

### I. Time-Dependent Perturbation Theory

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Ph 205a  ADVANCED QUANTUM MECHANICS

R.P. Feynman

I. Scattering

II. Solid State Physics

III. Relativistic Quantum Theory

Find a relativistically invariant S-equation. *

TIME-DEPENDENT PERTURBATION THEORY

A. FIRST-ORDER PERTURBATION THEORY

1) The time evolution of a state is given by

\[ \frac{-i}{\hbar} \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \]

Now suppose that somehow we can split up \( H(t) \) into a constant part and a small time-varying part:

\[ H(t) = H_0 + V(t) \]

First consider the related problem for \( V = 0 \)

\[ \frac{-i}{\hbar} \frac{d}{dt} |\psi(t)\rangle = H_0 |\psi(t)\rangle \]

* all problems are decided by \( A \), and are to worked without reference to books.

* here \( \hbar = c = 1 \)
Assume we can solve this by
\[ \Psi(t) = \sum_n c_n(t) \phi_n \]  
where the \( \phi_n \) are defined by the eigenvalue problem
\[ H_0 \phi_n = E_n \phi_n \]  
and
\[ c_n(t) = \langle \phi_n | \Psi(t) \rangle \]
If we plug our expansion (4) into (3) we find
\[ -\frac{i}{\hbar} \frac{d}{dt} \Psi(t) = E_n c_n(t) \]
and solving directly
\[ c_n(t) = a_n e^{-iE_n t} \]
Since this worked for \( V(t) = 0 \), suppose we try it again
\[ \text{for } V(t) \text{ small} \]
\[ -\frac{i}{\hbar} \frac{d}{dt} \Psi(t) = [H_0 + V(t)] \Psi(t) \]
and write again
\[ \Psi(t) = \sum_n c_n(t) \phi_n \]
Substitute into (8)
\[ \sum_n -\frac{i}{\hbar} c_n(t) \phi_n = \sum_n c_n(t) H_0 \phi_n + \sum_n c_n(t) V(t) \phi_n(t) \]
Take the scalar product with $\langle \phi_m \rangle$ and use orthogonality to find

$$-\frac{1}{i} \dot{C}_m(t) = E_m C_m(t) + \frac{\sum_n \langle m | V(t) | n \rangle}{V_{mm}(t)}$$  \hspace{1cm} (11)$$

Now if $V(t)$ is small, try a solution like

$$C_m(t) = e^{-i E_m t} \chi_m(t)$$  \hspace{1cm} (12)$$

---

Note: $|C_m(t)|^2 = |\chi_m(t)|^2 = \text{prob. of being in state } m \text{ at time } t$

i.e. $\chi_m(t)$ differs from $C_m(t)$ by only a phase factor.

Plug this into (11)

$$-\frac{1}{i} \dot{\chi}_m(t) = \frac{\sum_n \chi_n(t) e^{i(E_m - E_n)t} \langle m | V(t) | n \rangle}{V_{mm}(t)}$$  \hspace{1cm} (13)$$

$$= \frac{\sum_n V_{mn}(t) \chi_n(t)}{V_{mm}(t)}$$  \hspace{1cm} (14)$$

This is still exact and is just a system of first order ODE's. But far from solvable exactly.
Consider a special problem:

At $t = 0$ we know we have state $k$. We want the amplitude at time $T$ that we are in state $l$, i.e.

$$\chi_{lk} = \langle \phi | V(T) | \chi \rangle$$

Claim: we can solve this problem for given $k$ and $l$, we can solve it for any $k$ and $l$.

Thus our initial condition becomes

$$t = 0: \quad \chi_5(0) = \delta_{sk}$$

The idea is to use an iterative process, i.e., integrate (14) for $\chi_5(t)$ and use various approximations to the integrand obtained from a lower order iteration. Thus in general

$$\chi_5(t) = \chi_5(0) - i \int_0^t \int_0^t \int_0^t \left[ \chi_6(0) - i \int_0^t \int_0^t \int_0^t \chi_{nk}(t') \chi_{lk}(t'') dt'' dt' \right] dt$$

$$= \chi_5(0) - i \int_0^t \int_0^t \chi_{5k}(t') \chi_{5l}(0) dt' + (-i)^2 \int_0^t \int_0^t \int_0^t \int_0^t \chi_{5k}(t') \chi_{5l}(t'') \chi_{5m}(t''') \chi_{5n}(0) dt''' dt'' dt'$$

$$+ (-i)^3 \int_0^t \int_0^t \int_0^t \int_0^t \int_0^t \chi_{5k}(t') \chi_{5l}(t'') \chi_{5m}(t''') \chi_{5n}(t'''' \chi_{5p}(0) dt''''' dt'''' dt''' dt'' dt'$$

$$+ \ldots$$
Now using our i.c. \( \chi_{s}(0) = \xi_{sk} \) in our general "solution" for \( \chi_{m} \)

\[
\chi_{nk} = \langle l \mid U(t) \mid l \rangle
\]

\[
= \chi_{l}(T)
\]

\[
= \xi_{sk} + (-i) \int_{0}^{T} \frac{d}{dt} \chi_{nk}(t) \xi_{nh} dt + (-i) \int_{0}^{T} \frac{d}{dt} \chi_{nk}(t) \xi_{nh} dt + \cdots
\]

Now suppose our perturbation was constant, i.e. \( U(t) = U \). Then

\[
\chi_{mn}(t) = \chi_{mn} e^{i(E_{m} - E_{n}) t}
\]

and from (16),

\[
\chi_{nk} = \xi_{sk} + (-i) \int_{0}^{T} \chi_{nk} e^{i(E_{m} - E_{n}) t} dt + \cdots
\]

\[
= \xi_{sk} - e^{i(E_{m} - E_{n}) t} \frac{1}{E_{m} - E_{n}} V_{nk} + \cdots
\]
Thus
\[
\{ \text{prob. } k \to 0 \} = \frac{1}{| \lambda_k |^2}
\]
\[
= \frac{4 \sin^2 \left( \frac{E_0 - E_k}{2} \right) t}{(E_0 - E_k)^2} \quad |V_{ew}|^2 + \ldots
\]

What is wrong with this?
**SUMMARY TO DATE**

**Problem:**
\[-i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle\]

**Approach:** Let \( H(t) = H_0 + V(t) \)

\[\Rightarrow H_0 |\Phi_n\rangle = E_n |\Phi_n\rangle\]

Write \( |\Psi(t)\rangle = \sum_n \alpha_n(t) e^{-iE_n t} |\Phi_n\rangle\)

Then find
\[i \frac{d}{dt} \alpha_n(t) = \sum_n \beta_{nm} \alpha_n(t) \]

where \( \beta_{nm} = e^{iE_{nm}} \langle \Phi_m | V(t) | \Phi_n \rangle e^{-iE_{nl}} \)

\[= e^{i(E_n - E_m) t} \langle \Phi_m | V(t) | \Phi_n \rangle\]

**Solution:**
\[\alpha_m(t_f) = \sum_n \beta_{nm} \alpha_n(t_i)\]  \hspace{1cm} (19)

This is if \( \beta(t_i) \) is the coefficient \( t_i \) at initial time, then the coefficient \( t_f \) at final time is equal to the sum of each of the initial values \( \beta(t_i) \) times the amplitude that \( \beta \) state will go into \( \beta(t_f) \) at final time. Evidently, \( \beta_{nm} = \beta_{nm}(t_f, t_i) \)

We found
\[\beta_{nm}(t_f, t_i) = \delta_{nm} - i \int_{t_i}^{t_f} \beta_{mn}(t_i) \beta_{nm}(t_f) dt_i + (-i)^2 \sum_{t_f > t_i > t_2 > t_3} \beta_{nm}(t_i) \beta_{mn}(t_f) dt_i dt_2 dt_3 + ...\]  \hspace{1cm} (20)
Suggested Exercises

i.) play a few games with (20) until you understand it. Perhaps write it in a better notation.

ii.) study its properties - run backwards in time, say given V\textsuperscript{m}(t\textsubscript{i}) and see what you can find out about V\textsuperscript{m}(t\textsubscript{f}).

iii.) check fact that total probability = 1. How does this affect \textsuperscript{m}?

iv.) if $V(t) = V$, can we compare results to energy partitioning theory?

v.) make up several problems (mesons, etc.) and work through.

2.) The First Born Approximation

Now we will consider cases where just the first integral term in (20) is important. Then we will look into the higher order terms.

Return to the d.c. potential where

$$V_{mn}(t) = e^{i(E_m - E_n)\frac{t}{2}} V_{mn}$$

We find from (20)

$$\lambda_{mn} = \frac{e^{i(E_m - E_n)\frac{t}{2}} - e^{i(E_m - E_n)\frac{t\textsubscript{i}}{2}}}{(E_m - E_n)} V_{mn}$$

Last time we had

$$t\textsubscript{i} = 0$$
$$t\textsubscript{f} = T$$

Now setting $t\textsubscript{f} - t\textsubscript{i} = T$, we find

$$\text{Prob}_{n\rightarrow m} = |\lambda_{mn}|^2 = \left| \frac{4 \sin^2 \left( \frac{(E_m - E_n)T}{2} \right)}{(E_m - E_n)^2} \right| |V_{mn}|^2 \quad (21)$$

Note this depends only on the relative time $T$, not on when we turn $V$ on.
Of course we can see that if \( E_m - E_n \) is very large, \( \text{Prob} \) is very small.

Note if \( m = n \), we get a \( \text{Prob} \sim T^2 \left| V_{mn} \right|^2 \) dependence for small \( T \). For larger \( T \) we need higher order terms.

To handle closely spaced levels, i.e., levels which in the limit form a continuum, we reason as follows. Our notation (27) for \( \text{Prob} \) gets into trouble since it is too precise. We smear out our approximation. The first step is a little and acceptable the probability of going into a range of states. Then

\[
\text{Prob} = \sum_{n \geq m} \text{Prob}_{n \rightarrow m} = \sum_{m} \frac{4 \sin^2 \left( \frac{E_m - E_n}{2} \right) T}{\left( E_m - E_n \right)^2} \left| V_{mn} \right|^2
\]

Now look at these terms. As \( T \) gets large, the details get squeezed and do not matter very much, so we can take the value at the limit \( \ell \rightarrow \infty \) and all of the contributions to the integral comes from near \( E = E_n \):

\[
\int_{-\infty}^{\infty} \frac{4 \sin^2 \left( \frac{E - E_n}{2} \right) T}{\left( E - E_n \right)^2} \delta(E_n) \, dE = 2T \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx = 2\pi T
\]

Thus we can interpret the individual probabilities and rates as

\[
\text{Rate} = \lim_{n \rightarrow m} \frac{\text{Prob}_{n \rightarrow m}}{\text{Area}} = 2\pi \delta(E_m - E_n) \left| V_{mn} \right|^2
\]

(22)

Fermi's Golden Rule

This doesn't always hold (least \( 99.9\% \) of time). We can modify this to handle a continuous distribution of the initial states.
EXAMPLE: Coulomb Excitation

Consider an α-particle passing a nucleus (assume a straight-line trajectory $\alpha$).

The nucleus sees a time-dependent Coulomb potential $V(t)$ from the $\alpha$. Sometimes this potential can excite the nucleus ("Coulomb excitation").

Consider $t_i = -\infty$ before collision $t_f = +\infty$ after collision

[Now as $t_i \to -\infty, t_f \to +\infty$, the limit is frequently denoted by $\phi_{\infty}$ in the literature. From the origin of $S$-matrix terminology.]

For $m \neq n$,

\[
\phi_{mn} = -i \int_{-\infty}^{\infty} e^{i(Em-E_n)t} V_{mn}(t) \, dt \quad \text{from eqn. (20)}
\]

where $\langle \phi_m | V(t) | \phi_n \rangle = V_{mn}(t)$

But recall the Fourier transform is defined by

$\phi(\omega) = \mathcal{F} \{ \phi(t) \} = \int_{-\infty}^{\infty} e^{i\omega t} \phi(t) \, dt$

$\mathcal{F}^{-1} \{ \phi(\omega) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \phi(\omega) \, d\omega$
We could define the Fourier transform of an operator

\[ V^F_T(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} V(t) dt \]

Thus

\[ S_{mn} = -i \langle m | V^F_T(\omega) | n \rangle \]

\[ \omega = E_m - E_n \]

---

Try to write series (20) for \( \lambda_{mn} \) or \( S_{mn} \) as Fourier transforms.

---

But suppose we try a \( V(t) \) as shown. Then

\[ V^F_T(\omega) = \frac{e^{i\omega t}}{i\omega} \]

which is what we found earlier. Thus we can adapt the \( S \)-matrix to finite time intervals by inverting a \( V(t) \) which is turned on and off at finite times. If it makes the analytic work easier, we could smooth the square wave out.

---

Try to do the constant excitation calculation in detail (Pretty hard unless we assume straight trajectories).
3) Space-limited Central Potentials

What happens when a particle is deflected? We want the

\( \frac{d^2}{dt^2} \mathbf{r} = \mathbf{F} \)

Here we write

\[ H = \frac{\mathbf{p}^2}{2m} + V(r) = \frac{\hbar^2}{2m} \nabla^2 + V(r) \]

Our unperturbed eigenfunctions satisfy the free-wave

equation

\[ H_0 |\phi_n\rangle = \frac{\hbar^2}{2m} \nabla^2 |\phi_n\rangle \]

Thus we identify

\[ |\phi_n\rangle = |k\rangle \]

\[ |\phi_m\rangle = |k'\rangle \]

and in \( \mathbf{r} \) representation

\[ \langle \mathbf{r} | \phi \rangle = \psi_\phi (\mathbf{r}) = \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{\sqrt{\text{vol}}} \]

[Note we have used volume normalization of the plane wave: \( \psi_k (\mathbf{r}) \).]

\[ \langle \mathbf{r}' | \phi \rangle = \psi_\phi (\mathbf{r}') = \frac{e^{i \mathbf{k}' \cdot \mathbf{r}'}}{\sqrt{\text{vol}}} \]

Then

\[ H_0 |\phi\rangle = \frac{\hbar^2}{2m} |\phi\rangle = \mathbf{E} |\phi\rangle \]
Now we can compute

\[ V_{mn} = \langle \psi_m | V | \psi_n \rangle = \langle \mathbf{k}_m | V | \mathbf{k}_n \rangle \]

\[ = \int_{\text{vol}} \psi_{k_m}^* (\mathbf{r}) V(\mathbf{r}) \psi_{k_n} (\mathbf{r}) \, d^3r \]

\[ = \int_{\text{vol}} e^{-i \mathbf{k}_m \cdot \mathbf{r}} V(\mathbf{r}) e^{i \mathbf{k}_n \cdot \mathbf{r}} \, d^3r \]

But we can define a 3-dimensional Fourier transform by

\[ V^\text{FT}(\mathbf{q}) = \int_{\text{vol}} e^{i \mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}) \, d^3r \]

Thus

\[ V_{mn} = \frac{V^\text{FT}(\mathbf{k}_m - \mathbf{k}_n)}{\text{vol}} \]

Try to plug this back into our work and finish the problem. Find the apparent scattering cross-section \( \sigma \) [not easy]
Recall we had found
\begin{equation}
\frac{\text{Rate at which a system is transferred from state in zero by a perturbation}}{= 2\pi \delta(E_n - E_m) \left| \langle m | V | n \rangle \right|^2}
\end{equation}

We were considering scattering from a space-limited central potential $V(r)$

\[ V(r) \]

\[ E_n = \frac{\hbar^2 k^2}{2m} \quad \phi_n = \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{\sqrt{\text{vol}}} \]

Thus our matrix elements are
\begin{equation}
V_{mn} = \frac{1}{\text{vol}} \int e^{-i\mathbf{k}' \cdot \mathbf{r}} e^{i\mathbf{k} \cdot \mathbf{r}} V(r) \, d^3r = \frac{1}{\text{vol}} V^T(\mathbf{q})
\end{equation}

\[ \mathbf{q} = \mathbf{k}' - \mathbf{k} \]

**Example:** Coulomb potential \[ V(r) = \frac{Ze^2}{r} \]

\[ V^T(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}} \frac{Ze^2}{r} \, d^3r = \frac{4\pi Ze^2}{q^2} \]

\[ \neq \text{in charge} \quad \frac{e^2}{\hbar c} = e^2, \quad \frac{e^2}{mc} = \frac{1}{137.0365} \]
Thus using (1)

\[ \text{Rate} \mathbf{k} \rightarrow \mathbf{k}' \approx \frac{\pi}{2} \left[ \frac{k_x'^2}{2m} - \frac{k_z'^2}{2m} \right] \frac{1}{(\text{vol})^2} |U(q)|^2 \]

Note the S-fan implies \( |k| = |k'| \) \( \Rightarrow \) conservation of energy. However, we are actually interested in the probability of transition to some state in the neighborhood of \( k' \). Thus we need the state density about \( k' \) for a free particle.

We want to show

No. of states having \( k \) in range \( dk_x dk_y dk_z = \frac{d^3k}{(2\pi)^3} \text{ (vol)} \)

Consider a rectangular box and quantize the wave function as shown.

\[ \sin k_x x \Rightarrow k_x L_x = n_x \pi \]

\[ \sin k_y y \Rightarrow k_y L_y = n_y \pi \]

\[ \sin k_z z \Rightarrow k_z L_z = n_z \pi \]

Thus

\[ \text{No. of states in } \frac{dk}{dL} = \frac{dk_x}{L_x} \frac{dk_y}{L_y} \frac{dk_z}{L_z} = \frac{d^3k}{(2\pi)^3} \text{ (vol)} \]

But these are standing waves composed of two \( e^{i k_x x} \) waves which come from periodic \( b.c. \ Thus \frac{1}{2} \text{ factor in required expression.} \]
Thus

\[
\text{Rate} = 2\pi S \left[ \frac{k^2 - k'^2}{2m} \right] \frac{d^3k'}{\text{vol} (2\pi)^3} |V^*(q)|^2
\]

(4)

We are actually interested in scattering into a solid-angle d\Omega.
Thence we integrate (4) over \( k' \), using \( d^3k' = k'^2 dk'd\Omega \).

Use

\[
\int S[f(x)] \phi(x) dx = \frac{\phi(x_0)}{|f(x_0)|}
\]

(5)

where \( f(x_0) = 0 \)

Note: If \( f(x) = 0 \) at several points \( x_i \), we get a sum of terms.

\[
\text{Rate} = \frac{2\pi}{\text{vol}} \frac{k^2}{2m} d\Omega \ |V^*(q)|^2
\]

where \( q \to |k| = |k'| \)

We now introduce the concept of a cross-section: imagine an effective area which includes the vol of \( V(q) \). If we use

\[
\tau = \frac{dE}{dk} = \frac{k}{m}
\]

in a time \( T \), particle moves \( \tau T \). Thus the effective volume is \( \tau VT \).

But plane wave prob. in \( cc = \frac{1}{\text{vol}} \).

\[
\therefore \text{Prob} = \frac{\tau VT}{\text{vol}} \quad \text{Rate} = \frac{\tau V}{\text{vol}}
\]
Thus

\[ \psi d\sigma = \frac{\hbar k}{4\pi^2} 1V^*(q)^2 \]

Note: the 1 always cancels out. *FORGET IT* in the future and just normalize wave functions per unit vol.

Also, \( E \) & \( k \) always appear with \( \frac{1}{2\hbar} \). Also 2\( \pi \) \( S(\text{En-En}) \), since \( S \)-func involves \( \frac{1}{\hbar} \).

**Example**: Coulomb case

\[ V^* = \frac{4\pi Z e^2}{q^2} \]

\[ q^2 = 4 \sin^2 \theta (2k^2) \]

\[ = 2k^2 (1 - \cos \theta) \]

Thus

\[ \frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2} \frac{\hbar k}{V} \frac{16\pi^2 Z^2 e^4}{16 (\sin^4 \theta/2) k^4} \]

This is *Rutherford's law*. Just a coincidence that this agrees with classical result. Accident? minor miracle. But note:

classical works if \( V \) low, \( V \) large

first Born approx if \( V \) high, \( V \) low
4) Oscillating Perturbation

\[ \text{Prob}_{n \rightarrow m} = \left| \int_{t_i}^{t_f} \langle m | x(t) | n \rangle \, dt \right|^2 \]

Now take \( V(t) = 2V_0 \cos \omega t \)

Then

\[ \langle m | x(t) | n \rangle = e^{-i(E_n - E_m) t} 2 \omega \cos \omega t \langle m | V | n \rangle \]

Upon integrating we find things like

\[ \left[ \frac{e^{-i(E_n - E_m) t} e^{i(E_n - E_m) t}}{E_n - E_m} + \frac{e^{-i(E_n + E_m + \omega) t} e^{i(E_n + E_m + \omega) t}}{E_n - E_m + \omega} \right]^2 \]

For large \( T \) we can show cross-terms \( \sim 0 \) upon proper averaging.

For continuum transitions

\[ \text{Rate} = 2 \pi S(E_n - E_m) |V_{nm}|^2 + 2 \pi S(E_n + E_m) |V_{nm}|^2 \]

---

**Problem:** Consider an H atom in the ground state. Consider the

proof of exciting this atom and emitting an electron when

using an oscillating wave-like field [disregard quantun

nature of field -- assume it is very strong]. Do as

detailed as you can. Various directions, etc.
Comment: Say we wish to compute the rate of transitions between states $m$, e.g.

$$\text{Rate}_{n \rightarrow (n' \pm \mu)} = \sum_{n} \text{Rate}_{n \rightarrow m} = \sum_{n} 2\pi S(E_n - E_m) \left| \nu_{nm} \right|^2$$

$$\to \int dE_m \rho(E_m) 2\pi S(E_m - E_n) \left| \nu_{nm} \right|^2 = 2\pi \rho(E_n) \left| \nu_{nn} \right|^2$$

which is the form of Fermi's golden rule to which we are more accustomed.

Recall we had found

$$\text{Rate} = 2\pi S(E_n - E_{n' \pm \mu}) \left| \nu_{nm} \right|^2 + 2\pi S(E_n - E_{n' \pm \mu} + \hbar \omega) \left| \nu_{nm} \right|^2$$

We can see the prob. of excitation = prob. of emission if the state populations are identical.

Example: Consider an atom in an electric field

$$V = (2\cos \alpha + E_0) z$$

What is prob. of being excited to 1st excited state?

$$\text{Rate}_{nm} \rightarrow n' = 2\pi S(E_n - E_{n' \pm \mu} + \hbar \omega) \left| \nu_{nm} \right|^2 E_0^2$$
We use white light, we must integrate.

\[ \Delta \propto <E^2> = 2E_0^2 \quad \text{intensity} \]

\[
\text{Rate} = 2\pi \sqrt{8(E_n - E_n - \hbar \omega)} |Z_{nm}|^2 \frac{\delta(\omega)}{2} \text{ d}\omega
\]

\[ = |Z_{nm}|^2 \delta(\omega_n) \quad \omega_n = E_n - E_0 \]

5.) **Interaction with a Quantum Oscillator**

We now consider the interaction of a system with a quantum mechanical oscillator (distilled from a system being exposed to a classical perturbing field).

\[ H = H_{\text{osc}} + H_{\text{vac}} + H_{V} \]

\[ H_{\text{osc}} = \frac{p^2}{2\mu} + \frac{1}{2} \omega_0^2 q^2 \]

Use

\[
\text{Rate} = 2\pi \sqrt{8(E_n - E_n)} |Z_{nm}|^2
\]

We want

\[ H_0 |\phi_n\rangle = E_n |\phi_n\rangle \]

Suppose we know

\[ H_{\text{atom}} |\chi_a\rangle = |\chi_a\rangle \]
We know
\[ H_{\text{osc}} |u_n\rangle = -\hbar \omega (N+\frac{1}{2}) |u_n\rangle \]

Thus to find \( |\phi_n\rangle \), use the non-interaction of \( H_{\text{atom}} \) and \( H_{\text{osc}} \) to write
\[ |\phi_n\rangle = |X_a\rangle |u_n\rangle \]

\[ E_{\phi_n} = E_a + \hbar \omega (N+\frac{1}{2}) \]

Now plug into our expression
\[ \frac{\text{Rate}}{\langle \Phi_{\phi_M} | \phi_{\phi_M} \rangle} = 2\pi \delta (E_n - E_{\phi_n}) |\langle \phi_{\phi_M} | X_b | \phi_{\phi_M} \rangle|^2 \]

\[ = 2\pi \delta (E_a - E_b + \hbar \omega (N+\frac{1}{2})) \langle X_b | \langle \phi_{\phi_M} | X_b | \phi_{\phi_M} \rangle \rangle \]

To calculate this
\[ \langle \phi_{\phi_M} | X_b | \phi_{\phi_M} \rangle = \int \phi_{\phi_M}^* \delta \phi | X_b | \phi_{\phi_M} \rangle \delta r \text{ d}r \text{ d}q \]

\[ = \int \chi_b^*(\rho) \psi^* \psi \chi_a(\rho) \psi \psi \text{ d}r \text{ d}q \]

\[ = \left[ \int \chi_b^*(\rho) \chi a(\rho) \text{ d}r \right] \left[ \int \psi^* \psi \text{ d}q \right] \]

\[ = \langle X_b | X_a \rangle \langle u_n | q | u_n \rangle \]
Now we can show

\[ \langle n | q | n \rangle = 0 \quad \text{unless} \quad n = n+1 \]

\[ \langle n+1 | q | n \rangle = \sqrt{\frac{n}{2 \epsilon \omega}} \sqrt{n+1} \]

\[ \langle n-1 | q | n \rangle = \sqrt{\frac{n}{2 \epsilon \omega}} \sqrt{n} \]

Thus

\[ \text{Rate}_{a \to b} \bigg|_{N \to N+1} = 2\pi S(\epsilon_a - \epsilon_b - t\omega) |\hbar a|^2 \frac{t}{3\epsilon \omega} (N+1) \]

\[ \text{Rate}_{a \to b} \bigg|_{N \to N-1} = 2\pi S(\epsilon_a - \epsilon_b + t\omega) |\hbar a|^2 \frac{t}{3\epsilon \omega} N \]

Comparison with Classical Perturbation

Here \( H = H_{\text{Ham}} + \delta q(t) \)

\[ q(t) = 2Q_0 \cos \omega t \]

and

\[ \text{Rate}_{a \to b} = 2\pi S(\epsilon_a - \epsilon_b - t\omega) |\hbar a|^2 Q_0^2 \]

\[ \text{Rate}_{a \to b} = 2\pi S(\epsilon_a - \epsilon_b + t\omega) |\hbar a|^2 Q_0^2 \]

Suppose \( \langle Q^2 \rangle_{\text{Classical}} = 2Q_0^2 \). But for q.m. oscillator \( \langle W|Q|W \rangle = \frac{t}{\omega} (N+\frac{1}{2}) \)

\[ \therefore \text{Classical:} \quad \uparrow = \uparrow \quad \text{q.m.:} \quad \uparrow \rightarrow \downarrow \]
Try instead to compare with

\[ \langle \frac{\mu^2 Q^2}{\omega^2} \rangle = \frac{1}{2} \text{ excitation energy in oscillator} \]

Then the q.m. oscillator gives

\[ \text{Rate} \quad = 2\pi \delta(E_a - E_b - \hbar \omega) \left| k_{ao} \right|^2 \left[ \frac{1}{2} \langle Q^2 \rangle_{\text{excitation}} + \frac{\hbar}{2m\omega} \right] \]

\[ \text{Rate} \quad = 2\pi \delta(E_a - E_b + \hbar \omega) \left| k_{bo} \right|^2 \frac{1}{2} \langle Q^2 \rangle_{\text{excitation}} \]

Thus all but the \( \frac{\hbar}{2m\omega} \) term can be handled by using a classical oscillating perturbation. Thus we have a certain added rate for moving downhill which would be there even for no excitation. This is the "spontaneous emission" rate. It is the difference between q.m. and classical fields.

**Problem:** Consider a cavity with E.H. wave as shown.

Also a d.c. B0 field in z direction. Interaction of atoms of

\[ H = \mu e g (\mathbf{J} \cdot \mathbf{B}) \]

\[ = \mu e g J_z B_0 + \mu e g J_x B_y \]
Recall we had been considering

\[ H = H_{\text{atom}} + H_{\text{osc}} + V_q \]

**Example:** Consider the nucleus of an atom in a solid as it interacts with the quantized oscillator. We consider its interaction with an electron moving through a solid which is

\[ U(r - R_{\text{wden}}) \]

Locate the coordinate origin at the nucleus such that \( r = q \cdot z \).

Then we can linearize the interaction if \( q \) is small

\[ U(r - R_{\text{wden}}) \approx U(r_e) - \left( \frac{dU(r)}{dz} \right) q \]

We had found for the quantized oscillator interaction

\[ \text{Rate} = \sum_{n=m}^{N} \delta \left( E_n - E_m - \hbar \omega \right) \left| \phi_{nm} \right|^2 \frac{\hbar}{3\hbar \omega} (N+1) \]

\[ N \to N+1 \]

[Note this assumes \( T \) is large such that the states assume a continuum]
6. Nucleus in a Magnetic Field in a Cavity [NHR]

Let \( \mathbf{B} = B_0 \mathbf{e}_z + B_x \mathbf{e}_x \) [d.c. cavity mode]

Consider a typical cavity mode

\[ \mathbf{A} = A_y \mathbf{e}_y = q(t) \sin k_z e_y \]

\[ k_L \approx k \]

\[ E_y = \frac{\partial A_y}{\partial t} = \frac{q(t) \sin k_z}{k_L} \]

\[ B = \nabla \times \mathbf{A} \Rightarrow B_x = \frac{\partial A_y}{\partial z} = k q(t) \cos k_z \]

Then

\[ H = q \mu_n (\mathbf{J} \cdot \mathbf{B}) = q \mu_n J_z B_0 + q \mu_n J_x (k \cos k_z) q(t) \]

Now we know \( J_z = m_z \) \( m_z = -j \ldots +j \)

\[ \epsilon_{m_z} = q \mu_n B_0 m_z \]

Thus we can calculate

\[ \text{Rate} = \frac{2}{\pi} \int_{m_z}^{m_z} \int_{N}^{N+1} \left[ \mu_n B_0 \left( m_z - m_z \right) - \sin \right] \frac{1}{\mu_n} (N+1) \left( m_z \right) \left( J_x \right)^2 \left( B_0 \cos k_z \right)^2 q(t) \]
Now we want to deal with the cavity in such a way that we can associate with it the quantum oscillators treated earlier in our analysis. We work by analogy. Classically

\[ \text{Energy} = \frac{\varepsilon_0}{2} \int (E^2 + c^2 B^2) \, d\text{Vol} = \frac{\varepsilon_0 \omega_0}{2} \left[ \frac{\dot{q}^2}{2} + k^2 c^2 q^2 \right] \]

Now the usual simple harmonic oscillator obeys

\[ \text{Energy} = \frac{\mu}{2} \dot{q}^2 + \frac{\mu \omega_0^2}{2} q^2 \]

Thus we can identify \( \omega = k c \)

\[ \mu = \frac{\varepsilon_0 \omega_0}{2} = \varepsilon_0 \int (a \sin^2 k z) \, d\text{Vol} \]

**Assumption of Quantum Electrodynamics**: The classical EM cavity oscillations become simple harmonic oscillators in the quantum theory treatment. Thus

\[ H_{\text{osc}} = \frac{\mathbf{P}^2}{2\mu} + \frac{\mu \omega_0^2 q^2}{2} = -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial q^2} + \frac{\mu \omega_0^2}{2} q^2 \]

Note \( q \) is not a position variable, but merely a variable describing the excitation of the cavity.

**Example**: \( H_{\text{osc}} \) \( \psi \) yields the usual SHO wave function.

If we consider \( |\psi_0|^2 \), we can interpret it meaning that if we measured \( B \) when the cavity was empty, there would be a certain probability of finding \( B \) non-zero.
The energies are $E_N = \hbar \omega (N + \frac{1}{2})$

We can either say that the particle is in the 3rd excited state of the SHO or that we have 3 photons in the case of $N = 3$. Thus photons = oscillator excited states

Just remember however, $\phi$ is not a position variable.

Now we also know

$$\langle m_z | J_x | m_z' \rangle = \sqrt{\frac{j(j+1)-m_z(m_z+1)}{2\hbar^2 \omega}} \quad \text{for } m_z' = m_z \pm 1$$

$$= 0 \quad \text{otherwise}$$

Thus

$$\text{Rate} \propto \delta (\omega_0 - \omega) \frac{1}{N + \frac{1}{2}} \left( \frac{\gamma h}{\hbar} \right)^2 \left( \frac{g_{nm} \hbar \omega_0 \hbar \omega z_a}{2 \hbar^2 \omega} \right)^2 [j(j+1)-m_z(m_z+1)]$$

$$\omega_0 = g_{nm} \hbar \omega / h$$

= Larmor precession frequency

Thus no nuclear transitions unless cavity $\omega$ is at the Larmor precession frequency, i.e. resonances [N.M.R.]

Note: if the atom has only been in the field for a time $\frac{1}{T}$, we must go back to eqn. (21) on p. 8.
Interaction with a System of Oscillators

We now want to extend the analysis for many modes. Not too useful for cavities, but for QED or phonons we need it.

\[ H = H_{\text{atom}} + \sum_i H_{\text{osc}_i} + \sum_i \varepsilon_i q_i \]

Note we use the orthogonality of the \( \phi \) states to eliminate cross-terms (use normal coordinates otherwise). While

\[ H_{\text{osc}_i} = \frac{p_i^2}{2m} + \frac{\omega_i^2}{2} q_i^2 \]

\[ V = \sum_i V_i \]

Now return to the general theory.

Initial state \( n \): atom in state \( a \), 1st osc. in \( N_1 \), \( \ldots \)

\[ |n⟩ = |N_1, N_2, N_3, \ldots⟩ \quad [x_a(x_\bar{a}) \Phi_{N_1}(x_\bar{a}) \Phi_{N_2}(x_\bar{a}) \ldots] \]

\[ H_0 |n⟩ = \left[ \varepsilon_a + (N_1 + \frac{1}{2}) \hbar \omega + (N_2 + \frac{1}{2}) \hbar \omega + \ldots \right] |n⟩ \]

Final state \( m \): atom \( b \), \( N_1, N_2, \ldots \)

\[ |m⟩ = |N_1, N_2, b, \ldots⟩ \]

Now to calculate the matrix elements \( \langle m | V | n \rangle \)

\[ \langle m | V | n \rangle = \sum_i \langle m | V_i | n \rangle \]
\[ \langle m | V_j | n \rangle = \langle \psi_{j}(\pi)_{q_{j}} \mid q \rangle_{n_{1}n_{2}} \]

\[ = 0 \quad \text{unless} \quad n_{i} = n_{i}', \quad n_{j} = n_{j} \pm 1 \]

\[ = \langle b | V_{j} | a \rangle \frac{\hbar}{2\mu_{j} \omega_{j}} (N_{j} + 1) \quad N_{j} = n_{j} \pm 1, \quad n_{i} = n_{i}', \quad n_{j} = n_{j}' \]
PROBLEM: Consider states in the sense that the energy as a function of momentum is

$$\varepsilon(p) = \Delta + \frac{1}{2m} (p - p_0)^2$$

where $\Delta, p_0$ are constants. Now suppose

$$\Delta(p) = \Delta_0 + (p - p_0)d + O((p - p_0)^2)$$

What is the prob. of emitting a photon of momentum $\kappa$ in terms of $p_0, \Delta_0, d,$ and $c_5$? [Assume only longitudinal sound waves, and if it helps, set $p_0 = 0$.]

PROBLEM: Make up a "significant" problem involving the second order terms in our perturbation theory and work it out.

None.

Last time we had

$$H = H_{\text{atom}} + \sum_i ^i H_{i\text{on}} + \sum_i ^i \lambda_i \phi_i$$

where $H_{i\text{on}} = \frac{p_i^2}{2m_i} + \frac{q_i^2 c_i^2}{2m_i}$

Now we want the prob of

$$|n, \ldots \rangle \rightarrow |n', \ldots \rangle$$
We found
\[
\text{Rate}_{\alpha \rightarrow \beta} = \frac{2\pi S (E_a - E_b - h\nu_j) |\langle b | \gamma_j | a \rangle|^2 (N_j + 1)}{2\nu_j \omega_j}
\]
\[
\text{Rate}_{\alpha \rightarrow \beta} = \frac{2\pi S (E_a - E_b + h\nu_j) |\langle b | \gamma_j | a \rangle|^2 N_j}{2\nu_j \omega_j}
\]

Thus each oscillator interacts independently with the system. If we wanted just the rate of transition, regardless of which oscillator was involved, we need merely sum over the class \(N_j \rightarrow N_j \pm 1\) of the \(j\) oscillators.

8.) Atomic Transitions in a Cavity

**Example:** We consider the probability that a one-electron atom radiates into an empty box. [Since we may take the box as arbitrarily large, this is just the problem of an excited atom interacting with the \(\mathbb{Z}^3\) field. It is our first real example of quantum electrodynamics -- although in the non-relativistic approximation]

Assume when the atom is in state \(a\), no photons in the box. When it falls into state \(b\), a photon of energy \(\omega_j\) is emitted. This may be interpreted as exciting a mode in a certain direction.
Now to quantize the field, first set up the classical field as a superposition of modes

\[ \vec{A}(r, t) = A_1(t) \sin k_z x + A_2(t) \sin k_z y + A_3(t) \sin k_z z \vec{e}_z + \ldots \]

\[ k L_z = \frac{\omega}{c} \]

\[ \varepsilon \sum_{k, \vec{e}} \varepsilon_{k, \vec{e}} e^{i \vec{k} \cdot \vec{r}} \]

where \( \vec{k} \) is the wave vector
\( \vec{e} \) is the polarization vector

\[ \varepsilon \vec{k} = 0 \quad \text{transverse mode} \]

Note: (i) we have chosen the scalar potential \( \Phi = 0 \)
(ii) we need a \( \vec{E} \in \vec{B} \).

Now to find the classical field, use Maxwell's equations to find

\[ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \]

Plug in \((*)\) to find

\[ -k^2 c^2 \varepsilon_{k, \vec{e}} \dot{A}_{k, \vec{e}} - \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \]

Now to get the ged, find

\[ \dot{E} = -\frac{\partial \vec{A}}{\partial t} = \sum_{k, \vec{e}} \varepsilon_{k, \vec{e}} \dot{A}_{k, \vec{e}} e^{i \vec{k} \cdot \vec{r}} \]

(harmonic oscillator of \( \omega_{k, \vec{e}} = k c \))
\[ \mathbf{B} = \nabla \times \mathbf{A} = \sum_{k, \vec{e}} i (\mathbf{k} \times \vec{e}) Q_{k, \vec{e}} e^{-i \mathbf{k} \cdot \mathbf{x}} \]

Then
\[ \frac{\varepsilon_0}{2} \int (E^2 + c^2 B^2) \, dV = \frac{\varepsilon_0 \, V_{\text{vol}}}{2} \sum_{k, \vec{e}} \left( \frac{Q_{k, \vec{e}}^2}{\varepsilon_0 k_{\vec{e}}^2} + \frac{c^2}{k_{\vec{e}}^2} \right) \]

\[ M_{k, \vec{e}} = \frac{2}{\varepsilon_0 \, V_{\text{vol}}} = \mu \]

Now passing to q.e.d., use quantum oscillators
\[ H(\mathbf{\phi}, \mathbf{\phi}'') = \frac{1}{2} \sum_{k, \vec{e}} \left[ \frac{P_{k, \vec{e}}^2}{2m} + \mu (k_0 c)^2 Q_{k, \vec{e}}^2 \right] \]

\[ \mu = \frac{\varepsilon_0 \, V_{\text{vol}}}{2} \]

Now
\[ H_{\text{atom in a field}} = \frac{1}{2m} (\mathbf{p} - q_0 e \mathbf{A})^2 + \frac{q_0 e^2}{4\pi \varepsilon_0 v} + \mathbf{m} \mathbf{\dot{v}} \cdot \mathbf{B} + \ldots \]

\[ = H_{\text{atom}} + \left[ -\frac{q_0 e}{2m} (\mathbf{p} \cdot \mathbf{A}(\mathbf{v}) + \mathbf{\dot{A}}(\mathbf{v}) \cdot \mathbf{p} + M_0 \mathbf{\ddot{v}} \cdot (\nabla \mathbf{A}) \right] \]

\[ + \frac{q_0 e^2}{2m} \mathbf{A}(\mathbf{v}) \cdot \mathbf{\ddot{A}}(\mathbf{v}) \ldots \]

Now if \( \mathbf{v} \) is small, we drop the \( \mathbf{A} \cdot \dot{\mathbf{A}} \) term since it is 2nd order
( and our perturbation theory is only to first order)
\[
\text{We find: } \\
H_{\text{int}} = \sum_{k, \ell, \tau} \left\{ -\frac{q^2}{2\hbar} \left[ (p^2) e^{i k \ell \tau} + e^{i k \ell \tau} \right] + \mu \vec{\sigma} \cdot (\vec{p} \times \vec{e}) e^{i k \ell \tau} \right\} Q_{k, \ell, \tau} \\
\]

We now want

\[
\begin{align*}
\text{Rate} \\
\text{from } a \rightarrow b \\
\text{vacuum to 1 photon of } \vec{e} \\
\text{(all } N_i = 0 \text{ to all } N_i = 0) \\
\end{align*}
\]

\[
= 2\pi S (\varepsilon_a - \varepsilon_b - \hbar \omega) \left| \langle b | \hat{b}_{k, \ell, \tau}^+ | a \rangle \right|^2 \frac{N_{k, \ell, \tau} + 1}{2\hbar \omega} e_0 \\
[\text{Taking into account } \frac{1}{2} \text{ factor from } \text{Re} \{ ? \}] \\
\]

\[
\text{Show } \vec{p} e^{i k \ell \tau} = e^{i k \ell \tau} (\vec{p} + \vec{\ell}) \text{ and then } [\hat{\vec{p}}, \hat{\vec{A}}] = 0 \\
\]

\[
\text{Thus} \\
\text{Rate} \\
\text{a } \rightarrow b \\
\text{with } \vec{e} \\
\text{emitted photon} \\
\text{from } \vec{e} \\
\]

\[
= \frac{2\pi}{2\hbar \omega} S (\varepsilon_a - \varepsilon_b - \hbar \omega) 4\pi e^2 \left| \langle b | \hat{b}_{k, \ell, \tau}^+ | a \rangle \right|^2 \\
\]

\[\text{using } \frac{q^2 e^2}{4\hbar \omega e_0} = e^2 \]

and neglecting spin term.
Remember each time a photon couples with a system we get a $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{2\pi}$ factor.

Now, if we want the prob. of transition to $d^3k$, we sum, passing in the limit to

$$\sqrt{\int \frac{d^3k}{(2\pi)^3} \text{vol}}$$

If we integrate over constant $1|k|$, we find

$$\text{Rate} = \frac{2\pi}{2\omega} \frac{d^3z}{(2\pi)^3} 4\pi e^2 |\langle b| \left(\frac{e}{i}\right) e^{i\mathbf{k} \cdot \mathbf{r}} |a\rangle|^2$$

where $k = |\mathbf{k}| = \epsilon_a - \epsilon_b$

$$c = \hbar = 1$$
The Dipole Approximation

We had found

\[ \text{Rate of emission of electron with } k_r \text{ out of atom } a \to b \]

\[ = \frac{2\pi}{2\omega} S(e_a - e_b - i\omega) \cdot m e^2 |b_l \cdot \vec{e}_m e^{i k_r \cdot r} \cdot a|^2 \frac{e_b}{2\omega^2 m^2} \]

The \( e^{i k_r \cdot r} \) factor accounts for phase interference from different points in the atom. Now for light radiated by atoms, typically \( kr \approx \frac{\hbar}{\lambda} \ll 1 \). Thus we may approximate

\[ e^{i k_r \cdot r} \approx e^{i \vec{k}_r \cdot \vec{r}_m} \]

Now \( e^{i \vec{k}_r \cdot \vec{r}_m} \) is no longer an operator and can be treated as a term of the matrix elements. This is called the "electric dipole approximation." [Note it frequently doesn't work!]

Thus we can find

\[ \text{Rate into } a \to b = \frac{4\pi e^2}{2\omega} \frac{k_r^2 \cdot \sigma}{(2\pi)^3} |b_l \cdot \vec{e}_m \cdot a|^2 \]

Now if we consider \( \vec{p} \cdot \vec{e} = p_z \), then

\[ <b_l \cdot \vec{e}_m \cdot a> = <b_l | H_z - z | H_l a> = (e_b - e_a) <b_l \cdot \vec{e}_m \cdot a> \]

Thus

\[ \text{Rate} = 2\omega^2 |b_l \cdot \vec{e}_m \cdot a|^2 \frac{e_b}{4\pi} \]

Note all of this is exact within 1st order perturbation theory for non-relativistic theory.
Example: $2p \rightarrow 1s$

$l = 0 \ (\omega \theta \sim z) \quad <b | \xi | a > = 0$

$< b | y | a > = 0$

$< b | z | a > \neq 0$

Then

$$\text{Rate} = 2 \omega^2 e^2 | < b | z | a > |^2 \frac{\hbar}{i \hbar}$$

Now noting

The polarization out of page will vary.

Now

$$\text{Rate} \propto \omega^2$$

any direction

any polarization

Then

$$\text{Energy loss} = \frac{4 \omega^4}{3 c^3} \left[ \text{matrix element of dipole moment} \right]^2$$

---
We now compare this to a small classical oscillating dipole

\[ z = 2z_0 \cos \omega t \]

Dipole moment = \[ 2z_0 \cos \omega t \]

\[ (\text{Dipole moment})^2 = 2(z_0)^2 \]

\[ \frac{\text{Energy radiated}}{\omega} = \frac{2}{3} \frac{\omega^4}{c^3} (\text{Dipole moment})^2 \]

This is close to the quantum answer except for a factor \( z \).

Thus, an atom in empty space radiates as a dipole provided we use

\[ (\text{Dipole moment classical})^2 = 2 |\text{Dipole matrix element}|^2 \]

We will now prove that this rule holds for any cavity (even with holes and resistive walls).

We proceed for an atom radiating in a general cavity with holes in it. Assume the holes are small compared to the cavity size. Now surround the cavity by any absorbing wall. This has various modes [although very complicated]. For the indicated geometry, the modes resemble

\[ \text{resonance} \]

\[ \text{off resonance} \]

\[ \text{in between} \]
The superposition of these modes inside the box leads to the leakage from the box, thus a superposition of undamped modes yields a damped mode. If the box is large enough, (several light years), we needn't worry about reflection from the walls.

Suppose these modes are $\Phi_i(\vec{r})$. Then we use our result

$$\frac{\text{Rate}}{a+b} = 2\pi \delta(\vec{e}_a - \vec{e}_b - \vec{k} \cdot \vec{r}) \langle b|\Phi_i|a\rangle^2 \frac{1}{2\nu \omega}$$

Now $Hint = \hat{J} \cdot \hat{A}(\vec{r})$

Then expand $A(\vec{r}) = \sum \Phi_i(\vec{r}) Q_i(\vec{r})$

Thus

$$\langle \text{Rate} \rangle = \sum 2\pi \delta(\vec{e}_a - \vec{e}_b - \vec{k} \cdot \vec{r}) \frac{1}{2\nu \omega} |\Phi_i(\vec{r})|^2 |\text{j}(\vec{r})|^2$$

We will compare this to a classical problem

Classical: $\vec{J}(\vec{r})$ is the current in a dipole antenna located at $\vec{r}$. We have a coupling $\vec{j}(\vec{r}) \cdot \hat{A}(\vec{r})$. Now $A$ satisfies

$$\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = j(\vec{r}) = 8\pi \delta(\vec{r} - \vec{r}_0) j(\vec{r})$$

Now we $A = \sum \Phi_i(\vec{r}) Q_i(\vec{r})$ and plug in to find

$$\sum (\vec{a}_i + \omega^2 \vec{b}_i) \Phi_i(\vec{r}) = 8\pi \delta(\vec{r} - \vec{r}_0) j(\vec{r})$$
From orthogonality

\[ \phi_i'' + \omega^2 \phi_i = \Phi_i(\phi) j(t) \quad (\ast) \]

If \( j(t) \) varies as \( e^{j\omega t} \), then the electric field varies as \( E_{\omega} e^{j\omega t} \)

\[ E_{\omega} = \mathcal{Z}(\omega) j_{\omega} \]

where \( \mathcal{Z}(\omega) \) is the impedance

Using \( E_{\omega} = \sum \mathcal{Z}_i(\phi) \Phi_i(\phi) \)

we find, solving (\ast), that

\[ E_{\omega} = i\omega \sum_{i} \frac{1}{\omega^2 - \omega^* \omega^*} \Phi_i^2(\phi) j_{\omega} \]

\[ \mathcal{Z}(\omega) \]

Now return to our g.m. model where we found

\[ R_{\omega} \rightarrow b = \frac{2\pi}{a-b} \delta(\omega - \omega_m) \left| \left< b \mid j \right| a \right|^2 \Omega(\omega)^2 \left| \frac{1}{3\pi i \omega} \right| \]

Claim: \( R_{\omega} \rightarrow b = \frac{2}{\Omega_n \omega_n} \left[ \Re\{ \mathcal{Z}(\omega) \} \right] \left| \left< b \mid j \right| a \right|^2 \)
Proof: if we return to (1),

\[ f(\omega) = \frac{i\omega}{\omega_i^2 - \omega} \phi(\omega) \]

we realize we have no prescription for \( \lim_{\omega \to \omega_i} f(\omega) \).

Thus we use the analogy of a damped oscillator:

\[ \ddot{q} + \omega_0^2 q + \gamma \dot{q} = i \omega q \]

or \( (-\omega^2 + \omega_0^2 + i\omega) q = i \omega q \)

\[ q = \frac{i \omega}{\omega_i^2 - \omega^2 + i\omega} \]

we see that we can apply the condition of causality and the initial condition of \( q = 0 \) for \( t \leq 0^- \) to claim we should write 0/0:

\[ q(\omega) = \frac{1}{\omega_i^2 - (\omega - i\epsilon)^2} \phi(\omega) i \omega \]

Thus we assume no free waves on i.c. before turning on oscillator.

We eventually take \( \epsilon \to 0^+ \). Note this is just like "turning off the interaction" of \( q \) as \( \epsilon \to \pm 0 \) in scattering problems.

Before continuing, we need a little more with first,
Detour

Cauchy Principal Values and Generalized Functions (à la Feynman)

It is convenient to take as a prescription

$$\lim_{\varepsilon \to 0} \frac{1}{x+i\varepsilon} = \text{P.V.} \left( \frac{1}{x} \right) - i\pi \delta(x)$$

(3)

[Note: We also might have specified]

$$\lim_{\varepsilon \to 0} \frac{1}{x+i\varepsilon} = \text{P.V.} \left( \frac{1}{x} \right) + i\pi \delta(x)$$

or

$$\lim_{\varepsilon \to 0} \frac{1}{x+i\varepsilon} = \text{P.V.} \left( \frac{1}{x} \right)$$

More generally

$$\lim_{\varepsilon \to 0} \frac{1}{x+i\varepsilon} = \text{P.V.} \left( \frac{1}{x} \right) + \lambda S(x)$$

To "justify" (3), note

$$\frac{1}{x+i\varepsilon} = \frac{x-i\varepsilon}{x^2+\varepsilon^2} = \frac{x}{x^2+\varepsilon^2} - i \frac{\varepsilon}{x^2+\varepsilon^2}$$

The first term resembles

$$\frac{\varepsilon}{x^2+\varepsilon^2}$$

Now

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x} f(x) \, dx = \int_{\delta}^{\infty} \frac{1}{x} f(x) \, dx + \int_{-\infty}^{-\delta} \frac{1}{x} f(x) \, dx$$

By keeping equal distances we get cancellation.
For the second term

By definition\[ \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 + \epsilon} \, dx = \pi \]

**IMPORTANT POINT:** We don't, however, make a rule to take P.V. for all physical problems. The trick is to leave it unspecified until we know how to specify it.

**EXAMPLE:** Consider the Fourier transform of a step function \( \frac{1}{t-0} \)

The usual \( \mathcal{F} \) doesn't converge [at least in the usual sense]. Consider instead\[ \int_{-\infty}^{\infty} e^{-ix} \, dt = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-ix + \epsilon} \, dt \]

\[ = \lim_{\epsilon \to 0} \frac{1}{i\epsilon + \epsilon} = \lim_{\epsilon \to 0} \frac{-i}{x - \epsilon} \]

Note that how we take our prescription affects the inverse transform.
Returning now to our problem, we find:

\[ z(\omega) = i \omega \sum \frac{1 |\varphi_i|^2}{\omega_i^2 - \omega^2 + 2i \omega} \]

\[ = i \omega \sum \text{P.V.} \frac{1 |\varphi_i|^2}{\omega_i^2 - \omega^2} + i \omega (-i \pi \delta(\omega_i^2 - \omega^2) |\varphi_i|^2) \]

Note this is very close to our q.m. result. When we take Re \{z(\omega)\}, we get the result:

\[ \text{Rate}_{ab} = \frac{2}{\hbar} \text{Re} \left[ \frac{1}{\text{vol}} \left| \langle \text{b} | \text{j}_{ab} | \text{a} \rangle \right|^2 \right] \]

The difference of 2 between quantum and classical electrodynamics can be explained by noting that for q.m. we can have transitions both up and down:

\[ \text{down} \rightarrow \langle \text{a} | \text{j}_{ab}^+ | \text{a} \rangle = \frac{\xi}{\xi} \langle \text{a} | \text{j}_{bc}^+ | \text{c} \rangle \langle \text{c} | \text{j}_{1a} | \text{a} \rangle = \frac{\xi}{\xi} |\langle \text{a} | \text{j}_{1c} | \text{c} \rangle|^2 \]

\[ = \left| \langle \text{a} | \text{j}_{1b}^+ | \text{b} \rangle \right|^2 + \left| \langle \text{b} | \text{j}_{1a} | \text{a} \rangle \right|^2 \]
Comments from 10/18/66

We have now shown that the transition rate for an atom in an arbitrary cavity (with holes, resistance, etc.) is

$$\text{Rate}_{a \rightarrow b} = \frac{\gamma}{\hbar \omega_{ab}} \Re \left\{ \gamma(\omega_{ab}) \right\} |\langle a | j | b \rangle|^2$$

But for a perfect cavity the rate of transition with the energy going into the excitation of a well-defined mode $\phi_0$ is

$$\text{Rate}_{a \rightarrow b} = 2\pi S(\omega_{ab} - \omega_0) |\langle a | \mathbf{j} | b \rangle|^2 |\phi_0(\mathbf{r})|^2 \frac{\hbar}{\omega_{ab}}$$

To reconcile these expressions, we merely realize that the impedance of a perfect mode is

$$\gamma(\omega) = \lim_{\varepsilon \to 0} \frac{i\omega}{\omega^2 - \omega_0^2 + i\varepsilon \omega} |\phi_0(\mathbf{r})|^2$$

$$= \omega S(\omega^2 - \omega_0^2) |\phi_0(\mathbf{r})|^2$$

For the more general case of a lossy cavity, the resonance broadens about $\omega = \omega_0 + \Delta \omega$

$$\gamma(\omega) = \frac{i\omega}{\omega^2 - \omega_0^2 + i\varepsilon \omega} |\phi_0(\mathbf{r})|^2 = \frac{i\omega}{2\omega_0(\Delta \omega) + i\varepsilon \omega}$$

Thus

$$\text{Rate}_{a \rightarrow b} = \frac{2\pi}{(\omega - \omega_0)^2 + \varepsilon^2} \frac{(\beta^2)}{4} |\langle a | j | b \rangle|^2 |\phi_0(\mathbf{r})|^2 \frac{\hbar}{\omega_{ab}}$$
Claim: We now know quantum electrodynamics completely—

at least non-relativistically. The basic hypothesis—

that every classical EM mode of the field corresponded to a simple harmonic quantum oscillator—was made prior to 1920. The next 20 years work consisted of putting this in relativistically covariant form. We will now examine how this was accomplished.

10) Recasting of Classical Electrodynamics

The basic equations are

\[ \dot{B} = \nabla \times \dot{A} \]  

\[ \dot{E} = -\nabla \phi + \frac{\dot{A}}{cT} \]  

\[ \nabla \cdot \vec{E} = \gamma \epsilon_0 \]  

\[ \nabla \cdot \vec{B} = \gamma \epsilon_0 + \gamma \dot{\vec{E}} \]  

Now, of course, the gauge invariance is arbitrary, e.g.

\[ \vec{A}' = \vec{A} + \nabla \chi \]

It was customary to take a Lorentz gauge

\[ \nabla \cdot \vec{A} = \gamma \phi \]

since it was thought this preserved relativistic covariance.
However, any gauge preserves relativistic covariance—
the only difference is that the covariance is more obvious
in Lorentz gauge, or as J. Schwinger would say, the
equations are then “manifestly covariant.”

Thus, because it doesn’t hurt the relativity and may help
us in making the transition to QED, we choose to work
with the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0$$

[It took 20 years and a Nobel prize to get $$\nabla \cdot \mathbf{A} = 0$$ back to
relativistic covariance.]

We now begin to manipulate, to find, taking (5) into (6)

$$\nabla^2 \varphi = \frac{\rho}{\varepsilon_0}$$

Thus the scalar potential $$\varphi$$ is always determined by

$$\varphi(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}')}{r^2} dV'$$

Thus using $$e^2$$ is exact in the 2-sense. All we need to know
now is how it’s to fall in the $$\mathbf{A}$$. But (6) $$\Rightarrow$$
distributions propagate at infinite speed! Oh well, “Damn the torpedoes,
full-speed ahead!”
Use (4) in (7)

\[ \nabla \times (\nabla \times \mathbf{A}) - \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \partial \mathbf{A} = \mathbf{v} / \varepsilon_0 \]

or

\[ \nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{j} / \varepsilon_0 - \nabla (\nabla \cdot \mathbf{A}) \]

Then

\[ \nabla \cdot \mathbf{j} = \frac{\partial \mathbf{A}}{\partial t} \implies \frac{\partial \mathbf{A}}{\partial t} = \sqrt{\frac{\nabla \cdot \mathbf{j}(z)}{4\pi \varepsilon_0 r_z}} \]

Thus \( \mathbf{A} \) is completely determined by the current. This means we take into account the transverse part of the current so that \( \nabla \cdot \mathbf{A} = 0 \).

Now expand everything in plane waves

\[ \mathbf{j}(\mathbf{r}, t) = \sum_k \mathbf{j}(k; t) e^{i k \cdot \mathbf{r}} \]

\[ \mathbf{A}(\mathbf{r}, t) = \sum_k \mathbf{A}(k; t) e^{i k \cdot \mathbf{r}} \]
Now then from (9), (10), (11)

\[
\frac{d^2 \hat{A}(k, t)}{dt^2} + \hbar^2 c^2 \hat{A}(k, t) = \frac{\hbar}{2\pi} \tilde{f}(k, t)
\]

where \( \tilde{f}(k, t) = \tilde{f}(k) - \frac{(k \cdot \vec{J}) \vec{J}}{\hbar^2} \)

Equation (10) is now the classical mode which we will quantize in passing to QED.
Comments from 10/18/66

Return to our formulation of classical electrodynamics using the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. We found the basic equations to then be

$$\nabla^2 \Phi(r, t) = \rho(r, t) / \varepsilon_0$$

$$\nabla^2 \mathbf{A}(r, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(r, t) = \mathbf{j}(r, t) / \varepsilon_0 - \nabla \left( \frac{\rho(r, t)}{\varepsilon_0} \right)$$

$$\nabla \cdot \mathbf{j} = -\frac{\partial \Phi}{\partial t}$$

We solve immediately for

$$\Phi(r, t) = \int \frac{\rho(r', t) \, dV_2}{4\pi \varepsilon_0 \sqrt{(r - r_2)^2}}$$

This indicates one rule for quantization

1) Use the instantaneous Coulomb potential $\Phi(r, t)$ in the Schrödinger equation

In the non-relativistic limit this is all we need. However, to be relativistically correct, we must add in the effect of $\mathbf{A}$. To find this classically, we use Fourier transforms (momentum space)

$$\tilde{\mathbf{A}}(k, t) = \int e^{-i \mathbf{k} \cdot \mathbf{r}} \mathbf{A}(r, t) \, d^3r$$
Also \( \Phi(\vec{k}, t) = \Phi \{ \tilde{\Phi}(\vec{k}, t) \} \)

\( \Phi(\vec{k}, t) = \Phi(\tilde{\Phi}(\vec{k}, t)) \)

\( P(\vec{k}, t) = \Phi \{ p(\vec{k}, t) \} \)

Transforming our equations, the gauge becomes

\( \vec{k} \cdot \mathbf{A}(\vec{k}, t) = 0 \)

while the other ones are

\( k^2 \Phi = \frac{\mathcal{D}^2}{c^2} \)

\( \frac{\partial^2}{\partial t^2} \mathbf{A}(\vec{k}, t) + k^2 c^2 \mathbf{A}(\vec{k}, t) = \vec{J}_{\mathcal{L}} - i \vec{k} \cdot \left( \frac{\partial \Phi}{\partial t} \right) \)

\( i \vec{k} \cdot \vec{J} = -\frac{\partial P}{\partial t} \)

We can see \( \frac{\partial \Phi}{\partial t} = \frac{1}{k^2 c^2 0} \frac{\partial P}{\partial t} = -i \frac{\vec{k} \cdot \vec{J}}{k^2 c^2} \)

Thus we get

\( \frac{\partial^2}{\partial t^2} + k^2 c^2 \mathbf{A} = \frac{1}{c^2} \left( \vec{J} - \frac{k \cdot (\vec{k} \cdot \vec{J})}{k^2} \right) \) \hspace{1cm} (\ast)
We could expand \( \vec{a} \) in two transverse polarization modes

\[
\vec{a} = \hat{e}_1 \alpha_1(\vec{k},t) + \hat{e}_2 \alpha_2(\vec{k},t)
\]

\( \hat{e}_1 \cdot \hat{e}_2 = 0, \hat{e}_1 \cdot \hat{e}_1 = 1, \hat{e}_2 \cdot \hat{e}_2 = 0 \)

Plugging this into (*)

\[
\hat{e}_1 \left( \frac{\partial^2 \alpha_1}{\partial t^2} + \kappa^2 \alpha_1 \right) + \hat{e}_2 \left( \frac{\partial^2 \alpha_2}{\partial t^2} + \kappa^2 \alpha_2 \right) = \frac{i}{\varepsilon_0} \frac{\partial}{\partial t} \vec{J}
\]

Now dot with \( \hat{e}_i \) to find

\[
\frac{\partial^2 \alpha_1}{\partial t^2} + \kappa^2 \alpha_1 = \frac{\hat{e}_i \cdot \vec{J}}{\varepsilon_0}
\]

\[
\frac{\partial^2 \alpha_2}{\partial t^2} + \kappa^2 \alpha_2 = \frac{\hat{e}_i \cdot \vec{J}}{\varepsilon_0}
\]

Thus the current along \( \hat{e}_1 \) feeds the \( \alpha_1 \) mode, the \( \hat{e}_2 \) current feeds the \( \alpha_2 \) mode. Here \( \vec{J} \) is the projection of the current in the shape of the mode -- like \( (\vec{e}_i, \vec{J}(t)) \).

"This is the simplest form of electrodynamics that I could find."

Now to get QUANTUM ELECTRODYNAMICS

1.) use the instantaneous Coulomb potential \( \frac{\hat{e}_i \cdot e_1}{\varepsilon_0} \) in the S-eqn.

2.) put (***) into the form of quantum oscillators
DETUR: More Feynman guide and dirty methods.

Say we want to evaluate

$$\int e^{-i\mathbf{Q} \cdot \mathbf{r}} \varphi(\mathbf{r}) dV$$

We work backwards. Consider

$$\nabla^2 \left( \frac{1}{r} \right) = 4\pi \delta(\mathbf{r})$$

If we take the \( \mathcal{F} \{ \nabla^2(\mathbf{r}) \} = \mathcal{F} \{ \frac{1}{r} \} \Rightarrow 4\pi \mathcal{F} \{ \delta(\mathbf{r}) \} = 4\pi$$

$$\Rightarrow \int e^{-i\mathbf{Q} \cdot \mathbf{r}} \left( \frac{1}{r} \right) dV = \frac{4\pi}{Q^2}$$

Now consider

$$\left( \nabla^2 - m^2 \right) \varphi = 4\pi \delta(\mathbf{r})$$

From Yukawa, \( \varphi = \frac{e^{-mr}}{r} \)

Thus

$$\int e^{-i\mathbf{Q} \cdot \mathbf{r}} \frac{e^{-mr}}{r} d^3 \mathbf{r} = \frac{4\pi}{Q^2 + m^2}$$

"Whatever I can remember are my axioms!"
PROBLEM 3: For a long time astrophysicists wondered about continuum radiation from various nebulae. Some of this was due to synchrotron radiation. But not all. Consider an H atom (neglect spin)

\[ 2s \rightarrow 2p \]

Do something here, but neglect \( -2 \) if you wish.

Now we know \( 2s \rightarrow 1s \) transitions with one photon emitted are forbidden. However, these transitions can be induced by collisions or an electric field. However, in interstellar space collisions are effectively absent.

The alternative is the emission of 2 photons (although never seen in the lab). The 2-photon emission explains the continuum spectrum.

Do as much as you can with this. Figure out the spectrum shape. Maybe even do the integrals by Simpson's rule. Look sums over the higher matrix elements, so approximations are needed.

[This problem was first solved by E. Teller. The opposite problem is of interest today by causing two laser beams to intersect in H gas.]
B. Second-Order Time-Dependent Perturbation Theory

Recall we had

\[ S_{mn} = S_{mn} - i \int \tilde{V}_{mn}(t) \, dt + (i)^2 \int \int_{t_1 < t_2} \tilde{V}_{mk}(t_1) \tilde{V}_{kn}(t_2) \, dt_1 \, dt_2 + \ldots \]  

where \( \tilde{V}_{mn}(t) = e^{-i(E_m - E_n)t} \frac{\langle \Phi_m | V | \Phi_n \rangle}{V_{mn}} \)

1) Time-Independent Perturbation \( V \)

\[ V(t) = 0 \quad t < 0 \]
\[ = V \quad 0 \leq t \leq T \]
\[ = 0 \quad T < t \]

A long calculation then yields

\[ S_{mn} = S_{mn} - V_{mn} \left[ \frac{e^{-i(E_m - E_n)T}}{(E_m - E_n)} \right] + \sum_{k \neq n} \frac{V_{mk} V_{kn}}{(E_k - E_n)} \left[ \frac{e^{-i(E_k - E_n)T}}{(E_k - E_n)} - \frac{1}{E_k - E_n} \right] \]

\[ + V_{mn} V_{mn} \left[ \frac{e^{-i(E_m - E_n)T}}{(E_m - E_n)^2} - \frac{iT}{(E_m - E_n)} \right] + V_{mn} V_{mn} \left[ \frac{1 - i(E_m - E_n)T}{(E_m - E_n)^2} \right] \]
We now calculate $S_{\text{mm}}$, the amplitude to remain in the same state as time goes on.

$$S_{\text{mm}} = 1 - iT V_{\text{mm}} - \frac{T^2}{2} V_{\text{mm}}^2 - iT \sum_{k+m} \frac{V_{\text{km}} V_{\text{mm}}}{(E_k - E_m)} + \frac{(V_{\text{km}})^2}{(E_k - E_m)^2} [e^{-i(E_k - E_m)T} - 1]$$

We could have used time-independent perturbation theory for this problem, however, to find:

$$E_m' = E_m + V_{\text{mm}} + \sum_{k+m} \frac{V_{\text{km}} V_{\text{mm}}}{E_k - E_m}$$

shift in level $m$.

Now the state in various time as $e^{iE_m t}$. But our TDPT has taken out $e^{-iE_m t}$, e.g.

$$e^{-iE_m t} e^{iE_m t} = e^{-i(E_m' - E_m)T}$$

If we plug this in and take into account that when turning on $V$ we are left exactly in the right state, we get the above expression for $S_{\text{mm}}$. 
Comment: This rather involved lecture was presented by one of the physics department substituting for Tegman.

Review: We were considering

\[
i \frac{d}{dt} \vert \psi(t) \rangle = H(t) \vert \psi(t) \rangle
\]

We expanded \( \vert \psi(t) \rangle \) in states of the unperturbed \( H_0 \), \( \phi_n \). Using the summation convention throughout

\[
\vert \psi(t) \rangle = \sum_n c_n \vert \phi_n \rangle
\]

Now we put into "interaction picture" assume

\[
c_n = e^{-i\omega_n t} \phi_n
\]

Then find

\[
i \phi_n(t) = \psi_{nm} \phi_m \quad \text{where} \quad \psi_{nm} = V_{nm} e^{i(\omega_n - \omega_m)t}
\]

If we integrate this equation, we can expand

\[
y = e^{-i \beta(t)} \quad \text{using} \quad \phi_m(t) = \sum_n \phi_n(0)
\]

to find

\[
\psi_{nm} = \psi_{nm} - i \int V_{nm} \phi_m dt + \int \int V_{nm}(t_1, t_2) \phi_m(t_2) dt_1 dt_2 + \ldots
\]
Example: Consider tritium decay as an example of

\[ V = 0 \quad t < 0 \]
\[ = V_0 \quad t \geq 0 \]

Also assume electron doesn't have time to adjust, collect \( \Psi(0) = \Psi(0^+) \)

What is the prob for the electron to wind up in the s-state after the decay? Expand

\[ \phi_0 = a_n \chi_n \]

where \( |a_n|^2 \) is the prob.

Consider a 2-state system

We can solve this problem by stationary perturbation theory.

Note:

Energy is conserved in T.I.P.T.

"not" in T.D.P.T.
Now consider $S_{nm}$

$$S_{nm} = 1 - iT V_{nm} - \frac{T^2}{2} V_{nm}^2$$

$$- \frac{1}{T} \sum_{k} \frac{V_{nk} V_{km}}{(E_k - E_n)} + \frac{1}{T} \sum_{k} \frac{|V_{nk}|^2}{(E_k - E_n)^2} - \frac{i}{2} (E_n - E_k) T$$

Note we can write this as

$$S_{nm} = \exp \left\{ -iT \left[ V_{nm} - \frac{1}{T} \sum_{k} \frac{|V_{nk}|^2}{(E_k - E_n)^2} \right] \right\} \left\{ 1 + \frac{1}{T} \sum_{k} \frac{|V_{nk}|^2}{(E_k - E_n)^2} - \frac{i}{2} (E_n - E_k) T \right\} + \ldots$$

We can recognize the first term directly from TDPT. For the second term, just consider

$$|S_{nm}|^2 = 1 + 2 \text{Re} \left\{ \sum_{k} \frac{|V_{nk}|^2}{(E_k - E_n)^2} [e^{i(E_n - E_k) T} - 1] \right\}$$

$$= 1 - \sum_{k} \frac{4 \sin^2(E_n - E_k) T_k}{(E_n - E_k)^2} |V_{nk}|^2$$

first-order TDPT for transition prob.
2) Conservation of Probability (Unitarity)

\[ t = 0: \quad \chi_n(0) = 1, \quad \chi_m = 0 \quad m \neq n \]

\[ t = T: \quad \chi_m(T) = S_{mn} \chi_n(0) \]

Now for all \( t, \)

\[ \sum \limits_m |\chi_m|^2 = 1 \]

\[ \Rightarrow \chi^*_m(0) S_{mn} \chi_n(0) = \chi^*_m(T) \chi_n(T) \]

\[ = \chi^*_m(0) \chi_n(0) \quad \text{postulate} \]

In matrix notation, this implies

\[ \chi^* S^* T S \chi = \chi^* \chi \]

\[ \Rightarrow S^* T S = 1 \]

or

\[ S^* S = 0 \]

Thus conservation of probability \( \Rightarrow S \) is unitary

In real vector space, \( \Rightarrow S \) is orthogonal. Thus the S-matrix is made of a complex rotation, but it can't change length (by conservation of probability)
This is an extremely powerful principle. To apply it further, we define the T matrix by

\[ S = 1 + 2i T \]

Thus we can find our new unitarity condition

\[ S^* S = (1 - 2i T^*) (1 + 2i T) = 1 \]

\[ = 1 - 2i (T - T^*) - (2i)^2 T^* T = 1 \]

\[ \Rightarrow \quad \frac{T - T^*}{2i} = T^* T \]

**Applications:**

a. Optical Theorem:

Consider scattering

\[ A + B \rightarrow A + B \]

\[ \rightarrow C + D \]

\[ \rightarrow C' + D' + E' \]

To find the transition probability for forward elastic scattering, we look \( T_{ii} \) (state 0 to state 0)

\[ \frac{T_{ii} - T_{ii}^*}{2i} = T_{in}^* T_{ni} \]

\[ \text{Im} \{T_{ii}\} = T_{in}^* T_{ni} \]

\[ = \sum_n |T_{in}|^2 = (\text{kinematical factor}) \]

Note how similar this looks to our TDPT.
\[ |T_{ii}| > \text{Im}\{T_{ii}\} > T_{ii}^{*} T_{ii} = |T_{ii}|^2 \]

b.) Final State Theorem: (Watson)

Known: \( p^0 N \rightarrow p^0 N \) \( l = 0 \) (s-wave)

Want to know: \( \pi^0 N \rightarrow p^0 N \)

From time reversal we know \( T \) is symmetric.

\[ \text{Im}\{T_{ii}\} = T_{ii}^{*} T_{ii} + T_{ii}^{*} T_{ii} \]

If we assume \( 1 \) intermediate state is dominant (1st process is second order), then

\[ \Rightarrow \text{phase} (T_{ii}) = \text{phase} (T_{ii}) \]
3) **Scattering of Light from an Atom**

Classically one develops Thomson scattering.

By examining the terms in \( \frac{\sum m_n}{c^2} \) we can eventually arrive at quantum mechanics. This kind of classical fields to obtain quantum results has its analogue in particle physics today.

We will develop the quantum theory more directly.

**Part 1: The Student's Approach**

We need a current density. Perhaps we could try

\[
\tilde{J}(\vec{r},t) = \psi^*(\frac{\hbar}{im} \nabla \psi) + (\frac{\hbar}{im} \nabla \psi)^* \psi
\]

and then use

\[
\nabla^2 \tilde{A} = \tilde{J}(\vec{r},t)
\]

But look at this special case when the incident light is turned off. If \( \psi = \psi(\vec{r},t) \), we expand

\[
\psi(\vec{r},t) = \sum_n \phi_n(\vec{r}) e^{-iEt}
\]
Then

\[ J_n(r, t) = \sum_{n,m} C_n C_m^* e^{i(E_m - E_n) t} \left[ \frac{\partial^2 \Phi_n}{\partial r^2} + \frac{\partial \Phi_n}{\partial r} \right] \]

But if \( \Psi \) is an eigenstate, \( \Psi = C_n \Phi_n \) and \( \tilde{\Psi} = \tilde{\Psi}(t) \). Also

Energy radiated into \( (E_m - E_n) = 1 \) \( C_n C_m^* \) = \( |C_n|^2 \) \( |C_m|^2 \)

Thus if \( C_n = 0 \), all in the \( m \)-state so no spontaneous emission

**Problems:**

Point: if we think the other people are frequently go down the

same drawings

We might try to disregard the bad treatment of spontaneous emission and push ahead. Solve

\[ \frac{\partial \Psi}{\partial t} = (H_0 + V_0) \Psi \]

We can do it if we use both incoming and outgoing waves

(advanced and retarded potential)
The problem is that we have been dealing with the EM field classically. The N+1 \wedge N difference is needed for spontaneous emission.

How is one photon changed into another? Regard the atom as having a weak spring between the modes.

\[ A = \frac{1}{\hbar \epsilon} \hat{Q}_e e^{i(k\cdot \hat{r})} \]

\[ \text{Find } H = \frac{1}{\hbar} \left( \frac{\hbar^2}{2m} \hat{P}_e^2 + \frac{\mu_1}{2} Q_e^2 \right) + C_i Q_i \hat{Q}_j \]

\[ \text{Use } H_{\text{atom}} = \frac{1}{2m} (\hat{p} - \frac{\epsilon}{c} \hat{A}) \cdot (\hat{p} - \frac{\epsilon}{c} \hat{A}) + eV \]

\[ = \frac{1}{2m} \hat{p}^2 - \frac{\hbar^2}{2m} \frac{\hat{A} \cdot \hat{A}}{2m} + \frac{\hat{A} \cdot \hat{A}}{2m} + eV \]

\[ \frac{1}{2m} \hat{A}(\hat{r}) \cdot \hat{A}(\hat{r}) = \frac{1}{2m} \sum_i \sum_j \frac{Q_i e_i Q_j e_j}{e_i e_j} e^{i(k_i + k_j) \cdot \hat{r}} e^{i(k_i - k_j) \cdot \hat{r}} \]

Thus we suspect

\[ R_{ij} = 2\pi \delta(\epsilon_a + \omega_1 - \epsilon_b - \omega_2) |M_{ij}|^2 \]

\[ |M_{ij}| = \sum_k \langle b | \hat{E}_2 | Q_i Q_j C_j(\hat{r}) | 0, \hat{k} \rangle \]

\[ = \frac{\epsilon_1 \epsilon_2}{2m} \langle b l e^{i(k + \hat{\kappa} \cdot \hat{r})} | 0 \rangle \sqrt{\frac{\hbar}{2\mu_1 \epsilon_1}} \sqrt{\frac{\hbar}{2\mu_2 \epsilon_2}} \]

Note $\epsilon_1, \epsilon_2 = 0$, no scattering.
Part 2: Feynman's Approach

We were considering the scattering of photons off of atomic states. Let's consider the case where there is one photon of \( | \vec{k}_1, \vec{\omega}_1 \rangle \) in the initial state and two photons of \( | \vec{k}_2, \vec{\omega}_2 \rangle \) in the final state. The energy of the initial state is \( E_a + \omega_1 \), and the energy of the final state is \( E_b + \omega_2 \).

We had suggested that the scattering rate could be represented by:

\[
\text{Rate} = 2\pi \Im (E_f - E_i) \left| \langle \phi | M | \psi \rangle \right|^2
\]

where \( \langle \phi | M | \psi \rangle \) is the matrix element of the scattering operator. We will show later that:

\[
\langle \phi | M | \psi \rangle = -i \left[ \langle \phi | \psi \rangle + \sum_k \frac{\langle \phi | V_k \rangle \langle V_k | \psi \rangle}{E_i - E_k \pm i\gamma} + \ldots \right]
\]

Study this series, calculate higher terms, and interpret it.

Now, consider the Hamiltonian:

\[
H = \frac{\vec{p}^2}{2m} + V(r) - q \left( \frac{\vec{A} \cdot \vec{A} \cdot \varphi}{2m} \right) + \frac{q^2 \vec{A} \cdot \vec{A}}{2m} + H_{\text{interaction}}
\]

where we expand \( \vec{A} = \sum_{n \neq 0} \frac{e}{\sqrt{2\epsilon}} e^{i \vec{k}_n \cdot \vec{r}} \).

[Note: we have two quantizations -- that of the classical field modes and the true quantization of each of these field modes. Sometimes called second quantization.]
Now we had only been considering the $\tilde{A}$, $\tilde{A}$ form since it gave rise to a direct coupling between photon $k_1$ and photon $k_2$. We found

$$\langle \psi \mid \psi \rangle = \frac{\epsilon_1 \epsilon_2}{\sqrt{2} \gamma_1 \gamma_2} \left\langle b \big| e^{-i (k_1 - k_2) \cdot r} \big| a \right\rangle$$

But classically we found terms like $\frac{\epsilon_1}{\omega_1} \frac{\epsilon_2}{\omega_2}$. Why don't we get this by our q.m.?

We forget that indirect coupling of photons through intermediate states is also possible, order like $A$, $\pi$. We must then consider the probability that a photon interacts with the already $\tilde{A}$, $\pi$ and then the other re-admits a new photon again by $\tilde{A}$, $\pi$. This will involve the

$$\frac{1}{k} \frac{\langle k | \psi \mid k \rangle}{E_1 - E_k + i\eta}$$

Now our $V$ is

$$V = \sum_{k} \frac{\epsilon_1 \epsilon_2 e^{i k \cdot r}}{m \gamma_2} Q_{k \pi}$$

Now what intermediate states are possible? Certainly not all. Only those states for which $\langle k | V | k \rangle$ exist equivalently to (*)

Thus one must see where one can get from initial state via the perturbation. Only one of $\tilde{A}$ in Hint, so one photon process are all that are possible.
initial state
atom in a, all oscillators zero except $\vec{k}_1, \vec{e}_1$

possibly intermediate state
atom in c, all oscillators zero

$\hat{\bar{p}} \text{ atom in } \vec{k}_2, \text{ photon in } \vec{b}_2, \text{ photon in } \vec{b}_1$

atom in c, 2 photons in $\vec{k}_1$

Note: to calculate the matrix elements $\langle k1V\mid i \rangle$ using
the sudden approach [don't consider $\langle f1V\mid k \rangle$ yet]

$\langle k1V\mid i \rangle_0 = \sqrt{\frac{2}{\mu_0}} \langle b\mid \hat{\bar{p}} \mid e^{i \vec{b}_2 \cdot \vec{r}} | a \rangle$

$\langle k1V\mid i \rangle_2 = \sqrt{\frac{2}{\mu_0 \omega}} \langle b\mid \hat{\bar{p}} \mid e^{i \vec{b}_3 \cdot \vec{r}} | a \rangle$

$\langle k1V\mid i \rangle_3 = \sqrt{\frac{2}{\mu_0 \omega}} \langle b\mid \hat{\bar{p}} \mid e^{i \vec{b}_4 \cdot \vec{r}} | a \rangle$

Note that when a photon is entering, $e^{i \vec{k}_1 \cdot \vec{r}}$, and when it is split
out, $e^{-i \vec{k}_1 \cdot \vec{r}}$. Now for $\langle f1V\mid k \rangle$

$\langle f1V\mid k \rangle_0 = \sqrt{\frac{2}{\mu_0 \omega}} \langle b\mid \hat{\bar{p}} \mid e^{-i \vec{k}_2 \cdot \vec{r}} | c \rangle$

$\langle f1V\mid k \rangle_2 = \sqrt{\frac{2}{\mu_0 \omega}} \langle b\mid \hat{\bar{p}} \mid e^{i \vec{k}_3 \cdot \vec{r}} | c \rangle$

$\langle f1V\mid k \rangle_3 = 0 \quad \text{unless } \vec{k}_2 = \vec{k}_1, \text{ but we won't consider forward}
\text{scattering [do it yourself]}

Energy
$E_a + \omega_1$
$E_c$
$E_o + \omega_3 + \omega$
$E_c + 2\omega$
Thus we find finally

\[ 
\langle \text{sinh} \rangle = i \left\{ \frac{e_{1}e_{2}}{\sqrt{\omega_{1} \omega_{2}}} \langle b| e^{i(k_{1} - k_{2}) \cdot r}|a \rangle 
+ \sum_{c} \frac{q_{c}^{2}}{\sqrt{\omega_{1} \omega_{2} \omega_{3}}} \langle b| \frac{\tilde{p} c}{m} e^{-i \tilde{k}_{c} \cdot r} |c \rangle \langle c| \frac{\tilde{p} c}{m} e^{i \tilde{k}_{c} \cdot r} |a \rangle 
+ \frac{q_{c}^{2}}{\sqrt{\omega_{1} \omega_{2} \omega_{3}}} \langle b| \frac{\tilde{p} c}{m} e^{i \tilde{k}_{c} \cdot r} |c \rangle \langle c| \frac{\tilde{p} c}{m} e^{-i \tilde{k}_{c} \cdot r} |a \rangle \right\} 
\]

Now up to this point the analysis has been of a very general type. You will run into this quite frequently (photon scattering, etc.)

We will now make a few comments of specific interest in photon scattering.

1) violation of energy conservation ("Then that violate energy conservation the most is the least contributing.")

How to explain violation of energy conservation in barrier penetration to inquisitive students? Must use apparatus to measure the effect and then uncertainty principle fails you up.
If you don't like this explanation, just ask them to invent a physical experiment which shows a paradox.

ii) The atom may emit a photon before it absorbs the incident photon.

We can show this with Feynman diagrams (or as the master says, "we"-diagrams).

Normal process:

[Diagram]

[Diagram]
Problem: Derive a more specific theory for the S-matrix (use matrix algebra or such) - definition, more terms etc. Maybe just properties of S-matrix (time reversal).

Special Properties of Light Scattering:

We had found

\[
R_{\text{rate}} = \frac{4 \pi \delta}{2 \pi} \left( \varepsilon_\alpha + \omega_{\text{in}} - \varepsilon_\beta - \omega_{\text{out}} \right) \frac{k_{\text{fast}}^3}{2 \omega_{\text{in}} \omega_{\text{out}}} (4\pi)^2 |M|^2
\]

Note for each photon coupling \((4\pi e^2)\). Also

\[
M = \sum_c \frac{\langle b l | \hat{\sigma}^c \hat{e}^{-i \vec{k}_2 \cdot \vec{r}} | c \rangle \langle c | \hat{\rho}^c e^{i \vec{k}_1 \cdot \vec{r}} | a \rangle}{\varepsilon_c - \varepsilon_a - \omega_1 + \imath \eta}
+ \frac{\langle b l \hat{\sigma}^c \hat{e}^{i \vec{k}_2 \cdot \vec{r}} | c \rangle \langle c | \hat{\rho}^c e^{-i \vec{k}_1 \cdot \vec{r}} | a \rangle}{\varepsilon_c - \varepsilon_a + \omega_2 + \imath \eta}
+ \frac{1}{m} \langle b l | e^{i (\vec{k}_1 - \vec{k}_2) \cdot \vec{r}} | a \rangle \langle \hat{e}_1 | \hat{e}_2 \rangle
\]

See what the rules are for complex \( \hat{e}_1, \hat{e}_2 \) (circularly polarized light).

Consider atomic systems with more than one electron. Include spin.
Note that the outgoing photon may have a shifted frequency \( \hbar \omega \), and thus the atom is excited. In atomic
the electronic levels are largely spaced. However molecular
levels can show them some scattering [Raman scattering--
made Raman the hero of Indian and Indian science known
to the world].

We consider now elastic scattering from H-atom

\[ E_a + \vec{k}_1 \rightarrow E_a + \vec{k}_2 \]

[Of course we should really deal with 2 particle system
because of finite nuclear mass. Do this yourself for
a problem.]

