, and for each field such as $\Psi_{+\mu A}$ there is a complex conjugate $\Psi_{-\mu A}$. Thus, we are looking into four vector valued equations. It is somewhat tedious yet straight forward to write all four equations in components then solve for each $\Psi_{\pm\mu A}$. The result in the basis (3.6.17) is

$$\hat{\Psi}_{\mu A} = \frac{-i}{1-(l+1)^2}(-i(l+1)\delta_{AB} + \epsilon_{AB})\Big(- 2iD_\mu \tilde{\chi}_B - \frac{1-(l+1)^2}{2}\gamma_\mu \hat{\Psi}_B + i(l+1)\gamma_\mu \tilde{\chi}_B\Big), \tag{3.6.20}$$

and similarly for the conjugate field $\tilde{\Psi}_{\mu A}$. We will refer to (3.6.20) as the Rarita-Schwinger constraint. Note that it cannot be continued to $l = 0$ which we study separately.

We now insert the Rarita-Schwinger constraint (3.6.20) into the equations of motion (3.6.15) and (3.6.16). The first order derivative in (3.6.20) is acted on by further derivatives but the resulting second order term appears as a commutator that reduces to a curvature factor. The resulting equations are therefore of first order:

$$\left(\gamma^\mu D_\mu - (l+1)\right)\left[\hat{\Psi}_A + \frac{2}{(l+1)^2-1}(i(l+1)\delta_{AB} - \epsilon_{AB})\tilde{\chi}_B\right] = 0, \tag{3.6.21}$$

$$\left(\gamma^\mu D_\mu - (l+1)\right)(i(l+1)\delta_{AB} - \epsilon_{AB})\left[2\hat{\Psi}_B + \frac{1}{(l+1)^2-1}(i(l+1)\delta_{BC} - \epsilon_{BC})\tilde{\chi}_C\right] = 0. \tag{3.6.22}$$

The operator $(i(l+1)\delta_{AB} - \epsilon_{AB})$ can be inverted for $l \neq 0$, so we can decouple these into Dirac equations for $\hat{\Psi}_A$ and $\tilde{\chi}_A$:

$$\begin{aligned}(\gamma^\mu D_\mu - (l+1))\hat{\Psi}_A &= 0, \\ (\gamma^\mu D_\mu - (l+1))\tilde{\chi}_A &= 0.\end{aligned} \tag{3.6.23}$$

The conjugate equations similarly give

$$\begin{aligned}(\gamma^\mu D_\mu + (l+1))\tilde{\Psi}_A &= 0, \\ (\gamma^\mu D_\mu + (l+1))\hat{\chi}_A &= 0.\end{aligned} \tag{3.6.24}$$

At this point we have successfully decoupled all equations of motion with no constraints or gauge condition imposed.

3.6.4 Dualization.

We showed above that the field $\hat{\Psi}_{\mu A}$ is not independent from the spinors $\hat{\Psi}_A$ and $\tilde{\chi}_A$. However, we are going to fix a gauge and study supersymmetry variations that involve components of $\hat{\Psi}_{\mu A}$. So instead of throwing away the vector-spinors $\hat{\Psi}_{\mu A}$ we will dualize them into spinors in order to more precisely work with the Rarita-Schwinger constraint (3.6.20), gauge conditions, and variations.

We dualize $\hat{\Psi}_{\mu A}$ according to

$$\hat{\Psi}_{\mu A} = D_{(\mu)}\hat{\kappa}_A + \gamma_\mu \hat{\tau}_A. \tag{3.6.25}$$

Where $D_{(\mu)} = D_\mu - \frac{1}{2}\gamma_\mu\gamma^\nu D_\nu$. An analogous dualization is carried for $\tilde{\Psi}_{\mu A}$. Our field content is then the 16 components: $\hat{\kappa}_A, \hat{\tau}_A, \hat{\Psi}_A, \hat{\chi}_A$, with $A = 1, 2$ and their conjugates with tildes.

We can recast the Rarita-Schwinger constraint (3.6.20) as equations expressing the dual spinors introduced in (3.6.25) to other field components:

$$\begin{aligned}
\hat{\kappa}_A &= \frac{2}{1-(l+1)^2}(i(l+1)\delta_{AB} - \epsilon_{AB})\tilde{\chi}_B, \\
\hat{\tau}_A &= -\frac{i}{2}(i(l+1)\delta_{AB} - \epsilon_{AB})\hat{\Psi}_B, \\
\tilde{\kappa}_A &= \frac{2}{1-(l+1)^2}(i(l+1)\delta_{AB} + \epsilon_{AB})\hat{\chi}_B, \\
\tilde{\tau}_A &= \frac{i}{2}(i(l+1)\delta_{AB} + \epsilon_{AB})\tilde{\Psi}_B.
\end{aligned} \tag{3.6.26}$$

This is the dual form of the result that we can eliminate half of the initial field components and only work with the components $\hat{\Psi}_A, \tilde{\Psi}_A, \hat{\chi}_A, \tilde{\chi}_A$. This formulation will be useful in the following section.

3.6.5 Gauge Violating, Longitudinal, and Physical States.

We now impose Lorentz gauge on the on shell states we found and then construct pure gauge states.

The Lorentz gauge condition is $\Gamma^I \Psi_I = 0$. We write it in terms of 2D gamma matrices, insert the expansion of Ψ_I in spherical spinors, and dualize according to (3.6.25). The gauge condition in terms of 2D spinors is

$$\hat{\tau}_A - i\tilde{\chi}_A = 0, \qquad (3.6.27)$$
$$\tilde{\tau}_A + i\hat{\chi}_A = 0.$$

We already have expressed $\hat{\tau}_A$ and $\tilde{\tau}_A$ in terms of other fields in (3.6.26) so we can write the gauge condition in terms of $\hat{\Psi}_A, \tilde{\chi}_A$ and their conjugates

$$\tilde{\chi}_A = -\frac{1}{2}(i(l+1)\delta_{AB} - \epsilon_{AB})\hat{\Psi}_B, \qquad (3.6.28)$$
$$\hat{\chi}_A = -\frac{1}{2}(i(l+1)\delta_{AB} + \epsilon_{AB})\tilde{\Psi}_B.$$

After imposing the equations of motion and gauge condition there are four field components: $\hat{\Psi}_A, \tilde{\Psi}_A,\ A = 1, 2$.

We now look for pure gauge states. The supersymmetry variations of the 4D Rarita-Schwinger fields Ψ_{IA} are given by

$$\begin{aligned}\delta\Psi_{IA} &= (D_I\delta_{AB} - \frac{1}{4}\Gamma_{JK}F^{JK}_{AB}\Gamma_I)\theta_B \\ &= (D_I\delta_{AB} + \frac{i}{2}(1\otimes\gamma_S)\Gamma_I\epsilon_{AB})\theta_B.\end{aligned} \qquad (3.6.29)$$

In order to compute the supersymmetric variation of each mode we expand the spinor θ_A into partial waves in analogy with (3.6.6) - (3.6.7),

$$\theta_A = \theta_{+A}\otimes\eta + \theta_{-A}\otimes\gamma_S\eta, \qquad (3.6.30)$$

and rewrite the $\pm$ indices as the combinations $\hat{\theta}_A$ and $\tilde{\theta}_A$:

$$\hat{\theta}_A = \theta_{+A} - i\theta_{-A}, \qquad (3.6.31)$$
$$\tilde{\theta}_A = \theta_{+A} + i\theta_{-A}.$$

Note that the procedure here is in complete analogy with the bosonic sector: one writes the gauge variations then expands the parameters in partial waves. The next step is to find the constraints the gauge condition imposes on the supersymmetric parameters, that is, the residual gauge symmetry.

The preservation of the Lorentz gauge condition $\Gamma_I \Psi^I = 0$ constrains the 4D supersymmetric parameters to satisfy

$$\Gamma^I \big[D_I \delta_{AB} + \frac{i}{2}(1 \otimes \gamma_S)\Gamma_I \epsilon_{AB} \big] \theta_B = 0. \tag{3.6.32}$$

Expression (3.6.32) is once again decomposed into 2D conditions. The result are the constraints

$$\begin{aligned} (\gamma^\mu D_\mu + (l+1))\hat{\theta}_A = 0, \\ (\gamma^\mu D_\mu - (l+1))\tilde{\theta}_A = 0. \end{aligned} \tag{3.6.33}$$

The residual gauge symmetry has to satisfy (3.6.33) in order not to violate the imposed gauge.

We compute the supersymmetric variations of the dualized spinors in terms of the parameters $\hat{\theta}_A$, $\tilde{\theta}_A$, by expanding both sides of (3.6.29) in spinor harmonics, dualizing when needed, and comparing each variation in the (3.6.31) basis. We get

$$\begin{aligned}
\delta\hat{\kappa}_A &= \hat{\theta}_A, \\
\delta\tilde{\kappa}_A &= \tilde{\theta}_A, \\
\delta\hat{\tau}_A &= \frac{1}{2}(\gamma^\mu D_\mu \delta_{AB} + i\epsilon_{AB})\hat{\theta}_B, \\
\delta\tilde{\tau}_A &= \frac{1}{2}(\gamma^\mu D_\mu \delta_{AB} + i\epsilon_{AB})\tilde{\theta}_B, \\
\delta\hat{\Psi}_A &= \hat{\theta}_A, \\
\delta\tilde{\Psi}_A &= \tilde{\theta}_A, \\
\delta\hat{\chi}_A &= \frac{1}{2}(i(l+1)\delta_{AB} - \epsilon_{AB})\tilde{\theta}_B, \\
\delta\tilde{\chi}_A &= \frac{1}{2}(i(l+1)\delta_{AB} + \epsilon_{AB})\hat{\theta}_B.
\end{aligned} \tag{3.6.34}$$

We cannot remove $\hat{\Psi}_A$ and $\tilde{\Psi}_A$ with residual gauge transformations since their equations of motion (3.6.23)- (3.6.24) are inconsistent with (3.6.33).

As a clarifying example consider the 4D flat space case: supersymmetry transformations are given by $\delta\Psi_I = \partial_I\theta$ and the gauge condition $\gamma^I\Psi_I = 0$ requires θ to be massless. One cannot turn on pure gauge modes with a massive parameter θ since those would be gauge violating. An analogous situation is happening here. We cannot gauge away modes using the residual symmetry we have. Thus, there are no longitudinal modes.

The modes $\hat{\Psi}_A$, $\tilde{\Psi}_A$ with $A = 1, 2$, $l \geq 1$, and the masses reported in (3.6.23), (3.6.24) satisfy the gauge condition and are not gauge equivalent to vacuum. They are physical modes. This result agrees with [41] and with chapter 1.

3.6.6 l=0 Modes.

In this section we analyze the $l = 0$ sector. Two related issues that are special to $l = 0$ change the equations that apply: the $\Psi_{\pm A}$ components of the gravitino are not defined and also the equation of motion (3.6.16) does not apply. We are therefore left with (3.6.13) and

(3.6.15) which we write in the "hat-tilde" basis as

$$-\frac{i}{2}\gamma^{\mu}\hat{\Psi}_{A\mu} + \gamma^{\mu}D_{\mu}\tilde{\chi}_A + i\gamma^{\mu\nu}D_{\mu}\hat{\Psi}_{A\nu} + i\epsilon_{AB}\tilde{\chi}_B = 0. \tag{3.6.35}$$

$$\left(D_{\mu} - \frac{1}{2}\gamma_{\mu}\right)\tilde{\chi}_A = \frac{1}{2}(i\delta_{AB} + \epsilon_{AB})\hat{\Psi}_{\mu B}. \tag{3.6.36}$$

There are also analogous expressions for the conjugate field. Contracting these equations with the projection operators $(i\delta_{AB} \pm \epsilon_{AB})$ we find

$$(i\delta_{AB} + \epsilon_{AB})\Big[-\frac{i}{2}\gamma^{\mu}\hat{\Psi}_{B\mu} + i\gamma^{\mu\nu}D_{\mu}\hat{\Psi}_{B\nu} + (\gamma^{\mu}D_{\mu} - 1)\tilde{\chi}_B\Big] = 0, \tag{3.6.37}$$

$$(i\delta_{AB} + \epsilon_{AB})\Big[(D_{\mu} - \frac{1}{2}\gamma_{\mu})\tilde{\chi}_B - i\hat{\Psi}_{B\mu}\Big] = 0. \tag{3.6.38}$$

$$(i\delta_{AB} - \epsilon_{AB})\Big[-\frac{i}{2}\gamma^{\mu}\hat{\Psi}_{B\mu} + i\gamma^{\mu\nu}D_{\mu}\hat{\Psi}_{B\nu} + (\gamma^{\mu}D_{\mu} + 1)\tilde{\chi}_B\Big] = 0, \tag{3.6.39}$$

$$(i\delta_{AB} - \epsilon_{AB})(D_{\mu} - \frac{1}{2}\gamma_{\mu})\tilde{\chi}_B = 0. \tag{3.6.40}$$

We next impose Lorentz gauge in the form

$$\gamma^{\mu}\hat{\Psi}_{A\mu} = 2i\tilde{\chi}_A. \tag{3.6.41}$$

The gauge fixed gravitino equations then simplify to

$$(i\delta_{AB} \pm \epsilon_{AB})(D^{\mu} + \frac{1}{2}\gamma^{\mu})\hat{\Psi}_{A\mu} = 0. \tag{3.6.42}$$

We still have the equations of motion (3.6.38) and (3.6.40) for $\tilde{\chi}_A$.

In the sector with $(i\delta_{AB} + \epsilon_{AB})$ projection the equation of motion (3.6.38) and the gauge condition (3.6.41) combine to give

$$(i\delta_{AB} + \epsilon_{AB})(\gamma^{\mu}D_{\mu} + 1)\tilde{\chi}_A = 0. \tag{3.6.43}$$

Given a solution to this equation we can specify the gravitino $\hat{\Psi}_{A\mu}$ as in (3.6.38) and then the gauge condition and the gravitino equation (3.6.42) are all satisfied. Thus solutions to (3.6.43) parametrize the space of solutions to the full equations. It can be shown that all these solutions are pure gauge (up to normalization issues). We stress for later that in the special case where $\tilde{\chi}_A$ vanishes the gravitino $\hat{\Psi}_{A\mu}$ vanishes as well.

The sector with $(i\delta_{AB} - \epsilon_{AB})$ projection is more involved. Here (3.6.40) specifies $\tilde{\chi}_A$ as a Killing Spinor in AdS_2 with mass $+1$:

$$(i\delta_{AB} - \epsilon_{AB})(\gamma^\mu D_\mu - 1)\tilde{\chi}_A = 0. \tag{3.6.44}$$

The gauge condition (3.6.41) (which we could represent in terms of dual fields as in (3.6.27)) then gives the trace part of the gravitino but the traceless part remains unspecified. Rewriting the gravitino equation of motion (3.6.42) in terms of the dual spinor $\hat{\kappa}_A$ introduced in (3.6.25), we have

$$(i\delta_{AB} - \epsilon_{AB})\left([(\gamma^\mu D_\mu)^2 - 1]\hat{\kappa}_A - 4i\tilde{\chi}_A\right) = 0. \tag{3.6.45}$$

Given the Killing spinor $\tilde{\chi}_A$ this equation permits a particular solution for $\hat{\kappa}_A$. To this solution we can add solutions to the homogenous equation which we can represent as solutions to

$$(i\delta_{AB} - \epsilon_{AB})(\gamma^\mu D_\mu \pm 1)\hat{\kappa}_A = 0, \tag{3.6.46}$$

with either sign. In the special case where $\tilde{\chi}_A$ vanishes the traceless component of the gravitino is given by solutions to these equations.

The lightest fermion masses ± 1 are special in that they correspond to zero modes of the Dirac operator squared. The Euclidean version of these modes do not comprise a continuum of solutions of plane wave type but rather a discrete set of modes which are necessarily nonnormalizable. For this reason only the solutions with $\tilde{\chi}_A = 0$ are physical. After this normalizability condition is imposed the space of $l = 0$ modes that satisfy the equations of motion and the gauge condition reduces to the solutions of (3.6.46). Although these fields are also non-normalizable they are dual to physical gravitini according to

$$(i\delta_{AB}-\epsilon_{AB})\hat{\Psi}_{A\mu}=(i\delta_{AB}-\epsilon_{AB})D_{(\mu)}\hat{\kappa}_A=(i\delta_{AB}-\epsilon_{AB})(D_\mu\pm\frac{1}{2}\gamma_\mu)\hat{\kappa}_A. \tag{3.6.47}$$

These physical gravitini are normalizable, and they satisfy the equation of motion and the gauge condition. The spinors $\hat{\kappa}_A$ are such that $\gamma^\mu\hat{\Psi}_{A\mu}=0$.

We finally need to ask whether the remaining modes (3.6.47) are longitudinal. The pure gauge modes are

$$(i\delta_{AB}-\epsilon_{AB})\delta\hat{\Psi}_{B\mu}=(i\delta_{AB}-\epsilon_{AB})(D_\mu+\frac{1}{2}\gamma_\mu)\hat{\theta}_B. \tag{3.6.48}$$

with the residual SUSY transformation such that it preserves the gauge condition

$$(\gamma^\mu D_\mu+1)\hat{\theta}_A=0. \tag{3.6.49}$$

The mode that appears with upper sign in (3.6.46) is therefore pure gauge with the field and the gauge parameter coinciding $\hat{\kappa}_A=\hat{\theta}_A$ as we expected from (3.6.34). Since the gauge parameter is not normalizable the corresponding gravitino is physical even though it is formally pure gauge.

The mode that appears with lower sign in (3.6.47) is similarly nonnormalizable but corresponding to a normalizable gravitino. This mode is again formally pure gauge but with a transformation parameter that does not satisfy the condition (3.6.49) that the gauge is preserved. It is therefore not pure gauge because the would-be gauge transformation introduces a nonvanishing $\gamma^\mu\hat{\Psi}_{A\mu}$. It is possible to instead define a superconformal symmetry that leaves $\gamma^\mu\hat{\Psi}_{A\mu}$ invariant and consider this mode pure gauge with respect to this extended symmetry. Either way, it is a physical boundary mode.

Recall that the computation in this subsection focussed for definiteness on the $\hat{\Psi}_{A\mu}$, $\tilde{\chi}_A$ field components. It can be repeated for the conjugate fields $\tilde{\Psi}_{A\mu}$. $\hat{\chi}_A$. The analogue of

(3.6.47) in this sector is

$$(i\delta_{AB} + \epsilon_{AB})\tilde{\Psi}_{A\mu} = (i\delta_{AB} + \epsilon_{AB})D_{(\mu)}\tilde{\kappa}_A = (i\delta_{AB} + \epsilon_{AB})(D_\mu \pm \frac{1}{2}\gamma_\mu)\tilde{\kappa}_A, \tag{3.6.50}$$

with $\tilde{\kappa}_A$ such that $\gamma^\mu\tilde{\Psi}_{A\mu} = 0$. It is the opposite SUSY that gives rise to a boundary mode and it is now the lower sign that is a pure gauge mode while the upper is a superconformal extension.

In summary, there are no physical bulk modes at $l = 0$. However, each of the two SUSYs allow a nonnormalizable gauge parameter (and a superconformal analogue) that generates normalizable gravitini. This corresponds to four physical boundary modes.

3.7 Quantum Corrections to AdS$_2 \times S^2$ - Fermionic Sector.

In this section we compute the heat kernels for the fermionic sector of the gravity multiplet. An important preliminary result is the heat kernel of a free spin 1/2 fermion on the sphere S^2,

$$K_S^f = \frac{1}{4\pi a^2}\sum_{k=0}^{\infty} e^{-s(k+1)^2}(2k+2) = \frac{1}{4\pi a^2 s}\left(1 - \frac{1}{6}s - \frac{1}{60}s^2 + ...\right). \tag{3.7.1}$$

The AdS$_2$ heat kernel is obtained to the precision we need by flipping the sign of the terms that are odd in the curvature (with the overall sign changed due to fermion statistics)

$$K_A^f = -\frac{1}{4\pi a^2}\sum_{k=0}^{\infty} e^{-s(k+1)^2}(2k+2) = -\frac{1}{4\pi a^2 s}\left(1 + \frac{1}{6}s - \frac{1}{60}s^2 + ...\right). \tag{3.7.2}$$

As in (3.3.5) for bosons we compute the 4D heat kernels by summing over towers using

$$K_4^f = K_A^f \cdot \frac{1}{4\pi a^2}\sum_j e^{-m_j^2 s}(2j+2). \tag{3.7.3}$$

We are summing over each value of the angular momentum j on S^2 weighed by the effective AdS_2 masses.

3.7.1 Physical States.

The physical bulk spectrum summarized at the end of subsection 3.6.5 is four fermionic bulk degrees of freedom with masses $m^2 = (k+1)^2$ where $k > 0$. Hence, the 4D heat kernel is

$$\begin{aligned} K_4^{\text{bulk}} &= 4 \cdot K_A^f \cdot \frac{1}{4\pi a^2} \sum_{k=1}^{\infty} e^{-s(k+1)^2} (2k+2) \\ &= 4 \cdot K_A^f \cdot \frac{1}{4\pi a^2} \left(\sum_{k=0}^{\infty} e^{-s(k+1)^2} (2k+2) - 2e^{-s} \right) \\ &= -\frac{1}{4\pi^2 a^4 s^2} \left(1 - \frac{11}{180} s^2 + ... - 2s\big(1 - \frac{5}{6} s\big) + ... \right). \end{aligned} \tag{3.7.4}$$

We wrote the final line as the sum of the result we would get from four free fermionic degrees of freedom and a term we interpret as due to the couplings of the gravitino field.

3.7.2 Unphysical States.

The unphysical spectrum consists of twelve fermionic bulk degrees of freedom with masses $m^2 = (k+1)^2$ at $k \geq 0$. These modes were all established as unphysical either due to the Rarita-Schwinger constraint – which is a component of the equations of motion – or due to the gauge condition. No on-shell modes were removed by residual gauge symmetries. In our on-shell method we do not include contributions from any of these.

3.7.3 Boundary Modes.

The boundary modes are zero modes in AdS_2 while consisting of a full tower on S^2. Expression (3.7.3) for a 4D heat kernel is then modified to

$$K_4^{\mathrm{bndy}} = -\frac{1}{8\pi^2 a^4}\sum_j e^{-m_j^2 s}(2j+2), \tag{3.7.5}$$

where the contribution of the AdS_2 heat kernel is a factor of the regulated volume of AdS.

The boundary fields $\hat{\theta}_A, \tilde{\theta}_A$ each have a projection on the R index A but also a doubling due to conformal symmetry. Thus there are four towers of boundary states. We used the mass (3.6.33) to find the mass squared of $D_{(\nu)}\hat{\theta}_A$, $D_{(\nu)}\tilde{\theta}_A$, and then the heat kernel. The squared masses are given by

$$\begin{aligned}(\gamma^\mu D_\mu)^2 D_{(\nu)}\hat{\theta}_A &= D_{(\nu)}[(\gamma^\mu D_\mu)^2 - 1]\hat{\theta}_A \\ &= [(k+1)^2 - 1]D_{(\nu)}\hat{\theta}_A.\end{aligned} \tag{3.7.6}$$

The total heat kernel for the four boundary modes is

$$\begin{aligned}K_4^{\mathrm{bndy}} &= -\frac{4}{8\pi^2 a^4}\sum_{k=0}^{\infty} e^{-[(k+1)^2-1]s}(2k+2) \\ &= -\frac{4}{8\pi^2 a^4}\left(\frac{1}{s} - \frac{1}{6}\right)e^{-s} \\ &= -\frac{1}{4\pi^2 a^4}\left(\frac{2}{s} + \frac{5}{3} + \dots\right).\end{aligned} \tag{3.7.7}$$

3.7.4 Zero Modes.

Boundary states that are also zero modes on the S^2 are true zero modes of $\mathrm{AdS}_2 \times S^2$. Hence, the zero mode content can be read off from the spectrum of boundary states. The four fermionic zero-modes are the $k = 0$ entries in (3.7.6). As mentioned in the bosonic sector, zero-modes require special considerations discussed by [36, 9, 10].

In the naive treatment (3.7.7) each of the four zero modes contributes with $-\frac{2}{8\pi^2 a^2}$, but the correct contribution is larger. The correction due to zero-modes is

$$K_4^{zm,f} = -\frac{8}{8\pi^2 a^4}\left(\frac{3}{2} - \frac{1}{2}\right) = \frac{1}{8\pi^2 a^4} \cdot (-8). \tag{3.7.8}$$

3.7.5 Summary.

We add the fermionic contributions from bulk (4D), boundary (2D), and the zero-modes (0D),

$$K_4^f = -\frac{1}{4\pi^2 a^4 s^2}\left(1 + \frac{1309}{180}s^2 + \dots\right), \tag{3.7.9}$$

which is the total contribution from fermionic modes.

We finally add the total bosonic contribution (3.5.6) and the total fermionic contribution (3.7.9),

$$K_4^b + K_4^f = \frac{1}{4\pi^2 a^4}\left(\frac{1}{s} - \frac{23}{12} + \dots\right). \tag{3.7.10}$$

These are the quantum corrections to supergravity in $\mathrm{AdS}_2 \times S^2$.

Chapter 4

Divergences and Boundary Modes in N=8 Supergravity.

4.1 Motivation and Summary.

In this chapter we apply the quasi on-shell method we developed to address the possibility of divergences in $\mathcal{N}=8$ supergravity. The central point is an apparent incompatibility in the literature: on one hand it is known that supersymmetry mitigates divergences present in quantum gravity so effectively that for maximal $\mathcal{N}=8$ supergravity in four asymptotically flat dimensions it has not yet been established what divergences remain, if any [14, 15, 16, 17, 18, 19, 20]. On the other hand, it has long been known that in curved backgrounds, highly relevant for gravity, even the one-loop vacuum amplitude diverges [21, 22, 23, 24, 25, 26]. This apparent incompatibility between these results created controversy already in the 1980's [70, 27, 28, 71, 72, 73, 74]. In this chapter we revisit this tension from a modern perspective informed by the AdS/CFT correspondence [1]. Our goal is to unravel the role of boundary modes in this debate.

To exhibit the central issue in more detail it is convenient to focus on the anomalous contribution to the trace of the energy momentum tensor

$$\langle T^\mu_\mu \rangle_{\rm an} = \frac{1}{(4\pi)^2}\left(cW^2 - aE_4\right), \tag{4.1.1}$$

where the square of the Weyl tensor $W^2 = \mathrm{Riem}^2 - 2\mathrm{Ric}^2 + \frac{1}{3}R^2$ and the Euler density $E_4 = \mathrm{Riem}^2 - 4\mathrm{Ric}^2 + R^2$ encode dependence on the background geometry[1]. The coefficients c, a depend on the matter content of the theory and they have been studied in great detail; e.g. using perturbation theory in small curvature around flat space. Their values for fields with simple couplings to the background have long been established and are summarized in Table 4.1 (later). These well known coefficients are such that, for the field content of $\mathcal{N} = 8$ supergravity, their sum does not vanish. This fact establishes a divergence that is present already at one loop.

However, there are equally well established perturbative nonrenormalization theorems based on the helicity supertraces over the on-shell spectrum

$$\sum (-)^{2h} h^n = 0, \tag{4.1.2}$$

for $n < \mathcal{N} = 8$. These sum rules imply powerful cancellations for perturbative amplitudes in asymptotically flat space and related supertrace formulae are influential in particle phenomenology because they survive spontaneous breaking of supersymmetry. For us the important point is that the helicity sum rules establish one-loop nonrenormalization in $\mathcal{N} = 8$ AdS$_4$ supergravity (gauged $\mathcal{N} = 8$ supergravity) [71, 72, 75]. These cancellations even generalize to all massive levels obtained from Kaluza-Klein compactification of $\mathcal{N} = 1$ supergravity in 11 dimensions.

We will argue that despite appearances there is no contradiction, but rather a topological distinction encoded in the boundary conditions. The basis for the sum rules (4.1.2) is Lorentzian AdS$_4$ which, after Euclidean continuation, gives rise to $S^1 \times S^2$ boundary condi-

[1] We assume for simplicity a renormalization scheme where other possible terms are absent.

tions with the S^1 corresponding to Euclidean time. In this geometry the Euler characteristic

$$\chi = \frac{1}{32\pi^2} \int E_4 + \text{ bndy}, \tag{4.1.3}$$

vanishes. This is significant because the divergences uncovered by the curvature expansion are proportional to χ and so they are not captured by AdS_4 with $S^1 \times S^2$ boundary conditions. On the other hand, we will easily reproduce them from Euclidean AdS_4 with S^3 boundary conditions since this geometry has Euler invariant $\chi = 1$.

One might wonder if these divergences have any physical significance. We argue in the affirmative by computing a finite and nonvanishing one-loop correction to the cosmological constant in maximal AdS_4 supergravity. In this computation it is manifest that the helicity supertrace relations (4.1.2) are violated in spacetime with S^3 boundary conditions. Interestingly, the violation is rather mild so all power law corrections in fact cancel. Thus the cosmological constant acquires just logarithmic running. This feature is intriguing since it might offer a mechanism that could describe dark energy without sacrificing naturalness.

Our results are subject to an important subtlety that was noticed already in early studies of quantum fields in curved space: quantum inequivalence [27]. In our context an important example is the relation between a massless antisymmetric tensor and a scalar field. In the classical theory they are equivalent by a field redefinition but their quantum partition functions are related by a shift that is proportional to the Euler characteristic (4.1.3) [25, 26]. The coefficient of the logarithmic divergence we study therefore depends on the duality frame which becomes part of the data that defines the theory. We interpret this feature as a genuine physical effect: antisymmetric tensor fields support boundary modes that have no analogues in the corresponding scalar field theory.

In this chapter we primarily interpret $\mathcal{N} = 8$ AdS_4 supergravity as a low energy effective field theory in its own right but ultimately the UV completion involves the full string/M-theory. As an intermediate step we consider the theory as compactification of 11D supergravity on $\mathrm{AdS}_4 \times S^7$. This procedure defines a preferred duality frame for the massless fields and it happens that it is precisely the frame where all logarithmic divergences cancel [27]. In this setting boundary modes cancel bulk divergences.

There have been many other recent studies of quantum corrections to AdS spaces in various dimensions. A basic feature of this research is that divergences remain even when supersymmetry is maximal and those divergences are related to effects that are unambiguously physical in the dual theory. Some examples:

- In AdS_{d+1} with odd $(d+1)$ there are bulk divergences interpreted as finite quantum anomalies in the dual theory with even d. For example, in the case of $d = 4$ such anomalies are responsible for the shift $N^2 \to N^2 - 1$ that is expected and confirmed in $\mathcal{N} = 4$ SYM with $SU(N)$ gauge groups [76, 77].

- Quantum corrections to higher spin theories in AdS provide impressive evidence for higher spin holography. [78, 79, 80]

- The Bekenstein-Hawking area law for black holes is subject to $\log A$ corrections with coefficients determined by the low energy theory. For BPS black holes these coefficients are determined by divergences in AdS_2 and $\mathrm{AdS}_2 \times S^2$ which are generically nonvanishing (including for $\mathcal{N} = 8$), and their values are confirmed by the microscopic theory in cases where the latter has been established, [36, 37, 7, 9, 32, 10, 8], as well as the previous chapters of this work.

Our study of AdS_4 was motivated in part by these and related developments. Computations in these contexts share the techniques we employ and offer some confidence in their applicability.

4.2 One Loop Quantum Corrections in AdS$_4$.

In this section we employ heat kernel methods to compute the one loop contributions to the anomalous trace of the energy momentum tensor in AdS$_4$ from fields with various spins. We interpret the resulting divergences in the effective action as logarithmic running of the effective cosmological constant.

4.2.1 Notation and Review.

One loop quantum corrections in Euclidean quantum gravity are determined by a Gaussian path integral with the schematic form,

$$W = -\ln \int \mathcal{D}\phi \; e^{-\phi \Box \phi} = \frac{1}{2} \ln \det \Box = \frac{1}{2} \sum_i \ln \lambda_i, \tag{4.2.1}$$

where the ϕ denotes the collection of linearized fields, $\Box$ represents their kinetic operator, and λ_i are the eigenvalues of $\Box$. We represent the effective action W in terms of the heat kernel $D(t) = \sum_i e^{-t\lambda_i}$ as

$$W = -\int_{\epsilon^2}^{\infty} \frac{dt}{2t} D(t), \tag{4.2.2}$$

where ϵ is a UV regulator with dimension of length. It is customary to express results for heat kernels in terms of the (equal point) heat kernel density $K(t)$ expanded at small t

$$K(t) = \frac{1}{\mathrm{Vol}_{\mathrm{AdS}_4}} D(t) = \frac{1}{(4\pi t)^2} \Big(1 + a_2 t + a_4 t^2 + ... \Big). \tag{4.2.3}$$

Departures from the flat space limit are encoded in the two derivative correction a_2 proportional to the Ricci scalar and the four derivative correction a_4 that is a linear combination of Riemann squared, Ricci squared, and Ricci scalar squared.[2] Note that in this chapter as well as the next one we will be denoting the heat kernel parameter by t instead of s since

[2] The volume diverges, since AdS$_4$ is noncompact. We mostly consider local quantities in a homogeneous space and then the regulator details are unimportant. The standard renormalized value $\mathrm{Vol}_{\mathrm{AdS}_4} = \frac{4\pi^2 \ell_A^4}{3}$ will appear later from global considerations with explicit boundary terms.

we want to avoid any confusion with the spin s of a given field.

We divide the one loop effective action (4.2.2) into divergent contributions

$$W_{\rm div} = \frac{1}{32\pi^2}\left(-\frac{1}{2\epsilon^4} - a_2\frac{1}{\epsilon^2} + a_4 \ln\epsilon^2\right) {\rm Vol}_{{\rm AdS}_4}, \tag{4.2.4}$$

and a remainder that is finite. From either piece we can form the trace of the energy momentum tensor

$$T^\mu_\mu = \frac{2}{\sqrt{-g}} g^{\mu\nu}\frac{\delta W}{\delta g^{\mu\nu}}. \tag{4.2.5}$$

The logarithmic divergence of the effective action (4.2.4) gives an anomalous contribution that is conventionally presented as

$$\langle T^\mu_\mu\rangle_{\rm an} = \frac{1}{(4\pi)^2}a_4 = \frac{1}{(4\pi)^2}\left(cW^2 - aE_4\right). \tag{4.2.6}$$

In the nonconformal theories we consider there may be additional contributions to the trace of the energy momentum tensor.

The values of c and a have been computed perturbatively by many researchers using different methods and schemes [22, 81, 82, 44]. The values that are now standard (up to caveats discussed later in this section) are summarized in Table 4.1 below.