Assume wavelength \( \gg \) size of atom. [Ordinary
light scattering] Then in matrix elements we can take
\[ \vec{k}_1 \rightarrow 0 \]

and find

\[
M = \sum_c \langle c | \hat{p} | c \rangle \frac{\langle a | \vec{p} | c \rangle \langle c | \vec{p} | a \rangle}{E_c - E_a - \hbar \omega + i\eta} \\
+ \sum_c \langle c | \hat{p} | c \rangle \frac{\langle a | \vec{p} | c \rangle \langle c | \vec{p} | a \rangle}{E_c - E_a + \hbar \omega + i\eta} \\
+ \frac{1}{\hbar} \langle a | 11a \rangle \langle \hat{e}_1 \hat{e}_2 \rangle
\]
we find resonant scattering for \( \omega \sim E_2 - E_1 \)
difference and we can see a very large scattering.

Thus

\[ \langle c | \frac{\vec{P}}{m} | d \rangle = i \frac{\langle c | \hat{H} \hat{x} - \hat{x} \hat{H} | d \rangle}{\epsilon_c - \epsilon_a} \]

Then

\[ M = \sum_{\epsilon_c} \frac{i (\epsilon_a - \epsilon_c) \langle a | \hat{\sigma} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | d \rangle}{\epsilon_c - \epsilon_a - \omega} \]

\[ - \sum_{\epsilon_c} \frac{i (\epsilon_a - \epsilon_c) \langle a | \hat{\sigma} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | a \rangle}{\epsilon_c - \epsilon_a + \omega} \]

\[ + \frac{1}{m} \langle a | a \rangle \langle \hat{\epsilon}_z, \hat{\epsilon}_z \rangle \]

\[ = \frac{i}{\omega} \sum_{\epsilon_c} \left\{ \frac{\langle a | \hat{x} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | a \rangle}{\epsilon_c - \epsilon_a - \omega} + \frac{\langle a | \hat{\sigma} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | a \rangle}{\epsilon_c - \epsilon_a + \omega} \right\} \]

\[ - \frac{i}{\omega} \sum_{\epsilon_c} \left\{ \langle a | \hat{x} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | a \rangle - \langle a | \hat{\sigma} \hat{\epsilon}_z | c \rangle \langle c | \hat{\sigma} \hat{\epsilon}_z | a \rangle \right\} \]

\[ + \frac{1}{m} \langle a | a \rangle \langle \hat{\epsilon}_z, \hat{\epsilon}_z \rangle \]
Now use the matrix product
\[ \leq \langle a | A | c \rangle \langle c | B | b \rangle = \langle a | A B | b \rangle \]

Our second term becomes
\[ -i \left\{ \langle a | (x_i^\alpha \hat{\epsilon}_i^\beta) (p_i^\alpha \hat{\epsilon}_i^\beta) | a \rangle \right\} - \langle a | (p_i^\alpha \hat{\epsilon}_i^\beta) (x_i^\alpha \hat{\epsilon}_i^\beta) | a \rangle \]

Now, \[ p_i - q_i p = \frac{\hbar}{i} \]
\[ = -\frac{\hbar}{m} \langle a | a \rangle (\hat{\epsilon}_i^\alpha \hat{\epsilon}_i^\beta) \]
and cancels the third term.

\[ \begin{align*}
& \quad \text{Folks some about sum rules} \\
& \leq \frac{i}{\hbar} | X_{\lambda i} |^2 (E_n - E_k) = \frac{\hbar}{m} \\
\end{align*} \]

If we had also done the \[ \langle c | p_i^\alpha \hat{\epsilon}_i^\beta | a \rangle \] we would have arrived
\[ \leq \frac{i}{\hbar} (E_n - E_k) | \langle a | x_i^\alpha | c \rangle |^2 = \frac{1}{m} \]
Now in the first term in (\(*\)), we see that for small \( \omega \),
\( \varepsilon_0 - \varepsilon_0 - \omega \), are \( \sim \) small. Then we can do sums.
Thus sum rules are very useful.

**Roton Excitation Problem:**

\[ \varepsilon(p) = \Delta + \frac{1}{2m}(p-p_0)^2 \]

Here allow the density to vary,

\[ \varepsilon(p) = \Delta + \frac{1}{2m}(p-p_0)^2 + d(p-q) \]

If density varies in space, roton energy also varies. Then interaction
in our perturbation

The amplitude for finding the roton at \( r \) can be taken to be

\[ \Psi(r) = e^{i\vec{p}\cdot\vec{r}} \]

\[ \therefore \Psi(\text{int}) \sim e^{i\vec{p}\cdot\vec{r}} \Phi(\text{int}) \]

Thus we take

**Initial state:** roton \( \vec{p} \)

**Final state:** roton \( \vec{p}' \), photon \( k \)

Phonons are to sound what photons are to light. Expect sound modes
Classical

Displacement $\vec{u}$

$$\vec{u}(x,t) = \sum_k \frac{Q_k e^{-ikx}}{k} e^{ikx} \frac{\ddot{x}}{k}$$

Kinetic energy

$$T = \int \frac{1}{2} \rho (\frac{\ddot{x}}{k})^2 \, dx \, dt \approx \frac{1}{2} \rho \frac{k^2}{4} \left(\frac{\ddot{x}}{k}\right)^2$$

Potential energy

$$V = \frac{1}{2} \left( \rho - \rho_0 \right)^2$$

$$\Delta s = \Delta x + u(x,t) - u(x+\Delta x,t)$$

$$= \Delta x + \Delta x \frac{\partial u}{\partial x}$$

$$\rho \Delta s = \rho_0 \Delta x$$

$$\frac{\rho}{\rho_0} = 1 + \nabla \cdot \vec{u} \quad \Rightarrow \quad \rho_0 = \rho + \rho_0 \nabla \cdot \vec{u}$$

$$V \approx \frac{1}{2} \left[ \rho_0 \vec{u} \right]^2 = \frac{1}{2} \rho_0^2 \frac{Q_k^2}{k^2}$$

Total energy

$$E = \sum_k \frac{\rho_0}{2} \left( \frac{\dot{Q}_k^2}{k} + \gamma \rho k^2 \frac{Q_k^2}{k^2} \right)$$

Now $\omega = \gamma$ & $\gamma = \frac{\omega^2}{\rho_0}$
Quantum:

Set $Q_k$ become quantum oscillators

$$H = \frac{\hbar^2}{2m} \left( \frac{p_k^2}{2} + m \omega_k^2 Q_k^2 \right)$$

Thus

$$H = H_{ext} + H_{phonon} + H_{int}$$

$$= \varepsilon(p) + \frac{\hbar^2}{2m} \left( \frac{p_k^2}{2p_0^2} + \frac{m \omega_k^2}{2} Q_k^2 \right) + g(p'(r) - p)$$

$$H_{int} = \text{dip}_0 \sum_k Q_k e^{ikr}$$

Then

$$\text{Rate} = 2\pi S (\varepsilon(p) + \hbar \omega_k c - \varepsilon(p)) \frac{M^2}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}$$

Now

$$M = \langle \hat{p}' \hat{e} | \text{dip}_0 k Q_k e^{ik \cdot \hat{r}} | \hat{p}, 0 \rangle$$

$$= \frac{\text{dip}_0 k}{\sqrt{2p_0 \omega_k}} \langle \hat{p}' | e^{ik \cdot \hat{r}} | \hat{p} \rangle$$
\[ \langle \varphi' e^{-i\mathbf{p}' \cdot \mathbf{x}} \varphi \rangle = \sqrt{e^{-i\mathbf{p}' \cdot \mathbf{x}} e^{i\mathbf{p} \cdot \mathbf{x}}} \int d^3 \mathbf{r} \]

\[ = \delta(\mathbf{p}' - \mathbf{p} - \mathbf{k}) \]

\text{conservation of momentum}

\[ \text{Rate} = 2\pi \delta (\varepsilon(p') + \hbar c k - \varepsilon(p)) \frac{d^2 k^2 e^2_a}{2p_0 kc} \frac{\hbar^2 dk}{(2\pi)^3} \]

\text{Consider first forward emission}

\[ \begin{array}{c}
\mathbf{p} \\
\mathbf{p}' \\
\mathbf{k}
\end{array} \]

\[ \begin{array}{c}
\varepsilon(p) \\
\varepsilon(p') \\
\varepsilon(p)
\end{array} \]

\text{No good}

\[ \begin{array}{c}
\varepsilon(p) \\
\varepsilon(p)
\end{array} \]

\[ \begin{array}{c}
p_0 \\
p_0
\end{array} \]

\[ \text{Thus a critical } p_0 \text{ below which we cannot emit photons} \]

\[ \left. \frac{\partial \varepsilon(p)}{\partial p} \right|_{p_0} = \frac{\partial \varepsilon(p)}{\partial p_x} = c_s \]
In any system with $e = e(p)$

$$\frac{de}{dp} = \nu_{\text{group}}$$

Proof:

$$\Delta e = F \Delta x$$

$$F = \frac{dP}{dt}$$

$$\frac{\Delta e}{\Delta t} = \frac{dP}{dt} \nu$$

$$\frac{\Delta e}{\Delta P} = \nu$$

Thus, velocity of rotation must exceed speed of sound $c_s$ to leave plasma emission [same as bremsstrahlung]
3) Some Improvements to our Theory

We had found things like

$$0 \leq \frac{\langle b|\phi_1|c\rangle \langle c|\phi_2|a\rangle}{E_c - E_a - \omega_1}$$

But results like this are very similar to classical physics in which the atom is treated as a classical harmonic oscillator

$$m\left(x'' + \omega^2 x' + \omega^2 x\right) = E$$

This gives a broadening of the resonance.

We can reverse this argument to obtain absorption lines in solar spectra.

Now we would like to get such a line width into our quantum theory for very close to resonance (when we are not close, our new theory won't be quite right).

The trouble with the old theory arises from a degeneracy, e.g., a level \( n \) is degenerate with a class of states in the continuum

$$n \sim \frac{\omega}{\omega_0}$$

We will find the state \( n \) then has a finite lifetime and will eventually decay into the continuum states.
Recall we began our treatment of time-dependent perturbation theory by trying to calculate $g_m(t)$, the amplitude that we will find the system in state $m$ at time $t$. We found

$$i \dot{g}_m(t) = \sum_n V_{mn} \dot{g}_n(t)$$

$$= \sum_n V_{mn} e^{i(E_n - E_m)t} g_n(t)$$

(1)

Now let our system (say an atom) be in a state $n$ and let state $k$ be in a continuum (e.g., scattering of light from a particle that can disintegrate). Then (1) becomes

$$i \dot{g}_n(t) = \left( \text{terms relative to states other than (the continuum)} \right) + \sum_k V_{nk} e^{i(E_n - E_k)t} b_k(t)$$

(2)

If we assume $k$ interacts only with $n$, we can write

$$i \dot{b}_k(t) = V_{nk} e^{i(E_n - E_k)t} g_n(t)$$

(3)

We want to include the effects of $k$ close to $n$, thus we assume $V_{nk}$ are very small (so the system has at least a reasonable lifetime). Then we integrate (3) directly

$$b_k(t) = -i \int_0^t V_{nk} e^{i(E_n - E_k)t'} g_n(t') dt'$$

Substitute this into (2) to find

$$i \dot{g}_n(t) = (\text{other terms}) - i \sum_k \int_0^t V_{kn} e^{-i(E_n - E_k)t'} g_n(t') dt'$$

(4)
We rewrite an equation in terms of the $\xi_n$'s above. We have removed details of the box. We only remain to affect on the actual process we are interested in.

Now assume the $\xi_n$'s vary slowly and

$$\xi_n(t) \sim \xi_n(t')$$

Hence also consider time-independent contributions and note

$$\int_0^\infty du \, e^{-\xi_n(E_n - i\epsilon)} = \frac{1}{\xi_n + i(E_n - \epsilon)}$$

we can write (4) as

$$i \dot{\xi}_n(t) = (\text{other terms}) - \left[ \int_{-\infty}^{\infty} \frac{|\mathcal{M} n|_2^2}{E_n - E_{n+i\epsilon}} \right] \xi_n(t)$$

Now define

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{|\mathcal{M} n|_2^2}{E_n - E_{n+i\epsilon}} = \Delta_n + i \frac{\Gamma_n}{2}$$

so that (5) becomes

$$i \dot{\xi}_n(t) = (\text{other terms}) - (\Delta_n + i \frac{\Gamma_n}{2}) \xi_n(t)$$

Special Case: Solve (7) neglecting (other terms)

$$\xi_n(t) = \xi_n(0) e^{(\Delta_n + i \frac{\Gamma_n}{2})t}$$

But recall $\xi_n(t) = C_n(t) e^{E_n t}$
Thus \[ C_n(t) = e^{-i (E_n - A_n)t} e^{-\frac{\Gamma_n t}{2}} \]

We can see from this special case that the continuum interaction shifts the levels by \( A_n \) [for example, all levels of the \( S \)-state for atoms should be shifted because atoms can radiate into a continuum].

Also note

\[ |C_n|^2 = e^{-\Gamma_n t} \]

Thus the probability of remaining in the state \( n \) decays with a mean life \( \frac{1}{\Gamma_n} \).

In the remainder of our analysis we will forget the small shift correction \( A_n \) for a bit. The important physical effect for us is \( \Gamma_n \), the line width.

We now return to calculating \( \Gamma_n \) and \( A_n \) from (6).

Recall

\[ \lim_{\gamma \to 0} \frac{1}{x + i \gamma} = \text{PV.} \left( \frac{1}{x} \right) - i \pi S(x) \]

We then find

\[ A_n = \text{PV.} \int_{k_e} C_n^2 \frac{1}{E_n - E_k} \]

We then find

\[ \Gamma_n = 2\pi \int_{k_e} |C_n|^2 S(E_k - E_n) \]

Note to go from state \( n \) to anything (all) \[ \frac{\partial}{\partial t} \rho_n = -\Gamma_n \rho_n \]
To generalize these results we recognize that we get a Γ for each type of continuum (α decay, β decay, etc... — each decay "channel").

**Example:** Eight emission by an atom which itself can disintegrate

\[ n \quad m \quad \text{both } n \text{ and } m \text{ can disintegrate} \]

Set the atom initially in state \( n \), i.e. \( a_n(0) = 1 \).

The analogue to (5) becomes

\[ a_n(t) = -i \nu \nu_m(t) a_m(t) - \frac{\Gamma_n}{2} a_n(t) \]  
\[ a_m(t) = -i \nu \nu_n(t) a_n(t) - \frac{\Gamma_m}{2} a_m(t) \]  

where we have ignored \( \Delta n \) and \( \Delta m \). Ignoring \( \nu \nu_m \) in (24), we can solve directly for

\[ a_n(t) = e^{-\frac{\Gamma_n}{2} t} \]

We can then substitute this into (26) to find

\[ a_m(t) = e^{-\frac{\Gamma_m}{2} t} \left( -i \nu \nu_n(t) \right) e^{-\frac{\Gamma_n}{2} t} e^{\frac{\Gamma_n}{2} t} \int_0^t e^{-\frac{\Gamma_n}{2} t'} \, dt' \]

Set \( V = \text{const} \), \( \nu \nu_n = \nu \nu_m e^{-i(\mathcal{E}_n - \mathcal{E}_m)t} \)

Thus

\[ a_m(t) = \frac{\nu \nu_m}{i(\frac{\Gamma_n}{2}) + (\mathcal{E}_n - \mathcal{E}_m)} \left[ e^{-}\frac{\Gamma_n}{2} t - i(\mathcal{E}_n - \mathcal{E}_m) t e^{-\frac{\Gamma_n}{2} t} \right] \]
\[ \cdot P_n(t) = \frac{|V_{nm}|^2}{(E_n - E_m)^2 + (\frac{\Gamma_n - \Gamma_m}{4})^2} \left[ e^{-\frac{\Gamma_n t}{2}} - e^{-\frac{\Gamma_m t}{2}} - 2 \cos(E_n - E_m) t e^{-\frac{\Gamma_n t}{2}} e^{-\frac{\Gamma_m t}{2}} \right] \]

Suppose

state \( n \): atom in \( a \)

state \( m \): atom in \( b \), plasmon \( \omega \)

If we use \( \int_0^\infty \frac{dx}{x^2 + \frac{\gamma^2}{4}} = \frac{\pi}{\gamma} \) \( \int_0^\infty \frac{dx}{x^2 + \frac{\gamma^2}{4}} \cos x t = \frac{\pi}{\gamma} e^{-\gamma |t|} \)

we can find

\[ \cdot P_n(t) = \rho(E_n) |V_{nm}|^2 \frac{\pi}{\Gamma_n - \Gamma_m} \left[ e^{-\frac{\Gamma_n t}{2}} - e^{-\frac{\Gamma_m t}{2}} - 2 e^{-\frac{(\Gamma_{n} + \Gamma_{m}) t}{2}} e^{-\frac{\Gamma_n t}{2}} e^{-\frac{\Gamma_m t}{2}} \right] \]

\[ = \rho(E_n) |V_{nm}|^2 \frac{\pi}{\Gamma_n - \Gamma_m} \left[ e^{-\frac{\Gamma_n t}{2}} - e^{-\frac{\Gamma_m t}{2}} \right] \]

We can then this to the following problems ("decay bands")

\[ \cdot P_n(t) \quad \rho_n(t) = - \Gamma_n P_n(t) \]

\[ P_n(t) = \rho P_n(t) - \Gamma_n P_n(t) \]
PROBLEM: In strong interactions, Goldhaber coined up a quantity called strangeness \( S \), which is defined as \( \Delta S = 0 \).

\[ \pi^- + p \rightarrow \Lambda^0 + \bar{K}^0 \]

\[ S = 0 \quad S = 0 \quad S = 0 \quad S = 0 \]

or

\[ \pi^- + p \rightarrow p + K^- + \bar{K}^0 \]

\[ S = 0 \quad S = 0 \quad S = 0 \quad S = 0 \]

To distinguish, we note that in matter,

\[ \bar{K}^0 + p \rightarrow \Lambda^0 + \pi^+ \]

\[ \bar{K}^0 + p \rightarrow \Lambda^0 + \pi^+ \]

This shows the \( K^0 \) 's decay by weak interactions, leading to a continuum of states. Goldhaber and Pais found that for two states decaying into some continuum, the rate of finding a given \( K \) does not exponentially tend to zero with time, and we have thus measured the effect. [Feynman observed that this proves the principle of superposition of amplitudes.]
If you want to go further, you might look at

\[ \kappa^0 \rightarrow \pi^- + \mu^+ + \nu \]

rate = \( \frac{1}{600} \) of \( 2\pi \) decay

\[ \kappa^0 \rightarrow \pi^+ + \mu^- + \bar{\nu} \]

which makes a more realistic problem

we still need \( \mathbf{\rho} \rightarrow \langle \rangle = \langle \bar{\nu} \rangle \)

We return to our perturbation theory of states which can decay into something you are not interested in. Since the state doesn't have an infinite lifetime due to the decay, we know qualitatively from the uncertainty principle that it cannot have a definite energy, but must be regarded as an intermediate with a continuum of states, one of which was our earlier definite energy state. Thus, a kind of degenerate perturbation theory.

We had introduced these concepts by writing

\[ i \dot{a}_m = \sum_n V_{mn}(t) a_n(t) + \sum_k M_{km} b_k(t) e^{i (E_n - E_k) t} \]

\[ i \dot{b}_k = \sum_m M_{km} e^{-i (E_n - E_k) t} a_m(t) \]

We want to ignore the details of the \( b \)'s and worry only about the \( a \)'s. We found

\[ \dot{a}_m = -i \sum_n V_{mn}(t) a_n(t) - \sum_k M_{km} \int e^{-i (E_n - E_k) (t - t')} a_n(t') \, dt' \]

\[ + \sum_n \int M_{km} e^{-i (E_n - E_k) t'} e^{i (E_n - E_k) t} a_n(t') \, dt' \]
We are not interested in the last term since it won't affect us very much (no 'radial' or resonance effects), so we will ignore it.

If we assume constant perturbations and \( \mathbf{v}_n(t) \sim \mathbf{v}_n(t) \), we could integrate to find

\[
\dot{\mathbf{a}}_n = -i \sum_{m} \mathbf{v}_{nm}(t) \mathbf{a}_m(t) - i \left( \Delta_n - i \frac{\Gamma_n}{2} \right) \mathbf{a}_n(t)
\]

where

\[
\Delta_n - i \frac{\Gamma_n}{2} = \int \frac{|M_{nk}|^2}{E_n - E_k + i\epsilon}
\]

\[
\Gamma_n = 2\pi \int \frac{S(E_k, E_n)}{|M_{nk}|^2} dE_k
\]

If we ignore the shift \( \Delta_n \), we find

\[
\dot{\mathbf{a}}_n = -i \sum_{m} \mathbf{v}_{nm}(t) \mathbf{a}_m(t) - \frac{\Gamma_n}{2} \mathbf{a}_n(t)
\]

The probability of some disintegrated state from \( n \) state is

\[
P_{\text{dis}}(n) = \left| \sum_{k} b_{nk}^* b_k \right|^2
\]

Can show

\[
P_{\text{dis}}(n) = \Gamma_n |\mathbf{a}_n(t)|^2
\]

[Show it]
Last time we were considering an example

\[ a, \Gamma_a \]  
\[ n \text{ state: atomic excitation} \]

\[ b, \Gamma_b \]  
\[ n \text{ state: atomic } n, \text{ photon } \omega \]

\[ a_m = -i V_{mn} e^{i(E_m-E_n)t} a_n(t) - \frac{\Gamma_m}{2} a_m(t) \]

\[ a_n = -\frac{\Gamma_n}{2} a_n(t) \]

Can do this only to first order, by first solving

\[ a_n(t) = a_n(0) e^{-\frac{\Gamma_n}{2}t} = e^{-\frac{\Gamma_n}{2}t} \]

\[ a_m(0) = 1 \]

Can then find

\[ a_m(t) = -V_{mn} \left( e^{i(E_m-E_n)t - \frac{\Gamma_m}{2}t - \frac{\Gamma_n}{2}t} \right) \]

\[ \frac{E_m - E_n - i(\Gamma_n - \Gamma_m)}{E_m - E_n - i(\Gamma_n + \Gamma_m)} \]

\[ \rho_m = \left| a_m(t) \right|^2 = \left| V_{mn} \right|^2 \left( \frac{e^{\frac{\Gamma_m}{2}t} - e^{\frac{\Gamma_n}{2}t} e^{-\frac{\Gamma_n}{2}t} \cos(\omega t)}{(E_m - E_n)^2 + \left(\frac{\Gamma_n - \Gamma_m}{2}\right)^2} \right) \]

Now \( E_m - E_n = \epsilon_0 + \omega - \epsilon_0 = \omega - \omega_0 = \nu \)  

\[ \Gamma_m = \Gamma_b, \quad \Gamma_n = \Gamma_a \]
But we want prob. of finding photon of \( v \) and don't want
to worry about what happens to state \( b \).

\[
P_v(t) = P_m(t) + P_{\text{decays}}
\]

and

\[
P_{\text{decays}} = \Gamma_b P_m
\]

Recall

\[
P_m = \text{probability of photon emission of frequency } \omega + v = \omega \text{ and atom in } b
\]

\[
P_{\text{decays}} = \text{probability that atom in } b \text{ decays (photon still there)}
\]

\[
P_v = \text{probability the same photon of } v \text{ is present}
\]

Thus

\[
P_v(t) = P_m(t) + P_{\text{decays}}(t)
\]

Look at \( P_v(\infty) = P_m(\infty) + P_{\text{decays}}(\infty) \)

now \( P_m(\infty) = 0 \quad [\text{atom certainly has decayed by } t = \infty] \)

\[
P_{\text{decays}} = \Gamma_b \int_0^\infty P_m(t) \, dt
\]

Thus

\[
P_v(\infty) = \Gamma_b |V_{mb}|^2 \frac{1}{\omega^2 + (\frac{\omega - \Delta}{2})^2} \left[ \frac{1}{\Gamma_a} + \frac{1}{\Gamma_b} - \frac{2(\frac{\Delta + \omega}{2})}{\omega^2 + (\frac{\Delta + \omega}{2})^2} \right]
\]
Thus

\[ P_\nu(t) = \frac{\nu \nu' \nu''}{\Gamma_a} \left( \frac{\Gamma_a + \Gamma_b}{\nu^2 + \left(\frac{\Gamma_a + \Gamma_b}{2}\right)^2} \right) \]

"Lorentzian" or "Breit-Wigner" line shape

Note if our instruments aren't very accurate, this looks like

\[ P_\nu(t) \sim \frac{2\pi \nu \nu' \nu''}{\Gamma_a} S(\nu) \]

Thus the probability of any photon of any \( \nu \) can be found by summing over \( \nu \) (or integrating \( S(\nu) \) over \( \nu \))

\[ P = \sum_{\nu} P_\nu(t) \sim \int P_\nu(t) d\nu = \frac{2\pi \nu \nu' \nu''}{\Gamma_a} S(\nu) \]

---

Comment: We might arrive at (3) using a different argument.

Rate of emission of photon/sec = \( \frac{2\pi \nu \nu' \nu''}{\Gamma_a} \) \( S(\nu) \) \[ \text{(Prototype in upper state)} \]

Total prob. of emission = \( \int \text{Rate } dt = \frac{2\pi \nu \nu' \nu''}{\Gamma_a} S(\nu) \]

Note that if we replace \( S(\nu) \rightarrow \frac{\Gamma_a + \Gamma_b}{2\pi \left[ \nu^2 + \left( \frac{\Gamma_a + \Gamma_b}{2} \right)^2 \right]} \) we come back to (3).

Quite a few people use (3) to say that the line shape drops \( S(\nu) \) or absorption but decays as \( e^{-\Gamma_a t} \). However this is incorrect as we can see by considering a special case of...
transitions to a stable state b

\[ P_b(t) = P_m = |V_{m2}|^2 \frac{1 + e^{-\gamma_a t} - e^{-\gamma_b t}}{\gamma^2 + (\gamma_b/2)^2} \]

\[ \neq |V_{m2}|^2 \frac{1 - e^{-\gamma_a t}}{\gamma^2 + (\gamma_b/2)^2} \quad \text{as we would have expected from (5)} \]

Thus the line shape changes with time if we have decay. This has been verified by using Mössbauer effect with K-rays.

Now return to our light scattering problem. Consider only elastic scattering from a stable state a

\[ \Gamma_a = 0 \]

We can then set up the "Wigner–Weisskopf" equations

\[ \dot{a}_i = -i \, V_{ik} \, a_k(t) e^{-i(E_k - E_i) t} - \frac{\Gamma_a}{2} a_i \]

\[ \dot{a}_j = -i \, V_{jk} \, a_k(t) e^{-i(E_k - E_j) t} - \frac{\Gamma_a}{2} a_j \]

\[ \dot{a}_k = -i \, V_{ki} \, a_i(t) e^{i(E_k - E_i) t} - \frac{\Gamma_a}{2} a_k \]
Zero-order: \( a_i = 1 \)

First order: \( a_k = -ie^{-\frac{\Pi b t}{2}} \int e^{i(E_k - E_i)\frac{t}{2}} e^{-\frac{\Pi b t}{2}} dt \)

\[ = -V_{ki} \left[ \frac{e^{i(E_k - E_i)\frac{t}{2}} - e^{-\frac{\Pi b t}{2}}}{E_k - E_i - i\Pi b/2} \right] \]

We then find

\[ a_j = -i \frac{V_{jk} V_{ki}}{E_k - E_i - i\Pi b/2} \int e^{-i(E_k - E_j)\frac{t}{2}} e^{i(E_k - E_i)\frac{t}{2}} e^{i(E_j - E_i)\frac{t}{2}} e^{-\frac{\Pi b t}{2}} e^{-i(E_k - E_j)\frac{t}{2}} dt \]

\( \text{disappears for long scattering times} \)

Thus we find

\[ a_j(T) = \left\{ V_{ji} + \sum_k \frac{V_{jk} V_{ki}}{E_k - E_i - i\Pi b/2} \right\} e^{i(E_j - E_i)\frac{T}{2}} \int_0^T e^{i(E_j - E_i)\frac{T}{2}} e^{-i(E_j - E_i)\frac{T}{2}} dt \]

\[ = \frac{M_{ji}}{E_j - E_i} e^{i(E_j - E_i)\frac{T}{2}} \]

Thus

\[ P_{R_{ij}} = |M_{ji}|^2 \frac{\sin^2(E_j - E_i)\frac{T}{2}}{(E_j - E_i)^2} \]

Rate

\[ i \rightarrow j = 2n S(E_i - E_j) |M_{ji}|^2 \]

where \( M_{ji} = V_{ji} + \sum_k \frac{V_{jk} V_{ki}}{E_k - E_i - i\Pi b/2} \)

\( \text{replaces e in our light} \)
This is the desired correction to our second-order perturbation theory, since all real states have finite lifetimes, we always have a $\Gamma$ (Kubo resistance in circuit theory).

Long ago we found

$$\sum_{\nu \neq \mu} \lim_{\Delta \to 0} \frac{V_{\mu \nu} V_{\nu \mu}}{E_n - E_{\mu}} \left( \frac{e^{i(E_\mu - E_n)\Delta}}{E_\mu - E_n} - 1 \right) - \frac{e^{i(E_\mu - E_n)\Delta}}{E_\mu - E_n} - 1$$

Look in Hibbs and Feynman, p. 154 to get

$$\sum_{\nu \neq \mu} \lim_{\Delta \to 0} \frac{V_{\mu \nu} V_{\nu \mu}}{E_n - E_\mu - i\epsilon}$$
Trivials & Loops Rules

\[ \langle v \mid a_1 \times k \rangle < k_1 \times k_2 > \]

\[ \frac{\langle a_1 \times k \rangle < k_1 \times k_2 >}{c^2 + \alpha} = \langle a \mid x \frac{1}{H + \alpha} x \mid b > \]

very hard to invent however

Hints to get good \( \hbar / c \)

\[ \frac{e^2}{\hbar c} = \frac{1}{137} \]

[All theoretical physicists does before 137]

Length & time are equivalent units

\[ \text{Life time} = \frac{\text{length}}{c} \]

\[ \frac{1}{\text{length}} = \frac{1}{\text{time}} = \text{energy} = \text{momentum} = \text{mass} \]

Now \[ \frac{\hbar}{c} \]

\[ \text{mass} = \frac{\hbar / c^2}{\text{life time}} \]
C. **ALTERNATIVE FORMULATIONS**

Today we write the perturbation theory in self-adjoint form (not important, but necessary to read literature)

1) **Evolution Operator**
   We should be able to write

   \[ \psi(t_0) = U(t_0, t_a)|\psi(t_a)\rangle \]

   where \( U(t_0, t_a) \) is a linear operator. Now we define

   \[ i \frac{d\psi(t_0)}{dt} = H(t_a)|\psi(t_0)\rangle \]

   Thus

   \[ i \frac{dU(t_0, t_a)}{dt_0} = H(t_0)U(t_0, t_a) \]

   subject to \( U(t_a, t_a) \equiv 1 \)

   

Sometimes we use

\[ S = \lim_{t_a \to -\infty} \lim_{t_b \to \infty} U(t_b, t_a) \]

although, strictly speaking, the limit doesn't exist.

More trouble than it's worth, however.
Suppose \( H \) is independent of time

\[
U(t_b, t_a) = e^{-iH(t_b-t_a)} = e^{-iHT}
\]

\( T = t_b - t_a \)

What is \( e \) operation? Several possible (equivalent) definitions:

i.) \( H\ket{\phi} = E\ket{\phi} \), then

\[
e^{-iHT}\ket{\phi} = e^{-iET}\ket{\phi}
\]

ii.) powerseries expansion of \( e^{-iHT} \)

iii.) define by differential equation

\( \phi_n \) are base states of \( H\ket{\phi_n} = E_n\ket{\phi_n} \), then

\[
\langle \phi_m | U | \phi_n \rangle = \langle \phi_m | e^{-iHT} | \phi_n \rangle
\]

\[
= \langle \phi_m | \phi_n \rangle e^{-iE_n T} = \delta_{mn} e^{-iE_n T}
\]

Can now return and write

\[
U = \sum_{m,n} \langle \phi_m | e^{-iE_n (T_b-t_a)} | \phi_n \rangle \langle \phi_m | \phi_n \rangle
\]

\[
= \sum_{n} \langle \phi_n | e^{-iE_n (T_b-t_a)} | \phi_n \rangle
\]

If we can find \( \phi_n \),
2) Integral Formulations

We can now write $U$ in other representations, say $x$-representation.

$$\langle x' | U(t_b, t_a) | x \rangle = \mathcal{K}(x', x; t_b, t_a)$$

If $H$ is constant

$$\mathcal{K}(x', x; T) = \sum_n \phi_n(x') \phi_n^*(x) e^{-iE_nT}$$

Thus

$$\Psi(x', t_b) = \int \mathcal{K}(x', x; T) \Psi(x, t_a) dx$$

For many cases $\mathcal{K}(x', x; T)$ can be summed [free particle gives]

$$\mathcal{K}(x', x; T) = \sqrt{\frac{2\pi i m}{2\hbar T}} e^{-\frac{m[(x'-x)^2]}{2\hbar T}}$$

[the diffusion kernel]

Can sum for anything that has quadratic order.
3) Some properties of $U$

\[ U(t_0, t_a) = U(t_0, t_c) U(t_c, t_a) \quad t_a < t_c < t_0 \quad (\ast) \]

[Diagonalization in Hilbert space, so]

\[ U(t_0, t_i) = \prod_{i=0}^{N} U(t_{i+1}, t_i) \]

but relation not complete

Phys. Rev. 84, 108 (1951) says an ordering under

\[ U(t_0, t_i) = \prod_{i=0}^{N} U(t_{i+1}, t_i) \]

where \( U_{i+1} U_i \)

Thus, if you want \( B A \), write \( A_2 B \) or \( B A_2 \).

\( \text{Ex: old:} \quad \nabla x (U \times V) \)

\( \text{new:} \quad \nabla x (U \times V) \quad (??) \)

New using

\[ U(t + \Delta t; t) = e^{-i H(t) \Delta t} \]
we find

\[ U(t_b, t_a) = \lim_{\Delta t \to 0} e^{-i H(t_a) \Delta t} e^{-i H(t_{m-1}) \Delta t} \cdots e^{-i H(t_b) \Delta t} \]

\[ = \lim_{\Delta t \to 0, N \to \infty, N \Delta t \to t_b-t_a} e^{-i \int_{t_a}^{t_b} H(t) \, dt} \]

\[ = e^{-i \int_{t_a}^{t_b} H(t) \, dt} \]

But wrong because ordered limit has not been preserved.

Instead use

\[ U(t_b, t_a) = e^{-\int_{t_a}^{t_b} H(t) \, dt} \]

[Has not found a good use for this yet, but feels it will come in handy somewhere.]

Now from conservation of probability and causality

\[ [U(t_b, t_a)]^+ U(t_b, t_a) = 1 \]

\[ \Rightarrow U \text{ is a unitary operator} \]
Try to extend definition of $U$ such that

$$U(t_b, t_a) = U(t_b, t_c) U(t_c, t_a) \quad t_c \notin [t_a, t_b]$$

Suppose things in future can affect past, then

$U$ is not unitary and $(\star)$ doesn't work. However, we can fix $(\star)$ up with some new variables. Maybe causality isn't correct.

Ex: EH field can be turned on leaving only partly variables. Use highly abstracted and advanced mathematics. Feynman got stuck 3/4 through. Threat out differential

equation approach (he judged a 'price' for it)

However. D.E.'s are not useful relativistically.

All he did was use consider

$$H = \left( H^{(1)} + H^{(3)} \right) + \text{constants}$$

in

$$U^n = e^{-i \int H_0 \, dt} = e^{-i \int \left( H^{(1)} + H^{(3)} \right) \, dt} + e^{-i \int \frac{e^2}{r_{i+1} - r_i} \, dt}$$

$$= \left( 1 - \frac{r_{i+1} - r_i}{e^2} \right)$$

Doesn't quite work.
Recall we had defined:

\[ U(t_b,t_a) : \quad i \frac{dU}{dt_b} = H(t_b) U \quad \text{subject to} \quad U(t_a,t_a) = 1 \quad \text{(1)} \]

If \( H \) independent of time,

\[ U(t_b,t_a) = e^{-i H(t_b-t_a)} \]

\[ = \sum_n \langle \phi_n | e^{-i E_n(t_b-t_a)} | \phi_n \rangle \quad \text{(2)} \]

4. Neumann Series

If we can write \( H = H_0 + V \), we can set up a perturbation scheme. No physical significance of \( V \) (not "interaction" or such). Whatever is convenient.

Suppose

\[ i \frac{dU_0}{dt_b} = H_0(t_b) U_0(t_b,t_a) \quad \text{(2)} \]

We can integrate (1) and use (2) to write

\[ U(t_b,t_a) = U_0(t_b,t_a) - i \int_{t_a}^{t_b} U_0(t_b,t) V(t) U(t,t_a) dt \quad \text{(3)} \]

By iteration we can set up the Neumann series

\[ U(t_b,t_a) = U_0(t_b,t_a) - i \int_{t_a}^{t_b} U_0(t_b,t) V(t) U_0(t,t_a) dt + \sum_{n=2} \int_{t_{a+n-1}}^{t_b} U_0(t_b,t) V(t) U_0(t,t_{a+n-2}) V(t_{a+n-1}) V(t_{a+n-2}) \ldots dt \quad \text{(4)} \]
One can prove the convergence of this series but who cares? If it does not converge, it does not work, and we do it.

Now let H0 be small and simple enough so we know

\[ H_0 \phi_n = E_n \phi_n \]

Thus

\[ U_0(t_b,t_a) = e^{-iH_0(t_b-t_a)} \]

We can plug this into (4) to find

\[ U(t_b,t_a) = e^{-iH_0(t_b-t_a)} - i \int e^{-iH_0(t_b-t_1)} e^{-iH_0(t_1-t_2)} dt_1 dt_2 + \ldots \]

Now consider \( \langle m | U | n \rangle \). Note

\[ e^{-iH_0+} \langle n | = e^{-iE_0+} \langle n | \]

Thus our series becomes

\[
\langle m | U | n \rangle = \sum_n e^{-iE_n(t_b-t_a)} - i \int e^{-iE_n(t_b-t_1)} \langle m | U(t_1) | n \rangle e^{-iE_n(t_1-t_2)} dt_1 dt_2 \\
+ (i)^2 \int e^{-iE_n(t_b-t_1)} \langle m | U(t_1) e^{-iH_0(t_1-t_2)} U(t_2) | n \rangle e^{-iE_n(t_2-t_3)} dt_1 dt_2 \]

Use \[ \langle m | A | B | n \rangle = \sum_k \langle m | A_k | k \rangle \langle k | B | n \rangle \]
Thus write
\[
\langle \psi | V(t_1) e^{-i H_0 (t_1 - t_2)} V(t_2) | \psi \rangle = \sum_{b, e} \langle \psi | V(t_1) | b \rangle \langle b | e^{-i H_0 (t_1 - t_2)} V(t_2) e^{-i E_b (t_1 - t_2)} | e \rangle \\
\]

Thus the 3rd term becomes
\[
(-i)^2 \int \frac{d^4 k}{(2\pi)^4} \sum_{b, e} \langle \psi | V(t_1) | b \rangle e^{-i E_b (t_1 - t_2)} \langle b | V(t_2) | e \rangle e^{-i E_e (t_1 - t_2)} | e \rangle \\
\]

Now trade hagiography we calculated
\[
e^{-i E_b} e^{-i E_e} \langle \psi | \psi \rangle.
\]
Realizing this, we can recognize our earlier series.

Frequently people refer to using
\[
V(t) = e^{i (E_b - E_0) t} V(t)
\]
go the "interaction representation". Only a matter
of convenience ["one-up-manship"].
5. Interpretation by Feynman diagrams

\[ H_{0}, V \]

\[ -i \left[ U_{0}(t_{0}, t_{1}) V(t_{1}) U_{0}(t_{1}, t_{2}) V(t_{2}) U_{0}(t_{2}, t_{3}) \right] dt_{1} dt_{2} \]

\[ t_{0} \rightarrow t_{1} \rightarrow t_{2} \rightarrow t_{a} \]

\[ (-i)^{2} \sum U_{0}(t_{1}, t_{2}) V(t_{2}) U_{0}(t_{2}, t_{1}) \frac{1}{dt_{1} dt_{2}} \]

\[ t_{0} > t_{1} > t_{2} > t_{a} \]

Sometimes some people call \( U_{0} \) the propagator. Feynman first calls \( V \) the scatterer.

**Example:** Consider three \( V \)'s, \( V_{A} \) & \( V_{B} \). Barely interested in first order in \( V_{A} \), first order in \( V_{B} \).

\[ \sqrt{U_{0}(t_{1}, t_{2}) V_{A}(t_{2}) U_{0}(t_{2}, t_{1}) V_{A}(t_{1}) U_{0}(t_{1}, t_{2})} \frac{1}{dt_{1} dt_{2}} \]

\[ t_{1} < t_{2} \]

\[ t_{a} \rightarrow t_{1} \rightarrow t_{2} \rightarrow t_{A} \]
Can we have $y$?

\[ y \int_{b}^{t_1} dt_2 \]

Wrong

Instead reverse $B$ to $A$ to pick up a second term.

\[ (i)^2 \int\int u_0(b,1) v_A(1) u_0(1,2) v_B(2) u_0(2,a) dt_1 dt_2 \]

\[ +_{1} > _{2} \]

Actually this scheme isn't really dealing with relativistic

theory. This represents pair production, for instance. Need

a new type of perturbation in field

however. Need a more general

definition.

Comments:

1. Must add all terms together -- find phase interference.

2. To define $u_0$ for negative time,

\[ u_0(t_2, t_1) = 0 \quad t_2 < t_1 \]

Then restrictions on integration are automatic.
\[ U(t_b, t_a) = U_0(t_b, t_a) \cdot e^{-\frac{1}{2} \int_{U_0(t_b, t_1)} U(t_1, t_2) V(t_2) U(t_1, t_2) dt_1 dt_2} \]

- U_0(t_1, t_2) propagation
- \( iV(t_1) \) scattering amplitude per unit time

\[ U_{mn} e^{i(E_{m} - E_{n})t} = \delta_{mn} - \frac{1}{2} \int e^{-i(E_{m} - E_{n})t} V_{mn} dt + \]

\[ + \frac{1}{2} \int e^{-i(E_{m} - E_{n})t} V_{mk} e^{i(E_{k} - E_{m})t} V_{nk} dt_{1} dt_{2} \]

6) Physics of \( V \):

\[ H = H_0 + V \]

Is this only a mathematical convenience? It is not unique. But what is physics anyway? Can thinks of \( V \) connecting \( H_0 \) to the real world. \( H \) is very simple & solvable. \( H \) is dependent on an infinite number of variables. \( V \) connects the two.

We can't describe the world exactly. Cont...

\[ H = \sum_{j=1}^{23} \frac{\hbar^2}{2m} \Delta_j \frac{1}{2} + \sum_{i \neq j} V(r_{ij}) \]

We don't want the general solution. We want a specific solution for specific conditions.
The physics is the approximation we make to solve the problem. We can thus consider V as the correction to our approximation. In this sense it is physical.

"Everything is approximate."

To understand bound states, V appears again and again. They need the infinite series. Thus we need to analyze H by considering free p + e with V = coulomb interaction. Once we group bound states in H0, then we can apply perturbation of O(e^2)

for EM interactions.

The e^2 = \frac{1}{137} makes the treatment of QED feasible by this series. For much, a e^2 = 10, and series does not converge. [Field theory means perturbation theory], could try expansion in e^2.

People don't know where to make zero approximation. Note close enough to experience.

Problems may look the same, but have different numbers such that the same method won't work for both. Diff. between a bad and a good theoretical physical is between a guy who uses someone's method and between a guy who thinks for himself. Design methods for physics -- not vice versa.

[Don't conclude we don't understand superconductivity because BCS theory doesn't apply to modern structure.]

DON'T COPY METHODS. THINK FIRST

Decide what approximation will work for your problem, then think up a method.
Examples:

Propagator in state of energy $E_n$ is $e^{iE_n(t_2-t_1)}$

Scattering amplitude $= -iV_{mn}$

To find second order term

\[ (-i)^2 \int e^{-iE_n(t_2-t_1)} V_{mk} e^{-iE_k(t_2-t_1)} V_{kn} e^{-iE_n(t_1-t_0)} dt_1 dt_2 \]

There is also a space representation

Look for first order ray

\[ -i \int \kappa_0(x_0, x_1; t_0-t_1) \psi(x_1) \kappa_0(x_1, x_0, t_1-t_0) dx_1 \]
Still a third method [interaction representation]

Prop. a is $S_{nn} = 1$

Scatt. amplitude $= -i V_{mn}(+e^{i(E_n-E_k)t})^T$

Now recall

$U_{nn}e^{i(E_{te}-E_{ta})} = S_{nn} - i \int_0^T e^{-i(E_n-E_k)t} V_{mn}dt$ $\sum$

$+ (-i)^2 \int_0^T e^{-i(E_n-E_k)t} V_{mk}e^{i(E_n-E_k)t_2} V_{kn}dt_2$ $\sum$

We can use a shortcut by integrating over the difference assuming it is effective only over a finite time. Then we are left with

$S_{nn} - (-i) \int_0^T e^{-i(E_n-E_k)t} \frac{1}{En-E_k-\imath\epsilon} V_{mn} + \frac{1}{En-E_k-\imath\epsilon} V_{kn}$ $\sum$

Thus

$\text{Rate} = \frac{2\pi}{m_{nn}} S(E_n-E_k) \left| T_{nn} \right|^2$
7.) **T-MATRIX**

We have found

\[
\begin{align*}
\text{Rate} & \rightarrow 2\pi \delta(E_n - E) |T_{mn}|^2 \quad \text{Golden Rule \#2} \\
T_{mn} & = V_{mn} + \sum_k V_{mk} \frac{1}{E_n - E_{k+1}} V_{kn} + \sum_k V_{mk} \frac{1}{E_n - E_{k+1}} V_{ke} \frac{1}{E_k - E_{n+1}} V_{en} \quad \text{Golden Rule \#1}
\end{align*}
\]

These can also be written in different forms. First note for \(E = E_n\),

\[
T = V + V \frac{1}{E - H_{a+1}} V + V \frac{1}{E - H_{a+1}} V \frac{1}{E - H_{a+1}} V + \ldots \tag{3}
\]

where we have defined \(f(H_a)|\Phi_k\rangle = f(E_n)|\Phi_k\rangle\)

\[
[f(H_a)]_{kk} = f(E_n) \delta_{kk}
\]

Note we actually only want \([T(E_n)]_{mn}\) however. \(T(E)\) is "off the energy shell". Some people like to play with this, but not really very useful.

\[\]

If \(A\) and \(B\) are operators

\[
\frac{1}{A + B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} - \ldots
\]

\[\]

Proof: Set \(X = \frac{1}{A + B}\). Show \((A + B)X = 1\)

\[
AX = 1 - B \frac{1}{A} + B \frac{1}{A} B \frac{1}{A} - \ldots = 1 - B(\frac{1}{A} - B \frac{1}{A}) - \ldots = 1 - BX \Rightarrow (A + B)X = 1
\]
Now write

$$T = V + V W(E) V$$

where

$$W(E) = \frac{1}{E-H_0 + i\epsilon} + \frac{1}{E-H_0 - i\epsilon} \frac{1}{E-H_0 + i\epsilon} V + \cdots$$

$$= \frac{1}{E-H_0 + i\epsilon}$$

$$T(E) = V + V \frac{1}{E-H_0 + i\epsilon} V$$

$$= (H - H_0) + \frac{1}{E-H_0 + i\epsilon} (H - H_0)$$

Now suppose we want to study $\frac{1}{\mathcal{Q}}$. This means

$$\frac{1}{\mathcal{Q}} |\gamma_n\rangle = \frac{1}{\mathcal{Q}} |\gamma_n\rangle$$

so

$$\frac{1}{\mathcal{Q}} = \frac{\langle \gamma_n |}{\langle \gamma_n | \gamma_n \rangle}$$

Thus

$$\frac{1}{E-H_0 + i\epsilon} = \frac{\langle \gamma_n |}{\langle \gamma_n | \gamma_n \rangle} \frac{1}{E-E_n + i\epsilon} \langle \gamma_n |$$

where

$$H |\gamma_n\rangle = E_n |\gamma_n\rangle$$

Back where we started from.

Also mentions $e^{A B} = e^{A+B}$
Now in a system where \( H = H(t) \),

\[
U(t) = e^{-iHt}
\]

Now consider the Fourier transform

\[
\int_{-\infty}^{\infty} e^{iET} U(t) dt = \mathcal{F}(W(E))
\]

But note

\[
-\frac{i}{\hbar} \int_{\mathbb{R}} e^{i(E-H)t} dt = \frac{1}{E-H+i\hbar} = W(E)
\]

Thus \( W(E) = \mathcal{F}\{U(t)\} \)

We have been considering \( H = H_0 + V \). But now consider

**Example:** \( A + B \rightarrow C + D \), e.g. \( He^+ + H \rightarrow p + He \)

Write

\[
H = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y} + \frac{e_1 e_x}{\hbar x} + \frac{e_2 e_x}{\hbar x} + \frac{e_1 e_y}{\hbar y} + \frac{e_2 e_y}{\hbar y}
\]

\[
+ \frac{p_{x_1}^2}{2m_{x_1}} + \frac{p_{x_2}^2}{2m_{x_2}} + \frac{e_1 e_y}{\hbar y} + \frac{e_2 e_y}{\hbar y}
\]

becoming

\[
H = H_0 + V
\]

\[
H_0 = \frac{p_x^2}{2m_x} + \frac{p_{x_1}^2}{2m_{x_1}} + \frac{e_1 e_x}{\hbar x}
\]

\[
V = \frac{e_1 e_y}{\hbar y} + \frac{e_2 e_y}{\hbar y} + \frac{e_1 e_x}{\hbar x} + \frac{e_2 e_x}{\hbar x} + \frac{p_{x_1}^2}{2m_{x_1}} + \frac{p_{x_2}^2}{2m_{x_2}} + \frac{e_1 e_y}{\hbar y} + \frac{e_2 e_y}{\hbar y}
\]
outgoing: \( H = H_{II} + V_{II} \)

\[
H_{II} = \left( \frac{p_y^2}{2m_y} \right) + \left( \frac{p_z^2}{2m_z} \right) + \frac{e_x e_1}{v_{1y}} + \frac{e_x e_2}{v_{2y}} + \frac{e_1 e_2}{v_{12}}
\]

\[
V_{II} = \frac{e_1 e_1}{v_{xy}} + \frac{e_1 e_2}{v_{x2}} + \frac{e_2 e_2}{v_{12}}
\]

Which \( V \) does \( V_{II} \) in our formula? \( \text{to} \)

\[
T(E) = \left( H - H_0 \right) + \left( H - H_0 \right) \frac{1}{E - H + i\epsilon} \left( H - E_I \right)
\]

\[
= \left( H - E_I \right) + \left( H - E_{II} \right) \frac{1}{E - H + i\epsilon} \left( H - E_I \right)
\]

It makes no difference.

On the energy shell

\[
T_{\text{final}} = V_I + \frac{1}{E_I - H + i\epsilon} V_{II}
\]

\[
E_I = E_{II}
\]

Now

\[
\begin{align*}
T_{\text{final}} &= \langle M_{II} | V_I | N_I \rangle \odot \langle H_{II} | V_I | N_I \rangle \\
&= \langle M_{II} | H - H_0 | N_I \rangle \odot \langle H_{II} | H - H_0 | N_I \rangle
\end{align*}
\]

Chemists
8) $K^0, \bar{K}^0$ Decay Problem

\[ K^0 \rightarrow \pi^+ + \pi^- \quad \text{with} \quad M_{K^0,\pi^+\pi^-} \]
\[ \bar{K}^0 \rightarrow \pi^+ + \pi^- \quad \text{with} \quad M_{\bar{K}^0,\pi^+\pi^-} \]

\[ i \dot{a}_{K^0} = E_0 a_{K^0} + \sum \lambda K_{\pi^+\pi^-} b_{\pi^+\pi^-} \]
\[ i \dot{a}_{\bar{K}^0} = E_0 a_{\bar{K}^0} + \sum \lambda K_{\pi^+\pi^-} b_{\pi^+\pi^-} \]
\[ i \dot{b}_{\pi^+\pi^-} = E_{\pi^+\pi^-} b_{\pi^+\pi^-} + M^* a_{K^0} + M^* a_{\bar{K}^0} \]

Define \[ b_{\pi^+\pi^-}(t) = \sqrt{e^{i (E_{\pi^+\pi^-} - E_0) t}} M^* [a_{K^0}(t) + a_{\bar{K}^0}(t)] dt \]

Then

\[ i \dot{a}_{K^0} = -\sum \lambda K_{\pi^+\pi^-} \sqrt{e^{i (E_{\pi^+\pi^-} - E_0) (t+\tau)}} M^* [a_{K^0}(t) + a_{\bar{K}^0}(t)] dt \]

\[ i \dot{a}_{\bar{K}^0} = (\Delta - i \frac{\pi}{2}) [a_{K^0}(t) + a_{\bar{K}^0}(t)] \]

\[ i \dot{a}_{\bar{K}^0} = (\Delta - i \frac{\pi}{2}) [a_{K^0}(t) + a_{\bar{K}^0}(t)] \]

Solving

\[ a_{K^0} + a_{\bar{K}^0} = ce^{-i \Delta t} e^{-\frac{\pi}{2}} \]
\[ a_{K^0} - a_{\bar{K}^0} = D \]

For $a_{K^0} = 1, a_{\bar{K}^0} = 0$

\[ a_{K^0} = \frac{1}{2} \left[ 1 + e^{-\frac{\pi}{2}} e^{-i \Delta t} \right] \]
\[ a_{\bar{K}^0} = \frac{1}{2} \left[ 1 - e^{-\frac{\pi}{2}} e^{-i \Delta t} \right] \]
Thus \( |\alpha_{21}|^2 = \frac{1}{2} \left[ 1 + 2 \cos(\alpha t) e^{-\pi t/2} + e^{-\pi t} \right] \)

\[ |\alpha_{32}|^2 = \frac{1}{4} \left[ 1 - 2 \cos(\alpha t) e^{-\pi t/2} + e^{-\pi t} \right] \]

Thus population of \( |2\rangle \) in a chart

\[ \begin{array}{c|c}
 t & |2\rangle \\
 \hline
 0 & 1 \\
 \end{array} \]

The general effect is true for any system with two states, \( |\nu_0, \nu_0\rangle \)

\[ |1\nu_1\rangle = \frac{1}{\sqrt{2}} (|\nu_0\rangle + |\nu_0\rangle) \quad \alpha_{12} = \frac{\sqrt{2}}{2} (\alpha_{\nu_0} + \alpha_{\nu_0}) \]

\[ |1\nu_2\rangle = \frac{1}{\sqrt{2}} (|\nu_0\rangle - |\nu_0\rangle) \quad \alpha_{21} = \frac{\sqrt{2}}{2} (\alpha_{\nu_0} - \alpha_{\nu_0}) \]

Thus in our problem, \( \nu_2 \) cannot decay into \( \pi^+ + \pi^- \)

\( |\nu_1\rangle \) \& \( |\nu_2\rangle \) are best states to work with.

\( \Delta \) is mass difference between \( \nu_1 \) \& \( \nu_2 \), \( \Delta = 4 \times 10^{-5} \)

\( \sim 5-6 \text{ eV} \)

Uses this to argue against negative mass theory of antimatter.
But it is found $K^0$ does decay into $\pi^+, \pi^-$ at large times. This $\Rightarrow \bar{N}_{\pi^0} \neq N_{\pi^0} \Rightarrow \text{CP is not conserved.}$ One of the least understood problems today. Must break CP.

Mentioning "right force" theory. Maybe universe is asymmetrical. Theories explain this experiment, but predict weird things in other areas.

Thus, this is "impossible for this generation."
Angular Momentum

Problem: Extend this week's work on angular momentum as far as you can. (Bosonic transformations).

A. Rotation Groups

Most of what we do can be generalized to other groups than rotations. Also, directly of interest.

1.) Rotation of States

Consider an isolated system described by a state $|\Psi\rangle$. Now rotate the apparatus which prepares $|\Psi\rangle$. Then we produce a new rotated state $|\mathbf{R}\Psi\rangle$, with $\mathbf{R}$ would have been $|\Psi\rangle$, but had not rotated the apparatus.

[If we assume we rotate the observer instead we merely get sign changes.]

Now by superposition of states, we expand in base states,

$$|\Psi\rangle = \sum_i c_i |i\rangle$$

$$|\mathbf{R}\Psi\rangle = \sum_i c_i |\mathbf{R}i\rangle$$

But, if $\mathbf{R}$ is linear,

$$|\mathbf{R}\Psi\rangle = \sum_i c_i |\mathbf{R}i\rangle$$

Now we can also expand

$$|\mathbf{R}i\rangle = \sum_j D_{ij} |j\rangle$$

to yield

$$D_{ij} = \sum_k D^k_{ij} c_k$$
To every rotation there is associated a matrix $D^k$
Rotations form a "group" since

i.) first $R_1$, second $R_2 = R_3$ where $R_3 = R_2 R_1$

ii.) identity rotation

iii.) $(BA) = (CB) A$

This is called a "continuous group" since continuity of rotations corresponding to different angles. Actually a 3-parameter group.

Note note i.) means

$$\langle R_3 \phi_i \rangle = \langle R_2 (R_1 \phi_i) \rangle = \sum_j \langle R_2 \phi_j \rangle D^r_{ji}$$

$$= \sum_{i,k} D^r_{ki} D^r_{ji} \langle \phi_k \rangle$$

Thus

$$D^r_{ki} = \sum_j D^r_{kj} D^r_{ji}$$

or as a matrix product

$$D^r = D^r_2 D^r_1$$

to within a phase which can be approximately chosen.
To find how angles combine during rotations in a question of geometry, the same way

\[ \alpha \text{ about } x \text{ axis} \]

\[ x' = x \]
\[ y' = y \cos \alpha + z \sin \alpha \]
\[ z' = z \cos \alpha - y \sin \alpha \]

\[ \alpha' \text{ about } y \text{ axis} \]

\[ y'' = y' \]
\[ x'' = x' \cos \alpha' - z' \sin \alpha' \]
\[ z'' = z' \cos \alpha' + x' \sin \alpha' \]

Substitute in and use general transformations to do this.

Also note:

\[ x_i = \sum L_{ij} x_j \]
\[ L R^2 = L R R \]

An example with 3x3 matrices

2) Construction of Angular Momentum Operators (Generators)

We can now find our matrices by considering three infinitesimal rotations.

Take \( \epsilon \) about \( z \) axis. Then \( D^z \). Now for \( \alpha \) about \( z \),

\[ D^z D^z = D^z \text{ only needed to linear terms} \]

Thus write

\[ D^z = 1 + i \left( \frac{J_z}{\hbar} \right) \epsilon \]

where \( J_z \) is as yet unknown
Thus

\[ |\text{Rot}_\phi \psi \rangle = |\psi \rangle + \frac{i\phi}{\hbar} J_z |\psi \rangle \]

\( J_z \) is called the "generator of the rotation."

Now rotation of \( \phi \) about \( z \)

\[ \hat{\omega}^\phi = (1 + i\frac{\phi}{\hbar} J_z)(1 + i\frac{\phi}{\hbar} J_z) \cdots (1 + i\frac{\phi}{\hbar} J_z) \]

\[ = (1 + i\frac{\phi}{\hbar} J_z)^{\phi/\hbar} \]

\[ = 1 + i\phi J_z + \frac{(i\phi)^2}{2!} J_z^2 + \frac{(i\phi)^3}{3!} J_z^3 + \cdots \]

\[ = e^{i\phi J_z} \]

Thus to find \( \hat{\omega}^\phi \), we only need the generator \( J_z \).

Now support state is unchanged by rotation

\[ |\text{Rot}_\phi \psi \rangle = e^{i\phi \sigma_z} |\psi \rangle \]

But \( \text{Rot}_{\alpha_1} \text{Rot}_{\alpha_2} = \text{Rot}_{\alpha_1 + \alpha_2} \)

\[ \Rightarrow S(\alpha_1 + \alpha_2) = S(\alpha_1) + S(\alpha_2) \]

\[ \Rightarrow S = \max \alpha \quad \text{in constant of proportionality} \]

\( S \) is angular momentum about \( z \) axis.
Now suppose

\[ J_z |\psi\rangle = m |\psi\rangle \]

Then

\[ |\text{Rad} \psi\rangle = e^{i\theta} |\psi\rangle = e^{i\alpha m} |\psi\rangle \]

Why's this called angular momentum? Merely convenient since if we suppose a particle rotating in a track

\[ e^{i p_\theta} \text{ and distance } h = e^{i p_\theta} \frac{\hbar}{h} \]

\[ \therefore \frac{\hbar}{p} = \frac{\hbar}{h} \]

But \( p_\theta = \text{angular momentum} \)

Thus \( \chi \) corresponds to classical angular momentum

Suppose \( |\psi\rangle = \delta(x,y,z) \)

\[ |\text{Rad} \psi\rangle = \delta(x',y',z') = \delta(x+\epsilon y, y-\epsilon x, z) \]

\[ = \delta(x,y,z) + \epsilon \left( y \frac{\partial f}{\partial x} - x \frac{\partial f}{\partial y} \right) \]

Thus identity

\[ J_z = \frac{\hbar}{i} \left( \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) = y p_x - x p_y \]
3) Restrictions on Eigenvalues of $J$

We had found

$$|k\Psi\rangle = |\Psi\rangle + i e^{im\phi} |J_x|\Psi\rangle$$

We noted if $J_x|\Psi\rangle = m|\Psi\rangle$, then $e^{im\phi}$ must change.

Also found if our state is a function, then define $J_z \rightarrow L_z$

where

$$L_z = -i \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right)$$

Note we found this purely by rotations, with correspondence with classical mechanics.

Now if we have

$$L_z |\Psi\rangle = m |\Psi\rangle$$

then

$$L_z f(x,y,z) = m f(x,y,z) \Rightarrow L_z = \frac{\partial^2}{\partial \phi^2}$$

$$f(r,\theta,\phi) = F(r,\theta) e^{im\phi}$$

[Note this specific example concerns orbital angular momentum, $m$ any mom. due to rotation $(x,y,z)$]

---

What values do we have to leave? For 2-dimensional groups $m$ is arbitrary! This is so because we can put state back by properly defining our phase.

Need to go to 3-dimensional.
Three-dimensions

\[ |R_{\pm}^x| = |\Psi| + \imath \xi |J_2|\Psi\rangle \]
\[ |R_{\pm}^z| = |\Psi| + \imath \eta |J_3|\Psi\rangle \]

What about

\[ |\text{Rot}_{\eta}^y|\Psi\rangle = |\Psi| + \imath \eta |J_y|\Psi\rangle \]

Note

\[(\eta V_x \text{about} x)(\eta V_y \text{about} y)(\eta V_z \text{about} z) = (\eta \text{about} V) \quad \text{if} \eta \text{is very small} \]

Thus we can write this notation as

\[(1 + \imath \eta V_z J_z)(1 + \imath \eta V_y J_y)(1 + \imath \eta V_x J_x) |\Psi\rangle \]

The order makes no difference to \(O(\eta)\).

\[= 1 + \imath \eta(V_z J_z + V_y J_y + V_x J_x) \]

Thus the generator for infinitesimal rotations about \(V\) is

\[ \tilde{J}_V = V_x \tilde{J}_x + V_y \tilde{J}_y + V_z \tilde{J}_z = \tilde{V} \cdot \tilde{J} \]

[Note \(\tilde{J}\) is both a vector and an operator]
Now we must make sure that

\[ \text{first} R_1, \text{second} R_2 \rightarrow R_3 \]

\[ \Rightarrow \quad D_{R_2} D_{R_1} = D_{R_3} \]

What properties must the J's have so this holds? We studied continuous groups and found if we could put

\( (*) \) to work to second order, it will work for any order. Can see that

\[
\left( \mathcal{D}^{\nu_1 \nu_2} \mathcal{D}^{\nu_3 \nu_4} \ldots \mathcal{D}^{\nu_{2L}} \right) \left( \mathcal{D}^{\nu_{2L+1} \nu_{2L+2}} \ldots \mathcal{D}^{\nu_{2L+4L}} \right) = \frac{\mathcal{O}(\nu_2^{2L})}{o(\nu_2^{2L+2})}
\]

Note rotations do not commute

\((1^{st} \ \text{about } x) \ (2^{nd} \ \eta\ \text{about } y) \neq (1^{st} \ \eta\ \text{about } y) \ (2^{nd} \ \text{about } x)\)

Try to work it by applying them

\((3^{rd} - \text{about } x) \ (4^{th} - \eta\ \text{about } y)\)

This gives \( (1)(2)(3)(4) = (\eta\ \text{about } x) \)

This is a fact of geometry, not physics. [Ideas humans only understand 3-dimensional commutative groups. Bird and fish understand 3-dimensional]
Demonstration: Imagine the fields fixed on a sphere.

1. about x
2. η about y
3. - about x
4. - η about y

We can see that an rotation of η about z is necessary to return to the original η.

Thus we have found to second order

\[
(1 - i\eta \frac{\partial}{\partial x})(1 - i\eta \frac{\partial}{\partial y})(1 + i\eta \frac{\partial}{\partial y})(1 + \eta \frac{\partial}{\partial x}) = (1 + i\eta J_z)
\]
or expanding

\[
\frac{4\cos}{1 + (i\xi J_x + i\eta J_y - i\xi J_y + i\eta J_x - \xi^2 J_x^2 - \eta^2 J_y^2)}
\]

Thus

\[
1 + \eta (J_x J_y - J_y J_x) = 1 + i\eta J_z
\]

Therefore

\[ J_x J_y - J_y J_x = i J_z \]

must be obeyed by the generators of the group if

\[ \eta^2 D_z^2 = 2D_x. \]

[This is a much more logical derivation than the more elementary one from classical mechanics.]

We can also find

\[ J_y J_z - J_z J_y = i J_x \]

\[ J_z J_x - J_x J_z = i J_y \]
Thus \( D^{R_2} D^{R_3} = D^{R_3} \Rightarrow \)

\[ \begin{align*}
\text{i}) & \quad \mathbf{J}_V = \mathbf{\hat{J}} \\
\text{ii}) & \quad \mathbf{J}_x \mathbf{J}_y - \mathbf{J}_y \mathbf{J}_x = i \mathbf{J}_z \\
& \quad \mathbf{J}_y \mathbf{J}_z - \mathbf{J}_z \mathbf{J}_y = i \mathbf{J}_x \\
& \quad \mathbf{J}_z \mathbf{J}_x - \mathbf{J}_x \mathbf{J}_z = i \mathbf{J}_y
\end{align*} \]

These handle any angles or rotations

We now could use algebra to generate the e.o. and e.f. of the \( \mathbf{J}_i \)'s. Define

\[ \mathbf{J}_\pm = \mathbf{J}_x \pm i \mathbf{J}_y \]

Then using \( [\mathbf{K}, \mathbf{J}_i] = 0 \) where \( \mathbf{K} = \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2 \)

we consider

\[ \begin{align*}
\mathbf{J}_z |m\rangle &= m |m\rangle \\
\mathbf{K} |m\rangle &= k |m\rangle
\end{align*} \]

Our commutation rules become

\[ \begin{align*}
\mathbf{J}_z \mathbf{J}_+ - \mathbf{J}_+ \mathbf{J}_z &= \mathbf{J}_+ \\
\mathbf{J}_z \mathbf{J}_- - \mathbf{J}_- \mathbf{J}_z &= -\mathbf{J}_- \\
\mathbf{J}_+ \mathbf{J}_- &= \mathbf{J}_x^2 + \mathbf{J}_y^2 + \mathbf{J}_z^2 = \mathbf{K} - \mathbf{J}_z (\mathbf{J}_z - 1) \\
\mathbf{J}_- \mathbf{J}_+ &= \mathbf{K} - \mathbf{J}_z (\mathbf{J}_z + 1)
\end{align*} \]
Thus, \( J_{\pm} J_{\pm} |m\rangle = [\mathbb{I} - m(m+1)] |m\rangle \)

Applying
\[ J_z (J^+ |m\rangle) = J^+ |m\rangle + J_{\pm} J_z |m\rangle \]

Thus
\[ J^+ |m\rangle = (m+1) |J^+ |m\rangle \]

Thus
\[ J^+ |m\rangle = C_m |m+1\rangle \]
\[ \langle m| J^+ |m\rangle = C_m \delta_{m,m+1} \]

Also
\[ J^- |m\rangle = C_m^* |m\rangle \]
\[ \langle m| J^- |m\rangle = C_m^* \delta_{m,m+1} \]

Next
\[ J_{-} J_{+} |m\rangle = C_m \langle m^* | m\rangle \]

\[ \Rightarrow |C_m|^2 = \mathbb{I} - m(m+1) \]
\begin{align*}
\mathbf{J}_x \mathbf{J}_y - \mathbf{J}_y \mathbf{J}_x &= i \mathbf{J}_z \\
\mathbf{J}_y \mathbf{J}_z - \mathbf{J}_z \mathbf{J}_y &= i \mathbf{J}_x \\
\mathbf{J}_z \mathbf{J}_x - \mathbf{J}_x \mathbf{J}_z &= i \mathbf{J}_y
\end{align*}

\[ [\mathbf{J}_x, \mathbf{J}_y] = i \mathbf{J}_z \quad \text{etc.} \]

We had found \( \mathbf{J}_z |m\rangle = m |m\rangle \) by commutative algebra.

\[ \mathbf{J}_z |m\rangle = j(j+1) |m\rangle \]

\( m \) goes from \(-j\) to \(+j\)
in integer steps

\( \Rightarrow j \) is a half-integer

Also
\[
\langle m' | \mathbf{J}_+ | m \rangle = \delta_{m'+m, j(j+1)-m-m'}/\sqrt{j(j+1)-m-m'}
\]

\( m' = m+1 \)

\[
\langle m' | \mathbf{J}_- | m \rangle = \delta_{m-m', j(j+1)-m-m'}/\sqrt{j(j+1)-m-m'}
\]

\( m' = m-1 \)

We can then deduce
\[
\langle m' | \mathbf{J}_x | m \rangle = \delta_{m+m', j(j+1)-m-m'}/\sqrt{j(j+1)-m-m'}/2
\]

\[
\langle m' | \mathbf{J}_z | m \rangle = \delta_{m,m'} m
\]

Thus the \( |m\rangle \) states provide a representation for the group of rotations, so that
\[
\langle m' | \mathbf{O}^R | m \rangle = D^R_{m'm}
\]

\( \text{(might try to work them out)} \)
Other forms since
\[ \sqrt{j(j+1) - m(m+1)} = \sqrt{(j-m)(j+m+1)} \]

Thus
\[ \langle m | J_{\pm 1} | m \rangle = \sqrt{(j+m)(j-m)} \]

3) Could study these for special \( j \)'s

\( j = 0 \):
\( m = 0 \) \[ J_{\pm 1} | m \rangle = 0 \]
\( J_{\pm 1} | m \rangle = 0 \]
A rotation of any kind produces the same state \( | \psi \rangle = | \psi \rangle \)

\( j = \frac{1}{2} \):
\( m = + \frac{1}{2} \) \[ | 1 \rangle = 1 | x \rangle \]
\( m = - \frac{1}{2} \) \[ | 1 \rangle = 1 | y \rangle \]

Call \( \frac{1}{2} j = \hat{\gamma} \) for this special case.

\( S_z | 1 \rangle = \frac{1}{2} | 1 \rangle \]
\( S_z | 1 \rangle = - \frac{1}{2} | 1 \rangle \]
\( S_x | 1 \rangle = \frac{1}{2} | 1 \rangle \]
\( S_x | 1 \rangle = \frac{1}{2} | 1 \rangle \]
\( S_y | 1 \rangle = - \frac{1}{2} | 1 \rangle \]
\( S_y | 1 \rangle = \frac{1}{2} | 1 \rangle \]

Sometimes one uses Pauli matrices \( \frac{1}{2} \hat{\gamma} \)

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

\[ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \]
\[ \sigma_x \sigma_y - \sigma_y \sigma_x = 2i \sigma_z \]
\[ \sigma_x \sigma_y = i \sigma_z = - \sigma_y \sigma_x \]
Since these matrices and (6.1) can represent any
they are used quite frequently in 2-state systems.

In the beginning, only integral $j$ were allowed since if
we use $j = \frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$ yield the negative of the
state (e.g. for state (6.16) after rotating $360^\circ$, thus Pauli
used his matrices to handle 2 wave forms, $\psi^+$, $\psi^-$. Why can't we have spin $k$

$$|m\rangle \rightarrow e^{i\frac{1}{2}2\pi} |m\rangle = e^{i\delta} |m\rangle$$

Why must 2 rotations around $360^\circ$ return state to
original value, while one rotation doesn't have to.

Consider restoring rotations (path of successive rotations
which restore coordinate systems) = turn.

Consider 2 turns infinitesimally close to each other. We
suppose these turns are in the same class.

Let 0 turn have phase 0. Then consider class of turns
infinitesimally from 0. Assume these times have phase 0. These
does of turns have some phase. Will show

$$720^\circ \sim 0^\circ$$
$$360^\circ \sim 0^\circ \not\sim \text{not possible}$$

Will show, does $720^\circ$ have phase 0, does $360^\circ$ has
phase $\neq 0$. Thus $720^\circ$ is different than $360^\circ$ from a geometrical standpoint.
To prove this, he demonstrates that a class containing both $720^\circ$ and $0^\circ$ exists.

$$
720^\circ \quad 360^\circ \quad 180^\circ \quad 0^\circ
\quad 360^\circ \quad 180^\circ \quad 0^\circ
$$

He shows you can't do this for $360^\circ$. A curious property of 3-dimensional rotations.

---

\( J = 1 \): \( 1^+ \), \( 10 \), \( 1^- \) \( \rightarrow \)
\( J \rightarrow \ell \)

\[
\begin{align*}
    l_+ 1^+ &= 0 & l_+ 10 &= \sqrt{2}1^+ & l_+ 1^- &= \sqrt{2}10 \\
    l_- 1^+ &= \sqrt{2}10 & l_- 10 &= \sqrt{2}1^- & l_- 1^- &= 0
\end{align*}
\]

If we rotate coordinate system \( 1x \), \( 1y \), \( 1z \)\):

\[
\begin{align*}
    1^+ &= \frac{1}{\sqrt{2}} (1x - i1y) \\
    1^- &= \frac{1}{\sqrt{2}} (1x + i1y) \\
    10 &= 1z
\end{align*}
\]

A particle of spin 1 is called a vector particle in field theory since it behaves as a vector.
B. COMBINATION OF ANGULAR MOMENTA

Problem:

i) Anything you want

ii) Find the order in letting

\[ J_z = \frac{1}{2} [v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y}] + \gamma_z (v_x, v_y, v_z) \]

\[ J_y = \frac{1}{2} [v_x \frac{\partial}{\partial v_x} - v_z \frac{\partial}{\partial v_z}] + \gamma_x (v_x, v_y, v_z) \]

\[ J_x = \frac{1}{2} [v_y \frac{\partial}{\partial v_y} - v_z \frac{\partial}{\partial v_z}] + \gamma_y (v_x, v_y, v_z) \]

Let

\[ \gamma_z = \lambda v_z \]

\[ \gamma_x = -\lambda v_y \phi \]

\[ \gamma_y = \lambda v_x \phi \]

[These all commute correctly]

Claim: Total angular momentum be an integer

iii) Consider two connected objects (mudra, electron) with

\[ j_a, j_b \]. Assume neither object is excited. Assume

\[ H = A \frac{1}{j_a} \frac{1}{j_b} + B_z \left( \frac{\lambda}{j_d} \left( \frac{1}{j_a} \right) + \frac{\mu}{j_b} \left( \frac{1}{j_b} \right) \right) \]

a. In general for very small \( B \) find the energy levels of the

combined system, and find formula for magnetic moment

b. For special values of \( j_a, j_b \) find energy vs. \( B \) for arbitrary \( B \)
1. Introduction

Consider a system comprised of two points \( a \) and \( b \)

\begin{align*}
\mathbf{a} & : \quad |\psi_a\rangle \quad \text{any } \psi_a \\
W_a & = -j_a \psi_a \\
\mathbf{b} & : \quad |\psi_b\rangle \quad \text{any } \psi_b \\
W_b & = -j_b \psi_b
\end{align*}

Now try to represent the base states of the system

\[ |\psi\rangle = |\psi_a\rangle |\psi_b\rangle = |\psi_a\rangle |\psi_b\rangle \]

Thus \((2j_a+1)(2j_b+1)\) states.

**Example:** \( j_a = 1 \quad j_b = 1/2 \)

\[ |+\rangle |1/2\rangle, \quad |0\rangle |1/2\rangle, \quad |-\rangle |1/2\rangle, \quad |+\rangle |1-1/2\rangle, \quad |-\rangle |1-1/2\rangle \]

Suppose we want to form from these states, some states of definite total angular momentum \( J \), \( M \). What do we mean?

\[ \mathbf{R} | \psi \rangle = \mathbf{R}_a | \psi_a \rangle \mathbf{R}_b | \psi_b \rangle = (\mathbf{R}_a \otimes \mathbf{R}_b) | \psi_a \rangle | \psi_b \rangle \]

Let \( \mathbf{R} = \mathbf{e}_z \)

\[ |\psi\rangle + i \in J_z |\psi\rangle = \left( 1 + i \in J_z \right) |\psi\rangle \left( 1 + i \in J_z \right) |\psi\rangle \]

\[ \Rightarrow J_z |\psi\rangle = \left( J_z^a + J_z^b \right) |\psi_a \rangle \psi_b \rangle \]

\[ \Rightarrow \frac{1}{\sqrt{2}} = J_z^a 1^a + J_z^b 1^b \]
Try to find
\[ J_z |\psi\rangle = M |\psi\rangle = (J_z^a + J_z^b) |m_a\rangle |m_b\rangle \]
\[ = (m_a + m_b) |m_a\rangle |m_b\rangle \]

For our example, \( m = \frac{3}{2} \) \( \Rightarrow J = \frac{3}{2} \)
\[ J = \frac{1}{2}, -\frac{1}{2}, \frac{3}{2} \]
\[ +\frac{1}{2}, -\frac{1}{2} \text{ left over} \Rightarrow J = \frac{1}{2} \text{ state} \]
Thus don't get a definite \( J \) in general.

In general, a possible \( J = J_a + J_b \). Also can reason another possibility \( J = J_a + J_b - 1 \), \( J = J_a + J_b - 2 \).
Ensure \( J \) is \( J = J_a - J_b \). Thus
\[ |J_a - J_b| \leq J \leq J_a + J_b \] (Bowen rule)
\[ 2J + 1 \text{ state for each } J. \]

Now, what are these states, e.g. \( |J, M\rangle = \frac{3}{2}, \frac{3}{2} \rangle \)?

For the highest \( J \)
\[ |\frac{3}{2}, \frac{3}{2} \rangle = |+1\rangle_a |+\frac{1}{2}\rangle_b \]

But for
\[ |\frac{3}{2}, +\frac{1}{2}\rangle = a |0\rangle_a |\frac{1}{2}\rangle_b + b |1\rangle_a |-\frac{1}{2}\rangle_b \]

What combination will work? Find \( a, b \). These are Schmidt - Lanczos coefficients.

2) Method 1: Look in a table.
3) Method 2: Use

\[ J_- = J_a + J_b \]

We can start from the highest state, and work down. [For any five group, just work up or down from a known state]

\[ J_- |^{3/2, 3/2} = \sqrt{3} |^{3/2, 1/2} = (J_a + J_b)(|+1> + |+\frac{1}{2}> \]

\[ \sqrt{3} |^{3/2, 1/2} = \sqrt{2} |0> |1/2> + |+1> |-1/2> \]

\[ \Rightarrow a = \sqrt{3/3}, \quad b = \sqrt{3/3} \]

\[ J_- \left\{ \begin{array}{cccc}
+1/2 > 1 & 0 > \sqrt{12} & 1/2 > \sqrt{14} & 2 > \sqrt{14} \\
-1/2 > 1 & -0 > \sqrt{12} & -1/2 > \sqrt{14} & -2 > \sqrt{14} \\
1/2 > 1 & 1 > \sqrt{12} & 1/2 > \sqrt{14} & 1 > \sqrt{14} \\
-1/2 > 1 & -1 > \sqrt{12} & -1/2 > \sqrt{14} & -1 > \sqrt{14} \\
\end{array} \right. \]

Now do we reduce \( J \) now? Try to get \( |1, 3/2> \).

\[ |1, 3/2> = -\sqrt{3} |0> |1/2> + \sqrt{3} |1> |1/2> \]

use orthogonality

General method: 1) get \( J_{max} \), and use \( J_- \) to get \( |J, m> \)
2) use orthogonality to get \( |J, m> \)
4) Method 3: Straight Forward Method

The best way to generate C-G coefficients is to play with $J_\pm$. However, to evaluate one in the middle, say we want

$$|12,1\rangle = a |12a\rangle + b |12b\rangle + c |10\rangle + |1+1\rangle + |1-1\rangle$$

Want $J=2, M=1$, i.e.,

$$\left(J_{0z} + J_{bz}\right) |12,1\rangle = +1 |12,1\rangle$$

$$\langle J, J |12,1\rangle = J (J+1) |12,1\rangle = 6 |12,1\rangle$$

Now

$$\langle J, J |12,1\rangle = \left(J_a + J_b\right) \cdot \left(J_a + J_b\right) = J_a J_a + J_b J_b + 2 J_a J_b$$

Then

$$\langle J, J |J, M\rangle = J_a (J_a + 1) + J_b (J_b + 1) + 2 J_a J_b + J_a + J_b = J_a - J_b$$

or

$$\begin{bmatrix}
8 + 2 J_0 J_{bz} + J_a J_b - J_0 - J_{bz} \\
\end{bmatrix}
\begin{bmatrix}
|a |12\rangle + b |1+1\rangle + c |10\rangle + |1+1\rangle \\
\end{bmatrix}
= 6 a |12\rangle + 6 b |10\rangle + 6 c |1+1\rangle$$

$$|1+1\rangle = 4 a + 2 J_0 = 6 a \Rightarrow a = 52, b = 1$$

$$\begin{bmatrix}
8 a + (-2a) \end{bmatrix} |1+1\rangle + 2 J_0 |10\rangle + 2 J_0 |1+1\rangle + 2 J_0 |10\rangle$$

$$\begin{bmatrix}
8 b + 0 b \end{bmatrix} |10\rangle + 2 J_0 |10\rangle + 2 J_0 |10\rangle$$

$$< = 6 a |12\rangle + 6 b |10\rangle + 6 c |1+1\rangle$$

$$\begin{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix} = 6 a |12\rangle + 6 b |10\rangle + 6 c |1+1\rangle$$

$$\begin{bmatrix}
4 a + 2 J_0 = 6 a \Rightarrow a = 52, b = 1$$

$$\begin{bmatrix}
\end{bmatrix} = 6 b$$

$$\begin{bmatrix}
= -5 b$$

$$\begin{bmatrix}
\end{bmatrix} = 6 c$$

$$\begin{bmatrix}
\end{bmatrix} = -5 b$$

$$\begin{bmatrix}
\end{bmatrix} = 6 c$$
C. Another Example of Group Ideas

1) Isotopic Spin

Consider nucleons that can be in 2 states. If we transform all the nucleons in a nucleus,

\[|p\rangle = \alpha |p\rangle + \beta |n\rangle\]

\[|n\rangle = \beta^* |p\rangle - \alpha^* |n\rangle\]

such that \[|\alpha|^2 + |\beta|^2 = 1\]

This group now depends continuously on \(\alpha, \beta\). \(\alpha, \beta\) are parameters of the group. Now still have,

\[D^6 |\psi\rangle = |\psi\rangle\]

Again find \[D^6 \beta = D^6 \alpha \]

Can follow our work on angular momentum, since these are not groups. Take

\[\beta = 0, \quad \alpha = 1, \quad G_x\]

\[\beta = i, \quad \alpha = 1, \quad G_y\]

\[\beta = 0, \quad \alpha = 1 + i, \quad G_z\]

Again, \[D^6 = 1 + i \epsilon M_1\]?
It turns out this group is similar to groups for spin \( \frac{1}{2} \) objects. Thus we introduce isotropic spin, so we can use the words on \( \frac{3}{2} \) spin objects.

\[
\begin{align*}
T_3 & = (J_z) \\
T_+ & = \mathbb{J}(J^+) \\
T_- & = \mathbb{J}(J^-)
\end{align*}
\]

just a matter of convenience.

3) The Eightfold Way

**Problem:** Consider a system of 3 objects

\[
\begin{align*}
|\mathbf{1}\rangle & = \alpha |1\rangle + \beta |1\rangle + \gamma |1\rangle \\
|\mathbf{1}\rangle & = \alpha' |1\rangle + \beta' |1\rangle + \gamma' |1\rangle \\
|\mathbf{1}\rangle & = \alpha'' |1\rangle + \beta'' |1\rangle + \gamma'' |1\rangle
\end{align*}
\]

\(6\) independent numbers \([8 \text{ parameter group}]

The analogous to rotation groups anymore, but use some procedure. Representation

\[
1. \text{ singlet} \\
2. \text{ triplet} \\
3. \text{ etc.}
\]

This is called "8-fold way" in high energy physics.

\[
N, \rho, \Lambda, \Xi^+, \Xi^0, \Xi^-, \Omega^+, \Omega^0
\]
If we consider only strong interactions, then all objects have some energy. But nature is not limited to this group. This group is called SU\(_6\). Think now that all particles can be represented by such groups

\[
\begin{align*}
\text{SU}_2 & \quad 2 \text{ states} \\
\text{SU}_6 & \quad 3 \text{ states} \\
\text{SU}_{12} & \quad 6 \text{ states}
\end{align*}
\]

Actually \( \pi \) mesons correspond to isotopic spin \( T = 1 \) for the \( \pi^+, \pi^0, \pi^- \) states, \( T = 0 \) for the \( \pi^0 \) state.

(\( \frac{3}{2} \) are like integer \( \frac{1}{2} \) integer in angular momentum, have not seen any \( \frac{3}{2} \) yet.

People have tried to combine angular momentum with quality and found

\[
\text{SU}_6 \otimes \text{SU}_2 = \text{SU}_{12}
\]

Don't really know much about it yet. Some regularities appear, but may be coincident. \( \text{SU}_{12} \) has 6 states.
Special Case: spin $j = \frac{1}{2}$

Suppose we want

$\langle \frac{1}{2} | \overline{\frac{1}{2}} \rangle$ $\langle \frac{1}{2} | \overline{\frac{1}{2}} \rangle$ $\langle \frac{1}{2} | \overline{-\frac{1}{2}} \rangle$

$\langle -\frac{1}{2} | \overline{\frac{1}{2}} \rangle$ etc

We are calculating matrix elements between two states that differ only by the value of $m_z$ --- $j = \frac{1}{2}$ remains fixed. Thus two internal states, differs only by $m_z$.