Field	c	a	$c-a$
Real Scalar	$\frac{1}{120}$	$\frac{1}{360}$	$\frac{1}{180}$
Weyl Fermion	$\frac{1}{40}$	$\frac{11}{720}$	$\frac{7}{720}$
Vector	$\frac{1}{10}$	$\frac{31}{180}$	$-\frac{13}{180}$
Antisymmetric Tensor	$\frac{1}{120}$	$-\frac{179}{360}$	$\frac{91}{180}$
Gravitino	$-\frac{411}{360}$	$-\frac{589}{720}$	$-\frac{233}{720}$
Graviton	$\frac{783}{180}$	$\frac{571}{180}$	$\frac{53}{45}$

Table 4.1: Central charges c and a for minimally coupled massless fields. Each entry is a physical field with two degrees of freedom except the scalar and the antisymmetric tensor, which have just one degree of freedom.

4.2.2 Computations in AdS_4.

We now revisit these computations in the context of AdS_4. This geometry is conformally flat so the Weyl tensor vanishes and therefore the central charge c plays no role. Our focus on a is complementary to techniques that impose Einstein's equations in vacuum and identify just the Riemann-squared terms which have coefficient $c - a$.

The natural representations for fields in AdS_4 are the symmetric, transverse, and traceless (STT) tensors with spin s. The heat kernels for these fields were comprehensively analyzed by Camporesi and Higuchi [48, 49, 83] (and recently developed further [80]) both using explicit mode functions and also using group theory. We present their results for the AdS_4 heat kernel of a massive spin s field with conformal dimension Δ as[3]

$$K^{(s,\nu)}(t) = \frac{1}{\ell_A^4} \int_0^\infty d\lambda \; \mu_s(\lambda) \; e^{-\frac{t}{\ell_A^2}(\lambda^2+\nu^2)}, \tag{4.2.7}$$

where $\nu^2 = (\Delta - \frac{3}{2})^2$. The conformal dimension Δ is equivalent to the mass of the field and in the context of AdS_4 it is Δ that provides the simplest representation of this parameter. Crucially, the Plancherel measure $\mu_s(\lambda)$ for the integration over the continuous eigenvalues λ is different for bosons

$$\mu_s(\lambda) = (s + \frac{1}{2})\frac{\lambda^2 + (s+\frac{1}{2})^2}{4\pi^2}\lambda \tanh(\pi\lambda), \tag{4.2.8}$$

and for fermions

$$\mu_s(\lambda) = (s + \frac{1}{2})\frac{\lambda^2 + (s+\frac{1}{2})^2}{4\pi^2}\lambda \coth(\pi\lambda). \tag{4.2.9}$$

The distinction between bosons and fermions is inconsequential in the UV region where $\lambda \to \infty$ since then both $\tanh(\pi\lambda) \to 1$ and $\coth(\pi\lambda) \to 1$. It is instructive to evaluate the heat kernel (4.2.7) such that this common feature is manifest. For bosons we write

[3]We simplify notation by absorbing a numerical factor in the Plancherel measure.

$\tanh(\pi\lambda) = 1 - \frac{2}{e^{2\pi\lambda}+1}$ and then find

$$\begin{aligned}
K^{(s,\nu)}_{\rm boson}(t) &= \frac{s+\frac{1}{2}}{4\pi^2\ell_A^4} e^{-\frac{t\nu^2}{\ell_A^2}} \left[\int_0^\infty e^{-\frac{t\lambda^2}{\ell_A^2}} (\lambda^2 + (s+\frac{1}{2})^2)\lambda d\lambda - 2\int_0^\infty e^{-\frac{t\lambda^2}{\ell_A^2}} \frac{\lambda^2 + (s+\frac{1}{2})^2}{e^{2\pi\lambda}+1}\lambda d\lambda\right] \\
&= \frac{s+\frac{1}{2}}{8\pi^2\ell_A^4} e^{-\frac{t\nu^2}{\ell_A^2}} \left(\frac{\ell_A^4}{t^2} + \frac{\ell_A^2}{t}(s+\frac{1}{2})^2\right) - \frac{s+\frac{1}{2}}{8\pi^2\ell_A^4}\left(\frac{7}{480} + \frac{(s+\frac{1}{2})^2}{12}\right) \\
&= \frac{s+\frac{1}{2}}{8\pi^2\ell_A^4}\left[\frac{\ell_A^4}{t^2} + \frac{\ell_A^2}{t}\left((s+\frac{1}{2})^2 - \nu^2\right)\right] + \frac{s+\frac{1}{2}}{16\pi^2\ell_A^4}\left[\nu^4 - (s+\frac{1}{2})^2(2\nu^2+\frac{1}{6}) - \frac{7}{240}\right].
\end{aligned} \tag{4.2.10}$$

The first integral contains the UV terms that are common to bosons and fermions and is elementary for all t. The second integral is special to bosons. It is finite for small t so we evaluate it at $t=0$, omitting higher powers in t. It is evident from this structure that only the first integral contributes to the terms that are divergent in the UV limit $t \to 0$.

We next compare with the fermion heat kernel where we write $\coth(\pi\lambda) = 1 + \frac{2}{e^{2\pi\lambda}-1}$ and find

$$\begin{aligned}
K^{(s,\nu)}_{\rm fermion}(t) &= \frac{s+\frac{1}{2}}{4\pi^2\ell_A^4} e^{-\frac{t\nu^2}{\ell_A^2}} \left[\int_0^\infty e^{-\frac{t\lambda^2}{\ell_A^2}} (\lambda^2 + (s+\frac{1}{2})^2)\lambda d\lambda + 2\int_0^\infty e^{-\frac{t\lambda^2}{\ell_A^2}} \frac{\lambda^2 + (s+\frac{1}{2})^2}{e^{2\pi\lambda}-1}\lambda d\lambda\right] \\
&= \frac{s+\frac{1}{2}}{8\pi^2\ell_A^4}\left[\frac{\ell_A^4}{t^2} + \frac{\ell_A^2}{t}\left((s+\frac{1}{2})^2 - \nu^2\right)\right] + \frac{s+\frac{1}{2}}{16\pi^2\ell_A^4}\left[\nu^4 - (s+\frac{1}{2})^2(2\nu^2-\frac{1}{3}) + \frac{1}{30}\right].
\end{aligned} \tag{4.2.11}$$

Since the first integral is the same in the boson and fermion heat kernels (4.2.10, 4.2.11) these expressions have the same divergences in the UV limit $t \to 0$. It is for the same reason that they have the same dependence on conformal dimension appearing through $\nu^2 = (\Delta - \frac{3}{2})^2$. However, the two cases are of course different due to the second integral and this is reflected in the terms that are constant and independent of ν.

We are particularly interested in massless particles since those are the ones that appear in standard $\mathcal{N}=8$ supergravity. In AdS$_4$ masslessness is not well characterized by the absence of a mass term in the Lagrangian but rather by the reducibility of the field representation. Representations at spin s generally have dimension $2s+1$ but some special ones are reducible and allow decoupling of a ghost representation that has spin $s_{\rm ghost} = s-1$ and so dimension $2s_{\rm ghost}+1 = 2s-1$. This leaves two physical degrees of freedom for massless particles with spin, as expected. Group theory methods show that this reduction is possible precisely when the conformal dimension is $\Delta = s+1$ (and so $\nu = \Delta - \frac{3}{2} = s - \frac{1}{2}$) and also specify that the spin $s-1$ ghosts have $\Delta_{\rm ghost} = s+2$ [84]. These results do not strictly apply for the lowest spins $s = \frac{1}{2}, 0$ but we can apply them formally with the understanding that the ghost subtraction in fact enhances a real scalar to a complex representation.[4] These rules give

$$K^{(s,{\rm massless})}_{\rm boson}(t) = K^{(s,s+1)}_{\rm boson}(t) - K^{(s-1,s+2)}_{\rm boson}(t) = \frac{1}{16\pi^2\ell_A^4}\left(\frac{2\ell_A^4}{t^2} + \frac{8s^2\ell_A^2}{t} - 5s^4 + s^2 - \frac{2}{15}\right), \quad (4.2.12)$$

for a massless boson with spin s, and

$$K^{(s,{\rm massless})}_{\rm fermion}(t) = (-)\left[K^{(s,s+1)}_{\rm fermion}(t) - K^{(s-1,s+2)}_{\rm fermion}(t)\right] = \frac{1}{16\pi^2\ell_A^4}\left(-\frac{2\ell_A^4}{t^2} - \frac{8s^2\ell_A^2}{t} + 5s^4 - \frac{5}{2}s^2 - \frac{13}{240}\right), \quad (4.2.13)$$

for a massless fermion with spin s. We inserted a sign for the fermion by hand in order to take statistics into account.

The $t = 0$ poles in the massless heat kernels are the same for bosons and fermions (up to the sign that was inserted for fermions) as we expected since that is the case for each of the underlying massive representations. On the other hand, some of the terms that are finite as $t \to 0$ differ, also as expected. This feature is the origin of the apparent lack of pattern in the heat kernel coefficients that is evident when we consider the finite parts of $K_{\rm massless}$ for the first few spins in Table 4.2.

Our results for the finite parts of the heat kernel $K(t)$ in AdS$_4$ are identical to the a_4 coefficients introduced in (4.2.3) up to a factor $(4\pi)^2$. It can be further recast in terms of the a-anomaly introduced in (4.2.6) by noting that the Gauss-Bonnet density in AdS$_4$ is

[4] For spin $s = \frac{1}{2}$ the rule formally subtracts ghosts that have spin $s_{\rm ghost} = -\frac{1}{2}$ but that is inconsequential since this representation has dimension $2s_{\rm ghost}+1 = 0$. For spin $s = 0$ it formally subtracts a ghost with spin $s_{\rm ghost} = -1$ and dimension $2s_{\rm ghost}+1 = -1$ which effectively adds one degree of freedom, turning one boson into two, with conformal dimensions $\Delta = 1, 2$.

Spin	$16\pi^2\ell_A^4 K^{\text{finite}}_{\text{massless}}$	a
0	$-\frac{2}{15}$	$\frac{1}{180}$
$\frac{1}{2}$	$-\frac{11}{30}$	$\frac{11}{720}$
1	$-\frac{62}{15}$	$\frac{31}{180}$
$\frac{3}{2}$	$\frac{589}{30}$	$-\frac{589}{720}$
2	$-\frac{1142}{15}$	$\frac{571}{180}$

Table 4.2: The values of K_{massless} computed in AdS$_4$ and the corresponding a anomalies. All entries including the scalar $s = 0$ refers to two degrees of freedom.

$E_4 = 24/\ell_A^4$. We have included the a-anomaly computed this way in Table 4.2. These values agree perfectly with the results from the local expansion in curvature summarized in Table 4.1.

There is a caveat to this agreement. As we have stressed, our computation (which in fact closely follows Camporesi and Higuchi [85]) determines the a-anomaly unambiguously for all spin. In contrast, many researchers compute both c and a for low spin but results for $s = \frac{3}{2}, 2$ (and above) are not widely quoted and there is no obvious consensus on their values. This situation is tied with the background dependence of the linearized equations of motion for such fields. The most secure data points are for $c - a$ which is defined in Ricci flat backgrounds and a which, as we have stressed, is unambiguous in maximally symmetric spacetimes. For $s = \frac{3}{2}, 2$ the values of a, c given in Table 4.1 were obtained by combining the results for a given in Table 4.2 with the standard values of $c - a$.

4.2.3 Extended SUSY.

The t-poles in the heat kernels (4.2.12) and (4.2.13) correspond to power law divergences in the effective action. The boson and fermion contributions to these divergenes cancel when

$$\sum (-)^{2s} s^n = 0, \tag{4.2.14}$$

for $n = 0, 2$. The massless spectrum only comprises maximal helicity where $|h| = s$ so this condition is equivalent to the helicity sum rule (4.1.2) for $n = 0, 2$. This is satisfied for $\mathcal{N} \geq 3$ supergravity and we will focus on these theories.

Spin	**Conformal Dimension Δ**	$SO(8)$ **Multiplicity**
2	3	1
$\frac{3}{2}$	$\frac{5}{2}$	8
1	2	28
$\frac{1}{2}$	$\frac{3}{2}$	56
0	1	35
0	2	35

Table 4.3: The conformal dimensions and multiplicities of the massless multiplet in $\mathcal{N} = 8$ supergravity.

For maximal $\mathcal{N} = 8$ SUGRA the standard spectrum given in Table 4.3 satisfies the sum rule (4.2.14) even for $n = 4, 6$ yet the sum of the boson and fermion heat kernels do not vanish

$$K^{\text{total}}_{\mathcal{N}=8} = \langle T^{\mu}_{\mu} \rangle_{\text{ren}} = \frac{1}{16\pi^2 \ell_A^4}(-60). \tag{4.2.15}$$

This is possible because the bosonic and fermionic heat kernels (4.2.12-4.2.13) are different polynomials in the spin s.

We can represent the heat kernel result (4.2.15) for $\mathcal{N} = 8$ supergravity as an a anomaly for the entire multiplet,

$$a_{\mathcal{N}=8} = \frac{5}{2}. \tag{4.2.16}$$

Considering also the values of c from Table 4.1 we find that the central charge $c = 0$ for the full $\mathcal{N} = 8$ multiplet. We collect these results in Table 4.4.

The quantum effective action can be computed from the heat kernel (or, equivalently, from the trace of the energy momentum tensor) by the integral (4.2.2). We perform the integration with the dimensionless conformal weights Δ kept fixed. This is justified by the boundary perspective where the dual theory is conformal in the leading approximation and also from the bulk point of view where all fields are in the massless representations that do not even exist for other values of the conformal weights. Since we focus on theories with no power law corrections the integrand is a constant and, with the measure indicated in (4.2.2), the integral gives a logarithmically divergent term in the effective action.

Multiple research groups have reported that in fact the trace anomaly does vanish for $\mathcal{N}=8$ supergravity in AdS_4 [71, 72] and so there are no divergences. Those results refer to different boundary conditions where the spectrum is discrete and the helicity sum rule (4.1.2) applies for all $n<\mathcal{N}$. We will return to this in more detail in the next section.

	c	a	$c-a$
Massless $\mathcal{N}=8$ multiplet	0	$\frac{5}{2}$	$-\frac{5}{2}$

Table 4.4: Central charges c and a for $\mathcal{N}=8$ supergravity.

4.2.4 Interpretation of Quantum Corrections.

The anomalous contribution to the trace of the energy momentum tensor is independent of position because spacetime is homogeneous. A classical cosmological constant in the action similarly gives a constant contribution but the origin of the anomalous contribution is a divergence $W_{\mathrm{div}}=\frac{1}{2}D_0\ln\epsilon^2/\ell_0^2$ in the effective action that manifests itself in the renormalized action as a term that evolves logarithmically

$$W_{\mathrm{ren}}=-\frac{1}{2}D_0\ln\frac{x_{\mathrm{phys}}^2}{\ell_0^2}. \tag{4.2.17}$$

The renormalization scale ℓ_0 enters as an IR cutoff on the integral over the heat kernel. It is arbitrary but of order of the AdS-scale. The physical length scale x_{phys} depends on the process as usual and may be anywhere in the range from much smaller than the AdS scale (for UV processes) to much larger than the AdS scale (for the IR properties).

We interpret the scale dependent quantum effective action as a contribution $\delta\Lambda$ to the cosmological constant determined by

$$W_{\rm ren} = -\frac{{\rm Vol}_{{\rm AdS}_4}\delta\Lambda}{8\pi G}. \tag{4.2.18}$$

It is convenient to express the running in terms of the effective AdS scale $\ell_{\rm eff} = \sqrt{-3/\Lambda}$:

$$\frac{1}{\ell_{\rm eff}^2} = \frac{1}{\ell_A^2}\left[1 - \frac{4\pi G}{3\ell_A^2}(K_0\ell_A^4)\ln\frac{x_{\rm phys}^2}{\ell_0^2}\right]. \tag{4.2.19}$$

The combination $(K_0\ell_A^4)$ is a pure number that we have computed above for some specific fields. The most important part of this expression is the absence of power law corrections that would enter through the UV cutoff ϵ. This would signal dependence on unknown UV physics. Instead we have nontrivial logarithmic quantum corrections that are computable within the low energy theory. [5]

A good way to construct AdS supergravity is to gauge supergravity in flat space. This procedure identifies the gauge coupling constant as [88, 89]

$$e^2 = \frac{4\pi G}{\ell_A^2}. \tag{4.2.20}$$

This coupling constant is small $e^2 \ll 1$ when the AdS radius is much larger than Planck scale as we have implicitly presumed. Resumming the (possibly large) logarithms we can recast (4.2.19) as

$$e^2 = \frac{e_0^2}{1 + \frac{1}{3}e_0^2(K_0\ell_A^4)\ln\frac{x_{\rm phys}^2}{\ell_0^2}}. \tag{4.2.21}$$

Comparing with standard formulae we can write an effective β-function for these theories

$$\beta = -\frac{b}{16\pi^2}e^3. \tag{4.2.22}$$

[5]Logarithmic running of the cosmological constant was discussed also in [86, 87].

where

$$b = -\frac{1}{3}(16\pi^2 K_0 \ell_A^4). \tag{4.2.23}$$

The β-function determines the running of a dimensionless version of the cosmological constant through the usual renormalization group equations. The numerical coefficient $b = 8a$ is $b = 20$ for $\mathcal{N} = 8$ supergravity, $b = 8(1 + n_V/4)$ for $\mathcal{N} = 4$ supergravity with n_V matter multiplets, and similarly for other examples.

Our computations are all made in bulk and there is no reference to a boundary theory. This is a rather old fashioned point of view but it is worthwhile for interpreting the set up as a toy model for the physical cosmological constant. For this we imagine the signs such that the cosmological constant is positive and the running such that it becomes small at large distances. The dimensionless coupling e^2 would be tuned to take a tiny value, of order 10^{-120}. The absence of power law corrections would then ensure naturalness in the sense that the logarithmic running is so mild that quantum corrections would preserve the enormous hierarchy. This mechanism does not explain the smallness of the observed cosmological constant but it offers a viable scenario for its technical naturalness.