Recall any matrix for spin $\frac{1}{2}$ can be written with spinors

$$M = \alpha \mathbf{I} + \alpha_x \mathbf{S}_x + \alpha_y \mathbf{S}_y + \alpha_z \mathbf{S}_z$$

We can then write our matrix elements as

$$\langle m_2 | x | m_1 \rangle = \delta_{m_2, m_1}$$

Then

$$X = \mathbf{S}_0 \mathbf{I} + \mathbf{S}_x \mathbf{S}_x + \mathbf{S}_y \mathbf{S}_y + \mathbf{S}_z \mathbf{S}_z$$

$$Y = \mathbf{Y}_0 \mathbf{I} + \cdots$$

$$Z = \mathbf{Z}_0 \mathbf{I} + \cdots$$

Now since $X, Y, Z$ are components of a vector, the matrix must transform (rotate) like a vector. We are saying

$$\langle m_2 | \mathbf{R} | m_1 \rangle \propto \langle m_2 | \mathbf{I} | m_1 \rangle \begin{pmatrix} \mathbf{S}_+ & \mathbf{S}_- \\ \mathbf{S}_- & \mathbf{S}_+ \end{pmatrix}$$
Suppose we are calculating matrix elements relating to quadrupole moments. Then we want the matrix elements to transform when rotated such that they behave as tensors; i.e., the result of the rotation must behave like a tensor.

\[
\langle m_2 | \mathbf{1} \mathbf{r}_x \mathbf{1} \mathbf{m}_1 \rangle = \lambda \langle m_2 | \mathbf{1} \mathbf{r}_x \mathbf{1} \mathbf{m}_1 \rangle \left( \begin{array}{c} 5^{++} \\ 5^{-+} \\ 5^{+-} \end{array} \right)
\]

Choose new matrix element between scalars \( \mathbf{1} \mathbf{r}_x + \mathbf{y} \mathbf{r}_y + \mathbf{z} \mathbf{r}_z \) scalar

\[
\langle m_2 | \mathbf{1} \mathbf{r}_x \mathbf{1} \mathbf{m}_1 \rangle
\]

\[
\langle \mathbf{1} \mathbf{r}_y \mathbf{1} \rangle
\]

e.g., new matrix formed by taking linear combination -- get symmetric tensor of zero rank

\[
\begin{align*}
\lambda p_x & - \frac{1}{3} (\lambda p_x + y p_y + z p_z) \\
x p_y + y p_x & \\
y p_y & - \frac{1}{3} (x p_x + y p_y + z p_z) \\
y p_z + z p_y & \\
x p_z + z p_x &
\end{align*}
\]

take 5 so they will be independent
When doing calculation of matrix elements like \( \langle \mathbf{p}_1 \mathbf{p}_2 \rangle \), follow these steps:

i.) Take matrix elements of a set of quantities which is a scalar.

ii.) Follow the vector.

iii.) Follow the tensor.

Scalar component

\[
\langle m_1 | x_{\mathbf{p}_1} + y_{\mathbf{p}_2} + z_{m_2} \rangle = m_1 \langle m_1 | m_2 \rangle
\]

Vector components

\[
\langle m_1 | x_{\mathbf{p}_2} - y_{m_1} \rangle = \lambda \langle m_1 | \sigma_x \sigma_z \rangle
\]

Tensor components

\[
\langle m_1 | \text{tensor comp} | m_2 \rangle = \lambda \langle m_1 | \sigma_x \sigma_z - \frac{1}{2} (\sigma_x \sigma_x + \sigma_y \sigma_y + \sigma_z \sigma_z) | m_2 \rangle
\]

Tensor components are zero \((k=0)\) for spin \(\frac{1}{2}\) because there is only 2 unknowns. \(\Rightarrow\) tensor components are not independent, i.e.,

\[
\langle \psi_J | \mathbf{T}_I | \psi_J \rangle \quad J = J + 2, J + 1, \ldots, J - 2
\]

For \(J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, -\frac{5}{2}\)
We have been considering non-diagonal elements. Now let's try to get matrix elements for diagonal terms of angular momentum.

\[ \langle m_2 | P_{x} | m_1 \rangle \quad \langle m_2 | P_{y} | m_1 \rangle \quad \langle m_2 | P_{z} | m_1 \rangle \]

We have \( J_x, J_y, J_z \) \( \frac{4}{4 \text{ numbers}} \)

Matrix is

\[ \binom{2j + 1}{2j + 1} \]

Need \( (2j + 1)^2 \) matrix elements --

\[ \text{numbers -- in operator we know we only have 4 numbers} \]

But we also know \( J_x J_x, J_x J_y, J_x J_z, \ldots J_x J_x J_x \ldots \)

Can't include \( J_x J_y, J_x J_z \) because they have to be restricted to zero, we can't have \( \# \) of combinations? However, don't really have this many, e.g. for \( j = 1 \)

\[ (J_x^2 + 1) J_x (J_x - 1) |m\rangle = 0 \quad \text{for all } m = 0 \]

\[ J_x^2 = J_z \]

\[ J_x^3 - J_z = 0 \quad \text{can't allow } J_x J_x J_z \text{ or } J_x J_x J_x \text{ on } J_x J_x J_x, \]

Can't make combination of 3, so we cannot make a vector from 2J's

\[ \langle m_2 | P_{z} | m_1 \rangle = \lambda \langle m_2 | J_{z} | m_1 \rangle \]
Generally, for arbitrary \( \mathbf{J} \),

\[
\langle m_1 | \mathbf{I} \mathbf{I} | m_2 \rangle = a \langle m_1 | \mathbf{I} | m_2 \rangle
\]

vector

\[
\frac{1}{4}
\]

Tensor

\[
J_x^2 - \frac{1}{3}(J_x^2 + J_y^2 + J_z^2)
\]

For \( j = 1 \), can get higher than terms like \( J_x J_y + J_y J_x \)

Terms like \( J_x J_x J_x \) give 0 m.e. In general

\[
\langle J = 1 | \mathbf{I} \mathbf{I} | J = 1 \rangle = 0
\]

because \( \mathbf{1} + \mathbf{3} = \mathbf{4} \) so can have

\( J = 4, 3, 2 \)

Putting m.e. together, e.g.

\[
\langle m_2 | \mathbf{I} \mathbf{I} | m_1 \rangle = \langle m_2 | \mathbf{I} \mathbf{I} | m_1 \rangle - \frac{1}{3} (\mathbf{I} \mathbf{I} \mathbf{I} + \mathbf{I} \mathbf{I} \mathbf{I} + \mathbf{I} \mathbf{I} \mathbf{I}) \langle m_1 \rangle
\]

+ \frac{1}{3} \langle m_2 | \mathbf{I} \mathbf{I} | m_1 \rangle

Important point is a traceless tensor can be written as

\[
\langle m_2 | \mathbf{I} \mathbf{I} | m_1 \rangle = c \langle m_2 | J_x^2 - \frac{1}{3}(J_x^2 + J_y^2 + J_z^2) | m_1 \rangle
\]

+ \frac{1}{3} a \langle m_2 | \mathbf{I} | m_1 \rangle

= \langle m_2 | J_x J_x | m_1 \rangle + (\frac{1}{3} a - \frac{1}{3} c) \delta_{m_1, m_2}

traceless
Can prove that a nucleus of spin \( \frac{1}{2} \) cannot have a quadrupole moment -- no magnetic moment if spin is 0, because need

\[ \langle m_1, j_1 | m_1, j_1 \rangle \propto \langle m_2 | j_1 | m_1 \rangle \]

\( \propto \) vector \propto \) scalar

\[ \Rightarrow \text{scalar} = 0, \text{no mag. moment} \]

for \( j = 0 \) objects
To learn this field, one must read alot to gain experience of the various phenomena. Return to no books idea to develop theory.

Types of crystals

molecular: sugar, salt, etc.
covalent: diamond
ionic: NaCl
metallic:

We will be concerned with 2 topics: phonon interactions and conduction in crystalline material.
From statistical mechanics, if a crystal is at temperature \( T \) at equilibrium, the probability of being in state \( i \) is

\[
p_i \propto e^{-\frac{E_i}{kT}}
\]

\[
= \frac{Q}{e^{-\frac{E_i}{kT}}}
\]

where \( Q = \sum e^{-\frac{E_i}{kT}} \) is the partition function.

Can get all of thermodynamics from this:

\[
U = \text{internal energy} = \sum_i E_i p_i = \frac{\sum E_i e^{-\frac{E_i}{kT}}}{Q}
\]

If we know \( Q = Q(T) \)

\[
\frac{\partial Q}{\partial T} = -\frac{\sum E_i kT^2 e^{-\frac{E_i}{kT}}}{Q^2}
\]

\[
\Rightarrow U = \frac{RT^2}{Q} \frac{\partial Q}{\partial T}
\]

Can write

\[
Q = e^{-\frac{F}{kT}} \quad \text{where} \quad F = -kT \ln Q
\]

Then find

\[
U = -T^2 \frac{\partial}{\partial T} \left( \frac{F}{T} \right) = F - T \frac{\partial F}{\partial T}
\]

\[
\text{S entropy}
\]

\[
\therefore U + TS = F
\]
The energy of a molecule is a function of the position of the nuclei $U(R_1, R_2, \ldots)$. Thus

$$H \psi = \sum_i \left(-\frac{\hbar^2}{2M_i} \nabla_i^2 + V_i(r_i)\right) \psi(R_1, R_2, \ldots) + U(R_1, R_2, \ldots) \psi = -\frac{i}{\hbar} \frac{\partial}{\partial \psi}$$

**Problem:** Why is this equation right and how did we arrive at it? Assume that the correct S. -eqn is for Coulomb interactions and use mass differences of $N$, etc. try to separate electronic and nuclear motion.

[First consider case of fixed nuclei.

$$\sum_i \left(-\frac{\hbar^2}{2m_i} \nabla_i^2 + V_i(r_i)\right) \chi_i(r_i) = E \chi_i(r_i)$$

Find contact.

Note $E = E(R_1, \ldots)$ and idea is for $E = U$.

Want to show exact wave func

$$\Psi \approx \psi(R_1, R_2, \ldots) \chi_{R_1, R_2, \ldots}(r_2)$$]
PROBLEM: Find energy and harmonic oscillation of a system.

We were considering

\[
\left\{ -\frac{\hbar^2}{2\mu}\nabla^2 + U(R_1, R_2, \ldots) \right\} \psi(r) = -\frac{\hbar^2}{i} \frac{\partial \psi}{\partial t} = \varepsilon \psi(r) \tag{1}
\]

One of the laws of physics which seems to reproduce itself (like Newton's law \( F = ma \))

Since potential \( U \) varies very rapidly, we can assume that fields are pretty localized for vibration. Thus we can approximate (1) by expanding around equilibrium. Let \( x_i \rightarrow R_1, R_2, \ldots \) from equilibrium.

\[
-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x_i^2} + U(x_1, x_2, \ldots)
\]

Expand \( U(x_1, x_2, \ldots) + x_i \frac{\partial U}{\partial x_i}(x_1, x_2) + \frac{1}{2} \sum_{i,j} x_i x_j \frac{\partial^2 U}{\partial x_i \partial x_j}(x_1, x_2) \ldots \tag{2}

Redefine energy at \( U(x_1, x_2, \ldots) \), hence \( \varepsilon \) is zero.

Thus we have

\[
-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i,j} x_i x_j V_{ij} = \varepsilon
\]

Redefine \( \gamma_i = \sqrt{\mu x_i} \), let \( \hbar = 1 \)

\[
-\frac{\hbar^2}{2\gamma_i^2} + \frac{1}{2} \sum_{i,j} \gamma_i \gamma_j g_{ij} C_{ij} = H
\]

where

\[
C_{ij} = \frac{1}{\sqrt{\mu_1 \mu_2}} \frac{\partial^2 U}{\partial x_i \partial x_j}(x_1, x_2)
\]
This is just a set of coupled harmonic oscillators. First look at the classical analogue

\[ \dot{q}^2 + \frac{1}{2} \sum_j \frac{\partial_q}{\partial q_j} H \cdot \frac{\partial_q}{\partial q_j} = H \]

to find

\[ \ddot{q}_i = -\frac{1}{\Delta} \left( \text{Pot. Energy} \right) = -\frac{1}{\Delta} \sum_j C_{ij} q_j \]

Use \( q_i = \alpha_i e^{i\omega t} \)

\[ \omega q_i = \sum_j C_{ij} q_j \]

\[ \Rightarrow \Delta \left( C_{ij} - \omega^2 \delta_{ij} \right) = 0 \]

Find \( \omega \) roots yielding three \( q_i \) eigenvalues.

Also

\[ \sum_i \omega_0 \omega_0 = \delta \alpha \quad \text{since} \quad C_{ij} \quad \text{is symmetric} \]

Apply to \( \text{CO}_2 \) molecule

9 modes:

\[ \begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\end{array} \]

\( \omega = 0 \)

\( \omega \neq 0 \)

Also

\[ \sum_i q_i^{(n)} q_j^{(n)} = \delta_{ij} \]
We can reexpress \( q_i = \sum a_i (x) Q_{\alpha} \), i.e. \( q_i \) can be a sum of the eigenmodes. Can show

\[ Q_{\alpha} = \sum a_i (x) q_i \]

This separates the harmonic oscillators. Can also do this in q.m.
We were considering

\[ H = \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i,j} C_{ij} q_i q_j \]

and found

\[ \omega^2 a_{ij} = \sum_j C_{ij} a_{ij} \]

\[ \sum_j a_{ij} a_{ij} = 6 \delta_{ij} \]

To simplify, we substitute the normal coordinates

\[ Q_k = \sum_i a^{(\alpha)}_{ik} q_i \]

as new variables \( q_{\alpha} = \sum_k a^{(\alpha)}_{\alpha k} Q_k \) to find

\[ H = \frac{1}{2} \sum_k p_{\alpha}^2 + \frac{1}{2} \sum_k \omega_{\alpha}^2 Q_{\alpha}^2 \]

\[ = \sum_{\alpha} \hbar \omega_{\alpha} \] where \( \hbar = \frac{1}{2} p_{\alpha}^2 + \frac{1}{2} \omega_{\alpha}^2 Q_{\alpha}^2 \)

**Example:**

2) \( E_{\text{system}} = E_{\text{mc}} + E_{\text{mc}} + \ldots = \sum_{\alpha} E_{\alpha} \)

\[ = \hbar \omega_1 (n_1 + \frac{1}{2}) + \hbar \omega_2 (n_2 + \frac{1}{2}) + \ldots 
= \sum_{\alpha} \hbar \omega_{\alpha} (n_{\alpha} + \frac{1}{2}) \)
ii) Write wave functions as \( \phi_n(x) \) (Hermite polynomials)

\[
\Psi = \prod_n \phi_n(x_1) \phi_n(x_2) \quad \cdots
\]

\[
= \prod_n \phi_n(x)
\]

iii) Ground state of system: \( n_0 = 0 \), \( E_g = \frac{\hbar^2}{2m} q_0 a^2 + U(0,0,0) \)

\[
\psi_g = \prod_a e^{-\frac{1}{2} \hbar a_0 q_a^2} = e^{-\frac{1}{2} \hbar \omega_0 q_a^2}
\]

\[
= e^{-\frac{1}{2} \hbar \omega_0 q_i^2 q_i} q_i q_i = e^{-\frac{1}{2} \hbar \omega_{ij} q_i q_i}
\]

Might try to express \( \psi_{ij} = e^{\hbar \omega_{ij}} q_i q_i \) as a function of \( C \)

\[
\chi^2 = C \quad \left[ \sum_{n} n \kappa_{n} \psi_{ij} = C_{ij} \right]
\]

iv) \( \overline{n} = \frac{1}{e^{\frac{\beta \hbar a^2}{2T}} - 1} \) very useful

Thus internal energy

\[
\bar{U} = \sum_n \hbar \omega_0 \left( n + \frac{1}{2} \right) = \sum_n \frac{\hbar \omega_0}{2} + \frac{\hbar \omega_0}{e^{\frac{\beta \hbar a}{T}} - 1}
\]

To show \( \overline{n} \), consider probability of occupation of state \( \kappa_{n} \) i.e.

\[
P_n = \frac{1}{Q} e^{-\frac{\beta \hbar a}{n+\frac{1}{2}}}
\]

But \( \sum_{n} \overline{n} = 1 = \sum_{n} \frac{1}{Q} e^{-\frac{\beta \hbar a}{n+\frac{1}{2}}} \)

\[
\overline{n} = \sum_{n} n \overline{P_n} \sim 1 \cdot 1 \frac{1}{n} e^{-\frac{\hbar a}{n}} \frac{1}{n} = \frac{x}{1-x}
\]
Example:

Neutron scattering off of a nucleus in a solid, which is shown that neutron loses no energy?

Neutron-nucleus potential $V(R-\mathbf{q})$

is space limited, we use Born approx.

interaction $V(R-\mathbf{q}) \approx a\delta(R-\mathbf{q})$

initial state $e^{i\mathbf{P}_i\cdot\mathbf{R}} \Psi_i(\mathbf{q},\ldots) = \Psi(R,\mathbf{q})$

final state $e^{i\mathbf{P}_f\cdot\mathbf{R}} \Psi_f(\mathbf{q},\ldots) = \Psi_f^*\Psi_i(\mathbf{q},\ldots)$

Propt of transition $= -iM_{fi}^2$

$M_{fi} = \int \Psi_f^* e^{i\mathbf{P}_f\cdot\mathbf{R}} S(R,\mathbf{q}) e^{i\mathbf{P}_i\cdot\mathbf{R}} \Psi_i d^3rdq$

Define $K = \mathbf{P}_f - \mathbf{P}_i = \text{momentum transfer}$

$\int \Psi_f^*(\mathbf{q}) e^{iK\cdot\mathbf{q}} \Psi_i(\mathbf{q}) dq$

[i.e., sudden approximation to partial of $\Psi$, $\Psi_f = e^{iK\cdot\mathbf{R}+\text{part} \Psi_i}$]

or for $w$

$\int \Psi_f^* e^{iK\cdot\mathbf{q}} \Psi_i dq = \iint e^{-\frac{1}{2} \omega_0 \mathbf{Q}^2} e^{iK\cdot\mathbf{Q}+\text{part} \Psi_i(\mathbf{Q})} d\mathbf{Q} d\mathbf{Q}_f$
\[ T(e^{-\omega_0 \alpha^2 \Phi_0^2} \Phi_0 \Phi_0^* d^2 \Phi) = T e^{-\lambda(n/4\alpha)^2} \]

Thus
\[ \text{prob} = e^{-\lambda^2 s^2} \]
\[ s = \frac{1}{4\alpha} \left( \frac{a_0}{a_n} \right)^2 \]

**Problem:** One can find
\[ \langle q_k^2 \rangle = \int \psi_0 q_k^2 \psi_0 \partial \phi = s \]

Thus
\[ \text{prob} = e^{-\lambda^2 \langle q_k^2 \rangle / 4 s^2} \]

**Problem:** Prove the same formula for the system at temperature \( T \) i.e., that prob. of no energy is transferred to system is

\[ \text{prob} = e^{-\lambda^2 \langle q_k^2 \rangle / 4 t^2} \]

where now \( \langle q_k^2 \rangle \) is ave. at temp. \( T \)
We found

\[ \text{Prob. of going in ground state} = \text{Prob. of elastic scattering} = e^{-\frac{p^2}{2\text{M}^2}} \]

Debye-Waller Factor

Note we cannot scatter from zero-w modes (translational)
Always define elastic scattering in center of mass system to as
to throw zero translational mode.

**PROBLEM**: Suppose system in ground state. Show

\[ \langle \text{energy lost by neutron if fired} \rangle = \frac{p^2}{2\text{M}_n} \]

\[ \text{mean energy gained by crystal} \]
\[ \text{in matter distribution is } \epsilon \text{i} \text{p} x = \frac{p^2}{2\text{M}_n} \] (classical result)

i.) prove for ground state

ii.) for any state

iii.) steady system

\[ \text{Prob.} \]

\[ \text{cause a diffraction (only if we cannot distinguish which atom was hit). When is this true? When} \]

\[ \text{Prob.} > P \text{ zero point energy} \]
PROBLEM: Show that elastic scattering gives a diffraction. Show that inelastic scattering does not give a diffraction pattern.

Nucleus inside a crystal lattice at $k$. Expect a recoil (classically)

$E_f = E_0 - \frac{p^2}{2m}$

They could absorb this in a similar nucleus (resonance absorption)

Mössbauer discovered recoilless emission and absorption. Analogue to elastic neutron scattering. Very precise effect.
Back to Harmonic Oscillator

\[ H = \frac{1}{2} p^2 + \frac{1}{2} \xi^2 \]
\[ p = \frac{1}{i} \gamma \xi \]

Recall

\[ H \phi_n = \epsilon_n \phi_n \]
\[ \epsilon_n = (n + \frac{1}{2}) \]
\[ \phi_n(\xi) = \frac{1}{\pi^{1/4} \sqrt{2^n n!}} H_n(\xi) e^{-\frac{1}{2} \xi^2} \]

\[ \begin{align*}
  n=0 & \quad e^{-\frac{1}{2} \xi^2} \\
  n=1 & \quad \xi e^{-\frac{1}{2} \xi^2} \\
  n=2 & \quad (2\xi^2 - 1)e^{-\frac{1}{2} \xi^2} 
\end{align*} \]

Interesting that we can get a generating function

\[ e^{-\xi^2 + 2\xi \xi} = \sum_{n=0}^{\infty} \frac{H_n(\xi) \xi^n}{n!} \]

Also have

\[ \sum_{n} \frac{\phi_n(\xi) \xi^n}{\sqrt{n! \gamma^n}} = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2} (\xi - \gamma)^2} e^{\gamma \gamma} \]
Example:

\[ \gamma_{mn} = \int \phi_m^*(s) e^{i\tau s} \phi_n(s) ds \]

Calculate

\[
\frac{\sum_{m}^{n} s_{mn}^{T} i w}{\prod_{m}^{n} \sqrt{s_{mn}^{T} i w}} = \gamma(s, t)
\]

\[
= \frac{1}{i w^{n}} \int \frac{\prod_{m}^{n} \phi_{m}^*(s) e^{i\tau s} \phi_{n}(s) ds}{\prod_{m}^{n} i w^{m}}
\]

\[
= \frac{1}{i w^{n}} \int e^{-\frac{1}{2} (s-s)^2} e^{i\tau s} e^{-\frac{1}{2} (s-t)^2} e^{i\tau t} ds
gd\]
PROBLEM: A crystal at temperature $T$. Find the mean for scattering intensity (for given $p$) adiabatically.

Hint: If simple oscillator, $\rho_{\text{osc}} = \langle n | e^{i n \phi} | n \rangle$

$$\langle n \rangle = \frac{1}{\alpha^2} e^{-\alpha n / kT} |\langle n | e^{i n \phi} | 1 \rangle|^2$$

Then show

$$\rho_{\text{osc}} = e^{-\alpha^2 <q^2>}$$

PROBLEM: Show

$$\sum_n \rho_n(s) \varphi_n^*(s) e^{i \epsilon_n t}$$

$$= \left( \frac{m \omega}{2 \pi} \right)^{1/2} e^{i \frac{m \omega}{2 \pi} \epsilon^2} e^{-i \frac{m \omega}{2 \pi} \epsilon^2}$$

= \left( \frac{m \omega}{2 \pi \sin \omega t} \right)^{1/2} e^{i \frac{m \omega}{2 \pi \sin \omega t} \epsilon^2} e^{-i \frac{m \omega}{2 \pi \sin \omega t} \epsilon^2}

PROBLEM: Consider

$$\frac{d}{dt} \psi(s, t) = -\frac{i}{2} \frac{d^2}{ds^2} \psi(s, t) + \frac{1}{2} s^2 \psi(s, t)$$

Suppose $\psi(s, 0) = \delta(s - s_0)$. Try to solve the time-dependent eqn:

Hint: $\psi(s, t) = e^{-a(s) t^2} + b(s) t + c(t)$. 

\[ H = \frac{1}{2} \frac{d^2}{d\xi^2} + \frac{1}{2} \xi^2 \]
\[ p = \frac{1}{i} \frac{d}{d\xi} \]
\[ \phi_0(\xi) = e^{-\xi^2/2} \]
\[ \phi_n(\xi) \]

We can transform this to momentum space

\[ H = \frac{1}{2} p^2 - \frac{1}{2} \frac{d^2}{dp^2} \]

and find we get exactly the same equations and

\[ \psi_n(p) = i^n \phi_n(p) \]

\[ \text{A phase from } \psi(p) = \int e^{i p \xi} \phi(\xi) d\xi \]

Recall we had given

\[ \lambda_{mn} = \int \phi_m^*(\xi) e^{ip\xi} \phi_n(\xi) d\xi = \int \psi_m^*(p) \psi_n(p+p) dp \]

\[ -\frac{d^2}{dr^2} \left[ r^2 \psi \right] + \frac{1}{r} \left[ r \psi \right] = rE \left[ r \psi \right] \]

\[ -\frac{1}{i} \frac{d}{dp} (p^2 f(p)) + f(p) = -E \frac{\hbar}{m} \frac{df}{dp} \]

\[ \text{much easier} \]

Now return to

\[ H = \frac{1}{2} \frac{p^2}{\hbar^2} + \frac{m \omega^2}{2} q^2 \]
Now define
\[ a = \sqrt{\frac{\hbar}{2m}} (\omega q + i p/m) \]
\[ a^+ = \sqrt{\frac{\hbar}{2m}} (\omega q - i p/m) \]

Then
\[ q = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+) \]
\[ p = (-i) (a - a^+) \]

Note \( pq - qp = -i \). Forget them in a's

\[ a^+ a = \frac{\hbar}{\omega} H - \frac{1}{2} \]
\[ a a^+ = \frac{\hbar}{\omega} H + \frac{1}{2} \]

Thus \( a a^+ - a^+ a = 1 \)

Also \( H = \omega a^+ a + \frac{\omega}{2} \)

Excitation energy for harmonic oscillator

Can figure out eigenvalues and matrix elements

Sometimes define \( N = a^+ a \)
If \( N|n\rangle = n|n\rangle \), then
\[
|n+1\rangle = a^{+}|n\rangle = \sqrt{n+1} |n+1\rangle
\]
\[
|n-1\rangle = a|n\rangle = \sqrt{n} |n-1\rangle
\]
Thus, \( a^{+} \) creation operator, \( a \) annihilation operator:
\[
\langle m|a^{+}|n\rangle = \delta_{m,n+1} \sqrt{n+1}
\]
\[
\langle m|a|n\rangle = \delta_{m,n-1} \sqrt{n}
\]

Conclusion:
\[
(a^{+})^{n}|0\rangle = \frac{1}{\sqrt{n!}} |n\rangle
\]
where we define \( |0\rangle \) by
\[
a|0\rangle = 0
\]

Example:
\[
\delta_{mn} = \langle m|(a^{+}a)|n\rangle
\]
\[
= \langle 0|a^{m} (a^{+}a) a^{+n}|0\rangle \frac{1}{\sqrt{m!}} \frac{1}{\sqrt{n!}}
\]
Use \( \langle 0|a^{+} = 0 \) \( \langle 0|a = 0 \) to work out,
\[
\delta_{mn} = \langle 0| \frac{a^{m}}{\sqrt{m!}} e^{i \frac{\rho(a^{+}a)}{\sqrt{2}}} (a^{+})^{n}|0\rangle
\]
If we have \( N \) independent oscillators, define
\[
a^{+} = (\omega_{\alpha} a + \epsilon \frac{P_{\alpha}}{m}) \frac{1}{\sqrt{2 \omega_{\alpha}}} \\
\text{Commutators:}
\]
\[
a^{+} a - a^{+} a = \delta_{\alpha \beta} \\
a \cdot a^{+} a = a^{+} a \\
\text{Thus}
\]
\[
H = \frac{\hbar \omega_{\alpha}}{2} + \sum \omega_{\alpha} a^{+} a \\
\text{Energy}
\]
Can arbitrarily assign oscillators to six with state it has \( N \) "vibrations." Thus \( a^{+} \) a create and annihilate vibrations.

This is very convenient for more complicated problems.

- EM cavity: photons
- Crystal: phonons

"Calling a state of motion by a name," like "vortex." ["Can't step in same river twice." "Eating and anti-process"]

Humans as a pattern of motion.
Now return to our discussion of a polyatomic molecule
\[ H = \sum_{i} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i \neq j} C_{ij} q_i q_j \]
and try to extend it to a crystal.

Select location in a crystal

\( N \) cell

\( r \) atom in cell

If polyatoms in cell, \( r \) has 3 \( p \) values

\( a, b, c \) are directions which reproduce the crystal

Then to find \( N \)

\[ N = n_a a + n_b b + n_c c \]

Now \( H \) for the crystal becomes

\[ H = \sum_{N, r} \frac{P_{nr}^2}{2} + \frac{1}{2} \sum_{N, r, N', r'} C_{N, r, N', r'} q_{N, r} q_{N', r'} \]
Now \( C_{n,r,\mathbf{m},s} = \sum_{i,s} (\mathbf{M} - \mathbf{M}) \)

expressing fact that interaction doesn't depend on the absolute location of the cells, but only on relative distance between them.

We must solve for the frequencies using

\[
\omega^2 \mathbf{b}_{\mathbf{m},r} = \sum_{\mathbf{m},s} (\mathbf{M} - \mathbf{M}) \mathbf{b}_{\mathbf{m},s}
\]
\[ H = \sum_{n} \frac{p_{n, r}^2}{2} + \frac{1}{2} \sum_{r, s, m} \Gamma_{r,s}(n-m) q_{n,r} q_{m,s} \]

Now consider

\[ \omega_n^2 q_{n,r} = \sum_{m,s} \Gamma_{r,s}(n-m) q_{m,s} \]

**Example:**

\[ q_n \]

Choose as a potential

\[ \frac{A}{2} (q_n - q_{n+1})^2 + \frac{B}{2} (q_n - q_{n+2})^2 + \ldots \]

At first take \( B \) large; longer range terms vanish. Then

\[ H = \sum_{n} \frac{p_{n}^2}{2} + \frac{1}{2} \sum_{n} A (q_n - q_{n+1})^2 \]

\[ = \sum_{n} A q_n^2 - \sum_{n} A q_n q_{n+1} \]

\[ = \sum_{n,m} \Gamma(n-m) q_n q_m \]

\( \Gamma(0) = 2A, \quad \Gamma(n+1) = A \quad \Gamma(-1) = -A \)

Our e.o. problem becomes

\[ \omega_n^2 q_n = A(q_n - q_{n+1}) - A(q_{n-1} - q_n) \]

Let

\[
\begin{pmatrix}
2A - \omega^2 & -A \\
-A & 2A - \omega^2 & -A \\
& -A & 2A - \omega^2 & -A
\end{pmatrix}
\]

and as b.c. use \( q_{n, m} = \delta_{nm} \)
Rather than solve det $A$, we merely solve eqn by letting
\[ q_n = e^{i\alpha} \]

to find
\[ \omega^2 e^{i\alpha} = A[e^{i\alpha} - e^{i(n+1)\alpha}] - A[e^{i\alpha} - e^{i(n-1)\alpha}] \]

\[ \omega^2 = A[1 - e^{i\alpha} - e^{-i\alpha} + 1] = 2A(1 - \cos \alpha) \]

We also require
\[ q_{n+n} = q_n \]
\[ e^{iN\alpha} = 1 \quad \Rightarrow \quad \alpha = \frac{2\pi l}{N}, \quad l = 0, \ldots, N-1 \]
\[ \text{or} \quad l = \frac{N-1}{2}, \frac{N}{2} \]

Then we find $\omega^2$ for each $l$

If we included $B$, \[ 2A(1 - \cos \alpha) + 2(1 - \cos 2\alpha) \] which distorts the wave above
It is convenient to change $x$ to $k$, suggestive of a wave motion.

\[ k = \frac{\xi}{a} \quad \text{phase difference} \quad \xi = 2\pi. \]

Then

\[ \omega_k^2 = 2A(1 - \cos k\xi) \quad \text{spacing of } k = \frac{2\pi}{Na} = \frac{2\pi}{L} \]

For long wavelength

\[ \omega_k^2 \sim Aa^2k^2 \quad \Rightarrow \omega \sim k. \quad \text{Like sound waves } \omega = c \]

\[ \Rightarrow c_s = \sqrt{Aa^2} \]

Now for the normal modes

\[ Q_k = \sum a_i \phi_i = \sum e^{-ik\lambda q_n} \frac{1}{\sqrt{N}} \]

and $H$ becomes

\[ H = \sum \frac{p_{k_i}^2}{2k_i} + \sum \frac{\omega_k^2}{2} Q_k^2 \]

Now have 'phonons' of wavelength $k$. Thus sound has a maximum $\omega$ and min wavelength (corresponding to $\lambda$).
Now returning to our general problem
\[ q_{1s} = q_{1s} e^{-iM \cdot k} \]

Thus we take into account the phase difference between cells and between atoms. Thus
\[ \omega^2 q_{1s} e^{-iM \cdot k} = \sum_{M_S} \gamma_{1s}^{(M)} (M - M) e^{iK \cdot M} q_{1s} \]

Now define
\[ \sum_{N', N''} \gamma_{N'}^{(N)} e^{-iK \cdot N'} = \gamma_{N'}^{(N)} \]

(Kind of a Fourier transform)

Thus letting \( \omega^2 \rightarrow \omega_0^2 (K) \)
\[ \omega_0^2 (K) q_{1s} = \sum_{N'} \gamma_{N'}^{(N)} q_{1s} \]

Now this problem is just for atoms in cell and we can revert to our analysis of polyatomic molecules [we handle the infinite periodic unit cells by the \( K \)]

\( B \) has \( 3p \) value for each \( K \). Now spacing of \( \frac{k}{k_x} \), \( \frac{k_y}{k_y} \). . .

Thus
\[ \sum_{K} \rightarrow \sqrt{\frac{\omega_0^2 k \text{ Vol}}{2\pi^3}} \]

Where are repetition boundaries of \( K \) space? We'll return to this in a moment.
Consider a one dimensional $k_z$ plane where each unit cell has 1 atom - 3 modes per $k_z$.

Thus 3 types of sound waves - longitudinal and 2 transverse. [not actually either]

In other directions waves are different. Thus $c_s$ are different in different directions.

Acoustical / optical branches

\[ \omega \]

$\omega^2$
PROBLEMS:

1) One-dimensional line of atoms spacing a, limit case k\alpha \ll 1. Try to develop the q.m. of a continuous material (long waves). \sigma_n = \sigma(x). This is quantum field theory.

2) Find the frequency spectrum for some crystal. Specific heat. The cubic crystal.

Recall

\[ C_{N,M,M'} = \Gamma_{rs}(N-M) \]

\[ \sum \Gamma_{rs}(N)e^{i\mathbf{K} \cdot \mathbf{r}} = \Gamma_{rs}(\mathbf{K}) \]

Then solve

\[ \omega^2_{p}(\mathbf{K}) q'_{Br} = \sum_{s} \Gamma_{rs}(\mathbf{K}) q'_{Bs} \]

and

\[ q'_{r}(\mathbf{K}) = \sum_{r} \omega_{p}(\mathbf{K}) q_{br} \]

\[ q_{r}(\mathbf{K}) = \sum_{r,\mathbf{K}} \omega_{p}(\mathbf{K}) e^{-i\mathbf{K} \cdot \mathbf{r}} q_{br} \]
Continuous discussion: polyatomic crystals

Because of dipole moment, one can excite modes in optical branches with infrared light. Thus NaCl has numerous theoretical frequencies as $k_z \to 0$. Raman-like frequency, thus reflects monochromatic beam, would be an intense dye — if only not infrared — therefore look — like green dye which absorbs red light.

Can use good guesses of coupling constants to determine physical properties — or vice versa. Problem is essentially solved, can also get elastic constants (% for monodimic crystal).
Determination of $\omega$'s:

For each $i$, 3p frequency $\nu_{3p}(i)$
Not hard to show that if you bounce nearby from crystals

$$\Pi_{\text{in}} - \Pi_{\text{out}} = \Pi$$

Excite a mode of $\Pi$ if

$$\Pi = \Pi + \frac{\nu}{n_{\Pi}}$$

Can plot energy of reaction

By looking a peaks we can get $\omega$ directly.

Now in $\Pi$ space

$$\frac{E}{\Pi} \rightarrow \int_{R} \frac{\nu_{3p}^3 \text{vol}}{\nu_{3p}^3}$$

Now $\Pi = n_{a} a + n_{b} b + n_{c} c$

We can try to find $\Pi$, such that

$$\Pi' = \Pi + \Pi'$$

gives same answer.
\[ e^{i\mathbf{H} \cdot \mathbf{N}} = e^{i\mathbf{H} \cdot \mathbf{N}} \]

\[ \Rightarrow e^{i\mathbf{H} \cdot \mathbf{N}} = 1 \]

Thus

\[ (\mathbf{H} \cdot \mathbf{a}) = 2\pi \quad \text{(integer)} \]

\[ (\mathbf{H} \cdot \mathbf{b}) = 2\pi \quad \text{(integer)} \]

\[ (\mathbf{H} \cdot \mathbf{c}) = 2\pi \quad \text{(integer)} \]

\begin{align*}
\text{Let we know} \quad & (\mathbf{H} \cdot \mathbf{a}) = L_a \quad (\mathbf{H} \cdot \mathbf{b}) = L_b \quad (\mathbf{H} \cdot \mathbf{c}) = L_c \\
\text{Then one can show we can write the vector as} \quad & \mathbf{H} = \frac{L_a}{a} \mathbf{a} + \frac{L_b}{b} \mathbf{b} + \frac{L_c}{c} \mathbf{c} \\
\end{align*}

Here \( L_a, L_b, L_c = 2\pi \) integers. This forms a lattice work of points in \( \mathbf{K} \)-space analogous to the reciprocal of the original cubic crystal. Called the reciprocal lattice. Very important in diffraction studies (which is elastic).

In \( \mathbf{K} \)-space, can separate regions \( R \) which respect using reciprocal lattice. This is first Brillouin zone of \( \mathbf{K} \)-space.
Solid-state physicists (glorified chemists) deal with condensed crystals. True physicists always work with cubic crystals.

Specific Heats:

These nuclear vibrations are main contribution to specific heat in most problems.

\[ U(W) = \sum_{\mathbf{k}, \mathbf{q}} \frac{\hbar^2 \mathbf{q} \cdot \mathbf{\mathbf{q}}}{2 \hbar^2} \frac{\mathbf{q} \cdot \mathbf{\mathbf{q}}}{e^{\mathbf{q} \cdot \mathbf{\mathbf{q}}/kT} - 1} \]

\[ = \sum_{\mathbf{q}} \sqrt{\frac{\hbar^2 \mathbf{q} \cdot \mathbf{\mathbf{q}}}{e^{\mathbf{q} \cdot \mathbf{\mathbf{q}}/kT} - 1}} \frac{d^3 \mathbf{q}}{(2\pi)^3} \]

Can find

\[ C_V = \frac{\partial U}{\partial T}, \text{ etc.} \]
\[ \omega_x(n) = \text{mode frequencies} \]

\[ U = \frac{\text{internal energy}}{\text{kinetic energy}} = \frac{3}{2} kT \sqrt{\frac{3!}{(2\pi)^3}} \text{Vol} \left[ \frac{\hbar \omega_x(n)}{e^{\hbar \omega_x(n)/kT} - 1} + \frac{\hbar \omega_x(n)}{2} \right] \]

\[ C_v = \frac{\partial U}{\partial T} = kV \sqrt{\frac{3!}{(2\pi)^3}} \text{Vol} \left[ \frac{(\hbar \omega_x(n)/kT)^2 \hbar \omega_x(n)/kT}{(e^{\hbar \omega_x(n)/kT} - 1)^2} \right] \]

Phonon description is quite good for many crystals (not metals, magnets, etc., however). We can discuss \( C_v \)

**High temperatures:**

If \( \hbar T > \) highest frequency (usually at room temp)

\[ \frac{\hbar \omega_x}{e^{\hbar \omega_x/kT} - 1} \rightarrow kT \]

\[ U = kT \cdot \sqrt{\frac{3!}{(2\pi)^3}} \text{Vol} \approx 3NkT \]

\[ 3 \text{ No. of atoms in unit} \]

\[ C_v = 3Nk = 3R \text{ per } ^\circ \text{K, per mole} \]

\(~ 6 \text{ cal per } ^\circ \text{K, per mole} \)

Pretty good at high T
Low temperatures:

\[ kT < \text{any frequency} \Rightarrow \text{get } e^{-\frac{kT}{4}} \text{ - essential singularity} \]

But no lowest frequency

\[ e^\frac{2}{kT} \]

\[ \text{only these modes contribute} \]

Thus assume 3 values of \( \omega \) sum and

\[ \omega_{\text{ad}}(k) = C_0(i/k) \]

\[ U = kT \frac{C_0}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\text{re}^{i\omega_{\text{ad}}(k)/kT} - 1}{\text{re}^{i\omega_{\text{ad}}(k)/kT} - 1} \]

\[ = kT \frac{C_0}{2} \int \frac{d^3k}{(2\pi)^3} \frac{(kT)^3}{4 \pi} \frac{1}{2\pi^2} \int_0^\infty \frac{x^3 e^x}{e^x - 1} dx \approx \frac{\pi^4}{15} \]

\[ = (kT)^4 \frac{\pi^2}{10} \int \frac{d^3k}{(2\pi)^3} \frac{1}{3} \text{Vol} \]

\[ = \frac{(kT)^4}{10C_0^3} \]

\[ \text{are speed-unbound} \]

\[ \frac{1}{C_0} = \left< \frac{1}{(2\pi)^2} \right> \]

\[ C_V \sim T^3 \]

\[ C_V \]
The number of modes at low $T$ is roughly $k^3 \sim T^3$.

However, as we approach $0^\circ$ (10^-6 degrees), we find $C_V \approx T^3$. Why? Because $\approx \beta$ for very small crystals which are used at very low $T$. Then there is a lowest frequency $\omega$ with an essential singularity $\gamma$

Mention Debye theory, where $\omega = \frac{3kT}{\sqrt{\frac{\hbar^3 K}{(2\pi)^3} \frac{m v^3}{c^4}}}$

$$= \frac{(kT)^4}{(\pi \hbar)^3} \frac{\Theta/T}{10 \left(1 - e^{-x}\right)}$$

where $\Theta = \frac{K_{gy}T_0}{K_0}$

Debye temperature is a useful way to characterize a crystal.

But never plot $C_V$ vs $T$, unfortunately. Always plot $\frac{T}{\Theta_D}$ vs $T$. 
Neutron Scattering

\[
\langle \psi_1 \mid a_i e^{i p \cdot R_i} \mid i \rangle
\]

Actually this is gotten from,
\[
\langle e^{-i \mathbf{k}_i \cdot \mathbf{r}} \mid V(\mathbf{k}_i - \mathbf{R}_i) \mid e^{i \mathbf{k}_i \cdot \mathbf{r}} \rangle
\]

where we assume Fermi pseudopotential \( \equiv a_i(p) = \text{const.} \)

Now suppose elastic scattering \( i = f \)

\[
P_{\text{el}}^{f \rightarrow i} = | \langle i | \Psi | i \rangle |^2
\]

Put \( \mathbf{R}_i = \mathbf{R}_{\text{eq}} + \mathbf{q}_i \)

\[
P_{\text{el}}^{f \rightarrow i} = | \langle i | e^{i \mathbf{p} \cdot \mathbf{q}_i} e^{i \mathbf{p} \cdot \mathbf{q}_i} \rangle |^2
\]

If we take \( e^{i \mathbf{p} \cdot \mathbf{q}_i} \sim 1 \),

\[
P_{\text{el}}^{f \rightarrow i} = P_{\text{el}}^{f \rightarrow f} \]

However, carry \( e^{i \mathbf{p} \cdot \mathbf{q}_i} \) alone

\[
\leq \frac{e^{i \mathbf{p} \cdot \mathbf{q}_i}}{e^{-\frac{p^2 q_i^2}{2}}}; \quad \langle i | e^{i \mathbf{p} \cdot \mathbf{q}_i} | i \rangle
\]
Now for inelastic

\[ \langle \mathcal{E}_a | e^{i\mathbf{p} \cdot \mathbf{r}} | \mathcal{E}_a \rangle \]

For some reason we don't get any interference from inelastic scattering since we can always identify the excited atom.

Maybe a tiny bit of interference due to coupling between nuclei in crystal?

Try to work that for a linear chain.

Now \[ \psi_{\text{after}} = e^{i\mathbf{p} \cdot \mathbf{r}} \psi_{\text{before}} \] gives an impulse momentum \( \mathbf{p} \)

\[ \langle \psi_\alpha \mid H \mid \psi_\alpha \rangle - \langle \psi_\beta \ast H \mid \psi_\beta \rangle \]

\[ \left[ \frac{\mathbf{p}^2}{2m} + \text{other stuff} \right] \]

\[ \langle \psi_\beta \mid e^{-i\mathbf{p} \cdot \mathbf{r}} \left( \sum_{\alpha} e^{i\mathbf{p} \cdot \mathbf{r}} \right) - H \mid \psi_\beta \rangle \]

Now use \[ \frac{d}{dx} (e^{ax} f) = e^{ax} (\frac{d}{dx} + a) f \]

to find

\[ \langle \psi_\beta \mid (\mathbf{p} + \mathbf{P})^2 + \sum_{\alpha} - (\frac{\mathbf{p}^2}{2m} + \sum_{\alpha} \cdots) \mid \psi_\beta \rangle \]

\[ = \frac{\mathbf{P}^2}{2m} \langle \psi_\beta \mid \psi_\beta \rangle + \mathbf{P} \cdot \langle \psi_\alpha \mid \mathbf{P} \mid \psi_\alpha \rangle \]

when \( \langle \psi_\beta \mid \mathbf{P} \mid \psi_\alpha \rangle = 0 \), get classical result
The $|y> \cdot |p|_\mathbf{b} |n>_{\mathbf{a}}$ is kind of a Doppler correction.

Prob: No excitation from $e^{ip\mathbf{a}}$. If initially state is $|n>$, prob of no excitation is $1 - |<n|e^{i\mathbf{a} |n>|^2}$. At thermal equilibrium:

\[ \frac{\text{Prob of no excitation}}{\text{Prob of excitation}} = \frac{\sum_{n} e^{-E_n/k_B T}}{\sum_{n} e^{-E_n/k_B T}} <n|e^{i\mathbf{a} |n> |^2} \]

\[ = \sum_{n} e^{-E_n/k_B T} (1 - e^{-E_n/k_B T}) <n|e^{i\mathbf{a} |n> |^2} \]

First find $<n|e^{i\mathbf{a} |n>$ = $<0|\frac{a^n e^{ip\mathbf{a}\cdot \mathbf{x}}}{\sqrt{n!}}\frac{a^n}{n!}|0>$

\[ = \sqrt{\frac{\pi}{2}} \frac{i^n}{n!} <0|\lambda^n(a+ax)\mathbf{x}(ax)|n|e> \]

Use $a^na^\dagger = a^n a^n + n a^{n-1}$ — to derive.
\(\mu -\to \text{stimulated } p + p \rightarrow d + e^+ + \nu.\)

Except for your deepening. But \(\mu\) only lives \(\tau \approx 2.2 \times 10^{-6}\) sec, so drain cannot be sustained too long. But maybe another longer-lived heavy say. Charged particle.

At high \(T\), can getbarrier penetration for

\(p + p \rightarrow d + e^+ + \nu.\)

---

**PROBLEM:** Changing line of quantum to continuum

\[H = \frac{1}{2}p^2 + \frac{\alpha}{2}(\theta_n - \theta_{n+1})^2\]

\[\theta_n = \frac{\sqrt{N}}{\alpha} e^{i \frac{kx}{\alpha N}}\]

Set \(x_n = a_n\), thus \(q(x)\) replaces \(\theta_n\). Recall we found

\[\alpha^2 = 2\alpha (1 - \cos \alpha)\]

For large \(\alpha\), small \(q(x)\) continues

\[\alpha^2 = \alpha a^2 k^2\]

\[Q_n = \sqrt{q(x)} e^{i kx} \frac{dx}{a^2 N}\]

\[Q'_{n} = \sqrt{q(x)} e^{i kx} \frac{dx}{a N}\]

\[q(x) = \frac{Q_n}{2\pi}\]
\[ H = \int_{x_1}^{x_2} \left( \frac{p^2}{2m} + \frac{1}{2a} \left( \frac{\partial q}{\partial x} \right)^2 \right) \, dx \]

**K.E.**

\[ \mathcal{T}(x) = P_n a \]

\[ \phi_n P_n - P_n \phi_n = -i \hbar n \]

\[ \phi(x) \mathcal{T}(\phi(x)) \phi(x) = i \hbar \delta(x-x') \]

\[ H = \int \left[ \frac{1}{2} \mathcal{T}^2(x) + \frac{1}{2} \left( \frac{\partial q}{\partial x} \right)^2 \right] \, dx \]

Thus we have defined a field -- a q.m. field -- of two conjugate variables \( q(x), \mathcal{T}(x) \), [corresponds to vibrating string]. Note we can "diagonalize" \( H \) using

\[ q(x) = \int e^{i k x} Q(k) \frac{dk}{2\pi} \]

\[ \mathcal{T}(x) = \int e^{i k x} P(k) \frac{dk}{2\pi} \]

\[ \Rightarrow P(k) Q(k') - Q(k') P(k) = -2\pi i \delta(k-k') \]

\[ H = \frac{1}{2} \int \left| P(k) \right|^2 + \hbar^2 \frac{d^2 Q(k)}{dk^2} \]

\[ \Rightarrow \omega^2(x) = c k \]
We could have applied this to electrodynamics.

QED field theory, in 3-d

$\mathbf{Q}(x): \quad Q_x(x), \quad Q_y(x), \quad Q_z(x)$

$\text{K.E.} = \frac{1}{2} \int \frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\partial \mathbf{A}}{\partial t} \, d\text{Vol}$

$\text{P.E.} = \frac{1}{2} \int C_{ijkl} C_{ij} C_{kl} \, d^3x$
**Problem:** Diffusion of phonons \( \text{finite} \ m_T \) 

This is local diffusion. Need higher order 
interaction \( (q_i - q_{i+1})^3 \). Can do 
a perturbation theory, \( \text{(else} \ q_{\infty} \text{)} \) 
phonon-phonon scattering.

Undeformed process: \( \hbar k + \hbar k'' = \hbar k'' + \hbar k' \) because propagation 
moves at constant velocity \( \sqrt{\frac{\hbar^2}{m}} \). 

There is an additional 
diffusion, if second term \( \text{(e.g., diffusion 
process,} \ h^2 \text{)} \). At low-T not used. 

\[ \hbar k' + \hbar k'' = \hbar k'' + \hbar k' \]

\[ \hbar k' + \hbar k'' = \hbar k'' + \hbar k' \]

Problem: 
Some density of states for different 
phonons \( \text{we} \) can treat each 
phonon separately. Phasons will scatter 
\( \text{effect on thermal conductivity} \).

For small \( T \), how does 
thermal conductivity depend on \( T \) \( \text{and} \ n \)?
Continuum Approximation

\[ q(x), \pi(x) \Rightarrow \pi(x) q(x) - q(x) \pi(x) = -i \delta(x-x) \]

K.E. = \[ \frac{1}{2} \int \left( \frac{\partial q}{\partial x} \right)^2 dx = \frac{1}{2} \int [\pi(x)]^2 dx \]

P.E. = \[ \frac{c^2}{2} \int \left( \frac{\partial q}{\partial x} \right)^2 dx \]

Thus

\[ H = \frac{1}{2} \int [\pi(x)]^2 dx + \frac{c^2}{2} \int [\nabla q]^2 dx \]

Can build

\[ q(x) = \sum_{k} \frac{1}{\sqrt{2\omega_k}} \left[ (a^*_k + a_k) e^{-ikx} \right] \]

\[ \pi(x) = \sum_{k} \frac{i\sqrt{\omega_k}}{\sqrt{2}} \left[ (a^*_k - a_k) e^{ikx} \right] \]

Note we have \( a^*_k - a^*_l a_k = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{l}) \)

\[ \sum_{k} \Rightarrow \int \frac{d^3k}{(2\pi)^3} \]
He demonstrates

\[ [\pi(x), q(x)] = -i \sum_k e^{i k \cdot (x - x')} = -i \delta^3(x - x') \]

The representation (1) is very convenient for most physical problems. One finds

\[ H = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] + \omega_k k \]

We can rewrite it with (1). Note

\[ \langle \psi | A | \phi \rangle = \frac{d}{dt} \langle \psi | \psi \rangle \]

provided \( \psi, \phi \) are solutions

\[ = \langle \psi | \frac{dA}{dt} | \phi \rangle - i \langle \psi | A H | \phi \rangle + i \langle \psi | HA | \phi \rangle \]

\[ \Rightarrow \quad A = + \frac{dA}{dt} + i [H, A] \]

Thus

\[ \dot{\psi}(x) = i \left[ H, \psi(x) \right] \]

\[ = i \left[ \frac{1}{2} \left\{ \pi(x) \dot{\psi}(x) - \dot{\psi}(x) \pi(x) \right\} \right] \]

\[ = \int \left\{ \frac{\partial^2}{\partial x^2} \delta(x - x') \pi(x) - \delta(x - x') \pi(x) \frac{\partial^2}{\partial x^2} \right\} \pi(x) \pi(x) \]

\[ = \int \pi(x) \delta(x - x') d^3x = \pi(x) \]

\[ \therefore \quad \dot{\psi}(x) = \pi(x) \]
\[ \Pi(X) = i [H, \Pi] = i \sqrt{\int d^3 x \left\{ (\nabla \pi(x))^2 \pi(x) - \pi(x) (\nabla \pi(x))^2 \right\} \nabla_q(x) \left\{ \frac{\nabla_q(x) \pi(x) - \pi(x) \nabla_q(x)}{\nabla x [i \delta(x-x)]} \right\} + \ldots} \]

to find

\[ \Pi(X) = + c^2 \nabla^2 q(x) \]

Thus we find from (3) \& (4)

\[ \frac{\partial^2 \pi}{\partial t^2} = c^2 \frac{\partial^2}{\partial x^2} \pi(x) \]

This is called quantum field theory. [Variable at every point in space].

**Example:**

In elastic theory we need a vector field

\[ \phi_i(x), \Pi_i(x) \]

Now

\[ \Pi_i(x) \phi_i(y) - \phi_i(y) \Pi_i(x) = -i \delta_{ij} \delta(x-y) \]

\[ [\phi_i(x), \phi_j(y)] = 0 = [\Pi_i(x), \Pi_j(y)] \]
K.E. = \frac{1}{2} \left( \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial t} \right) dV = \frac{1}{2} \int \nabla \phi \cdot \nabla \phi \, dV

P.E. = \frac{1}{2} \int_{\Omega} \sum_{i, j, k, l} C_{ijkl} \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) \, dV

= \frac{1}{2} \int_{\Omega} \sum_{i} \nabla \times \nabla (\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)) \cdot \nabla (\frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)) \, dV

\text{using symmetry}

One can find an equation

\dot{\phi_i}(x) = \sum_{j, k, l} C_{ijkl} \frac{\partial u_j}{\partial x_k} \frac{\partial u_l}{\partial x_j} \phi_i(x)

\text{[wave equation in anisotropic media]}

Anisotropic media

\dot{\phi} = \lambda_1 \nabla^2 \phi + \lambda_2 \nabla (\nabla \cdot \phi)
ELECTRODYNAMICS

For free fields

\[ L = \frac{1}{2} \int \left( \mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B} \right) d^4x + \int \left( \mathbf{J} \cdot \mathbf{A} + e \varphi \right) d^4x \]

\[ = \frac{1}{2} \int \left( -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t} \right)^2 - (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}) \right) d^4x \]

\[ = \frac{1}{2} \int \left( \mathbf{A} \cdot \nabla \times \mathbf{A} - \nabla \cdot \mathbf{A} \right) - \nabla \phi \cdot \nabla \phi + \nabla \phi \cdot \nabla \phi \]

\[ \frac{\partial}{\partial t} (\nabla \phi) \]

Can evolve for \( \phi \) statically from \( \nabla^2 \phi = \rho \)

Now identify \( \mathbf{A} \) as an operator. Define a

\[ \Pi \mathbf{T}(x) \sim \frac{\partial \mathbf{A}}{\partial t} \]

\[ \Pi_i(x) A_j(x') - A_i(x') \Pi_i(x) = -i \left[ \mathbf{S}^3(x-x') \delta_{ij} \right] \]

\[ H = \frac{1}{2} \int \mathbf{T}(x) \cdot \Pi(x) d^4x + \frac{1}{2} \int (\nabla \times \mathbf{A})^2 d^3x \]
In classical mechanics we use

\[ \sqrt{L(q, \dot{q})} \, dt = S \]

and Euler equations (principle of least action). For \( E \neq M \) we use

\[ S = \frac{1}{2} \int \left[ \left( -\nabla \phi - \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right] \, d^3x \, dt \]

\[ + \int J_n(x(t)) \, A_n(x(t)) \, d^3x \, dt \]

Our variables corresponding to \( q, \dot{q} \) are \( \phi \) and \( A_\mu \).

We make our quantum theory in analogy only take out these. We use Coulomb gauge \( \nabla \cdot A = 0 \). Then no \( E \dot{B} + \text{c.c.} \) appears. Thus can determine \( \phi \) from

\[ \nabla^2 \phi = \rho \]

via instantaneous Coulomb interaction. In g.m. just use \( "\text{in S-}\text{eqn.}\)  

Thus in g.m. we get variables

\[ A(x) ; \quad \Phi(x) \]

and

\[ H = \frac{1}{2} \int (\Phi \cdot \Phi) \, d^3x + \int (\nabla \times A)^2 \, d^3x + \int J \cdot A \cdot B \, d^3x + \text{Matter} \]
Find
\[ [\Pi_i(x), A_j(y)] = -i \delta^3(x-y) S_{ij} \]

---

**Problem:** Derive Maxwell equations as operator equations.
\[ \ddot{\Pi} = i(\Pi \hat{\mathbf{H}} - \hat{\mathbf{H}} \Pi) \quad \ddot{A} = i(\hat{\mathbf{H}} A - A \hat{\mathbf{H}}) \]

**Problem:** Find
\[ [A_i(x,t), A_j(x',t')] \] for free fields

---

One can write \( A \) as a superposition of discrete waves (photons)
\[ A(x) = \sum_{k, \rho = 1,2} \frac{1}{\sqrt{2\omega_{\mathbf{k}}} } \left[ e_{\alpha} a_{\mathbf{k} \alpha} (x) e^{-i\mathbf{k} \cdot \mathbf{x}} + e_{\alpha}^* a_{\mathbf{k} \alpha}^* (x) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \]

\[ \Pi(x) = \text{---} \]

---

**Comments:**
1. Many more assumptions are needed in QED than in our crystal theory (2nd quantization).
2. The form of operators, 11. Very similar to elastic quantization:
   except for curl terms. There is a solid, which will reproduce \( E \) \& \( B \) -- see with problem Cij, 26.
This term is all the difference. In 19th century people didn't realize this, and since then we've been trying to understand waves, and since it turned out that light is a wave, we've been trying to study the nature of light. People like Newton, who got the right ideas, but most people got it wrong. We now think that people like this (Galileo), learn from mistakes. When we get stuck, it isn't because we haven't tried all of what was done before. Need new ideas in science; partly, every generation must invent its own methods.

PROBLEM: Surface interactions

Consider 2-dimensional gas. Can get two dimensional condensation

In 3-dimensional, 3 phases. In 2-D, 2 phases are claimed: condensed or expanded phase. Why? Is the condensed phase a solid or liquid?

Might-be suddenly metastable and have a deep diffraction pattern.

Sudden gas of \( c^2 = 0 \) in \( \rho \rho \phi \)

\[ \Rightarrow \ \text{we said?} \]

Try shear effects. Try strong, thin, transparent

(..., here, now, define a solid?)
ELECTRON THEORY OF METALS

Degenerate Electron Gas

Disregard ionic potential, Coulomb interaction. Assume they are free.

Consider a box of V with n electrons. What is behavior at 0 K finite T? 0 means lowest temp.

Fermi for k-space. Only discrete values for states.
Fill up to certain energy value of V

\[ V_{\text{max}} = k_F \]

Now \# of states filled is

\[ \frac{k_F^3}{(2\pi)^3} \frac{V_{\text{vol}}}{4\pi} \]

\[ = V \text{vol} \left( \frac{4\pi}{3} \right) \frac{k_F^3}{(2\pi)^3} \]

\[ = \frac{3}{2} \frac{k_F^3}{(2\pi)^3} \frac{\text{vol}}{4\pi} \]

\[ = \frac{4\pi}{3} \frac{k_F^3}{(2\pi)^3} \]

\[ \therefore n = 2 \left( \frac{4\pi}{3} \right) \frac{k_F^3}{(2\pi)^3} \]

Energy of electrons = \[ 2 \left( \frac{4\pi}{3} \right) \frac{k_F^3}{(2\pi)^3} \frac{V_{\text{vol}}}{2m} \frac{k_F^2}{2m} = \frac{3}{2} \frac{k_F^2}{2m} N \]

Energy/particle = \[ \frac{3}{2} k_F^2 \]

\[ \epsilon_F = \frac{k_F^2}{2m} \]

Pressure = \[ \frac{dF}{dV} \]

\[ \epsilon_f = \frac{k_F^2}{2m} \] (density = \( P \))

Crazy, but it works.
PROBLEMS: Determine the size of stars from your nucleus by balancing fermion repulsion against gravitational attraction. Find critical mass (gravitational collapse). Then find out what happens above critical mass (good luck). Take $T = 0°$, $p \sim n$. Use electrons or fermions (forget about Coulomb potential). $p = \rho^{5/3}$ — modify relativistically.

\[ \frac{dp}{dr} = \frac{GM(p)}{r^2} p - F(p) \]

\[ M(r) = \int p(r) \frac{dr}{r^2} \]

PROBLEMS: Find magnetic field of spin (Pauli’s paramagnetism). Disregard magnetic effect of spin in strong $B$. Consider Sommerfeld correction — find energy levels using fermions and plot $\chi = \frac{\omega}{2B}$.

\[ U = \int \epsilon(k) \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2m} \right] \]

From $p = \rho^{4/3}$ high density. Neglect mass of electrons (don’t correct $H(p)$ due to $U$). Ignore general relativity.
What are character of wave fns.

What is

\( \Psi(R_i, R_2, \ldots) \) ?

Use independent nature of electrons and consider only spin. Use exclusion principle (antiwavefn \( \psi \))

\[ \Psi(R_i, \ldots) = e^{i \mathbf{k}_{a_1} \cdot \mathbf{R}_1} e^{i \mathbf{k}_{a_2} \cdot \mathbf{R}_2} \ldots \]

\[ = e^{i \mathbf{k}_{a_3} \cdot \mathbf{R}_1} e^{i \mathbf{k}_{a_1} \cdot \mathbf{R}_2} \ldots \]

\[ \vdots \]

\[ = \frac{1}{\sqrt{N!}} \sum_{p} (-1)^p \prod_{i} e^{i \mathbf{k}_{a_p} \cdot \mathbf{R}_i} \]

\[ = \frac{1}{\sqrt{N!}} \det \begin{pmatrix} \Phi_1(R_1) & \Phi_2(R_2) & \cdots \\ \Phi_1(R_2) & \Phi_2(R_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \]

where \( \Phi_a(R) = e^{i \mathbf{k}_a \cdot \mathbf{R}} \)

Note this also means \( \mathbf{k}_{a_1} \neq \mathbf{k}_{a_2} \) or \( \Psi \equiv 0 \),

say we fixed electron at \( R_0 \). What is prob. finding

\[ P(r) \]

density of gas
Calculate instead prob. of finding \( e \) at \( R_a \) \& \( R_b \),
indeed calculate matrix elements
\[
\langle \psi | S(R_a - R_b) \rangle \psi
\]
\[
= \langle \psi | S(I_R - R_1) \rangle \langle \psi | S(I_R - R_2) \rangle
\]
\[
= \sum \left[ 1 - \sum e^{i(k_1 - k_2) \cdot (R_a - R_b)} \right]
\]
due to
\[
= 1 - \left| \frac{3}{\kappa^2} \sin\lambda \cdot \cos\lambda \right|^2
\]
\[
\kappa = \kappa_\lambda
\]
\[
\phi \quad \begin{array}{c}
\text{pass from} \\
\frac{3}{\kappa^2}
\end{array}
\begin{array}{c}
\rightarrow \\
\sqrt{\frac{\kappa}{\lambda}}
\end{array}
\]
\[
= 1 - \left\{ \frac{3}{\kappa^2} \left[ \sin\lambda \cdot \cos\lambda \right] \right\}^2
\]
\[
\lambda = \kappa R
\]
At high temp, the prob. of occupation of an electron state is

\[ N_e = e^{-\mu/kT} \]

\[ \mu = \text{Fermi energy or chemical potential}. \]

Can find at low temp, \( U \sim T^2 \), \( C \sim T \). At zero \( T \) we fill up the momentum sphere of radius \( \mu \).

Electrons are excited with energy \( \mu/kT \). How many can be excited? Only ones on surface of sphere.

No. of electron in region within \( \mu/kT \) of surface of Fermisphere

\[ \sim N \mu/kT \]

Thus \( U \sim (\mu/kT)^2 \sim T^2 \)

For phonons, \( N \sim kT \)

\[ \sim N \sim T^3 \]

\[ U \sim T^4 \]

Since working with new and need entire volume of sphere.

We can plot

\[ C \]

\[ \sim T^2 \]

\[ \sim T \]

\[ \text{Phonons} \]

\[ \text{Electrons} \]
Periodic Potentials

We now try to solve the Schrödinger equation for this [need Hamiltonian to get correct potential]

We consider bringing two potentials to get

First degeneracy is destroyed. For many atoms

Now \(-\frac{h^2}{2m} \nabla^2 \psi + V(R) \psi = E \psi\)

But \(V(R) = V(R + a)\)
Using "phoey-smeary" periodicity
\[ \psi(R + n a) = (\psi(R) e^{i Re}) \]

Can find

\[
\begin{array}{c}
\varepsilon_f \\
\varepsilon_d
\end{array}
\]

\[-\pi/a \quad \pi/a \quad \kappa \]

[Remember can only define \( \kappa \) well]

Frequently people use \( \psi(R) = e^{i k R} u_k(R) \) and \( u_k(R + a) = u_k(R) \)

Mention all of this as it applies to group theory

\[ T [S\text{-eqn}] = [S\text{-eqn}] \]

\[ \Rightarrow \quad T \psi(R) = \psi(R) \]
But this doesn't describe free propagation. We neglected higher states. Thus \( \tilde{p} \to 0 \) and free particle wave function.

\[ E_p \quad \rightarrow \quad 00V \to 0 \]

\[ E_s \quad \rightarrow \quad \text{Graph} \]
Weak Potentials

\[-\frac{\hbar^2}{2m} \nabla^2 \psi + V(R) \psi = \lambda \psi\]

\[V(R) = \sum_l V_l e^{i l \cdot R}
\]

\[\psi(R) = e^{i k \cdot R} \psi_l(R)
\]

\[\psi_l(R) = \frac{1}{l} \sum_m A_{lm} e^{i (k+l) \cdot R}
\]

as wave fun for one electron moving through a periodic potential

Hence

\[\left[\frac{\hbar^2}{2m} (k+l)^2 - \epsilon \right] A_{ll} = - \sum_{l'} V_{ll'} A_{ll'}
\]

Choose \(V_0 = 0\), all other \(V_l's\) small. First approximation

\(A_0 = 1\), other \(A_l's\) very small. Then \((1)\) for \(l = 0\) gives

\[\epsilon \approx \frac{\hbar^2}{2m} k^2 \]

\[l = 0
\]

\[A_{ll} = \frac{-V_0}{\frac{1}{2m} (k+l)^2 - \frac{1}{2m} k^2}
\]

\[l \neq 0
\]

\[\left[\text{Can solve for } \epsilon \text{ analytically for } \ldots \text{ or } \ldots \right]
\]

[Otherwise use a computer or approximations to solve]

For second order, use

\[\left(\frac{\hbar^2}{2m} k^2 - \epsilon \right) A_0 = - \sum_{l'} V_{ll'} A_{ll'} = \sum_{l'} \frac{1}{2m} \frac{|V_{ll'}|^2}{(k+l)^2 - k^2}
\]

\[\epsilon = \frac{1}{2m} k^2
\]

\[\epsilon \leq \frac{|V_{ll'}|^2}{2m (k+l)^2 - k^2}
\]
This is an exception when \( k \not\in \{ (k + l_1)^2 - h \}. \) Consider 
\[ k^2 = (k + l_1)^2 \]

get a gap now at edge of Brillouin zone.

If \( k \in \{ (k + l_1)^2 - h \}, \) we can start given by considering \( A_0 \) and \( A_1 \) not small -- all other \( A_i \) small. Then (1) gives 
\[
L = 0 \quad \left( \frac{\hbar^2}{\alpha m} k^2 - \epsilon \right) A_0 = -V_{-L1} A_{-L1}
\]

\[
L = 1 \quad \left( \frac{\hbar^2}{\alpha m} (k + l_1)^2 - \epsilon \right) A_1 = -V_+ A_0
\]

\[
\Rightarrow \left[ \frac{1}{2m} (k + l_1)^2 - \epsilon \right] \left[ \frac{1}{2m} k^2 - \epsilon \right] = |V_{L1}|^2
\]

Using \( V_{-L} = V_{+L} \)

Can find 
\[
\epsilon = \frac{w(k + l_1) + w(k)}{2} \pm \sqrt{\left( \frac{w(k + l_1) - w(k)}{2} \right)^2 + |V_{L1}|^2}
\]

If \( V_{L1} \ll \frac{w(k + l_1) - w(k)}{2} \)

\[
= w(k) - \frac{|V_{L1}|^2}{w(k + l_1) - w(k)}
\]

Thus a discontinuity
Strong Potentials:

\[ \psi \leq e^{i k \cdot N} f(r - in) \]

where \( f(r) \) is solutio to \( S \)-equ in one of these potentials

\[ -\frac{1}{2m} \nabla^2 \psi + u(r) \psi = E \psi \]

where \( u = \frac{\Phi}{\mu} \)

Can find

\[ \epsilon = \frac{\int \psi^* H \psi \, dV}{\int \psi^* \psi \, dV} = \frac{N}{D} \]

where \( \psi \) is trial fun
Now

\[ D = \sum_{\text{in}, \text{in'} } e^{i k \cdot (\text{in} - \text{in'})} \sqrt{f(\text{in} - \text{in'})} f(\text{in}) \, d^3 r \]

\[ = 1 + \sum_{\text{in} \neq 0 } e^{i k \cdot \text{in}} I(\text{im}) \]

\[ I(\text{im}) = \int f^*(\text{ir} - \text{im}) f(\text{ir}) \, d^3 r \]

\[ N = \sum_{\text{in}, \text{in'} } e^{i k \cdot (\text{in} - \text{in'})} \int f^*(\text{in} - \text{in'}) \left( -\frac{\hbar^2}{2m} \nabla + \frac{\xi^2}{\hbar^2} V(\text{ir} - \text{in'}) \right) f(\text{in}-\text{im}) \, d^3 r \]

\[ E_0 - V(\text{ir} - \text{in}) \]

\[ J(\text{im}) = \int f(\text{ir} - \text{im}) \left[ V(\text{ir}) - V(\text{ir}) \right] f(\text{ir}) \, d^3 r \]

\[ G = E_0 + \sum_{\text{in} \neq 0 } \frac{J(\text{im}) e^{i k \cdot \text{in}}}{1 + \sum_{\text{in} \neq 0 } I(\text{im}) e^{i k \cdot \text{in}} } \]

Note if \( \text{m} = 0 \)

\[ J(0) = \int |f(\text{ir})|^2 \left[ V(\text{ir}) - V(\text{ir}) \right] d^3 r \]

\[ J(\text{m}) \text{ falls off rapidly with m,} \]

\[ \text{average of all potentials leads to} \]

\[ f(\text{ir}) \]
Thus for say a cubic lattice

\[ E = E_0 + \frac{J_0 + 2J_{100}(\cos k_a + \cos k_y a + \cos k_z a)}{1 + 2J_{100}(\cos k_a + \cos k_y a + \cos k_z a)} \]
\[ \epsilon = E_0 + \frac{J_0 + 2J_{100}(\cos k_x a + \cos k_y a + \cos k_z a)}{1 + 2J_{100}(\cos k_x a + \cos k_y a + \cos k_z a)} \]

We have found really don't use this, but books rarely mention this.

Can get a "submodel" using s-states of helium as base states

\[ i \dot{a}_n = \gamma_0 a_n + A(a_{n+1} + a_{n-1} + a_{n+l} + \ldots) \]

\[ = \gamma_0 a_n + \sum_{n=1}^\infty A_M a_{n+n} \]

This model tells us all we need to know about bands. Could also set it up for s \& p states

\[ i \dot{a}_n = \gamma_0 a_n + \sum_{n=1}^\infty A_M a_{n+n} \]

\[ i \dot{b}_n = \gamma_0 b_n + \sum_{n=1}^\infty B_M b_{n+n} \]

H, Li, Be, B, C, N, O, F
N, Ne, Mg, Al, Si, P, S, Cl
Comments again on many views of band theory. Must consider electron interactions. Feynman suggests that if we could expand Li lattice it would eventually become an insulator.

Any substance at all squeezed hard enough becomes a conductor.

Methods of analyses in solid state physics rely too heavily on experimental information. Try to analyze a non-laboratory situation (what is generally useful). Solve problems: conduct and properties of materials of high pressures. Lots of experimental data.

**Problem:** How apply constant field

\[
\frac{d (k \mathbf{F})}{dt} = e (\mathbf{E} + \mathbf{V} \times \mathbf{B})
\]

\[
\mathbf{V} = \frac{\partial \mathbf{E}(k)}{\partial \mathbf{k}}
\]

Prove these.

Band theory works quite well for semi-conductors however.
Coulomb interaction in a Fermi gas

Energy per electron $/\text{per atom} = \frac{2.22}{r_s^2} - \frac{0.916}{r_s} + \varepsilon_c$

where $r_s = \frac{R_0}{\lambda} < \text{Bohr radius}$

$\frac{4\pi r_s^3}{3} = N_0 \text{ per shell}$

We use a uniform positive charge to neutralize atoms. The $\frac{0.916}{r_s}$ term is due to Coulomb interaction between $\text{shell atoms}$.

$\varepsilon_c = \text{"correlation energy"} = \text{all other terms}$

(Not physics to call it this)
Problem: Find the speed of sound in an ideal gas for long wavelengths. Now include effect of the Coulomb potential on $w(k)$

Conductivity of Metals:

Two origins of resistance due to collisions

1) imperfections in crystal

$$H = -\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{N} \sum V(r - \mathbf{R}_i) + \left[ u(r - \mathbf{R}_i) - u(r - \mathbf{R}_j) \right]$$

Here is to find scattering cross-sections and then to solve Boltzmann equation.

$$\langle \mathbf{k} | \mathbf{H} | \mathbf{k'} \rangle = \sqrt{\frac{1}{2\pi}} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\sqrt{2\pi\hbar}} e^{i\mathbf{k}\cdot\mathbf{r}}$$

Must include charge shielding effects however.

Very nice have been able to do this correctly.

The nightmare of the inner sphere Van Hove

[Still not finished. I don't think principle doesn't work to distinguish Zs from Is in Li, for instance]
11) Phonon scattering from atomic induction

Consider model with one-atom unit cell,

\[ \Xi V(\mathbf{r}-\mathbf{N}-\mathbf{q}) = \Xi V(\mathbf{r}-\mathbf{N}) - \Xi \phi_{\mathbf{N}} \left( \nabla V(\mathbf{r}-\mathbf{N}) \right) \]

\[ H_0 \]

\[ H_1 \]

Need

\[ \langle \text{phonon} | \frac{\Xi}{N} \phi_{\mathbf{N}} \nabla V(\mathbf{r}-\mathbf{N}) | \text{phonon} \rangle \]

\[ = \Xi \langle \text{phonon} | \phi_{\mathbf{N}} \nabla V(\mathbf{r}-\mathbf{N}) \rangle \frac{1}{\Xi} \frac{\Xi}{N} \phi_{\mathbf{N}} \left( \nabla V(\mathbf{r}-\mathbf{N}) \right) \Phi_{\mathbf{N}} e^{i \mathbf{k} \cdot \mathbf{r}} e^{i \mathbf{k} \cdot \mathbf{r}'} \]

where

\[ \Phi_{\mathbf{N}} = \int e^{i \mathbf{k} \cdot \mathbf{r}} \Phi_{\mathbf{N}}^*(\mathbf{r}) \nabla V(\mathbf{r}) \Phi_{\mathbf{N}}(\mathbf{r}) e^{i \mathbf{k} \cdot \mathbf{r}'}, \]

Use

\[ \Xi e^{i \mathbf{k} \cdot \mathbf{N}} e^{i (\mathbf{k}' \cdot \mathbf{N})} \rightarrow s(\mathbf{k} - \mathbf{k}' + \mathbf{k} + \mathbf{N}) \]

Saves vaguely like conservation of momentum mod \( \mathbf{L} \).
Can have electron-electron scattering using virtual phonons.

This is very important in superconductivity since it tends to bind electrons in pairs (BCS).
Creation Operators for Electrons

Recall we had defined for photons and phonons,

\[ a^*_{k, \beta} \quad a_{k, \beta} \]

\[ a^*_{i} a_{j} - a_{j} a^*_{i} = \delta_{ij} \]

\[ a^*_{i} a_{j} = a_{j} a^*_{i} = 0 \]

\[ a_{i} a_{j} = a_{j} a_{i} = 0 \]

Similar operators are used for electrons. We define these by \( c_{k, \beta} \).

The first time we found an amplitude

\[ c_{k, \beta} \quad \text{creates} \quad c_{k, \beta} \quad \text{annihilates electron of type } k \beta \]

Then in \( H \) we include an interaction term to describe this:

\[ H_{int} = \sum_{k, k', \beta} G_{k, k', \beta} \Phi_k c_{k, \beta}^* c_{k'} \]

This is just formal mathematics of course.
Actually to make H hermitian use

\[ H = \sum_{k, k', \ell} \sum_{\ell'} \alpha_{k k'} c^*_{k \ell} c_{k' \ell'} + \sum_{k, k', \ell} \sum_{\ell'} \alpha_{k k'} c^*_{k \ell} c_{k' \ell'} \]

Now consider the \( c^* \)'s in general.

\( c^* \) creates an electron in state \( i \).

Very useful since it is so easy and convenient

without tensoring mechanisms.

Exclusion principle allows only two states

\[ \text{No electron in } i > 10 > \]

\[ \text{One electron in } i > 11 > \]

Thus

\[ c^* 10 > = 11 > \]

\[ c^* 11 > = 0 \]

\[ 11 > = 10 > \]

\[ < 10 > = 0 \]

\( \Rightarrow \) properties of operator...
Now for correct adjoints
\[ <n'_1c^*|n'> = <n'_1c|n> \]

Applying this yields
\[ <1|cc^*|10> = 1 \]
\[ <01|cc|11> = 1 \]

so phase of states are properly fixed.

Now
\[ cc = 0 \]
\[ cc^* = 0 \]

Also
\[ c* c^* = 0 \]
\[ c^* c* 10> = 0 \]
\[ c^* c* 11> = 0 \]

Now
\[ cc^* |10> = 10> \]
\[ cc^* |11> = 0 \]
\[ c^* c|10> = 0 \]
\[ c^* c|11> = 11> \]

Thus we write
\[ c_i^* c_i = N_i \]
Now
\[(c^*c + cc^*) |0\rangle = |0\rangle\]
\[(c^*c + cc^*) |1\rangle = |1\rangle\]

Thus
\[c^*c + cc^* = 1\]
\[c^*c = 0\]
\[cc = 0\]

One could now work backwards and find the representations for these operators (eigenvalues and eigenstates) only from these equations.

Now consider different states

\[
\begin{array}{c|c}
 a & b \\
\hline
 c^a & c_{b}^* \\
 c_{a} & c_{b} \\
\end{array}
\]

\[
\begin{aligned}
 c^a c_{a} + c_{a} c_{a} &= 1 \\
 c_{b} c_{b} + c_{b} c_{b} &= 1 \\
 c^a c_{b} &= 0 = c_{a} c_{a} \\
 c_{b} c_{a} &= 0 = c_{b} c_{b}
\end{aligned}
\]

How do we conclude these?

\[c^* c_{b} = -c_{b}^* c_{a}\]

Since we are developing Fermi statistics,
Ca* also show:

\[ C_a^* C_a + C_a C_a^* = 0 \]

starting with:

\[ C_i^* C_i + C_i C_i^* = 1 \]

can superimpose state to get new basis:

\[ |c\rangle = \alpha |a\rangle + \beta |b\rangle \]
\[ |d\rangle = \beta^* |a\rangle - \alpha^* |b\rangle \]

Then:

\[ C_a^* C_a + C_a C_a^* = 0 \]
\[ C_d^* C_d + C_d C_d^* = 0 \]

Can find:

\[ C_c^* = \frac{1}{\sqrt{2}} (C_a^* + C_b^*) \]

Then use:

\[ 1 = (\alpha C_a^* + \beta C_b^*) (\alpha^* C_a + \beta^* C_b) (\alpha C_a + \beta C_b) \]
\[ = \alpha \alpha^* (C_a^* C_a + C_a C_a^*) + \beta \beta^* (C_b^* C_b + C_b C_b^*) \]
\[ + \alpha \beta^* (C_a^* C_b + C_b C_a^*) + \beta \alpha^* (C_a C_b^* + C_b^* C_a) \]

Now normalized state \( |\psi\rangle \) is

\[ \Rightarrow \quad |\psi\rangle = \frac{1}{\sqrt{2}} (|a\rangle + |b\rangle) \]

\[ \Rightarrow \quad C_a^* C_b + C_b C_a^* = 0 \]
\[ C_a^* C_a + C_a C_a^* = 0 \quad \text{in same way.} \]
For bosons

\[ a_i^* \text{ creates particle in state } i \]
\[ a_i a_j^* - a_j^* a_i = \delta_{ij} \quad (= \langle i | j \rangle) \]
\[ a_i^* a_j^* - a_j^* a_i^* = 0 \]
\[ a_i a_j - a_j a_i = 0 \]

For fermions

\[ c_i^* \text{ creates particle in state } i \]
\[ c_i^* c^* + c_j^* c_i = \delta_{ij} \]
\[ c_i^* c_j^* + c_j^* c_i^* = 0 \]
\[ c_i^* c_i = N_i = \text{no. of electrons in } i \]

Example: Consider \[ c_{p}^*\] creates electron with momentum \( p \) (including spin)

\[ c_{p}^* |\text{vacuum}\rangle = \text{state of electron } e^{-iP \cdot x} \]

Now create one electron which has amplitude \( \alpha_1 \) to have \( P_1 \)
and \( \alpha_2 \) to have \( P_2 \)

\[ c_{p_2}^* |\text{vacuum}\rangle = \text{state of one electron of } \alpha_1 e^{ip_1 \cdot x} + \alpha_2 e^{ip_2 \cdot x} \]

\[ \Rightarrow \quad c_{p_2}^* = \alpha_1 c_{p_1}^* + \alpha_2 c_{p_2}^* \]
\[ C^*_p \text{ create one electron which has amplitude } \phi(p) \text{ to be in}
\]
momentum state \( p \), i.e.,

\[ C^*_p |\text{vac}\rangle = \text{state one electron which has wave function}
\]
\[ \phi(p) e^{i p \cdot x} = \phi(p) \]

Thus

\[ C^*_p = \frac{\phi(p)}{p} \]

\[ = \langle x|p \rangle \langle p|\phi \rangle \]

\[ = \langle x|\phi \rangle \]

Now operator which creates electron at \( x_0 = \text{i}\langle x_0|\phi \rangle \)

\[ \text{i} \phi(x_0) |\text{vac}\rangle = \text{state one electron with wave function } \delta(x-x_0) \]

Thus

\[ \text{i} \phi(x_0) = \frac{\phi(p)}{p} \mathcal{C} p^* e^{-i p \cdot x_0} \]

Similarly

\[ \phi(x) = \sqrt{\frac{2\pi}{2\pi}} e^{i p \cdot x} \mathcal{C}_p \]

Note also

\[ \mathcal{C}_p = \sqrt{\phi^*_p(x) \phi(x) dx} \]
We expect them

\[ 4\Psi^*(x) \Psi(y) + \Psi^*(y) \Psi(x) = \delta^2(x-y) \]

\[ 4\Psi^*(x) \Psi^*(y) + \Psi(y) \Psi(x) = 0 \]

\[ \Psi^*(x) \Psi(x) = \text{"no of electrons at } x\text{"} = \text{density of electrons at } x \]

**EXAMPLE:** We return to bound theory. Recall we had for a single electron

\[ \frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = i\hbar \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + U(r) \psi \]

We found solutions

\[ \psi = e^{ikr} \text{ where} \quad \Psi_{ik}(r) = f_{ik}(r) \]

then

\[ \hbar \Psi_{ik}(r) = \epsilon_{ik} \Psi_{ik}(r) \]

Now to use our new notation let

\[ \Psi_{ik} \] create electron of propagation vector \( \textbf{k} \) with \( \Psi_{ik}(r) \)

\[ = \sqrt{\Psi_{ik}(r) \Psi_{ik}^*(r)} \text{ } d^3r \]

Now the \( \Psi_{ik} \) are states of indistinguishable electrons.

Hamiltonian \[ \sum_{ik} \epsilon_{ik} N_{ik} = \sum_{ik} \epsilon_{ik} \Psi_{ik}^* \Psi_{ik} \]

[Note how similar this formulation is to previous work]
Note

$$H_{\text{final}} = \Sigma_E G_E G^*_E = \int \psi^*(x) \psi(x) dx$$

First, let's use

$$\psi^*(x) = \sum_{\ell \ell} f^*(\ell, x) G^*_\ell$$

$$\psi(x) = \sum_{\ell} f_{\ell}(x) G^*_\ell$$

Then multiply by $$\psi^*(x)$$, integrate, and use separability to find $$\psi^*(x)$$.

We have

$$\text{amp} \Gamma_{\ell \ell'} |_{K} = \text{amp} \text{electron} H_{K} \rightarrow H'_{K}, \text{ describes} \text{photon}$$

We can then write $$H$$ for a system of electrons and photons as

$$H = \sum_{\ell K} G^*_\ell G_{\ell K} + \sum_{\ell \ell' \ell'' K} \omega_{\ell \ell'} a^*_{\ell K} a^*_{\ell' K} a_{\ell'' K} + \sum_{\ell \ell' K \ell''} \Gamma_{\ell \ell'}^{\ell''} G^*_\ell G_{\ell' K} a^*_{\ell'' K}$$

Note

$$\text{amp} \Gamma_{\ell \ell', K} = \text{amp} \text{direct} H_{K} \rightarrow H'_{K}, \text{ emitting} \text{photon}$$
This is the basis of the theory of conductivity and superconductivity.