4.3 Quantum Inequivalence and Boundary Modes.

In this section we discuss the interplay between the trace of the energy momentum tensor and quantum inequivalence between duality frames. We interpret quantum inequivalence as a physical effect due to boundary modes. We also show that the divergences and the boundary modes are both related to the topology of global AdS_4.

4.3.1 Quantum Inequivalence.

A massless antisymmetric tensor in four dimensions can be mapped into a massless scalar field via the classical duality transformation

$$H_{\mu\nu\sigma} = 3\nabla_{[\mu}B_{\nu\sigma]} = \epsilon_{\mu\nu\sigma\lambda}\nabla^{\lambda}\phi. \tag{4.3.1}$$

These fields are therefore classically equivalent. However, one loop corrections in curved space do not respect this equivalence. For example, the trace anomaly coefficients for these two fields differ as displayed in Table 4.5. This leads to the conclusion that these fields are quantum inequivalent [27]. However, in some sense the dual fields do not differ by terribly

	c	a	$c-a$
Antisymmetric Tensor	$\frac{1}{120}$	$-\frac{179}{360}$	$\frac{91}{180}$
Real Scalar	$\frac{1}{120}$	$\frac{1}{360}$	$\frac{1}{180}$
Antisymm Tensor $-\ \phi$	0	-$\frac{1}{2}$	$\frac{1}{2}$

Table 4.5: Central charges c and a for the 2-form, the real scalar, and their evanescent difference.

much. They have identical physical spectra as captured by propagating on-shell degrees of freedom: the "evanescent" field defined by their difference has no propagating degrees of freedom. Although the a-anomaly coefficients do indeed differ, the c-anomaly coefficients do not; and the a-anomaly is the coefficient of the Euler density which is topological. Many researchers therefore argue that these fields are equivalent, at least for all practical purposes [90, 73, 74].

Our discussion of divergences in $\mathcal{N} = 8$ supergravity (and related theories) is intertwined with quantum inequivalence. First of all, the divergence (4.2.16) is entirely an a-anomaly, the c-anomaly of $\mathcal{N} = 8$ supergravity vanishes. We nevertheless interpret this divergence physically in terms of the logarithmic evolution of the cosmological constant. This assigns physical significance to the a-anomaly even though it has a topological aspect.

Next, the value of the a-anomaly, and therefore its physical significance, depends on the duality frame. Concretely, one might choose to dualize any number of antisymmetric tensors into scalars, or vice versa, affecting the trace of the energy momentum tensor in the process. Therefore such dualizations are not symmetries.

In quantum field theory one must always ask whether the addition of a local counter term changes the situation. Presently the inequivalence is captured by a topological contribution to the a_4 coefficient of the logarithmically divergent term in the effective action (4.2.4). In conformal field theories such contributions are due to nonlocal terms in the effective action [91] and the geometric origin is the same here. The inequivalence is therefore robust under additions of local counter terms.

4.3.2 AdS$_4$ SUGRA from 11D.

The default spectrum of $\mathcal{N} = 8$ supergravity summarized in Table 4.3 comprises 70 scalars and no antisymmetric tensors. Comparing tables 4.4 and 4.5 we find that a duality frame where exactly five scalars are represented instead as antisymmetric tensors exhibits no trace anomaly.

It turns out that this precise number is a natural expectation when approaching supergravity in AdS$_4$ as compactification of 11D supergravity on S^7. The 11D 3-form with components a_{IJK} is reduced into various lower forms in 4D including 3-forms and 2-forms,

$$\begin{aligned} a_{\mu\nu\sigma}(x,y) &= b_{\mu\nu\sigma}(x)Y(y), \\ a_{\mu\nu p}(x,y) &= b_{\mu\nu}(x)Y_p^{(CE)}(y) + \tilde{b}_{\mu\nu}(x)Y_p^{(E)}(y). \end{aligned} \tag{4.3.2}$$

The AdS$_4$ coordinates are denoted by x and greek indices, while their S^7 counterparts are y coordinates and latin indices. The functions $Y(y)$, $Y_p^{(CE)}(y), Y_p^{(E)}(y)$ are spherical harmonics on S^7 that are respectively a scalar, a coexact 1-form, and an exact 1-form.

The 2-tensor $\tilde{b}_{\mu\nu}(x)$ is the coefficient of $Y^{(E)}(y) = dY(y)$ which is effectively a scalar on

S^7 so there is one of these modes, while $b_{\mu\nu}(x)$ is the coefficient of $Y^{(CE)}(y) = {}^*dY(y)$ which is effectively a transverse vector on S^7 with six modes. Thus there is a total of $1 + 6 = 7$ 2-tensors in the effective 4D theory as one would also expect from toroidal compactification of 11D supergravity to 4D. Classically these seven antisymmetric tensors can be dualized to seven scalars but in view of quantum inequivalence this must be done with care.

The 3-form tensor $b_{\mu\nu\sigma}(x)$ is the coefficient of the ordinary spherical harmonic so there is just one of these fields in four dimensions. A massless 3-form has no propagating degrees of freedom in four dimensions since the classical equations of motion force it to be constant. At the quantum level gauge fixing of the 3-form gives two 2-form ghosts with fermi statistics, three 1-form ghosts with bose statistics, and four scalar ghosts with fermion statistics. This counting gives $4 - 2 \cdot 6 + 3 \cdot 4 - 4 \cdot 1 = 0$ net components and so no propagating degrees of freedom, as expected. However, as we repeatedly stress, 2-forms must be handled with care at the quantum level and that applies also to the two ghosts that accompany the 3-form tensor. At the quantum level one 3-form tensor contributes with (-2) 2-forms that cannot be naively dualized to scalars.

In summary, the duality frame that arises naturally through the AdS_4 compactification of 11D supergravity on S^7 gives a net of five antisymmetric 2-tensors:

$$1 + 6 - 2 = 5. \tag{4.3.3}$$

In this duality frame the trace of the energy momentum tensor vanishes and there are no divergences [27].

This result does not invalidate our claim that there are divergences in $\mathcal{N} = 8$ supergravity. On the contrary, it implicitly confirms the notion that different duality frames are quantum inequivalent since otherwise the distinction between 2-forms and scalars would be meaningless and there would be no utility in counting 2-forms arising from Kaluza-Klein reduction of 11D supergravity. From the low energy effective field theory point of view it is legitimate to consider AdS_4 supergravity with any number of 2-forms, including none at all, although it must be understood that such theories might not arise in string theory [92] and they could be vulnerable to some subtle quantum inconsistency.

In this chapter we focus on massless fields in 4D but the computations can be generalized to the full KK tower of massive fields. All these contributions are again proportional to the Gauss-Bonnet invariant and, level by level, they are nonvanishing. One may sum over all KK towers and recast the remaining divergences in 11D where they become power law divergences. They generally appear at the four derivative order but in the duality frame favored by 11D supergravity they only appear at six derivative order. However, eleven is odd and in odd dimensions all these divergences are nonuniversal and scheme dependent so it is not clear that they are physical. The divergence that is definitely physical is again a logarithm which is due to zero-modes of the two form gauge symmetry. These zero modes were understood from the 11D perspective [54] and the resulting logarithmic correction agrees with the one expected from the solution of the dual ABJM theory via localization [93, 94, 95].

4.3.3 Boundary Modes in AdS_4.

The evanescent part of the 2-form — the quantum contribution of an antisymmetric tensor that is above and beyond that of its dual scalar field — is naturally interpreted as a boundary mode, at least in the context of AdS_4. A boundary mode is formally a pure gauge field configuration but it is physical because the putative gauge parameter is non normalizable and so the field configuration cannot be gauged away by any element of the symmetry group. This mechanism is unimportant in classical field theory but it matters in the quantum theory, as expected for a feature related to quantum inequivalence.

The boundary modes reside in the kernel of the classical duality transformation (4.3.1) between an antisymmetric tensor and a scalar. Their 3-form field strength vanishes identically in bulk, since they are formally pure gauge, and so they are not assigned a scalar dual

$$H_{\mu\nu\sigma}^{(\text{bndy mode})} = 0 = \epsilon_{\mu\nu\sigma\lambda}\nabla^{\lambda}\phi, \tag{4.3.4}$$

since a constant scalar ϕ is not normalizable in noncompact spacetimes. This is the source of quantum inequivalence from our point of view.

A priori any field with gauge symmetry might possess one or more boundary modes. For example, in global AdS_2 all fields with a gauge symmetry have them [49, 7] (also explicitly shown in previous chapters of this work). On the other hand, in AdS_{d+1} with higher d it was found by explicit construction in global AdS_{d+1} that boundary modes exist only for $\frac{d+1}{2}$-forms [96, 54]. In AdS_4 those are precisely the 2-forms that we are interested in.

To make the discussion explicit we write the background AdS_4 metric

$$ds_4^2 = \ell_A^2\left(d\rho^2 + \sinh^2\rho d\Omega_3^2\right). \tag{4.3.5}$$

We take the AdS_4 radius $\ell_A = 1$ in the remainder of this section to avoid cluttered formulae. The normalized boundary modes in this background are

$$\begin{aligned} B_{\rho i} &= \sqrt{\frac{k+1}{2}}\frac{1}{\sinh\rho}\tanh^{k+1}(\rho/2)\Theta_i^{(k,\sigma)}(\Omega_3), \\ B_{ij} &= \sqrt{\frac{1}{2(k+1)}}\tanh^{k+1}(\rho/2)[\tilde{\nabla}_i\Theta_j^{(k,\sigma)}(\Omega_3) - \tilde{\nabla}_j\Theta_i^{(k,\sigma)}(\Omega_3)]. \end{aligned} \tag{4.3.6}$$

for $k = 1, 2, \ldots$. The covariant derivative $\tilde{\nabla}$ refers to components along S^3 and latin indices represent these angular components. The 1-form field $\Theta_i^{(k,\sigma)}(\Omega_3)$ is a vector spherical harmonic with eigenvalue of the Hodge de Rham operator $= (k+1)^2$. The quantum numbers k, σ are analogous to the numbers lm used for scalar harmonics on S^2 but for vector harmonics $k = 0$ is excluded.

The antisymmetric 2-form with components (4.3.6) can be represented as pure gauge

$B = dA$ where the 1-form potential A has components

$$A_\rho = 0, \qquad (4.3.7)$$
$$A_i = \frac{1}{\sqrt{2(k+1)}} \tanh^{k+1}(\rho/2)\Theta_i^{(k,\sigma)}(\Omega).$$

This 1-form does not have finite norm

$$\int \sqrt{g}|A|^2 dV = \int \sinh^3 \rho (A^*_\rho A_\rho g^{\rho\rho} + A^*_j A_l g^{jl}) d\rho d\Omega$$
$$\propto \int_0^\infty \sinh \rho \tanh^{2k+2}(\rho/2) d\rho = \infty. \qquad (4.3.8)$$

The inverse metric g^{jl} contributes with a factor of $\sinh^{-2}(\rho)$ that dampens the radial integral at large ρ, but insufficiently to render it finite. However, the tensor $B = dA$ is normalizable for all $k = 1, 2, \ldots$.

$$\int \sqrt{g}|B|^2 dV = \int \sinh^3 \rho (2B^*_{\rho i} B_{\rho j} g^{\rho\rho} g^{ij} + B^*_{ij} B_{lk} g^{ik} g^{jl}) d\rho d\Omega$$
$$\propto \int_0^\infty \sinh^{-1}(\rho) \tanh^{2k+2}(\rho/2) d\rho < \infty. \qquad (4.3.9)$$

The index structure here gives enough factors of the inverse metric g^{jl}, contributing each with $\sinh^{-2}(\rho)$, such that their product with the field components is sufficient to overcome the volume factor. The normalization in (4.3.6) was chosen so that the integral (4.3.9) is unity. The 2-tensor has support in bulk but we interpret it as a boundary mode because it is locally pure gauge.

Once we have identified a 1-form A that gives rise to a 2-form boundary mode $B = dA$ we should note that gauge equivalent 1-forms $A' = A + d\Lambda$ give rise to the same boundary mode. The boundary modes thus belong to the two-form cohomology. In order to not overcount them we must impose a gauge condition, taken in (4.3.7) as $A_\rho = 0$.

In summary: while the 2-form modes (4.3.6) are formally pure gauge they are physical because the would-be gauge function is non normalizable. Therefore, they contribute to the quantum path integral. Moreover, we have argued that unlike all other modes of the massless 2-form field, the boundary modes are not captured by the scalar dual. We focus on the massless case for clarity but the quantum inequivalence between a massive 2-form and its (classically) dual massive vector is similarly due to boundary modes for the 2-form.

4.3.4 Counting Boundary Modes.

We can find the contribution of the boundary modes to the heat kernel and related quantities by explicitly counting modes, following [96, 54]. The wave function of each mode is normalized to unity so the total number of modes is

$$n_{\text{bndy modes}} = \sum_{\text{all modes}} \int d^4x\sqrt{g}|B|^2. \tag{4.3.10}$$

The sum in equation (4.3.10) is over the family of modes presented in (4.3.6) that is parametrized by the quantum numbers k, σ.

$$\begin{aligned}\int d^4x\sqrt{g}\sum|B|^2 &= \int d^4x\sqrt{g}\sum(2B^*_{\rho i}B_{\rho j}g^{\rho\rho}g^{ij} + B^*_{ij}B_{lk}g^{ik}g^{jl}), \\ &= \int d^4x\sqrt{g}\sum_{k,\sigma}2\frac{k+1}{2}\frac{\tanh^{2k+2}(\rho/2)}{\sinh^4\rho}|\Theta_i^{(k,\sigma)}(\Omega)|^2, \\ &+ \int d^4x\sqrt{g}\sum_{k,\sigma}\frac{1}{2(k+1)}\frac{\tanh^{2k+2}(\rho/2)}{\sinh^4\rho}|\tilde{\nabla}_i\Theta_j^{(k,\sigma)}(\Omega_3) - \tilde{\nabla}_j\Theta_i^{(k,\sigma)}(\Omega_3)|^2.\end{aligned} \tag{4.3.11}$$

We can simplify this sum using integration by parts on the angular dependence of the second term,

$$\begin{aligned}\int|\tilde{\nabla}_i\Theta_j^{(k,\sigma)} - \tilde{\nabla}_j\Theta_i^{(k,\sigma)}|^2 d\Omega_3 &= -2\int\Theta^{j(k,\sigma)*}\tilde{\nabla}^i(\tilde{\nabla}_i\Theta_j^{(k,\sigma)} - \tilde{\nabla}_j\Theta_i^{(k,\sigma)})d\Omega_3 \\ &= 2(k+1)^2\int\Theta^{j(k,\sigma)*}\Theta_j^{(k,\sigma)}d\Omega_3.\end{aligned} \tag{4.3.12}$$

In the last step we identified the operator acting on $\Theta_j^{(k,\sigma)}$ as minus the Hodge de Rham operator acting on vectors. We insert this result into (4.3.11), combining both contributions into one. One could evaluate the sum over modes at any point but, given that AdS_4 is homogeneous, it is sufficient to consider the origin $\rho = 0$ where only the $k = 1$ spherical harmonic contributes,

$$\begin{aligned}\sum_{\text{all modes}} |B|^2 &= \lim_{\rho\to 0}\sum_{k=1}\sum_{\sigma} 2(k+1)|\Theta_i^{(k,\sigma)}(\Omega)|^2 \frac{\tanh^{2k+2}(\rho/2)}{\sinh^4\rho}, \\ &= \frac{1}{4}\sum_{\sigma}|\Theta_i^{(1,\sigma)}(\Omega)|^2.\end{aligned} \tag{4.3.13}$$

The sum over $|\Theta_i^{(k,\sigma)}(\Omega)|^2$ for fixed k is proportional to the degeneracy of the S^3 vector spherical harmonics,

$$\sum_{\sigma} \Theta^{(k,\sigma)*i}(\Omega)\Theta_i^{(k,\sigma)}(\Omega)\bigg|_{k=1} = \frac{6}{\mathrm{Vol}_{S^3}} = \frac{3}{\pi^2}, \tag{4.3.14}$$

since there are $2k(k+2) = 6$ vector spherical harmonics on S^3 with $k = 1$. Collecting formulae, the number of boundary modes (4.3.10) becomes

$$n_{\text{bndy modes}} = \sum_{\text{all modes}} \int d^4x\sqrt{g}|B|^2 = \frac{3}{4\pi^2}\int d^4x\sqrt{g} = 1. \tag{4.3.15}$$

We used the standard regulated volume $\mathrm{Vol}_{\mathrm{AdS}_4} = \frac{4\pi^2}{3}$ since then the result looks nice and intuitive. However, in the current context of a noncompact and maximally symmetric space we should really focus on the density of modes. Indeed, the boundary modes have vanishing eigenvalue of the kinetic operator so they formally contribute by the "number" $D_0^{(\text{bndy})} = n_{\text{bndy modes}}$ to the constant part D_0 of the heat kernel $D(t)$ and this corresponds to the heat kernel density

$$K_0^{(\text{bndy})} = \frac{D_0^{(\text{bndy})}}{\mathrm{Vol}_{\mathrm{AdS}_4}} = \frac{3}{4\pi^2}, \tag{4.3.16}$$

independently of the value assigned to $\text{Vol}_{\text{AdS}_4}$. Comparing with the definition of a_4 in (4.2.3) and the introduction of the c, a anomaly coefficients in (4.2.6) we find

$$a^{(\text{bndy})} = -\frac{1}{2}. \tag{4.3.17}$$

since the Euler density $E_4 = 24$ in AdS_4 with unit radius $\ell_A = 1$.

The value (4.3.17) of the boundary anomaly agrees precisely with the a anomaly of the evanescent difference between a massless antisymmetric tensor and a scalar reported in Table 4.5. This quantitative agreement shows that the quantum inequivalence between an antisymmetric tensor and a dual scalar field is due to boundary modes. This in turn establishes a physical distinction between the inequivalent fields.

4.3.5 The Gauss-Bonnet Theorem in AdS_4.

We have emphasized the divergences that remain in AdS_4 even for maximal SUSY and their interrelation with quantum inequivalence, because these aspects are the most interesting to us and they have not been developed in recent literature. Another approach to one-loop effects that is closer aligned with conventional wisdom invokes reflecting boundary conditions on all modes [97, 98, 71, 72]. This leads to a discrete sum over modes, the helicity sum rule (4.1.2) applies in full, and there are no divergences at one loop (and well beyond). The relation between these apparently incompatible results involves global aspects of AdS_4, as captured by the Euler invariant. It is therefore instructive to evaluate the Euler invariant in detail.

The curvature tensor in a maximally symmetric spacetime is constant so the Gauss-Bonnet integral over the Euler density is proportional to the volume

$$\int E_4 = \int \text{Tr}\; \mathcal{R} \wedge^* \mathcal{R} = 24 \int e^{\hat{0}} e^{\hat{1}} e^{\hat{2}} e^{\hat{3}} = 24 \text{Vol}_{\text{AdS}_4}. \tag{4.3.18}$$

For global AdS_4 with metric (4.3.5) we regulate the volume by a surface at some constant value radial ρ_0 and find

$$\mathrm{Vol}_{\mathrm{AdS}_4} = 2\pi^2 \int_0^{\rho_0} d\rho \sinh^3\rho = 2\pi^2 \left(\frac{1}{3}\cosh\rho_0(\sinh^2\rho_0 - 2) + \frac{2}{3} \right). \tag{4.3.19}$$

Recall that we take $\ell_A = 1$ at this point of the chapter. The boundary term added when considering the Gauss-Bonnet theorem with a boundary is [81]

$$-2\int \epsilon_{abcd}\theta^a{}_b\mathcal{R}^c{}_d + \frac{4}{3}\int \epsilon_{abcd}\theta^a{}_b\theta^c{}_e\theta^e{}_d = -24\cdot\frac{1}{3}\cosh\rho_0(\sinh^2\rho_0 - 2)2\pi^2, \tag{4.3.20}$$

where the second fundamental form $\theta^a{}_b$ is essentially the connection 1-form and has nonvanishing components

$$\theta_{\hat{\rho}\hat{i}} = \omega_{\hat{\rho}\hat{i}} = -\frac{\cosh\rho}{\sinh\rho}e^{\hat{i}}. \tag{4.3.21}$$

The sum of the bulk and boundary terms then gives

$$\chi = \frac{1}{32\pi^2}\cdot 24\cdot\frac{2}{3}\cdot 2\pi^2 = 1, \tag{4.3.22}$$

after including the correct overall numerical factor already quoted in (4.1.3). The cancellation of the terms that diverge at large ρ_0 is guaranteed by topological invariance and the role of the boundary terms is to make this happen. The finite term that remains is essentially the regularized volume of AdS_4, except for the constant factor $E_4 = 24$.

In the context of AdS/CFT it is possible to add counter terms that are local on the boundary. However, such terms depend on the infra-red cut-off through the functional form $\sinh\rho_0$ taken to an odd power; they are therefore not able to change the finite value $\chi = 1$.

The important point is that AdS_4 with $S^1 \times S^2$ boundary works out qualitatively differently. The metric is thermal AdS_4

$$ds_4^2 = \cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2. \tag{4.3.23}$$

Taking the circumference of S^1 to be β, the bulk term (4.3.18) with a regulator in the new radial coordinate ρ gives

$$24\mathrm{Vol}_{\mathrm{AdS}_4} = 24\int_0^{\rho_0} \cosh\rho \sinh^2\rho d\rho \cdot \beta \cdot 4\pi = 32\pi\beta \sinh^3\rho_0, \tag{4.3.24}$$

and the boundary term is

$$-4\int \theta_{\hat{\rho}\hat{i}} R_{\hat{j}\hat{k}} \epsilon^{\hat{\rho}\hat{i}\hat{j}\hat{k}} + \frac{4}{3}\int \epsilon_{abcd}\theta^a{}_b\theta^c{}_e\theta^e{}_d = -8\sinh^3\rho_0 \cdot 4\pi\beta. \tag{4.3.25}$$

The sum vanishes,

$$\chi = 0. \tag{4.3.26}$$

The difference in topology is significant because the divergence and the corresponding physical logarithm depends on topology. We primarily study global AdS_4 with S^3 boundary conditions because for $\chi = 1$ there is a divergence. In thermal AdS_4 the boundary is $S^1 \times S^2$ and the S^1 guarantees a discrete spectrum. This gives technical simplifications but it also excludes the divergence altogether since then $\chi = 0$.

Quantum inequivalence between antisymmetric tensors and scalar fields also depends on the Euler number χ so similar comments apply. In AdS_4 with S^3 boundary conditions there is quantum inequivalence which we interpret as due to boundary modes. In AdS_4 with $S^1 \times S^2$ boundary there is quantum equivalence and no boundary modes. Thus it appears that there is a precise sense in which the number of boundary modes is $n^{\mathrm{bndy}} = \chi$ despite the subtleties due to noncompactness of AdS_4.

Chapter 5

Quantum Corrections to Massive Multiplets in $\mathcal{N} = 8$ Supergravity.

The goal of this chapter is to extend some of the calculations done in chapter 4 for the $\mathcal{N} = 8$ massless supergravity multiplet to massive multiplets. We work in the context of $\mathcal{N} = 1$ supergravity in $\mathrm{AdS}_4 \times S^7$ compactified around S^7. The massive multiplets arise as Kaluza-Klein towers, the spectrum of which can be obtained by diagonalizing the equations of motion of eleven dimensional supergravity using a suitable Freund-Rubin ansatz. The diagonalization of the equations of motion is detailed in [99].

We must warn the reader that this chapter ends in a puzzle: we expect, given our experience with the massless multiplet, that there will be no divergence in the duality frame arising from eleven dimensional supergravity. However, the boundary heat kernels we will compute contain singular terms that are not canceled by any of the terms in the bulk contribution. This departure from our understanding of the massless multiplet is the puzzle.

With that in mind we take a pragmatic route in the next few sections, reporting only the pieces that go into our computation, without spending much time interpreting open research results.

5.1 Heat Kernels for spin s fields.

To compute the quantum corrections to $\mathcal{N}=8$ supergravity in AdS$_4$ we follow the strategy of the previous chapters and employ heat kernel regularization. Hence we must define the heat kernel expressions for all fields present in the massive $\mathcal{N}=8$ multiplets.

In general one performs a large mass expansion to extract the leading behavior of a heat kernel. However, throughout the literature of supergravity a few different definitions of mass have been employed. To avoid ambiguities we choose to work with the conformal dimensions of the fields in AdS$_4$, as stablished by the formulae below.

The spin s transverse symmetric tensor $T_{\mu_1\mu_2\ldots\mu_s}$ in AdS$_{d+1}$ satisfies

$$\nabla^\mu T_{\mu\mu_2\ldots\mu_s} = 0, \tag{5.1.1}$$

$$(-\nabla_A^2 + \kappa) T_{\mu_1\mu_2\ldots\mu_s} = 0, \tag{5.1.2}$$

$$-\nabla^\mu[\nabla_\mu T_{\mu_1\ldots\mu_s} + \nabla_{\mu_1} T_{\mu_2\ldots\mu_s\mu} + \text{all cyclic permutations}] + m^2 T_{\mu_1\ldots\mu_s} = 0. \tag{5.1.3}$$

The quantities κ and m can be related by commuting covariant derivatives,

$$\kappa = m^2 + \frac{(d+s-1)s}{\ell_A^2}. \tag{5.1.4}$$

Here ℓ_A is the radius of AdS$_4$. The conformal dimension of the spin s tensor is related to the mass eigenvalues through

$$\begin{aligned}\Delta &= \frac{1}{2}\left(d + \sqrt{d^2 + 4(\ell_A^2\kappa + s)}\right), \\ &= \frac{1}{2}\left(d + \sqrt{(d+2s)^2 + 4\ell_A^2 m^2}\right).\end{aligned} \tag{5.1.5}$$

This relation can be presented alternatively as

$$\left(\Delta - \frac{d}{2}\right)^2 = \ell_A^2\kappa + s + \frac{1}{4}d^2. \tag{5.1.6}$$

Now that we have established that the conformal dimensions will play the role of the mass parameter, we write heat kernels as functions of them. The heat kernel of a field in AdS_{d+1} is defined in terms of $(\Delta - \frac{d}{2})$ as [85, 100]

$$K_A(t, s, \Delta) = \frac{c_{d+1}}{\Omega_d \ell_A^4}(2s+1)e^{-\frac{t}{\ell_A^2}(\Delta-\frac{d}{2})^2}\int_0^\infty d\lambda \mu_s(\lambda)e^{-\frac{t}{\ell_A^2}\lambda^2}, \tag{5.1.7}$$

Where $c_{d+1} = \frac{2^{d-1}}{\pi}$, ℓ_A is the radius of AdS, and Ω_d is the volume of a d-sphere. A fermionic heat kernel differs from a bosonic one only in the Plancherel measure $\mu_s(\lambda)$.

Note that in this chapter we represent the heat kernel parameter by t as opposed to s, since we are interested in having the spin s as an extra variable.

The last piece needed to compute AdS_4 heat kernels are the Plancherel measures $\mu_s(\lambda)$. For integer spin ($s = 0, 1, 2,$) the Plancherel measures are [85],

$$\mu_s^b(\lambda) = \frac{\lambda^2 + (s+\frac{1}{2})^2}{16}\pi\lambda\tanh(\pi\lambda), \tag{5.1.8}$$

while for half integer spin ($s = \frac{1}{2}, \frac{3}{2},$) they are [85],

$$\mu_s^f(\lambda) = \frac{\lambda^2 + (s+\frac{1}{2})^2}{16}\pi\lambda\coth(\pi\lambda). \tag{5.1.9}$$

We now have all the ingredients to compute spin s heat kernels on AdS. We finish this section by stating the following expansions, which will be useful in the evaluation of bosonic and fermionic heat kernels, respectively [7]:

$$\int_0^\infty d\lambda\lambda\tanh(\pi\lambda)e^{-t\lambda^2}\lambda^{2n} \tag{5.1.10}$$
$$= \frac{1}{2t^{1+n}}\Gamma(1+n) + 2\sum_{k=0}^{\infty} t^k \frac{(2k+2n+1)!}{k!}(2\pi)^{-2k-2n-2}(-1)^k(2^{-2k-2n-1}-1)\zeta(2k+2n+2),$$

$$\int_0^\infty d\lambda\lambda\coth(\pi\lambda)e^{-t\lambda^2}\lambda^{2n} \tag{5.1.11}$$
$$= \frac{1}{2t^{1+n}}\Gamma(1+n) + 2\sum_{k=0}^{\infty} t^k \frac{(2k+2n+1)!}{k!}(2\pi)^{-2k-2n-2}(-1)^k\zeta(2k+2n+2).$$

Through inspection of (5.1.10) and (5.1.11) we see that the first two terms are

$$K_A(t,s,\Delta) = \frac{1}{(4\pi t)^2}e^{-\frac{t}{\ell_A^2}(\Delta-\frac{3}{2})^2}\left((2s+1)+(2s+1)^3\frac{t}{4\ell_A^2}+...\right)+\mathcal{O}(t^0). \tag{5.1.12}$$

The first term in parenthesis counts the number of degrees of freedom of a spin s representation. This is a general feature of heat kernels that should be familiar to us from chapter 2.

5.1.1 Spectrum of Supergravity on AdS$_4 \times S^7$

The on shell spectrum of AdS$_4 \times S^7$ supergravity compactified on the round S^7 is obtained by solving the linearized equations of motion. This was originally done in [101], but the review [99] has a very pedagogical exposition. The spectrum is summarized in Table 5.1.

Field	Δ	$SO(8)$ content	Degeneracy	Range of n
Graviton	$3+\frac{n}{2}$	$(n000)$	$\frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)=1,8,35,112...$	$n\geq 0$
Gravitino$^{(1)}$	$\frac{5}{2}+\frac{n}{2}$	$(n001)$	$\frac{1}{90}(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)=8,56,224,672...$	$n\geq 0$
Gravitino$^{(2)}$	$\frac{7}{2}+\frac{n}{2}$	$(n-1010)$	$\frac{1}{90}n(n+1)(n+2)(n+3)(n+4)(n+5)=8,56,224...$	$n\geq 1$
Vector$^{(1)}$	$2+\frac{n}{2}$	$(n100)$	$\frac{1}{60}(n+1)(n+3)(n+4)^2(n+5)(n+7)=28,160,567,1568...$	$n\geq 0$
Pseudovector	$3+\frac{n}{2}$	$(n-1011)$	$\frac{1}{24}n(n+1)(n+3)^2(n+5)(n+6)=56,350,1296...$	$n\geq 1$
Vector$^{(2)}$	$4+\frac{n}{2}$	$(n-2100)$	$\frac{1}{60}(n-1)(n+1)(n+2)^2(n+3)(n+5)=28,160...$	$n\geq 2$
Spinor$^{(1)}$	$\frac{3}{2}+\frac{n}{2}$	$(n+1010)$	$\frac{1}{90}(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)=56,224,672,1680...$	$n\geq 0$
Spinor$^{(2)}$	$\frac{5}{2}+\frac{n}{2}$	$(n-1110)$	$\frac{1}{18}n(n+2)(n+3)(n+4)(n+5)(n+7)=160,840,2800...$	$n\geq 1$
Spinor$^{(3)}$	$\frac{7}{2}+\frac{n}{2}$	$(n-2101)$	$\frac{1}{18}(n-1)(n+1)(n+2)(n+3)(n+4)(n+6)=160,840...$	$n\geq 2$
Spinor$^{(4)}$	$\frac{9}{2}+\frac{n}{2}$	$(n-2001)$	$\frac{1}{90}(n-1)n(n+1)(n+2)(n+3)(n+4)=8,56...$	$n\geq 2$
Scalar$^{(1)}$	$1+\frac{n}{2}$	$(n+2000)$	$\frac{1}{360}(n+3)(n+4)(n+5)^2(n+6)(n+7)=35,112,294,672...$	$n\geq 0$
Pseudoscalar$^{(1)}$	$2+\frac{n}{2}$	$(n020)$	$\frac{1}{36}(n+1)(n+2)(n+3)(n+5)(n+6)(n+7)=35,224,840,2400...$	$n\geq 0$
Scalar$^{(2)}$	$3+\frac{n}{2}$	$(n-2200)$	$\frac{1}{18}(n-1)(n+2)(n+3)^2(n+4)(n+7)=300,1400...$	$n\geq 2$
Pseudoscalar$^{(2)}$	$4+\frac{n}{2}$	$(n-2002)$	$\frac{1}{36}(n-1)n(n+1)(n+3)(n+4)(n+5)=35,224...$	$n\geq 2$
Scalar$^{(3)}$	$5+\frac{n}{2}$	$(n-2000)$	$\frac{1}{360}(n-1)n(n+1)^2(n+2)(n+3)=1,8...$	$n\geq 2$

Table 5.1: Matter content of $\mathcal{N}=8$ Supergravity compactified on the seven sphere with the conformal dimensions, Dynkin labels, and degeneracies. The degeneracies values are shown explicitly up to $n=4$.

- The superscripts $^{(1)},^{(2)}$... label different towers whenever there is more than one.

- The SO(8) content is given in terms of the Dynkin labels $(p_1p_2p_3p_4)$, where all entries p_i are non negative integers. The degeneracy corresponding to the representation $(p_1p_2p_3p_4)$ is

$$\begin{aligned} d[(p_1p_2p_3p_4)] =&(1+p_1)(1+p_2)(1+p_3)(1+p_4)(1+\frac{p_1+p_2}{2})(1+\frac{p_2+p_3}{2})(1+\frac{p_2+p_4}{2}) \\ &(1+\frac{p_1+p_2+p_3}{3})(1+\frac{p_2+p_3+p_4}{3})(1+\frac{p_1+p_2+p_4}{3}) \\ &(1+\frac{p_1+p_2+p_3+p_4}{4})(1+\frac{p_1+2p_2+p_3+p_4}{5}). \end{aligned} \tag{5.1.13}$$

  It is explicit that $d[(p_1p_2p_3p_4)]$ is invariant under permutations of p_1, p_3, p_4. This is the famous SO(8) triality.

- The index n is an integer with the ranges specified in Table 5.1 and it labels each supersymmetric multiplet. The massless multiplet, studied in chapter 4, corresponds to $n = 0$. Not all towers are present at such lower values of n, a subtlety we will analyze later.

5.2 Heat Kernels in AdS$_4 \times S^7$ - Bulk Contribution.

To compute the bulk contribution to the heat kernel, we insert the conformal dimensions from Table 5.1 in the heat kernel expression (5.1.7), then we expand them in t using (5.1.10) and (5.1.11). In the next subsections we will go through a few cases studying their form.