[Note the a's and c's commute.]
We now discuss electron-electron interactions, say by $e^2/r_2$. Ordinarily we would write

$$H = \sum_i -\frac{e^2}{2m} \rho_i^2 + \sum_{i<j} V(x_i-x_j)$$

Consider first order scattering of electrons for $P_1 + P_2 = P_3 + P_4$.

Then for $U = 0$,

$$H = \sum_{P} \left( \frac{P^2}{2m} \right) C^*_P C_P + \sum_{P_1, P_2, P_3} \int \bar{\psi}_{P_1 P_2} \tilde{U} C^*_P C_P \psi_{P_3}$$

Now $H$ is quadratic in $\psi$. Thus we can represent interactions using the $C_P$ of $\psi$.

We can try to represent this in terms of $\psi^*$ $\psi(x)$'s.

$$C^*_P = \int e^{iP \cdot x} \psi^*(x) \psi(x) d^3x$$

The first term in (6) yields

$$\int \psi^*(x) \left[ -\frac{e^2}{2m} \nabla^2 \right] \psi(x) d^3x$$
\[ H_{\text{tot}} = \frac{1}{2} \sum \frac{p_i^2}{m_i} - \sum \frac{\mathbf{r}_{ij} \cdot \mathbf{p}_{ij}}{2 \mu_{ij}} + \sum \phi(x_i) \phi(x_j) \]

\[ = \int d^3x \left( \phi(x) \phi^*(x) V(x-y) \phi^*(y) \right) + \sum \phi(x_i) \phi(x_j) \phi(x_k) \]

\[ \text{Supposed we invert the order using} \]

\[ \int \phi^*(x) \phi^*(y) \phi^*(y) \phi^*(x) \phi^*(x) = \int \phi^*(y) \phi^*(x) \phi^*(y) \phi^*(x) \phi^*(x) + 8 \int \phi^*(x) \phi^*(y) \phi^*(y) \phi^*(x) \phi^*(x) \]

\[ H_{\text{tot}} = -\int \phi^*(y) \phi^*(x) \phi^*(y) \phi^*(x) \phi^*(x) + \int \phi^*(x) \phi^*(y) \phi^*(y) \phi^*(x) \phi^*(x) \]

\[ + \int \phi^*(y) \phi^*(x) \phi^*(y) \phi^*(x) \phi^*(x) \]

\[ \text{Classically we expect} \]

\[ \text{Energy} = \frac{1}{2} \int \phi(x) V(x-y) \phi(x) \phi^*(x) \phi(x) \phi^*(x) \phi^*(x) \]

Three troubles:

1. factor $\frac{1}{2}$$
2. aux
3. extra term
The last term is a self-energy term (energy of particle
acting on itself). This is true in classical the sense
(4) also indicates self-energy.

If we don't want self-energy, go back to (2)
[like with Coulomb interaction where \( V(0) = \infty \)]

\[ \frac{1}{2} \text{ comes from a mistake in original sum} \]

\[ \varepsilon \cdot V(0) \sum C_p^* C_p C_p C_p \]

Thus

\[ H_{\text{int}} = -\frac{1}{2} \int \psi^*(x) \psi^*(y) V(x-y) \psi(x) \psi(y) \, d^3x \, d^3y \]  \hspace{1cm} (5)

and

\[ H_{\text{int}} = +\frac{1}{2} \int \psi^*(x) \psi^*(y) V(x-y) \psi(x) \psi(y) \, d^3x \, d^3y \]  \hspace{1cm} (6)

The sign comes in also from initial ordering in \( \varepsilon \). Need

\[ \frac{1}{2} \sum_{P, P, P} C_{P}^* C_{P}^* C_P C_P \]

Have a negative self-energy term. Do you want it?

\( \text{or not?} \)
Thus neglecting self-energy, our H for an system of electrons is

\[ H = \int \psi(x) \left[ -\frac{\hbar^2}{2m} \nabla^2 + eA(x,1)^2 + e\phi(x,1) \right] \psi(x) \, d^3x \]

\[ + \frac{1}{2} \int \psi(x) \left( \frac{\psi^*(y)}{|x-y|} \frac{e^2}{|x-y|} \psi(y) \right) \psi^*(x) \sigma^0 \psi(x) \, d^3x \, d^3y \]

Note: Because of the double annihilation, this contains no self-energy term [consider H operating on a system of one electron]. Note also the number of electrons is not specified.

This is the H for say a quantum plasma, with fixed nuclei -- at least non-relativistically.

How do we use (7)? In the S-eqn

\[ -\hbar \frac{d}{dt} \Psi^0 = H \Psi \]

Now

\[ N = \int \psi^*(x) \psi(x) \, d^3x \]

Can show \([H, N] = 0\). This can show

\[ N \Psi^0 = n \Psi^0 \quad \text{where } n \text{ is an integer} \]
How do we deal with "holes" in our formal theory? Again use

$C_i^* \text{ creates electron in state } \phi_i(x)$

Then $H = \sum_i \epsilon_i C_i^* C_i + \sum_{i,j} \lambda_{ij} C_i^* C_j$

where $\lambda_{ij} = \int \phi_i^*(x) \phi_j(x) \phi_j(x) dx$

is our perturbing potential.

Suppose some empty levels above ground state
Try to imagine perturbation operating on hole in ground state. Define new indices as shown. Recall

$C_i^* C_i + C_i C_i^* = \delta_{ii}$

If $i = j$ write $C_i = C_j$

$i \neq k$ write $C_i = b_k \Rightarrow b_{kp} = C_i^*$

Ex: $G^* C_i + C_i C_i^* = 1 \Rightarrow b_k b_k^* + b_k^* b_k = 1 \text{ for } i \in S_k$

$C_i C_i = 0 \Rightarrow b_k^* b_k^* = 0$

$C_i^* C_i^* = 0 \Rightarrow b_k^* b_k = 0$
We now regard \( b^* \) as "creating a hole."

\( c_i^* \) creates an electron in state \( i \)

\( b_i^* \) creates a hole in state \( i \)

Then if we neglect perturbation

\[
H = \sum_i \varepsilon_i c_i^* c_i + \sum_{k} \varepsilon_k b_k^* b_k
\]

\[
\varepsilon \sum_i \frac{\Delta i}{i} c_i^* c_i - \sum_{k} \frac{\Delta k}{k} b_k^* b_k
\]

\( \frac{\Delta i}{i} \Delta j = 0 \)

Our perturbation becomes

\[
\sum_i \lambda_{ij} c_i^* c_j + \sum_{k,l} \lambda_{kl} b_k^* b_l
\]

\[
+ \sum_{i,j,k} \lambda_{ijk} c_i^* c_j b_k^* + \sum_{k,l,m} \lambda_{klm} b_k^* b_l b_m^*
\]

\[
- b_k^* b_k + \varepsilon b_k
\]

We now examine each term.
Use arrows to mark different types of particles.

\[ \pi_i ; i = \int \phi_i(x) \chi(x) \phi_i(x) \, dx \]

These arrows tell where to put states in matrix elements.

Can play now with more complicated interactions like

\[ C_i * C_i * C_i \]

Let things like

Can just picture this as bundling the diagrams (arrows) around.
These schemes are used very frequently in perturbation theory.

Consider now a second-order perturbation theory in this formalism.

\[ \text{amp} = \int \phi^+(x_b, t_b) \mathcal{K}(x_b, t_b) \mathcal{K}_0(x_0, t_0; x_{\alpha}, \tau_a) \]  
where the propagator is

\[ \mathcal{K}_0(x_0, t_0; x_{\alpha}, \tau_a) = -e^{-i(x_0 - x_{\alpha})} \mathcal{U}_1(x_b) \mathcal{U}_1^*(x_{\alpha}) H(t_b - t_0) \]

where \( H \mathcal{U}_1 = \epsilon_1 \mathcal{U}_1 \)

Now let \( \phi_1 \rightarrow \mathcal{U}_1(x_0) e^{-i \omega_{1} t_0} \quad \phi_2 \rightarrow \mathcal{U}_2(x_{\alpha}) e^{-i \omega_{2} t_0} \]

We then find

\[ \text{amp} = - \int e^{i \epsilon_{\alpha} t_0} e^{i \omega_1 t_0} e^{-i \epsilon_1 (t_b - t_0)} e^{-i \omega_2 (t_b - t_0)} e^{-i \omega_1 \tau_a} \mathcal{K}_0 \]  

\[ = 0 \quad \text{unless} \quad \epsilon_2 + \omega_2 = \epsilon_1 + \omega_1 \quad \text{as expected} \]

\[ = \int_{t_0}^{\infty} e^{-i \epsilon_1 (t_b - t_0)} e^{-i (\epsilon_1 + \omega_1) (t_b - t_0)} dt_b \quad \mathcal{K}_{x_2} (\mathcal{U}_1) \]
Then

\[ \text{Area} = \sum 2 \cdot \frac{\lambda_{2i}}{\lambda_{1i} - \lambda_{01} - i\eta} \]

Now can we write this allowing for holes? We guess by drawing pictures.

\[ \sum_{i} 2 \cdot \frac{-i}{\lambda_{2i} - \lambda_{01} - i\eta} \]

"Virtual pair production"

\[ \sum_{i} 2 \cdot \frac{\lambda_{k1}}{k - \lambda_{2k}} \cdot \frac{-i}{\epsilon_{1} - (\epsilon_{1} + \omega_{1}) - i\eta} \]

This is correct.
Can we combine these into (1) terms and return to our general perturbation theory? Write second term as

\[ \sum_{k} \frac{\lambda}{\lambda_k^2} \frac{-i}{\varepsilon_k - \varepsilon_{(\varepsilon + \nu)} - i \eta} \lambda_k. \]

Thus just let \( \nu \) range over both \( j \) and \( k \).
$$H = \sum_{i_1} \phi_i C_i + \sum_{i_1} b_{i_1} b_{i_2} + H_{\text{int}}$$

where \( H_{\text{int}} = \sum_{i_1} \phi_{i_1} \phi_{i_1}^* C_i + \sum_{i_1} \lambda_{i_1} b_{i_1} C_i + \sum_{i_1} \lambda_{i_1} C_i^* b^*_{i_1} \)

\[- \sum_{i_1, i_2} \lambda_{i_1} \lambda_{i_2} b_{i_1}^* b_{i_2} \]

Perturbation gives

\[ \lambda_2 = \frac{1}{\varepsilon_1 - \varepsilon_{\gamma_1}} \gamma_{\gamma_1} + \frac{\gamma_{\gamma_1}}{\varepsilon_1 - \varepsilon_{\gamma_1} - \varepsilon_{\gamma_2}} \gamma_{\gamma_2} \]

This can also be written

\[ \int \psi_k^*(x_1, t_1) \psi_{k_1}(x_1, t_1) \psi_{k_2}(x_1, t_1) \psi_{k_1}(x_1, t_1) dx_1 dt_1 dx_2 dt_2 \]

\[
\begin{cases}
K_+(2, 1) = \sum_{(k_2, k_1)} e^{-i \varepsilon_{k_2} (t_2 - t_1)} \phi_{k_2} \phi_{k_1}^* (x_1) \quad t_2 > t_1 \\
K_-(2, 1) = \sum_{(k_2, k_1)} e^{i \varepsilon_{k_2} (t_2 - t_1)} \phi_{k_2} \phi_{k_1}^* (x_1) \quad t_1 > t_2
\end{cases}
\]

[No violation of commutativity]
IV. RELATIVISTIC QUANTUM MECHANICS

A. INTRODUCTION

Some of the achievements of QM are:

i. Prediction of antimatter (CPT theorem)

ii. Can show particles with spin ½ obey Fermi statistics

iii. Fine structure of H-atom

These seem to be three ways to present this subject.

1) One way is easier to understand since it works from

\[ \frac{\partial}{\partial t} \psi = H \psi \]  

in form of Dirac eqn. However we lose the manifest covariance.

2) Relativistic invariance is obvious, but notation is

harder to understand (Feynmann, Schwinger, etc.)

3) Abstract formulation -- immediately implies position,

spin statistics, etc. [more recent formulation]

We will begin using Method 1, even though it is

very dull.
Example:

Maxwell Equations (Example of Method 1)

\[
\text{Action} = \sqrt{\left[ (-\nabla \cdot \mathbf{A})^2 - (\nabla \times \mathbf{A})^2 \right]} \, \text{dVol} \, dt + \sqrt{(\partial A_x - j\Omega A)^2} \, \text{dVol} \, dt
\]

Equivalent to (in vacuum, \( \nabla \cdot \mathbf{A} = 0 \) -- makes AEO no longer manifestly conserved)

1.) Instantaneous Coulombs: \( \mathbf{E} = \mathbf{F} \)

2.) \( \mathbf{H} = \sum \omega_n \mathbf{a}_n \mathbf{a}_n + \text{coupling} \)

where \( \omega_n = n \omega \)

\[
\text{coupling} \rightarrow \sqrt{\mathbf{J}(x) \cdot \mathbf{A}(x) \, d^3x}
\]

and

\[
\mathbf{A}(x) = \sum_{k} \frac{1}{\sqrt{2\omega_k}} \left[ e^{i k \cdot x} \mathbf{a}_{n,k,i} + e^{-i k \cdot x} \mathbf{a}_*^{n,k,i} \right]
\]
Recall in classical theory

\[ E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad P = \frac{mv}{\sqrt{1 - v^2/c^2}} \]

so that

\[ E^2 - P^2 = m_0^2 \quad c = 1, \quad \hbar = 1 \]

or

\[ E^2 - P^2 P = m_0^2 \quad m \text{ always refers to } m_0 \text{ rest mass in this notation} \]

Hereafter we denote 4-vectors by

\[ A_\mu = (A_t, A_x, A_y, A_z) \]

Also use summation convention

\[ A_\mu B_\mu = A_t B_t - A_x B_x - A_y B_y - A_z B_z \quad \text{is invariant} \]

We use also

\[ \varepsilon_{\mu \nu} = \begin{cases} 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1 \\ 0 & \mu \neq \nu \end{cases} \]

and note

\[ \varepsilon_{\mu \nu} A_\nu = A_\mu \]

Finally

\[ \nabla_\mu = \left( \frac{\partial}{\partial x^\mu} - \frac{1}{c^2} \frac{\partial E}{\partial x^\mu} \right) \]
\[ P_\mu = (E, p_x, p_y, p_z) \]
\[ k_\mu = (\omega, k_x, k_y, k_z) \]

De Broglie \( P_\mu = i\hbar k_\mu \) \( \rightarrow \) wave packet as \( e^{-i(k_x x + k_y y + k_z z)} \)

Also \( P^2 = m^2 \)

Note \( \omega = \sqrt{k^2 + m^2} \)

\[ V_{gr} = \frac{\partial \omega}{\partial k} = \frac{\partial E}{\partial p} = V_{el} \]

since
\[ V_{gr} = \frac{\partial \omega}{\partial k} = \frac{k}{\sqrt{k^2 + m^2}} = \frac{\hbar}{\omega} = \frac{p}{E} \]

A free particle of momentum \( p \) has
\[ e^{-i(P_\mu x \mu)} \]

Also
\[ \theta_p \mu = m^2 \]

These contain all of classical special relativity.
Now to formulate q.m., we need some axes to generate 
\[ e^{-i\theta_2 \phi} \] in r.q.m.

Now  
\[ \frac{-i}{\hbar} \frac{\partial}{\partial \phi} \psi = \frac{i}{\hbar} \frac{\partial}{\partial \phi} \psi \]  
\[ \text{held in n.r.q.m.} \]

Thus we seek a linear eqn. The first begins with  
\[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \phi^2} \psi = \sqrt{m^2 + \left( \frac{\hbar^2}{2m} \right)^2} \psi \]

But trouble with \( \sqrt{ } \). If we include vector potential this gets foiled up.

Start over again with soul

\[ E^2 - P \cdot P = m^2 \] leads to

\[ \left( -\frac{\hbar^2}{2m} \right)^2 \psi - \left( \frac{\hbar^2}{2m} \right) \psi = m^2 \psi \]

\[ \partial^2 \psi = m^2 \psi \]

\[ -\Delta \psi = m^2 \psi \]

Now we must put in an interaction, guess

\[ \left( -\frac{\hbar^2}{2m} - A \right)^2 \psi - \left( \frac{\hbar^2}{2m} \right)\left( \frac{\hbar^2}{2m} - A \right) \psi = m^2 \psi \]

\[ \partial \psi = m^2 \psi \]

\[ (i \nabla - eA) (i \nabla - eA) \psi = m^2 \psi \]
Do this eqn OK? Must test it

i.) cons. prob

ii.) reduce to $S$-eqn

iii.) H-atom

iv.) causality

PROBLEM: Try all of these to see if it works? iii) is most serious.

[Doesn't include spin]
\((i\partial_x - eA_x)(i\partial_y - eA_y)\psi = m^2\psi\)

Klein-Gordon Equation

But it cannot represent spin. Maybe it is OK for spinless particles. Dirac thought it was wrong and thought only a first order time derivative should appear.

\[ H\psi = -\frac{\hbar}{i} \frac{\partial \psi}{\partial t} \]

He then guessed only first-order derivatives in space appeared.

\[ H = \left( \alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z} + \beta m \right) \]

He also knew Pauli matrices. Thus he proposed \(\alpha\)'s for a free particle. He suggested the matrices. For potential

\[ (-\frac{\hbar^2}{2m} - eV)\psi = \left[ \alpha_x \left( \frac{\partial}{\partial x} - eA_x \right) + \alpha_y \left( \frac{\partial}{\partial y} - eA_y \right) \right. \]

\[ + \left. \alpha_z \left( \frac{\partial}{\partial z} - eA_z \right) + \beta m \right] \psi \]

wonder what a free particle would try

\[ \psi = u e^{-i(\hat{E} t - \hat{P} \cdot \hat{x})} \]

where \(\hat{\alpha}\)'s and \(\hat{\beta}\)'s are all only.
We substitute this in we find

\[ E u = \left[ (a \cdot a) + \beta m \right] u \]

Then we replace again

\[ E^2 u = \left[ (a \cdot a) + \beta m \right]^2 u \]

But we want \( E^2 = P^2 + m^2 \). Multiply out operators.

\[ E^2 u = a_x^2 P_y^2 + (dx dy + dy dx) P_y P_x + \]

\[ + (dx \beta + \beta dx) P_x m + \beta^2 m \]

Thus we choose \( \alpha, \beta \) as

\[
\begin{align*}
\alpha_x^2 &= 1 = \alpha_y^2 = \alpha_z^2 \\
\alpha_x dy + dy dx &= 0 \\
\alpha_y dz + dz dy &= 0 \\
\alpha_z dx + dx dy &= 0 \\
\beta dx + \beta dy &= 0 \\
\beta^2 &= 1
\end{align*}
\]

Thus 4 operators \( \alpha_x, \alpha_y, \alpha_z, \beta \) whose squares = 1 and which anticommute with each other. There are many different possible representations. Need 4x4 matrices to represent it up for a spin 1/2 object (leptons, muons, etc.)
\( \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \)

\( \alpha_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

Of course the \( \alpha \)’s are 4-component. For many cases consider splitting up into spinors

\( u = \begin{pmatrix} u_1 \\ u_2 \\ -u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_\alpha \end{pmatrix} \)

Then

\( \chi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Thus for a free particle

\( E^2 u = (\hbar^2 p + m^2) u \)
We can rewrite the Dirac equation in various ways:

\[ (-\frac{i}{\hbar}\frac{\partial}{\partial t} - eV)\psi = \alpha \cdot [\frac{-i}{\hbar}\gamma^\dagger A - A]\psi + \beta m \psi \]

Still another way is:

\[ \left\{ \beta \left( -\frac{i}{\hbar}\frac{\partial}{\partial t} - eV \right) - \beta \alpha \cdot [\frac{-i}{\hbar}\gamma^\dagger A - A] \right\} \psi = m \psi \]

We can define other matrices:

\[ \gamma_t = \beta \]
\[ \gamma_x = \beta a_x \]
\[ \gamma_y = \beta a_y \]
\[ \gamma_z = \beta a_z \]

Then Dirac equation becomes:

\[ \gamma_t (i \gamma^\dagger A - eA) \psi = m \psi \]

where:

\[ \gamma_t \gamma_\nu + \gamma_\nu \gamma_t = 2 \delta_{\nu t} \]

\[ \gamma_t^2 = 1 \quad \gamma_x^2 = -1 \quad \gamma_y^2 = -1 \quad \gamma_z^2 = -1 \]

This can be solved relativistically invariantly. Let support for this makes a calculation with:

\[ \tilde{\gamma} = \begin{pmatrix} \gamma_t & \gamma_\nu \\ \gamma^\dagger_\nu & \gamma_\nu \end{pmatrix} \]

This looks relativistically invariant.
But suppose someone migrates w.r.t. the rest

different f's. Do we still have covariance? Yes!

Commutation properties are invariant.

We can write the eigen-spinor form

\[ \psi = \begin{pmatrix} \psi_a(x,t) \\ \psi_b(x,t) \end{pmatrix} \]

and use

\[ \alpha \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \psi_b \\ \psi_a \end{pmatrix} \]

we can find

\[ \left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - eV \right) \psi_a = \alpha \left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - eA \right) \psi_b + m \psi_a \]

\[ \left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - eV \right) \psi_b = \alpha \left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - eA \right) \psi_a - m \psi_b \]

The argument that relativity + spin \( \Rightarrow \) spin is

circular. Relies upon assumption that \( \alpha \) can take the

form

\[ -\frac{i}{\hbar} \frac{\partial}{\partial t} = \gamma \psi \]

Only for spin \( \frac{1}{2} \) particles. For neutrinos

\[ -\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_a = \psi_a \left( -\frac{i}{\hbar} \right) \left( A \right) \psi_b \]

\[ -\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_b = \psi_a \left( -\frac{i}{\hbar} \right) \left( A \right) \psi_a \]

(Not valid for neutrinos, protons, etc.) Majorized on for leptons.
Suppose Klein-Gordon was right. Write

\[(i\frac{\partial}{\partial t} - eV)\psi = \chi\]

\[\left(i\frac{\partial}{\partial t} - eV\right)\chi = (-\frac{i}{\hbar}\nabla - eA)^2\chi - m^2\chi\]

We can then convert this into \(\psi = \frac{\chi}{\Psi}\)

\[H = eV + p + \hbar \left(\frac{-i}{\hbar}\nabla - A\right)^2\]

which looks first order in time but obviously isn't.
Indeed Dirac's eqn is actually 4-order in time. Thus Dirac's original logic was wrong.

Ideas: Frequently bad logic leads to good answer."
Dirac Equation (Cont.)

\[
\frac{-\hbar \alpha \beta}{i \hbar \partial t} = H \Psi
\]

\[
H = \beta m + eV + \alpha \cdot (p - \frac{e}{c} A)
\]

\[
p = \frac{\hbar}{i} \nabla
\]

\[\alpha \beta \mathbf{g} 4 \times 4 \text{ matrix which anticommutate squares = 1}\]

Eq:

\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 & \mathbf{g} \\ -\mathbf{g} & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}
\]

\[
\begin{pmatrix} -\frac{\hbar}{i \hbar} & -eV \end{pmatrix} \psi_a = \psi_0 \left( \mathbf{p} - \mathbf{A} \right) \psi_0 + m \psi_b
\]

\[
\begin{pmatrix} -\frac{\hbar}{i \hbar} & -eV \end{pmatrix} \psi_b = \psi_0 \left( \mathbf{p} - \mathbf{A} \right) \psi_a - m \psi_b
\]

To learn more about this equation, we will play with this.

Now His fermation. Because we show:

\[
\frac{1}{2} \int \psi \psi^* \text{d}V = 0
\]

Thus we normalize

\[
\int \psi \psi^* \text{d}V = 1
\]
What does Bessel mean? Write

\[ \Psi_1 = \Psi_1 \Psi_2 \Psi_3 \Psi_4 \]

to find inner product. Can also matrix multiply in usual manner

\[ \sqrt{\Psi_1 \Psi_1} = \sqrt{\Psi_1 \left( \frac{\psi_1}{\psi_1} \psi_1 \right)} + \sqrt{\left( \right)} \]

Now

\[ \langle 510011 \rangle = \sqrt{\Psi_1 \Psi_1} \psi_1 \text{ dual} \]

\[ \alpha_x = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \]

\[ \alpha_x \Psi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \]

\[ \langle 510011 \rangle = \sqrt{\det \left[ \psi_1 \psi_1 + \psi_2 \psi_2 + \psi_3 \psi_3 + \psi_4 \psi_4 \right]} \]

Thus all of this goes through in some way.

We can set up a probability density of current \( J \) to \( \Psi \)

\[ \Psi = -\nabla \cdot J \]
\[ p(x,t) = \psi^*(x,t) \psi(x,t) \]
\[ = \psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_3^* \psi_3 + \psi_4^* \psi_4 \]
\[ i = \psi^*(x,t) \sigma \psi(x,t) \]

(uniquness of \( \alpha \))

\[ \alpha x^1 = S^{-1} x^1 \leq 0 \text{ here \( S \) denotes } \psi \text{ and \( \sigma S \) denotes } \psi' = S \psi \]

4 components of Dirac wavefunctions with a 4-vector.

Example: in 3-d, we get 2-component spinor. Again no connection.

However, the matrices do form a 4-vector \([o; \psi]\).

Here use \( \psi_5 = \psi_a + \psi_b \), \( \psi_4 = \psi_a - \psi_b \) and find

\[ [(E-eV)^2 - (P-eA)^2 + e \sigma_0 (B+iE)] \psi_5 = m^2 \psi_5 \]

Two-component wave function.
What do 4 components mean? Can we get S-equ?

First order S-equ. Assume A, V, U time independent.

\[ E = m + U \quad \text{with } V \ll m \]

\[ \frac{p}{m} \sim V \quad \left( \frac{V}{c} \right)^2 \ll 1 \]

Writing out eoms

\[(m+U-V)\psi_a = \sigma \cdot (p-eA) \psi_a + m \psi_a\]

\[(m+U-V)\psi_b = \sigma \cdot (p-eA) \psi_b - m \psi_b\]

\[ \psi_b \sim \frac{1}{2m} \sigma \cdot (p-eA) \psi_a \]

Taking in

\[(W+U-V)\psi_a = \left[ \sigma \cdot (p-eA) \right] \left[ \sigma \cdot (p-eA) \right] \psi_a \]

Now \( \sigma \cdot a \neq \sigma \cdot a \)

But \( \sigma \cdot (A \cdot B) = A \cdot B + i \sigma \cdot (A \times B) \)

\[ \frac{[\sigma \cdot (p-eA)] [\sigma \cdot (p+eA)]}{2m} = \frac{1}{2m} \left[ (p-eA) \cdot (p+eA) \right] \]

\[ + \frac{e \mathbf{A}}{2m c} \sigma \cdot (\mathbf{A} \times \mathbf{A}) \]

"small amplitudes"
Thus we find the non-relativistic limit

\[ H \Psi_0 = \left[ eV + \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + \frac{e\hbar}{2mc} (\mathbf{r} \cdot \mathbf{B}) \right] \Psi_0 \]

\[ \Rightarrow \mu = \frac{e\hbar}{2mc} \]

They are known as magnetic quantum numbers for magnetic moment of electron. One of great triumphs of Dirac equation.

Thus we get a spin connection to S-wave.

Consider free particle at rest, we assume

\[ \Psi = u e^{-i(Et - p \cdot x)} \]

and sub into Dirac equation.

\[ H = \beta \mathbf{p} \]

\[ \Rightarrow E u = m \beta u \]

or \[ E u_1 = m u_1 \quad E u_2 = m u_2 \]

\[ E u_3 = -m u_3 \quad E u_4 = -m u_4 \]

Spin up, \( E = +m \)

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Spin down, \( E = +m \)

\[
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\]
(0) spin up \( E = -m \) 

(0) spin down \( E = -m \)

\[ \begin{array}{c}
E \\
0 \\
-m \\
\end{array} \]

Both ± energy.

Being proposed our world has filled band of electrons with \( E < 0 \). Can get holes however \( \Rightarrow \) positrons
Recall the Dirac equation in the form

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial t^2} = \hat{H} \psi$$

where

$$\hat{H} = \beta m + eV + \alpha \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A})$$

$$\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

If we multiply (1) by $\beta$, we note $\beta \cdot \beta = 1$

$$-\frac{\hbar^2}{2m} \frac{\partial \psi}{\partial t} = \left[ m + eV + \beta \alpha \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A}) \right] \psi$$

on regrouping

$$\left\{ \beta \left[ \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial t^2} - eV \right] - \beta \alpha \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A}) \right\} \psi = m \psi$$

Define the $\xi$ matrices which are conventionally used to make non-relativistic part obvious:

$$\xi_0 = \beta \quad \xi = \beta \alpha = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$$

Also define

$$\delta = \xi_0 B_0 - \xi \cdot \mathbf{B}$$
The Dirac equation then becomes
\[ [\not{p} - \frac{\gamma_0}{c} \not{A} - \not{m}] \psi = 0 \]

**Case I:** \( \not{A} = 0, \not{p} = 0 \) (free particle at rest)

\[ [p_0 \not{x}_0 - \not{m}] \psi = 0 \quad \text{where} \quad \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]

or
\[ \begin{pmatrix} -\frac{i}{c} \frac{\partial}{\partial t} & 0 \\ \frac{i}{c} \frac{\partial}{\partial t} & -m \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \]

This yields
\[ (E - m) \phi = 0 \]
\[ (-E - m) \chi = 0 \]

Thus
\[ \phi = (a) e^{-i \hbar \omega t} \]
\[ \chi = (a) e^{+i \hbar \omega t} \]

Note the two 2 component spinors are decoupled.
Case II: \( \dot{A} = 0 \), \( \dot{\rho} \neq 0 \) (moving free particle)

From

\[
[p - m] \psi = 0
\]

or

\[
[E \chi_0 - \dot{\rho} \sigma - m] \psi = 0
\]

or

\[
\begin{bmatrix}
E (1, 0) - (0, \rho \cdot \vec{\sigma}) - m (1, 0)
\end{bmatrix} \psi = 0
\]

or

\[
\begin{bmatrix}
E - m & -\rho \cdot \vec{\sigma} \\
\rho \cdot \vec{\sigma} & -(E + m)
\end{bmatrix}
\begin{bmatrix}
\phi \\
\chi
\end{bmatrix} = 0
\]

or

\[
(E - m) \phi - \rho \cdot \vec{\sigma} \chi = 0
\]

\[
\rho \cdot \vec{\sigma} \phi - (E + m) \chi = 0
\]

Now \( \phi \) and \( \chi \) are coupled. Solve for

\[
\chi = \begin{bmatrix}
\rho \cdot \vec{\sigma}
\end{bmatrix} \phi
\]

and then

\[
(E - m) \phi = \begin{bmatrix}
\rho \cdot \vec{\sigma} \\
E + m
\end{bmatrix} \phi
\]

Hence the energy must satisfy

\[
E^2 - m^2 - \rho^2 = 0
\]
Thus \( E \) can be \( + \) or \( - \):

\[
E^2 = \sqrt{\mathbf{p}^2 + m^2}
\]

To get the second solution as \( p \to 0 \), we need the negative sign for the square root. Then

\[
\psi = \begin{pmatrix}
\frac{\mathbf{i} \mathbf{p} \cdot \chi}{m} e^{i p^2 t + i \mathbf{p} \cdot \mathbf{x}} \\
\chi
\end{pmatrix}
\]

The positive sign gives:

\[
\psi_+ = \begin{pmatrix}
\phi \\
\frac{\mathbf{p} \cdot \nabla \phi}{E + m}
\end{pmatrix} e^{-i p^2 t + i \mathbf{p} \cdot \mathbf{x}}
\]

Since (1) implies negative energies, we try to understand why rotating solutions to try to get it to look like (2). Define

\[
\mathbf{p}' = -\mathbf{p} \quad \quad E' = +\sqrt{\mathbf{p}^2 + m^2}
\]

Then

\[
\psi_- = \begin{pmatrix}
\frac{\mathbf{i} \mathbf{p}' \cdot \chi}{E + m} \\
\chi
\end{pmatrix} e^{-i E' t + i \mathbf{p}' \cdot \mathbf{x}}
\]

This now looks like \( \psi_+ \) except with momentum reversed.
Drac Equation Including Electromagnetic Interactions

We first try to get the non-relativistic limit. Don't try to normalize \( \Psi \)

\[
[\frac{\partial}{\partial x} - \frac{e}{c} A - m] \Psi = 0
\]

writing it out

\[
\begin{bmatrix}
E - m - \frac{e}{c} V & (\vec{\sigma} - \frac{e}{c} \vec{A}) \cdot \vec{p} \\
-\vec{\sigma} \cdot (\vec{p} - \frac{e}{c} \vec{A}) & -(E + m + \frac{e}{c} V)
\end{bmatrix}
\begin{pmatrix}
\phi \\
\chi
\end{pmatrix}
= 0
\]

We can solve, taking \( \chi \) as the smallest component, to find

\[
\chi = \frac{-1}{(E + m + \frac{e}{c} V)} \vec{\sigma} \cdot (\vec{p} - \frac{e}{c} \vec{A}) \phi
\]

Then plugging back in

\[
\begin{bmatrix}
E - m - \frac{e}{c} V \\
E + m + \frac{e}{c} V
\end{bmatrix} \phi = \vec{\sigma} \cdot (\vec{p} - \frac{e}{c} \vec{A}) \left( \frac{1}{(E + m + \frac{e}{c} V)} \right)^2 \vec{\sigma} \cdot (\vec{p} - \frac{e}{c} \vec{A}) \phi
\]

\[
E = m + \frac{1}{2m + \frac{e}{c} V} \quad \vec{\pi} = \vec{p} - \frac{e}{c} \vec{A}
\]

To get the non-relativistic limit, assume potential energy small and expand

\[
\frac{1}{2m + \frac{e}{c} V} \sim \frac{1}{2m} - \frac{(\frac{e}{c} V)}{2m^2}
\]

Plugging this in to get

\[
(\frac{e}{c} V) \phi = \vec{\sigma} \cdot \vec{\pi} \left( \frac{1}{2m} \right)^2 \vec{\sigma} \cdot \vec{\pi} \phi
\]
alter as follows: \( \phi' = \frac{1}{2} \psi \)

\[
\left[ 1 + \frac{(\hat{\eta} \cdot \hat{\eta})^2}{8w^2} \right] (w \cdot \psi') \left[ 1 + \left( \frac{(\hat{\eta} \cdot \hat{\eta})}{8w^2} \right)^2 \right] \phi = \frac{1}{2w} (\hat{\eta} \cdot \hat{\eta}) \phi
\]

\[+ \frac{1}{8w^2} \left[ (\hat{\eta} \cdot \hat{\eta}) (w \cdot \psi) - 2 \hat{\eta} \cdot \hat{\eta} (w \cdot \psi) \hat{\eta} \cdot \hat{\eta} \right]
\]

\[+ (w \cdot \psi) (\hat{\eta} \cdot \hat{\eta})^2 \phi \]

(3)

Let

\[
\chi \equiv \left[ 1 - \frac{(\hat{\eta} \cdot \hat{\eta})^2}{8w^2} \right] \phi
\]

since to first order,

\[
\psi + \psi = \phi^+ \phi + \chi + \chi \sim \phi^+ \phi \left[ 1 + \left( \frac{(\hat{\eta} \cdot \hat{\eta})}{8w^2} \right)^2 \right]
\]

and to first order

\[
\phi \sim \chi - \frac{(\hat{\eta} \cdot \hat{\eta})^2}{8w^2} \phi
\]

\[\Rightarrow \psi + \psi \sim \chi^\dagger \chi
\]

Using this in (3) and using

\[
\left[ 1 + \frac{(\hat{\eta} \cdot \hat{\eta})^2}{8w^2} \right]^{-1} \sim \left[ 1 - \frac{(\hat{\eta} \cdot \hat{\eta})^2}{8w^2} \right]
\]

we find

\[
(w \cdot \psi') \chi = \frac{1}{2w} (\hat{\eta} \cdot \hat{\eta})^2 \chi - \frac{(\hat{\eta} \cdot \hat{\eta})}{8w^2} \chi
\]

\[+ \frac{1}{8w^2} \left\{ (\hat{\eta} \cdot \hat{\eta}) (w \cdot \psi) - 2 \hat{\eta} \cdot \hat{\eta} (w \cdot \psi) \hat{\eta} \cdot \hat{\eta} + (w \cdot \psi) (\hat{\eta} \cdot \hat{\eta})^2 \right\} \chi
\]

(4)
To rewrite the \{ \} expression, use the commutation relation


and let \( A = \frac{\hat{\sigma}}{\tau}, \quad B = \omega - V' \).

Now, since \( E \rightarrow \frac{\hat{\sigma}}{\tau} \), if \( A \) is time dependent, it would commute with \( E \). Assume

\[ \frac{\partial \hat{A}}{\partial t} = 0 \]

so that \( \hat{A} \) commutes with \( E, m, V' \). Now worry about \( \frac{\hat{\sigma}}{\tau} \), which commutes with \( E \). Thus

\[ [\frac{\hat{\sigma}}{\tau}, (\frac{\hat{\sigma}}{\tau} - \frac{\hat{e}}{\tau} \mathbf{A}), E - m - V'] \rightarrow [\frac{\hat{\sigma}}{\tau}, \frac{\hat{e}}{\tau}, - V'] = - i e \frac{\hat{\sigma}}{\tau} \cdot \mathbf{E} \]

Then in (4)

\[ \{ \} = \kappa [\frac{\hat{\sigma}}{\tau}, \frac{\hat{e}}{\tau} \cdot \mathbf{E}] \]

Now use

\[ (\frac{\hat{e}}{\tau} \cdot \mathbf{E})(\frac{\hat{\sigma}}{\tau} \cdot \mathbf{\pi}) = -(\frac{\hat{\sigma}}{\tau} \cdot \mathbf{\pi})(\frac{\hat{e}}{\tau} \cdot \mathbf{E}) \]

and \( (\frac{\hat{e}}{\tau} \cdot \mathbf{a})(\frac{\hat{e}}{\tau} \cdot \mathbf{b}) = \frac{\hat{e}}{\tau} \cdot \mathbf{a} + i \frac{\hat{\sigma}}{\tau} \cdot (\mathbf{a} \times \mathbf{b}) \)

to find

\[ \hat{E} \cdot \hat{\pi} + i \frac{\hat{\sigma}}{\tau} \cdot (\hat{E} \times \hat{\pi}) = \hat{\pi} \cdot \hat{E} - i \frac{\hat{\sigma}}{\tau} \cdot (\hat{\pi} \times \hat{E}) \]

\[ \hat{\rho} \times \hat{E} = - \hat{E} \times \hat{\rho} \]
Hence

\[ \{ \} = -e \left[ \nabla \cdot \hat{E} - 2 \mathbf{\sigma} \cdot (\hat{p} \times \hat{E}) \right] \]

and (4) becomes

\[ W \mathbf{\pi} = V' \mathbf{\pi} + \frac{1}{2m} (\mathbf{\sigma} \cdot \mathbf{\pi})^2 \mathbf{\pi} - \frac{(\mathbf{\sigma} \cdot \mathbf{\pi})^4}{8m^3} \mathbf{\pi} \]

\[ - \frac{e}{8m^2} \left[ \nabla^2 E + 2 \mathbf{\sigma} \cdot (\hat{p} \times \hat{E}) \right] \mathbf{\pi} \]

Use

\[ (\mathbf{\sigma} \cdot \mathbf{\pi})(\mathbf{\sigma} \cdot \mathbf{\pi}) = \mathbf{\pi} \cdot \mathbf{\pi} - e \mathbf{\sigma} \cdot \mathbf{B} \]

Then

\[ W \mathbf{\pi} = V' \mathbf{\pi} + \frac{1}{2m} \left[ \mathbf{\pi} \cdot \mathbf{\pi} - e \mathbf{\sigma} \cdot \mathbf{B} \right] \mathbf{\pi} \]

\[ - \frac{(\mathbf{\sigma} \cdot \mathbf{\pi})^2}{8m^2} \mathbf{\pi} - \frac{e}{8m^2} \left[ \nabla \cdot \hat{E} + 2 \mathbf{\sigma} \cdot (\hat{p} \times \hat{E}) \right] \mathbf{\pi} \]

1st order
relativistic correction

to K.E.

spin-orbital coupling
\[ W = \left\{ V + \frac{1}{2m} \left( P - \frac{e}{c} A \right)^2 - \frac{1}{8m^3} \left( P - \frac{e}{c} A \right)^4 \right\} \chi \]

Significance of terms:

\[ \frac{1}{8m^3} \left( P - \frac{e}{c} A \right)^4 \]

Consider

\[ E = \sqrt{m^2 + p^2} = m + \frac{p^2}{2m} = \frac{p^2}{2m} + \frac{p^4}{2m^3} \]

We can identify the terms directly.

\[ \frac{e}{2m} \left[ \delta \left( \frac{P - eA}{m} \cdot E \right) + \frac{1}{4m} V \cdot E \right] \]

Can understand \( \frac{d}{dx} (W \times E) \) as magnetic field seen by moving object. [But doesn't account for 1/3 term.]

Now \( E = \frac{Ze^2 R}{4m \hbar^3} \), thus get a \( \frac{1}{3} \) term --

"spin-orbit coupling." (Does example)

In an H-atom, \( \tilde{p} \cdot E \rightarrow 4 \pi \frac{Ze^2}{R^3} (1/R) \)
Pauli's Correction for the Proton (Anomalous moment)

The electron has a moment \( \frac{e}{2m} \), but the proton has \( \frac{2}{3} \left( \frac{e}{m} \right) \). Where does the extra 1.17 come from? (Anomalous magnetic moment). Pauli suggested an alternate equation by adding terms to Dirac equation

\[
[i \gamma_\mu (\gamma_\mu - e A) - m] \psi = \frac{i}{4} \gamma_\mu \gamma_2 \Gamma_{\mu \nu} \psi + \kappa_0 \nabla^2 \psi
\]

where \( \Gamma_{\mu \nu} \) is field tensor

\[
\Gamma_{\mu \nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}
\]

Multiply by \( i \) to find

\[
i \frac{d\psi}{dt} = \left[ i (e + \alpha, (1 P - \frac{e}{2} A)) \right] \psi + \mu B \left[ \gamma_3 B - \alpha \cdot \sigma \right] \psi
\]

In the non-relativistic limit this leads to

\[
\omega \chi = \left\{ \begin{array}{c} \psi + \frac{1}{2m} (1 P - \frac{e}{3} A)^2 - \frac{1}{3m^2} (1 P - \frac{e}{3} A)^4 + \\
+ (\frac{\mu}{2m}) (1 P - \frac{e}{4} A) \psi + (\mu + \frac{e}{4m}) \left[ \frac{1}{m} (1 P - \frac{e}{3} A) \psi + \frac{\psi \psi}{2m} \right] \end{array} \right\}
\]

where

\[
\psi_D = \begin{pmatrix} 0 \\ 0 \\ \sigma^z \end{pmatrix}
\]

spin operator for Dirac eqn

\[
\psi = i \alpha_x \alpha_y
\]
Note the \( (\mu + \frac{q}{4m}) = (\mu + \frac{q}{2m} - \frac{q}{4m}) - \text{term} \).

The extra \(-\frac{1}{2}\) term comes from Thomas precession.

**Thomas Precession**

\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}
\]

\[
\begin{aligned}
\begin{array}{c}
\text{Since Lorentz transformation does not commute, their commutator involves retardation operator.}
\end{array}
\end{aligned}
\]

At present we don't know what makes the \( \gamma \).

Current work points to quark (maybe mesons?) stuff.

"Ideal proton" satisfies Dirac eqn, while neutron does not.

Doesn't appear electron has quark, although it Dirac eqn predicts 1000 mesons.

Dune 1901 gives 1.001147, but 1.001167.

Thus neutron and quark are not satisfiers.

Thus Lorentz and nonrelativistic model satisfy the Dirac eqn -- at least to within experimental evidence.
We also have an "ideal" electron which satisfies

\[ \tau \left( i \vec{p} - e A_\mu \right) \psi = m \psi \]

and an ideal photon which satisfies Maxwell's equations. However, there is an interaction which gives rise to angular momentum. (Since radiation resistance term in classical EM)

\[ m \frac{d^2 \psi}{dt^2} = i \mu \psi = \psi \left( E_{\text{self}} + E_{\text{external}} \right) \]

No experiment has proved any failure in these equations.
Any disagreement can be attributed to lack of knowledge about strong interaction.

Saw of motion of electrons

Dirac 1928

Maxwell equations

Saw of motion of photons

Dirac 1928

Maxwell equations

Saw of motion of strongly interacted particles

(Snell-Maxwell, 1970)

Problem: Explain physical interpretation of \( \frac{1}{2m} \vec{P} \cdot \vec{E} \) term.

Problem: Can write Dirac Hamiltonian

\[ H_0 = \psi \left( \sigma_0 \left( \vec{p} - \frac{e}{c} \vec{A} \right) \right) + \mu m \]

But from Q.M.

\[ \dot{\psi} = i \left( H_0 \psi - \psi H_0 \right) \]
An 5. can we find \( \mathbf{x} = \frac{(\mathbf{p} - \mathbf{a})}{m} \) \( \mathbf{p} = -\nabla \mathbf{V} + \ldots \)

Try this game with Dirac eqns.

\[ \text{Problem:} \quad \text{Levi-Civita} \{ \alpha, \beta, \gamma, \delta \}; \text{Find how many are really different (to get used to commutation laws)} \]

---

Now we want to study free particle Dirac solutions to try to interpret components of Dirac spinor:

\( \Psi = (\psi_0, \psi_1) \) and \( \Psi = \exp(\mathbf{i}(\mathbf{E} - \mathbf{p}.\mathbf{x})) \psi_{a,b} \)

\( (\mathbf{E} + \mathbf{m}) \psi_b = (\mathbf{t}^a \mathbf{p}) \psi_a \)

\( (\mathbf{E} - \mathbf{m}) \psi_a = (\mathbf{t}^a \mathbf{p}) \psi_b \)

P, E are numbers

Combined

\( (\mathbf{E}^2 - \mathbf{m}^2) \psi_a = (\mathbf{t}^a \mathbf{p}) (\mathbf{E} + \mathbf{m}) \psi_b = (\mathbf{t}^a \mathbf{p}) (\mathbf{t}^b \mathbf{p}) \psi_b = (\mathbf{p} \cdot \mathbf{p}) \psi_b \)

\[ \Rightarrow \quad \mathbf{E}^2 - \mathbf{m}^2 = \mathbf{p} \cdot \mathbf{p} \]

\[ \mathbf{E} = \mathbf{E}_0 = \sqrt{\mathbf{m}^2 + \mathbf{p} \cdot \mathbf{p}} \]

\[ = -\mathbf{E}_0 \]
Suppose $E = E_p$ and solve
\[ \psi_0 = \frac{1}{E_p + m} (\gamma_0 \gamma_p) \psi_a \]

Consider case where $p \cdot \gamma = m \rightarrow p \cdot \gamma = 0$
Also
\[ \sigma_z \psi_a = +\psi_a \quad (\psi_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \]

Then
\[ \psi_b = \frac{1}{m + E_p} \sigma_z \gamma \psi_a = \frac{p}{E + m} \quad (\psi_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \]

\[ \sigma_z \psi_b = +1 \psi_b \]

Spin is in up direction.

But you still want to normalize $\psi_b$.

Then find
\[ \psi = \frac{1}{\sqrt{2E_p}} \begin{pmatrix} \sqrt{E_p + m} \\ 0 \\ 0 \\ \sqrt{E_p - m} \end{pmatrix} \]

Particle in $z$ direction, up spin.

\[ \sigma_z \psi = +1 \psi \]

Now for spin down
\[ \psi = \frac{1}{\sqrt{2E_p}} \begin{pmatrix} 0 \\ \sqrt{E_p + m} \\ 0 \\ -\sqrt{E_p - m} \end{pmatrix} \]

Particle in $z$ direction, down spin.
Now for spin in $x$-direction, recall we have:

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus we expect

$$\sigma^x \psi = \psi$$

$$\Psi = \frac{1}{\sqrt{2E_0}} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E_0 + m} \\ \sqrt{E_0 + m} \\ \sqrt{E_0 - m} \\ \sqrt{E_0 - m} \end{pmatrix}$$

But

$$\sigma^x \Psi = -\Psi$$

What's the matter? $\Psi$ is certainly a solution of the eigen.

But $\sigma^x$ does not commute with $H$. Mentions Hithal.

Left definite concept.
We have seen that $[\alpha, H_0] \neq 0$, thus no definite
ang. mom in x-direction.

Now say
P not in z-direction

For case of positive helicity, we require

$$\left( \frac{\mathbf{r} \cdot \mathbf{p}}{m} \right) \psi = +\psi$$

Now
$$\alpha \psi = +\psi$$
where $\psi = \left( \begin{array}{c} \cos \frac{\theta}{2} e^{i\phi_2} \\ \sin \frac{\theta}{2} e^{-i\phi_2} \end{array} \right)$

Thus in r.g.m.

$$\psi = \sqrt{2E_\rho} \left( \begin{array}{c} \sqrt{E_\rho + m} \cos \frac{\theta}{2} e^{i\phi_2} \\ \sqrt{E_\rho - m} \sin \frac{\theta}{2} e^{-i\phi_2} \end{array} \right)$$

For positive helicity, $\alpha = +1$
Now consider $E = -E_p$

In z-direction, $\sigma_z = +1$

\[
\frac{1}{\sqrt{2E_p}} \begin{pmatrix}
-\sqrt{E_p - m} \\
0 \\
+ \sqrt{E_p + m} \\
0
\end{pmatrix}
\]

$\sigma_z = -1$

\[
\frac{1}{\sqrt{2E_p}} \begin{pmatrix}
0 \\
+ \sqrt{E_p - m} \\
0 \\
+ \sqrt{E_p + m}
\end{pmatrix}
\]

Note 4: orbiting spins for free particle (4 base states).

These components cannot be easily interpreted.

Eq. (3) has no simple meaning.

Prior proposal: all negative states in vacuum are full. A "hole" is a positive.
**Example:** Scattering cross-section of an electron from a nucleus of $\frac{Z}{e}$, use Born approx.

\[
H_0 = \chi_0 p + Bm + \frac{1}{4}\frac{Z}{e} \quad \text{where } \chi_0
\]

\[
\text{Rate} = \sigma_{\text{Born}} = 2\pi S(E_f - E_i) \frac{d^3p_2}{(2\pi)^3} |\psi_f|^2
\]

[Note relativity only enters into $H_0$, not into state density]

\[
M_{fi} = \int \psi_f^*(x) \chi_0 \psi_i(x) \text{d}^3x
\]

Now use free wave-functions

\[
\psi(x) = e^{i\mathbf{p} \cdot \mathbf{x}} u
\]

where $u$ is a complex number (again)

\[
M_{fi} = \int u_2^* e^{-i\mathbf{p}_2 \cdot \mathbf{x}} \frac{Z\mathbf{e}^2}{r} e^{i\mathbf{p} \cdot \mathbf{x}} u_1 \text{d}^3x
\]

\[
= (u_2^* \gamma_\mu u_1) \int e^{i(\mathbf{p}_2 + \mathbf{p}) \cdot \mathbf{x}} \frac{Z\mathbf{e}^2}{r} \text{d}^3x
\]

\[
\approx \frac{4\pi Z\mathbf{e}^2}{Q^2} u_2^* \gamma_\mu u_1
\]
Note that except for \((u_2, u_1)\), everything is the same as m.m.p. M. However actually we have differences m.s. for

\[ \sigma = 2\pi \int S(\sqrt{m^2 + p_2^2} - \sqrt{m^2 + p_1^2}) \frac{p_2^2 dp_2 d\omega_2}{12\pi^3} \left( \frac{4\pi^2 e^2}{Q^2} \right)^2 \left| (u_2, u_1) \right|^2 \]

\[ = \frac{1}{(2\pi)^2} \int \frac{\sqrt{m^2 + p_2^2}}{p_2} \ d\omega_2 \left( \frac{E_2}{p_2} \right)^2 \]

[Note \( S(E_2 - E_1) \frac{dE_2}{dp_2} = S(E_2 - E_1) \frac{dp_2}{E_2} \) ]

Now \( \frac{E_2}{E_1} = \frac{1}{p_2} \Rightarrow 1p_2^2 = 1p_1^2 \]

\[ Q = 2p_2 \sin \theta_{1/2} \]

Now to calculate \((u_2, u_1)\), we need more information about initial and final spins.
in state 1: 
\[ u_1 = \frac{1}{\sqrt{2E}} \begin{pmatrix} \sqrt{E_1 + m} \\ 0 \\ 0 \end{pmatrix} \]

- helicity 
\[ u_1 = \frac{1}{\sqrt{2E_1}} \begin{pmatrix} 0 \\ \sqrt{E_1 - m} \\ 0 \end{pmatrix} \]

and state 2: 
\[ u_2 = \frac{1}{\sqrt{2E_2}} \begin{pmatrix} \frac{E_2 + m}{\sqrt{E_2}} \cos \frac{\theta}{2} \\ \frac{E_2 + m}{\sqrt{E_2}} \sin \frac{\theta}{2} \\ \frac{E_2 - m}{\sqrt{E_2}} \sin \frac{\theta}{2} \end{pmatrix} \]

- helicity 
\[ u_2 = \frac{1}{\sqrt{2E}} \begin{pmatrix} \frac{E_2 + m}{\sqrt{E_2}} \sin \frac{\theta}{2} \\ \frac{E_2 + m}{\sqrt{E_2}} \cos \frac{\theta}{2} \\ -\frac{E_2 - m}{\sqrt{E_2}} \cos \frac{\theta}{2} \end{pmatrix} \]

\[ (+ \rightarrow +) = \frac{1}{\sqrt{2E_2}} \left[ \sqrt{E_1 + m} \cos \frac{\theta}{2} + 0 + \sqrt{E_1 - m} \cos \frac{\theta}{2} + 0 \right] = \cos \frac{\theta}{2} \quad \text{(no spin flip)} \]

\[ (+ \rightarrow -) = \frac{1}{\sqrt{2E_2}} \left[ \sqrt{E_1 + m} \sin \frac{\theta}{2} + \sqrt{E_1 - m} \sin \frac{\theta}{2} \right] = -\frac{m}{E} \sin \frac{\theta}{2} \quad \text{(spin flip)} \]

\[ (+ \rightarrow \text{unknown helicity}) \]

\[ \sigma^\gamma = \frac{4Z^2e^4}{Q^4} \text{Ep} d\Omega \left( \cos^2 \frac{\theta}{2} + \frac{m^2}{E^2} \sin^2 \frac{\theta}{2} \right) \]

\[ \text{relativistic modification} \]
Note as \( NE \to 1 \) we get usual Coulomb scattering, i.e.

\[
\cos^2 \theta_2 + \frac{m^2}{E^2} \sin^2 \theta_2 = 1 - \frac{m^2}{E^2} \sin^2 \theta_2
\]

To get \( \theta \to \text{anything} \) just use parity invariance of EM interactions to get \( \theta \to -\theta \).
Quantum Mechanics + Relativity = Causality

R.P. Feynman

Consequences
1. Independence
2. Spin (Statistics added)
3. CPT invariance (in definite way)
4. No arbitrary functions
5. Discrepancy problems

Theorem: In general we can expand $S(t)$ in $\omega > 0$

$$f(t) = \int \phi(\omega) e^{-i\omega t} d\omega$$

uniquely from $f(t) \equiv 0$ over a finite $t$.

(Identity Theorem)

We want to include causality in QM. What one dreams future cannot affect the past. We imagine the state of spin at a certain time $|\psi\rangle$. Consider second order scattering in a potential.
If \( \psi \) is a particle only and scattering is local

\[
\Psi_t \propto e^{i \left[ \frac{(E_3 - E_1) t_a}{m^2 + p_0^2} - (P_3^2 - P_1^2) \cdot x_a \right]}
\]

\[
E_3 = \sqrt{m^2 + p_0^2}
\]

\[
\text{e.g., for microwaves, } \alpha, \beta = \text{const.}
\]

\[
\begin{align*}
\text{e}^{-i} &\rightarrow \sqrt{\frac{d^3 p_3}{2m^3 \cdot 2E_3}} \\
A \text{scat} &= p_0 e^{i \beta \cdot b} e^{-i \beta \cdot a} \quad N(b-a) \quad \text{for } t_b > t_a \\
N(x) &= \sqrt{\frac{d^3 p}{(2\pi)^3 \sqrt{2\pi^2 p^2}}} e^{-i \left[ \frac{(x - P_1) \cdot P_3}{m^2 + P_3^2} - (P_3^2 - P_1^2) \cdot x_a \right]}
\end{align*}
\]
Causality can't mean \( N = 0 \) across time line because a guy in another causal system wouldn't see it this way.

\[ e^{i\varphi_1 b} \text{ is this way.} \]

If \( t_b > t_a \), this doesn't help matters.

If \( t_b < t_a \), it is undefined because \( t_b \geq 2t_a \).
Consider emission from a pt. We expect velocity of particle \( \leq c \)

\( i.e. \) no influence outside light cone. But if we try to

\[
\langle \psi | e^{i p \cdot x} e^{-i \sqrt{s} p^0} | \psi \rangle
\]

\[
\mathcal{S}(\mathcal{A}) = \int e^{i p \cdot x} d^4 p e^{-i \sqrt{s} p^0}
\]

But by our theorem, \( \mathcal{S} \) can't be zero for a finite
range, and locality is not compatible with velocity

(assumption).

Only one way out, and that is to take \( t_1 < t_2 \)

\[
\begin{align*}
\langle \psi | & e^{i P_2 \cdot b} e^{-i [E(t_2-t_1) - \mathbf{P}_2 \cdot (x_b - x_a)]} e^{-i P_1 \cdot a} \\
& = \langle \psi | e^{i P_2 \cdot b} e^{-i [E(t_2-t_1) - \mathbf{P}_2 \cdot (x_b - x_a)]} e^{-i P_1 \cdot a} \\
& \text{if } b \text{ is outside lightcone of } a,
\end{align*}
\]

we find form \( \langle x_1 | e^{i P_2 \cdot b} e^{-i [E(t_2-t_1) - \mathbf{P}_2 \cdot (x_b - x_1)]} e^{-i P_1 \cdot y_2} \)

to get invariant result & causality.
In vacuum, amps for pair produced in vacuum

\[ \text{pair produced} + \text{pair at rest} \rightarrow \text{amps} = \frac{1}{2} (x + i\alpha^1) |\alpha|^2 \]

\[ \text{prob} = 1 - x|\alpha|^2 \]

Note it doesn't add up, unless we assume amp for pair produced in vacuum is less than in presence of first pair.

\[ \text{Prob something happens} = |\alpha|^2 + x|\alpha|^2 + z|\alpha|^2 \]

\[ \Rightarrow \text{base statistics} \quad \text{(since we assumed no spin)} \]
"One man's particle is another man's antiparticle."

Outside light cone we have antiparticle. No freedom to vary from this idea.

This leads to \( C = PT \) invariance.

Not quite as direct, to get spin 1 statistics. Basic idea is unitarity \( \mathbb{E} \left( \langle a \rangle^2 \right) = 1 \). Consider first

\[
\begin{align*}
\text{Prob. scatt} &= C |a|^2 \\
\text{Prob. not scatt} &= 1 - \frac{1}{2} C |c + ic| |a|^2 \\
\end{align*}
\]

\( \mathbb{E}(\text{prob}) = 1 \) if \( C = C \).
Spin $\frac{1}{2}$: For particle standing still

\[ |+0> \]

\[ |-0> \]

For particle moving, use Lorentz transformations (lost me here). Shows spin $\frac{1}{2} \Rightarrow \text{pseudo-scalar}$.  

Now, pseudoscalar property happens

\[ c|\alpha|^2 + \chi|\alpha|^2 - 8|\alpha|^2 \]

\[ 1 - c|\alpha|^2 + 8|\alpha|^2 - \chi|\alpha|^2 \]

He has tried to invent a non-local theory (doesn't assume causality). Difficult.  

If locality fails, it will fail for all theories.

But CPT, causality looks OK so far.
In order to get rid of negative energy states, we must introduce antiparticles. Consider again 2\textsuperscript{nd} order scattering and sum over intermediate states.

\[ e^{i\beta b} e^{-i\beta b} e^{i\alpha a} e^{-i\alpha a} \]

valid only for $t_b > t_a$.

This amplitude should invariant, but the condition $t_b > t_a$ is not invariant. One way to avoid this is to remove the restriction $t_b > t_a$. Then we can use the diagram below.

But if we use it as we did the earlier diagram we find $E = E_i - E_b + E_a$ may be negative. But we cannot have $\varepsilon < 0$ since then vacuum would be unstable. Thus we must try something else.
Thus we must extend the formula (x) for negative times.

The correct way out is to notice in the sum (x) that

\[ N(k,t) = \frac{\alpha^2}{\sqrt{2\pi m^2 \hbar^2}} e^{-i(\sqrt{k^2 + \hbar^2} - \sqrt{\hbar^2 k^2} t)} \]

By \( R = \sqrt{k^2} > t, \) \( N \) real

Thus we can proceed for \( t_0 < t_a \) to use the same formula as for \( t_0 > t_a \)

\[ (x) = \ldots e^{-i(\mathcal{E}_3 t_0 - \mathcal{E}_2 t_a + i(\mathcal{E}_3 t_0 - \mathcal{E}_2 t_a))} + \ldots \]

except

\[ (x) = \alpha \delta \ldots e^{i(\mathcal{E}_3 t_0 - \mathcal{E}_2 t_a)} + \ldots \]

then the diagram is

Now causality requires us to continue to allow "backward scat" -- pair production -- for all situations (doesn't change about annihilation when pair is produced)

Now the formula (x) is relativistically invariant for \( t_0 > t_a \),

because \( N(k,t) \) is real in space-like regions.

The new diagram gives us new physics -- antiparticles.
Problem: Do Compton effect

Note: We consider only

\[ N = + + + \rightarrow \]

Actually, we must integrate

\[
\sqrt{d\tau d^3x_0 d^3x_1} \sum_{f_p} e^{i (P_f X_0) - i P_f (X_0 - X_0)} \left[ e^{i P_f X_1} - e^{-i P_f X_1} \right] \chi (X_{b, t_b}) \chi (X_{t_a})
\]

\[
\left[ e^{i (P_f X_1)} + e^{-i (P_f X_1)} - i P_f X_1 \right]
\]

\( t_b > t_a \)

\( t_b < t_a \)
To describe spin-\(3/2\) particles without hypothesizing the Dirac equation (which presumes antimatter), we have:

\[ [J_x, J_y] = i J_z \]

Also Lorentz generators \(L_i = i \gamma_i\):

\[ [J_x, N_3] = 0 \]

\[ [N_3, N_4] = -i J_z \]

\[ [N_3, J_y] = i N_2 \]

Also a momentum operator \(P\):

\[ [J_z, P_x] = i P_y \]

\[ [J_x, P_x] = 0 = [J_z, P_z] \]

\[ [N_3, P_z] = i P_z \]

\[ [N_2, P_z] = i P_z \quad \text{etc.} \]

Can try to find representation for these groups.

However, consider a particular case with the notation \(\gamma_i\):

State \(10, i\gamma\)
Consider uniform motion
\[ e^{i \tau N_z} |0, i\rangle \equiv |p_z, i\rangle \]

create state of uniform motion
\[ \tan \theta = \frac{P}{E} \]
\[ \sin \theta = \frac{P}{m} \]
\[ \cosh \theta = \frac{E}{m} \]

Can prove
\[ P_z |p_x p_y p_z, i\rangle = |p_z, p_x p_y p_z, i\rangle \]

and
\[ P_z e^{i N_z \phi} = e^{i N_z \phi} \left[ P_z \cosh \phi + P_z \sinh \phi \right] \]

The indices i give the spin in the rest system only.

A rotation
\[ J_z = i \left[ P_x \frac{\partial}{\partial P_y} - P_y \frac{\partial}{\partial P_x} \right] + j_z \]

and
\[ N_z = i \left[ \frac{P_x}{\mathcal{E}} \frac{\partial}{\partial P_x} - \frac{P_y}{\mathcal{E}} \frac{\partial}{\partial P_y} \right] + \left( \frac{\mathcal{I} \times \mathcal{P}}{\mathcal{E} + \mathcal{P}^2} \right)_z \]

He then used these to set up a system of states for spin \( \frac{1}{2} \) particle.
Now we want to apply this to scattering problems

\[ |\psi\rangle = \sum_{p} \left[ c_p |p\rangle + d_p \langle p| \right] \]

For elastic scattering

\[ |p_1\rangle \rightarrow |p_2\rangle \]

Use interaction \( f(p_1, p_2) + g \cdot F(p, p) \)

which includes the field.

We now consider scattering from a scalar potential.

Composition under \( N \cdot J \). E.g., \( f(p, p) \).

Finds out this is enough to determine 2 solutions.

\[
\frac{(E_1 + m)(E_2 + m) - p_1 \cdot p_2 - q^2 (p_1 \times p_2)}{\sqrt{2m(E_1 + m)} \sqrt{2m(E_2 + m)}} \quad \text{scalar}
\]

and

\[
\frac{(E_1 + m) (q \cdot p_2) - (E_2 + m) (q \cdot p_1)}{\sqrt{2m(E_1 + m)} \sqrt{2m(E_2 + m)}} \quad \text{pseudo-scalar}
\]
Now considered 2nd order scattering.

$$\frac{(E_3 + \mu)(\sigma_P \rho) - (E_3 + \mu)(\sigma_0 R)}{\sqrt{2\mu (E_3 + \mu) 2\mu (E_3 + \mu)}}$$

annihilation

$$\frac{(E_3 + \mu)(\sigma_0 F_0) - (E_3 + \mu)(\sigma_0 R)}{\sqrt{2\mu (E_3 + \mu) 2\mu (E_3 + \mu)}}$$

But turns out these against "fermion." Then this carries Fermi statistics. Note also these formulas look quadratic even through potential was scalar. Means positron has negative parity from statistics.

He said it for general $i$, $\psi_j$ even all $D_i$. We even $\mu$, e.g., are fermion, $\psi_j$ odd all powers of $D_i$ are odd. Non-fermion $\mu$ even statistics.
A simpler (and physically reasonable) argument comes when

\[ |+\rangle = e^{i\theta} |\downarrow\rangle \]

(\theta \text{ phase})

However \( T \) is stronger because it \( \frac{3\pi}{2} \) causes

\[ T \{ \alpha |a\rangle + \beta |b\rangle \} = \alpha^* T |a\rangle + \beta^* T |b\rangle \]

Also

\[ T |+\rangle = \eta |+\rangle \]

Actually, we fix either \( \eta = 1 \) or \( \eta = \frac{1}{2} \).

\[ T |+\rangle = |\uparrow\rangle \]

\[ T |\downarrow\rangle = \eta |+\rangle \]

Now

\[ T |+\rangle = e^{i\theta} \{ |+\rangle \} \]

\[ \frac{1}{\sqrt{2}} T \{ |+\rangle + |\downarrow\rangle \} = \frac{e^{i\theta}}{\sqrt{2}} \{ |+\rangle - |\downarrow\rangle \} \]

\[ |\downarrow\rangle + e |+\rangle = e^{i\theta} \{ |+\rangle - |\downarrow\rangle \} \]

\[ \Rightarrow T |+\rangle = |\downarrow\rangle \]

\[ T |\downarrow\rangle = -|+\rangle \]

(2 spin \( \frac{1}{2} \) states)

(2 times repeated discussion)


double time reversal theorem in a - phase for spin $\frac{1}{2}$

\[ T(T\uparrow\uparrow) = T\uparrow\uparrow = -|\uparrow\rangle \]

\[ T(T\downarrow\downarrow) = T(-|\uparrow\rangle) = -|\downarrow\rangle \]

Ospin

\[ T|0\rangle = e^{i\phi}|0\rangle \]

\[ T(T|0\rangle) = T(e^{i\phi}|0\rangle) = e^{-i\phi}T|0\rangle = |0\rangle \]

$\Rightarrow$ + phase for integer spin
COMPTON SCATTERING

This first calculation will be rather lengthy as we learn more we will streamline our calculations.

Photon scattering off of a free electron

\[ \gamma + e^- \rightarrow \gamma + e^- \]

First make a picture / define a coord system. We take lab system

Now

\[ p_1 = (m,0,0,0) \]

\[ \mathbf{k}_1 = \omega_1 (1,0,0,1) \]

\[ \mathbf{k}_2 = \omega_2 (1, \sin \Theta, 0, \cos \Theta) \]

\[ \mathbf{p}_2 = (E_2, p_2 \sin \varphi, 0, p_2 \cos \varphi) \]

Note

\[ \mathbf{p}_2 = \mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2 \]

Squaring

\[ m^2 = m^2 + 0 + 0 + 2 \mathbf{p}_1 \cdot \mathbf{p}_2 - 2 \mathbf{k}_1 \cdot \mathbf{k}_2 - 2 \mathbf{k}_2 \cdot \mathbf{k}_2 \]
or using our 4-vectors

\[ 0 = m \omega_1 - m \omega_2 - \omega_1 \omega_2 (1 - \cos \theta) \]

or

\[ \frac{1}{\omega_2} - \frac{1}{\omega_1} = \frac{1}{m} (1 - \cos \theta) \]

This was all just kinematics and is always done first.

Also add in polarizations

\[ e_1 = (0, 0, 1, 0) + \text{plane} \]
\[ = (0, 1, 0, 0) \text{ in plane} \]

\[ e_2 = (0, \cos \theta, 0, -\sin \theta) \text{ in plane} \]
\[ = (0, 0, 1, 0) + \text{plane} \]

[To get recoil electron, square

\[ \delta_2 = \delta_1 + \delta_1 - \delta_2 \]

to get \( 0 = \ldots \). etc.]
Now develop Klein-Nishina formula for $\sigma$:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi S(E_{\text{in}} - E_{\text{out}}) \frac{e^2 k_2}{(2\pi)^3}}{M^2} |M|^2$$

Now recall

$$H = \beta m + \alpha \cdot (p - \frac{e}{c} A) = H_0 + H_{\text{int}}$$

where

$$H_0 = \beta m + \alpha \cdot p$$

$$H_{\text{int}} = \frac{e}{c} \alpha \cdot A$$

Now recall

$$A = \frac{1}{\sqrt{1 - \beta^2}} \left( \psi_{\text{in}} e^{i k \cdot x} + a_{\text{out}} e^{-i k \cdot x} \right)$$

Then

$$M = \frac{\langle 2 | H_{\text{int}} | i \rangle \langle i | H_{\text{int}} | 1 \rangle}{E - E}$$

Write up to

$$\frac{d\sigma}{d\Omega} = \frac{2\pi S}{\sqrt{(\omega_2^2 - (1 - \gamma^2) k_2^2)^2 + (1 - \gamma^2) k_2^2}} \frac{k_2^2 \text{d}k_2 \text{d}^3x}{(2\pi)^3} |M|^2$$

$$\left( \sqrt{\omega_2^2 + k_2^2} \right)$$

We use

$$\int S[f(x)] \text{d}x = \frac{1}{f(0)}$$
\[
\frac{d}{d\mu_2} \left( \frac{\mu_2 - \mu_1 \cos \theta}{\sqrt{E_2}} + 1 \right) = \frac{\mu_2 - \mu_1 \cos \theta + E_2}{E_2} = \frac{E_1 + \omega_1 - \omega_1 \cos \theta}{E_2} = \frac{m + \omega_1 \sin (\frac{\theta_2 - \frac{1}{2})}{E_2}}{E_2} = \frac{m \omega_1}{E_2 \omega_2}
\]

Thus integrating over \( \mu_2 \):

\[
\text{Predicted} = \sigma_T = \sigma_C = \omega_2^2 \frac{\omega_1 E_2}{m \omega_1} \left( \frac{1 + \gamma E_2}{2m^2} \right) M^2
\]

Now for matrix element, we consider possible intermediate states:

\[
\langle \text{photon 2 scattered}\rangle \langle \text{photon 1 unscattered}\rangle \quad A
\]

\[
+ \langle \text{photon 1 scattered}\rangle \langle \text{photon 2 unscattered}\rangle \quad B
\]
Then
\[ A = \frac{\langle 2 | \alpha_1 \cdot e^{i k_2 x} | i \rangle \langle i | \alpha_0 e^{i k_1 x} | 1 \rangle}{E_1 + \omega_1 - E_1} \]
\[ B = \frac{\langle 2 | \alpha_2 e^{i k_2 x} | j \rangle \langle j | \alpha_0 e^{i k_1 x} | 1 \rangle}{E_1 - \omega_2 - E_3} \]

Can associated pictures with these

\[ \text{[Have actually calculated] } M \left( \frac{e}{\sqrt{2}} \right)^2 = A + B \]

For more elaborate problems we need is for references.

Note we have made these calculations in a conventional non-relativistic way -- and they certainly hint non-relativistically plane wave states.
Can write using Dirac plane waves.

\[ A = \sum_j \frac{\langle u_1^* | \alpha_2 | u_1 \rangle \langle u_1^* | \alpha_1 | u_1 \rangle}{E_1 + \omega_1 - |E'_j|} + \sum_j \frac{\langle u_2^* | \alpha_2 | u_1 \rangle \langle u_1^* | \alpha_1 | u_1 \rangle}{E_1 + \omega_1 + |E'_j|} \]

\[ B = \sum_j \frac{\langle u_1^* | \alpha_1 | u_1 \rangle \langle u_1^* | \alpha_2 | u_1 \rangle}{E_1 - \omega_2 - |E'_j|} + \sum_j \frac{\langle u_2^* | \alpha_1 | u_1 \rangle \langle u_1^* | \alpha_2 | u_1 \rangle}{E_1 - \omega_2 + |E'_j|} \]

But all negative \( E_j \) states are full. Need 2 new possibilities —

"point theory." (Full era.)
Poincare

\[
C = \begin{cases} 
1st & \text{photon } \Omega \text{ boosts electron in sea (neg. } k) \text{ to a positive energy state } \theta, \text{ leaving a hole at } k \\
2nd & \text{photon } \theta \text{ makes electron } \Omega \text{ go into negative hole } k_1.
\end{cases}
\]

\[
C = -\frac{1}{\hbar} \frac{\langle 2 | \alpha_1 e^{-i\hbar \omega_1 t/k} | k \rangle \langle k | \alpha_2 e^{-i\hbar \omega_2 t/k} | \rangle}{E_1 + \omega_1 - (E_1 + E_2 + 1E_1)}
\]

Finding \( F_2 = P_2 - \Pi \), \( P_1 - \Pi = \Pi \) before

Can also show denominator = \( \omega_2 - E_1 - 1E_1 \)
\[ \text{Rate} = \sigma \omega = 2\pi S(E_{\text{out}} - E_{\text{in}}) \frac{\hbar^3 k^2}{(2\pi)^3} |M|^2 \]

\[ \sqrt{2\omega_1 \omega_2} \frac{1}{e^2} M = A + B \]

\[ A = \sum_{\text{apm}} \frac{(u_2^* d_2 u_1^*)(u_1^* d_1 u_1)}{E_1 + \omega_2 - |E_2|^2 + i\epsilon} \]

\[ B = 2 \leftrightarrow 1 \]

\[ \nu_1, \omega_1 \leftrightarrow -\nu_2, -\omega_2 \]

Technically, the main difference in the last theory is in the \( i\epsilon \). Only important for resonances, but since degeneracy can't vanish in Compton scattering, we disregard this.

Now can pick polarization of electron + photons and calculate + write some algebra. We will go on to the short cuts however.
\[ (u^* v^i) (u_i^* w) \]

\[ \sum_{\alpha \beta} u^*_\alpha u_\beta u^*_\alpha u_\beta \]

\[ \lambda_{ab} = \sum_{\alpha \beta} u^*_\alpha u_\beta \]

\[ \exists (u_2^* \delta^* u_i) (\eta_i^* \delta u_i) \rightarrow (u_2^* \delta^* \lambda, \alpha_i^* u_i) \]

This if we can find \( \lambda \) we shall use to const. We will work it out \( p \) to make \( z \)

\[ E = \sqrt{\rho^2 + \rho^2} \]

\[ u = \frac{1}{\sqrt{2E}} \begin{pmatrix} \sqrt{E + \rho} \\ 0 \\ \sqrt{E - \rho} \end{pmatrix} \]

\[ u \lambda u^*_\theta = \frac{1}{2E} \begin{pmatrix} \begin{matrix} \sqrt{E + \rho} \\ 0 \\ \sqrt{E - \rho} \end{matrix} \end{pmatrix} = \frac{1}{2E} \begin{pmatrix} \frac{E + \rho}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E - \rho & 0 \\ 0 & -\rho & 0 & E - \rho \end{pmatrix} \]

\[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \lambda = \frac{1}{2E} \begin{pmatrix} E + \mu & 0 & 0 & 0 \\ 0 & E + \mu & 0 & -\rho \\ -\rho & 0 & E - \mu & 0 \\ 0 & 0 & 0 & E - \mu \end{pmatrix} \]

\[ = \frac{1}{2E} (E_1 + \mu \beta + \alpha_2 \rho) \]

\text{In general,}

\[ \lambda = \frac{1}{2E} (E_1 + \mu \beta + \alpha_2 \rho) \]

\text{Then}

\[ A = \frac{u_2^* \alpha_2 (|E_i| + \mu \beta + \omega \circ P_i) \alpha_i U_i}{2 |E_i| (E_i + \omega_i - |E_i|)} \]

\[ + \frac{u_2^* \alpha_2 (E_i - \mu \beta - \omega \circ P_i) \alpha_i U_i}{2 |E_i| (E_i + \omega_i + |E_i|)} \]

\[ E_i = \sqrt{\mu^2 + (\rho_i + \kappa_i)^2} \]
Note we might have deduced this by using

$$\langle 2|A_1|1\rangle \langle 1|B_1|1\rangle = \langle 2|A B|1\rangle$$

But note also

$$\sum (u_2 \times N) \left( \frac{H_{12} + (E_{11}) u_1}{2E_{11}} \right) (u_1 \times N u_1)$$

$$= \sum (u_2 \times N u_1) (u_1 \times N u_1)$$

$$= \frac{u_2 \times N (E_1 + H) N u_1}{2E_1}$$

This is what we could have done to get $\eta$, used concept of perturbation operator $|E_1 + H, \eta\rangle$. (Namir)

Next add terms to find

$$A = \frac{u_2 \times \vec{a}_2 \left( E_1 + \omega, -i\beta - \alpha \cdot \vec{P}_i \right) \chi_1 u_1}{(E_1 + \omega)^2 - |E_1|^2}$$

We now find a way to write this directly.
\[ A = \frac{\alpha_2 \epsilon_2 (E_1 + \omega_1 + \alpha \Omega (p + ik_x) + i \beta) \alpha_1 \epsilon_1}{(E_1 + \omega_1)^2 - (p + ik_x)^2 - \mu^2} \]

Note how this appears relativistically. How can we make the invariance obvious by cleaning up Dirac notation.

Recall we had:

\[ \gamma + = \beta \]

\[ \gamma x = \beta \alpha_x \quad \gamma y = \beta \alpha_y \quad \gamma z = \beta \alpha_z \]

so we can rewrite Dirac equations

\[ (E \gamma - \beta \alpha_0 \gamma_0) u = m u \]

\[ P_+ \gamma \alpha \]

or

\[ P_+ \gamma \alpha \quad u = m u \]

Thus put A in terms of \( \gamma \)'s.
\[ A = \frac{(u_1 + \kappa \delta_3)(\rho_\mu + \kappa \mu)\delta_\mu + \mu}{(\rho_\mu + \kappa \mu)(\rho_\mu + \kappa \mu) - \mu^2} \]

Can assume \( \gamma \cdot e_i = \gamma \mu \varepsilon_i \) where \( \varepsilon_i = (0, \ldots, 0) \)

Define \( u^* \beta = \bar{u} \)  

relativistic conjugate

\[ u^* = \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \]

\[ \bar{u} = u_1^* u_2^* - u_2^* - u_4^* \]

and

\[ \gamma \mu \varepsilon_i = \phi \]

Then

\[ A = \frac{\bar{u}_2 \delta_2 (\rho_\mu + \kappa \mu)\delta_\mu + \mu}{(\rho_\mu + \kappa \mu)^2 - \mu^2} \]

Note: "strength of intermediate state depends on"

\[ \frac{1}{(\rho_\mu + \kappa \mu)^2 - \mu^2} \]
Problem:
Positron annihilation \( e^+ + e^- \rightarrow 2\gamma \) (denucleon-nucleon)
Estimate lifetime of pionium
Dineutron polarization charachteristics

Compton Effect (continued)

Note \( B \) is topologically different from \( A \). Now write amplitudes in new notation

\[
A = \frac{(\bar{u}_2 \gamma^2 (\not{p}_1 + \not{p}_2 + m)) \gamma \not{u}_1}{(\not{p}_1 + \not{p}_2)^2 - m^2}
\]

\[
B = \frac{(\bar{u}_2 \not{e}_1 (\not{p}_1 - \not{p}_2 + m) \not{e}_3 u_1)}{(\not{p}_1 - \not{p}_2)^2 - m^2}
\]

Can we figure out a way to write these directly? Recall notation

\[
g_{\mu \nu} = a_{\mu} a_{\nu} = a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 - a_5 b_5
\]

\[
\gamma_\mu = (\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5)
\]

\[
\not{u}_\mu = (\beta, \beta v_1, \beta v_2, \beta v_3, \beta v_4)
\]

\[
\gamma_\mu \not{u}_\nu = a = a_1 \gamma_1 \gamma_4 v_1 - a_2 \gamma_2 \gamma_4 v_2 - a_3 \gamma_3 \gamma_4 v_3
\]

\[
= \beta (a_+ - \not{a} \cdot \not{a})
\]
\[ A = \nabla \psi = \psi + \frac{\partial \psi}{\partial t} + \frac{i}{\hbar} \frac{\partial \psi}{\partial x} + \frac{i}{\hbar} \frac{\partial \psi}{\partial y} + \frac{i}{\hbar} \frac{\partial \psi}{\partial z} \]

\[ = \beta [\frac{\partial}{\partial t} + \omega \cdot \nabla] \]

Now
\[ \phi^2 = g \mu \cdot g \]
\[ \phi = \psi \Rightarrow g \mu = b \mu \]

Sometimes he uses \( \bar{\phi} \) for 4-vectors also \((\mu, \bar{\phi})\)
\[ \phi \bar{\psi} = \frac{1}{2} (a \cdot b) - b \bar{\phi} \]
[expresses commutation laws of \( \psi \)'s]
\[ \phi \bar{\psi} + \bar{\psi} \phi = 2a \]
\[ \bar{\psi} \psi + \psi \bar{\psi} = 2a \]

The Dirac equation now becomes
\[ (i \nabla - \frac{\gamma}{\hbar} \nabla - m) \psi = 0 \]

Ex: \( \psi = -i \chi \)
\[ \bar{\psi} = m \phi \text{, etc.} \]

Current Density
\[ j_\mu = \bar{\psi} \gamma_\mu \psi \]

\[ \bar{\psi} = \psi^* \phi^* - \psi^* \phi \]

\[ \psi^* \phi^* \text{ density} \]
\[ \psi^* \phi \text{ current} \]
Now our matrix element looks relativistic invariant.

1.) But $U_1$ and $U_2$ are not strictly invariant because of normalization.

2.) Formula for proof is: $\rho = \text{space density of}\ e$

3.) Real issue of $e$

1.) Normalization: we have been looking

$$ u^* u = 1 = \bar{u} \cdot u = a \cdot u $$

This rep. 1 particle/unit vol. But not invariant.

Our new normalization will be:

$$ \bar{u} \cdot u = 2m $$

[2m instead of 1 to get rid of $k^2$ factor in front of operator]

$$ \bar{u} \cdot (\gamma + u) = 2E = u^* u $$

Also change $M$ as

$$ M = M \sqrt{2E \sqrt{\frac{2E}{2m}, \sqrt{2m}} } $$

$\sqrt{2E}$ energy of every free particle
2) Our old formula was

\[
\text{Rate} = 2\pi \delta(E_{p^+} - E_i) \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_1 E_2 \omega_1 \omega_2} |M|^2
\]

\[
= \frac{(2\pi)^4 \delta(E_{p^+} - E_i) S^2 (P_2 - P_1) \frac{d^3k_2}{(2\pi)^3 2\omega_2}}{2E_1 \omega_1} \frac{d^2P_2}{(2\pi)^3 2\omega_2} \frac{d^2P_2}{(2\pi)^3 2E_2} \frac{1}{|M|^2}
\]


\[
\frac{dP_2}{E} = \frac{dP_1}{E} \quad \text{for} \quad P_2 \neq P_1
\]

Actually rate isn't conserved due to \( \frac{1}{E \omega} \) for excited states

\[
2\pi \delta(p^2 - m^2) \frac{d^4p}{(2\pi)^4} \quad \text{as} \quad \text{invariant differential}
\]

This in general

\[
\text{Rate} = \frac{1}{(2\pi)^4 \delta(E_{p^+} - E_i) S^4 (4 \text{momentum in} - 4 \text{momentum out})}
\]

\[\begin{array}{c}
\frac{d^3\text{Real part}}{2E_1 	ext{in and out}} \quad \frac{d^3\text{Born}}{(2\pi)^3 2E_2 \text{in and out}} \frac{1}{|M|^2}
\end{array}\]

\[\text{End. materialism}\]
Rate = \left( \frac{1}{2m} \right) \left( \frac{1}{2E_{\text{in}}} \right) \cdot \text{Density of initial states} \cdot \text{in forward range} \cdot 2\pi \cdot \left| W \right|^2

= \left( \frac{1}{2m} \right) \left( \frac{1}{2E_{\text{in}}} \right) (2\pi)^4 \cdot \delta^4 \left( \Delta p_{\text{in}} - \Delta p_{\text{out}} \right) \cdot \int_{\text{out}} 2\pi S(p^2-m^2) \frac{d^3p}{(2\pi)^3} \cdot \left| W \right|^2

\text{where} \quad 2\pi S(p^2-m^2) \frac{d^3p}{(2\pi)^3} = \frac{d^3p}{(2\pi)^3}

\text{Compton scattering yields}

M = \frac{\bar{\mu}_2 \phi_2 \left( p + k - \omega \right) \phi_1 U_1}{(p + k - \omega)^2 - m^2} + \frac{\bar{\mu}_2 \phi_1 \left( p - k + \omega \right) \phi_2 U_1}{(p - k + \omega)^2 - m^2}

The main idea was to write the rate in a manner in which the relativistic invariance is obvious. Note for 2 particles in

Rate = 0 \quad \text{Total} \quad \text{Total} = |V_1 - V_2|

The cross-section should be invariant. Note from earlier this means \text{Total} E_1E_2 is invariant since the rate transforms as \( \vee E_1E_2 \). Can try to demonstrate this.