5.2.1 Large values of n.

At large n all 15 towers are present. We will combine the contribution for each field in the following way,

$$K_{\text{Total}} = \sum_{\sigma=0,1,2}\sum_{\Delta} d[(p_1p_2p_3p_4)]K_A^b(t,s,\Delta) - \sum_{s=\frac{1}{2},\frac{3}{2}}\sum_{\Delta} d[(p_1p_2p_3p_4)]K_A^f(t,s,\Delta). \qquad (5.2.1)$$

$d[(p_1p_2p_3p_4)]$ are the degeneracies listed on Table 5.1, and the sums run over all the towers present. The minus sign of the second term was added by hand to account for fermi statistics. The heat kernels $K_A^b(t,s,\Delta)$ and $K_A^f(t,s,\Delta)$ were defined in (5.1.7).

Before computing the complete result, it is instructive to look at some of the individual contributions. The heat kernel corresponding to gravitons on AdS_4 is given by only one tower with conformal weight $\Delta = 3+\frac{n}{2}$, degeneracy $d[(n000)]$, and $n \geq 0$,

$$\begin{aligned}
K_{\text{Graviton}} &= d[(n000)]K_A^b(t,2,3+\frac{n}{2}) \qquad (5.2.2)\\
&= \frac{3}{4\pi^2\ell_A^4}\cdot d[(n000)]\cdot 5\cdot\Bigg(\frac{1}{12}\frac{\ell_A^4}{t^2} - \frac{(-16+6n+n^2)}{48}\frac{\ell_A^2}{t}\\
&+ \frac{(-5792-2880n+60n^2+180n^3+15n^4)}{5760}\\
&+ \frac{(613888+729792n+303072n^2+37800n^3-6300n^4-1890n^5-105n^6)}{483840}\frac{t}{\ell_A^2} + \mathcal{O}(t^2)\Bigg).
\end{aligned}$$

We expanded the expression $K_A^b(t,2,3+\frac{n}{2})$ using (5.1.10). The overall factor of $\frac{3}{4\pi^2\ell_A^4}$ is the inverse regulated volume of AdS_4.

The explicit $n-$dependence allows us to set n to any value $n \geq 0$ and obtain the graviton contribution to an arbitrary massive multiplet ($n > 0$) or to the massless multiplet[1] ($n = 0$). The n-dependence on the $\mathcal{O}(t^{-2})$ term is exclusively due to the degeneracy factor

$$d[(n000)] = \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5), \qquad (5.2.3)$$

[1]In order to correctly account for the 2 polarizations of the massless graviton one needs to subtract the contribution of a vector ghost as an extra step. We will do so later.

while the n-dependence on the next terms receives contributions from $e^{-\frac{t}{\ell_A^2}(\Delta-\frac{3}{2})^2} = e^{-\frac{t}{\ell_A^2}(\frac{3}{2}+\frac{n}{2})^2}$. We factored out the degeneracy factor to keep the expression cleaner.

The next simplest example is the gravitino contribution, which consists of two towers:

$$\begin{aligned} K_{\text{Gravitino fields}} &= d[(n001)]K_A^f(t, \frac{3}{2}, \frac{5}{2}+\frac{n}{2}) + d[(n-1010)]K_A^f(t, \frac{3}{2}, \frac{7}{2}+\frac{n}{2}) \\ &= \frac{3}{4\pi^2\ell_A^4} \cdot d[(n000)] \cdot 4 \cdot \left(\frac{2}{3}\frac{\ell_A^4}{t^2} - \frac{-12+6n+n^2}{6}\frac{\ell_A^2}{t} - \frac{169}{90} - \frac{5n}{2} + \frac{n^2}{3} + \frac{n^3}{4} + \frac{n^4}{48} \right. \\ &\left. + \frac{43360+230832n+129192n^2+7560n^3-8820n^4-1890n^5-105n^6}{60480}\frac{t}{\ell_A^2} + \mathcal{O}(t^2) \right). \end{aligned} \tag{5.2.4}$$

It is noteworthy that although the individual gravitino towers have degeneracies

$$d[(n001)] = \frac{1}{90}(n+1)(n+2)(n+3)(n+4)(n+5)(n+6) \tag{5.2.5}$$

$$d[(n-1010)] = \frac{1}{90}n(n+1)(n+2)(n+3)(n+4)(n+5) \tag{5.2.6}$$

when combined they are proportional to the degeneracy $d[(n000)]$ (5.2.3). This is true for all spins: while individual towers have different n-dependence, their combinations into sets of the same spin are all proportional to $d[(n000)]$.

We summarize the heat kernels for all spins in Table 5.2.

Spin	$\frac{3}{4\pi^2\ell_A^4}K(t)/d[(n000)]$
2	$5\cdot\left[\frac{1}{12}\frac{\ell_A^4}{t^2} - (-16+6n+n^2)\frac{1}{48}\frac{\ell_A^2}{t} + (-5792-2880n+60n^2+180n^3+15n^4)\frac{1}{5760}\right]$
$\frac{3}{2}$	$4\cdot\left[\frac{2}{3}\frac{\ell_A^4}{t^2} - (-12+6n+n^2)\frac{1}{6}\frac{\ell_A^2}{t} - \frac{169}{90} - \frac{5n}{2} + \frac{n^2}{3} + \frac{n^3}{4} + \frac{n^4}{48}\right]$
1	$3\cdot\left[\frac{9}{4}\frac{\ell_A^4}{t^2} - (-80+54n+9n^2)\frac{1}{16}\frac{\ell_A^2}{t} + (-1376-2880n+1140n^2+540n^3+45n^4)\frac{1}{640}\right]$
$\frac{1}{2}$	$2\cdot\left[4\frac{\ell_A^4}{t^2} + (20-3n(6+n))\frac{1}{3}\frac{\ell_A^2}{t} - \frac{29}{15} + (n(n+6)(-4+n(n+6)))\frac{1}{8}\right]$
0	$\left[\frac{7}{2}\frac{\ell_A^4}{t^2} - 7(-16+18n+3n^2)\frac{1}{24}\frac{\ell_A^2}{t} + 7(-224+180n^2+60n^3+5n^4)\frac{1}{320}\right]$

Table 5.2: Heat kernel expansions for fields in $N=8$ supergravity compactified around S^7 up to the term constant in t, with $n \geq 2$.

We now compute the contribution due to all bosons and fermions separately. The bulk bosonic contribution is

$$
\begin{aligned}
K_{\text{Bosons}} =& \sum_{\text{Spin }2} d[(p_1p_2p_3p_4)]K_A^b(t,2,\Delta) + \sum_{\text{Spin }1} d[(p_1p_2p_3p_4)]K_A^b(t,1,\Delta) \\
&+ \sum_{\text{Spin }0} d[(p_1p_2p_3p_4)]K_A^b(t,0,\Delta) \\
=& \frac{3}{4\pi^2\ell_A^4}\cdot d[(n000)]\Bigg[\frac{32}{3}\frac{\ell_A^4}{t^2} - \frac{8(-8+n(n+6))}{3}\frac{(a_4)^2}{t} \\
&- \frac{737}{45} + \frac{n(n+6)(-8+n(n+6))}{3} + \left(\frac{8698}{945} - \frac{n(n+6)(-977+5n^2(n+6)^2)}{180}\right)\frac{t}{\ell_A^2} + \mathcal{O}(t^2)\Bigg].
\end{aligned} \tag{5.2.7}
$$

Each of the sums in the first line collects the individual contribution of towers with that spin; the spin two contribution is due to only one tower, while the spin one and zero contributions are due to three and five towers respectively. Not all bosonic towers are defined for $n = 0, 1$. Thus, this result as it stands is only valid for $n \geq 2$.

The bulk fermionic contribution is similarly computed,

$$
\begin{aligned}
K_{\text{Fermions}} =& \sum_{\text{Spin }\frac{3}{2}} d[(p_1p_2p_3p_4)]K_A^f(t,\frac{3}{2},\Delta) + \sum_{\text{Spin }\frac{1}{2}} d[(p_1p_2p_3p_4)]K_A^f(t,\frac{1}{2},\Delta) \\
=& \frac{3}{4\pi^2\ell_A^4}\cdot d[(n000)]\Bigg[\frac{32}{3}\frac{\ell_A^4}{t^2} - \frac{8(-8+n(n+6))}{3}\frac{\ell_A^2}{t} \\
&- \frac{512}{45} + \frac{n(n+6)(-8+n(n+6))}{3} + \left(\frac{6808}{945} - \frac{n(n+6)(-752+5n^2(n+6)^2)}{180}\right)\frac{t}{\ell_A^2} + \mathcal{O}(t^2)\Bigg].
\end{aligned} \tag{5.2.8}
$$

The two fundamental orders in t are identical to the bosonic contribution which implies that these orders will vanish in the full multiplet result, when relative signs due to statistics are taken into account.

We combine the results (5.2.7) and (5.2.8) to find the total contribution,

$$
K_{\text{Total,n}\geq 2}(t) = \frac{3}{4\pi^2\ell_A^4}\cdot d[(n000)]\Bigg[-5 + \frac{8+30n+5n^2}{4}\frac{t}{\ell_A^2} + \mathcal{O}(t^2)\Bigg]. \tag{5.2.9}
$$

We explicitly noted this as a $n \geq 2$ result since some of the towers are not defined at $n = 0, 1$.

The lower values of n are treated next.

The leading divergences cancel as expected, and furthermore the n-dependence of the full result was simplified, since the $\mathcal{O}(t^0)$ term is a 6th order polynomial in n as opposed to a 10th order polynomial as in the contributions (5.2.7) and (5.2.8).

5.2.2 Low values of n.

We compute the heat kernels for $n = 1$ and $n = 0$, which are multiplets with fewer towers present. At $n = 1$, we have nine modes:

Field	Δ	$SO(8)$ content	Degeneracy
Graviton	$\frac{7}{2}$	(1000)	8
Gravitino$^{(1)}$	3	(1001)	56
Gravitino$^{(2)}$	4	(0010)	8
Vector$^{(1)}$	$\frac{5}{2}$	(1100)	160
Pseudovector	$\frac{7}{2}$	(0011)	56
Spinor$^{(1)}$	2	(2010)	224
Spinor$^{(2)}$	3	(0110)	160
Scalar$^{(1)}$	$\frac{3}{2}$	(3000)	112
Pseudoscalar$^{(1)}$	$\frac{5}{2}$	(2020)	224

Table 5.3: $n = 1$ multiplet.

This is a massive multiplet so we can compute the heat kernel for the full multiplet the same way we did for $n \geq 2$,

$$K_{\text{Total,n=1}}(t) = \frac{3}{4\pi^2\ell_A^4}\bigg(-40 + 86\frac{t}{\ell_A^2} + \mathcal{O}(t^2)\bigg). \tag{5.2.10}$$

This is the $n \to 1$ continuation of the result found for $n \geq 2$.

At $n = 0$ there are six modes:

This is the massless multiplet, so we have to account for ghosts. We find the ghost

Field	Δ	$SO(8)$ content	Degeneracy
Graviton	3	(0000)	1
Gravitino$^{(1)}$	$\frac{5}{2}$	(0001)	8
Vector$^{(1)}$	2	(0100)	28
Spinor$^{(1)}$	$\frac{3}{2}$	(1010)	56
Scalar$^{(1)}$	1	(2000)	35
Pseudoscalar$^{(1)}$	2	(0020)	35

Table 5.4: Massless multiplet.

conformal dimensions according to [102]:

$$\Delta_{\text{ghost}} = \Delta_{\text{field}} + 1. \tag{5.2.11}$$

We start with the contribution due to the graviton. The massive graviton has 5 components, so we must subtract the contribution of a vector field to obtain the two components of a massless graviton. We then evaluate the heat kernel of a spin two object with $\Delta = 3$ minus the heat kernel of a spin one object with $\Delta = 4$,

$$\begin{aligned} K_{\text{n=0 Graviton}}(t) &= K^b_A(t,2,3) - K^b_A(t,1,4), \\ &= \frac{3}{4\pi^2\ell_A^4}\left(\frac{1}{6}\frac{\ell_A^4}{t^2} + \frac{8}{3}\frac{\ell_A^2}{t} - \frac{571}{90} + \frac{4918}{945}\frac{t}{\ell_A^2} + \frac{5441}{1890}\frac{t^2}{\ell_A^4}\right) + \mathcal{O}(t^3). \end{aligned} \tag{5.2.12}$$

The massive gravitino has 4 components, which is two more than a massless gravitino. We then subtract the contribution of a spinor field with $\Delta = \frac{7}{2}$ from the $\Delta = \frac{5}{2}$ gravitino,

$$\begin{aligned} K_{\text{n=0 Gravitino}}(t) &= K^f_A(t,\frac{3}{2},\frac{5}{2}) - K^f_A(t,\frac{1}{2},\frac{7}{2}), \\ &= \frac{3}{4\pi^2\ell_A^4}\left(\frac{1}{6}\frac{\ell_A^4}{t^2} + \frac{3}{2}\frac{\ell_A^2}{t} - \frac{589}{360} + \frac{1405}{1512}\frac{t}{\ell_A^2} - \frac{3277}{10080}\frac{t^2}{\ell_A^4}\right) + \mathcal{O}(t^3). \end{aligned} \tag{5.2.13}$$

The massive vector has 3 components, which is one too many for a massless vector field. We must subtract from this $\Delta = 2$ vector field the contribution of a scalar field with $\Delta = 3$,

$$\begin{aligned} K_{\text{n=0 Vector}}(t) &= K^b_A(t,1,2) - K^b_A(t,0,3), \\ &= \frac{3}{4\pi^2\ell_A^4}\left(\frac{1}{6}\frac{\ell_A^4}{t^2} + \frac{2}{3}\frac{\ell_A^2}{t} - \frac{31}{90} + \frac{26}{189}\frac{t}{\ell_A^2} - \frac{13}{270}\frac{t^2}{\ell_A^4}\right) + \mathcal{O}(t^3). \end{aligned} \tag{5.2.14}$$

The massless spinor needs no ghost subtractions, it is given by

$$\begin{aligned} K_{\text{n=0 Spinor}}(t) &= K^f_A(t,\frac{1}{2},\frac{3}{2}) \\ &= \frac{3}{4\pi^2\ell_A^4}\left(\frac{1}{6}\frac{\ell_A^4}{t^2} + \frac{1}{6}\frac{\ell_A^2}{t} + \frac{11}{360} - \frac{31}{7560}\frac{t}{\ell_A^2} + \frac{41}{30240}\frac{t^2}{\ell_A^4}\right) + \mathcal{O}(t^3). \end{aligned} \tag{5.2.15}$$

The two massless scalar modes contribute with

$$\begin{aligned} K_{\text{n=0 Scalars(t)}} &= K^b_A(t,0,1) = K^b_A(t,0,2) \\ &= \frac{3}{4\pi^2\ell_A^4}\left(\frac{1}{12}\frac{\ell_A^4}{t^2} - \frac{1}{180} + \frac{2}{945}\frac{t}{\ell_A^2} - \frac{1}{1260}\frac{t^2}{\ell_A^4}\right) + \mathcal{O}(t^3). \end{aligned} \tag{5.2.16}$$

Both scalars contribute identically in spite of their different conformal dimensions[2]: the dependence on Δ is in the factor $e^{-\frac{t}{\ell_A^2}(\Delta-\frac{3}{2})^2}$ which is the same for $\Delta = 1$ and $\Delta = 2$.

We assemble the contributions due to all massless modes according to their multiplicities from group theory,

$$K_{\text{Total,n=0}}(t) = \frac{3}{4\pi^2\ell_A^4}\Bigg(-5 + 2\frac{t}{\ell_A^2} + 4\frac{t^2}{\ell_A^4} + \mathcal{O}(t^3)\Bigg). \tag{5.2.17}$$

This is the $n \to 0$ continuation of the result found for $n \geq 2$. Moreover this is the result reported in 4 in which just the massless multiplet was analyzed.

Finally, the bulk contribution to the heat kernel of the n-th $\mathcal{N} = 8$ supergravity multiplet

[2]The a anomalies of the conformal scalar and minimally coupled scalar are the same, though there might be subtleties related to anomalies proportional to R^2.

is

$$K(t) = \frac{3}{4\pi^2 \ell_A^4} \cdot d[(n000)] \Big[-5 + \frac{8 + 30n + 5n^2}{4} \frac{t}{\ell_A^2} + \mathcal{O}(t^2) \Big]. \tag{5.2.18}$$

5.3 Boundary Modes.

We want to analyze the boundary mode contribution to the one loop determinant. Recall that boundary modes are formally pure gauge field configurations with a non normalizable gauge parameter. The non normalizable nature of the gauge parameter implies that no ghost subtractions cancel the boundary mode contribution, since that would require non normalizable ghosts. Thus, their contribution is physical even though they are pure gauge.

Schematically, we are looking for some p-form A_p such that

$$A_p = d\lambda_{p-1}, \tag{5.3.1}$$

with

$$\begin{aligned} \int |A_p|^2 dV &< \infty, \\ \int |\lambda_{p-1}|^2 dV &= \infty. \end{aligned} \tag{5.3.2}$$

A priori any field with gauge symmetry[3] could present boundary mode configurations, but in practice the conditions 5.3.2 can only be satisfied for a subset of the gauge fields in a given theory.

In the present context we are interested in boundary modes living in AdS_4. The available fields with continuous symmetry are those inherited from the eleven dimensional supergravity action prior to any dualizations: a graviton, gravitini, and p-forms with $1 \leq p \leq 3$ arising

[3]Gravitons are good candidates for boundary modes, which would be pure diffeomorphism configurations. Here we use the term gauge symmetry in a broad sense that includes diffeomorphisms.

from the KK compactification of the 11D 3-form.

Camporesi and Higuchi [96] show that the only p-form boundary modes in AdS_N are harmonic forms with $p = \frac{N}{2}$. Moreover, Bhattacharyya et al. [54] also argue that there are no graviton or gravitini boundary modes in AdS_4. This leaves us with 2-forms as the only possible source of boundary modes in AdS_4. This is a great simplification since we do not have to worry about off-shell actions for all the other fields. Furthermore, we can restrict ourselves to harmonic modes.