For only one particle in,

Rate = \frac{\gamma}{c}, \quad \gamma = \text{lifetime}
In our expression for $M$, \( \frac{p_1 + k_1 + \mu \gamma}{(p_1 + k_1)^2 - m^2} \) is called the propagator of a virtual photon. In general for an electron propagator,

\[
\frac{\gamma + \mu \gamma}{\gamma^2 - m^2 + i\epsilon}
\]

For interaction with a photon of polarization \( \epsilon \),

\[
\omega = \sqrt{4\pi \epsilon} \mu
\]

The amplitude for a free electron to enter is \( \mu \), \( \bar{\psi} \), and leave is \( \psi \).

Thus we can write the matrix element directly:

\[
\langle \text{amp. for electron} | \text{amp. for electron} | \text{prop. for electron \& \gamma} | \text{phot.} | \text{amp. for electron} \rangle
\]

\[
\bar{\psi}_2 \gamma_2 \frac{(p_1 + k_1 + \mu \gamma)}{(p_1 + k_1)^2 - m^2} \bar{\psi}_1 \psi_1
\]
Note the propagator is usually written as
\[ \frac{1}{\xi - m} \]
on the reciprocal of a number

Must solve \((\xi - m)^2 = 1\)

\[
\frac{(\xi - m)(\xi + m)}{\xi^2 - m^2} = \frac{\xi^2 m^2}{\xi^2 - m^2} = 1
\]

\[ \Rightarrow \frac{1}{\xi - m} = \frac{\xi + m}{\xi^2 - m^2 + i \epsilon} \]

Thus the propagator is in general the reciprocal of a Dirac operator. Consider

\[(i \mathcal{D} - m) \psi = S \]
as Dirac eqn for free particle

Take F.T.

\[(\xi - m) \psi = S \]
or \[ \psi = \frac{1}{\xi - m + i \epsilon} S \]

Thus a mnemonic is to regard each interaction as a source of wave

\[ \overline{U}_2 \phi_2 \frac{1}{\xi + \mathbf{k}_2 - m} \phi_1 U_1 \]
Consider
\[(i\nabla - k \cdot \mathbf{m}) \psi = 0\]
or
\[(i\nabla - \mathbf{m}) \psi = k \psi\]

Solve by Neumann series
\[\psi_0 = \phi, \quad \text{with approx } \approx (i\nabla - \mathbf{m}) \phi = 0\]

Next solve
\[(i\nabla - \mathbf{m}) \psi_1 = k \phi_1\]

\[\psi_1 = \phi_1 + \frac{1}{i\nabla - \mathbf{m}} k \phi_1\]

and so on until
\[\psi = \phi + \frac{1}{i\nabla - \mathbf{m}} k \phi + \frac{1}{i\nabla - \mathbf{m}} k \frac{1}{i\nabla - \mathbf{m}} k \phi + \ldots\]

This is just a perturbation expansion.

Thus it is important to learn these tricks because they make the calculations so much simpler.

Also idea of representing a scattering as a 2-nd order process
\[\ldots = 0\]
In general for \( A_\mu = 0 \), \( \varphi = 0 \)

\[
A_\mu = e_\mu e^{-i k \cdot x} = e_\mu e^{-i (\omega t - k \cdot x)}
\]

\( \nabla \cdot A_\mu = 0 \Rightarrow \nabla \cdot k = 0 \quad \varphi = 0 \Rightarrow \epsilon_+ = 0 \quad \epsilon_+ \epsilon_- = 1 \)

But note

\[
\epsilon_\mu k_\mu = \epsilon_\mu \omega = \epsilon_\mu k = 0
\]

\[
\epsilon_\mu \epsilon_\nu = -1
\]

But expect that we could Lorentz transform to a frame where \( \epsilon_+ \neq 0 \). How do we get right answer? This has to do with an additional property of electrodynamics and concerns gauge invariance.

\[
A''_{\mu} = A_{\mu} + \partial_{\mu} \chi
\]

Has nothing to do with Lorentz invariance. It is true for Dirac Eq.

\[
(i\gamma^\mu \nabla - \mu)\psi = 0
\]

\[
(i\gamma^\mu \nabla - \mu \nabla \chi \cdot \gamma^\nu \nabla - \mu)\psi'' = 0
\]

Hence \( \psi'' = e^{i\chi} \psi \) we get same physics. Thus gauge invariance introduces a phase.
Substituting in
\[ \xi_\mu (i \gamma_\mu - A_\mu') e^{iX} \psi - m e^{-iX} \psi = 0 \]

\[ e^{-iX} [ \xi_\mu (+\nabla_\mu + i \partial_\mu - A_\mu') \psi] = m e^{-iX} \psi \]

or
\[ \xi_\mu [i \partial_\mu - (A_\mu' - \nabla_\mu)] \psi = m \psi \]

\[ \underline{\underline{A_\mu}} \]

Thus rule is to use
\[ A''_\mu = A_\mu + \nabla_\mu x \]

\[ \psi'' = e^{iX} \psi \]

Thus support we have phase wave
\[ A_\mu = e_{\mu} e^{-i k_{\mu'} x} \]

\[ A''_\mu = e''_{\mu} e^{-i k_{\mu'} x} = A_\mu + \nabla_\mu x \]

\[ = e_{\mu} e^{-i k_{\mu'} x} + b_{\mu} e^{-i k_{\mu'} x} \]

Thus \( e''_\mu = e_{\mu} + b_{\mu} \) without changing physics.

Can always use the gauge invariance to get rid of the time component \( e_t \). Thus can always wipe out the scalar potential.

Kettler demonstrated this on Compton's muon decay.
\[ M^{new} = M^{old} + \alpha \bar{u}_2 \bar{k}_2 \frac{1}{P_1 + K_1 - \mu} \bar{u}_1 K_1 \bar{u}_1 + \bar{u}_2 k_1 \frac{1}{P_1 - K_2 - \mu} \bar{u}_1 K_1 \bar{u}_1 \]

\[ \bar{u}_2 \bar{k}_2 \frac{1}{P_1 + K_1 - \mu} \bar{u}_1 K_1 \bar{u}_1 - \bar{u}_2 \bar{k}_2 \frac{1}{P_1 - K_1 - \mu} \bar{u}_1 K_1 \bar{u}_1 = 0 \]

\[ \Rightarrow M^{new} = M^{old} \Rightarrow \text{formula is gauge invariant} \]

Note both terms in the matrix element are of equal significance in determining gauge invariance. Thus it seems that our diagram is "larger" than another one we are considering.

In general, a simple diagram is relativistic invariant with not gauge invariance.

Need all diagram for gauge invariance.
We now double arithmetically of the Complex formula.

\[ p_1 = \omega(1,0,0) \]
\[ p_2 = k_2 = \omega_2(1,0,0,1) \]
\[ k_1 = \omega_1(1,0,0,0) \]

Recall \( \frac{1}{\omega_2} - \frac{1}{\omega_1} = \frac{1}{w} (1 - \cos \theta) \)

Also need coord:
\[ p_2 + k_2 = p_1 + k_1 \implies \omega_2 = \omega_1 (E_2 - P_2 \cos \theta) \]

Now recall
\[
M = \frac{\bar{u}_2 x (p_1 + k_1 + w)}{(p_1 + k_1)^2 - w^2} + \frac{\bar{u}_1 x (p_1 - k_1 + w)}{(p_1 - k_1)^2 - w^2}
\]

\[
\tilde{E}_N = \frac{\bar{u}_2 x (p_1 + k_1 + w) x_1 u_1}{\omega_1} - \frac{\bar{u}_1 x (p_1 - k_1 + w) x_2 u_1}{\omega_2}
\]

Possible cases:
\[ u_{1+} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ pos z-spin} ; \quad u_{1-} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ neg z-spin} \]

\[ u_{2+} = \begin{pmatrix} \sqrt{E_{-x+y} \cos \theta} \sin \theta \delta_2 \\ \sqrt{E_{-x+y} \cos \theta} \cos \theta \sin \theta \\ \sqrt{E_{-x+y} \cos \theta} \sin \theta \delta_2 \\ \sqrt{E_{-x+y} \cos \theta} \cos \theta \sin \theta \end{pmatrix} \text{ positive} ; \quad u_{2-} = \begin{pmatrix} \sqrt{E_{-x+y} \sin \theta} \delta_2 \\ \sqrt{E_{-x+y} \cos \theta} \sin \theta \delta_2 \\ \sqrt{E_{-x+y} \sin \theta} \sin \theta \delta_2 \\ \sqrt{E_{-x+y} \cos \theta} \cos \theta \sin \theta \end{pmatrix} \text{ negative} \]

\[ \theta_{ab} = \delta_x \quad \theta_{bb} = \delta_y \]

\[ \theta_{2a} = \delta_x \cos \theta + \delta_z \sin \theta \quad \theta_{2b} = \delta_y \]

(\text{plane pol}) (\text{plane pol})
\[ g_1 = \omega t \quad g_1 = \omega_1 (x_1 - x_2) \quad g_2 = \omega_2 (x_1 - x_2 (\cos \theta - x_2 \sin \theta)) \]

Thus we find \( M \) for 16 different cases. We do the case \( u_1, u_2, e_1, e_2 \).

\[ A = \frac{U_{21} (x_1 \cos \theta - x_2 \sin \theta)(x_1 + x_2 + x_1 \cos \theta + x_1 \sin \theta)}{\omega_1} \]

\[ = \frac{U_{21} (x_1 \cos \theta - x_2 \sin \theta)(x_1 + x_2 + x_1 \cos \theta + x_1 \sin \theta)}{\omega_1} \]

Prepare:

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \]

\[ \gamma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \gamma_1 \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \gamma_2 \gamma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad i \vec{1} \gamma_1 = \vec{1} \gamma_1 = \vec{1} \gamma_2 = \vec{1} \gamma_3 = \vec{1} \gamma_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \]

\[ \left( i \vec{1} \gamma_1 \gamma_2 \gamma_3 \right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{where} \quad \vec{1} = i \vec{1} \gamma_1 \gamma_2 \gamma_3 \gamma_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \]

\[ A = \frac{U_{21} (x_1 \cos \theta - x_2 \sin \theta)(x_1 + x_2 + x_1 \cos \theta + x_1 \sin \theta)}{\omega_1} \]

Note: \( g_1 u_1 = u_1 \)

\[ = \frac{U_{21} (x_1 \cos \theta - x_2 \sin \theta)(-1 + x_2)}{\omega_1} u_1 + \]

\[ = \frac{U_{21} (-\cos \theta - x_2 \sin \theta)(-1 + x_2)}{\omega_1} u_1 + \]

\[ = \frac{U_{21} \left( \sqrt{E_{21} \cos \theta - x_2 \sin \theta} \right) \left( \sqrt{E_{21} \cos \theta - x_2 \sin \theta} \right)}{\omega_1} \left( \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) \]

\[ = \frac{U_{21} \left( \cos \theta + \sqrt{E_{21} \cos \theta - x_2 \sin \theta} \right) \left( \sqrt{E_{21} \cos \theta - x_2 \sin \theta} \right)}{\omega_1} \left( \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) \]

\[ = \left( \sqrt{E_{21} \cos \theta - x_2 \sin \theta} \right) \left( \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right) \sqrt{2} \mu \]
Unillatiztize Form of Q.E.D. (many electron systems) 4/23/67 Lecture

\[ H = \frac{1}{8n} \int \left( E^2 + B^2 \right) dV + \int \left[ \Phi^*(x) \left[ \beta m + e \phi(x) + \alpha^+ \left( \frac{\hbar}{i} \vec{\nabla} - e \vec{A} \right) \right] \Phi(x) \right] d^3x \]

\[ \text{Energy of electron} \]

\[ = \frac{1}{8n} \int \left( E^2 + B^2 \right) dV + \int \left[ \Phi^*(x) \left[ \beta m + \alpha^+ \left( \frac{\hbar}{i} \vec{\nabla} - e \vec{A} \right) \right] \Phi(x) \right] d^3x \]

\[ \text{Interaction of electron with photon} \]

\[ + \sqrt{\int \Phi^*(x) \left[ \Phi(x) - \alpha^+ A \right] \Phi(x) d^3x} \]

Hint of vanishing field with matter

Now assume \( \nabla \cdot A = 0 \)

\[ \left[ \nabla^2 \phi = e \frac{\phi}{\hbar} \right] \]

Then write

\[ A(x) = i \frac{\Phi}{\hbar} \left( e_{i \lambda} \alpha_{\lambda} e^{iK \cdot x} + e^{*}_{i \lambda} \alpha^{\ast}_{\lambda} e^{-iK \cdot x} \right) \]

\( e_{i \lambda} \) = polarization vector of photon of norm \( K \)

\( i = \) index of photon pol. (2 values)

\( \alpha_{\lambda} \) = annihilation ops. for photons
Then

\[ H = \sum_{\text{Hilb}} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Phi_{\text{Hilb}} + \int \Phi^* \left[ \psi_{\text{Hilb}} + q - \frac{\partial}{\partial \rho} \right] \Phi \, ds \]

\[ - \frac{e^2}{2} \int \frac{\psi^* \psi \psi^* \psi}{\mathbf{r}_{xy}} \, d^3 \mathbf{r} \]

\[ - e \int \frac{\psi^* \psi \cdot \mathbf{A}}{\hbar c} \, d^3 \mathbf{r} \]

Now write

\[ \Psi(x) = \sum_{\mathbf{p}, \mathbf{r}} C_{\mathbf{p}, \mathbf{r}} U_{\mathbf{p}, \mathbf{r}} e^{i \mathbf{p} \cdot \mathbf{x}} \]

\[ \Psi^*(x) = \sum_{\mathbf{p}, \mathbf{r}} C^*_{\mathbf{p}, \mathbf{r}} U_{\mathbf{p}, \mathbf{r}}^* e^{-i \mathbf{p} \cdot \mathbf{x}} \]

where \((\psi_{\text{Hilb}} + q \cdot \mathbf{A}) U_{\mathbf{p}, \mathbf{r}} = \mathcal{E} U_{\mathbf{p}, \mathbf{r}}^* \)

where \( \mathcal{E} = +E_0 \) or \(-E_0 \)

Use indeed

\[ U_{\mathbf{p}, \mathbf{r}} = \text{eigen functions for (energy solution) \& spin index \& values} \]

\[ \mathcal{E}_{\mathbf{p}, \mathbf{r}} = \text{(- energy solution)} \]
Then

\[ H_{d, pos} = \sum_{P_k} E_k C_{IP_k} C_{IP_k} \]

\[ = \sum_{P_k} E_k C_{IP_k} C_{IP_k} - \sum_{P_j} E_j C_{IP_j} C_{IP_j} \]

Define

\[ C_{IP_k} = \text{means amplitudes of罗马文字 up to \( IP_k \)} \]

\[ C_{IP_j} = d^*_{IP_j} \text{ position of \( IP_j \)} \]

\[ d_{IP_j} = \text{amplitudes of \( IP_j \)} \]

\[ H_{d, pos} = \sum_{P_k} E_k C_{IP_k} C_{IP_k} + \sum_{P_j} E_j d^*_{IP_j} d_{IP_j} - \sum_{P_j} E_j \]

Now

\[ H_{d, pos, phases} = -e^2 \sum_{P_k, P_j, \phi, \theta, \omega} \left[ \frac{\hbar^2}{2m} \left[ \begin{array}{c} u_{IP_k}^* \phi \theta \omega \end{array} \right] \begin{array}{c} C_{IP_k} \end{array} \right] \begin{array}{c} C_{IP_k} \end{array} \cdot \begin{array}{c} C_{IP_j} \end{array} \cdot e^{iP_{IP_k} K} \cdot \delta_{(P_{IP_k} - K \omega)} \] (1)

\[ -e^2 \sum_{P_k, P_j, \phi, \theta, \omega} \left[ \frac{\hbar^2}{2m} \left[ \begin{array}{c} u_{IP_k}^* \phi \theta \omega \end{array} \right] \begin{array}{c} C_{IP_k} \end{array} \right] \begin{array}{c} C_{IP_k} \end{array} \cdot \begin{array}{c} C_{IP_j} \end{array} \cdot e^{iP_{IP_j} K} \cdot \delta_{(P_{IP_j} + K \omega)} \] (2)

[3 more terms below this, very diagrammatical. Diagrammatically]
Now have 16 terms from Heaviside with diagrams like
Pair Production

Our diagrams are

$$\sqrt{4\pi \alpha^3} \left( \bar{U}_2 U_1 \right) \frac{1}{q^2} \left( U_2 U_1 \right)^{1/2} \rho R_{K-m}$$

For considering greens of proton, replace ($\bar{U}_2 U_1$) by

$$\bar{p} \left( g + \frac{m}{\alpha \pi} (g - g_K) F_2(q) \right) n_1$$

where is current operator

Bremsstrahlung

Assume fixed potential $V(R) = \frac{Z e^2}{R}$ usually
since most amp. comes from small $q^2 \Rightarrow$ nucleus not affected

Also a diagram:

Diagram 1:

Diagram 2:
\[
\frac{\frac{eQ_i}{2m_i} - \frac{1}{2} \left( \frac{\langle \eta_2 \rangle_{\eta_{\mu_i}}}{(q + x - q)^2} \right)}{P_i + q - H} \times (4\pi e^2)^{3/2}
\]

Note prominent formula is same as demonstrating.

P. D.

Mentioning others or cascade problems.
\[ \frac{4\pi e^2 (\vec{u}_3 \cdot \vec{u}_4) + \vec{u}_5 \cdot \vec{u}_4 + \vec{u}_3 \cdot \vec{u}_5}{(P_2 - P_1)^2} \]

and so on.

If position error, expect some in final order. If potential acts as additional friction.

[signature]
Bremsstrahlung (from potential scattering \( \sigma_I (\omega) \))

\[
\begin{align*}
q_b &= p_2 + k_z - p_1 \\
\end{align*}
\]

[Scattering with no emission]

\[
\begin{align*}
\text{Rate} &= \frac{2\pi}{(2\pi)^3 2E_1 2E_2} \left| \frac{\langle \bar{u}_2 \sigma_m u_1 \rangle}{(\omega^2 - Q^2 + i\epsilon)} \right|^2 \\
\end{align*}
\]

For both processes above

\[
\begin{align*}
\text{Rate} &= 2\pi S(E_2 + \omega - E_1) \frac{d^3 P_2}{(2\pi)^3 2E_2 2E_1} \frac{d^3 k}{2\omega (2\pi)^3} |M|^2 \\
M &= u_2 \phi \frac{1}{p_2 + k - \omega} \phi u_1 + u_2 \Phi \frac{1}{p_1 - k - m} \phi u_1 \sqrt{4\pi\epsilon^2} \\
&= u_2 \phi \frac{Q^2 + k + \omega + m}{2p_2 \cdot k} \phi u_1 + \bar{u}_2 \Phi \frac{Q - k + \omega}{-2p_1 \cdot k} \phi u_1 \\
\end{align*}
\]

We want to consider low \( \omega \) behavior (\( \omega \ll p_2 + m \))
Using commutation law \[ A \cdot \phi = -i \hbar \partial \phi - 2i \mathbf{a} \cdot \mathbf{b} \]

\[
W = \frac{\langle \mathbf{u}_2 \cdot \partial_x \rangle e^{i P_1 \cdot x} + 2P_1 \cdot e + \phi(x) \phi(x)}{2 P_1 \cdot k} \mathbf{u}_1 + \langle \mathbf{u}_2 \phi \mathbf{u}_1 \rangle \frac{2 P_1}{-2P_1 \cdot k}
\]

\[
= \langle \mathbf{u}_2 \phi \mathbf{u}_1 \rangle \left\{ \frac{P_2 \cdot e}{P_2 \cdot k} - \frac{P_1 \cdot e}{P_1 \cdot k} \right\} \sqrt{4\pi e^2}
\]

This is the matrix element for scattering without emission of photons.

Thus:

\[
\text{Rate} = \frac{2\pi \delta(E_2 - E_1) \left( \frac{3\hbar}{2m_0^2 E_2^2 E_1} \right) (\mathbf{u}_2 \phi \mathbf{u}_1)^2}{(2\pi)^3 \left( \frac{\hbar}{P_1 \cdot k} \right)^2 \left( \frac{\hbar}{P_2 \cdot k} \right)^2}
\]

Treat with no photon emitted.  

\[ \text{Prob. that if}\]

\[ \text{scattering occurs between}\]  

\[ \text{the two photon,}\]  

\[ \text{it is just a coll.}\]  

\[ \text{photon}.\]

\[ \text{Prob.} \]

\[ \text{it is}\]  

\[ \text{independent of process}\]  

\[ \text{Prob.}\]
To analyze in more detail, consider coordinates

\[ \mathbf{p}_1 \cdot \mathbf{k} = E_1 \mathbf{\Omega} - \mathbf{p}_1 \mathbf{\omega} \cos \theta_1 = \omega E_1 (1 - \mathbf{v}_1 \cos \theta_1) \]

where \( \mathbf{v}_1 = \frac{\mathbf{p}_1}{E} \)

\[ \mathbf{p}_1 \cdot \mathbf{e} = -\mathbf{p}_1 \cdot \mathbf{e} = -E \mathbf{v}_1 \cdot \mathbf{e} \]

Then

\[ \left\{ \begin{array}{c}
\mathbf{v}_1 \cdot \mathbf{e} \\
\mathbf{v}_2 \cdot \mathbf{e}
\end{array} \right\} = \frac{\mathbf{v}_1 \cdot \mathbf{e}}{1 - \mathbf{v}_1 \cos \theta_1} - \frac{\mathbf{v}_2 \cdot \mathbf{e}}{1 - \mathbf{v}_2 \cos \theta_2} \]

\[ = \frac{\mathbf{v}_1 \sin \theta_1}{1 - \mathbf{v}_1 \cos \theta_1} = \frac{\mathbf{v}_2 \sin \theta_2}{1 - \mathbf{v}_2 \cos \theta_2} \]

pol. in plane of collision

Can get same result from classical physics

PROBLEM: Emission of two independent photons.
Classical Bremsstrahlung:

Instead of using the Ward-Wilson potentials, do it a different way. Consider a path in 4-space. Now to write current density

\[ J_\mu(x) = e^{iZ_\mu(s)} \delta^4(x - Z(s)) ds \]

Now must solve \( \Box^2 A_\mu = j_\mu \). Also define F.T.

\[ j_\mu(k) = \sqrt{A_k(x)} e^{-ik \cdot x} d^4 x \]

Now take F.T. of \( j_\mu \),

\[ \hat{J}_\mu(k) = \sqrt{e^{-ik \cdot x} j_\mu(x)} d^4 x \]

\[ = \int e^{-ik \cdot x} \int \widehat{Z}_\mu(x) \delta^4(x - Z(s)) ds \] \( d^4 x \)

\[ = \int \widehat{Z}_\mu(k) e^{-ik \cdot Z(s)} ds = j_\mu(k) \]

For \( \alpha < 0 \), \( Z_\mu(\alpha) = P_\mu \alpha \)

\( \alpha > 0 \), \( Z_\mu(\alpha) = P_\mu \alpha \)

Thus

\[ j_\mu(k) = \int_{-\infty}^{0} P_\mu e^{-ik \cdot P_\mu \alpha} d\alpha + \int_{0}^{\infty} P_\mu e^{-ik \cdot P_\mu \alpha} d\alpha \]
\[ J_A(k) = i \left( \frac{p_1 \mu}{p_1 \cdot k} - \frac{p_2 \mu}{p_2 \cdot k} \right) \]

which agrees with what we found from Q.E.D.

Classical expected energy in \( dk \) = \( \epsilon C(\omega) d\omega \). In q.m., mean energy liberated is \( \langle \epsilon \rangle C(\omega) d\omega \). But mean no. of photons is \( \epsilon < C(\omega) > \). Like Poison distribution:

\[ \frac{(\langle n \rangle)^n}{n!} e^{-\langle n \rangle} \]
Group

\[ U(x,y) = \xi(x,y) \frac{\partial f}{\partial x} + \eta(x,y) \frac{\partial f}{\partial y} \]

Once-extended group

\[ U(x,y,y') = \xi(x,y) \frac{\partial f}{\partial x} + \eta(x,y) \frac{\partial f}{\partial y} + \eta'(x,y,y') \frac{\partial f}{\partial y'} \]

where

\[ \eta'(x,y,y') = \frac{D}{Dx} \eta(x,y) - y' \frac{D}{Dx} \xi(x,y) \]

\[ = \frac{\partial \eta}{\partial x} + \left( \frac{\partial y'}{\partial y} - \frac{\partial \xi}{\partial x} \right) y' - \frac{\partial \xi}{\partial y} y'^2 \]

The general form of the ODE is

\[ \xi(x,y,y') = 0 \quad \text{--- a surface} \]

We want \( U \xi = 0 \) for invariance

\[ \xi \frac{\partial \xi}{\partial x} + \eta \frac{\partial \xi}{\partial y} + \eta' \frac{\partial \xi}{\partial y'} = 0 \quad \text{(1)} \]

We know there can be two independent integrals of \( U(x,y,y') \), \( U(x,y,y') \). The general solution is some

\[ W(u,v) = 0 \]

More explicitly \( W(u,v) = v - w(u) = 0 \)
We are free to choose

\[ u(x, y, y') = u(x, y) \iff \delta \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} = 0 \]

so

\[ \delta \frac{dx}{x} = \frac{dy}{y(x'c)} \Rightarrow u(x, y) = \text{const.} = c \]

\[ \Rightarrow y = y(x; c) \]

Suppose path curves \( u(x, y) = \text{const.} \) are common. Then we can solve

\[ \frac{dx}{x} = \frac{dy'}{y_x + (\eta_y - \delta x)y' - \delta y' y^2} \Rightarrow u(x, y'; c) = c, \]

or

\[ \frac{dy'}{dx} = \frac{\delta y'}{\delta x} x + \frac{1}{\delta (\eta_y - \delta x)} y' - \frac{\delta y'}{\delta y} y^2 \]

\[ = F(x; c) + F_1(x; c) y' + F_2(x; c) y^2 \]

\[ = \text{General form of Riccati equation} \]

Then we write

\[ y'(x) = y_0(x) + \frac{1}{g(x)} \]

where \( y_0(x) = \frac{\eta}{\delta} \) is particular solution

\[ \delta y' - \eta = 0 \]

or

\[ (\delta x + y'' \delta y) y' + \delta \frac{dy'}{dx} - (\eta x + y' y_x) = 0 \]
Now if we substitute \((x)\) into \((x)\) we get a linear equation for \(g(x)\):

\[
\frac{d}{dx} g(x) = -F_1(x,c)g(x) - 2y_p F_2(x,c)g(x) - F_2(x,c)
\]

This can be integrated by quadratures to find \(g(x)\).

---

An alternative method is to introduce canonical coordinates: \(v(x,y), s(x,y) \Rightarrow \nabla f = \frac{\partial f}{\partial x} = 0\).

Now ODE is \(W(v,s, \frac{ds}{dr}) = 0\). To admit translation group we require

\[
\frac{ds}{dr} - F(s) = 0
\]

Now, go back to \((x,y)\), we\(\text{func}\)

\[
\frac{s_x + s_y y'}{r_x + r_y y'} - F(r_{xy}) = 0
\]
Example: Similarity Transformation \( Uf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \)

Then \( \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 0 \frac{\partial f}{\partial y} = 0 \)

\( Uf = \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \) as extended group

Now to find \( u(x,y), v(x,y) \) which satisfy

\[ x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 0 \frac{\partial f}{\partial y} = 0 \]

\( \frac{dx}{x} = \frac{dy}{y} \Rightarrow u = \frac{y}{x} = \text{const.} \)

By inspection take \( v = s(y) = y', \) for instance.

\( \Rightarrow \quad y' = \frac{dy}{dx} = F\left(\frac{y}{x}\right) \)

is general form of ODE admitting \( U. \)

Example: Rotation Group \( Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \)

\( Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} + (1+y^2) \frac{\partial f}{\partial y} \)

To find invariants

\( \frac{dx}{-y} = \frac{dy}{x} = \frac{dy'}{1+y^2} \)

find \( v(x,y) = x^2+y^2 \)

take \( \tan^{-1} y' - \sin^{-1} \frac{y}{x} = \tan^{-1} v \)
\[ v = \tan(\tan^{-1} y - \tan^{-1} \frac{y'}{x}) = \frac{y' - y}{1 + y'y/x} = \frac{x y' - y}{x + y y'} = f(x + y) \]

In this way you can use different groups to generate a table of 1st order ODE which admit these groups.
Second Order ODE's

Now two parameter family of curves
(2 initial conditions or b.c.)

\[ \omega(x, y; a, b) = 0 \]

where \( a, b \) are essentially independent

Invariant under point transformations

\[ x_1 = \varphi(x, y) \]
\[ y_1 = \psi(x, y) \]

\[ T[\omega] = \omega(x_1, y_1; a, b) = 0 \]

Example: all st. lines in plane

\[ \omega(x, y; a, b) = y - ax - b = 0 \]

Should be invariant under rotation:

\[ x = x_1 \cos \theta + y_1 \sin \theta \]
\[ y = y_1 \cos \theta - x_1 \sin \theta \]

Can verify this by direct calculation.
Now consider \( \omega(x, y; a, b) = 0 \)

\[
\begin{align*}
\frac{\partial}{\partial x} \omega &= \omega_x + \psi_y y' = 0 \\
\frac{\partial^2 \omega}{\partial y^2} &= \omega_y y'' = 0
\end{align*}
\]

can eliminate a \( \frac{1}{b} \) from these to obtain some 2nd order ODE

\[
\Omega(x, y, y', y'') = 0
\]

Cannot talk about existence of this ODE \( \exists \)

\[
\Omega(x, y, y', y'') = 0 \quad \text{[\( \exists \) is some function \( \Psi \)]}
\]

Now use extended group concept.

\[
\begin{align*}
x_1 &= \Phi(x, y) \\
y_1 &= \Psi(x, y) \\
y_1' &= \frac{\Phi_x + \Psi_y y'}{\Phi_x + \Psi_y y'} = \lambda(x, y, y') \\
y_1'' &= \frac{\Phi'' + 2\Phi_y y' + \Psi_y y''}{\Phi_x + \Phi_y y'}
\end{align*}
\]

Need also

\[
y'' = \frac{\Phi''}{\Phi_x} = \frac{\Phi''}{\Phi_x + \Phi_y y'}
\]
Need to construct twice extended infinitesimal transformation

$$U'' f(x, y, y', y'')$$

$$U'' f = \xi(x, y, y') \frac{\partial f}{\partial x} + \eta(x, y, y') \frac{\partial f}{\partial y} + \eta'(x, y, y') \frac{\partial f}{\partial y'} + \eta''(x, y, y, y'') \frac{\partial f}{\partial y''}$$

Now

$$\eta' = \frac{D_x \eta - y' \frac{D_x}{D_y}}{D_x}$$

Can also write

$$\eta'' = \frac{D_x \eta' - \eta'' \frac{D_x}{D_y}}{D_x}$$
Example - $\mu^-$ scattering

\[ P_2 = (m_\mu, 0, 0, 0) \]
\[ P_1 = (E, p, 0, 0) \]
\[ P_4 + P_3 = P_1 + P_2 \]
\[ P_1^2 = m^2 \]
\[ P_2^2 = m^2 \]
\[ P_3^2 = m^2 \]
\[ P_4^2 (\lambda_1, \mu_1, \nu_1, \theta_1, 0) \]

Then
\[ R = \frac{1}{2 \omega E} \sum_{\Delta m} (2\pi)^4 \delta^{(4)}(E_{\mu} - E_{e\nu}) \delta (P_3^2 - m^2) \delta (P_4^2 - m^2) \frac{1}{(2\pi)^4} \frac{d^4 P_4}{(2\pi)^4} \]

\[ \frac{1}{m^2} \]
The rate for emission of some soft photons

\[ \text{Rate} = \frac{3\pi}{(2\pi)^3} \frac{\left( \frac{1}{2} g_{\mu\nu} \right)^2 \tilde{C}_{\mu\nu}^2}{2E_0} \]  

\[ \text{Rate depends on}\]  
\[\text{radiation coupling}\]

A more elaborate calculation gives

\[ \sigma_{\text{photo}} = C \]  

\[ \sigma_{\text{photo}a} = \sigma_{\text{photo}a} \left( \frac{g_{\mu\nu}}{\alpha} \right)^2 \]  

\[ \sigma_{\text{photo}a, b, c} = \sigma_{\text{photo}a} \int \left( \frac{g_{\mu\nu}}{\alpha} \right)^2 f(\theta) d\theta \sum \frac{1}{\alpha} \text{ for soft photons} \]

\[ \text{Tract integral here} = \sigma_{\text{hard}c} c \]

Now will find

\[ c = e^{-\sqrt{\alpha} f(\theta) d\theta} \]  

\[ \text{for soft photons} \]

\[ \text{As } \alpha \to 0, \text{ integral diverges} \]

However for an experiment with finite parameters, find

\[ \sigma_{\text{photo}} \sim \sigma_{\text{DISORDER}} \text{ where } E > A \]

\[ \sigma_{\text{photo}} \sim \sigma_{\text{DISORDER}} c = \alpha \left( \frac{E}{A} \right) \]

\[ \alpha \sim \frac{1}{137} \]

This was originally called the 'infrared catastrophe'...
c is merely the result of a dumb question.

**BACK TO DIAGRAMS**

Consider e-e scattering

\[ M = \sqrt{4\pi e^2} \frac{(u_4 \Gamma_{2ud}) (u_3 \Gamma_{2ud}^*)}{(p_1-p_3)^2} \]

But there is an exchange possibility. Must also consider

\[ -\sqrt{4\pi e^2} \frac{(u_3 \Gamma_{2ud}) (u_4 \Gamma_{2ud}^*)}{(p_1-p_4)^2} \]

Now consider something cute, e+e- scattering

\[ M = \sqrt{4\pi e^2} \frac{(u_4 \Gamma_{2ud}) (u_3 \Gamma_{2ud}^*)}{i(p_1-p_3)^2} \]
But also can have annihilation + pair production

\[ W = \sqrt{-m^2 - (p_2 \cdot k_1 + k_2 \cdot p_1)} \]

Thus matrix elements are the same for e-e+ as for e-e- scattering.

Kind of suggests a generalization of the exclusion principle.

---

**Technical Trick**

Suppose we don't want to measure exit spin. Then we need

\[ \sum_\text{2pions} (U_2 N u_1)^* = \sum_\text{2pions} (U_2 N u_1)^* (U_2 N u_1) \]

Now define \( \bar{N} = (\bar{g} N f)^* = (\bar{f} N g)^* \) (relativistic definition)

Then

\[ \sum_\text{2pions} (U_i \bar{N} U_2) (U_2 N u_i) \]

\[ \lim \mu \]
\[ |\bar{u}_2 N u_1|^2 = (\bar{u}_1 N (\rho + m) N u_1) \]

Can also write
\[ |(\bar{u} N u_1)|^2 = Tr \left[ (\rho + m) N (\rho + m) N \right] \]

To find \( \bar{N} \) where
\[ N = \chi B \theta \]
\[ \bar{N} = \frac{1}{\chi} u^* \phi^* \]
Recall

\[ H_0 = \beta m + \alpha \cdot (\mathbf{P} - \mathbf{A}) + V \]

Now calculate velocity operator

\[ \hat{x} = i (H_0 \hat{x} - \hat{x} H_0) = \alpha \hat{x} \]

But \( \alpha \hat{x} \hat{x} = 1 \). \( \Rightarrow \) e.v. are ±1. Thus if you measure velocity in \( x \)-direction, you get ±\( \pm c \)?

Dirac's explanation using momentum measurements: If you make measurements over long times,

\[ \langle (\mathbf{P} + \mathbf{T} - \mathbf{P}_0 - \mathbf{T})^2 \rangle \sim \frac{p}{\sqrt{p^2 + m^2}} \quad \text{if} \quad T > \frac{\hbar}{mc^2} \]

Whole idea is that Dirac eqn does not specify motion of a single electron, but rather a combination of electron & position motion.

No one has been able to interpret \( \alpha \hat{x} \alpha \hat{x} = 1 \) directly. Other interpretations trouble with

\[ \frac{\hbar}{2} (\alpha \alpha \alpha \alpha) \], etc.

Only one interpretable is \( \hat{P} = \hat{E} + \alpha \times \mathbf{B} \)

No one has ever made a detailed study of how to interpret the Dirac eqn. Not just a one-electron answer.
How do we know Dirac cone is relativistically invariant.

\[ \gamma_\mu (\gamma_\mu - \alpha_\mu - m) \psi = 0 \]

If the \( \psi \) is used \( \Rightarrow \) and if the \( \psi \) is used
\[ \gamma_x' = \gamma_x \]
\[ \gamma_y' = \gamma_y \]
\[ \gamma_z' = \gamma_z \]
\[ \gamma_t' = \gamma_t \]

Then all measurements will be ok.

Suppose Mr. Professor uses
\[ \gamma_x' = \gamma_x \]
\[ \gamma_y' = \gamma_y \]
\[ \gamma_z' = \gamma_z \]
\[ \gamma_t' = \gamma_t \]

Must show Mr. Professor gets same answer as Mr. Why. This is challenge.

How unique are vertexes \( \gamma \).

Set
\[ \gamma_x \Rightarrow \gamma_x' \]
\[ \gamma_y \Rightarrow \gamma_y' \]
\[ \gamma_z \Rightarrow \gamma_z' \]
\[ \gamma_t \Rightarrow \gamma_t' \]

and ask question to whether physics can be maintained.

Note have same problem in rotations.
Problem: mass corrections to electron m from higher
order self-interaction

\[ 2m \Delta m = \left( \frac{e^2}{2m} \right) \sqrt{\frac{C \gamma_\mu (P - k + m) \gamma_\mu (u)}{(P - k)^2 - m^2}} \frac{d^4 V}{k^2 (2\pi)^4} \]

Try to derive this from non-relativistic thinking

\[ \Delta E = \frac{\varepsilon}{n} \frac{H_{nm} H_{ni}}{E_n - E_i} \]

[Easy to do in relativistic notations]

---

Basis to our examination of \( \Phi \)-matrices. Consider

\[ [\gamma_\mu (i\gamma_\nu - A_\nu) + m] \psi = 0 \]

Consider \( \psi'' = S \psi \quad S = S(\psi') \)

Then

\[ [\gamma_\mu (i\gamma_\nu - A_\nu) + m] \psi'' = 0 = [\gamma_\mu (i\gamma_\nu - A_\nu) + m] S \psi = 0 \]

or

\[ [S^{-1} \gamma_\mu S (i\gamma_\nu - A_\nu) + m] \psi = 0 \]

\[ \Rightarrow \] physics one comes \( \Phi \)

\[ \gamma_\mu'' = S \gamma_\mu S^{-1} \]
That is, all transformation matrices will give same physics. All commutation relations hold if
\[ S = S^{-1} \] "rotationally unitary"

where \( S \) defined by

\[ (\mathbf{M} g)^* = (g^t \mathbf{M}^g) \] \text{relativistic adjoint}

\[ \Rightarrow \quad \mathbf{M} = g \mathbf{M}^t g^t \]

In the standard representation of Dirac and
\[ \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \overline{\mathbf{M}} = \begin{pmatrix} a^t & -c^t \\ -b^t & d^t \end{pmatrix} \]

Note these means
\[ \overline{\overline{\mathbf{x}}} = \mathbf{x} \] \( \mathbf{x} \text{ matrices are self-adjoint relativistically} \)

Example with
\[ x^+ = (\cosh w) x^+ - (\sinh w) x^- \]
\[ x^x = x^x \]
\[ x^y = x^y \]
\[ x^z = (\cosh w) x^z - (\sinh w) x^+ \]

where
\[ \frac{1}{c} = \tanh w \]
\[ \cosh w = \frac{1}{1-w^2} \quad \sinh w = \frac{w}{\sqrt{1-w^2}} \]
Fluid \( S = e^{\frac{1}{2} \mathbf{v} + \mathbf{g}} \)

Recall in rotation:
- \( \mathbf{y}' = \mathbf{y} \)
- \( \mathbf{x}' = \mathbf{x} \cos \theta + \mathbf{y} \sin \theta \)
- \( \mathbf{y}' = -\mathbf{x} \sin \theta + \mathbf{y} \cos \theta \)
- \( \mathbf{z}' = \mathbf{z} \)

Conserved

\[ J_z = \left( p \times \mathbf{r} \right)_z + \frac{1}{2} \mathbf{v} \times \mathbf{v} \cdot \mathbf{v} = L_z + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \]
Techniques

If we have to sum

$$\sum_{\psi_i} (\bar{u}_3 A u_i) (\bar{u}_3 B u_3)$$

Note that we had to sum over

$$\sum_{\psi_i} (\bar{u}_3 A u_i) (\bar{u}_3 B u_3) = (\bar{u}_3 A B u_3)$$

$$\sum_{\psi_i} (\bar{u}_3 A u_i) (\bar{u}_3 B u_3) = \bar{u}_3 (A (p_1 + m) B) u_3$$

$$\sum_{\psi_i} |(\bar{u}_3 H u_i)|^2 = \sum_{\psi_i} (\bar{u}_3 M (p_1 + m) M (p_2 + m) u_3$$

$$= 8p [M (p_1 + m) M (p_2 + m)]$$

Now

$$M : \text{ reverse order of } \psi_i$$

Change applied $i \rightarrow \bar{i}$

Useful information: $2p \Gamma_1 = 4 \quad 2p \Gamma_4 = 0 \quad \Gamma_5 = 0$
\[ S_p[AB] = S_p[BA] \]

\[ S_p[ABC] = S_p[CAB] + S_p[CBA] \]  \[ \text{[must maintain cyclic order]} \]

\[ S_p[a] = 0 \]  \[ \phi = \phi \text{ mod } 2 \pi \]

\[ S_p[a \cdot b] = 4(a \cdot b) \]

\[ S_p[\phi \cdot \phi \cdot \phi] = 0 \]

\[ S_p[\phi \cdot \phi \cdot \phi] = 4 \sum (a \cdot b)^2 + (b \cdot c)^2 + (c \cdot d)^2 = (a \cdot c)(b \cdot d) \]

\[ \sum \text{ over all possible polynomial } \phi \]

\[ \sum_{\phi} S_p[\phi \cdot \phi \cdot \phi] = \frac{1}{2} \sum (a \cdot b)(c \cdot d) + (b \cdot c)(a \cdot d) - (a \cdot c)(b \cdot d) \]

\[ \text{from gauge invariance} \]
Summary of Rules for Perturbation Calculations

\[ \text{Product transition} = \frac{2\pi}{\alpha c} \langle 2E \rangle \int \frac{d^4p}{(2\pi)^4} \langle 2E \rangle \] \[ |M|^2 \]

\[ = (2\pi)^4 \delta^4(\vec{p} - \vec{p}') \int \frac{d^4p}{(2\pi)^4} 2\pi S(p^2 - m^2) \frac{1}{(2\pi)^4} \]

Now for calculation of matrix element $M$

$\epsilon_i \epsilon'_i = -1$

$a_i = 2m$

Sum over states because for one electron incident

$2p \left[ (p^2 + m^2) \bar{M} (p^2 + m^2) \right]$

Electron propagator \[ \frac{1}{p^2 - m^2 + i\epsilon} \]

Photon propagator \[ \frac{1}{p^2 + i\epsilon} \]

Meson propagator (technically a Smatrix) \[ \frac{1}{p^2 - m^2 + i\epsilon} \]
Complains

Real photon \phi

Potential \phi(x) \equiv \sqrt{e^{i \phi x} A(x) \, d^4 x}

Closed loop (fermion) \quad - 2p

Virtual photon \quad \frac{\epsilon}{\bar{\epsilon}} \quad - \frac{\epsilon}{\bar{\epsilon}}

\sqrt{4 \pi e^2} \text{ for each photon coupling}

\sqrt{4 \pi g^2} \text{ for each meson coupling}

Pseudoscalar meson \quad \bar{\psi} \psi

Scalar meson \quad \bar{\psi} \psi

Sum each indeterminate momentum by \sqrt{4 \pi p}

To get phases right:

\begin{align*}
&-i \quad \text{each electron propagator} \\
&-i \quad \text{each photon} \\
&+i \quad \text{spin zero meson} \\
&-i \quad \text{photon coupling}
\end{align*}

Energy \quad 2E = \text{expectation values of } \epsilon \bar{\epsilon} \quad \text{W}
In normal per. th.

\[ M_{si} = \frac{H_{in} H_{ni}}{E_i - E_n} \]

Also

\[ \Delta E_i = M_{ir} = \frac{H_{in} H_{ni}}{E_i - E_n} \]

Thus if we adapt rate formulas to calculate diagonal element, fluid

\[ \Delta \text{Energy} \cdot 2E = \text{correction value of } i \text{ M} \]
Self-Energy of Electron

While moving, a lepton can change its energy due to an interaction with E \& H field.

Note momentum \( \mathbf{p} \) is not determinable.

\[
2E \Delta E = \sqrt{\frac{\mathbf{k} \cdot \mathbf{v}}{(2\pi)^4}} \frac{\mathbf{v} - \mathbf{v} - M + i \epsilon}{\mathbf{p} - \mathbf{p} - M + i \epsilon} \frac{1}{\mathbf{k}^2 + i \epsilon} \frac{4\pi e^2}{(\mathbf{k}^2 + i \epsilon)^2}
\]

\((-1)(-1)(-1)^2 i\ i\)
PROBLEM: $\beta$-decay

$\mu \rightarrow e + \bar{\nu}_e + \nu_e$

Suppose a 4 pt. interaction

\[ \mathcal{M} = \frac{\overline{u} d g a u}{(\bar{d} c f)} \sqrt{g} \]

where \( a = \frac{1 + i \delta}{2} \)

From experiment

\[ \frac{G M_p^2}{\hbar c} = 1.01 \times 10^{-5} \]

For our example

\( (\overline{\nu}_e) (\overline{\nu}_\mu) = (\overline{\nu}_e \nu_a u_e) (\overline{\nu}_\mu \nu_a u_\mu) \)

But this leads to $\mu + \bar{\mu} \rightarrow e + \bar{e}$, however, using antiparticle we get right result
Find spectrum & lifting of $\mu$-decay, and also demonstrate non-conservation of parity.

Notice that this is lowest order contribution from process -- we suppress $\pi$'s, etc. Actually if we do not assume a contact interaction, get a intermediate interaction.

Suppose we can represent this by current operators

$$M = G_F \, J \mu^\dagger \, J \mu$$

where

$$J \mu = (\bar{u}_\mu \gamma_\mu u_e) + (\bar{u}_\mu \gamma_\mu u_e)$$

$$J^* \mu = (\bar{u}_\mu \gamma_\mu u_e) + (\bar{u}_\mu \gamma_\mu u_e)$$

Note there introduce a number of other processes like

$$e + \nu_e \rightarrow e + \nu_e$$

$$e + \bar{\nu}_e \rightarrow e + \bar{\nu}_e$$

$$e + \bar{\nu}_e \rightarrow e + \nu_e$$

$$e + \bar{\nu}_e \rightarrow e + \nu_e$$
We need the strong for $N \rightarrow P + e + \bar{\nu}_e$. Use things like:

\[
\langle \text{proton} | J_\mu | \text{medium} \rangle = (\bar{u}_p \left[ \frac{\kappa u + 1.26 i \kappa u}{2} \right] u_n)(1.97)
\]

Also, E-H conserving of $O(1\%)$.
\[ \Delta(m^2) = -4\alpha e^2 i \int \left( \frac{1}{p - k - m + i\epsilon} \frac{\gamma \mu}{k^2 - m^2 + i\epsilon} \right) \frac{d^4 k}{(2\pi)^4 k^2 + i\epsilon} \]

\[ = -4\alpha e^2 i \int \frac{2m^2 + 2m k}{k^2 - 2p k} \frac{d^4 k}{k^2 (2\pi)^4} \]

\[ \sim \frac{2m^2 d^4 k}{k^4} + \frac{4(\alpha e)^2}{k^2} \frac{d^4 k}{k^4} \]

\[ \sim \int \frac{3m^2 d^4 k}{k^4} \]

Logarithmic integral

This would cut this integration off at \( \Lambda \)

\[ \sim \frac{3\alpha^2}{\pi} \ln \frac{\Lambda}{m} \]

Infinite correction to QED. End of theory.

How can we keep going for calculations? Any problem with potential indeterminant momentums gives us's. Some ways to stop integration.

Multiply integrand by \( C(k^2) = C \rightarrow 0 \) for \( k > \Lambda \).

E.g.

\[ C(k^2) = -\frac{\Lambda^2}{k^2 - \Lambda^2} \]
First that all physically measurable processes do not depend on $\Lambda$, this can take $\Lambda \to 0$ and get invariant predictions of QED.

Here is to use

$$m^2 = m_0^2 + \frac{3e^2 m_0}{\pi} \ln \frac{\Lambda}{m} + \ldots$$

to cancel odd logs before taking limit $\Lambda \to 0$.

[Mass renormalization]. Current state of research.
Scattering in a potential to $O(e^2)$

\[ \phi(q), \quad q = q_2 - q_1 \]

\[ I = \int \left( \bar{u}_2 \frac{\delta}{\delta \phi} \phi \right) \left( \frac{1}{p_2 - k - m} \right) \left( \frac{1}{p_1 - k - m} \right) \frac{d^4 k}{k^2 (2\pi)^4} \]

Complications arising for any free particle problem

\[ II = \bar{u}_2 \left( \frac{d}{dp_1} \frac{4\pi e^2}{p_1 - m} \int \frac{1}{p_1 - k - m} \frac{d^4 k}{(2\pi)^4} \right) \frac{C(k^2)}{12 \pi^4} \]

\[ \equiv \bar{u}_2 \delta \phi \frac{p_1 + m}{(M^2 - m^2)} \]

\[ \equiv \bar{u}_2 \delta \phi \frac{M_{\text{eff}} - m^2}{M_{\text{eff}}^2} \]

\[ \equiv \bar{u}_2 \delta \phi \frac{p_1 + m}{(M_{\text{eff}}^2 - m^2)} \]

\[ \equiv \bar{u}_2 \delta \phi \frac{p_1 + m}{(M_{\text{eff}}^2 - m^2)} \]

\[ \equiv \bar{u}_2 \delta \phi \frac{p_1 + m}{(M_{\text{eff}}^2 - m^2)} \]
\[ 4\pi e^2 \left( \overline{u}_2 \right) \frac{1}{q^2} \int d^4p \left[ \frac{1}{p-m} \frac{1}{p+q-m} \phi \right] \frac{d^4p}{(2\pi)^4} \]

Need a new type of cutoff (dimensional regularization)

Subtract 3 terms

\[ 2p \left[ \frac{1}{p-M} \frac{1}{p+q-M} \phi \right] - 2p \left[ M_2 \right] - 8p \left[ M_3 \right] \]

As before, this is

\[ \begin{array}{c}
\text{Use} \\
\frac{e^2_{\text{ep}}}{1 - (\ln 4\pi) e^2_{\text{nu}}} \\
\text{as charge renormalization}
\end{array} \]

See any book.
We were calculating

\[ r = \frac{4\pi e^2}{(2\pi)^2} \sqrt{\frac{\mu}{P - \mu m}} \frac{P^2}{m} \left( \frac{1}{P - \mu m} \right) \frac{d^2 k}{k^2} C(k^2) \bigg|_{n = u} \]

Consider the probability of the virtual process for long-time propagation

\[ \frac{1}{1 + \frac{1}{P - m_0} (a \phi + b) \frac{1}{P - m_0} + \ldots} = \frac{1}{P - m_0 - (a \phi + b)} \]

Want to examine this near pole

\[ \frac{1}{P - M} \]

where

\[ f = \frac{m_0 + b(M^2)}{1 - a(M^2)} \]

Examine

\[ \frac{1}{1 - a(\rho)} \]

\[ \frac{1}{P - f(\rho)} \]

\[ \frac{1}{1 - a} \left[ \frac{p + f(p)}{p^2 - f(p)^2} \right] \]

Suppose \( M \) satisfies \( M = f(M^2) \). For \( a, b \) small

\[ M = m_0 + b(m_0^2 + m_0 a(m_0^2)) = m_0 + [m_0 a(m_0^2) + b(m_0^2)] \]

just old formula for mass correction
Use \( f(p^2) = M^2 + 2M(p^2 - M^2)f'(M^2) + \ldots \)

\[
= \frac{1}{1 - \alpha} \frac{p^2 + f'(M^2)}{(p^2 - M^2)[1 - 2Mf'(M^2)]} = (1 + r) \frac{1}{\frac{p^2}{M^2} - 1}
\]

where

\[
1 + r = \frac{1}{[1 - \alpha] [1 - 2Mf'(M^2)]}
\]

\[
\Rightarrow r \sim \alpha + 2M(b + Ma)
\]

\[
= \left. \frac{\partial}{\partial \frac{p^2}{M^2}} \left[ \frac{a(p^2)p + b(p^2)}{p^2} \right] \right|_{p^2 = M^2}
\]

To differentiate \( \frac{1}{A} \), use

\[
\frac{1}{A + \Delta A} = \frac{1}{A} - \frac{1}{A} \Delta A + \ldots
\]

Rather than using a new propagator, (1 + \( r \)) \( \frac{1}{\frac{p^2}{M^2} - 1} \) is easier to include (1 + \( r \)) in expanding as

\[
\sum_{n=0}^{\infty} \frac{e^n}{n!} (1 + r)
\]

Thus shift in pole \( \Rightarrow \) shift in \( \sum_{n=0}^{\infty} \frac{e^n}{n!} \)

Change in strength of pole \( \Rightarrow \) change in coupling

\( r \), unfortunately is divergent
\[ 4\pi e^2 \sqrt{\frac{\hbar}{\ell^2 - \cos \theta}} \int_0^1 \frac{d\rho}{\ell^2 - \rho^2} = \frac{\hbar}{\ell^2 - \rho^2} \int_0^1 \frac{d\rho}{\ell^2 - \rho^2} \]

\[ = 4\pi e^2 \sqrt{\frac{\hbar}{\ell^2 - \cos \theta}} \int_0^1 \frac{d\rho}{(\ell^2 - \rho^2)(\ell^2 - 2\rho\cos \theta)} \int_0^1 \frac{d\rho}{\ell^2 - \rho^2} \]

\[ \text{Let's impose infrared divergences on loop diagrams using ultraviolet}
\]
\[ \text{divergence technique to handle infrared problems. Also, a}
\]
\[ \text{we have } \frac{\ell^2 - \rho^2}{\ell^2 - \rho^2} \]

\[ \text{Returns Pauli correction, we have }
\]
\[ \left[ i\hbar \nabla \psi - eA \psi - \frac{\hbar^2}{2m} (\partial_\mu \psi \partial^\mu \psi) \right] \psi = m \psi
\]

\[ i\nabla \psi = i(\gamma^0 \partial_t - \gamma^i \partial_i)\psi = i(\partial_t - \partial_x)\psi
\]

After adding all above diagrams, we get an expansion in powers of
\[ \psi
\]
\[ \left[ \phi \psi + \gamma^0 (\gamma^0 \psi - \gamma^0 \phi) + (\gamma^2 \partial_2 \psi + \gamma^1 \partial_1 \psi) + \cdots \right] \psi
\]
Firstly, we need only calculate integral to first order, thus we examine a technique for doing integrals.

\[
\int \frac{1}{(k^2 - l)^3} \frac{d^4 k}{(2\pi)^4} = \frac{1}{32\pi^2 L}
\]

\[
\int \frac{1}{(k^2 - 2p \cdot k + \Delta)^3} \frac{d^4 k}{(2\pi)^4} = \int \frac{1}{(k^2 - p)^2 - (\Delta - p)^2} \frac{d^4 k}{(2\pi)^4}
\]

\[
= \frac{1}{32\pi^2 (p^2 + \Delta)}
\]

\[
\int \frac{k_x}{(k^2 - L^2)^3} \frac{d^4 k}{(2\pi)^4} = 0
\]

\[
\int \frac{k_{2m}}{(k^2 - 2p \cdot k - \Delta)^3} \frac{d^4 k}{(2\pi)^4} = \frac{1}{32\pi^2} \frac{P_m}{p^2 + \Delta}
\]

\[
\int \frac{1}{(k^2 - 2p \cdot k + \Delta_1)(k^2 - 2p \cdot k + \Delta_2)} \frac{d^4 k}{(2\pi)^4}
\]

\[
= \frac{1}{\sqrt{(k^2 - 2p \cdot \Delta_1)^2}} \int \frac{dx}{(k^2 - 2p \cdot x + \Delta_2)^2}
\]

\[
P_x = xP_1 + (1-x)P_2
\]

\[
\Delta_x = x\Delta_1 + (1-x)\Delta_2
\]

Using
\[
\frac{1}{a \cdot b} = \sqrt{\frac{1}{[ax + b(1-x)]^2}}
\]

very useful
\[ \Pi = (\bar{u}_i u_i) \frac{4 \pi e^2}{g^2} \int 2p \left[ \frac{1}{p^2 - m^2} \gamma_i \frac{1}{p^2 + q^2 - m^2} \right] (-1) \frac{4p}{(2\pi)^4} \]

\[ \text{Vacuum polarization form} \]

\[ \Gamma = 4\pi e^2 \int \frac{d\mu}{p^2 - \mu} \left( \frac{p^2}{m^2} \right) \frac{1}{p^2 - \mu} \gamma^\mu \frac{1}{k^2} \gamma^\nu C(k^2) \]

Roughly

\[ p_i = p_0 + \Delta p \]

\[ \gamma^\mu \frac{1}{p_0 + \Delta p - m} \gamma^\nu = \gamma^\mu \frac{1}{p_0 - k - m} \gamma^\nu + \gamma^\mu \frac{1}{p - k - m} \Delta p \frac{1}{k - m} \gamma^\nu \]

\[ \left[ \frac{1}{A \cdot B} = \frac{1}{A} - A \cdot \frac{B}{A} + A \cdot \frac{B}{A} \cdot A + \cdots \right] \]
add virtual photons $e^2$
Problem: Neutral K decay

Background: A K meson is a bodon with mass more than three times the mass of the more common pi meson. There are four K mesons — two neutral ones, one each with positive and negative charge. These particles are produced in strong interactions of the form \( \pi^- p \rightarrow \Lambda^0 k^0 \).

It was found experimentally that if the initial state contained only pions and nucleons, measurements of reactions with two particle final states showed only \( k^+ \) particles, never \( k^- \). Gell-Mann came up with the idea of assigning to particles an additional quantum number called strangeness to predict this pattern.

\[
S(\Lambda^0) = S(\Xi^+ - \Xi^-) = S(\Xi^0) = -1
\]

\[
S(k^+) = S(k^-) = +1 \quad S(k^0) = S(\Xi^0) = -1
\]

If we postulate that strangeness be conserved in strong (i.e., fast) reactions, we predict

\[
\pi^- p \rightarrow \Xi^- k^+
\]

\[
\pi^- p \rightarrow \Xi^- k^+
\]

To date predictions of this sort have always worked. However, the theory is rather peculiar in that it predicts two neutral K particles with opposite values of strangeness. If there were no way of distinguishing between this case and the case of only one neutral K, there would be questions raised about the theory. One way of distinguishing the two cases is to compare the rates of reactions between different members of the same multiplet. The ratios depend only on the isospins of the different constituents — thus they will be different if the K forms two isospin doublets or one isospin triplet.

Another way of distinguishing the cases is in the \( k^0 \) decay. If only one \( k^0 \) were present, it would decay exponentially with time in the usual fashion. Your problem is to figure out what happens when two \( k^0 \)'s exist.

**Statement of Problem:**

\[
K^0 \rightarrow \pi^+ \pi^- \quad \bar{k}^0 \rightarrow \pi^+ \pi^-
\]

Assume that the two neutral K's have the same mass and the same matrix element for this
decay. Assume that this is by far the most important reaction in the decays.

Given a $K^0$ all alone in a box at $t=0$, what is the probability as a function of time of finding a $K^0$ in the box for $t > 0$?

For people who worry about infinities:

There are some reactions which distinguish between $K^0$ and $\bar{K}^0$ although they have small probability.

\[ K^0 \rightarrow \pi^- + \left( \frac{\mu^+}{e^+} \right) + \nu \]
\[ K^0 \rightarrow \pi^- + \left( \frac{\mu^-}{e^-} \right) + \bar{\nu} \]
\[ \bar{K}^0 \rightarrow \pi^- + \left( \frac{\mu^-}{e^-} \right) + \nu \]
\[ \bar{K}^0 \rightarrow \pi^- + \left( \frac{\mu^+}{e^+} \right) + \bar{\nu} \]

MORE ON THE THEORY OF LINE WIDTHS

Suppose \[ H = H_0 + H_1 \]

Consider the following model:

$H_0$ has as its eigenstates both

1) a collection of discrete states $\alpha^0, \alpha^1, \alpha^2, \ldots$
   with energies $E^i$

2) a continuum. Each state in the continuum consists of one $\alpha^i$ state, and $n$ particles of a different type called $b$. Each $b$ particle has an energy $E_{b_i}$. The $\alpha^i$ state or particle is assumed to always have energy $E^i$ (i.e. it never gets kinetic energy in this model).

Spectrum of $H_0$

\[ \begin{array}{ccc}
E_{\alpha^0} & E_{\alpha^0+2\lambda_0} & \text{Energies of } \alpha^{0+2\lambda_0} \\
E_{\alpha^1} & E_{\alpha^1+m_b} & \text{Energies of } \alpha^{1+m_b} \\
 & \text{Minima} & \text{Minima} & \text{etc}
\end{array} \]

$H_1$, as usual, induces transitions between the states of $H_0$.

Because the states of the discrete spectrum are usually simpler to analyze, it is desirable to eliminate the other states from our equations.

Label the coefficients of the discrete spectrum by $a_{m}$; those of the continuum states by $b_{\lambda}$

\[ i \dot{a}_m(t) = \sum_{m} \nu_{mm}(t) a_m(t) + \sum_{\lambda} \nu_{m\lambda}(t) b_{\lambda}(t) \]
Assume for ease of computation that the perturbation connecting continuum and discrete states is roughly independent of time (in practice this is almost always true - most of the "leakage" to continuum states is not connected with the turning on and off of laboratory apparatus).

Further, assume that at \( t=0 \) you have no continuum states populated. Then the \( b_n^\ast \) are small compared to the \( a_m \); the feeding of continuum states may be assumed to come mostly from the discrete ones. Hence we arrive at

\[
\begin{align*}
\dot{a}_m(t) &= \sum_{n} v_{nm}(t) a_m(t) + \sum_{k} M_{nk} b_k(t) e^{-i(E_n - E_k)t} \\
\dot{b}_k^\ast(t) &= \sum_{m} M_{km} a_m(t) e^{-i(E_m - E_k)t}
\end{align*}
\]

Integration gives

\[
\begin{align*}
\hat{a}_m &= -i \sum_{n} v_{nm}(t) a_m(t) - \int_{0}^{t} \sum_{k} |M_{nk}|^2 e^{-i(E_n - E_k)(t-t')} a_m(t') dt' \\
&\quad - \int_{0}^{t} \sum_{k} M_{nk} M_{km}^* e^{-i(E_m - E_k)t} e^{-i(E_m - E_k) t'} a_m(t') dt' 
\end{align*}
\]

The last piece has no average effect over large periods of time. Hence we may neglect it.

(General edification: a small perturbation which produces a large effect after a long time is called a secular disturbance)

As shown in last week's notes we may then define quantities \( \Delta + \Pi \) such that

\[
\begin{align*}
\dot{a}_m &= -i \sum_{n} v_{nm}(t) + \Delta m \delta_{m,m'} \hat{a}_m(t) - \Pi_m \frac{1}{\omega} \hat{a}_m(t) \\
\Pi &= 2m \sum_{k} \int (E_k - E_m) |M_{km}|^2 
\end{align*}
\]

Define \( \nu_{nm}(t) + \Delta \delta_{nm} = \nu_{nm'}(t) = \nu_{nm'}(t) e^{-i(E_n - E_m) t} \)

Last week we computed the probability for light emission by an atomic transition between states with widths. If it began in state \( n \) with the atom in a

and went to state \( m \) with atom in \( b \), and photon of energy \( \nu \)

then

\[
P_m = |a_m(t)|^2 = \frac{\left| \nu_{nm} \right|^2 \left[ e^{-\frac{\gamma_n t}{2}} + e^{-\frac{\gamma_m t}{2}} - 2 e^{-\frac{(\gamma_m + \gamma_n) t}{2}} \cos (E_n - E_m) t \right]}{2 \left( (E_m - E_n)^2 + \left( \frac{\gamma_n - \gamma_m}{2} \right)^2 \right)}
\]

is the probability to be in state \( m \) at time \( t \); this goes to 0 as \( t \) goes to infinity.

Suppose you just want the total probability at time \( t \) to find a photon of frequency \( \nu \) which came from this transition. This is

\[
P_m + \text{probability that this took place and then the atom in state } b \text{ decayed}
\]
As $t$ goes to infinity, this total probability approaches $P_{\text{decayed}}$

Rate of decay = $\frac{1}{dt} \left[ \text{Probability that the state has decayed} \right]$

$\phi = \Gamma_b \int_0^\infty P_m(t) dt \ \{ \text{From rate formula} \}$

$P_{\text{decay}} = \Gamma_b \int_0^\infty P_m(t) dt$

Assuming $V_{mn}(t) = V_{mn}$, we find $(V_{mn} = V_{nm}, m \neq n)$

$$P_V(\omega) = \left| V_{mn} \right|^2 \frac{\Gamma_b}{\sqrt{\gamma^2 + \left( \frac{\Gamma_a - \Gamma_b}{2} \right)^2}} \left[ \frac{1}{\Gamma_a} + \frac{1}{\Gamma_b} - \frac{2}{\sqrt{\gamma^2 + \left( \frac{\Gamma_a + \Gamma_b}{2} \right)^2}} \right]$$

$$= \left| V_{mn} \right|^2 \left[ \frac{1}{\Gamma_a} + \frac{1}{\Gamma_b} \right] \frac{1}{\Gamma_a} \frac{1}{\sqrt{\gamma^2 + \left( \frac{\Gamma_a + \Gamma_b}{2} \right)^2}}$$

Notice that the line shape as $t\to\infty$ is a "Lorentzian curve" with a width $\Gamma_a + \Gamma_b$.

If the resolution of our apparatus is crude, the line looks sharp enough that we may make the replacement

$$\frac{\Gamma_a + \Gamma_b}{\sqrt{\gamma^2 + \left( \frac{\Gamma_a + \Gamma_b}{2} \right)^2}} \to 2\pi \delta(\gamma)$$

thus

$$P_V(\omega) = \left| V_{mn} \right|^2 \frac{2\pi}{\Gamma_a} \delta(\gamma)$$

and the total probability of emitting some photon is

$$\int P_V(\omega) \frac{d^3h}{(2\pi)^3} = \frac{2\pi}{\Gamma_a} \left| V_{mn} \right|^2 \rho(E_a - E_b)$$

This looks different from what we had before, because of the $\frac{1}{\Gamma_a}$

However, it is easily explained:

Rate = $2\pi \left| V \right|^2 \delta(\Delta E) \times \text{Probability to be in upper state}$

$$= 2\pi \left| V \right|^2 \delta(\Delta E) e^{-\Gamma_a t}$$

Total probability of emission =

$$\int \text{Rate} \ dt = 2\pi \left| V_{mn} \right|^2 \rho(E_a - E_b) \frac{1}{\Gamma_a}$$

$$= \frac{\Gamma_a T_e \times \text{mean life time}}{\Gamma_a}$$

Notice that the line shape changes as a function of time. In fact, if you look at only part of the line, the intensity is expected to change with $t$. This has been experimentally detected using the Mossbauer effect. Hence one must be careful in making arguments about probabilities and rates.

To see how the inclusion of widths removes certain infinities in the scattering problems.
Previously in the second order terms we had factors like \( \frac{1}{E_c - E_d + \omega} \)

Clearly difficulties might arise for \( \omega = \Delta \varepsilon \)

Example of Present Formalism:

Work out the case of light scattering by an atom with stable ground state but unstable excited states,

\[
\hat{a}_q = -i \sum_k V_{qk} a_k(t) e^{-i(E_k - E_q) t} \\
\hat{a}_e = -i \sum_m V_{em} a_m(t) e^{-i(E_m - E_e) t} \\
\hat{a}_g = \hat{a}_g(0) - i \int_0^t \sum V_{qk} a_k(t) e^{-i(E_k - E_q) t} dt)
\]

Assume we start with \( \hat{a}_q = 1 \), all other states unpopulated.

\[
\hat{a}_e \text{ the amplitude to reach some excited state is to a first approximation}
\]

\[
\hat{a}_e = -i \int_0^t \sum V_{qk} a_k(t) e^{-i(E_k - E_q) t} dt
\]

\[
\hat{a}_e = e^{-i \epsilon t/2} f \quad f = -i \int_0^t e^{i \epsilon t/2} V_{qk} e^{-i(E_k - E_q) t} dt
\]

\[
\hat{a}_e = \frac{V_{qk}}{E_q - E_k + i \epsilon/2} \left[ e^{-i(E_k - E_q) t} - e^{-i \epsilon t/2} \right]
\]

Then the transitions to another stable state are given by

\[
\hat{a}_j = -i \int_0^t \sum V_{jg} a_g(0) e^{-i(E_g - E_j) t} dt
\]

\[
= -i \int_0^t \sum_{g \neq q} \frac{V_{jg} V_{qg}}{E_g - E_j + i \epsilon t/2} \left[ e^{-i(E_g - E_j) t} - e^{-i \epsilon t/2} \right] dt
\]

\[
= -i \int_0^t \sum_{g \neq q} V_{jg} a_g(0) e^{-i(E_g - E_j) t} dt - i \int_0^t \sum V_{jg} a_g(0) e^{-i(E_g - E_j) t} dt
\]

For light scattering \( V_{jg}, V_{jg} \) are second order, \( a_j \) is first order; \( a_q = 1 + \hat{f} \)

\[ f \text{ is first order.} \]

Hence keeping only second order terms gives

\[
\hat{a}_j \rightarrow -i \left[ V_{jg} + \sum_{g \neq q} \frac{V_{jg} V_{qg}}{E_g - E_j + i \epsilon t/2} \right] \int_0^t e^{-i(E_g - E_j) t} dt
\]

\[
+ i \int_0^t \sum_{g \neq q} \frac{V_{jg} V_{qg}}{E_g - E_j + i \epsilon t/2} e^{-i(E_g - E_j) t} dt
\]

Explicit calculation shows that the last terms can be neglected compared to the other terms at long \( t \).

Hence if \( M_{ji} = V_{ji} + \sum_k \frac{V_{jk} V_{ki}}{E_i - E_j + i \epsilon t/2} \), we find

\[
M_{ji} = V_{ji} + \sum_k \frac{V_{jk} V_{ki}}{E_i - E_j + i \epsilon t/2}
\]
Probability = \left| M_{ji} \right|^2 \frac{\sin^2 \left( \frac{E_j - E_i}{2} \right)}{(E_j - E_i)^2}

Rate = 2\pi \int \frac{1}{\left( E_i - E_j + i\frac{\Gamma_j}{2} \right)} \left| M_{ji} \right|^2

Near the resonance the scattering amplitude looks like

\frac{1}{E_i - E_j + i\Gamma_j}

...a Breit-Wigner form

(this method of treating widths is sometimes called the Wigner-Weisskopf method)
Last week we obtained the **GOLDEN RULE**

\[
R_{m \rightarrow n} = 2\pi \frac{\delta}{E_m - E_n} |T_{mn}|^2
\]

where

\[
T_{mn} = V_{mn} + \sum_{k} V_{mnk} \frac{1}{E_m - E_{k\ell} + i\epsilon} V_{k\ell n} + \sum_{k, \ell} \frac{V_{mnk}}{E_m - E_{k\ell} + i\epsilon} \frac{1}{E_{k\ell} - E_{\ell m} + i\epsilon}
\]

Rehash of derivation:

Draw all graphs possible for the path between \( n \) and \( m \), with this perturbation

\[
\begin{align*}
\begin{tikzpicture}
  \draw[thick,->] (0,0) -- (1,0) node[midway,above] {\( \cdot \)};
  \draw[thick,->] (1,0) -- (2,0) node[midway,above] {\( \cdot \)};
  \draw[thick,->] (0,1) -- (0,0) node[midway,left] {\( \cdot \)};
  \draw[thick,->] (0,1) -- (1,1) node[midway,right] {\( \cdot \)};
  \draw[thick,->] (1,1) -- (1,0) node[midway,above] {\( \cdot \)};
  \draw[thick,->] (1,1) -- (2,1) node[midway,right] {\( \cdot \)};
  \draw[thick,->] (2,1) -- (2,0) node[midway,above] {\( \cdot \)};
\end{tikzpicture}
\end{align*}
\]

Remember that the energies involved are those of the unperturbed Hamiltonian. Then think what a graph actually means

\[
\begin{align*}
\begin{tikzpicture}
  \node at (0,0) {$\tilde{t}_0$};
  \node at (1,0) {$\tilde{t}_1$};
  \node at (2,0) {$\tilde{t}_2$};
  \node at (3,0) {$\tilde{t}_3$};
  \draw[thick,->] (0,0) -- (1,0) node[midway,above] {\( \cdot \)};
  \draw[thick,->] (1,0) -- (2,0) node[midway,above] {\( \cdot \)};
  \draw[thick,->] (2,0) -- (3,0) node[midway,above] {\( \cdot \)};
\end{tikzpicture}
\end{align*}
\]

System moves \( \psi_0 \) from \( n \) to \( 1 \) in eigenstate \( |\tilde{t}_2, \tilde{t}_3, \tilde{t}_0 \rangle \) of the unperturbed system. At \( 1 \) it gets bumped into state \( |0 \rangle \) by the perturbation. It moves from \( 1 \) to \( k \) in state \( |\tilde{t}_1, \tilde{t}_3, \tilde{t}_0 \rangle \), then it is bumped into state \( |\tilde{m}, \tilde{t}_3, \tilde{t}_0 \rangle \), etc.

Associate a time with each corner. Along a given straight line,

state \( |\tilde{t}_2, \tilde{t}_3, \tilde{t}_0 \rangle \) at \( \tilde{t}_0 \), \( n \) becomes state \( |\tilde{t}_1, \tilde{t}_3, \tilde{t}_0 \rangle \) at \( \tilde{t}_1 \).

Hence the graph, which might be written

\[
(-i)^2 \langle \tilde{t}_0, \tilde{t}_2 | V | \tilde{t}_1, \tilde{t}_3 \rangle \langle \tilde{t}_1, \tilde{t}_3 | V | \tilde{t}_0, \tilde{t}_2 \rangle
\]

takes on the form

\[
(-i)^2 \langle \tilde{t}_0, \tilde{t}_2 | V | \tilde{t}_1, \tilde{t}_3 \rangle \langle \tilde{t}_1, \tilde{t}_3 | V | \tilde{t}_0, \tilde{t}_2 \rangle e^{-i E_2 (t_2 - t_0)} e^{-i E_3 (t_1 - t_0)} e^{-i E_1 (t_0) - (E_m - E_n)}
\]

Since \( t_0 \) is just a constant, set it equal to 0 (this won't influence the rate)

\[
(-i)^2 V_{mn} \langle \tilde{t}_2, \tilde{t}_3 | e^{-i (E_2 - E_1)(t_2 - t_1)} | \tilde{t}_1, \tilde{t}_3 \rangle e^{-i (E_m - E_n)}
\]

The perturbation could act at any points \( t_0 \), \( t_1 \), \( t_2 \), \( t_3 \) \( t_0 \)

To sum the contributions, integrate over \( t_0 \)

Change of variables gives

\[
(-i)^2 \int_{t_1}^{t_2} \int_{t_3 - t_1}^{t_2 - t_1} V_{mn} (x + t_1) V_{n,m} (t) e^{-i (E_m - E_n)} d x d t_i
\]

Consider first

\[
\int_{t_3 - t_1}^{t_2 - t_1} V_{mn} (x + t_1) e^{-i (E_m - E_n)} d x
\]
In most cases $\sqrt{t}$ varies very slowly with time compared to the exponential. Thus we have

$$V_{m_2} (t_1) \int_0^t e^{-i(E_m - E_{m_2} - i\epsilon)} dt_1$$

$$V_{m_2} (t_1) \left[ \frac{e^{-i(E_m - E_{m_2} - i\epsilon)(t_2 - t_1)} - 1}{-i(E_\mu - E_{m_2} - i\epsilon)} \right]$$

It is reasonable to replace $t_2$ by $\infty$ in most situations. Then the final result is

$$\left( \frac{-i}{E_m - E_{m_2} - i\epsilon} \right) \int_0^\infty e^{-i(E_m - E_{m_2}) t_1} dt_1$$

Notice that all terms in the expansion of $T$ have the same number of $i$'s

$$\left( \frac{i}{E_m - E_{m_2} - i\epsilon} \right) \left( \frac{i}{E_m - E_{m_2} - i\epsilon} \right) = -i \times 1 = -i$$

where $H_\epsilon$ is diagonal, $e^{-iE_\mu t} = e^{-iH_\epsilon t}$

Thus we may write

$$\int_0^\infty e^{iE_\mu t} e^{-iH_\epsilon t} dt = \frac{1}{E - H_\epsilon + i\epsilon}$$

Pursuing this leads to the formal expression

$$T(E) = V + V^* \frac{1}{E - H_\epsilon + i\epsilon} + V + \ldots$$

In interpreting operator inverses like the above, keep in mind that

$$\frac{1}{A + B} = G_{\alpha} T c \leftrightarrow (A + B)(G_{\alpha} T c) = (G_{\alpha} T c) ((A + B)^{-1})\alpha = \frac{1}{A + B}$$

where $A, B, 1 \in G_{\alpha} T c$ are all operators.

$$G_{\alpha} T c = \frac{1}{A} - \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} - \ldots$$

works here. Thus we set it equal to

$$\frac{1}{A + B}$$

By use of this trick, $T = V + V^* w(E) + V$

with $w(E) = \frac{1}{E - H_\epsilon + i\epsilon} + \frac{1}{E - H_\epsilon + i\epsilon} V E - H_\epsilon + i\epsilon + \ldots$

may be written

$$w(E) = \frac{1}{E - H_\epsilon + i\epsilon} = \frac{1}{E - H + i\epsilon}$$

This expression as it stands is absolutely exact. However, it is also almost absolutely useless. For suppose you tried to find $T_{m,n}$ for $m, n$ eigenstates of $H_\epsilon$.

The evaluation of $\langle \begin{pmatrix} \lambda \Omega \\ 0 \end{pmatrix} | E - H + i\epsilon | \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \rangle$ is non-trivial.

Hence one is forced to try approximation schemes.
Example #1

\[ \langle \psi_m | T | \psi_n \rangle = \langle \psi_m | V | \psi_m \rangle + \sum_k \frac{\langle \psi_m | V | \psi_k \rangle}{(E - E_k + i\varepsilon)} \langle \psi_k | V | \psi_n \rangle \]

Here \( | \psi_k \rangle \) is an eigenstate of \( H_{\text{full}} \). Of course you don't know what it is, but you can try to get close to it by different methods:

a) perturbation theory and the unperturbed eigenstates

b) sometimes the wave functions can be approximated more accurately by some physical guesswork than by blind series summing. For example, it might be convenient to use the variational method to get \( | \psi_{hi} \rangle \) and \( E_{ki} \).

Example #2

In certain contexts it is useful to replace \( V \) by \( H - H_0 \).

Consider a two-body process \( A + B \rightarrow C + D \)
like \( H^+ + \bar{H} \rightarrow p + \bar{p} \).

It is only reasonable to split up the Hamiltonian in a different way in the initial state than in the final one:

\[ H_{\text{full}} = K_1 (A_{\text{mic}}) + K_2 (A_{\text{elec}}) + K_3 (B_{\text{mic}}) + K_4 (B_{\text{elec}}) + C_1 (A_{\text{mic}} - B_{\text{mic}}) + C_2 (A_{\text{elec}} - B_{\text{elec}}) + C_3 (B_{\text{mic}} - A_{\text{mic}}) + C_4 (B_{\text{elec}} - A_{\text{elec}}) + C_5 (B_{\text{mic}} - A_{\text{mic}}) \]

where \( K \) is kinetic energy; \( C \) is Coulomb energy.

\[ V^I = C_1 + C_2 + C_5 + C_6 \]
\[ V^{\text{II}} = C_1 + C_2 + C_4 \]

The \( V \)'s contain those pieces not in the individual Hamiltonians of the ions and atoms.

Then the formula for \( T \) becomes

\[ \frac{1}{(H - E_I)} + \frac{1}{(H - E_{II})} \left( \frac{1}{E - H + i\varepsilon} \right) \frac{1}{(H - E_I)} \]

\[ V^I + V^{\text{II}} \left( \frac{1}{E_I - H + i\varepsilon} \right) V^I \]

To apply this, one must still approximate \( H \).

Notice that

\[ \langle \psi^I_{II} | V^{\text{II}} | \psi^I_{II} \rangle = \langle \psi^I_{II} | H - E_1 | \psi^I_{II} \rangle \]
\[ \langle \psi^I_{II} | V^{\text{II}} | \psi^I_{II} \rangle = \langle \psi^I_{II} | H - E_2 | \psi^I_{II} \rangle \]

These matrix elements are the same because the \( T \) matrix is multiplied by

\[ \delta(E_1 - E_2) \]
$K^0$ decay problem

\[ i \dot{\alpha}_{K^0} = E_i \alpha_{K^0} + \frac{e}{\hbar} M_{K^0, \pi^+} b_{\pi^+} \]

\[ i \dot{\alpha}_{\bar{K}^0} = E_i \alpha_{\bar{K}^0} + \frac{e}{\hbar} M_{\bar{K}^0, \pi^+} b_{\pi^+} \]

set the energy scale such that $E_i = \Delta$

\[ b_{\pi^+} = e^{-i(E_{\pi^-} - E_i)t} \int_0^t e^{i(E_{\pi^-} - E_i)t'} M_{K^0, \pi^+} \left[ \alpha_{K^0}(t') + \alpha_{\bar{K}^0}(t') \right] dt' \]

\[ i \dot{\alpha}_{K^0} = \sum_{\pi^\pm} M_{K^0, \pi^\pm} e^{-i(E_{\pi^-} - E_i)t} \int_0^t e^{i(E_{\pi^-} - E_i)t'} M_{K^0, \pi^\pm} \left[ \alpha_{K^0}(t') + \alpha_{\bar{K}^0}(t') \right] dt' \]

and

\[ \left[ \Delta - i/2 \right] \left[ \alpha_{K^0}(t) + \alpha_{\bar{K}^0}(t) \right] \]

Solving gives

\[ \alpha_{K^0} + \alpha_{\bar{K}^0} = e^{-2i\Delta t - \pi t} \]

\[ \alpha_{K^0} = \alpha_{\bar{K}^0} = 1 \]

\[ |\alpha_{K^0}|^2 = \frac{1}{4} \left[ 1 + e^{-2\pi t} + 2 \cos(2\Delta t) e^{-\pi t} \right] \]

\[ |\alpha_{\bar{K}^0}|^2 = \frac{1}{4} \left[ 1 + e^{-2\pi t} - 2 \cos(2\Delta t) e^{-\pi t} \right] \]

It is clear that $2\Delta$ represents a mass difference between the states $\alpha_{K^0} \leftrightarrow \alpha_{\bar{K}^0}$ and $\alpha_{K^0} \leftrightarrow \alpha_{\bar{K}^0}$. The relative size of $\Delta < \pi$ determines whether it is possible to detect the wiggles. The wiggles can be seen and $\Delta$ comes out to be $\sim 4.5 \pi$.

For $\gamma \sim \nu$, this gives an energy difference of $\sim \mu e^\nu$.

This mass split is of interest because:

1) it is really there. Hence there can be no question that a perturbation changes the mass in the way indicated by the formula. (in certain other applications of perturbation theory only the final (perturbed) mass is physical - hence the formal manipulations required seem much less meaningful)
ii) It is small. If the $\bar{K}^0$ had a negative mass compared to $K^0$ this mass difference, as measured by the wigglers, would be enormous ($\sim 2 \times 10^{-4}$ eV).

Thus we can dispose of speculations that antiparticles have negative masses.

$a_{K^0} - a_{\bar{K}^0}$ are eigenstates of the operator CP (which takes an antiparticle into its particle and vice versa). Our result showed that one CP eigenstate decays into two $\pi^+$'s (also an eigenstate of CP), whereas the other doesn't.

For some time after its discovery, the long lived eigenstate $K^0_L$ was observed to go only into $\pi^0$'s and lepton pairs (lepton pairs aren't CP eigenstates, so the $K^0_L$'s go into them as well). Hence it was believed that CP was conserved in all interactions.

It is now known that $K^0_L$ does go into two $\pi^0$'s, but with a very low rate. So far, no theory has managed to a) explain all observed phenomena and b) maintain CP conservation. Thus CP appears to be broken, although with very small probability.

Furthermore, it appears that this CP nonconservation cannot be studied very well in any presently observed reactions other than $K^0$ decay. Hence it will be difficult to fit it into the total picture of things until there are new developments (either in theory of experiment).
Now we have been alluding to the excited state as a plane wave. This is not quite correct, however. Due to the range of the Coulomb potential, we must actually use the Coulomb wave function [confined hypergeometric function].
**Problem:** Consider transitions in He and that the energy as a function of momentum is

\[ E(p) = \Delta + \frac{1}{2M} (p - p_0)^2 \]

where \( \Delta \) and \( p_0 \) are constants. Suppose also

\[ \Delta(p) = \Delta_0 + (p - p_0)^2 + O((p - p_0)^2) \]

What is the probability of emitting a photon of momentum \( \mathbf{k} \) in a box of side \( L \)?

Consider the He contained in a box of volume \( V \). Now, the density in the box obeys the wave equation

\[ \nabla^2 \rho - \frac{\rho}{\rho_0} = 0 \quad \text{classically.} \]

We wish to quantize this fluid field. First, if we imagine hexagonal b.c., we can represent it as the sum of plane waves

\[ \rho(p, t) = \sum Q_{\mathbf{k}}(t) e^{ik \cdot \mathbf{r}} \]

If we plug these into (3) we find

\[ -\hbar^2 \frac{\partial^2 \rho}{\partial \mathbf{r}^2} - \frac{\rho}{\rho_0} = 0 \]

**Hint:** Show that \( \rho = \hbar^2 \)
Now to pass to q.m., we find the energy
\[ E = \int \rho \, d\mathbf{r}. \]

Now we can write the interaction by writing
\[ \psi(p) = \frac{1}{2m} (p \cdot \mathbf{p})^2 + \Delta_0 + \left( \mathbf{p} - \mathbf{p}_0 \right) \cdot \mathbf{p} \]

interaction term

Thus
\[ H_{\text{proj}} = \frac{1}{2m} (\mathbf{p} - \mathbf{p}_0)^2 + \Delta_0 - \mathbf{p}_0 \cdot \mathbf{p} + \sum_k \frac{\mathbf{p}_k \cdot \mathbf{Q}_k}{\mathbf{p}^2} \]

where
\[ H_{\text{proj}} = \frac{\mathbf{p}^2}{2m} + \mu \frac{\mathbf{Q}^2}{\mathbf{p}^2} \]

\[ \mu = \frac{\rho \mathbf{Q}}{Z} \]

Thus our unperturbed Hamiltonian is
\[ H_{\text{proj}} = \frac{1}{2m} (\mathbf{p} - \mathbf{p}_0)^2 + \Delta_0 - \mathbf{p}_0 \cdot \mathbf{p} \]

Note these will give motion states corresponding to that of a free particle
since \( \Delta_0 - \mathbf{p}_0 \cdot \mathbf{p} = \text{const} \), e.g. \( \mathbf{p} = \text{const} \) in the S-eq. With these,

\[ \langle \phi \mid \mathbf{p} \rangle = \frac{e^{i k \mathbf{p} \cdot \mathbf{r}}}{\mathbf{p}} \]
\[
\frac{R_{p}^{2}}{\rho + \rho} = 2m \epsilon (\frac{\rho^2}{2m} - \frac{\rho^2}{2m} - \hbar^2c^2) \left| \langle \rho | \frac{\rho}{\hbar} | \rho \rangle \right|^2 \frac{1}{2m \hbar^2 \rho^2}
\]

The remainder of the analysis should be shown. Note that \( P_0 \) means that the photon will eventually decay to a ground state energy of \( \frac{P_0^2}{2m} + \Delta \). One should be able to say something about the behavior of \( \text{He} \) from this.
3. Our problem is now to consider

\[ \begin{align*}
  25 & - \quad \text{2 photon emission} \\
  15 & -
\end{align*} \]

This immediately suggests the use of 2nd order perturbation theory, since a 2 photon process is 2nd order. 

Thus we try to generalize our result on p. 31. This corresponds to the

\[ \frac{\alpha^2}{2} \hat{A}(\vec{r}_1) \cdot \hat{A}(\vec{r}_2) \]

term for the field interaction. The field, of course, is

\[ H = \frac{1}{2} \frac{\alpha^2}{\hbar c^2} \left[ \frac{p^2}{2m} + m \left( \frac{\beta^2}{2} \right) \right] \]

\[ \text{Interaction} = \frac{\alpha^2}{\hbar c^2} \hat{Q} \hat{\Phi} + \frac{\alpha^2}{\hbar c^2} \frac{e^{i(k_1 - k_2) \cdot \vec{r}}}{\vec{k}_1 \cdot \vec{k}_2} \]

where \( \hat{Q} \) is the first order operator on p. (34).

Now since the first order breather is forbidden, we can conclude

\[ \langle s | P | 2s \rangle = 0 \quad \text{[parity violated]} \]

What we now need is a nice dandy dandy formula like

\[ \text{Res} = 2\pi \sum_{\alpha \neq \beta} \left( \epsilon_\alpha - \epsilon_\beta - n\hbar c \right) \left| \langle b | \psi_\beta \rangle \right|^2 \frac{\alpha^2}{2 e^2 \hbar^2 \beta} \]

for 2nd order theory. I'll wait for \( \beta \) to derive it... if he doesn't then I'll do it myself.


COMMENTS ON TIME-DEPENDENT PERTURBATION THEORY

We wish to solve

\[-\frac{i}{2\hbar} \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle\]

Write

\[H(t) = H_0 + V(t)\]

and expand

\[|\Psi(t)\rangle = \sum_n C_n(t) |\phi_n\rangle\]

Plug into (1)

\[\sum_n -\frac{i}{\hbar} \frac{d}{dt} C_n(t) |\phi_n\rangle = \sum_n C_n(t) E_n |\phi_n\rangle + \sum_n C_n(t) V(t) |\phi_n\rangle\]

Take a scalar product with $<\phi_m |$

\[-\frac{i}{\hbar} \frac{d}{dt} C_m(t) = E_m C_m(t) + \sum_n C_n(t) <\phi_m | V(t) |\phi_n\rangle\]

This suggests we try

\[C_m(t) = e^{-iE_m t} \chi_m(t)\]

Then (2) becomes

\[\dot{\chi}_m(t) = -i \sum_n \chi_n(t) e^{-iE_n t} <\phi_m | V(t) |\phi_n\rangle e^{iE_n t} \]

\[= i \sum_n \chi_m(t) \chi_n(t)\]

(What relationship does $\chi_m(t)$ have to $\chi_n(t)$?) Now we want

\[\chi_m(t_f) = \sum_n \Lambda_{mn}(t_f, t_i) \chi_n(t_i)\]

Thus $\chi_m(t)$ is some integral of $\chi_m(t_f)$. 

\[\sum_n \chi_m(t_f) \chi_n(t_i)\]
To evaluate the \( \gamma_{mn} \) amplitude that a state is in \( |\phi_n(t)\rangle \) will be in \( |\phi_m(t)\rangle \), we take a specific case of \( \gamma_m(t) \) and compute \( \gamma_m(t) \) by integrating (5) and iterating

\[
\gamma_m(t) = \gamma_{mn} + (i \hbar) \int_t^{t_f} \gamma_{mn}(t) dt + (2i \hbar)^2 \int_t^{t_f} \int_t^{t_f} \gamma_{mn}(t_1) \gamma_{mn}(t_2) dt_1 dt_2 ...
\]

and then identify \( \gamma_m(t_f) = \lambda_{mn}(t, t_f) \) such that

\[
\lambda_{mn}(t, t_f) = \gamma_{mn} + (i \hbar) \int_t^{t_f} \gamma_{mn}(t) dt + (2i \hbar)^2 \int_t^{t_f} \int_t^{t_f} \gamma_{mn}(t_1) \gamma_{mn}(t_2) dt_1 dt_2 ...
\]

If we consider a constant perturbation \( V \) applied at time \( t_i \), we find

\[
\lambda_{mn}(t, t_f) = \lambda_{mn}(t_i-t_f) = \gamma_{mn} + (i \hbar) \int_{t_i}^{t_f} \gamma_{mn}(t) dt + ...
\]

\[
= \gamma_{mn} + \frac{e^{-i(E_m - E_n)T} - 1}{(E_m - E_n)} + ...
\]

The idea here is that if we can represent the state at \( t_i \) as a pure state \( |\phi_m\rangle \) of the unperturbed Hamiltonian \( H_0 \), then the probability the state will be in an unperturbed state \( |\phi_n\rangle \) at \( t_f \) is \( \gamma_{mn} \). Note that since \( \gamma_{mn}^2 \leq 0 \) in general, the perturbation \( V \) mixes the pure state \( |\phi_m\rangle \) and requires us to represent \( \langle \psi(t) | \rangle \) as a superposition of the \( |\phi_m\rangle \)'s. Of course this merely expresses the fact that \( |\phi_n\rangle \) is not a C.O.M. for the perturbed system.
Now we can define the S-matrix as

\[ S_{mn} = \chi_{mn}(-\infty, +\infty) \]

Then

\[ S_{mn} = S_{mn} + (-i) \int_{-\infty}^{\infty} e^{i(E_n - E_n)t} V_{mn}(t) dt + (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \left| \frac{e^{i(E_n - E_n)t_1}}{E_n - E_n} V_{mn}(t_1) \right|^2 \]

\[ + \ldots \]

\[ = S_{mn} + (-i) \sum_{\ell} V_{mn}^{F,T}(E_n - E_\ell) + (-i)^2 \sum_{\ell} V_{mn}^{F,T}(E_n - E_\ell) V_{mn}^{F,T}(E_\ell - E_n) \]

\[ + \ldots \]

Thus we can express the Born expansion of \( S_{mn} \) as the Fourier transforms of the matrix elements of the perturbation.

Now suppose again at \( t = t_i \), \( \{ \Psi(t) \} = |\psi_n(t)\rangle e^{-iE_n t_i} \). Then we require by conservation of probability,

\[ \sum_{\ell} |C_{\ell n}(t_i)|^2 = 1 \]

or since \( C_{\ell n} = e^{-iE_n t_i} \chi_{\ell n} \),

\[ \sum_{\ell} |\chi_{\ell n}(t_i)|^2 = 1 \]

Thus

\[ \sum_{\ell} |\chi_{\ell n}(t_i)|^2 = 1 \]

\[ \sum_{\ell} \left| \int_{t_i}^{t_f} \psi_{\ell n}(t) dt + (-i)^2 \int_{t_i}^{t_f} \chi_{\ell n}(t) dt_1 \int_{t_i}^{t_f} \chi_{\ell n}(t_1) \psi_{\ell n}(t_2) \right|^2 = 1 \]

This seems pretty unlikely.
Instead try a different test. Consider the first term of (1) as an integral equation

\[ X_m(t) = X(t) + (-i) \int_{t}^{+} V_{mn}(t_i) X_n(t_i) \, dt_i \]  \hspace{1cm} (10)

Define the integral operator

\[ \mathcal{L}(t, t_i) = (-i) \int_{t}^{+} V_{mn}(t_i) \, dt_i \]  \hspace{1cm} (11)

so that (10) becomes

\[ X_m(t) = X_m(t_i) + \mathcal{L} \{ X_m(t) \} \]

or

\[ X_m(t) = \left[ 1 - \mathcal{L}(t, t_i) \right]^{-1} X_m(t_i) \]  \hspace{1cm} (12)

Thus we can identify an "evolution operator" \( U_m(t, t_i) \)

\[ U_m(t, t_i) \equiv \left[ 1 - \mathcal{L}(t, t_i) \right]^{-1} \]  \hspace{1cm} (13)

such that

\[ X_m(t) = U_m(t, t_i) X_m(t_i) \]

It would even be better to define a \( U(t, t_i) \) such that

\[ |\Psi(t)\rangle = U(t, t_i) |\Psi(t_i)\rangle \]  \hspace{1cm} (14)
Central Potential Scattering:

\[ H = \frac{p^2}{2m} + V(r) \]

Set \[ H_0 = \frac{p^2}{2m} \]

Thus we use plane wave forms

\[ H_0 \phi_n \rangle = H_0 \phi_k \rangle = \frac{k^2}{2m} \phi_k \rangle \]

Thus \[ \phi_n \rangle = \phi_k \rangle \]

Thus we expand

\[ \langle \bar{\phi} | \phi \rangle = \frac{e^{ik \cdot \bar{r}}}{\sqrt{Vol}} \]

Actually we usually consider scattering from one continuum \( \phi_k \rangle \)

to another continuum \( \phi_{\bar{k}} \rangle \). Using our work on first Born approx

\[ \frac{\text{Rate}}{k \to \bar{k}} = 2\pi S(k^2 - \bar{k}^2) | V_{k\bar{k}} |^2 \]

Note the \( S(k^2 - \bar{k}^2) \) merely ensures conservation of energy.

Where \( V_{k\bar{k}} = \langle \bar{k} | V(r) | k \rangle \)

\[ = \int e^{ik \cdot r} V(r) \frac{d^3r}{Vol} = V(\hat{r}) \]

\[ = 2\pi \int_{-1}^{+1} d\theta \int_0^{2\pi} r dr e^{i\theta s} V(r) \]

Where \( s = \bar{k} - k \).
Problem: Due first class of second term

Choose one of the following:

a) Something you thought of yourself

b) It is commonly claimed in classes like Ph 205 that one can prove from the commutation relations for the rotation group that only half integer spins will appear.

Is this true?

The following has been suggested as a counterexample:

Take operators

\[ \hat{S}_x = \frac{1}{2} \left[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right] \]
\[ \hat{S}_y = \frac{1}{2} \left[ \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right] \]
\[ \hat{S}_z = \frac{1}{2} \left[ \frac{\partial}{\partial z} - \frac{\partial}{\partial x} \right] + \hat{S}_z \phi \]

\[ \phi = \arctan \frac{\hat{S}_x}{\hat{S}_y} \]

It is claimed that these commute like rotation group generators, but the \( \hat{S}_z \) associated with them is not half integral.

Where does the inconsistency lie?

c) Suppose a system in which two spins are combined, with a Hamiltonian of the form

\[ H = \alpha \hat{J}_x \hat{J}_y + B_z \left[ \frac{1}{j_a} (\hat{J}_x)^2 + \frac{1}{j_b} (\hat{J}_y)^2 \right] \]

where B is a fixed field.

1) For small \( B \), general spins, what are the energy levels of the combined system?

2) Find the energy as a function of \( B \) for arbitrary \( B \), in some special cases

\( \text{e.g.} \ \hat{J}_x = \hat{f}_1 \hat{f}_2 = \frac{1}{2} \)

or something similar

(of course, if you can solve the problem in complete generality....)

Combination of angular momentum states

Suppose there are two particles of spins \( \frac{1}{2}, \frac{1}{2} \) together in a system. The spin of each individual particle tells how it transforms when a rotation of the coordinates is made. Now we must determine how the entire system transforms.

\[ |\Psi\rangle = |\Psi_1\rangle |\Psi_2\rangle \]
Symbolize the rotation operation by \( R \)

\[
R |\psi\rangle = R |\psi_1\rangle = R |\psi_2\rangle
\]

\[
R |\psi_1\rangle = \mathcal{D}_j (R) |\psi_1\rangle
\]

\[
R |\psi_2\rangle = \mathcal{D}_{j_2} (R) |\psi_2\rangle
\]

where \( \mathcal{D}_j \) is a matrix which acts on the spin indices of the wave functions.

Thus we presumably have some matrix \( \mathcal{D} \) such that

\[
R |\psi\rangle = \mathcal{D} (R) |\psi\rangle
\]

To see what size \( \mathcal{D} \) should have:

\[
|\psi_1\rangle \text{ has } (2j_1 + 1) \text{ components}
\]

\[
|\psi_2\rangle \text{ has } (2j_2 + 1) \text{ components}
\]

\[
|\psi\rangle \text{ has } (2j_1 + 1)(2j_2 + 1) \text{ components}
\]

However, \( \mathcal{D} \) is not necessarily one of the fundamental ("irreducible") representations of rotations - i.e. it does not necessarily correspond to one particular spin.

Because it is a representation, it can be broken up into a direct sum of these irreducible representations.

This means that \( |\psi\rangle \) can be written as \( \sum c_i |\psi_i\rangle \) where the \( |\psi_i\rangle \) correspond to each to a particular spin, and the \( c_i \) are completely determined by \( |\psi_1\rangle \) and \( |\psi_2\rangle \):

\[
|\psi_1\rangle \quad |\psi_2\rangle = \sum c_i |\psi_i\rangle
\]

The problem is to determine:

a) Which \( j_i \) will appear

b) What are the \( c_i \)

Both of these questions are answered by mechanically applying rotation operators and seeing how the state transforms. It may seem dull to go through this machinery, because in practice the \( c_i \) (Clebsch-Gordan coefficients) are usually found by looking in a table.

However, other continuous groups are sometimes relevant, and not all the coupling coefficients have been calculated. Thus it is worthwhile knowing what to do.

**ONWARD!**

I) Apply an infinitesimal rotation about the z axis \( (1 + \varepsilon \mathcal{J}_z) |\psi\rangle \rightarrow (1 + \varepsilon \mathcal{J}_z) |\psi\rangle \)

\[
= (1 + \varepsilon \mathcal{J}_z) |\psi_1\rangle \quad |\psi_2\rangle = (1 + \varepsilon \mathcal{J}_z) |\psi_1\rangle \quad |\psi_2\rangle
\]

Here \( \mathcal{J}_z \) is just the appropriate operator in \( \mathcal{D} \)

\[
\Rightarrow \mathcal{J}_z |\psi\rangle = m |\psi\rangle \quad m = m_1, +m_2
\]
II) To find which \( j \) appear

As \( m \) ranges over its \( 2j+1 \) values

\[ M = \sum_{m=0}^{M} 2j - 1 \]

\[ \text{Max}(m_1 + m_2) = j_1 + j_2 \]

Hence this is the maximum value of \( J \) allowed in the decomposition.

If \( \langle \Psi | J_{-} | \Psi \rangle \) is then a new thing whose possible \( \text{Max} J \) is \( j_1 + j_2 - 1 \) etc.

This goes on until \( \langle \Psi \rangle \) is exhausted.

to see how many steps are allowed, do a little calculation:

\[ (2j_1 + 1)(2j_2 + 1) = \sum_{n=0}^{M} 2(j_1 + j_2 - n + 1) \]

\[ = 2(j_1 + j_2)(M+1) - M^2 + 1 \]

\[ \Rightarrow M = 2j_1 \text{ or } 2j_2 \]

\( j_1 + j_2 - 2j_1 = j_2 - j_1 \)

\( j_1 + j_2 - 2j_2 = j_1 - j_2 \)

Then the minimum \( J \) is \( j_1 - j_2 \) or \( j_2 - j_1 \) whichever is physical

Rule 2) The \( J \)'s allowed vary in steps of 1 from \( j_1 + j_2 \) to \( j_1 - j_2 \)

(the steps are 1 because \( m \) steps necessarily come in ones)

III) To determine the \( a_j \)

Clearly it makes no difference whether we begin with a basis \( |j,m> \)

and expand it as \( \sum b_{j_1,j_2,j,m} |j_1,m_1> |j_2,m_2> \)

or start with \( |j_1,m_1> |j_2,m_2> \) = \( \sum a_j |j,M> \)

It is most convenient to find the \( b_{j_1,j_2} \)

\[ |j,M> = \sum b_{j_1,j_2} |j_1,m_1> |j_2,m_2> \]

Clearly \( |-j,M> = |j_1,j_2> |j_2,j_1> \) \( \sqrt{2j_1+1} \sum M = j_1 + j_2 \)

We then just apply \( J_{-} \) to both sides. Use of the relations derived last week \( \sum |j,M> = c_j |j,M> \) will then completely specify the \( b_{j_1,j_2} \)

A word of caution is in order here:

In the most general case it is quite possible that beginning at \( \Lambda \) and applying

might lead to different phases for the \( b_{j_1,j_2} \) than beginning at the top and

working down. For the rotation group it happens that the states can be chosen
such that the phases for the two cases come out the same. However, not all Clebsch-Gordan tables use the same phase convention. Thus, unless you generate all your own Clebsch-Gordan coefficients it is absolutely essential never to use more than one table of C-G coeffs. at a time (i.e. within a given problem).

Assume all the states \(|\sum_{m', M} M'\rangle\) have been determined by this method. We next must find \(|\sum_{m', m''} M'\rangle\). To do this, notice that \(|\sum_{m, m} M\rangle\) is orthogonal to \(|\sum_{m', m''} M'\rangle\) and that there are only two possible states with \(M = \sum M'\) made up of \(|j_1, m_1\rangle |j_2, m_2\rangle\).

Then just ladder down from this new state, etc.

Suppose you just want some state which would be in the middle of one of the above series and can't bear the thought of constructing all the others.

Then the following trick may help:

\[
|\sum_{m', m''} M'\rangle = \sum b_{m', m''} |j_1, m_1\rangle |j_2, m_2\rangle
\]

\[
|\sum_{m'} M\rangle = \sum (\sum b_{m', m''}) |j_1, m_1\rangle |j_2, m_2\rangle
\]

\[
= \left(\sum b_{a}^2 + \sum b_{b}^2 + 2 b_{a} b_{b} \sum \sum \right) i |j_1, m_1\rangle \sum b_{m', m''} |j_2, m_2\rangle
\]

Comparison of the equations

\[
|\sum_{m'} M\rangle = \sum b_{m', m''} |j_1, m_1\rangle |j_2, m_2\rangle
\]

\[
|\sum_{m'} M\rangle = \frac{1}{\sum |j_1, m_1\rangle |j_2, m_2\rangle}
\]

will often allow you to determine the b's

Suppose you start out with n objects of spin \(\frac{1}{2}\), all spinning up \(\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow\) and then ladder down.

The state is initially symmetric under permutations of the spinning objects. The ladder operator is also symmetric. Hence all the states of this J, the highest one possible for combinations of these spinors, are symmetric (This is one of those things its useful to have filed away somehow)

It happens that all the representations of the rotation group can be found by appropriately combining spinors in this way.

i.e. the representation with \(J = m'\) is obtained from a symmetric combination of n spinors
Isotopic Spin

One of the symmetries first noticed in nuclear physics is that the nuclear forces between neutron and proton appear to be the same as those between neutron and neutron, and proton and proton.

Further, one can even apply a transformation of the sort

\[ |\rho\rangle \rightarrow \alpha|\rho\rangle + \beta|\eta\rangle \]
\[ |\eta\rangle \rightarrow -\beta^*|\rho\rangle + \alpha^*|\eta\rangle \]

\[ |\alpha|^2 + |\beta|^2 = 1 \]

without changing any of the forces.

The pair \( \alpha/\beta \) determines an element of this group of transformations.

The matrix \( \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \) acts on the pair \( |\rho,\eta\rangle \) in exactly the same way as an element of the rotation group acts on an object of spin \( \frac{1}{2} \). Hence the set \( |\rho,\eta\rangle \) is called an isotopic spinor; and all the Clebsch-Gordanry involved with isotopic spin is identical to that for rotations.

If the only representation of isotopic spin present in nature were the neutron-proton pair, this analogy would not be particularly useful. However, many isotopic families have now been identified:

- **spin 0 representation**
  \[ \Lambda \]

- **spin \( \frac{1}{2} \) representations**
  \[ \frac{m}{2} \rightarrow \frac{1}{2} \]

- **spin 1 representations**
  \[ k^I \rightarrow k^0 \]

- **spin 3/2 representation**
  \[ \Delta - \Delta^0 \Delta^+ \Delta^{++} \]

are a few examples.

Notice that the rules derived above for coupling angular momentum already tell us something about strong interaction forces:

if we wish to make a \( \Lambda \) particle as a bound state of 2 particles, we must use a \( \Sigma^I \) combination - a \( \Omega^I \) one will not work (\( T = 0 \) cannot couple to \( T = 1 \)).

The fact that \( \Delta \) has \( T = \frac{3}{2} \) makes its decay into \( \Omega^I \) easily classified.
SU(3)

Both isotopic spin and rotations can be built up from a group of transformations on spinors, where the matrices acting on the spinors have the form

\[
\begin{pmatrix}
  \alpha & \beta \\
  -\beta^* & \alpha^*
\end{pmatrix}
\]

\[|\alpha|^2 + |\beta|^2 = 1\]

This is the group of special (i.e., determinant = 1) unitary transformations on two dimensions SU(2).

Suppose that instead of two "fundamental" objects there were three. Then we would have SU(3) as the matrix group.

We could go ahead and combine the fundamental 3 dimensional representation with itself to make more representations, as we did with the two dimensional spinors for SU(2).

Unfortunately it is not possible to get all the representations of SU(3) this way.

One other three dimensional representation must be introduced. Then products of \( \bar{3} \otimes \bar{3} \) (the two 3 dimensional reps.) will generate all the representations.

It is easy to classify particles into isotopic spin multiplets because the masses of particles within a multiplet are very close together (they would be the same if there were no electromagnetic perturbations acting).

Classifying particles into SU(3) representations is more difficult because the symmetry is not exact (see discussion of Wigner-Eckart Theorem below). This means that the particles which belong to a given multiplet may have masses which differ by several hundreds of Mev. Nevertheless several representations have been identified:

**Baryons**

Octet Spin \( \frac{1}{2} \) Parity +

\[
\begin{align*}
\Sigma^+ & \quad \Sigma^0 & \quad \Lambda & \quad \Sigma^- \\
\Xi^0 & \quad \Xi^- & \\
\end{align*}
\]

Decuplet Spin 3/2 Parity +

\[
\begin{align*}
\Xi^0 & \quad \Xi^- \quad \Xi^+ & \quad \Xi^{*-} & \quad \Xi^{*0} \quad \Xi^{*+} \\
\end{align*}
\]

**Mesons**

Octet Spin 0 Parity -

\[
\begin{align*}
K^- & \quad \bar{K}^0 & \quad \bar{K}^+ \\
\eta^- & \quad \eta^0, \eta & \quad \eta^+ \\
\end{align*}
\]
Octet  Spin 1  Parity -

\[ K^- \quad K^0 \]
\[ \rho^- \quad \sigma^0 \quad \rho^+ \]
\[ K^{*0} \quad K^{*+} \]

No one has yet found members of the \( \frac{1}{2} \) or \( \frac{3}{2} \) representations. In most theories these are expected to have the following properties:

- **Particle a)** strangeness -1  isospin 0  charge 0
- **Particle b)** strangeness 0  \( T = \frac{1}{2} \)  charge 1/3
- **Particle c)** strangeness 0  \( T = \frac{3}{2} \)  charge -2/3

Lists the particles in one rep. The ones in the others are charge conjugates of these. b) and c) would be easily spotted (we hope) due to their fractional charge. People have looked for these in quite a number of experiments and have not yet found anything that is statistically believable.

All of the physics involved with the above groups has to do with calculating matrix elements of operators between states which belong to various representations of the group. The most powerful tool for doing this uses the transformation properties of the operators under generators of the group. In many books this method can be found masquerading behind the name **WIGNER ECKART THEOREM**

1) Given any operator, decompose it into pieces according to its transformation properties under spatial rotations:

- scalars: \( \vec{X}, \vec{\rho} \) and other dot products
- vectors: \( \vec{X}, \vec{\rho}, \vec{X} \times \vec{\rho}, \ldots \) etc.
- tensors \( \vec{\vec{X}}, \vec{\vec{\rho}} \)

2) Re-express these so that they are separated into things with exact spin transformation properties:

\[ \vec{\vec{X}}, \vec{\vec{\rho}} \] is an exact scalar which transforms like spin 0

\[ \vec{\rho} \] is an exact vector with transforms like spin 1

What do we mean "transforms like spin 1?"

The spherical harmonics \( Y_5^m \) are eigenfunctions of total angular momentum with \( z \) projection \( M \). Under a rotation of the coordinates, the \( Y_5^m \) for a given \( J \) transform among themselves according to the appropriate \( \vec{D}^J_5 \)

\[ Y_5^m (\theta_2) = \sum_{m'} \vec{D}^J_5 (\theta_2, \theta_1) Y_5^{m'} (\theta_1) \]
Any wave function with these properties transforms like Spin \( J \).

Now look at operators of the type shown above.

The components of \( \eta_\lambda^\mu \) can be made to correspond to certain combination of \( a_\lambda \) for a vector \( \sqrt{2} \sum a_\lambda \hat{e}_\lambda \)

\[
\gamma_{1,1} \leftrightarrow -\sqrt{2} (\hat{e}_x + i \hat{e}_y) \quad \gamma_{1,1} \leftrightarrow \sqrt{2} (\hat{e}_x - i \hat{e}_y)
\]

\[
\gamma_{1,0} \leftrightarrow \hat{e}_z
\]

But \( \gamma_2 \) has only 5 components, whereas the spatial tensor \( \hat{e}_i \hat{e}_j \)
has 9. Hence \( \vec{\lambda} \vec{\rho} \) cannot transform exactly like spin 2.

To figure out the proper spin decomposition of something like \( \vec{\lambda} \vec{\rho} \), notice the following:

a) a generalization of the arguments for spinor representations made above shows that every integral spin \( n \) representation can be formed from a symmetric combination of \( n \) spin 1 objects.

Hence if we have two spin 1 objects (\( \vec{\lambda}, \vec{\rho} \)), the spin two piece (highest spin formed) must not contain any antisymmetric combinations.

symmetric piece

| \( \hat{e}_x \hat{e}_x \) | \( \hat{e}_x \hat{e}_y + \hat{e}_y \hat{e}_x \) | \( \hat{e}_z \hat{e}_z \) |
| \( \hat{e}_x \hat{e}_y + \hat{e}_y \hat{e}_x \) | \( \hat{e}_y \hat{e}_z + \hat{e}_z \hat{e}_y \) | \( \hat{e}_z \hat{e}_z \) |
| \( \hat{e}_x \hat{e}_z + \hat{e}_z \hat{e}_x \) | \( \hat{e}_y \hat{e}_z + \hat{e}_z \hat{e}_y \) | \( \hat{e}_z \hat{e}_z \) |

There are 6 independent symmetric pieces. We still can't hope to put this in one-to-one correspondence with a spin 2 object.

antisymmetric piece

\[
\begin{align*}
\hat{e}_x \hat{e}_y - \hat{e}_y \hat{e}_x & \quad \hat{e}_x \hat{e}_z - \hat{e}_z \hat{e}_x \\
\hat{e}_y \hat{e}_z - \hat{e}_z \hat{e}_y & \quad \hat{e}_z \hat{e}_z \quad \hat{e}_z \hat{e}_z
\end{align*}
\]

There are three independent elements here.

We recognize this as \( \vec{\lambda} \vec{\rho} \) a vector and thus a spin 1 object.

\( \vec{\lambda} \vec{\rho} \) can contain only spins 2, 1, 0 because \( \vec{\lambda} \) and \( \vec{\rho} \) are spin 1. The only remaining problem is to separate the spin 0 from the spin 2.

spin 0 looks like \( \vec{\lambda}, \vec{\rho} \) and thus corresponds to \( \hat{e}_x \hat{e}_x + \hat{e}_y \hat{e}_y + \hat{e}_z \hat{e}_z \)

Therefore any piece of our tensor of the form \( a \hat{e}_x \hat{e}_x + b \hat{e}_y \hat{e}_y + c \hat{e}_z \hat{e}_z \)
is a linear combination of spin 2 and spin 0.

It is clear that there are many ways to separate out the two, but to preserve the
symmetry under interchange of $x, y$ and $z$ one writes
\[ \delta_x \delta_y \left[ a - \frac{1}{3} (a + b + c) \right] + \delta_y \delta_z \left[ b - \frac{1}{3} (a + b + c) \right] + \delta_z \delta_x \left[ c - \frac{1}{3} (a + b + c) \right] + \frac{(a + b + c)}{3} \left[ \delta_x \delta_y + \delta_y \delta_z + \delta_z \delta_x \right] \]

Spin decomposition of

| spin 0 | $\frac{1}{\sqrt{2}} (x, y)$ |
| spin 1 | $\frac{1}{\sqrt{2}} (x, -y)$ |
| spin 2 | $\frac{1}{\sqrt{2}} (x + i y, z)$ |

Having explicitly constructed all 5 independent elements of the spin 2 tensor, we put them in one-to-one correspondence with the $\frac{\gamma_{12}^M}{\sqrt{5}}$

Define

\[
\begin{align*}
X_i^1 &= -\frac{1}{\sqrt{2}} (x + i y) \\
X_i^{-1} &= \frac{1}{\sqrt{2}} (x - i y) \\
X_i^0 &= \frac{1}{\sqrt{5}} \left( x + y, z \right)
\end{align*}
\]

Define

\[
\begin{align*}
P_i^1 &= -\frac{1}{\sqrt{2}} (p_x + i p_y) \\
P_i^{-1} &= \frac{1}{\sqrt{2}} (p_x - i p_y) \\
P_i^0 &= \frac{1}{\sqrt{5}} (p_x + i p_y + p_z)
\end{align*}
\]

Solving backwards gives

\[
\begin{align*}
X_i^1 P_i^1 &= \frac{1}{\sqrt{2}} \left( i (p_x + x p_x) + x p_x - y p_y \right) \\
X_i^{-1} P_i^{-1} &= \frac{1}{\sqrt{2}} \left( i (p_x + x p_y) + x p_x - y p_y \right) \\
X_i^0 P_i^0 &= \frac{1}{\sqrt{5}} \left( x + y, z \right)
\end{align*}
\]

\[
\begin{align*}
X_i^1 P_i^1 + X_i^{-1} P_i^{-1} + 2X_i^0 P_i^0 &= \frac{1}{\sqrt{5}} \left( x + y, z \right) \\
X_i^0 P_i^0 &= \frac{1}{\sqrt{5}} \left( x + y, z \right)
\end{align*}
\]

Now refer to a table of Clebsch-Gordan coefficients and construct spin 2 from two spin 1 objects

\[
\begin{align*}
(1 1 1 1 2 2) &= 1 \\
(1 1 1 -1 1 2 -2) &= 1 \\
(1 1 1 -1 1 2 0) &= \frac{1}{\sqrt{6}} \\
(1 1 1 -1 0 1 2 0) &= \frac{1}{\sqrt{6}} \\
(1 1 0 1 2 -1) &= \frac{1}{\sqrt{2}} \\
(1 1 0 1 0 2 -1) &= \frac{1}{\sqrt{2}} \\
(1 1 0 1 2 1) &= \frac{1}{\sqrt{2}} \\
(1 1 0 1 1 2 1) &= \frac{1}{\sqrt{2}}
\end{align*}
\]

Hence

\[
\begin{align*}
\gamma_2^2 &= X_i^1 P_i^1 \\
\gamma_2^{-2} &= X_i^{-1} P_i^{-1} \\
\gamma_2^1 &= \frac{1}{\sqrt{2}} (X_i^0 P_i^1 + P_i^0 X^{-1}) \\
\gamma_2^{-1} &= \frac{1}{\sqrt{2}} (X_i^0 P_i^1 + P_i^0 X^{-1}) \\
\gamma_2^0 &= \frac{1}{\sqrt{6}} \left( X_i^0 P_i^1 + P_i^0 X^{-1} + 2 X_i^0 P_i^0 \right)
\end{align*}
\]

Thus certain linear combinations of the $X_i^0 P_i^1$ correspond exactly to spherical harmonics.
In general the rule is:

Take the operator in question and break it up into pieces which transform under rotation like the spherical harmonics. Call these pieces

\[ Y^m_j \]

Then the rule for matrix elements is simple.

\[ \langle \alpha j_1 m_1 | Y^m_j | \alpha j_2 m_2 \rangle = C_{\Delta} \langle j_2 j_1 m_1 m_2 \rangle \]

where \( \Delta \) specifies additional quantum numbers of the system.

Here \( C_{\Delta} \) is a number that cannot be obtained by angular momentum arguments.

To prove this:

\[ Y^m_j | \alpha j_2 m_2 \rangle \]

transforms under rotation like

\[ \bar{\text{\( \Delta \)}} \alpha j_{\bar{\text{\( \Delta \)}}} \]

where the \( \alpha_j \) are the Clebsch-Gordan coefficients.

A matrix element is just a number; hence it must be a rotational scalar. We can form this only if

\[ \langle j_1 m_1 | = \langle j_2 j_1 m_1 m_2 \rangle \text{ same } j_2, \bar{\text{\( \Delta \)}} \]

As far as the rotational character of the states and operators is concerned, \( \alpha_j \) completely specifies the matrix element.

Notice that it is not necessary to think of operators in terms of spherical harmonics - you can do exactly the same thing by considering only the \( \overrightarrow{\text{\( \Delta \)}} \) operator, as follows:

\[ [ \overrightarrow{x}, \overrightarrow{p}, \overrightarrow{\text{\( \Delta \)}} ] = 0 \]

an operator's angular momentum properties are determined completely by its commutation relations with \( \overrightarrow{\text{\( \Delta \)}} \). Because \( \overrightarrow{x}, \overrightarrow{p} \) commutes with \( \overrightarrow{\text{\( \Delta \)}} \) in the same way as the unit operator, we know

\[ \langle m_1 j_1 | \overrightarrow{x}, \overrightarrow{p} | m_2 j_2 \rangle = \alpha \langle m_1 j_1 | 1 | m_2 j_2 \rangle = \alpha d_{m_1 m_2 j_1 j_2} \]

where \( \alpha \) has no dependence

\[ [ \overrightarrow{\text{\( \Delta \)}} j_1 j_2 ] = \alpha^{\overrightarrow{\text{\( \Delta \)}} j_1 j_2} P_n \]

where the \( \alpha^{\overrightarrow{\text{\( \Delta \)}} j_1 j_2} \) are the same as in

\[ [ \overrightarrow{\text{\( \Delta \)}} j_1 j_2 ] = \alpha^{\overrightarrow{\text{\( \Delta \)}} j_1 j_2} \overrightarrow{\text{\( \Delta \)}} \]

Hence

\[ \langle m_1 j_1 | \overrightarrow{p} | m_2 j_2 \rangle = \alpha \langle m_1 j_1 | \overrightarrow{p} | m_2 j_2 \rangle \]

The operator \( \overrightarrow{x}, \overrightarrow{p} \) as one would expect from the above discussion, does not have exactly the same commutation relations as the most simple tensor of the form \( \overrightarrow{\text{\( \Delta \)}} \overrightarrow{\text{\( \Delta \)}} \).

Exercise (for fun) do the operator decomposition of \( \overrightarrow{x}, \overrightarrow{p} \) that corresponds to our spherical harmonic decomposition.
It is often helpful in computation to realize that algebraic simplifications exist for a system of given spin.

Example

\[
(J_2 + iJ_3) J_2 \begin{pmatrix} J_2 & -i \end{pmatrix} \mid 1, m \rangle = 0
\]

allows you to reexpress products \( J_2^3 \) in terms of \( J_2^2 + J_2 \).

Similar things can be done for \( J_x + J_y \).

APPLICATIONS OF WIGNER-ECKART THEOREM

Directly:

a nucleus of spin 0 has no magnetic moment

a nucleus of spin \( \frac{1}{2} \) has no quadrupole moment

Suppose the Hamiltonian is invariant under rotation. It is thus a scalar \( \left( C J_3 \right) \).

Hence it cannot alter the \( J \) of \( H \) of a system \( \langle j_1, m_1, \rangle \big/H \big/ j_2, m_2 \rangle \sim \delta_{j_1, j_2} \delta_{m_1, m_2} \).

Angular momentum is thus conserved in interactions governed by this Hamiltonian.

When viewed in the light of isotopic spin, this argument is repeated as follows:

If \( H \) is invariant under rotations in isotopic spin space, then it cannot connect states of different isotopic spin and \( z \) component of isospin, \( T_z \) must then be conserved.

The energy of a particle at rest is its mass; this is just the expectation value of the Hamiltonian in that state.

\( H \) a scalar in isospin space implies

\[
\langle \alpha | T_z | \beta \rangle \big/H \big/ \langle \alpha | \tilde{T_z} \rangle = \langle \alpha | T \cdot O_T \cdot T \rangle \langle \alpha | T \rangle
\]

which is independent of \( T_z \). Thus states which differ only in \( T_z \) should have the same mass.

The strong interaction Hamiltonian is believed to be invariant under rotations in isospin space. It follows that if this were the only force acting, one would expect that all particles in a given isospin multiplet (which differ only by \( T_z \)) should have the same mass.

Neutron and proton have masses differing by about 1.3 Mev. They differ only in \( T_z \). However there is another piece in our total Hamiltonian, formed by the
electromagnetic interaction.

The electromagnetic energy must have a piece proportional to the charge. In the neutron-proton system this would be written \( k(1+\hat{S}_z) \). But \( T_\perp \propto \hat{S}_z \).

Thus the Hamiltonian contains an isospin vector. It is this piece which (in a general sense) accounts for the mass difference.

Suppose that the Strong Interaction Hamiltonian were invariant under rotations in \( SU(3) \) space. Then all the elements of an \( SU(3) \) multiplet would be expected to have the same masses (The \( \text{Wigner-Eckart Theorem} \) generalizes easily to a group like \( SU(3) \)).

However it is observed that the particles assigned to a multiplet do not all have the same mass. Hence the first conclusion to draw is that \( SU(3) \) is only an "approximate" symmetry of the Hamiltonian.

In fact, the "broken" symmetry is much more interesting in one sense than a complete symmetry would be. This is because we can expand the Hamiltonian in the \( SU(3) \) equivalent of \( \sum_{\lambda} \hat{S}_\lambda \).

\[ H = \sum_{\lambda} A \hat{S}_\lambda \]

Study of the mass differences and coupling constants indicates that most of the non-symmetric piece belongs to one representation of \( SU(3) \) (the octet representation).

This is analogous to the \( \frac{1}{4} \hat{S}_z \) electromagnetic term; but so far the explanations of "octet dominance" are not as simple to understand as is the presence of electric charge.

An elegant and reasonably complete discussion of the properties of operators under rotation (and decomposition into irreducible tensor operators) can be found in Messiah Vol. II (first chapter and appendix).

In practice one usually ends up constructing things explicitly out of vectors by use of Clebsch-Gordon coefficients, as shown above.
Solid State Physics

There is a rich variety of phenomena associated with materials in the solid state. A very general approach is likely to lose this variety, so it is wise to think in terms of specific examples.

There are many places to read about the phenomena (eg., the Scientific American). The best way to understand how a phenomenon works is to figure it out for yourselves. The second-best way is to read a book. I suggest


In books and courses, there is an (unhealthy?) emphasis on methods. Examples, however, are essential, because very different approximations are used for different solids.

Types of Solids

1) Groups of atoms form molecules, and periodic arrangements of molecules form a **molecular crystal**. The molecules have no permanent electric dipole moment and are held together by Van der Waals forces.

   Ex. Octane.

2) Another type of solid is known as a **covalent crystal**. The molecules are held together by covalent bonds (electron sharing) between molecules.

   Ex. Diamond.

3) **Ionic Crystal**. Ex. NaCl.

   Each atom in the crystal has a net charge and is held in place by electrostatic attraction.
4) **Metals.** Certain electrons are no longer even approximately localized at definite atomic sites.

Ex: Liquid mercury is a metal.

Metals are relatively soft. That is because the atoms can be moved with respect to each other (as in bending) without serious modification of the electron wavefunctions. Distortion of the perfect lattice generally results in increased hardness.

We will develop two topics in our excursion into solid state and molecular physics:

1) Vibrational motions of atoms about equilibrium positions.

2) Phenomena associated with the movement of electrons through the metallic lattice.

First, we will consider briefly the effect of temperature.

Consider an arbitrary crystal (or molecule). There exist energy levels $E_i$ for the crystal, and the relative occupation probability $p_i$ at temperature $T$ is given by the Boltzmann law:

$$p_i = e^{-E_i/kT}$$

Normalization: $\sum_i p_i = 1 \Rightarrow Q = \sum_i e^{-E_i/kT}$

Then the average energy of the crystal is

$$U = \sum_i p_i E_i = \frac{1}{Q} \sum_i E_i e^{-E_i/kT}$$
There are common mathematical tricks that are useful in calculations at temperature $T$:

1) If you've found $Q = \sum_i e^{-E_i/kT}$

Then $\frac{\partial Q}{\partial T} = \frac{1}{kT^2} \sum_i E_i e^{-E_i/kT} = \frac{UQ}{kT^2}$

So that $U = \frac{kT^2 \partial Q}{Q}$.  $Q$ is called the Partition Function.

2) Let $Q = e^{-F/kT}$

then $F = -kT \log Q$  

$F$ is called the Free Energy.  

(A note of caution: there are several different kinds of "free energy" in use.)

$$U = -T \frac{\partial}{\partial T} \left( \frac{F}{T} \right)$$

$$= F - T \frac{\partial F}{\partial T}$$  

$T \frac{\partial F}{\partial T}$ is entropy.

1) Vibrations of Nuclei in a Crystal (or Polyatomic Molecule)

Problem: The Schrödinger Equation for the whole system (nuclei + electrons) is

$$\left[ \sum_j \frac{-\hbar^2}{2M_j} \frac{\partial^2}{\partial r_j^2} + V(R_1, R_2, \ldots, r_{x1}, r_{x2}, \ldots) \right] \psi = \frac{-\hbar^2}{i} \frac{\partial \psi}{\partial t}$$

Where $R_k$ is the position vector of the $k^{th}$ nucleus, 

$r_{x_k}$ """"""""axth electron,

and $V = \sum_{i,j=\neq k} \frac{Z_i Z_j e^2}{r_{ij}}$  

$r_{ij} = |R_i - R_j|$ or $|r_{x_i} - r_{x_j}|$, etc.
The problem is to show that the following equation is a good approximation for the state of lowest energy:

$$\sum_{\text{orbitals only}} -\frac{\hbar^2}{2m_e} \frac{\partial^2}{\partial x^2} \chi(x) + V(x) \chi(x) = \frac{i}{\hbar} \frac{\partial \chi}{\partial t} = E \chi$$

where the $R_i$ are fixed so in such a way that $E = E(R_i)$ is a minimum.

Show that to an excellent approximation

$$\Psi = \Psi(R_i) \chi(x_1, x_2, \ldots)$$

Discuss the terms that make the equation an approximation and estimate the size of the errors involved.

Exercise: (Don't hand in)

Find $F$ and $U$ for a harmonic oscillator with natural frequency $\omega$ and mass $m$ at temperature $T$.

The problem says that a Schrödinger Equation is approximately valid for the electron motion with fixed nuclei. Consider now the motion of the nuclei as small vibrations about the equilibrium configuration. Suppose the nuclei alone approximately obey a Schrödinger equation with hamiltonian $H$,

$$H = \sum_{i} -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + U(x_i)$$

where $(x_1, x_2, x_3) = \text{displacement vector 1st nucleus}$

$$(x_4, x_5, x_6) = \text{displacement vector of 2nd nucleus}$$

Then $U(x_i) = U(0) + \sum_i \frac{\partial U}{\partial x_i} \bigg|_{0} + \frac{1}{2} \sum_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} \bigg|_{0}$.
Choose the zero of energy so that \( U(0) = 0 \) for convenience.

At the equilibrium configuration, \( \frac{\partial U}{\partial x_i} \bigg|_0 = 0 \).

Consider the displacements to be small (valid for many applications).

Then
\[
H = \sum_i -\frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{ij} x_i x_j V_{ij}
\]

where \( V_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j} \bigg|_0 \).

Let \( q_i = \sqrt{m_i} x_i \) and \( C_{ij} = \frac{1}{\sqrt{m_i m_j}} \left( \frac{\partial^2 U}{\partial x_i \partial x_j} \right) \bigg|_0 \).

Then
\[
H = \sum_i \frac{p_i^2}{2m_i} + \frac{1}{2} \sum_{ij} q_i q_j C_{ij},
\]

where \( p_i = \frac{1}{i} \frac{\partial}{\partial q_i} \).

This problem can be analyzed classically and this is worthwhile because the results are rather similar to the quantum-mechanical case.

Classically,
\[
\ddot{q}_i = -\frac{\partial V}{\partial q_i} = -\sum_j C_{ij} q_j.
\]

A possible solution (periodic motion, really constant) is
\[
q_i = a_i e^{i \omega t}
\]

Then we obtain the equation
\[
\omega^2 a_i = \sum_j C_{ij} a_j.
\]

This is an eigenvalue equation and has solutions in \( \omega^2 \) where
\[
\det (C - \omega^2 I) = 0.
\]
This equation is of the 11th degree in \( w^2 \), and has \( n \) real solutions (real because \( c_{ij} \) real and \( c_{ij} = c_{ji} \)).

Label these solutions \( w^2 = \alpha \), \( \alpha = 1, \ldots, n \) \( (n = \text{number of coordinates}) \)
\( = 3N, \) \( N = \text{no. of nuclei} \)

Let the corresponding eigenvectors be designated \( a_i^{(w)} \).

i.e., \( w^2 a_i^{(w)} = \sum c_{ij} a_j^{(w)} \)

\( \alpha \) is known as the mode number.

We can choose normalization of the \( a_i \)’s so that

\[ \sum a_i^{(w)} a_i^{(w)} = \delta_{\alpha\beta} \]

Then it is also true that \( \sum a_i^{(w)} a_i^{(w)} = \delta_{\alpha\beta} \) (corresponding to different eigenvalues)

(That the \( a_i \)’s are orthogonal is a well-known property of the eigenvectors of hermitian matrices).

Example: \( \text{CO}_2 \) molecule.

\[ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \]

If you push one of the atoms, the other two atoms will respond and a complex motion will ensue.

Guess at the normal modes:

\[ x_1 = x_2 = x_3 = \frac{1}{\sqrt{3}} \]

3 modes of frequency \( \omega \) (no restoring force)
(same in \( y, z \) directions for other 2 modes)
(2) \[ x^{(1)} = -x^{(2)} = -\frac{1}{\sqrt{2}} \quad x^{(3)} = 0 \] other displacements 0.

The same in the other 2 directions yields 3 modes of this type. All three frequencies are the same.

(3) \[ \frac{1}{\sqrt{6}} \quad \frac{2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \]

+ same in other directions 3 modes of same frequency.

(The numbers were chosen so as to ensure orthogonality and normalization to unity.)

Quantum-mechanically, instead of talking about displacements, we write the motion as a superposition of the different modes, giving the amplitude to be in each mode.

Let us make a linear transformation of coordinates using the \( a^{(i)} \) we found classically,

\[ q_i = \sum_a a^{(i)} \xi_a \]

Then \[ Q_\alpha = \sum_i a^{(i)} q_i \]

The modes separate when we use these coordinates.

Let \[ P_\alpha = \frac{1}{i} \frac{\partial}{\partial Q_\alpha} \]
Then \( \frac{\partial}{\partial q_i} = \sum_{\alpha} \frac{\partial a_{\alpha}^{(i)}}{\partial q_i} \frac{\partial}{\partial a_{\alpha}} = \sum_{\alpha} a_{\alpha}^{(i)} \frac{\partial}{\partial a_{\alpha}} \)

\[ p_i = \sum_{\alpha} a_{\alpha}^{(i)} p_{\alpha} \quad \text{(and, incidentally, } p_0 = \sum_{i} a_{\alpha}^{(i)} p_i)\]

\[ \frac{1}{2} \sum_i p_i^2 = \frac{1}{2} \sum_{i, \beta} a_{i}^{(\alpha)} p_{i} a_{\beta}^{(\beta)} p_{\beta} \]

\[ = \frac{1}{2} \sum_{\alpha, \beta} \delta_{i, \beta} p_{i} p_{\beta} = \frac{1}{2} \sum_{\alpha} p_{\alpha}^2 \]

And \( \sum_{i, j} c_{ij} q_i q_j = \frac{1}{2} \sum_{i, \beta} c_{ij} a_{i}^{(\alpha)} a_{j}^{(\beta)} a_{\alpha} a_{\beta} \)

\[ = \frac{1}{2} \sum_{\alpha, \beta} \omega_{\beta}^2 a_{i}^{(\alpha)} a_{i}^{(\beta)} a_{\alpha} a_{\beta} \]

\[ = \frac{1}{2} \sum_{\alpha} \omega_{\alpha}^2 q_{\alpha}^2 \]

So that \( H = \sum_{\alpha} (H_{\alpha})_{\text{osc.}} \), where \( (H_{\alpha})_{\text{osc.}} = \frac{p_{\alpha}^2}{2} + \frac{1}{2} \omega_{\alpha}^2 q_{\alpha}^2 \)

First, what does a particular energy level look like?

\[ E = \sum_{\alpha} (n_{\alpha} + \frac{1}{2}) \hbar \omega_{\alpha} \quad n_{\alpha} \text{ are nonnegative integers.} \]

Each choice of the set of \( n_{\alpha} \) corresponds to a possible state of the system.

Wavefunctions \( \psi \equiv \prod_{\alpha} \phi_{n_{\alpha}}(Q_{\alpha}) \)

where \( \phi_{n_{\alpha}}(Q_{\alpha}) \) is the wavefunction for the \( n_{\alpha} \)-th excited state of a harmonic oscillator.

\[ \phi_{0}(x) \propto e^{-\frac{x^2}{2}} \quad (= e^{-\frac{p_{\alpha}^2}{2\hbar}} \text{ is a harmonic for remembering}) \]
Ground state: All $n_x = 0$

$$E_{\text{ground state}} = \sum_{2} \frac{1}{2} \hbar \omega x^2 + U(0)$$

And wavefunction $\Psi = \prod_{i} e^{-\frac{1}{2} \hbar \omega x^2} = e^{-\frac{1}{2} \sum_{i} \hbar \omega x^2}$

where $\chi_i = e^{-\frac{1}{2} \hbar \omega z^2}$

Exercise: Show $\gamma = C$.

At temperature $T$, the average excitation of a harmonic oscillator is

$$\bar{n} = \frac{1}{1 - e^{-\frac{\hbar \omega}{kT}}}$$

At temperature $T$, internal energy of vibration

$$U = \sum_{n} \frac{\hbar \omega}{2} (n + \frac{1}{2})$$

$$= \sum_{n} \frac{\hbar \omega}{2} + \sum_{n} \frac{\hbar \omega}{2} \frac{1}{1 - e^{-\frac{\hbar \omega}{kT}}}$$

And to obtain the specific heat, merely calculate $\frac{dU}{dT}$

**Theorem:** Average degree of excitation of an oscillator at temp $T$

$$\bar{n} = \frac{1}{1 - e^{-\frac{\hbar \omega}{kT}}}$$

**Proof:** $p_n = e^{-\hbar \omega / kT} e^{-\frac{1}{2} \hbar \omega / kT}$, But $\sum_{n} p_n = 1 \Rightarrow \bar{n} = \frac{e^{-\frac{1}{2} \hbar \omega / kT}}{1 - e^{-\hbar \omega / kT}}$
So that
\[ H = \sum \frac{n \cdot P_n}{n} \]
\[ = (1 - e^{-\frac{\bar{u} v}{kT}}) \sum \frac{n \cdot x^n}{n} \]
\[ = (1 - x) \frac{d}{dx} \left( \frac{1}{1-x} \right) \]
\[ = \frac{1}{1 - e^{-\frac{\bar{u} v}{kT}}} \]

Comment: \[ U = \frac{\frac{1}{2} \omega}{e^{\frac{\bar{u} v}{kT}} - 1} \]

This gives a specific heat curve as a function of temperature like

\[ \text{Specific Heat (one mode)} \]

This freezing-out of the contribution of the modes as \( T \to 0 \)
was the first indication of the failure of classical mechanics, since
this was discovered experimentally but not predicted theoretically.
This discovery was made in the 19th century by Maxwell.

Application: Scattering of a neutron by a molecule.

Question: Suppose the molecule is in the ground state before the collision.
What is the probability that after the collision the molecule remains
the ground state?

As an interaction, take \( V(R) = \alpha \frac{q^2}{R} \delta(R - q_k) \)

Initial state \( \psi_i(R, q) = e^{i \Phi_{in} \cdot k} \psi_i(q_k, \Phi_{in}) \)

Final state \( \psi_f(R, q) = e^{i \Phi_{out} \cdot k} \psi_f(q) \)
Probability $\propto |Me|^{2}$

Make Born approximation for $M$:

$$M \propto \int \psi_{f}^{*} e^{i(\mathbf{p}_{f} - \mathbf{p}_{i}) \cdot \mathbf{R}} s(\mathbf{R} - \mathbf{q}_{e}) \psi_{i} \ d^{3}R \ dq$$

$$\propto \int \int \cdots e^{-\frac{\varepsilon_{i}^{2} \mathbf{q}_{e}^{2}}{2\sigma_{i}^{2}}} e^{i\frac{\varepsilon_{f}^{2} \mathbf{q}_{e}^{2}}{2\sigma_{f}^{2}}} \ d\mathbf{q}_{e} d\mathbf{q}_{e}$$

$$\propto \frac{1}{2 \sqrt{\hbar \omega}} \ e^{-\frac{\varepsilon_{i}^{2} \mathbf{q}_{e}^{2}}{4\hbar \omega}} \ e^{-s^{2}}$$

where $s = \langle \mathbf{q}_{e}^{2} \rangle$.

Office hours, second term:

L. Jones  Friday 12-1.
G. Luxton  Monday 11-12.
WAVE ON INTERACTIONS OF MULTIASTATOMIC SYSTEMS

Last week we found that the probability for a polyatomic system in the ground state to gain momentum \( p \) in some elastic collision (i.e., neither system changes energy state) is

\[
\mathcal{C}^{-1} e^{-\rho^2 \langle q_k^2 \rangle}
\]

where \( \langle q_k^2 \rangle \) is the average square displacement of the oscillator.

\( p \) is here measured in the center of gravity system. Hence this is quite different from what one would expect classically, for a classical elastic collision between an incident neutron and an atom in the lattice would impart just one definite momentum to the lattice.

Some things, however, are similar to the classical case:

(Do any one of the exercises listed here or in the Jan 20 Lecture)

EXERCISE A:

Show that if the system is initially in the ground state, the mean value of the energy lost by a neutron is \( \frac{1}{2} p_{osc}^2 \), for a disturbance with matrix element

\[
\langle \phi_f | e^{ip\cdot p} | \phi_i \rangle
\]

It is only because the momentum transferred does not have a definite value that one can obtain a diffraction pattern from the crystal. Diffraction patterns are possible only if it is impossible after the scattering to tell which atom the neutron hit. Hence the scattering must be elastic as inelasticity would lead to changes in the crystal which could be detected. (See the sophomore physics text if you don't really understand this).

The oscillator has, along with its position \( q_k \), a momentum \( p_k \). To ensure elastic scattering we would expect \( p < |p_k| \approx \frac{\hbar}{l} |q_k| \) which agrees with \( e^{-\frac{|p|^2}{2l^2}} \)

If the momentum transfer is too big the probability of elastic scattering is small and hence one cannot achieve a good diffraction pattern.

EXERCISE B: Show that elastic scattering from a crystal gives a diffraction pattern related to the structure of the crystal and that inelastic scattering does not. Disregard the fact that the momentum of the neutron is changed in the scattering.
Construct a theory of inelastic scattering from a crystal or a polyatomic molecule.

The probability \( e^{-\rho^2 q^2} \) derived above is called the Debye-Waller factor.

We calculated it only for the simple case where the scattering is off only one particular kind of atom in the crystal. Figure out how to take into account the fact that there are different types of atoms in each unit cell.

Consider further implications of the fact that the lattice system can emit or gain momentum \( \vec{p} \) and still remain in the ground state of energy.

Suppose a nucleus in a crystal undergoes some transition and emits a gamma ray of momentum \( \vec{p} \). Classically you would expect that

\[
E_f = \text{change of nuclear energy} = (\Delta E)_{\text{transition}} - \frac{\vec{p}^2}{2M_{\text{nucleon}}}
\]

This is, in fact, true in a gas where the atoms are not bound. To detect the emitted gamma ray, one would like to have it induce the inverse process in some other nucleus. However, the recoil energy is so big that the energy expected of the gamma ray is shifted far from the transition energy, even considering its width.

Thus if one wants to do the experiment

\[
\frac{E}{p} \rightarrow \frac{\gamma}{p} \rightarrow \text{\gamma ray energies in here will induce the desired reaction}
\]

one is forced to shift the effective \( \Delta E_{\text{II}} \) in order that the lines overlap. This is accomplished by moving the second system. Then the Doppler effect shifts the gamma ray frequency seen by the system to the proper level.

This method of detecting resonance absorption has actually been carried out by putting one system on a spinning wheel.

However, if you think of the nucleus involved as simply one of the masses in a system of coupled harmonic oscillators, you see that there is a finite probability that the energy of this system will not change when the gamma ray is emitted. Hence the energy of the gamma ray will be exactly that of the nuclear transition. To detect this recoilless emission (Mossbauer effect), one slowly moves the receiver back and forth.
and notices that the peak absorption is at zero relative velocity (i.e. no shift).
The lines are so closely matched that a relative velocity of \( \frac{1}{\text{mm/ sec}} \) is sufficient
to put you off resonance.

**PROPERTIES OF THE HARMONIC OSCILLATOR**

"Why do we spend so much time on the harmonic oscillator, Daddy?"

1) because it is one of the few problems in quantum mechanics which can be solved
   exactly by practically every method
2) because of the great similarities between quantum mechanics and classical mechanics
   here
3) because whenever you expand a system about equilibrium you get a harmonic oscillator
4) because it is used all the time in talking about solid state, light, particles...
   etc.

For convenience set \( m = 1 \), \( \omega = 1 \)

\[
H = \frac{1}{2} p^2 + \frac{1}{2} \xi^2
\]

\[
P = \frac{i}{\hbar} \frac{\partial}{\partial \xi}
\]

\[
H \phi_n = \epsilon_n \phi_n
\]

\( \epsilon_n = n + \frac{1}{2} \) are the eigenvalues

\[
\phi_n = \frac{1}{\sqrt{n!}} \frac{H_n(\xi)}{\sqrt{\frac{\pi^2}{m^2}}} e^{-\frac{1}{2} \xi^2}
\]

**Examples:**

\[
H_0 = 1 \quad H_1 = 2 \xi
\]

\[
H_2 = 4 \xi^2 - 2
\]

They can all be written compactly in terms of a generating function

\[
e^{-\frac{1}{2} \xi^2 + 2 \xi S^2} = \sum_n \frac{H_n(S)}{n!} S^n
\]

Hence

\[
\sum_m \frac{\phi_m(S) S^m}{\sqrt{m!}} = \frac{e^{-\frac{1}{2} S^2}}{\sqrt{\frac{\pi^2}{m!}}} \sum_n \frac{H_n(S)}{m!} \left( \frac{S}{2} \right)^m = \frac{e^{-\frac{1}{2} \left(3 - S\right)^2 + \frac{3}{4} S^2}}{\sqrt{\frac{\pi^2}{1! \cdot 1!}}}
\]

Sometimes you can make up other generating functions, for fun and profit.

**Example:** to compute the general matrix element

\[
\lambda_{m,m'} = \int \phi_{m'}^* (S) e^{i\rho S} \phi_m (S) \, dS
\]
Define \[ f(s, t) = \sum_{m=0}^{\infty} \frac{2^{m} s^{m} t^{m}}{\sqrt{2^{m} 2^{m} m! m!}} \]

It happens to be simpler to compute \( f \) and expand in \( s \) and \( t \) to get \( \lambda_{m,n} \) than to do the explicit integral every time

\[
f = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} s^{m} t^{m} e^{i p \cdot \mathbf{s}} \left( \mathcal{Q}_{m}(\mathbf{s}) \mathcal{Q}_{n}(\mathbf{s}) \right) d\mathbf{s}
\]

\[
= \int \frac{e^{i p \cdot \mathbf{s}}}{\sqrt{2^{m} 2^{m}}} e^{-\frac{1}{2} \left( 1 - s \right)^{2} + \frac{1}{2} \left( 1 - t \right)^{2} + \frac{1}{4} \mathbf{c}^{2}} d\mathbf{s}
\]

\[
= e^{-\frac{p^{2}}{4}} \sum_{m} \frac{e^{i \rho^{*} s}}{\sqrt{2^{m} m!}} e^{i \rho^{*} t} e^{i \rho \mathbf{c}} = e^{-\frac{p^{2}}{4}} \sum_{m} \frac{(p \rho)^{m}}{2^{m} m!} \frac{(p \rho \mathbf{c})^{k}}{2^{k} k!} \frac{(st)^{m}}{m! 2^{m}}
\]

Thus

\[
\lambda_{i,j} = e^{-\frac{p^{2}}{4}} \sum_{m} \frac{(i \rho)^{i} 1 + j - 2m}{(i - n)! (j - m)!} \frac{2^{m}}{m! 2^{m}}
\]

EXERCISE C: For a crystal at temperature \( T \), find the probability for elastic scattering with a given p.

Hint: if we had just a single oscillator, the answer would look like

\[
\langle \text{Normalization} \rangle \left| <m| e^{i n \mathbf{S}} |m> \right|^{2} e^{-E_{n}/kT}
\]

because \( e^{-E_{n}/kT} \) is the probability to be in the \( n \)th state.

One should find that the probability distribution for the energy lost again looks like

\[ e^{-\frac{p^{2}}{4} <q^{2}> T} \]

EXERCISE D:

Prove

\[
\sum_{m} \mathcal{Q}_{m}(\mathbf{s}) \mathcal{Q}_{m}^{*}(\mathbf{s}^{'}) e^{i \rho \mathbf{c}} = \sqrt{\frac{m \omega}{2 \pi \omega}} \frac{\Gamma(k \omega)(\frac{1}{2} - 3^{2} - 23^{2})}{\omega}
\]

EXERCISE E: Unlike any other problem in quantum mechanics (with the exception of the free particle), the time dependent equation can be solved exactly.

Solve:

\[
\frac{\partial}{\partial t} \psi(s, t) = -\frac{1}{2} \frac{d^{2}}{d s^{2}} \psi(s, t) + \frac{1}{2} \frac{32}{3} \psi(s, t)
\]

for some set of initial conditions - e.g., \( t=0 \quad \psi = \delta(3 - z_{0}) \)

Hint: Try \( \psi(s, t) = e^{at} 3^{2} + b(t) 3 + c(t) \)
Another nice property of the harmonic oscillator equation is that it looks exactly the same in momentum space as in configuration space. Hence the normalized eigenfunctions must be the same up to phase. The phase is defined by Fourier transform from the configuration space functions.

\[ \psi_\ell (\rho) = \int e^{-i\rho s} \phi(s) ds \]

(Feynman's guess is that \( \psi_\ell (\rho) = \lambda_\ell \phi_\ell (\rho) \). What do you get?)

It may on occasion be more convenient to use the momentum space wave functions

\[ \lambda_{\ell m} = \int \phi^{\star}_\ell (s) e^{i\rho s} \phi_m (s) ds \]

\[ = \lambda_{\ell m} = \int \psi^{\star}_\ell (\rho) \psi_m (\rho + \mathcal{P}) d\rho \]

shows explicitly that momentum \( \mathcal{P} \) has been transferred to one of the oscillators.

**ASIDE:**

Notice that a lot of these techniques which suggest themselves naturally in the harmonic oscillator problem can be used to advantage elsewhere. Consider the hydrogen atom:

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{l(l+1)}{r^2} R + \frac{2m}{\hbar^2} \left[ E - \frac{Z}{r} \right] R = 0 \]

For simplicity take \( l = 0 \)

\[ 2m\omega^2 = 1 \]

\[ r \mathcal{P} = f \]

\[ -r \frac{d^2}{dr^2} f + f = E\mathcal{P}f \]

This can be easily transformed to momentum space

\[ -\frac{m}{i} \frac{d}{d\rho} \left( \frac{\rho^2}{2} \phi^{\star}(\rho) \right) + \phi(\rho) = -\frac{E \hbar}{i} \frac{d\phi}{d\rho} \]

Now it is a first order differential equation, which is presumably easier to solve than the previous second order differential equation.

This equation (with \( \mathcal{P} \) left in) can also be factorized into "creation" and "annihilation" operators in a manner not too unlike the one used below for the harmonic oscillator. These can then be used to find the eigenvalues. Interested persons should consult Morse and Feshbach pp. 270, 729

**CREATION AND ANNIHILATION OPERATORS FOR THE HARMONIC OSCILLATOR**

\[ \Pi = \frac{1}{2} \mathcal{P}^2 + m\omega^2 q^2 \]

\[ a = \sqrt{\frac{m}{2\omega}} (\omega q - i\frac{\mathcal{P}}{m}) = \frac{1}{\sqrt{2}} \left( \mathcal{P} - \frac{i}{\hbar} \frac{d}{dq} \right) \]

\[ a^+ = \sqrt{\frac{m}{2\omega}} (\omega q + i\frac{\mathcal{P}}{m}) = \frac{1}{\sqrt{2}} \left( \mathcal{P} + \frac{i}{\hbar} \frac{d}{dq} \right) \]
\( a^\dagger \) is called a creation operator and \( a \) an annihilation operator
\[
[ p, q ] = -i \quad \Rightarrow \quad a \; a^\dagger - a^\dagger a = 1
\]
\[
a \; a^\dagger = \frac{1}{\sqrt{2}} \; H^+ \; \frac{1}{\sqrt{2}}
\]
\[
a^\dagger a = \frac{1}{\sqrt{2}} \; H^- \; \frac{1}{\sqrt{2}}
\]
The most useful form is \( H = \frac{1}{2} a^\dagger a + \frac{1}{2} \), following a general rule of thumb wherein all the annihilation operators are written to the right of all the creation operators (normal ordering)
\[
a \mid 0 \rangle = 0 \quad \Rightarrow \quad H \mid 0 \rangle = \frac{1}{\sqrt{2}} \mid 0 \rangle
\]
and \( \frac{1}{\sqrt{2}} \) can thus be interpreted as the energy of the ground state; \( \frac{1}{2} a^\dagger a \) as the energy above the ground state
\[
N = a^\dagger a \quad \text{has integer eigenvalues} \quad N \mid m \rangle = m \mid m \rangle
\]
on which the creation and annihilation operators act like
\[
a^\dagger \mid m \rangle = \sqrt{m+1} \mid m+1 \rangle \quad a \mid m \rangle = \sqrt{m} \mid m-1 \rangle
\]
Hence
\[
\frac{(a^\dagger)^m \mid 0 \rangle}{\sqrt{m!}} \quad = \quad m \langle
\]
All matrix elements written in terms of \( p \)'s and \( q \)'s can be reexpressed in terms of creation and annihilation operators (although this is not necessarily worthwhile)
\[
q_{mn} = \langle m \mid \frac{a^\dagger a}{\sqrt{2}} \mid n \rangle = \langle 0 \mid \frac{a^m}{\sqrt{m!}} \frac{(a^\dagger a)^n}{\sqrt{n!}} \mid 0 \rangle
\]
This can then be calculated explicitly by commuting until all operator products are in normal order form
\[
q_{01} = \langle 0 \mid \frac{a_1 a^\dagger}{\sqrt{2}} + \frac{1}{\sqrt{2}} \langle 1 \mid a^\dagger a \rangle \mid 0 \rangle = \frac{1}{\sqrt{2}} e^{i \theta c}
\]
Thus
\[
\lambda_{mn} = \langle 0 \mid \frac{a^m}{\sqrt{m!}} e^{i \rho (\theta + \phi)} \frac{(a^\dagger)^n}{\sqrt{n!}} \mid 0 \rangle
\]
If several independent oscillators are involved
\[
a^\dagger \quad = \quad \frac{1}{\sqrt{2 \omega^0}} \left( \omega_d \; q^\dagger_a + i \; \frac{P^a}{m_a} \right)
\]
\[
a^\dagger \quad = \quad \frac{1}{\sqrt{2 \omega^a}} \left( \omega_d \; q^\dagger_a + i \; \frac{P^a}{m_a} \right)
\]
\[
\alpha^2 a^\dagger \; a^\dagger a \; q^\dagger a = \delta_{\alpha \beta}
\]
\[
[ a^\dagger, a^\dagger ] = 0 \quad \quad [ a^\dagger, a^\dagger ] = 0
\]
\[
H = (\text{ground state energy}) + \sum \omega_0 \; a^\dagger a
\]
\( \omega_0 \) notice the embarrassing fact that if there are an infinity of oscillators the ground state energy is infinite

In usage the harmonic oscillator formalism drags along some terminology. This can be
put in general form as follows:

when the oscillator is in the \( n \) th state it contains \( n \) vibratons

\( a^+ \) creates a vibraton

\( a \) takes one away

The word vibraton is then replaced by phonon, roton, phton, meson, nucleon, etc. depending on your audience.

HOW TO APPLY ALL THIS TO CRYSTALS

Last week we discussed the Hamiltonian

\[
\hat{H} = \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i,j} C_{ij} q_i q_j
\]

for a small set of coupled oscillators (e.g. in a polyatomic molecule).

The point is to change coordinates by an orthogonal transformation which diagonalizes the \( C_{ij} \); the new coordinates are called normal modes of vibration and in terms of them

\[
\hat{H} = \frac{p_i^2}{2m} + \frac{1}{2} \sum q_i^2
\]

Diagonalizing \( C_{ij} \) can be done by brute force for the case of molecules which contain only a few atoms; however in the case of a crystal with \( 10^{23} \) or so atoms some other approach must clearly be used.

The most obvious distinguishing characteristic of crystals is their periodic structure.

By exploiting this symmetry we are able to reduce the potentially complicated problem of crystal vibrations to one with as few independent factors as the polyatomic molecule case.

Most of the definitions which follow are simply bookkeeping procedures to this end.

BASIC CRYSTAL STRUCTURE

The lattice is a neatly stacked array of unit cells

Each cell is identified by a vector \( \vec{N} = m_a \vec{a} + m_b \vec{b} + m_c \vec{c} \)

where \( \vec{a}, \vec{b}, \vec{c} \) are fundamental vectors characteristic of the crystal lattice and \( (m_a, m_b, m_c) \) is a triplet of integers characterizing the cell.

Within each cell there is the same (usually complicated) arrangement of atoms and molecules. Label the units within a cell by \( \vec{r} \)

Then the total Hamiltonian for the system takes the form

\[
\sum_{\vec{N}, \vec{r}} \frac{p_{\vec{N}, \vec{r}}^2}{2m} + \frac{1}{2} \sum_{\vec{N}, \vec{r}, \vec{M}} C_{\vec{N}, \vec{r}, \vec{M}, \vec{S}} q_{\vec{N}, \vec{r}} q_{\vec{M}, \vec{S}}
\]
\[ C_{n',n} \frac{r_{n',n_0}}{r_{n,n_0}} = \prod_{n, s} (\vec{N} - \vec{M}) \]

i.e., the forces between atoms depend only on the differences between the cell positions, not on the individual cell positions.

As before we can write the equations
\[ \omega_0^2 q_{n,n} = \sum_{M,s} \prod_{n,s} (\vec{M} - \vec{N}) q_{n,s} \]

but the method of solution used before is no good.

To see how we will use the periodicity of the lattice, consider a one dimensional crystal

Let \( q_n \) be the displacement of the \( n \)th atom from equilibrium.

Then the potential energy takes the form
\[ \sum_{m} \frac{A}{2} (q_m - q_{m+1})^2 \]

Differentiate with respect to \( q_m \) to get the force
\[ \omega_0^2 q_m = A(q_m - q_{m+1}) - A(q_m - q_{m-1}) = A(2q_m - q_{m+1} - q_{m-1}) \]

To patch up the end points we define the conditions
\[ \prod_{-1} = A \quad \prod(0) = 2A \]

Then our previous techniques would be to find the determinant of

\[
\begin{vmatrix}
2A - \omega^2 & -A \\
-A & 2A - \omega^2 - A \\
-A & 2A - \omega^2 - A \\
\end{vmatrix}
\]

If we wished to consider interactions not only of nearest neighbors, but also of next nearest neighbors, the Hamiltonian would have an additional term of the form
\[ \sum_{m} \frac{A}{2} (q_m - q_{m+2})^2 \]

and the matrix would become correspondingly more complicated.

There is a theorem which tells you that it is possible to find the determinant of a matrix with the above form, regardless of its size. But all this says is that it is possible to find the solution of a linear homogeneous differential equation which involves only differences between the displacements (note that the differential equation involved has time as the variable - this has already been factored out).
We know this already (by intuition). Therefore, try
\[ q_m = e^{i m \delta} \]
\[ \omega_n^2 e^{i m \delta} = A \left[ 2 e^{i m \delta} - e^{-i (m+1) \delta} - e^{-i (m-1) \delta} \right] \Rightarrow \omega_n^2 = 2 A \left[ 1 - \cos \delta \right] \]
At first sight one feels that \( \delta \) may be arbitrary and hence that there is a continuum of solutions for \( \omega_n \). However, this is not the case because of the periodic boundary conditions imposed (from the matrix above we see that we have in effect hooked the first atom in the chain onto the last one). The imposition of periodic boundary conditions is one quite standard technique in these crystal problems. It means that if \( \omega_n \) is the number of atoms in the crystal, then
\[ q_{\omega} + \omega \vdash = q \vdash \]
For our example this means
\[ e^{i \omega \delta} = 1 \]
Hence \( \delta = \frac{2 \pi \ell}{\omega} \) where \( \ell \) is an integer from 0 to \( \omega / 2 \) (or it may assume an equal number of values in an interval symmetric about zero).

Only discrete values corresponding to the integer \( \ell \)'s are allowed. This treatment is really valid only if a large number of cells is involved, however, so no harm is done in drawing the curve as a continuum. (If there were not large numbers of cells involved, the surface effects would become important and our imposition of periodic boundary conditions would be dubious).

Another interesting property is the number of vibrational frequencies at a given frequency (i.e., the density of points along the curve.) For fun, figure out how to calculate this and check the curve drawn in class.

If \( \delta \) continued outside this fundamental interval, the frequencies would simply repeat.

Hence we always limit ourselves to the first zone constructed in the way.

If \( \beta \) should be non-zero, it is easy to see that the answer will have the form
\[ \omega = 2 \beta (1 - \omega_0 \delta) + 2 \beta (1 - \omega_2 \delta) \]
and so forth for more distant interactions.
Solid state physicists do not in general talk about $\mathbf{r}$, however. The quantity normally used is $k = \frac{\delta}{a}$ = phase difference/cm. For a three dimensional crystal, $k$ becomes a vector $\mathbf{k} = (\frac{\delta}{a} , \frac{\delta}{b} , \frac{\delta}{c} )$ in a space called reciprocal lattice space (since $\frac{1}{a} , \frac{1}{b} , \frac{1}{c}$ are the units of length along the respective crystal directions).

Because the spacing of $\mathbf{r}$ is $\frac{2\pi}{a}$, the spacing of $k$ is $\frac{2\pi}{La} = \frac{2\pi}{Le_n a} \mathbf{y}$.

For very small $k$ (i.e., very long waves) we get from above that $\omega^2 \propto \frac{k^2}{a^2}$, $\omega \approx \frac{c}{k}$.

where $c$ may be thought of as the speed of sound. Hence at long wavelengths the speed of sound is very nearly independent of the wavelength.

Given an allowed $k$, define a mode of vibration by $\mathbf{Q} = e^{i\mathbf{Q} \cdot \mathbf{r}}$.

Then the Hamiltonian assumes the form $H = \sum_{\text{atoms}} \left( \frac{p^2}{2} + \frac{\omega^2}{2} Q^2 \right)$

where $\omega^2(k) = 2k \left( 1 - \frac{\omega}{c} k a \right)$.

Notice again that just as there is only a region of $\mathbf{r}$ which is significant, there is only a region of $k$ which is significant. For $k$ too large, the wavelengths get smaller than the spacing between the atoms

and these can't have any extra significance for the vibrations of the atoms. Thus there is only a fundamental zone of $k$ which is important (a region in reciprocal lattice space called the first Brillouin zone). After this point things just repeat.

Thus sound has a maximum frequency of importance (minimum wavelength).

---

For a general problem we conclude that the solution should have a quite simple form: the first unit cell has some disturbance, and the same thing should appear in the next cell only slightly out of phase.

Inserting this $\omega^2 q_r \frac{1}{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}} = \sum_{\mathbf{r}} \frac{1}{\mathbf{r}} \frac{q_r}{\mathbf{q}} \frac{1}{\mathbf{r}} e^{i\mathbf{q} \cdot \mathbf{r}}$.

But $\sum_{\mathbf{q}} \frac{1}{\mathbf{q}} \frac{q_r}{\mathbf{q}} \frac{1}{\mathbf{q}} = \delta_{r,s}(\mathbf{q})$ is just a Fourier transform of the $\sum_{\mathbf{q}} \frac{1}{\mathbf{q}} \frac{q_r}{\mathbf{q}}$.

In this form it is clear that it is only necessary to solve the determinant within a single cell - a problem of the same order of magnitude as encountered in the case of polyatomic molecules, and one which can conceivably be done on the computer.

In computational problems, one should keep in mind that the spacings of $k$ (box type
normalization) are \( \frac{2\pi}{L_a} \) \( \frac{2\pi}{L_b} \) \( \frac{2\pi}{L_c} \). Any sum over \( k \) goes over naturally to an integral

\[
\int d^3k \left( \frac{\text{volume}}{(2\pi)^3} \right)
\]

For a three dimensional crystal with one atom per unit cell the simple generalization of our calculation for a one dimensional crystal yields

\[
\begin{array}{c}
\omega \\
\hline
kz \\
\hline
kx \\
\end{array}
\]

(projections on various axes in \( k \) space need not look alike unless the crystal is isotropic)

It is possible to prove that at \( k=0 \), \( \omega \sim k^2 \) for all cases of physical interest.

The three modes shown above can be approximately classified as 1 longitudinal mode and two transverse modes, although when the crystal is anisotropic there is no need for the sound vibrations in the longitudinal mode to be exactly along \( k \), or for the transverse mode to be exactly transverse (compare propagation of polarized light in an anisotropic crystal)

---

If there are two atoms in the unit cell then the number of degrees of freedom is multiplied by 2 and analysis shows that there are indeed 6 possible modes of vibration. However, as \( k \to 0 \), it is not necessary that all frequencies go through zero. One finds in fact

\[
\begin{array}{c}
\omega \\
\hline
\text{optical branch} \\
\hline
\text{acoustical branch} \\
\end{array}
\]

In the acoustical branch one might have, for example, the two atoms moving together

\[
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]

with the pattern slightly displaced from cell to cell

For the optical branch they might move \( \leftrightarrow \) with a much higher frequency and hence energy, but the entire pattern may be only slightly displaced from cell to cell (this is what determines \( k \))
Additional Problems (Do one or more):

1. Consider a one-dimensional line of atoms with spacing $a$. Develop the theory of small vibrations in the limiting case $a \to 0$. (This is a special example of a one-dimensional quantum field).

2. Find the frequency spectrum for the normal modes for a particular crystal, or find the specific heat of the crystal.

Simplest crystal is cubic:

\[ \begin{array}{c|c}
\text{N} & \text{K} \\
\hline
\text{K} & \\
\end{array} \]

Find the high frequency branch (optical modes) as well. Can you determine the elastic constants?

In the acoustical branch, $\omega \to 0$ as $k \to 0$ because the forces depend only on relative displacement of the nuclei, and $k \to 0$ corresponds to all the atoms moving together. On the other hand, $k \to 0$ can yield a dipole moment and this can act as a restoring force to generate frequencies $\neq 0$, and this is the optical branch. (As $k \to 0$, there are 3 optical and 3 acoustical frequencies).

The coupling constants of lattice vibrations to external disturbances are needed for calculations. Experimentally, some of these are determined from elastic constants which are either measured or inferred from the speed of sound.

3. Show how to find the elastic constants by measuring the speed of sound.
(4) Neutron Scattering  

Let \( P = P_{\text{out}} - P_{\text{in}} \)

Show that this will excite a mode of wavevector \( \mathbf{k} \) where

\[
\mathbf{k} = \mathbf{L} + \mathbf{\ell}
\]

where \( \mathbf{L} \) is a reciprocal lattice vector. (see below)

![Graph showing probability of scattering with peaks labeled elastic and neutron out]

The positions of the peaks are determined by the modes, and provide a means of measuring the frequencies of the modes.

**Reciprocal Lattice:**

\[
\sum_{\mathbf{k}} \rightarrow \int \frac{d^3 \mathbf{k}}{(2\pi)^3}
\]

We are not usually interested in all \( \mathbf{k} \) values.

If the 3 fundamental translation vectors of the lattice are \( \mathbf{a}, \mathbf{b}, \mathbf{c} \), then the phase in the \( n \) th cell is

\[
e^{i \mathbf{k} \cdot \mathbf{c}}
\]

for a mode with wave vector \( \mathbf{k} \).

Now if \( \mathbf{k}' = \mathbf{k} + \mathbf{\ell} \)

then \( e^{i \mathbf{k} \cdot \mathbf{c}} = e^{i \mathbf{k}' \cdot \mathbf{c}} \) if \( e^{i \mathbf{k} \cdot \mathbf{\ell}} = 1 \)

This will be insured if

\[
\mathbf{\ell} = L_a \frac{b \cdot c}{|a \cdot b \cdot c|} + L_b \frac{c \cdot a}{|a \cdot b \cdot c|} + L_c \frac{a \cdot b}{|a \cdot b \cdot c|}
\]

where \( L_a = \mathbf{L} \cdot \mathbf{a} = 2\pi n_1 \),

\( L_b = 2\pi n_2 \), \( L_c = 2\pi n_3 \)

\( n_1, n_2, n_3 \) are integers.
\( l_a, l_b, l_c \) are the components of \( \mathbf{L} \) along the three directions perpendicular to the 3 planes formed by choosing 2 of the 3 vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \).

\[ \mathbf{L}_a, \mathbf{L}_b, \mathbf{L}_c = 2\pi \text{ (integer)} \]

The set \( \mathbf{L}_a, \mathbf{L}_b, \mathbf{L}_c \) form a lattice, called the reciprocal lattice. The name 'reciprocal lattice' is justified by the fact that the reciprocal lattice of the reciprocal lattice is the original lattice.

Construct the 'Brillouin zone' from this.

**Specific Heat of a Crystal due to Atomic Vibrations**

Modes \( \omega_\beta (k) \)

At temperature \( T \),

\[ U = \sum_{\beta} \sum_k \frac{\hbar \omega_\beta (k)}{e^{\frac{\hbar \omega_\beta (k)}{k_B T}} - 1} \]

\[ = V \sum_{\beta} \int_{R} \frac{d^3k}{(2\pi)^3} \frac{\hbar \omega_\beta}{e^{\frac{\hbar \omega_\beta}{k_B T}} - 1} \]

\[ C_v = \frac{\partial U}{\partial T} = k_B V \sum_{\beta} \int_{R} \frac{d^3k}{(2\pi)^3} \left( \frac{\hbar \omega_\beta}{k_B} \right)^2 \frac{e^{\frac{\hbar \omega_\beta}{k_B T}}}{\left( e^{\frac{\hbar \omega_\beta}{k_B T}} - 1 \right)^2} \]

The integration \( \int d^3k \) is only over a single cell in the reciprocal lattice. This is because the other cells give the same frequencies and energies of the harmonic oscillators of the atomic vibrations, and the level of excitation of a particular oscillator depends only on the temperature.

The approximations made in obtaining \( C_v \) are not valid for metals or magnets. In metals, there is energy in the electronic motions, in magnets, there is energy in the spin orientations of the particles.

In many solids, the approximation that only phonons occur is a good one.
For high $T$:

$$kT > \hbar \omega_k (\mathbf{k}) \quad \text{for all } \mathbf{k}$$

Then

$$U \approx kT \sum_{\text{modes}} \int \frac{d^3k}{(2\pi)^3} V$$

$$V = \text{volume of unit cell} = 12 \cdot \mathbf{k} \cdot x \leq 1$$

$$\int \frac{d^3k}{(2\pi)^3} V = 1$$

$$U \approx k_b T \left( \text{no. of modes} \right) = 3Nk_b T$$

$N = \text{no. of atoms in crystal}$

$$C_v = 3R \quad \text{per degree Kelvin per mole}$$

$$= 6 \text{ calories per deg. per mole at high } T.$$

For low $T$:

$$\omega_k (\mathbf{k}) = C_k \beta \mathbf{k} \quad ; \quad C_k \text{ depends slightly on direction of } \mathbf{k};$$

$$\text{only 3 d's exist for low } T - \text{ those for which } \omega_k \to 0 \text{ in rectangular branch.}$$

Then

$$U = k_b T \sum_{\mathbf{k}} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\hbar C_k \beta \mathbf{k}}{k_b T} \right) \frac{1}{\exp \left( \frac{\hbar C_k \beta \mathbf{k}}{k_b T} \right) - 1}$$

Let

$$x = \frac{\hbar C_k \beta \mathbf{k}}{k_b T}$$

Then

$$U = k_b T \sum_{\mathbf{k}} \frac{\hbar C_k \beta \mathbf{k}}{k_b T} \left( \frac{k_b T}{\hbar C_k \beta \mathbf{k}} \right)^3 \frac{1}{(2\pi)^2} \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

only a small error is introduced if we take the upper limit as $\infty$, and then the integral $= \frac{\pi^4}{15}$

Then

$$U = (k_b T)^4 \frac{\pi^2}{10 C_0^3 \hbar^3}$$

where

$$\frac{1}{C_0^3} = \frac{1}{3} \sum \frac{1}{C_k^3}$$

$$C_v \propto T^3 \quad \text{as } T \to 0$$

$C_0$ is some average speed of sound.
For very small $T$ (say, $10^{-6}$ °K.), $C_v$ no longer goes as $T^3$, and that is because of the discreteness of the modes. $C_v$ then depends on the volume.