We can see how 2-forms arise by revisiting the compactification of an 11 dimensional 3-form a_{IJK} around S^7:

$$
\begin{aligned}
a_{\mu\nu\rho}(x,y) &= b_{\mu\nu\rho}(x)Y(y) \\
a_{\mu\nu p}(x,y) &= b_{\mu\nu}(x)Y_p^{(CE)}(y) + \tilde{b}_{\mu\nu}(x)Y_p^{(E)}(y) \\
&\dots
\end{aligned}
\tag{5.3.3}
$$

We used greek indices and x coordinates for AdS_4 as well as lowercase latin indices and y coordinates for S^7. The functions $Y_p^{(E)}(y), Y_p^{(CE)}(y)$, and $Y(y)$ are respectively an exact 1-form, coexact 1-form and scalar spherical harmonics on S^7.

The AdS_4 3-form $b_{\mu\nu\rho}(x)$ does not produce boundary modes itself, but its quantization will involve the subtraction of two 2-form ghosts, that can produce boundary modes. The definitive list of boundary modes for 11D supergravity compactified around S^7 is then $b_{\mu\nu}(x), \tilde{b}_{\mu\nu}(x)$, and the 2-form ghosts of $b_{\mu\nu\rho}(x)$.

5.3.1 Where are the anomalies?

Following the discussion established in chapter 4, when analyzing the on-shell spectrum in Table 5.1 we must look for any dualization that was done without proper concern to anomalies. To do that one goes back to the equations of motion [101] and keeps track of the 2- and 3-form fields that were present in the beginning. The sources of anomalies in the on-shell spectrum from Table 5.1 are

- Both vector towers, which came from the mixing of a dualized 2-form and a true vector.
- The scalar towers (1) and (3), which came from the mixing of a dualized 3-form with three true scalars (plus constraints).

The classical duality relations between the partition functions of 3-forms, 2-forms and their duals are given in [25, 26]:

Massive fields:

$$Z_3 = Z_0 - \chi, \tag{5.3.4}$$

$$Z_2 = Z_1 + \chi. \tag{5.3.5}$$

Massless fields:

$$Z_3 = -2\chi, \tag{5.3.6}$$

$$Z_2 = Z_0 + \chi. \tag{5.3.7}$$

Where Z_p is the partition function of a p-form field, and χ here is the anomaly related to the Euler characteristic. For massless fields, our understanding of these anomalies is that they arise from boundary modes of the 2-forms. Concretely, the field strength $H = dB$ is identically zero for a pure gauge configuration,

$$0 = H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma}\partial^\sigma \phi, \tag{5.3.8}$$

corresponding to constant dual scalar, which is non normalizable in a non compact space. Since boundary modes are physical albeit pure gauge, they only are counted on the 2-form side of the duality. This is the source of quantum inequivalence in our point of view.

3-forms have no boundary modes of their own in AdS_4. However their quantization encompasses 2-forms, which we represent mnemonically as

$$\text{Physical 3-form} = 1(\text{3-form}) - 2(\text{2-form}) + 3(\text{1-form}) - 4(\text{0-form}). \tag{5.3.9}$$

That is, the quantization involves subtraction of two 2-form ghosts, such that the anomaly of a 3-form is -2χ, where χ is the anomaly observed for a 2-form.

In the context of the massive forms, the dualities change. Boundary modes are still exclusive to 2-forms, but now the dualization of a 2-form B to a 1-form A is

$$B = \star dA \rightarrow B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\nabla^\rho A^\sigma. \tag{5.3.10}$$

Normalizable pure gauge configurations of the 2-form $B = d\lambda$ with a non-normalizable gauge parameter λ are now dual to the field strength of a 1-form A. This implies that A is also non-normalizable; we don't count them when evaluating the path integral. Quantum inequivalence in the context of massive forms is in a sense more straightforward: non-normalizable 1-forms show up in both sides of the duality, on the 2-form side they are merely gauge parameters and do generate physical configurations, while on the 1-form side they are the field configurations themselves and are not counted.

The next step is to make sense of how the 3-form anomaly $-\chi$ is generated. We know that any anomalous contribution has to come from 2-forms, and that these will show up in the quantization of the 3-form (that was the case for massless forms). We approach the correct ghost counting prescription by looking at how degrees of freedom are counted. A standard trick to count degrees of freedom (dof) for massive N dimensional forms is to look at N+1 dimensional massless forms. We get

$$\begin{aligned}\text{5D massless physical 3-form} &= 1(\text{3-form}) - 2(\text{2-form}) + 3(\text{1-form}) - 4(\text{0-form}) \quad (5.3.11)\\ &= 1(10) - 2(10) + 3(5) - 4(1) = 1 \text{ dof},\\ \text{5D massless physical 2-form} &= 1(\text{2-form}) - 2(\text{1-form}) + 3(\text{0-form}) \quad (5.3.12)\\ &= 1(10) - 2(5) + 3(1) = 3 \text{ dof}.\end{aligned}$$

However, this trick can be quite opaque when one is already working within a compactification. The formal procedure involves repeating the ghost counting in [70] for massive fields, but for simplicity, we present here the counting of ghosts and support it with an example.

Using the same mnemonic strategy as before, the ghost counting is given by

$$\begin{aligned}\text{4D massive physical 3-form} &= (\text{3-form}) - (\text{2-form}) + (\text{1-form}) - (\text{0-form}) \quad (5.3.13)\\ &= 1(4) - (6) + (4) - (1) = 1 \text{ dof},\\ \text{4D massive physical 2-form} &= 1(\text{2-form}) - (\text{1-form}) + (\text{0-form}) \quad (5.3.14)\\ &= 1(6) - 1(4) + (1) = 3 \text{ dof}.\end{aligned}$$

This argument that the ghosts for massive forms appear with $-1, +1$ alternating factors instead of $-2, +3, -4...$ matches what one would expect for a 3-form in N dimensions

$$\begin{aligned}
\text{(N+1)-dim massless physical 3-form} &= 1(\text{3-form}) - 2(\text{2-form}) + 3(\text{1-form}) - 4(\text{0-form}) \\
&= 1\left(\frac{(N+1)N(N-1)}{3!}\right) - 2\left(\frac{(N+1)N}{2!}\right) + 3(N+1) - 4(1) \\
&= \frac{N^3}{6} - N^2 + \frac{11N}{6} - 1 \text{ dof,}
\end{aligned} \tag{5.3.15}$$

$$\begin{aligned}
\text{N-dim massive physical 3-form} &= 1(\text{3-form}) - (\text{2-form}) + (\text{1-form}) - (\text{0-form}) \\
&= \frac{N(N-1)(N-2)}{3!} - \frac{N(N-1)}{2} + N - 1 \\
&= \frac{N^3}{6} - N^2 + \frac{11N}{6} - 1 \text{ dof,}
\end{aligned} \tag{5.3.16}$$

(5.3.17)

a similar check is true for 2-forms.

The upshot is that this ghost counting implies that only one 2-form ghost shows up in the quantization of a massive 3-form, giving our expected $-\chi$ anomaly.

5.3.2 Heat kernels for Boundary Modes.

Boundary modes arising on $\mathrm{AdS}_4 \times S^7$ live on the sphere, and therefore their heat kernel is of lower dimension when compared to bulk modes. In the previous section we computed the bulk contribution to each multiplet according to their label n. Those heat kernels had the general form

$$K_A(t, s, \Delta) = \frac{1}{(4\pi t)^2} e^{-\frac{t}{\ell_A^2}(\Delta - \frac{3}{2})^2} \left((2s+1) + (2s+1)^3 \frac{t}{4\ell_A^2} + ... \right) + \mathcal{O}(t^0), \tag{5.3.18}$$

This is a 4-dimensional heat kernel -as denoted by the leading divergence $\frac{1}{4\pi t^2}$- where the sphere dependence is encoded in the conformal dimension Δ, which is n dependent. If one chooses to sum over n, with a suitable regularization scheme, the result is an 11-dimensional heat kernel.

The boundary mode heat kernels are in this setting just a polynomial in n. A regularized sum over n would yield a 7-dimensional heat kernel, which is what one naively expects from a field living on S^7. The upshot is that we are looking at each individual multiplet from the point of view of AdS_4, and in boundary modes are zero-dimensional fields in this context.

The off-shell spectrum of harmonic 2-forms in $\mathrm{AdS}_4 \times S^7$ supergravity is given in Table 5.5.

Mode	Off shell mass squared	Degeneracy	Range
$b_{\mu\nu}$	$k(k+6)+5$	$\frac{1}{60}k(k+2)(k+3)^2(k+4)(k+6)$	$k=1...$
$\tilde{b}_{\mu\nu}$	$k(k+6)$	$\frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5)$	$k=1...$
Ghosts	$k(k+6)$	$\frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5)$	$k=0...$

Table 5.5: Off shell spectrum of the harmonic 2-forms.

The contribution of each boundary mode is

$$\begin{aligned} K^{\text{bndy}}_{b_{\mu\nu}} &= \frac{1}{\mathrm{Vol}_{11\mathrm{D}}} \sum_{k=1} e^{-\frac{t}{\ell_S^2}(k+1)(k+5)} \frac{1}{60} k(k+2)(k+3)^2(k+4)(k+6) \\ &= \frac{3}{4\pi^2 \ell_A^4} \cdot \frac{1}{(4\pi t)^{7/2}} e^{\frac{4t}{\ell_S^2}} \left(6 - \frac{24}{\ell_S^2} t + \frac{72}{5\ell_S^4} t^2\right) + \frac{3}{4\pi^2 \ell_A^4} \cdot \frac{3}{\pi^4 \ell_S^7}. \end{aligned} \tag{5.3.19}$$

$$\begin{aligned} K^{\text{bndy}}_{\tilde{b}_{\mu\nu}} &= \frac{1}{\mathrm{Vol}_{11\mathrm{D}}} \sum_{k=1} e^{-\frac{t}{\ell_S^2}k(k+6)} \frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5) \\ &= \frac{3}{4\pi^2 \ell_A^4} \cdot \frac{1}{(4\pi t)^{7/2}} e^{\frac{9t}{\ell_S^2}} \left(1 - \frac{2}{\ell_S^2} t + \frac{16}{15\ell_S^4} t^2\right) - \frac{3}{4\pi^2 \ell_A^4} \cdot \frac{3}{\pi^4 \ell_S^7}. \end{aligned} \tag{5.3.20}$$

$$
\begin{aligned}
K^{\text{bndy}}_{\text{ghost}} &= \frac{1}{\text{Vol}_{11\text{D}}} \sum_{k=0} e^{-\frac{t}{\ell_S^2}k(k+6)} \frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5) \qquad (5.3.21)\\
&= \frac{3}{4\pi^2\ell_A^4} \cdot \frac{1}{(4\pi t)^{7/2}} \left[1 + \frac{7}{\ell_S^2}t + \frac{707}{30\ell_S^4}t^2 + \frac{501}{10\ell_S^6}t^3 + \frac{2943}{40\ell_S^8}s^4 \cdots \right].
\end{aligned}
$$

Here we represented the radius of AdS_4 and S^7 as ℓ_A and ℓ_S, respectively.

We close this chapter restating the puzzle advertised in the introduction: our experience with the massless multiplet tells us that combination of the bulk heat kernels with the boundary mode heat kernels plus zero modes yields the total contribution. In addition to that, the duality frame arising from eleven dimensional supergravity has vanishing anomaly. One would then expect the massive multiplet result parallel that behavior. As it stands the boundary mode heat kernels produce terms singular in t that are not canceled by any term in the bulk heat kernel (5.2.18).

One possibility is that we are missing some extra boundary mode contribution, that when combined with the ones we presented yields a total boundary heat kernel with no leading divergences. In fact, in the characterization of the boundary modes for the AdS_4 supergravity theory, we used results from [96] claiming that the only p-forms that produce boundary modes are 2-forms and from [54] claiming that gravitini and gravitons produce no boundary modes in our setting. It is conceivable that some special class of boundary modes was overlooked in the literature, and that this is our missing piece.

Further investigation of these statements is a direction currently being explored.

Appendix A

Generalized Eigenvectors.

Repeated eigenvalues and generalized eigenvectors play an important role in our solutions so here we review a few of their features.

An elementary example with an eigenvalue that is repeated twice is the nonhermitean 2×2 matrix

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \tag{A.0.1}$$

with two eigenvalues identical to 2. There is only one true eigenvector

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.0.2}$$

but there also a generalized eigenvector

$$\eta_2 = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix}, \tag{A.0.3}$$

that satisfies the generalized eigenvalue equation

$$(M - \lambda I_2)^2 \eta_2 = 0, \tag{A.0.4}$$

with eigenvalue $\lambda = 2$. The generalized eigenvector η_2 is not a true eigenvector since

$$(M - \lambda I_2)\eta_2 = \begin{pmatrix} 0 & 1 \\ & \\ 0 & 0 \end{pmatrix} \eta_2 = \eta_1. \tag{A.0.5}$$

However, the generalized eigenvalue equation (A.0.4) follows because η_1 is a true eigenvector. Importantly, the determinant det M $= 2 \cdot 2 = 4$ is the product of eigenvalues even though one appearance of the repeated eigenvalue $\lambda = 2$ only allows a generalized eigenvector.

Generalized eigenvectors are ubiquitous in our setting because the linearized equations of motion have kinetic terms and mass-matrices that cannot be simultanously diagonalized. For example, the AdS$_2$ volume mode $H^{(00)}{}_{\rho}{}^{\rho}$ and the S^2 volume mode $h_{\alpha}{}^{\alpha} = \pi^{(00)}$ couple through the Lagrangean

$$\mathcal{L}^{l=0}_{\text{scalar}} = -\frac{1}{8} H^{(00)}{}_{\rho}{}^{\rho} (\nabla_A^2 - 2)\pi^{(00)} - \frac{1}{4}\pi^{(00)2}. \tag{A.0.6}$$

In the given basis the mass matrix is diagonal but the kinetic matrix is not. There is no basis where both are diagonal. The equations of motion are naturally presented in a form where $\pi^{(00)}$ sources ${H^{(00)}}_{\rho}^{\ \rho}$ but not the other way around

$$\nabla_x^2 \begin{pmatrix} {H^{(00)}}_{\rho}^{\ \rho} \\ \\ \pi^{(00)} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ \\ 0 & 2 \end{pmatrix} \begin{pmatrix} {H^{(00)}}_{\rho}^{\ \rho} \\ \\ \pi^{(00)} \end{pmatrix}. \tag{A.0.7}$$

The mass matrix is similar to (A.0.1) and the eigenvalue problem is analogous to the elementary one discussed above. $\pi^{(00)}$ is a true eigenvector but ${H^{(00)}}_{\rho}^{\ \rho}$ is just a generalized eigenvector satisfying

$$(\nabla_x^2 - 2)^2 {H^{(00)}}_{\rho}^{\ \rho} = 0. \tag{A.0.8}$$

We consider one additional example from our setting: the fields $b_{\parallel}^{(lm)}$, $B_{\parallel}^{(lm)}$, $\tilde{b}^{(lm)}$ for $l \geq 1$. The equations of motion (3.4.30):

$$(\nabla_x^2 - l(l+1)) \begin{pmatrix} B_{\parallel}^{(lm)} \\ \\ b_{\parallel}^{(lm)} \\ \\ \tilde{b}^{(lm)} \end{pmatrix} = \begin{pmatrix} 2 & 2 & -2 \\ \\ 2l(l+1) & 0 & 0 \\ \\ 4+2l(l+1) & 4 & -4 \end{pmatrix} \begin{pmatrix} B_{\parallel}^{(lm)} \\ \\ b_{\parallel}^{(lm)} \\ \\ \tilde{b}^{(lm)} \end{pmatrix}. \tag{A.0.9}$$

The 3×3 matrix on the RHS of (A.0.9) has one eigenvalue $\lambda = -2$ and also a repeated eigenvalue $\lambda = 0$. There are two conventional (true) eigenvectors and one generalized eigenvector; they are presented in Table A.1.

Mode	Mass	Comment
$b_{\parallel}^{(lm)} + B_{\parallel}^{(lm)} - \tilde{b}^{(lm)}$	$m^2 = l(l+1) - 2$	Conventional.
$b_{\parallel}^{(lm)} + 2B_{\parallel}^{(lm)} - \tilde{b}^{(lm)}$	$m^2 = l(l+1)$	Conventional.
$b_{\parallel}^{(lm)} + l(l+1)B_{\parallel}^{(lm)}$	$m^2 = l(l+1)$	Generalized.

Table A.1: Conventional and generalized eigenvectors for the 3×3 block with modes $b_{\parallel}^{(lm)}$, $B_{\parallel}^{(lm)}$, $\tilde{b}^{(lm)}$.

The generalized eigenvector satisfies

$$[\nabla_A^2 - l(l+1)](b_\parallel^{(lm)} + l(l+1)B_\parallel^{(lm)}) = -(b_\parallel^{(lm)} + 2B_\parallel^{(lm)} - \tilde{b}^{(lm)}). \tag{A.0.10}$$

The RHS is a true eigenvector of $[\nabla_A^2 - l(l+1)]$ with eigenvalue $\lambda = 0$ so the higher order operator $[\nabla_A^2 - l(l+1)]^2$ annihilates the generalized eigenvector $b_\parallel^{(lm)} + l(l+1)B_\parallel^{(lm)}$.

The contribution to the functional determinant from these fields is computed correctly by multiplication of all eigenvalues whether they are repeated or not. Thus, the complications due to generalized eigenvectors are not an issue as far as the heat kernels are concerned.