So the ideal specific heat curve looks like

$$
\begin{align*}
\text{C}_v & \\
& \ \text{vs.} \ T
\end{align*}
$$

We can see that for particular crystals $C_v$ vs. $T$ needn't be so simple. For example, if we had a crystal in which strongly bound molecules are weakly bound to each other, then we would obtain

$$
\begin{align*}
\text{C}_v & \\
& \ \text{vs.} \ T
\end{align*}
$$

There is a theory due to Debye, and this consists of first writing

$$ W_2 = C_0 k \quad \text{for all } k $$

and considering that there are 3 such modes.

Use the same $C_0$ for each mode because other approximations used are more severe.
Then
\[ U = 3k_B T \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k_c k} \exp \left( \frac{h c \omega}{k_B T} \right) - 1 \]

Use the approximation of replacing the unit cell in the reciprocal lattice by a sphere of radius \( k_{\text{max}} \)

Then
\[ \int \frac{d^3 k}{(2\pi)^3} \to 4\pi \int_{0}^{k_{\text{max}}} \frac{k^2}{(2\pi)^3} \frac{h c \omega}{k_B T} \frac{1}{\exp \left( \frac{h c \omega}{k_B T} \right) - 1} \]

and
\[ U = \left( \frac{k_B T}{\hbar c \omega} \right)^4 \frac{3}{2\pi^2} \int_{0}^{\Theta/T} \frac{x^3 \, dx}{e^x - 1} \quad \Theta = \frac{k_{\text{max}} h c \omega}{k_B} \]

For \( T \to 0 \), we recover previous formula.

For \( T \gg \Theta \), (\( \Theta \) is called the Debye temperature, it is a useful quantity to know in the case of a particular crystal)

\[ U = \frac{k_B^4 T}{(\hbar c \omega)^3} \frac{\Theta^3}{2\pi^2} \Rightarrow C_v = \text{constant independent of } T \]

and independent of \( c_0 \).

Hard crystals usually generally have high \( \Theta \) (diamond)
Soft crystals generally have low \( \Theta \) (Argon).

This theory of specific heat has been widely abused. Rather than plotting \( C_v \) vs. \( T \), with \( \Theta \) as an adjustable parameter, and comparing with experimental \( C_v \) vs. \( T \), people have plotted \( \Theta_0 \) vs. \( T \) as a way of reporting measurements of the specific heat.

One obtains in this manner foolish curves like

\[ \Theta_0 \rightarrow T \]
Problem: Evaluate \( \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(x') e^{-iE_n t} \)

where \( E_n \) is the energy and \( \phi_n(x) \) the wavefunction for the \( n \)th excited state of a harmonic oscillator with frequency \( \omega \).

Solution: \( \phi_n(x) = \frac{1}{\sqrt{\pi n!}} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x) \)

where \( H_n(x) = (-)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \)

(A word of caution: different books use different normalizations for the Hermite polynomials. Our normalization is fixed by the normalization of \( \phi_n(x) \) and the expression for \( \phi_n^*(x) \) in terms of \( H_n(x) \))

\[ \sum = \sum_{n=0}^{\infty} \frac{e^{ix^2}}{\sqrt{\pi}} \frac{e^{ix'^2}}{\sqrt{2^n n!}} \left( \frac{d^n}{dx^n} e^{x^2} \right) \left( \frac{d^n}{dx^n} e^{-x^2} \right) e^{-i\omega t} e^{-i\omega t} \]

\[ \frac{d^n}{dx^n} e^{-x^2} = \frac{1}{\sqrt{\pi}} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} e^{ixy} e^{-y^2/4} dy \]

\[ = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (iy)^n e^{ixy} e^{-y^2/4} dy \]

\[ \sum = \frac{e^{-i\omega t}}{2\pi i^n} \frac{e^{ix^2}}{\sqrt{\pi}} \frac{e^{ix'^2}}{\sqrt{2^n n!}} \int_{-\infty}^{\infty} e^{ixy} e^{-y^2/4} \int_{-\infty}^{\infty} e^{ix'y} e^{-y'^2/4} \sum_{n=0}^{\infty} \frac{(-y)^n (y')^n}{2^n n!} e^{-i\omega t} e^{-i\omega t} \]

\[ = \frac{e^{-i\omega t}}{2\pi i^n} \frac{e^{ix^2}}{\sqrt{\pi}} \frac{e^{ix'^2}}{\sqrt{2^n n!}} \int_{-\infty}^{\infty} e^{ixy} e^{-y^2/4} \int_{-\infty}^{\infty} e^{ix'y} e^{-y'^2/4} \exp(-yy'e^{-i\omega t}) \frac{1}{2} \]

The evaluation of the integrals is straightforward (though slightly tedious) the result is

\[ \sum_{n=0}^{\infty} = \frac{2}{\sqrt{\pi}} \frac{\exp \left( -\frac{x^2}{2} + x' e^{i\omega t} + 2xx' e^{-i\omega t} \right)}{(1-e^{-2i\omega t})^{1/2}} \]
More on Neutron Scattering

The amplitude for the process takes the form

\[ \langle f | \sum_{j} V_{i}^{*} (r_{i} - \mathbf{R}_{i}) e^{i \mathbf{k}_{f} \cdot \mathbf{r}} \rangle \]

where the sum is over the lattice sites

\[ V_{i} \] has different values for the different types of atoms involved

\[ \sum_{i} e^{-i \mathbf{k}_{f} \cdot \mathbf{R}_{i}} e^{i \mathbf{k}_{i} \cdot \mathbf{r}_{i}} \langle \mathbf{r}_{i}^{*} | \mathbf{V}_{j} | \mathbf{r}_{j} \rangle d^{3} \mathbf{r}_{i} d^{3} \mathbf{R} = \langle f | \sum_{j} V_{j}^{*} (\mathbf{P}) e^{i \mathbf{R}_{j} \cdot \mathbf{P}} | f \rangle \]

Provided the wavelength for \( \mathbf{P} \) is long compared to other dimensions, the dipole approximation can be used to evaluate the integrals and \( V_{j}^{*} (\mathbf{P}) \) is approximately independent of \( \mathbf{P} \)

If the wavelength is comparable with the dimensions involved, then \( V_{j}^{*} (\mathbf{P}) \) is a form factor

If the final lattice state is the same as the initial state, the matrix element looks as follows:

\[ \langle i | \sum_{j} e^{i \mathbf{P} \cdot \mathbf{q}_{j}} e^{i \mathbf{P} \cdot \mathbf{R}_{j}} | i \rangle \]

where \( \mathbf{q}_{j} \) is the oscillating coordinate and \( \mathbf{R}_{j} \) is the equilibrium position

Hence we find

\[ \sum_{j} V_{j}^{*} (\mathbf{P}) e^{i \mathbf{P} \cdot \mathbf{R}_{j}} \langle i | e^{i \mathbf{P} \cdot \mathbf{q}_{j}} | i \rangle \]

this was previously evaluated to give \( e^{-\mathbf{P}^{2} q_{j}^{2}}/2 \)

It is obvious that if \( \frac{q_{j}^{2}}{\mathbf{P}^{2}} \gg 1 \), there will be almost no elastic scattering

The probability is then just

\[ \left| \sum_{j} V_{j}^{*} (\mathbf{P}) e^{i \mathbf{P} \cdot \mathbf{R}_{j}} \right|^{2} e^{-\mathbf{P}^{2} q_{j}^{2}} \]

\( V_{j}^{*} (r_{i} - r_{j}) \) is the same for all atoms of a given type inside the crystal

Hence \( V_{j}^{*} (\mathbf{P}) \) is the same for all atoms of the type

Thus we get

\[ \sum_{\nu \mu} V_{\nu}^{*} (\mathbf{P}) \sum_{\mu} e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}} \]

for one type this is

\[ V (\mathbf{P}) \equiv \sum_{\mu N} e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}} \]

where \( \mathbf{R}_{\mu}, \mathbf{R}_{\nu}, \mathbf{R}_{\lambda} \) are the components along the directions in reciprocal lattice space

\[ \sum_{\nu}^{N} e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}} = \left( 1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}} \right) \left( 1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\nu}} \right) \]

Hence we arrive at

\[ V (\mathbf{P}) \left[ \frac{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}}}{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\nu}}} \right] \left[ \frac{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\nu}}}{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\lambda}}} \right] \left[ \frac{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\lambda}}}{1 - e^{i \mathbf{P} \cdot \mathbf{R}_{\mu}}} \right] \]

The absolute square gives

\[ |V (\mathbf{P})|^{2} e^{-\mathbf{P}^{2} q_{j}^{2}} \]

\[ \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}} \right\} \]

\[ \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}} \right\} \]

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\[ \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}} \right\} \]

\[ \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}} \right\} \]

\[ \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\nu}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}} \right\} \left\{ \frac{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\lambda}}{S_{\nu}^{2} \mathbf{P} \cdot \mathbf{R}_{\mu}} \right\} \]
This approaches a product of \( N \) functions as \( N \to \infty \): 
\[
\sigma \left( \frac{p_a N}{2} \right) = \prod \sigma \left( \frac{p_{A_i} N}{2} \right)
\]
Thus \( \mathbf{P} \) must be a reciprocal lattice vector to have any appreciable contribution.

Now consider the case of inelastic scattering:
\[
\sum_j \left| V_j \right| \left| \mathbf{P}_j \right| \left< f | e^{i \mathbf{P}_j \cdot \mathbf{r}_j} | i \right> = e^{i \mathbf{P}_j \cdot \mathbf{r}_j}
\]
In general the crystal will make transitions to all sorts of final states. These final states are clustered close together in energy because the spacing of the \( k \) for the modes is so small.

Hence our plan of attack should be to:

a) Find the transition amplitude to a final state at energy \( E_0 \):
\[
T_{fi} (E_0) = \int \frac{d^3 k_F}{(2\pi)^3} \delta (E_{k_F} - E_0) \left< f | e^{i \mathbf{P}_j \cdot \mathbf{r}_j} | i \right>
\]

b) Sum individual probabilities over \( E_0 \) to find the total probability:
\[
\sum_{E_0} | T_{fi} (E_0) |^2
\]

Why this gives an answer with no hint of a diffraction pattern is left as an exercise to the student.

Demonstration that the average energy lost by the neutron to the crystal is just the classical energy:
\[
\Psi_{\text{after}} = e^{i \mathbf{P} \cdot \mathbf{r}} \Psi_{\text{before}}
\]
\[
\left< \Psi_{\text{after}} | H | \Psi_{\text{after}} \right> - \left< \Psi_{\text{before}} | H | \Psi_{\text{before}} \right>
\]
\[
H = \mathbf{P}^2_{2m} + \mathbf{P} \cdot \mathbf{P} + V
\]
\[
\left< \Psi_{\text{after}} | e^{i \mathbf{P} \cdot \mathbf{r}} H e^{i \mathbf{P} \cdot \mathbf{r}} - H | \Psi_{\text{after}} \right> = \mathbf{P}^2_{2m} + \mathbf{P} \cdot \mathbf{P} - \left< \Psi_{\text{before}} | \mathbf{P}^2_{2m} | \Psi_{\text{before}} \right>
\]

because:
\[
\mathbf{P} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{P}
\]

Provided the mean momentum of the system is 0, the second term is zero. If there is mean momentum, we get a Doppler type correction.

Question from the class: how do you calculate the probability of "recoilless emission" at finite temperature?

This involves sums of the form:
\[
\sum_n e^{-\beta m} |<m|e^{i \mathbf{P} \cdot \mathbf{r}|m>|^2
\]
Feynman was unable to solve this problem in class and returned at the next meeting with
the news that the answer he had predicted is probably incorrect.

Beware: in doing the sums required for crystal statistical mechanics, one encounters the
operators
\[ q_m = \text{displacement} = \frac{1}{\sqrt{\eta}} \sum q_i e^{i k x} \]
where \( \eta \) is the number of atoms.

Do not use arguments based on the fact that \( \frac{1}{\sqrt{\eta}} \) is a very small number, for you will
almost always discover that there is a very large number of these to be added up.
(This is the way the answer suggested in previous notes was arrived at).

QUANTUM FIELD THEORY: Developed as the continuum limit of a line of oscillators
\[ H = \sum_{m} \left[ \frac{1}{2} p_m^2 + \frac{1}{2}\left(q_m - q_{m+1}\right)^2 \right] \]
where \( x_m = \alpha m \) is the actual position in space of the atom.

For long wavelengths \( \omega^2 = 2 \alpha (1 - \cos k \alpha) \) becomes \( \omega^2 \propto \alpha \alpha^2 k^2 \)
\( k \alpha \) small implies \( q_m \) may be replaced by the continuous \( Q(x) \)
\[ Q_n = \int \frac{dx}{\alpha} e^{ikx} Q(x) \]
defines the normal modes.

Generally one redefines \( Q' = \int \frac{dx}{\alpha} e^{ikx} q(x) \)
\[ q(x) = \int Q' \frac{dx}{\alpha} e^{-ikx} \]
Then the formula for the energy becomes
\[ H = \sum \left[ \frac{1}{2} \left( \frac{\partial^2 q(x)}{\partial x} \right)^2 \frac{dx}{\alpha} + \frac{1}{2} \alpha^2 \left( \frac{\partial^2 q(x)}{\partial x^2} \right)^2 \frac{dx}{\alpha} \right] \]
\( \rho = \frac{1}{\alpha} \) is the density, because each atom had unit mass.

This is not the Hamiltonian, in a strict sense. The standard Hamiltonian is a form in terms
of \( \bar{\Pi}(x) \) and \( q(x) \). To get \( \bar{\Pi}(x) \), set \( \bar{\Pi}(x) = \rho_n \alpha \)
\( \bar{\Pi} q_m \bar{\Pi} q_m = -i \delta \delta_{m,n} \rightarrow \bar{\Pi}(x) q(x') \bar{\Pi}(x) = -i \alpha \int (x - x') \]
\[ H = \int \frac{dx}{\alpha} \left[ \frac{\rho^2}{2} \bar{\Pi}^2 + \frac{c^2}{2} \bar{\Pi}^2 \right] \]
\( q(x) \left( \bar{\Pi}(x') \right) \)
because they give a scalar operator for each value in \( x \), are quantum mechanical
scalar fields.

Notice that our Hamiltonian is not expressed in normal form because of the \( (\bar{\Pi} q)^2 \) terms.
Before substituting the $Q_{\lambda}$'s, define a Fourier transform of the conjugate momentum

$$ p_{\pi}(x) = \int e^{-i\hbar k} \frac{p(\hbar k)}{2\pi} dk $$

$$ P(K_1) Q(K_2) - Q(K_2) P(K_1) = -i \Delta Q(K_1 - K_2) $$

$$ H = \int \frac{dk}{2\pi} \left[ -\frac{1}{2} p^2 + \frac{1}{2} k^2 c^2 Q(\hbar k)^2 \right] $$

which is a normal mode form.

Keep in mind that this is the form used for expansion of the Hamiltonian when we can't see the atoms, but only the waves.

This was the procedure followed for quantum electrodynamics. But we have no idea of the scale $a$; this may be why there are high energy difficulties with QED.

In practice, of course, one works in three dimensions and must take into account the anisotropy of the crystal.

At each point in space there is a displacement vector field, not just a scalar

$$ KE = \frac{1}{2} \int \frac{\partial Q}{\partial t} \cdot \frac{\partial Q}{\partial t} d\Omega $$

$$ PE = \frac{1}{2} \int \epsilon_{ij} \epsilon_{jk} \epsilon_{ki} e_{ij} e_{jk} e_{ki} dx $$

The tensor $\epsilon_{ij}$ hides all the elastic constants of the crystal.

For your amusement:

How many elastic constants are there for a general crystal? (21)

Show that there are three elastic constants for a cubic crystal.

Show that an isotropic medium has only two elastic constants: the compression and shear moduli.

Problem: Let $p$ and $q$ be the coordinates of a harmonic oscillator.

Then $q(t)$ can be expressed in terms of $p(0)$ and $q(0)$.

Find $[q(t_0), q(t_1)]$

(notice that changes of $q$ with time look like momenta, so there is no reason for this commutator to be 0.)

Hint: use the representation in which operators change with time

$$ q(t) = U^{-1} q(0) U $$
PHONON INTERACTIONS

So far we have talked only about modes of vibration of an entire perfect crystal. Independent excitation of these can't possibly explain thermal conductivity. To get at thermal conductivity, it is necessary to realize that real phonons have a finite mean free path; hence they can diffuse.

Derivation of this mean free path depends on including higher order terms in the expansion of the potential; for example third order pieces in $a$ and $a^*$. When the processes are virtual, it is possible to have contributions from all of these terms.

Problem: calculate the mean free life of a photon. Explain thermal conductivity on this basis.

This finite lifetime can be detected experimentally by neutron scattering. Recall that subsidiary peaks are spaced from the elastic peak by wave vector $k_f$. Finite lifetime implies that there will be a spread in $k$.

At the low $k$ end of the spectrum, we can approximate reality by the previously developed continuum formalism, and try using $\left( \frac{\partial \omega}{\partial x} \right)^3$ as a perturbation.

We can, using the field expansion for soft phonons, write down and calculate diagrams for all sorts of processes. Phonon-phonon scattering, for example

There is, however, one large difference between this treatment of phonons and the corresponding treatment of photons. This comes in the definition of the momentum assigned to the quantum of excitation.

For a photon the momentum is well defined by its interactions with other particles. For a phonon, however, the wave number vector (which people like to call the momentum)
can be defined only up to a reciprocal lattice vector (because if $\overrightarrow{u}$ is a reciprocal lattice vector, $e^{i\overrightarrow{x} \cdot \overrightarrow{u}} = 1 \ \forall \overrightarrow{x}$ indicating lattice sites in the crystal).

Thus the momentum conservation laws are not as cut and dried as in electron-photon interactions. With the $k'$s all cut back to members of the first Brillouin zone, the law becomes

$$\overrightarrow{k'} + \overrightarrow{k''} = \overrightarrow{k'''} + \overrightarrow{u}$$

for vertex $k'$ and $k''$, $k'''$.

For very small $k$, no $\overrightarrow{u}$ will be required. If $\overrightarrow{u}$ is required, the process is called an umklapp process.

Hence we may restate the above.

Problem: Explain carefully how the umklapp process gives you finite thermal conductivity.

For very small $k$, due to lack of umklapp, there is an exponentially small thermal resistance. The formula written down in class was

$$e^{-\frac{h}{kT}[\text{Energy at top of acoustic branch}]}$$

How is this obtained?

Problem: It is difficult to understand the idea of some process taking place at one atom in a lattice imparting a momentum to the entire crystal. Use the uncertainty principle to explain the recoil momentum of the whole crystal in (for example) the Mössbauer effect

Problem: Invent a way to measure the momentum of a phonon.

Aside:

i) For an amorphous medium there are quantities analogous to phonons, but they scatter very fast and very randomly.

ii) For waves the simple minded rule of thumb

$$\frac{k}{\nu} = \frac{\text{momentum}}{\text{energy}}$$

will almost always give the same answers as more elaborate analyses.

Problem: Influence of impurities on the properties of crystals

Take a low density $\rho$ of atoms with different masses. Assume each flaw acts independently scattering phonons into other directions.

As $T \to 0$, for density $\rho$, how does the thermal conductivity depend on $T$ and how on $\rho$?

(give just a power law dependence, since that would be the first thing you'd measure)
We found that we can treat the quantum-mechanical vibrations of a one-dimensional line of atoms in the limit of zero spacing between atoms as a quantum field.

If \( p_n, q_n \) are the momentum and position operators respectively of the \( n^{th} \) atom, then the transition to the continuum is achieved by

\[
q_n \rightarrow q(x) \\
\frac{p_n}{\hbar} \rightarrow \pi(x)
\]

then \[ \frac{\hbar}{\pi} \left[ q_n, p_m \right] = i \delta_{nm} \]

\[ \Rightarrow \left[ \pi(x), q(x') \right] = -i \delta(x-x') \]

(1)

In terms of \( q(x), \pi(x) \),

kinetic energy \[ \frac{1}{2} \int \left( \frac{\partial q}{\partial x} \right)^2 dx = \frac{1}{2} \int \pi^2(x) dx \]

potential energy \[ \frac{\hbar}{2} \int \left( \frac{\partial q}{\partial x} \right)^2 dx \]

i.e., \[ H = \frac{1}{2} \int \left[ \pi^2(x) + \frac{\hbar}{2M} \left( \frac{\partial q}{\partial x} \right)^2 \right] dx \]

(2)

Write \[ q(x) = \sum_k \frac{i}{\sqrt{2\Omega_k}} \left( q^+_k + q^-_k \right) e^{-ikx} \]

\[ \pi(x) = \sum_k \frac{i \hbar}{2} \left( q^+_k - q^-_k \right) e^{-ikx} \]

(3)

then with the substitution

\[ \sum_k \rightarrow \frac{i}{(2\pi)^{1/2}} \int dk \]
the commutation rules of the \( a \) 's are found to be:

\[
[a_k, a^*_l] = \delta(k-l),
\]

\[
[a_k, a_l] = 0 = [a^*_k, a^*_l].
\]  

(3)  

One may verify the equivalence of (4) and (1) by using (4) and (3) to obtain the CR's of \( \pi(x), \pi(y) \):

\[
\pi(x) \pi(y) - \pi(y) \pi(x) = \sum_k \frac{i}{\sqrt{2\pi}} \left( a^*_k + a_k \right) (a^*_k - a_k) \delta(x - y) = \int dk e^{-ik(x-y)} = i \delta(x-y).
\]

And clearly, \( [\pi(x), \pi(y)] = 0 = [\pi(x), \pi(y)] \).

The Hamiltonian may also be evaluated in terms of the \( a \) 's. It is a simple matter to show

\[
H = \frac{i}{\hbar} \int \left( \pi^2 + c^2 \partial_x q^2 \right) dx = \sum_k \hbar \omega_k (a^*_k a_k + \frac{1}{2})
\]

To obtain the equations of motion, recall that for any operator \( A \) that does not depend explicitly upon the time

\[
-i \dot{A} = [H, A]
\]

So that

\[
\dot{q}(y) = -i \int dx \left[ \pi^2(x) + c^2 \partial_x q^2 \right],
\]

\[
=q(y) \pi(x) + \pi(x) q(y) - \pi(x) \delta(x-y) + \pi(x) q(y) \pi(x) - \pi(x) \delta(x-y)
\]

\[
=q(y) \pi(x) - i \delta(x-y) + (-i) \delta(x-y) \pi(x)
\]

\[
=q(y) \pi(x) - i \delta(x-y) + (-i) \delta(x-y) \pi(x)
\]
or, \( \dot{q}(y) = \Pi(y) \) as expected.

Also
\[
\dot{\Pi}(y) = i \{ H, \Pi(y) \} = c^2 \frac{\partial^2 \dot{q}}{\partial y^2}
\]
upon integrating by parts.

For a 3-dimensional scalar field (only one disturbance at each point in space)
\[
\Pi = c^2 \nabla^2 \dot{q} = \dddot{q}
\]

**Vibrations of a three-dimensional solid:**

There are 3 different disturbances (3 independent displacements) at each point — label by \( i \).

Then
\[
\{ \Pi_i(x), q_j(y) \} = -i \delta^3(y - x) \delta_{ij}
\]

**Kinetic energy**
\[
\frac{1}{2} \int \frac{\partial \dot{q}}{\partial x} \cdot \frac{\partial \dot{q}}{\partial x} \, dx
\]

**Potential energy**
\[
\frac{1}{2} \int \sum_{i,j,k,l} C_{i,j,k,l} \frac{\partial q_i}{\partial x_j} \frac{\partial q_k}{\partial x_l} \, dx
\]

where the coefficients \( C_{i,j,k,l} \) are symmetric under any of the following interchanges:
\[
i \leftrightarrow j
\]
\[
k \leftrightarrow l
\]
\[
ij \leftrightarrow kl
\]

This gives the equations of motion
\[
\dddot{q}_i = \sum_{j,k,l} C_{i,j,k,l} \frac{\partial^2}{\partial x_j \partial x_k} q_j(x)
\]
The general case is rather formidable. A simpler case would be

\[ \ddot{\phi} = \lambda \nabla^2 \phi + \lambda_2 \nabla (\nabla \cdot \phi) \]

with \( \lambda, \lambda_2 \) being elastic constants characteristic of the solid.

Free Fields

In classical physics we obtain the equations of motion of a free field not by commuting the fields with the Hamiltonian, but from the action principle. In general, \( S = \int L \, dt \) \( x \), \( L \) may depend on \( \frac{\partial L}{\partial \dot{x}} \).

then \( S = \int L \, dt \) \( x \) is an extremum for arbitrary variations \( \delta \phi \) in the fields \( \phi \).

the Euler-Lagrange equations are then

\[ \frac{\partial L}{\partial \phi} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) + \sum_k \frac{\partial}{\partial \phi_k} \left( \frac{\partial L}{\partial \phi_k} \right) \]

Example: For \( L = \frac{1}{2} \phi^2 - \frac{c^2}{2} \phi_0^2 \phi_0 \dot{\phi}^2 \)

the equation of motion is \( \ddot{\phi} = c^2 \frac{\partial^2 \phi}{\partial x^2} \).

For classical electromagnetism (free field),

\[ L = \frac{1}{2} (E^2 - B^2) \]

(in presence of sources, add \( J \cdot A - \partial \cdot E \phi \))
Writing $L$ in terms of $A$, $\phi$, the action principle becomes

$$\frac{1}{\hbar} \delta \left\{ \left[ \frac{\partial A}{\partial t} + \nabla \phi \right]^2 - \left[ \nabla \times A \right]^2 \right\} d^3x = 0$$

Expanding and integrating by parts

$$\delta \int \left\{ \left( \frac{\partial A}{\partial t} \right)^2 + (\nabla \phi)^2 - 2 \phi \frac{\partial}{\partial t} \nabla A - (\nabla \times A)^2 \right\} d^3x = 0$$

Choose gauge gauge so that $\nabla \cdot A = 0$ (Coulomb gauge)

Integrating $(\nabla \phi)^2 = -\nabla^2 \phi + \nabla \cdot (\phi \nabla \phi)$ by parts and making use of the result $\nabla^2 \phi = 0$ for free fields as a shortcut

$$\delta \int \left( \frac{\partial A}{\partial t} \right)^2 - (\nabla \times A)^2 \right\} d^3x = 0 \Rightarrow \Delta \phi - \nabla^2 \phi = 0$$

Let $\pi = \frac{\delta A}{\delta t}$

Then applying the quantum mechanical condition is

$$[\pi_i(x), A_j(y)] = -i \delta \frac{\partial}{\partial x} \delta_{ij} \delta (x-y)$$

and

$$H = \int A \cdot d^3x = \frac{1}{2} \int \left( \nabla^2 \phi \right)^2 + (\nabla \times A)^2 \right\} d^3x$$

or, when sources are present

$$H = \frac{1}{2} \int \left[ \pi^2 + (\nabla \times A)^2 + J \cdot A \right] \phi \frac{\partial}{\partial t} \phi \frac{\partial}{\partial t} d^3x$$
Problems (Do not hand in — no assignment this week)

(1) Derive Maxwell's Equations as operator equations from

\[ \Pi = i [H, \Pi] \]
\[ \dot{A} = i [H, A] \]

(2) a. Evaluate \[ [A_k(x,t), A_m(x',t')] \] for free fields.

b. Write \[ A_k(x) = \sum_{k,\alpha = \pm} \frac{1}{\sqrt{2W_k}} \left( q_k^\alpha e^{ik\cdot x} + q_k^* e^{i-k\cdot x} \right) \]

and interpret the \( \alpha \)'s. (Find CR's, etc.)

Remarks:

In our treatment of elasticity, we began with real atoms and the Schrödinger equation, noting that in certain situations a quantum field-theoretic description held. In electromagnetism, on the other hand, we began by assuming a field structure. The question arises as to whether there exists a solid that imitates electrodynamics. The answer at present is no.

Historically, the wave-like properties of light caused people to treat light as the elastic vibrations of a solid, since that was where wave-like behaviour first arose. Mculloch was the first to consider light independently of the model. His work was not recognized as correct until later because scientists' minds were so tuned to thinking of light in terms of the model that they couldn't accept it.
For amusement, consider the following:

In 3 dimensions, as everybody knows, there are 3 phases of matter — solid, liquid, gas. Does the same thing hold in 2 dimensions? There is some experimental evidence that this is not the case.

If you decide that there are only 2 phases, is the second the analog of the liquid or the solid phase of 3 dimensions? Elaborate — consider neutron scattering in two dimensions, for example.

Elections in Metals

Let us work out the consequences of making the following wild approximations:

1) Disregard the potential at the electron due to the ions.
2) Disregard electron-electron interactions.

Let there be $N$ electrons in volume $V$. Density $n = \frac{N}{V}$.

The electrons are taken to be independent except for the Pauli Exclusion Principle.

$$E = \frac{k^2}{2m}$$

3) All occupied states are within a sphere of radius $k_{max} (= k_f)$ in $k$-space.

Then $N = \frac{4}{3} \pi \frac{d^3 k}{(2\pi)^3} \text{ Vol.} = \frac{2}{3} \frac{V}{\pi^2} \int k^3 f(k)\,dk$.

For two possible momenta, we have $n = \frac{k_f^3}{3 \pi^2}$. 

And the total energy
\[ E = 2 \int_{\text{sphere}} \frac{d^3k}{(2\pi)^3} \frac{V \cdot k^2}{2m} = \frac{k^5 V}{1071 m} \]

This is a completely degenerate electron gas — electron gas at $0$ temperature.

Energy/unit volume $\propto n^{5/3}$

Problems (Do one, due February 20)

1) Stars burn up. An ash remains which then contracts. What is the maximum size of the star? What is the distribution of mass inside a star that is smaller than the critical size (which corresponds to a critical mass $M_c$)? What happens if $M > M_c$?

   Use Newtonian Gravity.

   Suggestions for a model:

   $T = 0$ (the temperature assumption is not important)

   $n_e = n_p$ (no. of neutrons = no. of protons)

   Electrons get pushed up as the nucleons are pulled together, so that they are approximately free

   $p_{\text{pressure}} = \frac{dP}{d\xi} = \frac{G}{(\text{Mass inside radius } r)} \rho \quad \text{Max} = \int r^2 c(r) x^2 dx < \infty$

   Find $p = p(c)$

2) Weak magnetic field $B$ on a system of electrons. Find the magnetic effect of the spins. (This is easy — phenomenon called Pauli paramagneti

   Neglect the effect on the spins — there is an effect on the orbits.

   Strong $B$ — what happens?

   Find energy $E$ vs. field strength $B$. $X = \frac{dE}{dB} = \text{magnetic susceptibility}$
We obtained the energy of a system of free electrons. What is the wave-function \( \Psi(R_1, R_2, \ldots, R_n) \)?

These are non-interacting e's. Consider electrons with spin "up", wavefunction is

\[
\Psi = e^{i \mathbf{k}_1 \cdot \mathbf{R}_1} e^{i \mathbf{k}_2 \cdot \mathbf{R}_2} \ldots
\]

Let there be \( N \) electrons with spin up, then the requirements of Pauli principle are satisfied if

\[
\Psi = \frac{1}{\sqrt{N!}} \sum_{\text{permutations}} (-1)^P \prod_{i} e^{i \mathbf{k}_{\alpha_i} \cdot \mathbf{R}_i}
\]

where \( P \) is the permutation that takes \( \alpha_i \) into \( \alpha_{\alpha_i} \).

This can also be written as a (Slater) determinant:

\[
\Psi = \frac{1}{\sqrt{N!}} \det \begin{pmatrix}
\phi_{\alpha_1}(R_1) & \phi_{\alpha_2}(R_2) & \cdots & \phi_{\alpha_n}(R_n) \\
\phi_{\alpha_2}(R_1) & \phi_{\alpha_3}(R_2) & \cdots & \phi_{\alpha_n}(R_n) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{\alpha_n}(R_1) & \phi_{\alpha_n}(R_2) & \cdots & \phi_{\alpha_n}(R_n)
\end{pmatrix}
\]

Incidentally, all the \( \phi_{\alpha_i} \)'s are different — otherwise \( \Psi \) would vanish.

What is the probability of finding an electron at \( R_b \) if we find one at \( R_a \)? We expect this to be a function of \( r = |R_a - R_b| \) only, starting at 0 for \( r = 0 \) and leveling off to a constant as \( r \to \infty \)

\[
\frac{H(r)}{r}
\]

Find instead the probability of finding one e at "a" and one electron at "b". This is proportional to finding the probability of finding one at "b"
When one has been found at "b",

\[ \text{Prob. (one e at \text{a}, one e at \text{"b"})} \]

\[ = \left< \prod_i \delta(k_a - R_i) \right> \prod_j \delta(k_b - R_j) \frac{1}{\mathcal{V}} \]

Typical term in this matrix element is

\[ \left< e^{i k_a \cdot R} e^{i k_b \cdot R} \ldots e^{i k_n \cdot R} \right| \delta(k_a - R_i) \delta(k_b - R_j) \left| e^{i k_a \cdot R} e^{i k_b \cdot R} \ldots \right> \]

Integrating over all variables except \( R_1, R_2 \)

obtain either \( k_1 = R_1 \)

\[ k_2 = R_2 \]

Contributes matrix element = 1

Contribution to matrix element

\[ = -e^{i (k_1, k_2, k_3, k_4)} \cdot (R_a - R_b) \]

\[ = -e^{i \mathbf{k} \cdot x} \quad \mathbf{k} = k_a - k_4, \quad \mathbf{x} = R_a - R_b \]

So that probability \( \propto \int d^3k \left( e^{i \mathbf{k} \cdot \mathbf{x}} - 1 \right)^2 \)

Integrate over \( \mathbf{k} \)

\[ K_f = \text{femto energy} \]

\[ \text{Prob.} \propto \left| \int d^3k \left( \frac{2K_f^2}{4\pi} \frac{1}{K_f} \right) \left( e^{i \mathbf{k} \cdot \mathbf{x}} - 1 \right)^2 \right|^2 \]

Notice that as \( R \to 0 \), prob. \( \to 0 \)

Let \( x = 2K_f R \)

Then prob. \( \propto \left( \frac{x}{2} \right)^2 \left[ 1 + \frac{x}{3} (x \cos x - \sin x) \right]^2 \)

Normalize so that prob. \( \to 1 \) as \( x \to \infty \)

\[ \text{Prob.:} \]

\[ \frac{1}{K_f \mathcal{R}} \to \]
So far we have discussed the problems of putting electrons in a three dimensional box: a) how the exclusion principle governs the filling up of discrete energy levels and b) what the antisymmetrization of wave functions tells you about the form of the electron distribution. We now want to go into further properties of many electron systems.

In order to understand the electrical conductivity properties of solids one must know what the energy levels of electrons inside them look like, and how they are populated as a function of temperature.

Hence the basic outline of the next few lectures is as follows:

1) How is the quantum statistics of particles in a box affected by the fact that they are fermions with the exclusion principle acting?

2) How can the theory of particles in a box be applied to electrons in a solid with periodic potential?

Quantum Statistics of Fermions

Oversimplified explanation to get the picture

Assume a flock of fermions trapped in a potential well of depth $V$

\[
\begin{array}{c}
E_0 - E' \\
E_0 - \phi \\
\end{array}
\]

Let $\phi$ be the energy necessary to remove a particle at the Fermi surface from the metal.

Recall that we defined the Fermi energy to be the maximum energy reached by particles when they are thrown into a box a $T=0$ with regard to the exclusion principle. Thus any unoccupied state below the Fermi energy, or occupied state above it, implies an excitation of the system. The weighting factor in quantum statistics is $e^{-\frac{E}{kT}}$, where $E$ is the excitation above ground state. Hence to compute this exponent we must measure energy levels relative to the Fermi surface.

\[
(E_0 - \phi') - (E_0 - \phi) = -\phi' + \phi
\]

Weighting factor is

\[
e^{\frac{(\phi' - \phi)}{kT}}
\]

\[
\phi = E_0 - \phi_0 , \quad \phi' = E_0 - \phi', \quad \epsilon_a = E_0 - \phi
\]

\[
c^{\frac{(\phi' - \phi)}{kT}} = e^{\frac{(E_0 - \phi)}{kT}}
\]
The above is not strictly correct - the energy levels should be measured relative to the chemical potential \( \mu \), not the Fermi energy. For metals near room temperature, the two are almost the same. At zero \( T \) the two are exactly the same. Unfortunately it is difficult to understand what the chemical potential is until the Fermi-Dirac statistics has been developed more fully. For the time being let \( \Phi_f \) be the energy necessary to remove some "average" electron from the metal; \( \mu_T = \nu - \Phi_f \); weighting factor \( e^{-\beta(\epsilon_a - \mu_T)} \).

We will soon discover just what is meant by "average" in this context; and that \( \Phi_f \) is just the work function of the metal involved.

Suppose that there are \( N_a \) particles in energy state \( \epsilon_a \), for a given configuration \( i \) of the \( N \) electrons in the system. Then the probability of this configuration is \( e^{-\beta \sum_i \epsilon_a i} \) and the net probability that the system is in some state is \( \sum_{i \text{ and } N \text{ particles}} e^{-\beta \sum_i \epsilon_a i} \).

It is extremely hard to do the sum, taking care of the constraint on the number of particles. Hence a mathematical device is used: write \( \epsilon_a \Rightarrow \epsilon_a - \mu \) for some \( \mu \) and sum as if there were no constraint on the total number of particles in the system. \( \mu \) is then picked to make the total number come out right, i.e. from now on all energies are written relative to \( \mu \) and all sums are unrestricted. (Students who want to really understand what is going on here should trace this through by use of Lagrange multipliers - see a statistical mechanics text.)

Thus the normalization for all statistical quantities is written as \( \sum_i e^{-\beta \sum_i \epsilon_a i (\epsilon_a - \mu)} \equiv e^{-\beta \sum \epsilon_a} \).

A bit of experimentation will show that if you have no constraint on the total number of particles then

\[
\sum_{\text{configurations}} e^{-\beta \sum_i \epsilon_a i (\epsilon_a - \mu)} = \prod_a \left[ \sum_{m_a} e^{-\beta(m_a i)(\epsilon_a - \mu)} \right]
\]

each configuration has either \( m_a = 0 \) or \( 1 \). Then this becomes

\[
\prod_a \left[ 1 + e^{-\beta(\epsilon_a - \mu)} \right]
\]

From

\[
e^{-\beta \sum_{\epsilon_a} \left[ 1 + e^{-\beta(\epsilon_a - \mu)} \right]} = \prod_a \left[ 1 + e^{-\beta(\epsilon_a - \mu)} \right]
\]

we find

\[
\beta - \frac{1}{\beta} = \sum_{\epsilon_a} \left[ 1 + e^{-\beta(\epsilon_a - \mu)} \right]
\]

The sum is then approximated by an integral. (Prove to yourselves that this is a good approximation and that we do not lose anything by setting \( \epsilon_a = \frac{p^2}{2m} \) for the electron.)

\[
\beta = \frac{1}{\beta} \int d\epsilon_a \left[ 1 + e^{-\beta(\epsilon_a - \mu)} \right] = \frac{2 \pi^2}{(2\pi \hbar)^3} \text{ Volume}
\]

Phase space for electrons of momentum \( p \). The 2 is because each direction of motion is accounted for.
Calculation of Statistical Quantities

\[ \langle N \rangle = \text{average number in the box} = \sum_{i} \left[ \frac{\frac{\partial}{\partial \mu} \sum_{\alpha} e^{-\beta \varepsilon_{\alpha}} \delta(\varepsilon_{\alpha} - \mu)}{\sum_{\alpha} e^{-\beta \varepsilon_{\alpha}}} \right] \]

\[ = \frac{1}{\beta} \frac{\partial}{\partial \mu} e^{-\beta \mu} = -\frac{2\beta}{\beta} \frac{\partial}{\partial \mu} \]

This shows how the number in the box is related to the chemical potential.

We can also calculate the density \( \rho = \frac{\langle N \rangle}{V} \) and the average number in a given energy state

\[ \langle m_{\alpha} \rangle = \frac{\sum_{i} m_{\alpha} e^{-\beta m_{\alpha} (\varepsilon_{\alpha} - \mu)}}{\sum_{i} e^{-\beta m_{\alpha} (\varepsilon_{\alpha} - \mu)}} = \frac{\rho + e^{-\beta (\varepsilon_{\alpha} - \mu)}}{1 + e^{-\beta (\varepsilon_{\alpha} - \mu)}} \]

\[ \langle m_{\alpha} \rangle = \frac{1}{1 + e^{\beta (\varepsilon_{\alpha} - \mu)}} = \left[ 1 + e^{\beta (\varepsilon_{\alpha} - \mu)} \right]^{-1} \]

It is easy to remember the + sign in the denominator, because \( \langle m_{\alpha} \rangle \) must be a number between 0 and 1.

We can now define "average" energy as that energy at which \( \langle m_{\alpha} \rangle = \frac{1}{2} \). This defines \( \mu_{T} + \beta \).

Notice that if \( T = 0 \quad \beta \to \infty \quad \epsilon < \mu \to e^{\beta (\varepsilon_{\alpha} - \mu)} = 1 \quad \epsilon > 0 \to e^{\beta (\varepsilon_{\alpha} - \mu)} \to \infty \)

\[ \langle m_{\alpha} \rangle \]

\[ \frac{\varepsilon}{\mu} = \varepsilon_{T} \to \varepsilon_{\alpha} \]

As the temperature gets higher and higher, \( \langle m_{\alpha} \rangle \sim \frac{1}{2} e^{\beta (\varepsilon_{\alpha} - \mu)} \), which has the shape \( e^{-\beta \varepsilon_{T}} \) that you'd expect if you knew only about Maxwell-Boltzmann statistics.

The curve is symmetric in the sense that the shape of the pieces above and below \( \mu \) is the same

\[ \frac{1}{1 + e^{\beta (\mu - \varepsilon_{T})}} = 1 - \frac{1}{1 + e^{\beta (\mu - \varepsilon_{T})}} \]

\[ |A| = |B| \]

This will be useful in doing integrals.

PROBLEM (If you wish this can be handed in)

a) Density of states as a function of energy (plot this as \( T = 0 \) and see what it looks like)

b) Energy density as a function of \( T \)

c) Pressure as a function of \( T \)
In order to do b) and c), it is necessary to do integrals of the form
\[ I = \int_0^\infty \frac{g(\varepsilon)}{1 + e^{\beta(\varepsilon - \mu)}} \, d\varepsilon = \int_0^\infty f(\varepsilon) \, d\varepsilon \]

There is a handy way to get at these:
\[ I = \int_0^\mu f(\varepsilon) + \int_\mu^\infty f(\varepsilon) = \int_0^\mu g(\varepsilon) \, d\varepsilon - \int_0^\mu \frac{g(\varepsilon) \, d\varepsilon}{1 + e^{\beta(\varepsilon - \mu)}} + \int_\mu^\infty \frac{g(\varepsilon) \, d\varepsilon}{1 + e^{\beta(\varepsilon - \mu)}} \]
\[ \int_\varepsilon^\mu \frac{g(\varepsilon) \, d\varepsilon}{1 + e^{\beta(\varepsilon - \mu)}} = \frac{1}{\beta} \int_0^\beta \frac{g(u - \varepsilon/\beta) \, dy}{1 + e^{\beta u/\beta}} \]
\[ \int_\mu^\infty \frac{g(\varepsilon) \, d\varepsilon}{1 + e^{\beta(\varepsilon - \mu)}} = \frac{1}{\beta} \int_0^\infty \frac{g(u + \varepsilon/\beta) \, d\varepsilon}{1 + e^{\beta u/\beta}} \]

For most applications
\[ \int_0^\beta \frac{g(u - \varepsilon/\beta) \, dy}{1 + e^{\beta u/\beta}} \approx \int_0^\beta \frac{g(u - \varepsilon/\beta) \, dy}{1 + e^{\beta u/\beta}} \]
\[ g(u - \varepsilon/\beta) \approx g(u - \varepsilon/\beta) \]
\[ g(u + \varepsilon/\beta) \approx g(u) + \frac{\varepsilon}{\beta} g'(u) \]

To do \( \int_0^\infty \frac{x \, dx}{1 + e^x} \), expand \( \frac{xe^{-x}}{1 + e^{-x}} \) in a power series and sum. You should find
\[ \int_0^\infty \frac{x \, dx}{1 + e^x} = \pi^2 / 12 \]

If you use these tricks in b), you should find that the total energy looks like
\[ U = U_0 + \gamma T^2 \]

then
\[ C_V = \frac{\partial U}{\partial T} = 2\gamma' T \]

Hence the specific heat of a metal takes the form
\[ C_V = 2\gamma' T + \alpha T^3 \]
\[ 2\gamma' T + 3R \]

in both cases the electron contribution is detectable.

If you go to temperatures that are too high, it is necessary to include higher terms in the expansion of \( g(\varepsilon) \) and the result will have a different \( T \) dependence.

It is fun to see that this qualitative behavior can be found by quite simple basic arguments.

Electron contribution: At low \( T \), we fill up all the states in \( k \) space to the Fermi surface. Then only those electrons within energy \( kT \) of the Fermi energy will be excited.

Electrons in a spherical shell = \( 4\pi k^2 \, dk = 4\pi k \, E \, dE = 4\pi \hbar c \varepsilon \varepsilon (kT) \)

Phonon contribution: The number of phonons which will contribute to the specific heat is the number of lattice vibrations that can be excited = all those up to \( \hbar \varepsilon = kT \).
Look in \( k \) space for the phonons. We know have the whole volume of the sphere
\[
\sim k_{\text{outside}}^3 \sim \frac{\hbar^3}{4 \pi \mathcal{V}^2} \sim \frac{1}{\mathcal{V}^3}
\]

We have sketchily derived the basic effects of Fermi-Dirac statistics on the electron distribution within a box. For any reasonable \( T \), most of the electrons are huddled in the states at the bottom of our potential well. The states just below the chemical potential energy are not fully populated: these particles are in the state above \( \mu \).

Now the point is to apply this box to the physical case of a solid. What we have to begin with is a number of positive ions held together in a lattice
\[
\text{Coulomb potential from ions seen by electrons}
\]

The innermost electrons in the atoms will in general stay attached to the ions, and we can forget about them for the moment. Let's worry only about the outer electrons. These are much more loosely bound to the ions and are thus the ones which will best reflect the peculiar properties of solids.

Before we begin to do detailed calculations, let's try to understand what happens on a qualitative level. Most students have done the following problem at some time in elementary quantum mechanics courses:

**Begin with 2 identical potential wells**

Each alone has a certain set of energy levels, \( \varepsilon_d \). Now consider the system of the two together and compute the energy levels of the systems as a function of \( r \). One finds that for large \( r \) the levels are roughly those of the individual atoms; but as \( r \) decreases they tend to split - each into 2 new energy levels

where the energy \( \delta \) the new levels depends on the symmetry of the new wave functions.

Repeating this with enough atoms, you can assemble a whole crystal lattice. Each of the atomic states characterized by \( n \) and \( l \) now splits into \( 2N \) levels.

It happens that this splitting is so small that there is usually still space between the
resulting energy bands.

Within a band the relative energies of the levels depend on the phase difference of the wave functions from one cell to the next. The lowest energy state will look something like

\[ \text{Phase difference} = 0 \]

The highest energy state will have a phase difference of \( \pi \).

PROBLEM: Why does a phase difference of \( \pi \) always give the same energy as \(-\pi\)?

PROBLEM: From some sort of knowledge about the shapes of wave functions within a given cell, plus phase differences from cell to cell, explain why for one band the energy versus \( k \) curve looks like \( E \) whereas for the next band it may look like \( E \).

It also happens that within a given band the electrons feel a certain average potential due to the collection of ions and other electrons, which is as though the electrons were held in a box with this average potential. Thus the distribution among levels in the solid follows the laws of Fermi-Dirac statistics that we found above.

Now to get down to brass tacks, we go into the mathematics of wave functions for electrons in a periodic potential. Once again we find that taking advantage of the periodicity saves quite a bit of work.

Begin by writing down the Schrödinger equation for the system:

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r})
\]

\[ V(\vec{r}) = V(\vec{r} + \vec{N}) \]

where \( \vec{N} \) is any lattice vector.

Substitute \( \vec{r}' = \vec{r} + \vec{N} \)

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r} + \vec{N}) \right] \psi(\vec{r} + \vec{N}) = E \psi(\vec{r} + \vec{N})
\]

Hence \( \psi(\vec{r} + \vec{N}) \) is a solution with the same energy as \( \psi(\vec{r}) \), for any lattice vector \( \vec{N} \).

Define \( \mathcal{M}(\vec{r}) = \sum_{\vec{N}} \psi(\vec{r} + \vec{N}) e^{-i \vec{r} \cdot \vec{N}} \)

This is then a solution of the Schrödinger equation with the same energy as \( \psi(\vec{r}) \)

It has the property \( \mathcal{M}(\vec{r} + \vec{N}_0) = \mathcal{M}(\vec{r}) e^{i \vec{r} \cdot \vec{N}_0} \)

Thus solutions of the Schrödinger equation with a periodic potential can be constructed to satisfy \( \psi(\vec{r} + \vec{N}) = \psi(\vec{r}) e^{i \vec{r} \cdot \vec{N}} \vee \vec{N} \).

Then Floquet's theorem says that the solutions take the form

\[ \psi(\vec{r}) = u(\vec{r}) e^{i \vec{K} \cdot \vec{r}} \]
Expand the \( u(\vec{r}) \) in a Fourier series

\[
\frac{\vec{L}}{L} \text{ is a vector in the reciprocal lattice space.) Then plug this back into}
\]

\[
\sum_{L} \left[ \frac{\hbar^2}{2m} \nabla^2 + V_0(\vec{r}) \right] u(\vec{r}) e^{i\vec{K} \cdot \vec{r}} = \epsilon u(\vec{r}) e^{i\vec{K} \cdot \vec{r}}
\]

We then find

\[
\sum_{L} \left[ \frac{\hbar^2}{2m} (\vec{K} + \vec{L})^2 + V_0(\vec{r}) - e \right] \hat{A}_L e^{i(\vec{K} + \vec{L}) \cdot \vec{r}} = \sum_{L} \hat{A}_L u(\vec{r}) e^{i(\vec{K} + \vec{L}) \cdot \vec{r}}
\]

Hence

\[
\sum_{L} \left[ \frac{\hbar^2}{2m} (\vec{K} + \vec{L})^2 - e \right] \hat{A}_L = -\sum_{L} V_0 \hat{A}_L
\]

The point is to solve this for the energy levels \( \epsilon \) and the wave function pieces \( \hat{A}_L \).

Up to this point we haven't made approximations. But, as usual, to get a solution it is necessary to approximate.

**APPROXIMATION 1: Weak Binding**

\[
V_0(\vec{r}) \rightarrow V_0 \text{ small,} \quad V_0 = 0
\]

This just sets the constant part of the potential equal to 0; we are just raising or lowering the overall scale.

First approximation to the wave function

Weak potential implies

\[
\psi(\vec{x}) \approx \hat{A}_0 e^{i\vec{K} \cdot \vec{x}}, \quad \hat{A}_0 \approx 0 \quad \vec{L} \neq 0
\]

\[
\vec{L} = 0 \Rightarrow \epsilon \approx \frac{\hbar^2 k^2}{2m}
\]

\[
\vec{L} \neq 0 \Rightarrow \hat{A}_L = -\frac{V_0}{\left(\vec{K} + \vec{L}\right)^2 - \vec{K}^2}
\]

We can substitute this back to find the correction to \( \epsilon \)

\[
\epsilon = \frac{k^2 L}{2m} - \sum \frac{V_0 L^2}{\left(\vec{K} + \vec{L}\right)^2 - \vec{K}^2}
\]

However for \( \vec{L} \neq 0 \quad \left(\vec{K} + \vec{L}\right)^2 \approx \vec{K}^2 \rightarrow \hat{A}_L \text{ is large and not } \approx O \text{ as we assumed. Thus in this case the approximation breaks down and the analysis has to be redone.}

The trouble comes for

\[
2 \vec{K} \cdot \vec{L} + \vec{L}^2 = 0 \quad \vec{L} \cdot (K^2 + \frac{L^2}{2}) = 0
\]

This equation defines a set of troublesome vectors \( \vec{L} \) - a plane of \( \vec{K} \)'s which goes through \( \frac{L}{2} \).

Hence each \( \vec{L} \) gives a plane of singular points in k space. The set of all such planes divides k space up into Brillouin zones. Thus we see that at the edge of the Brillouin zone
the energy will probably deviate from \( \frac{k^2}{2m} \).

To find out quantitatively what is happening, pick one particular \( L_1 \) of

\[
(k + L_1)^2 - k^2 \sim \nabla
\]

Then \( A_0, A_{L_1} \) are not small, but all the other terms are. Returning to the original set of equations gives

\[
\begin{bmatrix}
\frac{k^2}{2m} - \epsilon \\
\frac{k^2}{2m} (k + L_1)^2 - \epsilon
\end{bmatrix} A_0 = -\nabla L_1 A_{L_1},
\]

\[
L_1 A_{L_1} = -\nabla L_1 A_0
\]

(to have a real potential). Define \( \psi(\rho) = \frac{k^2}{2m} \).

Then equations can be solved only for

\[
\epsilon = \frac{\psi(k + L_1) + \psi(k)}{2} \pm \sqrt{\left( \frac{\psi(k + L_1) - \psi(k)}{2} \right)^2 + |\nabla L_1|^2}
\]

Small \(|\nabla L_1|^2 \Rightarrow \epsilon = \psi(k) - \frac{|\nabla L_1|^2}{2 \psi(k + L_1) - \psi(k)}\]

Thus the edge of the zone leads to a gap in the energy curve.

\[\text{gap}\]

\[\text{ASIDE: ONE DIMENSIONAL MODELS OF SOLIDS TO PLAY WITH}\]

\[\text{a) Cosine potential - leads to Mathieu's equation}\]

\[\text{b) } \uparrow V(x) \rightarrow x\]

The Kronig-Penny model of a solid.

This is quite useful because you can vary the heights and widths of the holes and hence can see what happens if one of the atoms is different.

\[\text{STONG BUNDING APPROXIMATION}\]

Here the potential wells are taken as relatively deep and far apart.

If \( f(r) \) is the solution around one of the wells, try as a solution

\[
\sum_N e^{i \mathbf{k} \cdot \mathbf{r}} \cdot \psi(\mathbf{r} - \mathbf{R}_N)
\]

It there is only one atom per unit cell of a given kind,

\[
\psi = \sum_N \mathbf{\nabla} \cdot \mathbf{u} (\mathbf{R} - \mathbf{N}) \uparrow -\frac{1}{2m} \nabla^2 \psi + \mathbf{u}(\mathbf{r}) \psi = \epsilon \psi
\]

To get an idea of the energy levels, use the formula

\[
\mathcal{E} = \int \frac{1}{D} = \int \frac{\psi^* \psi}{\psi^* \psi} dV
\]
\[ D = \sum_{N \neq N'} e^{i \mathbf{u} \cdot (\mathbf{r} - \mathbf{r}')} \int f^* (\mathbf{r} - \mathbf{r}') f (\mathbf{r} - \mathbf{r}') d^3 r \]

\[ = 1 + \frac{1}{N} \sum_{N' \neq N} e^{i \mathbf{u} \cdot (\mathbf{r} - \mathbf{r}')} \int f^* f d^3 r = 1 + \sum_{M} e^{i \mathbf{u} \cdot M} I (M) \]

\[ I (M) = \int f^* f d^3 r \]

should probably be computed for only those \( M \) corresponding to nearest neighbors

\[ N = \sum_{N \neq N'} e^{i \mathbf{u} \cdot (\mathbf{r} - \mathbf{r}')} \int f^* f \left[ \frac{-\frac{\hbar^2}{2m} \nabla^2 + \sum_{l} \nabla V (\mathbf{r} - \mathbf{r}')} \right] f d^3 r \]

Then

\[ E = E_0 + N D \]

Define

\[ J (M) = \int f^* f \left[ V (\mathbf{r}) - N (\mathbf{r}) \right] f d^3 \mathbf{r} \]

\[ E = E_0 + \left[ \frac{J_0 + \sum_{M \neq 0} J (M) e^{i \mathbf{u} \cdot M}}{1 + \sum_{M \neq 0} I (M) e^{i \mathbf{u} \cdot M}} \right] \]

In practice it is difficult to get numbers out of this. However we can understand what is going on by assuming the \( J \)'s go down so fast that only the nearest neighbors contribute. Then

\[ E = E_0 + \left[ \frac{J_0 + 2 J_{00} \left[ \cos k_x a + \cos k_y a + \cos k_z a \right]}{1 + 2 J_{00} \left[ \cos k_x a + \cos k_y a + \cos k_z a \right]} \right] \]
CREATION AND ANNihilation OPERATORS FOR ELECTRONS

Up to this point we have been using a formalism in which bosons (photons, phonons, etc.) are described by creation and annihilation operators, while the electrons in the system have been treated by standard wave function methods. One might expect that a certain elegance would result from use of a formalism which also contained creation and annihilation operators for fermions. However, the main reason for applying this language to electrons is not elegance, but rather the fact that certain phenomena, such as pair creation, seem much more reasonable when approached in this manner.

What we want is some sort of operator \( c \) such that \( c_{k,i}^* \) creates an electron with momentum \( k \) in spin state \( i \), and \( c_{k,i} \) removes such an electron from the picture. (To simplify things, the spin index will be left out; any student who is confused by this should go through and reinsert all the indices.) If we had such an operator, we could write down interaction Hamiltonians which would in first order produce any desired process.

Example:

Electron \( k \) scatters to electron \( k' \) by absorbing phonon \( h \)

This is produced by an interaction like

\[
\sum \langle P_{k,k',k'} | a_{k} c_{k'}^* c_{k} \rangle
\]

However, if this is part of a Hamiltonian operator, its Hermitian conjugate must be included also

\[
\langle P_{k,k',k'} | a_{k} c_{k'}^* c_{k} + \sum P_{k,k',k'} a_{k'}^* c_{k'}^* c_{k} \rangle
\]

Knowledge of the \( P \)'s would then allow perturbation theory calculations of all sorts of processes, with \( P \) as perturbation.

To perform such calculations, one must know the properties of these \( c \)'s as operators. To begin with, suppose a universe of one electron state. It can either be empty or filled with one electron. Hence we arrive at

\[
\begin{align*}
  c^{*} | 1 \rangle &= 1 | 1 \rangle \\
  c^{*} | 1 \rangle &= 0 \\
  c^{*} c^{*} &= 0
\end{align*}
\]

By taking the adjoint,

\[
\begin{align*}
  c | 1 \rangle &= 10 \rangle \\
  c c &= 0
\end{align*}
\]
\[ \Rightarrow \quad C|0\rangle = 0 \]

Similarly \[ CC^\dagger + C^\dagger C = 1 \]
(test by applying to all possible states)

and \[ C_i^\dagger C_i = N_i \]
(\text{where } N_i \text{ is either } 0 \text{ or } 1)

Puzzle: Suppose you know only that an operator \( C \) obeys
\[ C^\dagger C^\dagger = 0 \quad C C = 0 \quad C C^\dagger + C^\dagger C = 1 \]

Find a representation for the operator. Is it unique?

When the universe is expanded to include two electrons, a little bookkeeping becomes necessary. Let electron \( \#1 \) be the one which is created first.
\[ C_a^\dagger C_b^\dagger |0\rangle = 1 \alpha \beta \]

means electron \( \#1 \) is in state \( b \); electron \( \#2 \) is in state \( a \).

The electron wave functions must be antisymmetric under interchange of electrons 1 and 2
\[ |ab\rangle = -|ba\rangle \quad \Rightarrow \quad C_a^\dagger C_b^\dagger |0\rangle = - C_b^\dagger C_a^\dagger |0\rangle \]

In fact the same relation holds for states other than the vacuum and we have the operator equations
\[ C_a^\dagger C_b^\dagger = -C_b^\dagger C_a^\dagger \]
\[ C_a C_b = -C_b C_a \]

The next thing to check is clearly \[ C_a C_b^\dagger = -C_b C_a^\dagger \]

There is a clever way to determine this:

The equation \[ C_a^\dagger C_b^\dagger + C_b^\dagger C_a^\dagger = 1 \]
must hold for the creation operator for any electron state. By the principle of superposition, if \( |a\rangle \) and \( |b\rangle \) are states, a perfectly allowable state is
\[ \alpha |a\rangle + \beta |b\rangle = 1 c \quad \exists \quad |\alpha|^2 + |\beta|^2 = 1 \]

\[ C_c^\dagger = \alpha C_a^\dagger + \beta C_b^\dagger \]

Hence
\[ (\alpha C_a^\dagger + \beta C_b^\dagger)(\alpha^* C_a + \beta^* C_b) + (\alpha^* C_a + \beta^* C_b)(\alpha C_a^\dagger + \beta C_b^\dagger) = 1 \]

Expand. Use \( |\alpha|^2 + |\beta|^2 = 1 \)

Then
\[ \beta \alpha^* C_a^\dagger C_b + \alpha \beta^* C_b^\dagger C_a + \beta^* \alpha^* C_b C_a^\dagger + \alpha \beta C_a C_b^\dagger = 0 \]

\[ C_b^\dagger C_a + C_a^\dagger C_b = 0 \]
We thus have a set of anticommutation relations for fermions which parallel the commutation relations previously derived for bosons
\[ [c_a, c_b^*]_+ = 0 \quad [c_a^*, c_b^*]_+ = 0 \]
and
\[ [c_a, c_b^*]_+ = d_{ab} \]
if \( a \neq b \) are orthogonal.

Exercise: figure out what these relations are if \( a \) and \( b \) are not orthogonal.

Above we determined the operator \( c_\psi^* = \sum \phi(p) c_p^* \) which creates an electron with amplitude \( \lambda \) to be in state \( \alpha \) and amplitude \( \beta \) to be in state \( \beta \). Likewise,
\[ c_\psi^* = \sum \phi(p) c_p^* \]
creates an electron with amplitude \( \phi(p) \) to have momentum \( p \).

This means its wave function must have the form \( \sum \phi(p) e^{ip \cdot x} = \psi(x) \)
and \( c_\psi^* = \sum c_p^* \langle \rho | \psi \rangle \). If we symbolize the creation operator by \( \langle \mathbf{k} | \# \rangle \)
then \[ c_\psi^* = \sum \phi(p) \langle \rho | \psi \rangle \langle \rho | \psi \rangle = \langle \mathbf{k} | \# \rangle \]
This sort of formalism makes it obvious that the manipulations performed here with operators are the same as usually performed with the states they create.

Let \( \psi(x_0) \) be an operator which creates an electron at \( x_0 \).
Then \[ \psi(x_0) = \sum \phi(p) \langle \rho | x_0 \rangle = \sum c_p^* e^{-ip \cdot x_0} \]
\[ \psi(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} c_p \]
Exercise: using the properties of the \( c_p^* \)'s, convince yourself that
\[ \psi^*(x) \psi(y) + \psi(y) \psi^*(x) = \delta^3(x-y) \]
These \( \psi(x) \) operators are called field operators. Because \( \psi^*(x) = \int \frac{d^3p}{(2\pi)^3} e^{-ip \cdot x} c_p^* \)
and \[ c_\psi^* = \sum c_p^* \langle \rho | \# \rangle \]
Then \[ c_\psi^* = \int \frac{\psi^*(x)}{\text{wave}} \delta(x) d^3x \]
The field operator may be thought of as just a device which changes wave functions into the corresponding operators.

These methods can equally well be applied to solid state physics. If \( \psi_k(r) = e^{ik \cdot r} u_k(r) \)
\[ f_k(r) = e^{ik \cdot r} u_k(r) \]
is the wave function for an electron of propagation vector \( k \), then the operator which creates such a particle is \[ c_\psi^* = \int \psi_k^*(R) d^3R \]
Thus, for an arbitrary number of non-interacting electrons, the Hamiltonian of the system can be written as
\[ \mathcal{H} = \sum \epsilon_k n_k = \sum \epsilon_k \sum \psi_k^* \psi_k \]
But \[ \sum \psi_k^* \psi_k^* f_k(R) \]
and if \( h \) is the Hamiltonian for the individual electron wave function, then
\[ h \sum \psi_k^* \psi_k^* f_k(R) = \sum \epsilon_k \sum \psi_k^* \psi_k^* f_k(R) \]
\[ \int \psi^+ \frac{i}{\hbar} \psi d^3x = \int d^3x \sum \xi \xi' \epsilon_{\xi} \xi' \xi'. f^{\dagger}_{\xi}(k) \xi' \xi \xi. f_{\xi}(k) \]

\[ = \sum \xi \epsilon_{\xi} \xi' \xi'. f^{\dagger}_{\xi}(k) \xi' \xi \xi. f_{\xi}(k) \]

Thus the Hamiltonian operator for the entire system can be expressed in terms of the Hamiltonian for an individual wave function and the field operator by

\[ H = \int \psi^+ (x) \hbar \psi (x) d^3x \]

Convince yourself that the creation operators for electrons must commute with all the operators for phonons. Then it is easy to see that a sample Hamiltonian for an interacting system of phonons and electrons might be written

\[ H = \sum \xi \epsilon_{\xi} \xi' \xi'. f^{\dagger}_{\xi}(k) \xi' \xi \xi. f_{\xi}(k) \]

\[ + \sum \xi \xi' \xi' \xi'. \left[ f^{\dagger}_{\xi}(k) \xi' \xi \xi. f_{\xi}(k) \xi' \xi + f^{\dagger}_{\xi'}(k) \xi' \xi \xi. f_{\xi'}(k) \xi' \xi \right] \]

This formalism can be used to treat any system in which electrons interact only by exchange of phonons. If, in addition to this, there is some direct electron-electron interaction, a further development is necessary.

If the electrons interact through some potential \( V(|R_i - R_j|) \) then the amplitude for scattering by this potential will be proportional to \( V(Q) = \int V(k) e^{-iQ.k} d^3k \)

Then the definitions of the creation operators imply that the Hamiltonian should contain terms of the form

\[ \frac{-1}{\rho} \left( \frac{\partial^2}{\partial x^2} \right) c_p \psi_k + \sum V(Q) \int (p_1 p_2 - p_1 - p_2) c_{\psi_k} \psi_k \]

Use of the relation \( c_{\psi_k} \psi = \int e^{i\psi_k \cdot x} \psi_k^*(x) dx \)

allows us to rewrite the interaction term as

\[ \int \int \psi^+ (x) \psi^+ (y) V(x-y) \psi (x) \psi (y) d^3x d^3y = \int \psi^+ (x) \psi (x) V(0) d^3x \]

\[ - \int \int \psi^+ (x) \psi (x) V(x-y) \psi (y) \psi (y) d^3x d^3y \]

To understand this, we compare it with the classical expression for interaction of two charge densities, \( C_{\text{classical}} = \frac{1}{2} \int \int \rho(x) V(x-y) \rho(y) d^2xd^2y \)

By doing this we discover:

a) we should have defined the interaction as \( \frac{1}{2} \sum V(Q) \int (c_{\psi_k} \psi_k \psi_k \psi_k \psi_k \psi_k \psi_k \psi_k \psi_k \psi_k) \)

Else we count each interaction twice

b) There seems to be an extra term \( \int \psi^+ (x) \psi (x) V(0) d^3x \)

This means that our quantum interaction contains no self-energy. This is exactly what we want (especially for things like a Coulomb potential where the self-energy
is infinite)

c) There is a difference of a - sign between the classical result and the one we found.

But we can fix this just by permuting the fermion creation operators in the definition.

So this is not serious or important.

Hence the Hamiltonian for a system of any number of electrons, interacting through the Coulomb potential, can be expressed as

\[ H = \int \psi^*(x) \left[ -\frac{\hbar^2}{2m} \left( -\nabla - e/\epsilon A(x,t) \right)^2 + e\phi(x,t) \right] \psi(x) \, d^3x \]

\[ + \frac{1}{2} \int \int \psi^*(x) \psi^*(y) \frac{e^2}{|x-y|^3} \psi(y) \psi(x) \, d^3x \, d^3y \]

where the \( \psi \)'s are field operators.

With this Hamiltonian, the Schrödinger equation for such a system is

\[ -\frac{\hbar}{i} \frac{\partial}{\partial t} \Psi = H \Psi \]

Notice that there is no reason for the solution \( \Psi \) to be composed of a definite number of electrons. However, because this particular Hamiltonian commutes with the particle number operator

\[ \hat{N} = \int \psi^* \psi \, d\tau \]

it is possible to choose solutions which do have the property \( \hat{N} \Psi = n \Psi \)

In general, this interaction will not mix subspaces with different \( n \).
Let $c_k^*$ be the operator that creates an electron in state $k$. Then $\sum_k c_k c_k^* = \delta_{kk}$ and $\sum_k c_k c_k^* = 0$.

A simple Hamiltonian that permits discussion of electron scattering in a crystal is

$$H = \sum_k \epsilon_k c_k^* c_k + \sum_{kk'} \lambda_{kk'} c_k^* c_{k'}$$

An unfilled electron state in an energy band is called a "hole." Electron creation operators are hole destruction operators and vice versa.

The formal identification is $c_k = b_k^*$, consequently

$$\sum \epsilon_k c_k b_k^* = \delta_{kk} \epsilon_k, \quad \sum \epsilon_k b_k b_k^* = 0.$$ 

Rewrite the Hamiltonian:

$$H = \sum_k \epsilon_k c_k^* c_k + \sum_\ell \epsilon_\ell b_\ell^* b_\ell$$

$$+ \sum_{kk'} \lambda_{kk'} c_k^* c_{k'} + \sum_{kk'} \lambda_{k'k} c_{k'}^* b_k$$

$$+ \sum_{\ell\ell'} \lambda_{\ell\ell'} b_\ell^* c_{\ell'} + \sum_{\ell\ell'} \lambda_{\ell'\ell} b_{\ell'}^* b_\ell$$

where the sum over all electron states $k$ has been written as the sum over the sets of states.

$$\sum \epsilon_\ell b_\ell b_\ell^* = -\sum \epsilon_\ell b_\ell^* b_\ell + \sum \epsilon_\ell$$

The last term in the right is a zero point energy which may be neglected by changing the zero of the energy.
The 4 terms bilinear in the creation and destruction operators yield the following processes in first order:

\[ \text{Convention: An arrow pointing in the direction of increasing time indicates an electron; an arrow pointing in the direction of decreasing time indicates a hole.} \]

2\textsuperscript{nd} order contributions

Consider the following

\[ \text{Diagram 2 diagrams that contribute in 2}\textsuperscript{nd}\text{ order:} \]

\[ \text{Now to conserve energy, in the initial state there is a phonon of energy } \omega_1 \text{, and in the final state there is a phonon of energy } \omega_2 \text{, where} \]

\[ \epsilon_1 + \omega_1 = \epsilon_2 + \omega_2 \]

\[ \epsilon_1 = \text{energy of electron } 1, \epsilon_2 = \text{energy of electron } 2. \]
From the perturbation theory of last term, the 2nd-order contribution to the amplitude is

\[ \sum \frac{\langle f|V|r\rangle\langle r|V|i\rangle}{E_i - E_r + i\varepsilon} \]

where the sum runs over all intermediate states.

Evaluating (A) and (B):

(A):

\[ \sum_k \frac{\langle 2|\lambda_{2k}^* c_i^k c_{-i}^k |k\rangle\langle k|\lambda_{1k}^* c_{-i}^k c_i^k |1\rangle}{E_i - E_k + i\varepsilon} \]

= \sum_k \frac{\lambda_{2k}\lambda_{1k}^*}{E_i + \omega_1 - E_k - i\varepsilon}

(B):

\[ \sum_k \frac{\langle 2|\lambda_{1k} b_{-i}^k c_i^k |k\rangle \langle k|\lambda_{2k}^* b_{-i}^k c_i^k |1\rangle}{E_i + \omega_1 - (E_i + \varepsilon_1 + \varepsilon_2 + \varepsilon_1 - E_k) + i\varepsilon} \]

= \sum_k \frac{\langle 0|c_i^k b_{-i}^k |k\rangle \langle k|b_{-i}^k c_{-i}^k |0\rangle}{E_k - (E_2 + \omega_1) + i\varepsilon}

= \sum_k \frac{\lambda_{2k}\lambda_{1k}^*}{E_k - \omega_1 - E_i + i\varepsilon} = \sum_k \frac{\lambda_{2k}\lambda_{1k}^*}{E_i + \omega_1 - E_k - i\varepsilon}

So that the contribution of (B) is the same as that of (A) except for the sign of i\varepsilon in the denominator.

Notice that we have not evaluated the complete 2nd-order amplitude so we have omitted
The contribution of the holes may be included as a modification of the propagator:

Let \( \lambda_{k,2} = \int \phi_{k}^{*}(x) \lambda(x) \phi_{k}(x) \, dx \)

then the contribution of (A) may be written

\[
\int dx_{2} \, dx_{1} \, dt_{2} \, dt_{1} \, \phi_{2}^{*}(x_{2}) \lambda(x_{2}) K(2,1) \lambda(x_{1}) \phi_{1}(x_{1})
\]

and if we redefine \( K_{+}(2,1) = \Theta(t_{2} - t_{1}) \sum_{\text{holes}} e^{-i \varepsilon_{k} (t_{2} - t_{1})} \phi_{k}^{*}(x_{2}) \phi_{k}(x_{1}) \)

if we define for \( t_{2} < t_{1} \):

\[
K_{+}(2,1) = -\sum_{\text{holes}} e^{-i \varepsilon_{k} (t_{2} - t_{1})} \phi_{k}^{*}(x_{2}) \phi_{k}(x_{1})
\]

(A) + (B) may be written in the same form as (A) alone:

\[
\int dx_{2} \, dx_{1} \, dt_{2} \, dt_{1} \, \phi_{2}^{*}(x_{2}) \lambda(x_{2}) K(2,1) \lambda(x_{1}) \phi_{1}(x_{1})
\]
Relativistic Quantum Mechanics

There is no known violation of the invariance principle of relativity. Two consequences of putting together relativity and quantum mechanics that are important in elementary particle theory are: (1) The existence of antiparticles. Moreover, the dynamics of antiparticles are determined from those of the dynamics of the corresponding particles. This is known as the CPT theorem.

(2) The connexion between spin and statistics, namely that integer spin particles obey Bose statistics and that half integer spin particles obey Fermi statistics. We will give a demonstration of this result later.

The Schrödinger equation \( H\psi = i\frac{\partial \psi}{\partial t} \) is unsymmetrical in its treatment of time and space.

Basically there are 3 ways of doing relativistic quantum mechanics:

(1) A method which is easy to understand, but in which the relativity is not manifestly obvious.

(2) A method which is less easy to understand, in which the relativity is obvious.

(3) Abstract, elegant way.

We will follow method (1). In some problem, approach (2) is simpler.

Maxwell’s equations are relativistically invariant, but their effect may be treated in a non-relativistic manner in some applications (Compton scattering, photoelectric effect).

Maxwell’s Eq’ns may be derived from a Lagrangian \( L \) that gives the action

\[
\text{action} = \int \left( \frac{-\nabla \cdot \mathbf{E}}{\varepsilon_0} + \mathbf{E} \cdot \mathbf{D} - \mathbf{H} \times \mathbf{B} - \mathbf{J} \cdot \mathbf{E} \right) d^3x + \int \left( \frac{\varepsilon_0}{2} \nabla \times \mathbf{B} \cdot \nabla \times \mathbf{B} - \mu_0 \nabla \times \mathbf{E} \cdot \nabla \times \mathbf{E} \right) d^3x
\]
Interaction with matter is handled by adding the additional term $\int g_{\mu}(x)A_\mu(x)\,d^3x$ to the Lagrangian.

In the Lorentz-invariant non-covariant Coulomb gauge $D.A=0$, the interaction of charged particles through the electromagnetic field is equivalent to

$$H = \sum_k W_k a^+_k a_k + \int g_{\mu}(x)A_\mu(x)\,dx + H_{\text{matter}}$$

$$A(x) = \frac{1}{\sqrt{2w_k}} \sum_{k,i} \left[ e_{k,i} e^{ik\cdot x} a_k + e_{k,i}^* e^{-ik\cdot x} a_k^* \right]$$

Let us write down the classical relativistic relations for particles:

$$E = m_0 \gamma, \quad \mathbf{p} = m_0 \gamma \mathbf{v}, \quad \beta = \frac{\gamma - 1}{\gamma}$$

(C taken to be 1)

and

$$E^2 = p^2 + m_0^2$$

so

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

or

$$p_\mu = m^2 \gamma$$

where $p_\mu = (E, \mathbf{p})$

and if $A_\mu = (A_0, \mathbf{A}), \quad B_\mu = (B_0, \mathbf{B})$

then $A \cdot B = A_\mu B_\mu = A_0 B_0 - \mathbf{A} \cdot \mathbf{B}$

Definition: $\delta_{\mu,\nu} = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu = 0 \\ -1 & \mu = \nu = 1, 2, \text{or} 3 \end{cases}$

Let $V_\mu = \left( \frac{\partial}{\partial t}, \frac{-\partial}{\partial x}, \frac{-\partial}{\partial y}, \frac{-\partial}{\partial z} \right) = \left( \frac{\partial}{\partial t}, -\mathbf{V} \right)$

(Notice the minus sign in the definition of $V_\mu$.)

Another useful vector is $X_\mu = (t, x, y, z)$.
Non-relativistically, the phase of a free particle of momentum $k$
frequency (energy) $E$ is
$$e^{i(p \cdot x - Et)} \quad \text{or} \quad e^{i(k \cdot x - Et)}$$

This may be written as a Lorentz invariant as
$$e^{-ip \cdot x}$$
we will take this over to the relativistic case (first done by de Broglie)

Then, using the relativistic expression: $E = \sqrt{k^2 + m^2}$

we obtain
$$\frac{dW}{dt} = \frac{dE}{dp} = \text{classical} = \frac{p}{E}$$

The Schrodinger $\Psi$'s is of first order in time and permits the
principle of superposition which we want to retain. Let us
look for a relativistic wave equation that is first order in time

$$E = \sqrt{p^2 + m^2}$$

suggests
$$i \frac{\partial \Psi}{\partial t} = \sqrt{m^2 - p^2} \Psi$$

which, horrible as it looks, can be made to work out

But the equation in the presence of an external field:

$$(i \frac{\partial}{\partial t} - eA_t) \Psi = \sqrt{m^2 - (p - ieA)^2} \Psi$$

presents difficulties.

a second-order equation suggests itself

$$\left[ (i \frac{\partial}{\partial t} - eA_t)^2 - \delta \frac{\partial}{\partial x} (-i \frac{\partial}{\partial x})^2 \right] \Psi = m^2 \Psi$$
This is a serious contender for a relativistic wave equation. There are possible difficulties:

1. Does it conserve probability?
2. Does it reduce to a correct non-relativistic equation (in the non-relativistic limit)?
3. Energy levels of the hydrogen atom.
4. Does it violate causality?

Problem (due first week next term): Test out one of these questions.

The most serious test is (3) — the others are basically theoretical prejudices.

The Dirac Equation

Dirac looked for an equation of the form $H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$, where he permitted the wave-function $\Psi$ to have several components, and the Hamiltonian was permitted to be a matrix operator on these components.

Guessed

\[
H = \alpha \frac{1}{i} \frac{\partial}{\partial x} + \beta \frac{1}{i} \frac{\partial}{\partial y} + \gamma \frac{1}{i} \frac{\partial}{\partial z} + \beta m
\]

Where $\alpha, \beta$ are matrices.

I.e.,

\[
H = \alpha \cdot \mathbf{p} + \beta m
\]

$\alpha, \beta$ hermitean

with an external

an external

with an electrostatic field present,

\[
H = \mathbf{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m
\]

$\mathbf{p} = -i\hbar \frac{\partial}{\partial t}$

and $i\hbar \frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t}$.
Look for a sol’n of the form \( \psi = u e^{-ik \cdot x} \) (free particle)

then \( u \) satisfies the equation

\[
E \ u = (\alpha \cdot k + \beta m) u
\]

\[
E^2 u = (\alpha \cdot k + \beta m)^2 u
\]

\[
E^2 = \mathbf{k}_{1}^2 + \mathbf{k}_{2}^2 + \mathbf{k}_{3}^2 + m^2
\]

\[
\Rightarrow \quad \alpha_1^2 = 1 = \alpha_2^2 = \alpha_3^2
\]

\[
\beta \alpha_1 \beta + \alpha_1 \beta = 0, \text{ etc.}
\]

\[
\beta = 1
\]

In 4 dimensions an explicit set of \( \alpha \)'s and \( \beta \) that satisfy these equations is

\[
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
= \begin{pmatrix} 0 & 6x \\ 6x & 0 \end{pmatrix}
\]

\[
\alpha_2 = \begin{pmatrix} 0 & 6y \\ 6y & 0 \end{pmatrix}
\]

\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Write the 4 component wavefunction

\[
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} e^{-ik \cdot x} \quad \text{as} \quad \begin{pmatrix} u_0 \\ u_5 \\ u_6 \end{pmatrix} e^{-ik \cdot x}
\]

where \( u_0, u_5, u_6 \) are 2 component objects
In an external field \( \frac{i}{\hbar} \left( \frac{\partial}{\partial t} - e \phi \right) \psi = \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla - e A \right) \psi + \beta m \psi \). Multiply by \( \beta \):

\[
\beta \left( \frac{i}{\hbar} \left( \frac{\partial}{\partial t} - e \phi \right) \psi \right) = \beta \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla + e A \right) \psi = m \psi
\]

Define another set of matrices:

- \( \chi_0 = \gamma_0 = \beta \)
- \( \chi = \beta \chi_0 \)

Then the equation becomes

\[
\gamma_\mu \left( \frac{1}{i \hbar} \nabla_\mu - e A_\mu \right) \psi = m \psi
\]

where \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu \nu} \)

These commutation rules are invariant under Lorentz transformations.

Let \( \psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \)

Then

\[
\left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - e \phi \right) \psi_a = \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla - e A \right) \psi_a + m \psi_a
\]

and

\[
\left( -\frac{i}{\hbar} \frac{\partial}{\partial t} - e \phi \right) \psi_b = \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla - e A \right) \psi_b - m \psi_b
\]

For a real spinor, both equations are

\[
-\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_a = \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla \right) \psi_b
\]

\[
-\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_b = \frac{\hbar}{c} \left( \frac{1}{i \hbar} \nabla \right) \psi_a
\]
DIRAC EQUATION

\[-\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = H \psi \]

where

\[ H = \beta m + eV + \vec{A} \cdot (\vec{p} - e \vec{A}) \]

in order to a) have an equation linear in time derivatives and

b) have time and space derivatives enter on an equal footing.

It is found that in order for this to work, different \( \alpha \)'s must anticommute, and

\[ \alpha \beta = -\beta \alpha \]

One representation of these commutation relations is

\[ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \]

Hence if \( \psi \) is written in the form of a pair of 2 component spinors \( \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \)

then the Dirac equation gives a set of 2 coupled equations for the two component spinors.

Because \( \overrightarrow{\psi} \) is Hermitian, \( \frac{\partial}{\partial t} \int \overrightarrow{\psi} \overrightarrow{\psi} \, d\nu = \int \frac{\partial \psi^* \psi}{\partial t} + \frac{\partial \psi^* \psi}{\partial t} \, d\nu = 0 \)

Thus \( \psi^* \psi \) can be thought of as a probability density.

Expectation values take the usual form \( \langle \alpha \rangle = \int \psi^* \alpha \psi \, d\nu \)

Notice that none of the answers will change if we change the representation (i.e. if we pick some different form which satisfies the commutation relations) for the \( \alpha \) and \( \beta \) matrices. Then the new representation can be related to the old as follows:

\[ S S^* = 1 \quad \psi' = S \psi \quad \psi'^* = \psi^* S^{-1} \]

\[ \alpha' = S \alpha S^{-1} \quad \beta' = S \beta S^{-1} \]

Check: \( \alpha' \psi' = S \alpha S^{-1} S \psi = S (\psi \alpha) \)

\[ \alpha' \beta' = S (\alpha \beta) S^{-1} \]

Change of representation sometimes makes the physics more transparent. For example, the Dirac equation written in terms of \( \psi_a \) and \( \psi_b \) looks like

\[ (E - eV) \psi_a - \vec{p} \cdot \vec{A} \psi_b = m \psi_a \]

\[ (E - eV) \psi_b - \vec{p} \cdot \vec{A} \psi_a = -m \psi_b \]

for \( \psi_s = \psi_a + \psi_b \quad \psi_d = \psi_a - \psi_b \)

these may be rewritten as

\[ (E - eV) \psi_s - \vec{p} \cdot \vec{A} \psi_b = m \psi_s \]

\[ (E - eV) \psi_s + \vec{p} \cdot \vec{A} \psi_b = m \psi_s \]
\[(E - eV + \frac{\mathbf{p}^2}{2m}) (E - eV - \frac{\mathbf{p}^2}{2m}) \psi_s = m^2 \psi_s\]
\[(E - eV - \frac{\mathbf{p}^2}{2m}) (E - eV + \frac{\mathbf{p}^2}{2m}) \psi_d = m^2 \psi_d\]

Thus in this representation it is easy to uncouple the equations (although the equation has become quadratic in the energy rather than linear, in the process).

Because the electron in $\beta$ decay is created as pure $\psi_s$, this is the natural representation to use to calculate the effects of electromagnetism on it as it shoots out of the nucleus.