Appendix B

Tensor Modes on the Boundary.

We want to identify residual diffeomorphisms that are not fixed by our gauge. A 2D diffeomorphism generated by ξ_μ gives rise to a traceless symmetric tensor

$$H_{\{\mu\nu\}} = \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu - g_{\mu\nu} \nabla_\rho \xi^\rho. \tag{B.0.1}$$

The gauge condition $\nabla^\mu H_{\{\mu\nu\}} = \frac{1}{2} \nabla_\nu \pi$ with the 2D scalar π invariant is preserved iff the vector ξ_μ satisfies

$$(\nabla_A^2 - 1)\xi_\mu = 0. \tag{B.0.2}$$

For Kähler metrics on the disc we can rewrite the holomophic component of (B.0.2) as

$$2g^{z\bar{z}} \nabla_{\bar{z}} \nabla_z \xi_z = 0. \tag{B.0.3}$$

The covariant derivative is $\nabla_{\bar{z}} = \partial_{\bar{z}}$ when acting on an object with lower holomorphic indices so the solutions are those where $\nabla_z \xi_z$ are holomorphic. The induced tensor $H_{\{\mu\nu\}}$ is therefore a *quadratic holomorphic differential.*

We consider the holomorphic differential $\nabla_z \xi_z = z^{n-2}$ with $n \geq 2$. The holomorphic derivative is

$$\nabla_z \xi_z = g_{z\bar{z}} \partial_z (g^{z\bar{z}} \xi_z) = g_{z\bar{z}} \partial_z \xi^{\bar{z}}, \tag{B.0.4}$$

so

$$\partial_z \xi^{\bar{z}} = \frac{1}{2a}(1-|z|^2)^2 z^{n-2}, \tag{B.0.5}$$

and upon integration we find

$$\xi^{\bar{z}} = \frac{1}{2a}\left(\frac{1}{n-1}z^{n-1} - \frac{2\bar{z}}{n}z^n + \frac{\bar{z}^2}{n+1}z^{n+1}\right). \tag{B.0.6}$$

This explicit form shows that we must indeed take $n \neq 0, \pm 1$. For $n \geq 2$ the vector exists but it is not normalizable

$$\int_{|z|\leq 1} |\xi_z|^2 \sqrt{g} d^2 z = \int_{|z|\leq 1} |\xi_z \xi_z^*| \; d^2 z = \int_{|z|\leq 1} |g_{z\bar{z}}\xi^{\bar{z}}|^2 \; d^2 z \to \infty, \tag{B.0.7}$$

since $g_{z\bar{z}}$ diverges as $|z| \to 1$ while $|\xi^z|$ remains finite.

Importantly the quadratic holomorphic differential generated by the non-normalizable vector is finite

$$\int_{|z|\leq 1} |\nabla_z \xi_z|^2 \; \sqrt{g} d^2 z = \int_{|z|\leq 1} g^{z\bar{z}} |z|^{2(n-2)} \; d^2 z < \infty, \tag{B.0.8}$$

for $n \geq 2$ since $g^{z\bar{z}} = \frac{1}{2a}(1-|z|^2)^2$ is perfectly well behaved near the boundary at $|z| = 1$. We introduce the tensor modes

$$w_{zz}^{(n)} = \sqrt{\frac{|n|(n^2-1)}{2\pi}} z^{|n|-2}, \tag{B.0.9}$$

normalized such that

$$\int |w_{zz}^{(n)}|^2 \sqrt{g} d^2 z = 1. \tag{B.0.10}$$

With this normalization the sum over all tensors give

$$\begin{aligned} \sum_{n=2}^{\infty} |w_{zz}^{(n)}|^2 + \text{c.c.} &= \frac{1}{2a^2} \sum_{n=-1}^{\infty} (1-|z|^2)^4 \cdot \frac{n(n^2-1)}{2\pi} \cdot |z|^{2(n-2)} \\ &= \frac{1}{4\pi a^2}(1-x)^4 \partial_x^3 \frac{1}{1-x} = \frac{3}{2\pi a^2}. \end{aligned} \tag{B.0.11}$$

This is three times the corresponding value for the normalized vector field derived from a non-normalizable scalar. In that case we referred to a single boundary mode so we interpret the result for the tensor as three boundary modes. There are of course infinitely many boundary modes enumerated by the index n but there are three per unit volume.

Appendix C

Gravitino Modes on the Boundary.

We want to find normalizable pure gauge gravitini constructed out of non normalizable spinor parameters. We start in analogy with the tensor boundary modes, studying the non normalizable solutions to Dirac's equation in AdS_2.

We choose the same gamma matrices as Sen [7] for easy reference:

$$\begin{aligned} \gamma^{\hat{\theta}} &= -\sigma^2, \\ \gamma^{\hat{\eta}} &= \sigma^1. \end{aligned} \tag{C.0.1}$$

We compute the twisted derivatives

$$\begin{aligned} D_\eta + \frac{1}{2}\gamma_\eta &= \partial_\eta + \frac{1}{2}\sigma^1, \\ D_\theta + \frac{1}{2}\gamma_\theta &= \partial_\theta + \frac{i}{2}\cosh\eta\sigma^3 - \frac{1}{2}\sinh\eta\sigma^2. \end{aligned} \tag{C.0.2}$$

The Dirac operator in the coordinates (3.2.19) with the gamma matrices (C.0.1) is

$$\not{D} = -\sigma^2 \frac{1}{\sinh\eta}\partial_\theta + \sigma^1\partial_\eta + \frac{1}{2}\sigma^1\coth\eta \tag{C.0.3}$$

We will work with $a = 1$ for now and restore it later. Camporesi and Higuchi [103], found the solutions

$$\chi_k^{\pm}(\lambda) = e^{i(k+\frac{1}{2})\theta} \begin{pmatrix} i\frac{\lambda}{k+1}\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}F(k+1+i\lambda, k+1-i\lambda; k+2; -\sinh^2\frac{\eta}{2}) \\ \\ \pm\cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}F(k+1+i\lambda, k+1-i\lambda; k+1; -\sinh^2\frac{\eta}{2}) \end{pmatrix} \tag{C.0.4}$$

and

$$\eta_k^{\pm}(\lambda) = e^{-i(k+\frac{1}{2})\theta} \begin{pmatrix} \cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}F(k+1+i\lambda, k+1-i\lambda; k+1; -\sinh^2\frac{\eta}{2}) \\ \\ \pm i\frac{\lambda}{k+1}\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}F(k+1+i\lambda, k+1-i\lambda; k+2; -\sinh^2\frac{\eta}{2}) \end{pmatrix} \tag{C.0.5}$$

which satisfy

$$\not{D}\ \chi_k^{\pm}(\lambda) = \pm i\lambda\chi_k^{\pm}(\lambda), \tag{C.0.6}$$

$$\not{D}\ \eta_k^{\pm}(\lambda) = \pm i\lambda\eta_k^{\pm}(\lambda). \tag{C.0.7}$$

The label k is a non-negative integer. The continuous spectrum is given by λ real and positive. However, these are not all the modes of the Dirac operator, for there are non normalizable discrete modes corresponding to $\lambda = i$. In this case the hypergeometric functions in (C.0.4) and (C.0.5) simplify,

$$\chi_k^{\pm}(i) = e^{i(k+\frac{1}{2})\theta} \begin{pmatrix} -\sinh\frac{\eta}{2}\tanh^k\frac{\eta}{2} \\ \\ \pm\frac{1}{2\cosh\frac{\eta}{2}}(1+2k+\cosh\eta)\tanh^k\frac{\eta}{2} \end{pmatrix}, \tag{C.0.8}$$

$$\eta_k^{\pm}(i) = e^{-i(k+\frac{1}{2})\theta} \begin{pmatrix} \frac{1}{2\cosh\frac{\eta}{2}}(1+2k+\cosh\eta)\tanh^k\frac{\eta}{2} \\ \\ \mp\sinh\frac{\eta}{2}\tanh^k\frac{\eta}{2} \end{pmatrix}, \tag{C.0.9}$$

For $k \geq 0$. These are solutions to the Dirac equation on AdS$_2$ with $m^2 = 1$. From now on we will refer to the solutions (C.0.8) and (C.0.9) as $\chi_k^{\pm}$ and $\eta_k^{\pm}$ for simplicity, since we only care about $\lambda = i$.

Using the complex coordinates defined in (3.2.19), the solutions (C.0.8) and (C.0.9) are

$$\chi_k^{\pm} = \begin{pmatrix} -(1-|z|^2)^{-\frac{1}{2}}|z|^{\frac{1}{2}} & \pm(1-|z|^2)^{\frac{1}{2}}|z|^{-\frac{1}{2}}(k+\frac{1}{1-|z|^2}) \end{pmatrix} z^{k+\frac{1}{2}} \tag{C.0.10}$$

$$\eta_k^{\pm} = \begin{pmatrix} (1-|z|^2)^{\frac{1}{2}}|z|^{-\frac{1}{2}}(k+\frac{1}{1-|z|^2}) \\ \\ \mp(1-|z|^2)^{-\frac{1}{2}}|z|^{\frac{1}{2}} \end{pmatrix} \bar{z}^{k+\frac{1}{2}} \tag{C.0.11}$$

The normalization condition for the spinors (C.0.10) and (C.0.11) is

$$\int \left[\frac{|z|}{1-|z|^2} + \frac{1-|z|^2}{|z|}\left(k+\frac{1}{1-|z|^2}\right)^2\right]|z|^{2k+1}\frac{2}{(1-|z|^2)^2}d^2z = \infty. \tag{C.0.12}$$

These are non normalizable modes. We want to construct gravitini solutions that are pure gauge with gauge function proportional to the discrete modes (C.0.10) and (C.0.11).

To construct the gravitini solutions we write the derivatives

$$zD_z = z\partial_z + \frac{1}{4}\frac{1+|z|^2}{1-|z|^2}\sigma^3, \tag{C.0.13}$$

and the holomorphic gamma matrix,

$$z\gamma_z = \frac{|z|}{1-|z|^2}(\sigma^1 + i\sigma^2). \tag{C.0.14}$$

Evaluation of the twisted holomorphic derivative yields

$$(D_z + \frac{1}{2}\gamma_z)\chi_k^+ = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} k(k+1)\Big(\frac{1-|z|^2}{|z|}\Big)^{\frac{1}{2}} z^{k-\frac{1}{2}}. \tag{C.0.15}$$

(C.0.15) is explicitly convergent at $|z| \to 1$. Since the normalization integral for gravitini can be evaluated with the unit metric on the disk, we already know (C.0.15) is normalizable. This is an advantage of working with complex coordinates. We compute the norm of (C.0.15),

$$\begin{aligned} \int k^2(k+1)^2\Big(\frac{1-|z|^2}{|z|}\Big)|z|^{2k-1}d^2z &= 2\pi k^2(k+1)^2 \int_0^1 \Big(\frac{1-x}{\sqrt{x}}\Big)x^{k-\frac{1}{2}}dx \\ &= 2\pi k(k+1). \end{aligned} \tag{C.0.16}$$

The normalized gravitino boundary mode is

$$\Psi_z = \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \sqrt{\frac{k(k+1)}{2\pi}}\Big(\frac{1-|z|^2}{|z|}\Big)^{\frac{1}{2}} z^{k-\frac{1}{2}}. \tag{C.0.17}$$

The gravitini Ψ_z are given for $k > 0$, since $k = 0$ is explicitly zero. The solutions (C.0.17) are normalizable modes that are pure gauge with a non normalizable gauge parameter. They are gravitino boundary modes.

Through a similar computation one finds the modes $(D_z + \frac{1}{2}\gamma_z)\chi_k^-$ to be non normalizable. Also, if one computes the norms of $(D_z - \frac{1}{2}\gamma_z)\chi_k^{\pm}$ in analogy with the previous case, one finds that the gravitini $(D_z - \frac{1}{2}\gamma_z)\chi_k^+$ are non normalizable, while $(D_z - \frac{1}{2}\gamma_z)\chi_k^-$ are.

This is easily seen by noting that

$$\chi_k^+ = \sigma^3 \chi_k^- . \tag{C.0.18}$$

Also, according to (C.0.1),

$$\begin{aligned} [D_\mu, \sigma^3] &= 0, \\ \{\gamma_\mu, \sigma^3\} &= 0. \end{aligned} \tag{C.0.19}$$

So that going from $(D_z + \frac{1}{2}\gamma_z)$ to $(D_z - \frac{1}{2}\gamma_z)$ can be achieved by multiplication with σ^3, which takes χ_k^+ into χ_k^- and vice-versa. In fact, $(D_z - \frac{1}{2}\gamma_z)\chi_k^-$ are given by

$$(D_z - \frac{1}{2}\gamma_z)\chi_k^- = \begin{pmatrix} 0 \\ \\ -1 \end{pmatrix} k(k+1)\left(\frac{1-|z|^2}{|z|}\right)^{\frac{1}{2}} z^{k-\frac{1}{2}}. \tag{C.0.20}$$

These are the modes (C.0.15) up to a multiplicative constant. Thus, one should not count them as additional modes.

We find the action of the antiholomorphic twisted derivative on χ_k^+ to vanish:

$$(D_{\bar{z}} + \frac{1}{2}\gamma_{\bar{z}})\chi_k^+ = 0. \tag{C.0.21}$$

When building gravitini out of the $\eta_k^\pm$ solutions, we find

$$\begin{aligned} (D_{\bar{z}} + \frac{1}{2}\gamma_{\bar{z}})\eta_k^+ &= \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} k(k+1)\left(\frac{1-|z|^2}{|z|}\right)^{\frac{1}{2}} \bar{z}^{k-\frac{1}{2}}, \\ (D_z + \frac{1}{2}\gamma_z)\eta_k^+ &= 0. \end{aligned} \tag{C.0.22}$$

and $(D_{\bar{z}} + \frac{1}{2}\gamma_{\bar{z}})\eta_k^-$ are non normalizable. The normalized antiholomorphic modes are

$$\bar{\Psi}_{\bar{z}} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} \sqrt{\frac{k(k+1)}{2\pi}} \left(\frac{1-|z|^2}{|z|}\right)^{\frac{1}{2}} \bar{z}^{k-\frac{1}{2}}, \tag{C.0.23}$$

for $k > 0$. The modes $\eta_k^- = \sigma^3 \eta_k^+$ are once again just (C.0.22) up to a phase.

In summary, the boundary modes we need to account for are (C.0.17) and (C.0.23). One important property of these modes is that they are (anti-)holomorphic differentials:

$$\begin{aligned} D_{\bar{z}}\Psi_z &= 0, \\ D_z\bar{\Psi}_{\bar{z}} &= 0. \end{aligned} \tag{C.0.24}$$

We have encountered a similar dependence for the tensor modes in (B.0.9). The gravitini modes are different in that they are not powers of z or $\bar{z}$, but instead have a $|z|$ dependent prefactor that is canceled by the spin connection.

Finally, we sum over all values of k in our boundary modes.

$$\begin{aligned} \sum_{k=1}^{\infty} |\Psi_z|^2 + |\bar{\Psi}_{\bar{z}}|^2 &= 2\sum_{k=1}^{\infty} \frac{k(k+1)}{2\pi a^2} \frac{(1-|z|^2)^3}{2} |z|^{2k-2}, \\ &= \frac{1}{2\pi a^2} \sum_{k=-1}^{\infty} (1-x)^3 \partial_x^2 x^{k+1}, \\ &= \frac{2}{2\pi a^2}. \end{aligned} \tag{C.0.25}$$

In the second equality we used the variable $x = |z|^2$, and added the empty entries $k = 0, -1$. In the last step we evaluated the geometric series and the partial derivatives. We have one mode per unit volume for the holomorphic gravitino (C.0.17) and one other mode for the antiholomorphic gravitino (C.0.23).

The four boundary modes accounted for in Section 7 are the modes in (C.0.25) times two supersymmetries.

Appendix D

Conventions for Gamma-matrices.

In this appendix we summarize conventions used in chapter 3, notations, and properties of gamma-matrices.

The upper case Γ_I refers to the 4D gamma matrices, while the lower case γ^μ, γ^α refer to AdS_2 and S^2, respectively. They satisfy:

$$
\begin{aligned}
\{\Gamma^I, \Gamma^J\} &= 2g^{IJ}, \\
\Gamma^\mu = \gamma^\mu \otimes \gamma_S, &\quad \Gamma^\alpha = 1 \otimes \gamma^\alpha, \\
[\gamma^\mu, \gamma^\alpha] &= 0,
\end{aligned}
\tag{D.0.1}
$$

Chiral projection operators in 4D and 2D, along with their relations:

$$
\begin{aligned}
&\Gamma_5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \gamma_A \otimes \gamma_S, \\
&\gamma_A = \gamma^{01}, \quad \gamma_S = i\gamma^{23}, \\
&[\gamma_A, \gamma_S] = 0, \\
&\gamma_A^2 = \gamma_S^2 = 1.
\end{aligned}
\tag{D.0.2}
$$

Conventions on orientation (all indices are local)

$$\epsilon^{0123} = +1,$$
$$\epsilon^{01} = +1, \quad \epsilon^{23} = +1. \tag{D.0.3}$$

Some useful identities,

$$\Gamma^{IJKL} = -i\Gamma_5\epsilon^{IJKL}, \quad \Gamma^{IJK} = -i\Gamma_5\epsilon^{IJKL}\Gamma_L,$$
$$\gamma_A\epsilon^{\mu\nu} = \gamma^{\mu\nu}, \quad \gamma_S\epsilon^{\alpha\beta} = i\gamma^{\alpha\beta}. \tag{D.0.4}$$

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