**Change of Notation**

To make the equation more obviously similar in time and space components, multiply through by $\beta$

\[\left\{ \beta \left[ -\frac{\mathbf{p}}{\gamma t} - eV \right] - \beta \mathbf{\alpha} \cdot (\mathbf{p} - e/\gamma \mathbf{A}) \right\} \psi = m \psi\]

Define $\gamma_0 = \beta$ \[\gamma^2 = \beta \mathbf{\alpha} = \left( \begin{array}{cc} \gamma_t & \gamma_s \\ \gamma_s & \gamma_t \end{array} \right)\]

because $\gamma_0 = -\frac{\mathbf{p}}{\gamma t}$ we can write

\[\left[ \gamma_0 (\gamma_0 - eV) - \mathbf{\alpha} \cdot (\mathbf{p} - e/\gamma \mathbf{A}) \right] \psi = m \psi\]

For any four vector $\mathbf{B}$, define \[\mathbf{B} = \gamma_0 \beta_0 - \mathbf{B} \cdot \mathbf{\alpha}\]

then the Dirac equation takes the form

\[\left( \gamma_0 - \frac{e}{\gamma} \mathbf{A} - m \right) \psi = 0\]

i.e., because $\gamma_0 = e \mathbf{p}/\gamma_t$, \[\mathbf{p} = \gamma_t \frac{\mathbf{p}}{\gamma_t}\] define \[\nabla = \left( \begin{array}{cc} \gamma_t \frac{\partial}{\partial \xi} & \gamma_s \frac{\partial}{\partial \eta} \end{array} \right)\]

then \[\left( i \mathbf{\alpha} - \frac{e}{\gamma} \mathbf{A} \right) \psi = m \psi\]

**Solutions of the Dirac Equation**

a) Particle at rest, no field

\[(\gamma_0 \beta_0 - m) \psi = 0 \Rightarrow \left( \begin{array}{cc} 0 \\ -m \end{array} \right) \psi = 0\]

For $\gamma_0$, $E = m$

\[\psi = \left( \begin{array}{c} \phi \\ 0 \end{array} \right) e^{-imt}\]

For $\gamma_0$, $E = -m$

\[\psi = \left( \begin{array}{c} \phi \\ 0 \end{array} \right) e^{imt}\]

Here $\phi$ and $\chi$ can be any 2 component spinor \[\left( \begin{array}{c} \psi \\ \bar{\psi} \end{array} \right)\] for general constants $a$ and $b$.

b) No field, particle moving

It is much easier to solve the problem in momentum space. Then one can always Fourier transform back by multiplying by \[e^{-iE t + i\mathbf{p} \cdot \mathbf{x}}\]

\[\left( \gamma_0 - m \right) \psi = 0\]
\[(E - m)\phi - \vec{p} \cdot \vec{\sigma} \phi = 0\]
\[\vec{p} \cdot \vec{\sigma} \phi - (E + m) \phi = 0\]

Solution 1)
\[\phi = \left( \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \right) \phi\]

\[\Psi = \left( \begin{array}{c}
\phi \\
\frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi
\end{array} \right)\]

By substituting this back we see that
\[(E - m)\phi - \vec{p} \cdot \vec{\sigma} \left( \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \right) \phi = 0\]

Hence in position space
\[\Psi = \left( \begin{array}{c}
\phi \\
\frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi
\end{array} \right) e^{i(\vec{p} \cdot \vec{x} - E t)} + i \vec{p} \cdot \vec{x}\]

Choose the sign such that as \(\vec{p} \to 0\), this approaches
\[\left( \phi \right) e^{-i \sqrt{\vec{p}^2 + m^2} t + i \vec{p} \cdot \vec{x}}\]

Solution 2)
\[\phi = \frac{\vec{p} \cdot \vec{\sigma} \vec{X}}{E - m}\]

Here again \(\vec{X}^2 = \vec{x}^2 + \vec{p}^2\), but we choose the square root such that as \(\vec{p} \to 0\),
the solution approaches
\[\left( \phi \right) e^{i \vec{m} \cdot \vec{x}}\]

Thus we get
\[\left( \frac{-\vec{p} \cdot \vec{x}}{m + \sqrt{\vec{p}^2 + m^2}} \right) e^{i \sqrt{\vec{p}^2 + m^2} t + i \vec{p} \cdot \vec{x}}\]

or, if \(\vec{p}' = -\vec{p}\),
\[\left( \frac{\vec{p}' \cdot \vec{x}}{m + \sqrt{\vec{p}'^2 + m^2}} \right) e^{i \sqrt{\vec{p}'^2 + m^2} t - i \vec{p}' \cdot \vec{x}}\]

which is sometimes written
\[\left( \frac{\vec{p}' \cdot \vec{x}}{E + m} \right) e^{-i \vec{m} \cdot \vec{x}}\]

Thus if we associate this second solution with \(\vec{p}'\), it looks just like the first
solution except that the "large" and "small" components of the spinor have been inter-
changed, and there is an overall - sign in the exponent. Some people like to think of
this - sign as a reversal of \(\vec{x}\) and \(t\); this at least is a mnemonic.

For the moment, ignore the problem of normalizing these,
It is much more complicated to find solutions for \(A \neq 0\). For the present, lets forget about
this and just try to understand the nonrelativistic limit of the Dirac equation
in a field
\[(\mathbf{p} - e\mathbf{A} - m)\phi = 0 \quad \text{reduces to the set of coupled equations}\]
\[(E - m - eV)\phi - (\mathbf{p} - e\mathbf{A}) \cdot \nabla \phi = 0\]
\[
\mathbf{p} \cdot (\mathbf{p} - e\mathbf{A}) \phi - (E + m - eV)\phi = 0
\]
\[
\phi = \left(\frac{E + m - eV}{(E + m - eV)^2}\right) \frac{1}{2} \mathbf{p} \cdot (\mathbf{p} - e\mathbf{A}) \phi
\]

Substitute this into the first equation. It is important to preserve the order of operators, because the \(\mathbf{p}\) and \(E\) are differential operators which don't commute with \(\phi\).

\[(E - m - eV)\phi = \left(\mathbf{p} - e\mathbf{A}\right) \cdot \nabla \phi - \frac{1}{2} \mathbf{p} \cdot (\mathbf{p} - e\mathbf{A}) \phi\]

Use \(E \equiv M + W\)

\[
\frac{1}{2M + W - eV} \approx \frac{1}{2M} - \frac{(W - eV)}{(2M)^2}
\]

\[
(W - eV)\phi = \frac{1}{2M} \mathbf{p} \cdot \nabla \phi - \frac{(W - eV)}{(2M)^2} \mathbf{p} \cdot \nabla \phi
\]

Rewrite the equation by adding and subtracting the same thing on both sides

\[
\left[1 + \frac{(B_0^2)^2}{M^2}\right] (W - V) \left[1 + \frac{(B_0^2)^2}{M^2}\right] \phi = \frac{1}{2M} \left(\frac{\mathbf{p} \cdot \nabla}{M}\right)^2 \phi + \frac{1}{2M} \left[\frac{(B_0^2)^2 (W - V)}{M} - 2 \mathbf{p} \cdot \nabla (W - V) \mathbf{p} \cdot \nabla + (W - V) \left(\frac{\mathbf{p} \cdot \nabla}{M}\right)^2\right] \phi
\]

To really be able to consider the non-relativistic limit, we should write everything in terms of a non-relativistic wave function. \(\phi\) is not the non-relativistic wave function in the case where the fields are present. Define \(\bar{\phi} = \left[1 + \frac{(B_0^2)^2}{M^2}\right] \phi\)

Then to lowest non-trivial order \(\phi = \bar{\phi} - \frac{(B_0^2)^2}{M^2} \bar{\phi}\)

hence \(\bar{\phi} + \psi = \phi^* \phi + \bar{\phi} \bar{\phi} = \phi^* \phi\left[1 + \frac{(B_0^2)^2}{M^2}\right] \approx \bar{\phi}^* \bar{\phi}\)

\(\bar{\phi}\) contains the major effects of the "small" components as well as the large ones. Thus we can think of \(\bar{\phi}\) as a non-relativistic wave function and imagine the problem of finding the energy eigenvalues to be the same as that of minimizing

\[
\int \bar{\phi}^* \left(\text{mass}\right) \bar{\phi} \text{dvol} - \int \bar{\phi}^* \bar{\phi} \text{dvol}
\]

Upon substituting this expression for \(\phi\) we obtain an equation of the form

\[
\left[1 + \frac{(B_0^2)^2}{M^2}\right] (W - V) \bar{\phi} = \frac{6}{5} \phi \text{dch}
\]

which we unravel again to first order by

\[
(W - V) \bar{\phi} \approx \frac{6}{5} \phi \text{dch} - \frac{(B_0^2)^2}{M^2} \frac{6}{5} \phi \text{dch}
\]

Keeping lowest order terms, we arrive at

\[
(W - V) \bar{\phi} = \frac{1}{2M} \left(\frac{B_0^2)^2}{M^2}\right) \bar{\phi} - \frac{(B_0^2)^2}{M^2} \bar{\phi}
\]

\[
+ \frac{1}{2M} \left[\frac{(B_0^2)^2 (W - V)}{M} - 2 \mathbf{p} \cdot \nabla (W - V) \mathbf{p} \cdot \nabla + (W - V) \left(\frac{\mathbf{p} \cdot \nabla}{M}\right)^2\right] \bar{\phi}
\]
For any two operators $A$ and $B$,
\[ A^2 B - 2 A B A + B A^2 = \{ A, [A, B] \} \]
set $A = \vec{\sigma}, \vec{\pi}$
$B = \vec{W} - \vec{V} '$
Calculate $\left[ \vec{\sigma} \cdot \vec{\pi}, E-M-eV \right] = \left[ \vec{\sigma} \cdot (\vec{P} - M \vec{A}), E-m-eV \right]$
Assume for simplicity that $\frac{d}{dt} (\vec{A})$ (This is not necessary. For fun and practice try working the whole mess through without assuming this).
Thus we get
\[-e \left[ \vec{\sigma} \cdot \vec{\pi}, V \right] = -e \vec{\sigma} \cdot \left[ \vec{p}, V \right] = -i e \vec{\sigma} \cdot \vec{E} \]
\[\{ \} = \left[ \vec{\sigma} \cdot \vec{\pi}, -i e \vec{\sigma} \cdot \vec{E} \right] \]
\[-i e \left[ \vec{\sigma} \cdot \vec{p}, \vec{\sigma} \cdot \vec{E} \right] + i \frac{\hbar}{c} \left( \vec{\sigma} \cdot \vec{\pi} \vec{\sigma} \cdot \vec{E} - \vec{\sigma} \cdot \vec{E} \vec{\sigma} \cdot \vec{A} \right) \]
use
\[\vec{\sigma} \cdot \vec{\pi} \vec{\sigma} \cdot \vec{b} = \vec{\sigma} \cdot \vec{b} + i \vec{\sigma} \cdot (\vec{\sigma} \times \vec{b}) \]
\[\left[ \vec{\sigma} \cdot \vec{A}, \vec{\sigma} \cdot \vec{E} \right] = 2 i \vec{\sigma} \cdot (\vec{A} \times \vec{E}) \]
\[\left[ \vec{\sigma} \cdot \vec{p}, \vec{\sigma} \cdot \vec{E} \right] = -i \vec{\sigma} \cdot \vec{E} + i \vec{\sigma} \cdot (\vec{p} \times \vec{E}) - i \vec{\sigma} \cdot (\vec{E} \times \vec{p}) \]
\[\vec{\sigma} \cdot (\vec{p} \times \vec{E}) f = E \frac{\partial}{\partial x} \vec{\sigma} \cdot \vec{p} f \]
\[= -\vec{\sigma} \cdot (\vec{E} \times \vec{p}) f + \vec{\sigma} \cdot \left[ \frac{\partial}{\partial x} E - \frac{\partial}{\partial x} E \right] f \]
But
\[E = -e \frac{\partial \phi}{\partial x} j \implies \vec{p} \times \vec{E} = -\vec{E} \times \vec{p} \]
and
\[-i e \left[ \vec{\sigma} \cdot \vec{\pi}, \vec{\sigma} \cdot \vec{E} \right] = -e \left[ \vec{\nabla} \cdot \vec{E} + 2 \vec{\sigma} \cdot (\vec{E} \times \vec{A}) \right] \]

In similar fashion we calculate
\[\vec{\sigma} \cdot \vec{\pi} \vec{\sigma} \cdot \vec{\pi} = \vec{\pi} \cdot \vec{\pi} + i \vec{\sigma} \cdot (\vec{p} \times \vec{\pi}) \]
\[\vec{\pi} \times \vec{\pi} = e i c \vec{\nabla} \times \vec{A} = i \vec{\nabla} \vec{A} \]
Hence we obtain (to lowest order)
\[\mathcal{W} = \mathcal{W} + \frac{1}{2M} \left( \vec{\pi} \cdot \vec{\pi} \right) - \frac{\hbar}{2M} \vec{\sigma} \cdot \vec{B} \]
\[-\frac{e}{\hbar M} \left\{ \vec{\nabla} \cdot \vec{E} + 2 \vec{\sigma} \cdot (\vec{E} \times \vec{A}) \right\} \]
The individual terms can now be interpreted
\[\mathcal{W} \quad \text{ordinary potential energy} \]
\[\frac{\vec{p} \cdot \vec{\pi}}{2M} \quad \text{non-relativistic kinetic energy} \]
\[-\frac{\hbar}{2M} \vec{\sigma} \cdot \vec{B} \quad \text{Pauli spin effect (magnetic moment $-\frac{\hbar}{2M}$ due to spin)} \]
\[-\frac{(\vec{p} \times \vec{p})^2}{8M^2} \quad \text{first relativistic correction to the kinetic energy} \]
\[\left( \sqrt{p^2 + m^2} = m \sqrt{1 + \frac{p^2}{m^2}} = m \left[ 1 + \frac{1}{2} \frac{p^2}{m^2} - \frac{1}{8} \frac{p^4}{m^4} + \ldots \right] \right] \]
We are left only with the term \( \frac{-e}{2m} \left\{ \frac{\vec{V}.\vec{E}}{4m} + \frac{\vec{E} \times (\vec{E} \times \vec{\mu})}{2m} \right\} \)

\( \vec{\mu} \) is like \( \vec{p} \)

a particle in motion in an electric field sees a magnetic field

\[-\frac{\vec{V}^2}{c} = \frac{\vec{E} \times \vec{\mu}}{m} / \]

Then the energy of interaction of the electron's spin magnetic moment with this field will be \( \frac{-e}{2m} \frac{\vec{E} \cdot (\vec{E} \times \vec{\mu})}{m} \)

This looks like the term found above but differs from it by a factor of 2 (see below)

(Another way to get the same result is to recall that a moving magnetic moment produces an electric dipole moment of the form \( -\vec{u} \times \vec{u} \) which then interacts with the electric field via \( -\vec{u} \times \vec{u} \cdot \vec{E} \))

Within an atom, the electric field of the nucleus has the form \( \frac{\vec{e}}{r^3} \)

Then \( \vec{E} \cdot (\vec{V} \times \vec{E}) = \frac{2e^2}{4m^2} \left( \frac{\vec{r} \times \vec{p}}{r^3} \right) \cdot \vec{E} \)

\( \vec{r} \times \vec{p} = \vec{r} \) the orbital angular momentum

\( \vec{L} \cdot \vec{S} / r^3 \) produces \( \vec{L} \cdot \vec{S} \) \( \langle 1/r^3 \rangle \)

\( \vec{L} \cdot \vec{S} = 2\vec{L} \cdot \vec{S} = j(j+1) - L(L+1) - s(s+1) \) splits apart energy levels of different \( j \)

\( \vec{L} \cdot \vec{S} \) is, however, 0 for \( L=0 \) states

But for \( \vec{E} = \frac{ze^2}{r^3} \), \( \vec{V} \cdot \vec{E} = \frac{4\pi}{r^3} \frac{ze^2}{r^3} \)

This contributes only to \( s \) states (the only ones which are non-zero at the origin)

Thus the combination of terms in \( \{ \} \) can be thought of as a spin-orbit contribution, for every \( l \).

Not all spin \( \frac{1}{2} \) particles found in nature have magnetic moments equal to the Dirac moment \( \frac{g}{2m} \). In fact only the electron and the muon appear to obey the unmodified Dirac equation (in any approximation). To account for these "anomalous" magnetic moments, an additional term is added to the original equation (Pauli's idea).

Of course, you could add lots of different terms to the original equation and still have it remain Lorentz covariant, but it is conventional to choose a particular form which accounts for the observations (and introduces as few derivatives as possible. This second criterion will become reasonable in a month or two when we study various divergences)
The new equation is

\[
\left[ F - eA - \frac{\hbar}{2} \sigma \cdot \mathbf{\gamma} \right] \mathcal{V} = 0
\]

where

\[
\sigma_{\mu \nu} = \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right]
\]

\[
\Pi_{\mu \nu} = \frac{\partial}{\partial x_\mu} A_\nu - \frac{\partial}{\partial x_\nu} A_\mu
\]

Figure out how to work this through to a non-relativistic equation and show that

\[
\mathbf{\omega} = \mathbf{\nu} + \frac{1}{2m} \left( \mathbf{E} \times \mathbf{B} \right) + \left( \mu - \frac{e}{2m} \right) \mathbf{\sigma} \cdot \mathbf{\beta} \frac{\mathbf{P}^2}{\mathbf{E}^2 M^2}
\]

This form shows that the factor of 2 which we previously had trouble with in the term is not really an overall multiplier. Rather there are two effects

\[
\frac{1}{2} \left( \mu - \frac{e}{2m} \right) \left\{ \mathbf{\gamma} \cdot \mathbf{E} + 2 \mathbf{\sigma} \cdot (\mathbf{E} \times \mathbf{B}) \right\}
\]

This part was explained above.

The second term is called the Thomas term. Its presence may be understood as follows:

We calculated the effect on the electron due to precession of the spin magnetic moment in the magnetic field seen instantaneously. This would be ok if the electron were an inertial frame. But it is constantly being accelerated by a force proportional to the charge. Thus we must correct for this (since the acceleration is not proportional to magnetic moment, we don't expect the correction term to involve \( \mu \)).

The correction is actually calculated in the Ph 209 book. You might be able to find a dimpler and clearer explanation of the numbers on your own.

When the magnetic moments of the electron and muon are actually measured, they are found to be slightly different from \( \frac{e}{2m} \). The discrepancy can, however, be completely accounted for by considering quantum effects of the electromagnetic field.

The anomalous magnetic moments of other spin \( \frac{1}{2} \) particles are presumably due to their interactions with mesons. One way of looking at the situation is to say that the physical proton (neutron, lambda, etc.) is made up of an ideal Dirac proton plus a lot of mesons, which are continually being emitted and absorbed. These mesonic interactions alter the Dirac moment to its anomalous value. At present there is no good (i.e. precise) way to theoretically calculate anomalous moments.
Anomalous magnetic moment of the electron: An electron is described by a wave function $\psi$ that satisfies the Dirac Equation:

$$\gamma_{\mu} \left( i \gamma_{\nu} \partial_{\nu} - e A_{\mu} \right) \psi = m \psi \quad \text{(1)}$$

where $A_{\mu} = A_{\mu}^{\text{total}}$ is made up of an external field $A_{\mu}^{\text{ext}}$ together with $A_{\mu}^{\text{em}}$, the field due to the electron itself.

$$A_{\mu} = A_{\mu}^{\text{ext}} + A_{\mu}^{\text{em}}$$

Classically, an electron's trajectory satisfies

$$m \frac{d^2 x_{\mu}}{dt^2} = \frac{1}{\gamma_{\mu}^2} F_{\mu\nu}$$

where

$$F_{\mu\nu} = F_{\mu\nu}^{\text{ext}} + F_{\mu\nu}^{\text{em}}$$

Then

$$m \frac{d^2 x_{\mu}}{dt^2} - \frac{1}{\gamma_{\mu}^2} F_{\mu\nu}^{\text{em}} = \frac{1}{\gamma_{\mu}^2} F_{\mu\nu}^{\text{ext}}$$

and the term $\frac{1}{\gamma_{\mu}^2} F_{\mu\nu}^{\text{em}}$ is called the force of radiation resistance.

Equation (1) with $A_{\mu} = A_{\mu}^{\text{ext}}$ is analogous to the classical equation in which the force of radiation resistance is neglected.

If we put in the full $A_{\mu}$ instead of $A_{\mu}^{\text{ext}}$ into eq. (1) we obtain the anomalous magnetic moment of the electron as a term in the non-relativistic reduction of eq. (12) hamiltonian derived from eq. (1). The term would look like

$$\frac{e}{2m} g_\text{\mu} \tilde{F}_{\mu\nu}^{\text{ext}}$$

where the anomalous moment $g_\mu$ is determined from the radiation resistance. The calculation of this quantity using standard techniques yields divergences which may be removed by the method of "mass and charge renormalization".
One compares the calculated value of the anomalous moment with experiment by observing the hyperfine structure of the energy levels of atomic hydrogen. One also wants to correct for effects due to a space charge distribution within the proton when making the comparison.

There is no experiment today that absolutely requires a modification of the scheme outlined above.

Problems

1. In the non-relativistic reduction of the Dirac equation with the Pauli term there appeared the term

$$\frac{\mathbf{e}}{\mathbf{8m}} \nabla \boldsymbol{E}$$

which came mathematically from $[\mathbf{6p}, \mathbf{6E}]$.

   Explain the physical origin of the term.

2. For entertainment, you might try to examine the Dirac equation in Hamiltonian form with

$$\mathcal{H}_D = \mathbf{\varepsilon} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m$$

and using the operator equation of motion

$$\Theta^* = i \mathbf{[H, \Theta]}$$

evaluate $\Theta^*$ and $\mathbf{p}$.

   Find the velocity operator for the Dirac electron.

3. How many different matrices can you make from products of $\mathbf{S}$'s and $\beta$ (The purpose of this exercise is to gain familiarity with the commutation laws).
Solutions to the Dirac Equation for a Free Particle

Let \( \Psi = e^{-i(Et - px)} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \); \( \psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \)

Then with our choice of \( \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)
the Dirac equation becomes

\[
(E + m) \psi_b = \beta \cdot p \psi_a \\
(E - m) \psi_a = \beta \cdot p \psi_b
\]

\[
(E^2 - m^2) \psi_a = (E^2 - m^2) \beta \cdot p \psi_b = \beta \cdot p (E + m) \psi_b = p \psi_a
\]

so that \( E^2 = m^2 + p^2 \) or \( \psi = 0 \)

\( \Rightarrow E = \pm E_p \) where \( E_p = \pm \sqrt{E^2 + m^2} \)

A. For \( E = E_p \)

\[
\psi_b = \frac{\beta \cdot p \psi_a}{m + E_p}
\]

Let us choose our axes so that \( p \) is in \( \hat{z} \) direction.

Spin up (positive helicity)

then \( \beta_3 \psi_a = + \psi_a = \delta(1) \)

\( \delta = \) some number chosen for convenient normalization

then \( \psi_b = \frac{\beta \cdot p \psi_a}{m + E_p} = \frac{\delta \psi_a}{E + m} (0) \)

Choosing the normalization \( \psi^* \psi = \psi^* \psi_a + \psi^* \psi_b = 1 \)

we have \( \gamma = \sqrt{\frac{2E}{E + m}} \) or \( \delta = \sqrt{\frac{E + m}{2E}} \)

or \( \gamma = \sqrt{\frac{E + m}{2E}} \begin{pmatrix} 0 \\ \frac{E + m}{2E} \end{pmatrix} = \frac{1}{\sqrt{2E}} \begin{pmatrix} \frac{E + m}{E_p + m} \\ \frac{E_p - m}{E_p} \end{pmatrix} \)
Spin down
(negative helicity)

Then \( \gamma_y \gamma_b = -\gamma_b \)

and

\[
\psi = \frac{1}{\sqrt{2E_e}} \begin{pmatrix} 0 \\ VE_e + m \\ 0 \\ -VE_e + m \end{pmatrix}
\]

Now, if we want an eigenstate of \( \gamma_x \) instead of \( \gamma_y \), we try

\( \gamma_x \gamma_a = \gamma_a \)

\( \gamma_a = \frac{1}{\sqrt{2}} (1) \)

but

\( \gamma_x \gamma_b = \frac{p}{\sqrt{2E_e (E_e + m)}} (-1) \)

This is because the operator \( \begin{pmatrix} \gamma_x & 0 \\ 0 & \gamma_x \end{pmatrix} \) does not commute with the Hamiltonian. Only the operator \( \begin{pmatrix} \gamma_y & 0 \\ 0 & \gamma_y \end{pmatrix} \) commutes with \( H \) when the momentum \( \mathbf{p} \) is in the \( z \) direction.

The form of the 4x4 matrices that represent the spin operators are

\[
\begin{pmatrix} \gamma_x & 0 \\ 0 & \gamma_x \end{pmatrix}
\]

more generally,

\[
\begin{align*}
\gamma_x &= \begin{pmatrix} \gamma_x & 0 \\ 0 & \gamma_x \end{pmatrix} \\
\gamma_y &= \begin{pmatrix} \gamma_y & 0 \\ 0 & \gamma_y \end{pmatrix} \\
\gamma_z &= \begin{pmatrix} \gamma_z & 0 \\ 0 & \gamma_z \end{pmatrix}
\end{align*}
\]

Momentum in arbitrary direction

Suppose the spherical angles describing the direction of \( \mathbf{p} \) relative to fixed coordinate axes are \( \theta, \phi \). Then \( \gamma_x \gamma_z = \mathbf{p} \gamma_z \) where

\[
\gamma_z = \left( \begin{array}{c} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{array} \right)
\]
Then $\psi = \chi$ and $\chi = \begin{pmatrix} \psi_m \cos \theta e^{-i\phi} \\ \psi_m \sin \theta e^{i\phi} \end{pmatrix}$

(positive helicity spinor)

and negative helicity spinor is $\chi = \begin{pmatrix} -(\psi_m \sin \theta e^{-i\phi}) \\ (\psi_m \cos \theta e^{i\phi}) \end{pmatrix}$

$\begin{pmatrix} -\psi -m \sin \theta e^{-i\phi} \\ \psi -m \sin \theta e^{i\phi} \\ +\psi -m \cos \theta e^{-i\phi} \\ -\psi -m \cos \theta e^{i\phi} \end{pmatrix}$

B. For $\mathbf{E} = -\mathbf{E}_P$

Let $\mathbf{P}$ be in $\hat{z}$ direction. For helicity ($h$) = + 1, $\chi = \begin{pmatrix} -\psi -m \\ 0 \\ \psi +m \end{pmatrix}$

$\begin{pmatrix} 0 \\ \psi +m \\ 0 \end{pmatrix}$

$h = -1, \chi = \begin{pmatrix} 0 \\ \psi +m \\ 0 \end{pmatrix}$

Dirac proposed that the negative energy states exist but are all filled with electrons. This invites the interesting possibility that a photon will excite an electron from a negative energy state to a positive energy state. The appearance of an electron with positive energy and the absence of an electron with negative energy from the negative energy sea of $\text{f}$ electrons is physically the same as the creation of an electron and a positron, both with positive energy.

Example of the use of a Dirac Equation: Scattering in a Coulomb Potential (first Born approximation)

The scattering cross section $\sigma$ is given by

$$6 \pi \sigma = \frac{2 \pi \delta(E - E_i) d \Omega}{E_{r}^{2} - m_{f}^{2}}$$
\[ M_{ci} = \int \psi_i^* V(x) \psi_i(x) \, d^3x \]

\[ \psi_i(x) = e^{-i(B \cdot x - E_i t)} u \]

\[ M = \int u^+_i e^{-i(B - p) \cdot x} \frac{Ze^2}{\sqrt{Q}} u_1 \, d^3x \]

\[ = \begin{pmatrix} u^+_i u_1 \end{pmatrix} \frac{4\pi Ze^2}{Q} \delta = p_i - p_1 \]

\[ \delta(E_f - E_i) \quad \text{ensures that} \quad |p_i| = |p_1| = p \]

Then,

\[ 6 V_i = \frac{2\pi}{(2\pi)^2} \int \frac{dp'}{dp} \delta \left( \sqrt{p^2 + m^2} - \sqrt{p'^2 + m^2} \right) p'^2 \, dp' \left( \frac{4\pi Ze^2}{Q} \right)^2 \left| u_i^* u_1 \right|^2 \]

\[ = (2\pi)^{-2} \left( 16\pi^2 \right) \frac{2e^2}{Q^4} \frac{E_f}{E_i} \int d\Omega \left| u_i^* u_1 \right|^2 \]

\[ 6(\theta) = \frac{4Ze^2 (m^2 + p^2) |u_i^* u_1|^2}{16p^4 \sin^4 \theta} \]

We must still evaluate \( u_i^* u_1 \).

There are 4 cases:

- \( \hbar = + \rightarrow k + \text{ or } k = - \)
- \( \hbar = - \rightarrow k + \text{ or } k = - \)

\[ u_1: \quad \hbar = + \quad u_1 = \begin{pmatrix} \frac{V_{Em}}{V_{E-M}} & 1 \end{pmatrix} \frac{1}{V_{B-E}} \quad \hbar = - \quad u_1 = \begin{pmatrix} 0 \\ \frac{V_{E-M}}{V_{Em}} \end{pmatrix} \]

\[ u_2: \quad \hbar = + \quad u_2 = \begin{pmatrix} \frac{V_{Em}}{V_{E-M}} & 0 \\ \frac{V_{E-M}}{V_{Em}} \end{pmatrix} \sin \theta \quad \hbar = - \quad u_2 = \begin{pmatrix} \frac{V_{Em}}{V_{E-M}} & 0 \\ \frac{V_{E-M}}{V_{Em}} \end{pmatrix} \cos \theta \]
\[ h^+ \rightarrow h^+ , \quad u^+ u_1 = \cos \frac{\Theta}{2} \]
\[ h^+ \rightarrow h^- , \quad u^+ u_1 = -\frac{m}{E} \sin \frac{\Theta}{2} \]
\[ h^- \rightarrow h^- , \quad u^- u_1 = \cos \frac{\Theta}{2} \]
\[ h^- \rightarrow h^+ , \quad u^- u_1 = \frac{m}{E} \sin \frac{\Theta}{2} \]

If the helicity of the incoming electron is +, then the scattering cross section is:

\[
\frac{Z^2 e^4 (m^2 p^2)}{4 p^4 \sin^2 \frac{\Theta}{2}} \left( E^2 \cos^2 \frac{\Theta}{2} + m^2 \sin^2 \frac{\Theta}{2} \right)
\]

\[
= \frac{Z^2 e^4}{4 p^4 \sin^4 \frac{\Theta}{2}} \left( E^2 \cos^2 \frac{\Theta}{2} + m^2 \sin^2 \frac{\Theta}{2} \right) \left( 1 + \frac{p^2 \cos^2 \frac{\Theta}{2}}{E^2} \right)
\]

The cross-section is the same with incoming electron of \( h^- \), and also the same as the cross-section for scattering unpolarized electrons.

**Problem**: Calculate the cross-section for the Compton effect using the solutions of the Dirac equation for a free particle and perturbation theory.

**Compton effect**: incoming photon \( \vec{k}_1 \) \( \rightarrow \) outgoing photon \( \vec{k}_2 \)

incoming electron \( \vec{p}_1 \) \( \rightarrow \) outgoing electron \( \vec{p}_2 \)

(a special coordinate system)
CALCULATION OF THE COMPTON EFFECT

\[ \gamma + e^- \rightarrow \gamma + e^- \]

(notice that the process \( \gamma + e^- \rightarrow e^- \) can't be a real process, because it is impossible to conserve energy and momentum. Hence Compton scattering is the simplest process involving electrons and photons)

Choose to work this in the lab system

\[ \text{wiggly lines are photons; straight ones are electrons} \]

Write down all the 4 vectors \( (A_0, A_x, A_y, A_z) \) associated with the problem:

\[
\begin{align*}
P_1 &= (m, 0, 0, 0) \\
h_1 &= v_i (1, 0, 0, 1) \\
P_2 &= (E_2 - m c^2 \cos \phi, 0, 0, m c \sin \phi) \\
h_2 &= \omega_2 (1, \sin \Theta, 0, \cos \Theta)
\end{align*}
\]

Final state quantities are constrained by

\[
\begin{align*}
P_1^+ + h_1^+ &= P_2^+ + h_2^+ \\
\text{energy-momentum conservation}
\end{align*}
\]

\[
\begin{align*}
P_2^+ &= m^2 = (P_1^+ + h_1^+ - h_2^+)^2 = m^2 + 2 P_1 \cdot h - 2 P_1 \cdot h_2 - 2 h_1 \cdot h_2
\end{align*}
\]

This gives the well known formula for change of \( \gamma \) ray frequency

\[
\gamma = \frac{m c}{\omega_1} \left( \frac{m}{\omega_2} \right) = \frac{m c}{\omega_2}
\]

Must also consider the possible polarizations of the \( \gamma \) ray

\[
\begin{align*}
\perp & \quad \text{plane of scattering} \ (0, 0, 1, 0) \\
\parallel & \quad \text{plane of scattering}
\end{align*}
\]

Possible for both \( E_1 \) and \( E_2 \)

\[
\begin{align*}
e_1 \cdot h &= 0 \\
e_1 &= (0, 1, 0, 0) \\
e_2 &= (0, c \cos \theta, 0, -c \sin \theta)
\end{align*}
\]

The cross section formula we are about to derive was first obtained by Klein and Nishina. When they did it, it was considered a difficult and complicated problem; with the computational tricks to be taught in this course plus a little practice the average Ph 205 student will be able to complete problems of this
difficulty in less than \( \frac{1}{2} \) hour.

\[
\text{Probability of transition/sec} = 2\pi \int \left( E_0 \delta - E_i \right) \frac{d^3k_2}{(2\pi)^3} |M|^2
\]

(Notice: this formula always holds, relativistic kinematics or no relativistic kinematics. It is the change in the way you write \( E \) inside the \( \delta \) function which changes the ultimate form for the phase space)

\[
\hat{H} = \beta \hat{m} + \frac{1}{2} \left( \vec{p} - \frac{\omega_c}{c} \hat{A} \right) = H_0 + \hat{H}_{\text{int}} \quad \hat{H}_{\text{int}} = -\frac{e}{c} \frac{\omega_c}{c} \hat{A}
\]

(notice that Dirac theory contains no \( \hat{A}, \hat{A}^\dagger \) terms)

\[
\hat{A} = \frac{1}{\sqrt{2\omega}} \sum_{k} \left[ a_k^i \hat{e}^i_k e^{i\vec{k} \cdot \vec{x}} + \hat{a}_k^i e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k} \cdot \vec{x}} \right]
\]

sum is over 2 different possible polarizations for each \( k \)

Since Compton scattering requires the annihilation of one photon and the creation of another, the first contribution in perturbation theory comes from second order.

\[
\text{Prob/sec} = 2\pi \int \left( \sqrt{m^2 + (\vec{k}_1 \cdot \vec{k}_2)^2 + \omega_2} - m - \omega_1 \right) \frac{d^3k_2}{(2\pi)^3} d\Omega_2 |M|^2
\]

If we cancel the \( \delta \) function against \( dk_2 \) we get an angular distribution for the emitted photons; if we cancel it against \( d\Omega_2 \) we get a frequency spectrum (there are of course related by *)

Generally one measures angular distributions.

use \( \delta(f(x)) = \frac{\delta(x-a)}{|f'(a)|} \)

Here

\[
f(k_2) = \sqrt{m^2 + \omega_1^2 + \omega_2^2 - 2\omega_1\omega_2} \quad \text{and} \quad \omega_2 - m - \omega_1
\]

\[
f'(k_2) = \frac{\omega_2 - \omega_1 + \omega}{E_2} + 1
\]

This must be evaluated at \( \omega_2 = m + \omega_1 - E_2 \)

\[
f'(a) = \frac{\omega_2 - m}{\omega_2 E_2}
\]

Thus

\[
\text{Prob/sec} = \sigma C = \frac{\omega_2^3 E_2}{m \omega_1 (2\pi)^2} \left| \langle M \rangle \right|^2 = \text{cross section} \times \text{relative velocity}
\]

where \( E_2 = m + \omega_1 - \omega_2 \) and \( 1 - \cos \theta = \frac{m}{\omega_2} - \frac{\omega_1}{\omega_2} \)

allow one to get the whole mass solely in terms of the angles and incident energy

Now for the matrix element. From first term we recall that there are two contributions to the second order matrix element

A: first annihilate \( \delta_1 \), then create \( \delta_2 \)

\[
\left. \begin{array}{c}
\sum_{i} \left< 2 \mid \hat{E}_2 \hat{e}^i \hat{e}^{-i\vec{k}_2 \cdot \vec{r}} \mid 1 \right> \times \hat{e} \hat{e}^{-i\vec{k}_1 \cdot \vec{r}} \mid 1 \right>
\end{array} \right| E_i + \omega_1 - E_i
\]
\[ B: \text{first create } \psi_1, \text{then annihilate } \psi_1, \frac{\sum_j \langle \psi_1 | \phi_j \rangle < \phi_j | \gamma_1 \rangle}{E_1 + \omega_i - (E_j + \omega_j + \omega_2)} \]

\[ |E_i| = \sqrt{P_i^2 + M^2} \]

from the way we got the formula

All that remains is to stick in the wave functions and get the numbers. Each electron
wave function has the form \( u e^{i \mathbf{k} \cdot \mathbf{x}} \), where \( u \) is a \( 4 \)-component spinor. Everyone knows
what to do with the \( e^{i \mathbf{k} \cdot \mathbf{r}} \) pieces (integral over \( x \) just gives momentum conservation) so
we forget about this for now and concentrate on the matrix part.

Each of the sums must be split into a sum over positive energy intermediate states and
over negative energy states

\[ A = \sum_{2 \text{spins}} < \psi_2 | d_2 | u_i > < \psi_1 | d_1 | u_i > \] \[ \frac{1}{E_1 + \omega_i - 1E_1 - \epsilon} \] \[ + \sum_{2 \text{spins}} < \psi_2 | d_2 | u_i > < \psi_1 | d_1 | u_i > \] \[ \frac{1}{E_1 + \omega_i + 1E_i - \epsilon} \]

\[ B = \sum_{2 \text{spins}} < \psi_2 | l_1 | u_i > < \psi_1 | d_2 | u_i > \] \[ \frac{1}{E_1 - \omega_i - 1E_1 - \epsilon} \] \[ + \sum_{2 \text{spins}} < \psi_2 | l_1 | u_i > < \psi_1 | d_2 | u_i > \] \[ \frac{1}{E_1 - \omega_i + 1E_i - \epsilon} \]

Keep in mind that the states \( j \) have different momentum from those labelled \( i \).

Physically, however, we do not allow the possibility of negative energy states. We do
have the new possibility of pair creation. This means we should eliminate all sums
over negative energy states and add in terms for diagrams like

\[ \text{add to } B \]

The idea expressed in terms of hole theory is as follows:

You can't knock an electron into a negative energy state, because they are full.

However, a photon can excite an electron out of one of these states, leaving a hole
in the sea. Then the next photon can make one of the electrons drop back into the
negative energy state.

New term to be added to \( B \) has energy denominator

\[ E_1 + \omega_i - (E_1 + 1E_k + E_2) - \epsilon \]

But overall contribution must be multiplied by a \(-\) sign relative to the first term in \( B \)
because of the following argument
We should really put in our matrix element the grand wave function of all electrons in the world, even though the Hamiltonian acts only on one particular electron. The incident g.v.f. is the same for both pictures. But \( B_2 \) differs from \( B_1 \) in the final state, because in \( B_1 \) the final state is the same electron as initially whereas in \( B_2 \) the initial electron has been exchanged for one in the sea. The g.v.f. is totally antisymmetric under interchange of any two electrons.

Hence the only overall change is in the sign of the \( i \epsilon \) contribution to the denominator

\[
\frac{\sqrt{2\omega_1 \omega_2}}{e^2} A = \sum \frac{\langle u_2 | d_2 | u_1 \rangle}{E_1 + \omega_1 - i \epsilon} \frac{\langle u_1 | d_1 | u_i \rangle}{E_i - i \epsilon} + \sum \frac{\langle u_2 | d_2 | u_i \rangle}{E_1 + \omega_1 - i \epsilon} \frac{\langle u_1 | d_1 | u_i \rangle}{E_i - i \epsilon}
\]

\[
\frac{\sqrt{2\omega_1 \omega_2}}{e^2} B = \sum \frac{\langle u_2 | d_2 | u_j \rangle}{E_1 - \omega_2 - i \epsilon} \frac{\langle u_j | d_j | u_i \rangle}{E_i - i \epsilon} + \sum \frac{\langle u_2 | d_2 | u_j \rangle}{E_1 - \omega_2 - i \epsilon} \frac{\langle u_j | d_j | u_i \rangle}{E_i - i \epsilon}
\]

When the arrow on the solid line is running backward in time you have a positron (hole).

It is possible (although difficult!) to do an experiment in which all the spins of incident and final particles are polarized. However, one must always sum over the spins of the internal states. When one contemplates just how many matrix elements have to be summed if the external particles are unpolarized, the problem that faced Klein and Mischina becomes quite clear.

Casimir invented an improvement to do the sum over intermediate states

\[
\sum_i (u_{2i}^* d_2 u_i^+) (u_i^+ d_1 u_1) = \sum_i (\sigma^+)_a (u_i^+)_a (u_i^+)_b \beta (\omega)_b
\]

\[
= \lambda^+ \lambda^+ \omega
\]

where

\[
\lambda^+_{\alpha \beta} = \sum_{\text{energies}} (u_i^+)_a (u_i^+)_b
\]

Explicit computation with the \( u_i \)'s shows that

\[
\lambda^+ = \frac{1}{2 \epsilon} \left[ E + m \beta + \frac{\sigma}{\beta} \right] \quad \lambda^- = \frac{1}{2 (1 + \beta)} \left[ 1 - E - m \beta - \frac{\sigma}{\beta} \right]
\]

These can be obtained more simply:

\[
\sum_{\text{all states}} u_i^+ u_i^+ = \frac{1}{2}
\]

\[
\sum_{\text{energies}} u_{2i}^* \frac{N (H + E_i)}{2 E_i} u_i^+ M u_1 = \sum_{\text{all energies}} u_{2i}^* \frac{N (H + E_i)}{2 E_i} u_i^+ M u_1
\]

\[
= \frac{u_{2i}^* N (H + E_i)}{2 E_i} u_i^+ M u_1
\]
Likewise \[ \tilde{\lambda} = \frac{H - iE_i}{2E_i} \leq \frac{1}{2} (E_i + H) \]

We thus see that any sum over positive energy intermediate states is got by inserting \( \lambda^+ \); similarly for \( \lambda^- \) and negative energy intermediate states.

Hence we have

\[
A = \frac{u_2 \Lambda_2 \left( |E_i| + m \beta + \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{2 |E_i| (E_i + w_1 - |E_i|)} + \frac{u_2 \Lambda_2 \left( |E_i| - m \beta - \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{2 |E_i| (E_i + w_1 + |E_i|)}
\]

\[
= \frac{u_2 \Lambda_2 \left( |E_i| + w_1 + m \beta + \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{(E_i + w_1)^2 - (\vec{p}^2 + \vec{k}^2) - m^2} = \frac{u_2 \Lambda_2 \left( |E_i| + w_1 - m \beta - \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{(\vec{p}^2 + \vec{k}^2) - m^2}
\]

Define \( u^* \beta = \tilde{u} \)

Thus if \( u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \)

then \( \tilde{u} = (u_1^*, u_2^*, -u_3^*, -u_4^*) \)

The final form for our A matrix element is then

\[
\frac{u_2 \Lambda_2 \left( |E_i| + w_1 - m \beta - \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{(\vec{p}^2 + \vec{k}^2) - m^2}
\]

Likewise B is

\[
\frac{u_2 \Lambda_2 \left( |E_i| - w_1 + m \beta - \alpha \cdot \vec{p} \right) \vec{p}^* u_1}{(\vec{p}^2 + \vec{k}^2) - m^2}
\]

Feynman's contribution was to arrive at this stage and then figure out rules by which you could write these answers down without going through all the intermediate steps.

Aside B1: Effect of two time inversions on a spin J system

\[
\begin{align*}
T^2 | J = m \frac{1}{2} > & = -1 \frac{1}{2} > (m = + \text{ spin rev}) \\
T^2 | J = m > & = +1 \frac{1}{2} > \\
\end{align*}
\]

How to find answer:

\[
\begin{align*}
\text{from elem. } \frac{1}{J} & \text{ phase. } \frac{1}{J} - m \\
\text{ from mech. } & \\
\text{ a) } T|J, m > & = \text{ phase. } |J, -m > \\
\text{ b) } T(\alpha |1 > + \beta |0 >) = \alpha^* T|1 > + \beta^* T|2 > \\
\text{ c) } \text{Integral spin, mass } \neq 0 \Rightarrow \text{ study } \exists \text{ } S_z = 0 \\
T|0 > & = e^{i\delta} |0 > \\
T^2 |0 > = e^{-i\delta} T|0 > & = |0 > \\
\text{ d) } \text{Half integral spin} \\
T|+ > = e^{i\delta} |+ > \\
T|- > = e^{i\delta} |+ > \\
\text{ must be spin down in } x \times y \text{ direction} \\
\frac{1}{\sqrt{2}} T \left[ |+ > + |- > \right] & = \sqrt{2} \left[ e^{i\delta} |+ > + e^{i\delta} |- > \right] \\
& = \frac{e^{i\delta}}{\sqrt{2}} \left[ |+ > + e^{i(\delta - \delta')} |- > \right] \\
& \Rightarrow e^{i(\delta - \delta')} = -1
\end{align*}
\]
Aside #2: Because the electron has only two possible helicities, it should be possible to describe it by just a 2 component wave function, rather than a 4 component one. One way to do this is to start with the two component wave function for an electron at rest and then transform it to the desired momentum.

For a state at rest \( |0, i\rangle \) where \( i \) describes the spin state

\[
e^{i \sqrt{\hbar} N_x} |0, i\rangle = | m \text{spin}, 0, 0, 0, i\rangle = |3 \text{momentum}, i\rangle
\]

The general "boost" operator has the form

\[
e^{i \sqrt{\hbar} \mathbf{N} \cdot \mathbf{V}}
\]

Other operators work as expected

\[
\mathcal{P}_z |p_x, p_y, p_z, i\rangle = \mathcal{P}_z |p_x, p_y, p_z, i\rangle
\]

\[
t \sinh \theta = \frac{p}{m} \quad \sinh \theta = \frac{p}{m}
\]

defines the parameter \( v \) for the transformation

\[
p_t / o_i\rangle = m / o_i\rangle
\]

The operator \( p_t^2 - p_x^2 - p_y^2 - p_z^2 \) is an invariant for a given representation. Hence every state can be described in terms of a momentum and a spin state at rest.

The generators of the transformation can be written

\[
\mathcal{J}_z = i \left[ p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} \right] + \mathcal{J}_z \rightarrow (\text{intrinsic spin})
\]

\[
N_{\mathcal{E}} = i \left[ p_t^2 \frac{\partial}{\partial p_z} - p_z \frac{\partial}{\partial p_t} \right] + \left[ \frac{p_x^2 + p_y^2}{E + m} \right] 2
\]

The most general wave function can then be written in the form

\[
|\Psi\rangle = \sum_P \left( a_p |1_p^+\rangle + a_p^- |1_p^-\rangle \right)
\]

and the most general scattering operator can be put as

\[
f(e_1, e_2) + \frac{\phi}{\sqrt{2}} \left( \mathcal{P}_1, \mathcal{P}_2 \right)
\]

For fun, figure out from conventional theory and/or this theory why the two component scattering amplitude operator in a scalar potential would take the form

\[
\frac{\sqrt{2m(E_1+m)}}{2m(E_2+m)} \mathcal{P}_1 \mathcal{P}_2 + i \frac{\phi}{\sqrt{2}} (\mathcal{P}_1 \times \mathcal{P}_2)
\]

and in a pseudoscalar potential would look like

\[
\frac{\phi}{\sqrt{2m(E_1+m)}} \left( E_1 + m \right) - \frac{\phi}{\sqrt{2m(E_2+m)}} \mathcal{P}_2 (E_1 + m)
\]

Question: If you scatter twice in a scalar potential according to the graph

\[
\text{can you prove from the amplitude resulting that electron and positron must have opposite intrinsic spin?}
\]
Differential cross section for Compton scattering in lab ($\beta^2 = 0$):

$$
\frac{d\sigma}{d\Omega} = \frac{1}{16mE_2 \omega_1 \omega_2} \frac{1}{(2\pi)^2} \frac{1}{m^2} \frac{mE_2}{m+2m\sin^2 \theta/2} \, d\omega_2 \, d\Omega = \text{solid angle of outgoing photon}.
$$

$$
= \frac{d\omega_2}{6}\frac{\omega_2}{m^2} \frac{1}{m^2} \frac{E_2}{m^2} \frac{m^2}{m+2m\sin^2 \theta/2}
$$

From kinematics,

$$
\frac{\omega_2}{\omega_1} = \frac{m}{m+2m\sin^2 \theta/2}
$$

$$
\frac{d\sigma}{d\Omega} = \frac{1}{6}\frac{1}{m^2} \frac{1}{m^2} \frac{m^2}{m^2} \frac{\omega_2^2}{\omega_1} = \frac{\omega_2^2}{\omega_1} \quad \text{(1)}
$$

This cross section refers to a definite process in which the initial and final states are completely specified, defined “pure” states.

Now there are 2 possible helicities for the electron in the initial state, as well as 2 possible helicities for the electron in the final state. In addition, there are 2 possible polarization states for the photon in both the initial and final states. There are therefore 16 polarized differential amplitudes \( M \) that occur in Compton scattering.

**Kinematics of scattering in the lab system**

Initial state spinors

\[ u_1 = u_{1+} = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_1 = u_{1-} = \sqrt{2m} \begin{pmatrix} 1 \\ i \end{pmatrix} \]

\[ u_2 = u_{2+} = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad u_2 = u_{2-} = \sqrt{2m} \begin{pmatrix} 0 \\ i \end{pmatrix} \]

\[ u_3 = u_{3+} = \sqrt{2m} \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{or} \quad u_3 = u_{3-} = \sqrt{2m} \begin{pmatrix} -i \\ i \end{pmatrix} \]

\[ u_4 = u_{4+} = \sqrt{2m} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{or} \quad u_4 = u_{4-} = \sqrt{2m} \begin{pmatrix} i \\ i \end{pmatrix} \]

\( \text{Recall: normalization is 2E particles/unit volume} \)
Final state spinors:

\[ \mathbf{U}_2 = \mathbf{U}_{2-} = \begin{pmatrix} \sqrt{E+m} \cos \alpha e_x \\ -\sqrt{E+m} \sin \alpha e_y \\ \sqrt{E-m} \cos \alpha e_z \\ -\sqrt{E-m} \sin \alpha e_w \end{pmatrix}, \quad \text{or} \quad \mathbf{U}_2 = \mathbf{U}_{2+} = \begin{pmatrix} \sqrt{E+m} \sin \alpha e_x \\ \sqrt{E+m} \cos \alpha e_y \\ -\sqrt{E-m} \sin \alpha e_z \\ -\sqrt{E-m} \cos \alpha e_w \end{pmatrix} \]

\[ \mathbf{U}_{2-} = (\sqrt{E+m} \, e_x, \sqrt{E+m} \, \Gamma, \sqrt{E-m} \, e_x, \sqrt{E-m} \, \Gamma) \]

\[ \mathbf{U}_{2+} = (\sqrt{E+m} \, \Gamma, -\sqrt{E+m} \, e_x, -\sqrt{E-m} \, \Gamma, \sqrt{E-m} \, e_x) \]

where \[ \Gamma = \cos \frac{\beta}{2} \]

There are 2 possibilities for \( \epsilon_1 \), the polarization of the incoming \( \gamma \)
- (i) polarized in plane of scattering, \( \epsilon_1 = (1, 0, 0; 0) = \epsilon_{1a} \)
- (ii) polarized perpendicular to plane of scattering, \( \epsilon_1 = (0, 1, 0; 0) = \epsilon_{1b} \)

Similarly for \( \epsilon_2 \):
- (i) in plane of scattering, \( \epsilon_2 = (0, \cos \theta; 0, -\sin \theta; 0) = \epsilon_{2a} \)
- (ii) perpendicular to the plane of scattering, \( \epsilon_2 = (0, 1, 0; 0) = \epsilon_{2b} \)

Now we carry out the computation of \( \mathbf{M} \) for the 16 different processes:

In each case, \( \mathbf{U} \),

\[ \mathbf{M} = \frac{\mathbf{U}_2 \cdot (\mathbf{p}_1 + \mathbf{k}_1 + \mathbf{m}) \cdot \mathbf{U}_1 + \mathbf{U}_2 \cdot (\mathbf{p}_1 - \mathbf{k}_2 + \mathbf{m}) \cdot \mathbf{U}_1}{(\mathbf{p}_1 + \mathbf{k}_1)^2 - m^2} + \frac{\mathbf{U}_2 \cdot (\mathbf{p}_1 - \mathbf{k}_2 + \mathbf{m}) \cdot \mathbf{U}_1}{(\mathbf{p}_1 - \mathbf{k}_2)^2 - m^2} \]

\[ = \mathbf{U}_2 \left( \frac{\mathbf{p}_1 \cdot \mathbf{p}_1 + \mathbf{p}_1 \cdot \mathbf{m} + \mathbf{m} \cdot \mathbf{m}}{2m \mathbf{w}_1}, \frac{\mathbf{p}_1 \cdot \mathbf{p}_1 + \mathbf{m} \cdot \mathbf{m}}{2m \mathbf{w}_2} \right) \mathbf{U}_1 \]

since \( \mathbf{p}_1 \cdot \mathbf{p}_1 = -\mathbf{p}_1 \cdot \mathbf{p}_1 \)

\( \mathbf{M} \)

\[ \chi_1 = \mathbf{w}_1 (\chi_1 \cdot \mathbf{p}_1) \]

\[ \chi_2 = \mathbf{w}_2 (\chi_1 - \chi_1 \cos \theta - \chi_1 \sin \theta) \]
Case 1: \[ u_1 = u_1^* \quad \quad u_2 = u_2^* \]
\[ e_1 = e_1^* \quad \quad e_2 = e_2^* \]  

Then \[ -\gamma_1^* = \gamma_1 \quad \quad \gamma_2^* = xC - \gamma_2 S = xC - yS \]

according to the abbreviations:
\[ \gamma_t \rightarrow x \quad \gamma_y \rightarrow y \quad \gamma_z \rightarrow z \quad \gamma_t \rightarrow t \]

Then \[ M_1 \cdot \frac{2m}{4\pi e^2} = U_2 \left[ (xC - yS)(t - z)x + x(t - zC - xS)(xC - yS) \right] u_1 \]
\[ = U_2 \left[ tC - zC - xzS - xS + tC + txS - zC^2 - yS^2 \right] \]
\[ = U_2 \left[ 2xC - xS - z(1+C) \right] u_1 \]

To evaluate \[ 2xC - xS - z(1+C) \] \[ u_1 \], we make use of the table of \( \gamma \)-matrices on page 4.

Obtain \[ M_1 \cdot \frac{2m}{4\pi e^2} = U_2 \begin{pmatrix} 2C & 0 \\ 0 & 1/C \end{pmatrix} \begin{pmatrix} 2m \end{pmatrix} = \begin{pmatrix} 2m \end{pmatrix} \begin{pmatrix} tC + \sqrt{E-m} \gamma_1 - \gamma_2 \sqrt{E-m} \gamma_2 - \sqrt{E-m} \gamma_1 \gamma_2 \end{pmatrix} \begin{pmatrix} 2C \\ 0 \\ 1/C \end{pmatrix} \]

\[ = U_2 \begin{pmatrix} 2C & 0 \\ 0 & 1/C \end{pmatrix} \begin{pmatrix} 2m \end{pmatrix} \]
\[ = U_2 \begin{pmatrix} 2m \end{pmatrix} \begin{pmatrix} tC + \sqrt{E-m} \gamma_1 - \gamma_2 \sqrt{E-m} \gamma_2 - \sqrt{E-m} \gamma_1 \gamma_2 \end{pmatrix} \begin{pmatrix} 2C \\ 0 \\ 1/C \end{pmatrix} \]
\[ = 2 \sqrt{E+m} \cos \theta \cos \phi \gamma_x - \sqrt{E-m} \cos \theta \gamma_x - \sqrt{E-m} \cos \phi \gamma_2 \]

or, \[ M_1 \frac{2m}{4\pi e^2} = \begin{pmatrix} 2 \sqrt{E+m} \cos \theta \cos \phi \gamma_x - \sqrt{E-m} \cos \theta \gamma_x - \sqrt{E-m} \cos \phi \gamma_2 \end{pmatrix} \]

\[ S = \sin \theta \]
Table of Independent $\gamma$-matrices:

We write this out in a particular representation defined by

$$\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $\gamma = \beta \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\gamma_x \gamma_y = -i \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_y \gamma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_y \gamma_t = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\gamma_t \gamma_x = \gamma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_t \gamma_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_y \gamma_t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_t \gamma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_t \gamma_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_t \gamma_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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$$\gamma_t \gamma_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_t \gamma_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\gamma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
2. \[ u_{1+} \rightarrow u_{2+} \]
\[ \theta_{1a} \rightarrow \theta_{2b} \]
\[ -\xi_1 = \xi_2 \]
\[ -\xi_2 = \xi_3 \]

\[ M_{2} \frac{\frac{2m}{4\pi e}}{4\pi e} = \overline{u}_2 \left[ \frac{1}{2} y (t-z)x + x(t-z(C-xS))y S^2 \right] u_1 \]
\[ = \overline{u}_2 \left[ \frac{1}{2} t y x - y y x - t y x + y t y z C + y S^2 \right] u_1 \]
\[ = \sqrt{\frac{m}{e}} \begin{pmatrix} 0 \\ i c \end{pmatrix} = i \sqrt{\frac{2m}{e}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} C - C - C S^2 \end{pmatrix} \]

\[ M_{2} \frac{\frac{\sqrt{2m}}{4\pi e}}{4\pi e} = i \sqrt{\frac{2m}{e}} \begin{pmatrix} \cos \frac{\theta}{2} - \cos \left( \theta - \frac{\theta}{2} \right) \end{pmatrix} \]

3. \[ u_{1+} \rightarrow u_{2-} \]
\[ \theta_{1a} = \theta_{1a} \rightarrow \theta_{2b} = \theta_{2b} \]
\[ -\xi_1 = \xi_3 \]
\[ -\xi_2 = \xi_3 \]

\[ M_{3} \frac{\frac{2m}{4\pi e}}{4\pi e} = \overline{u}_2 \left[ \frac{1}{2} (C-zS)(t-z)x + x(t-z(C-xS))(zS) \right] u_1 \]
\[ = \overline{u}_2 \left[ \frac{1}{2} t C - z C - t x y S - x S + t C - z C^2 + x C S + t x S \right] \]
\[ -x C S - z C^2 \]
\[ = \overline{u}_2 \left( 2 t C - z C - z C - x S \right) u_1 \]

\[ M_{3} \frac{\frac{\sqrt{2m}}{4\pi e}}{4\pi e} = \overline{u}_2 \begin{pmatrix} 2C \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \sqrt{\frac{2m}{e}} \cos \theta \sin \frac{\theta}{2} + \sqrt{\frac{2m}{e}} \sin \left( \theta + \frac{\theta}{2} \right) \end{pmatrix} \]
\[ + \sqrt{\frac{2m}{e}} \sin \left( \theta - \frac{\theta}{2} \right) \]

4. \[ u_{1+} \rightarrow u_{2-} \]
\[ \theta_{1a} = \theta_{1a} \rightarrow \theta_{2b} \]
\[ -\xi_1 = \xi_3 \]
\[ -\xi_2 = \xi_3 \]

\[ \frac{M_{3} \frac{\frac{\sqrt{2m}}{4\pi e}}{4\pi e}}{4\pi e} = -i \sqrt{\frac{2m}{e}} \begin{pmatrix} \sin \left( \theta - \frac{\theta}{2} \right) + \sin \left( \theta - \frac{\theta}{2} \right) \end{pmatrix} \]

5. \[ u_{1-} \rightarrow u_{2-} \]
\[ \theta_{1a} = \theta_{1a} \rightarrow \theta_{2b} \]

\[ M_{5} \frac{\frac{\sqrt{2m}}{4\pi e}}{4\pi e} = \begin{pmatrix} 2 \sqrt{\frac{2m}{e}} \cos \theta \cos \frac{\theta}{2} - \sqrt{\frac{2m}{e}} \cos \left( \theta + \frac{\theta}{2} \right) - \sqrt{\frac{2m}{e}} \cos \left( \theta - \frac{\theta}{2} \right) \end{pmatrix} \]
\[ + \sqrt{\frac{2m}{e}} \cos \left( \theta - \frac{\theta}{2} \right) \]
Notice that $M_5 = M_1$. This can be seen to be a consequence of the invariance of the coupling $g\mu A_\mu$ under parity transformation $P$.

Under $P$, $\vec{S}$ (spin), like $\vec{l}$ (orbital angular mom.) is invariant. $\vec{P}$, however, changes sign, so $\vec{S} \cdot \vec{P}$ also changes sign and we have the result that the helicity of an electron changes sign under parity.

From the form of the amplitude, it is obvious that reversing the directions of polarization of both photons leaves the lowest order amplitude.

Putting these together, we see that

$$M_1 (u_1^+ \rightarrow u_2^+) = M_2 (u_1^- \rightarrow u_2^-) = M_5 (e_{1a} \rightarrow e_{2a})$$

Similarly, $M_6 (u_1^- \rightarrow u_2^-) = M_7 (u_1^+ \rightarrow u_2^+)$

$$M_7 (u_1^- \rightarrow u_2^+) = M_5 (e_{1a} \rightarrow e_{2a})$$

$$M_8 (u_1^- \rightarrow u_2^+) = M_4 (u_1^- \rightarrow u_2^-)$$

9. $u_1^+ \rightarrow u_2^+$
$$e_{1b} \rightarrow e_{2b}$$

$$\phi_1 = \phi_2$$

$$\phi_2 = \phi_1$$

$$M_9 \frac{\sqrt{2m}}{4\pi e} = \left\{ \frac{2\sqrt{2}m \cos \theta - \sqrt{E - m} \cos \varphi - \sqrt{E - m} \cos (\theta + \phi)}{4 \pi e} \right\}$$

10. $\bar{u}_1^+ \rightarrow \bar{u}_2^+$
$$\bar{e}_{1b} \rightarrow \bar{e}_{2b}$$

$$-\phi_1 = \phi_2$$

$$-\phi_2 = \phi_1$$

$$M_{10} \frac{\sqrt{2m}}{4\pi e} = \frac{1}{\sqrt{2m}} \bar{u}_2 (x y z - y z - x y) u_1 = i \sqrt{E - m} (\cos \varphi - \cos (\theta + \phi))$$
11. \( U_1 - \rightarrow U_2 - \)
   \( E_{1b} \rightarrow E_{2b} \)  
   By parity  \( M_{11} = M_9 \)

12. \( U_1 - \rightarrow U_2 - \)
   \( E_{1b} \rightarrow E_{2a} \)  
   By parity  \( M_{12} = M_{10} \)

13. \( U_1 + \rightarrow U_2 - \)
   \( E_{1b} \rightarrow E_{2b} \)

\[ M_{13} = M_{14} \]

\[ \frac{\sqrt{2} m}{4 \pi e^2} M_{13} = 2 \sqrt{E+m} \sin \theta \frac{\sin \phi}{2} + \sqrt{E-m} \sin \theta \frac{\sin \phi}{2} + \sqrt{E-m} \sin (\theta + \phi) \]

15. \( U_1 + \rightarrow U_2 - \)
   \( E_{1b} \rightarrow E_{2a} \)

16. \( U_1 - \rightarrow U_2 + \)
   \( E_{1b} \rightarrow E_{2a} \)

\[ M_{15} = M_{16} \]

\[ \frac{\sqrt{2} m}{4 \pi e^2} M_{15} = \frac{1}{\sqrt{2}} (x_1 y_2 - y_1 x_2) u_1, \quad \text{(Some form 10.)} \]

\[ = e \left( \sqrt{E-m} \sin (\theta + \phi) - \sqrt{E-m} \sin \phi \right) \]

The various polarized cross sections are obtained by
the replacement  \( M \rightarrow M_k \),  \( k = 12, \ldots, 16 \) in Equation (1).
In this section we will work out the matrix elements from standard nonrelativistic perturbation theory, using all the field theory learned to date. Despite the fact that none of the intermediate steps look relativistically invariant, the answer will appear in a relativistically invariant form. We will then formulate rules for writing down the relativistically invariant form at once from the diagrams, and will never again work through the mess that is displayed below.

\[ H = \frac{i}{8\pi} \int (E^2 + B^2) d\text{vol} + \int \psi^* (x) \left[ \beta m + e \phi (x) + \frac{e}{c} \left( \frac{\nabla^2}{c^2} \psi \right) \right] \psi (x) d^3 x \]

Energy of Free Electromagnetic Field

where the \( \psi \)'s are operators which create and annihilate electrons.

This may be broken down further into the energy of the free electron-positron field

\[ \int \psi^* (x) \left[ \beta m + \frac{e}{c} \left( \frac{\nabla^2}{c^2} \right) \psi \right] \psi (x) d^3 x \]

and a term which represents the interaction between the electric field and the matter fields

\[ e \int \psi^* (x) \left[ \phi (x) - \frac{e}{c} \left( \frac{\nabla^2}{c^2} \right) \psi \right] \psi (x) d^3 x = \int \psi^* (x) A_\mu (x) \partial_\mu \psi (x) d^3 x \]

where \( \partial_\mu = e \psi^* \beta \gamma_\mu \psi = e \overline{\psi} (x) \gamma_\mu \psi (x) \)

and it is important to keep in mind that each of \( \psi \), \( \psi^* \), \( A_\mu \) is linear in the appropriate creation operators.

Now we ruin the manifest covariance in order to relate pieces to previously learned physics.

Choose a gauge such that \( \nabla \cdot A = 0 \) and \( \nabla^2 \phi = \rho \)

(a point charge has \( \phi (r') = \frac{e Q}{4 \pi r} \)

Thus the \( e \int \psi^* \phi (x) \psi (x) d^3 x \) term can be written

\[ \frac{e^2}{2} \int \frac{\psi^* (x) \psi (x) \psi^* (y) \psi (y) d^3 x d^3 y}{r_{xy}} \rightarrow \infty \]

and we have managed to express this piece of the interaction without use of photon creation and annihilation operators. The \( \infty \) mentioned above comes from the \( \int \) function in the anti-commutation relations for \( \psi^* (x) \) and \( \psi (y) \).

It is a self-energy term, of the type discussed before in this course. The thing to notice here is that this infinity seems to involve only spatial coordinates and
thus is not obviously relativistically covariant (in another system the self-energy subtracted might be different). This was one of the difficulties of formulating the theory this way. Later on in this course the removal of certain relativistically covariant infinities will be discussed.

Then the interaction of the electron-positron field with photons is given by
\[-e \int \nabla \psi^+(x) \cdot \mathbf{A}(x) \, d^3x\]
where
\[\mathbf{A}(x) = \sum \mathbf{e}_{\nu u} \frac{\hbar}{2 \omega_u} \left[ e_{\nu u}^+ a_{\nu u} e^{i \mathbf{k} \cdot x} + e_{\nu u}^- a_{\nu u}^+ e^{-i \mathbf{k} \cdot x} \right]\]
where \(e_{\nu u}^\pm\) is the polarization vector for a photon of polarization type \(\pm 1\) (helicity) and
\[\psi(x) = \sum_{\nu, u} u_{\nu, u} c_{\nu, u} e^{i \mathbf{p} \cdot x}\]
\(u_{\nu, u}\) is a 4 component spinor for a solution of the Dirac equation of momentum \(p\), type \(\nu\) (\(\nu\) runs over 4 possibilities — positive energy spin up and spin down, and negative energy spin up and spin down).

\[\left(\beta \mathbf{p} + \mathbf{A} \cdot \mathbf{p}\right) u_{\nu, u} = \mathcal{E} u_{\nu, u}\]
\[\mathcal{E} = \pm \mathcal{E}_\pm = \pm \sqrt{\mathbf{p}^2 + m^2}\]
Likewise
\[\psi^+ = \sum_{\nu, u} c_{\nu, u}^+ u_{\nu, u} e^{-i \mathbf{p} \cdot x}\]
Change notation:
let \(u_{\nu, \kappa}\) stand for those solutions such that \(\mathcal{E} = \mathcal{E}_{\nu, \kappa}\)
where \(\kappa\) is a spin index with two possible values
\(u_{\nu, \kappa} = u_{-\nu, \kappa}\) for those solutions with \(\mathcal{E} = -\mathcal{E}_\nu\)
Then we have
\[\left(\beta \mathbf{p} + \mathbf{A} \cdot \mathbf{p}\right) u_{\nu, \kappa} = \mathcal{E}_\nu u_{\nu, \kappa}\]
\[\left(\beta \mathbf{p} - \mathbf{A} \cdot \mathbf{p}\right) u_{\nu, \kappa} = -\mathcal{E}_\nu u_{\nu, \kappa}\]
And the free Hamiltonian takes the form
\[\hat{H}_{\text{free}} = \sum_{\nu, \kappa} E_\nu c_{\nu, \kappa}^+ c_{\nu, \kappa} = \sum_{\nu, \kappa} E_\nu \left[ c_{\nu, \kappa}^+ c_{\nu, \kappa} - c_{\nu, \kappa}^- c_{\nu, \kappa}^+ \right]\]
where \(c_{\nu, \kappa}^+ c_{\nu, \kappa}^+ + c_{\nu, \kappa}^- c_{\nu, \kappa}^- = \sum_{\nu, \kappa} d_{\nu, \kappa} d_{\nu, \kappa}\)
In accordance with our new terminology for the spinors, define a new terminology for the annihilation operators of negative energy states
If \(c_{\nu, \kappa}\) annihilates an electron of momentum \(p\), spin type \(\kappa\), then \(c_{\nu, \kappa} = d_{-\nu, \kappa}\)
creates a positron with momentum \(-p\).

Then
\[\hat{H}_{\text{free}} = \sum E_\nu \left[ c_{\nu, \kappa}^+ c_{\nu, \kappa} + d_{\nu, \kappa}^+ d_{\nu, \kappa} \right] - \sum E_\nu\]

another \(\mathcal{E}\)
This is the energy of the full sea of negative energy states. We measure relative energies away from it.

Now \( \psi = \frac{1}{\sqrt{h_0}} \left[ c_{\rho, k} \, \psi_{\rho, k} \, e^{i p \cdot x} + d_{\rho, k}^\dagger \, \psi_{\rho, k}^\dagger \, e^{-i p \cdot x} \right] \)

notice that creation of positron in initial state (i.e. annihilation of positron in final state) gives a column vector \( \psi_{\rho, k} \) whereas annihilation of electron in final state (by \( \psi_{\rho, k}^\dagger \)) gives a row vector \( \psi_{\rho, k}^\dagger \).

Hence a positron in the initial state leads to \( \bar{\psi} \) on the left side of the matrix element, whereas an electron in the initial state leads to \( \psi \) on the right side of the matrix element.

Use this to expand out the interaction Hamiltonian

\[
A = \sum \int \frac{\psi^\dagger(x) \cdot \bar{\psi}(x) \, d^3x}{2 \omega_k}
\]

\[
\psi = \sum \left[ c_{\rho, k} \, \psi_{\rho, k} \, e^{i p \cdot x} + d_{\rho, k}^\dagger \, \psi_{\rho, k}^\dagger \, e^{-i p \cdot x} \right]
\]

to obtain

\[
\sum \frac{1}{\sqrt{\omega_k}} \int \frac{\left[ c_{\rho, k}^\dagger \, \psi_{\rho, k}^\dagger \, e^{-i p \cdot x} + d_{\rho, k}^\dagger \, \psi_{\rho, k}^\dagger \, e^{-i p \cdot x} \right] [ e^{i p \cdot x} \cdot \bar{\psi}_{\rho, k} \, e^{i k \cdot x} + e^{i p \cdot x} \cdot \bar{\psi}_{\rho, k} \, e^{i k \cdot x} ]}{2 \omega_k} \]

\[
A \left[ c_{\rho, k} \, \psi_{\rho, k} \, e^{i p \cdot x} + c_{\rho, k}^\dagger \, \psi_{\rho, k}^\dagger \, e^{-i p \cdot x} \right]
\]

In every case the spatial integration will give only a \( \int \) function of the momenta (because the exponentials are the only functions of \( x \) present)

We then obtain a sum of the following pieces:

\[
\sqrt{\frac{4 \pi e^2}{2 \omega_k}} \left( c_{\rho, k} \, a_{\rho, k}^* \, c_{\rho', k'}^* \, u_{\rho, k} \cdot \bar{\psi}_{\rho', k'} \, \int \left( -p + k + \rho' \right) \right)
\]

\[
\sqrt{\frac{4 \pi e^2}{2 \omega_k}} \left( c_{\rho, k} \, a_{\rho, k}^* \, d_{\rho, k'}^* \, u_{\rho, k} \cdot \bar{\psi}_{\rho', k'} \, \int \left( -p + k - \rho' \right) \right)
\]

\[
\sqrt{\frac{4 \pi e^2}{2 \omega_k}} \left( c_{\rho, k} \, a_{\rho, k}^* \, c_{\rho, k'}^* \, u_{\rho, k} \cdot \bar{\psi}_{\rho', k'} \, \int \left( p + k - \rho' \right) \right)
\]

\[
\sqrt{\frac{4 \pi e^2}{2 \omega_k}} \left( c_{\rho, k} \, a_{\rho, k}^* \, d_{\rho, k'}^* \, u_{\rho, k} \cdot \bar{\psi}_{\rho', k'} \, \int \left( p + k + \rho' \right) \right)
\]
\[
\sqrt{\frac{4\pi e^2}{2w_k}} \, d\rho_k \, a_{\rho_k} \, c_{\rho_k,1} \, V_{\rho_1}^* \, \vec{e}_{\rho_1} \cdot \vec{a} \, \psi_{\rho_1} \, \delta(p + k + \rho')
\]

\[
\sqrt{\frac{4\pi e^2}{2w_k}} \, d\rho_k \, a_{\rho_k}^* \, c_{\rho_k,1} \, V_{\rho_1}^* \, \vec{e}_{\rho_1} \cdot \vec{a} \, \psi_{\rho_1} \, \delta(p - k + \rho')
\]

\[
\sqrt{\frac{4\pi e^2}{2w_k}} \, d\rho_k \, a_{\rho_k} \, c_{\rho_k,1}^* \, V_{\rho_1} \, \vec{e}_{\rho_1} \cdot \vec{a} \, \psi_{\rho_1} \, \delta(p + k - \rho')
\]

\[
\sqrt{\frac{4\pi e^2}{2w_k}} \, d\rho_k \, a_{\rho_k}^* \, c_{\rho_k,1}^* \, V_{\rho_1} \, \vec{e}_{\rho_1} \cdot \vec{a} \, \psi_{\rho_1} \, \delta(p - k - \rho')
\]

The arrows show which is the entrance state and which the exit state in the matrix element. In writing a matrix element down from a picture start at the beginning of the arrow and the right of the matrix element; then follow the arrows and move to the left in the matrix element, putting in interaction matrices and spinors as they arise. Any line which points backwards in time belongs to a positron; those which point forward in time belong to electrons (It is best not to put arrows on boson lines except possibly to remind yourself of the sign of the momentum).

The 16 terms which arise from the Coulomb interaction can also be expressed in terms of pictures, even though the intermediate lines weren't obtained by explicit creation and annihilation operators.

For fun and practice with the operator anti-commutation relations, see what the equation of motion \[ i \frac{\partial \psi}{\partial t} = \mathcal{H} \psi \] tells you about the operator \( \gamma \).
Previously we obtained the electron propagator to be \( \frac{1}{p^2 - m^2} \)

It should be possible to obtain the photon propagator from the pieces of matrix elements above.

Consider scattering to order \( \alpha^2 \)

\[ \langle p_f | \sqrt{\gamma} | p_i \rangle \]

for \( \mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2 \) the contributions are

1) Coulomb interaction

\[ \frac{4\pi e^2}{Q^2} \left( \bar{u}_3 u_1 \right) \left( \bar{u}_4 u_2 \right) \]

2) Virtual transverse photons

\[ \frac{4\pi e^2}{2\omega_q} \left( \bar{u}_3 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_1 \right) \left( \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 \right) \]

\[ \frac{1}{E_1 + E_2} \left( E_1 + E_2 + i\epsilon \right) \]

\[ \sum_{\text{internal polarizations}} \]

\[ \sum_{\text{internal polarizations}} \]

Notice that if there were 3 directions of polarization (two \( \perp \bar{q} \) and one \( \parallel \bar{q} \) ) then

\[ \sum_{\text{transverse}} \left( \mathcal{A} \cdot \mathcal{E} \right) \left( \mathcal{B} \cdot \mathcal{E} \right) = \mathcal{A} \cdot \mathcal{B} \]

Hence

\[ \left( \mathcal{A} \cdot \mathcal{E} \right) \left( \mathcal{B} \cdot \mathcal{E} \right) = \mathcal{A} \cdot \mathcal{B} - \mathcal{E} \cdot \mathcal{Q} \mathcal{B} \cdot \mathcal{Q} \]

Thus combination of the two transversely polarized pieces gives

\[ \frac{4\pi e^2}{2\omega_q} \left[ \left( \bar{u}_3 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_1 \right) \left( \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 \right) \right] \]

But

\[ \frac{1}{E_1 - E_2 - i\epsilon} + \frac{1}{E_2 - E_4 - i\epsilon} = \frac{2\omega_q}{E_2^2} \]

Hence the sum of these terms gives

\[ 4\pi e^2 \left\{ \frac{1}{Q^2} \left( \bar{u}_3 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_1 \right) \left( \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 \right) \right\} \]

But \( \bar{q}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 = \bar{u}_4 \left( q_4^2 - q_2^2 \right) \bar{u}_2 = \bar{u}_4 \left( m - m \right) \bar{u}_2 = 0 \)

(current conservation)

Hence

\[ \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 = q_4 \bar{u}_4 \gamma_\perp \mathcal{B} \bar{u}_2 \]

and we arrive at

\[ \sum_{\text{internal polarizations}} - \frac{1}{Q^2} \left( \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 \right) \left( \bar{u}_3 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_1 \right) + \frac{1}{Q^2} \left( \bar{u}_4 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_2 \right) \left( \bar{u}_3 \gamma_\perp \mathcal{E} \cdot \mathcal{B} \bar{u}_1 \right) = 0 \]

But

\[ \frac{1}{Q^2} + \frac{q_4^2}{Q^2 - q_2^2} = 0 \]
Hence we have shown that all the lowest order interactions due to charge coalesce into one manifestly covariant matrix element
\[-\frac{4\pi e^2}{\hbar c} (U_{14} U_{23} U_{14}) (U_{3} U_{1})\]
which can be symbolized by the picture
\[\sqrt{4\pi e^2} \gamma \mu\]
For the charge current \( j_{\mu} = U_{23} \delta_{\mu} u_{1} \), we have
\[\Gamma A_{\mu} = j_{\mu} \quad A_{\mu} = \frac{1}{q^2} j_{\mu}\]
Interaction with \( j'_{\mu} = U_{4} \delta_{\mu} u_{2} \) is by means of \( j'_{\mu} A_{\mu} = j'_{\mu} \frac{1}{q^2} j_{\mu}\)

It is especially important to remember that when you use these rules with the propagators \( \frac{1}{\beta - m} \) or \( \frac{1}{\beta} \), it is no longer necessary to draw two graphs which differ by time ordering
\[\text{Just indicates intermediate state}\]
It is only necessary to draw one graph \( \text{This is always the convention used in applications.} \)

**Applications and Discussion**

It happens that \( \mu \) mesons obey (to the best of our present knowledge) exactly the same equations as electrons, with only the mass changed.

Suppose a stationary electron target, with \( \mu \) particles shot in (in the lab frame). The \( \mu \) and \( e \) will scatter into new momentum states because of the interaction of their charges
\[\begin{array}{c}
\mu \\
\hline
\rho_4
\end{array} \quad \begin{array}{c}
e \\
\hline \mu \rho_3
\end{array}\]

\[\text{Rate} = \sigma U_{\mu} = (2\pi)^4 \delta(\rho_4 + \rho_3 - \rho_2 - \rho_1) \frac{d^4\rho_4}{(2\pi)^4} \frac{1}{2m_e} \frac{1}{2E_{\mu}} |M|^2\]

The \( \delta \) functions may then be unraveled against the differentials in any order you find convenient (hint: the answer is usually pretty messy no matter what order you use, but there may be fewer messy steps in one sequence than in another)

Because the \( \mu \) and electron have the same interaction with the electromagnetic field, we can write down the matrix element by using the same rules as for electron-
It is easy to see that the largest contributions to the cross section come from 
\( p_2 \approx p_1 \) (forward scattering in the center of mass system). When \( \vec{q} \) is small, the 
interaction radius (the Fourier transform conjugate of \( \sqrt{\vec{q}} \)) is large. This means 
most of the cross section is obtained from cases where the particles aren't close 
together. Hence this is not a good experiment to do if you want to test the laws 
of electrodynamics - we know already that they work fine for large separations because 
this is the classical limit. What we want to examine closely is whether the laws 
brake down when the particles get close together. This would be found by looking 
at large \( \vec{q}^2 \) behavior of the matrix element. But \( \vec{q}^2 \) for this particular 
experiment is extremely small even for high energy \( \mu^+ \)'s

**Question:** If it were possible to have a target of \( \mu^+ \) and shoot electrons at them, 
would it be easier to explore the large \( \vec{q}^2 \) limit?

**For education:** Is it true that to lowest order the scattering of \( \mu^+e^- \) can be obtained 
from the scattering of \( \mu^-e^- \) by the replacement \( \frac{1}{q_{\mu}} = -\frac{1}{q_{\mu}} \)?

**Why?**

One of the most precise experiments to date along this line is the scattering 
\( e^-p \rightarrow e^-p \). Here the matrix element is again proportional to 
\( \frac{J_{\mu} \bar{u}_1 \gamma_\mu u_1}{(p_2 - p_f)^2} \), where \( J_{\mu} \) now represents the electromagnetic current of the proton.

It can be shown that the most general form of \( J_{\mu} \) is

\[ J_{\mu} = \bar{u}_3 \left[ e^- F_1 \gamma_\mu + F_2 (\gamma_\mu \gamma_5 - \gamma_5 \gamma_\mu) \right] u_1 \]

where \( u_3 \) and \( u_1 \) are nucleon plane wave spinors and \( F_1 \) and \( F_2 \) are assumed for 
thetical reasons to be functions only of the exchanged 4 momentum transfer 
squared (\( \vec{q}^2 \))

(Justify this form for \( J_{\mu} \) by fiddling with others)

It is found empirically that both \( F_1 \) and \( F_2 \) have the shape 
\( \frac{k_{\mu} \bar{u}_1}{(q^2 - x^2)^2} \) where \( x^2 \)
is a universal constant. One could, therefore, get the same result just by modifying the photon propagator. It is now considered more acceptable to ascribe the form factors $F_i$ to mesonic substructure of the nucleons, rather than to a breakdown of qed.
Differential cross-section for Compton Scattering in lab ($\theta^2 = 0$):

$$d\sigma = \frac{1}{16\pi E_2} \frac{1}{(2\pi)^2} \frac{1}{m^2} \frac{1m^2}{m^2 + 2m^2\sin^2\theta \gamma} d\Omega$$

$$= \frac{d\Omega}{6\pi} \frac{w_2}{m\gamma} \frac{1m^2}{m^2 + 2m\sin^2\theta \gamma}$$

From kinematics, $\frac{w_2}{w_1} = \frac{m}{m + 2m\sin^2\theta \gamma}$

$$\therefore \frac{d\sigma}{d\Omega} \text{ lab} = \frac{1}{6\pi m^2} \frac{1m^2}{m^2} \frac{w_2^2}{w_1^2} \quad \text{(1)}$$

This cross-section refers to a definite process in which the initial and final states are completely specified, defined "pure" states.

Now there are 2 possible helicities for the electron in the initial state, as well as 2 possible helicities for the electron in the final state. In addition, there are 2 possible polarization states for the photon in both the initial and final states. There are therefore 16 polarized differential amplitudes $M$ that occur in Compton Scattering.

**Kinematics of scattering in the lab system**

![Kinematics diagram]

**Initial state spinors**

$$u_1 = u_{1+} = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_1 = u_{1-} = \sqrt{2m} \begin{pmatrix} 0 \\ i \end{pmatrix}$$

(Recall: normalization is 2E particles/unit volume.)
For the state spinors:

\[ \psi_2 = \psi_2^+ = \begin{pmatrix} \sqrt{E+\text{m}} \cos \phi_2 \\ -\sqrt{E+\text{m}} \sin \phi_2 \\ \sqrt{E-\text{m}} \cos \phi_2 \\ -\sqrt{E-\text{m}} \sin \phi_2 \end{pmatrix}, \quad \text{or} \quad \psi_2 = \psi_2^- = \begin{pmatrix} \sqrt{E+\text{m}} \sin \phi_2 \\ \sqrt{E+\text{m}} \cos \phi_2 \\ -\sqrt{E-\text{m}} \sin \phi_2 \\ -\sqrt{E-\text{m}} \cos \phi_2 \end{pmatrix} \]

\[ \psi_2^- = (\sqrt{E+\text{m}} \ 6, \sqrt{E+\text{m}} \ \Gamma, \sqrt{E-\text{m}} \ 6, \sqrt{E-\text{m}} \ \Gamma) \]

\[ \psi_2^+ = (\sqrt{E+\text{m}} \ \Gamma, -\sqrt{E+\text{m}} \ 6, -\sqrt{E-\text{m}} \ \Gamma, \sqrt{E-\text{m}} \ 6) \]

where \[ \begin{align*}
6 &= \sin \phi_2 \\
\Gamma &= \cos \phi_2 
\end{align*} \]

There are 2 possibilities for \( \phi_1 \), the polarisation of the incoming \( \psi \):

(i) polarised in plane of scattering, \( \phi_1 = (1, 0, 0) \) = \( \phi_{1a} \)
(ii) polarised perpendicular to plane of scattering, \( \phi_1 = (0, 1, 0, 0) \) = \( \phi_{1b} \)

Similarly for \( \phi_2 \):

(i) in plane of scattering, \( \phi_2 = (\cos \Theta, 0, -\sin \Theta, 0) \) = \( \phi_{2a} \)
(ii) perpendicular to plane of scattering, \( \phi_2 = (0, 1, 0, 0) \) = \( \phi_{2b} \)

Now we carry out the computation of \( M \) for the 16 different processes.

In each case, \( M \) is

\[ M \left(\frac{\text{tev}}{2}\right) = \frac{\psi_2 \ 6 \ \phi_1 (\phi_1^+ + \phi_1^-) \psi_1 \ U_1 + \psi_2 \ \phi_2 (\phi_2^+ - \phi_2^-) \psi_1 \ U_1}{(\phi_1^+ + \phi_1^-)^2 - \text{m}^2} \]

\[ = \frac{\psi_2 (\phi_2^+ \phi_1 + \phi_2^\phi_1^-) \ U_1}{2\text{m} \psi_1}, \quad \text{since} \quad \phi_1 \phi_2 = -\phi_1 \phi_2 \\
\phi_1 \phi_2 = -\phi_2 \phi_1 \\
\phi_1^+ \phi_1 = -\phi_1 \phi_1^- \\
\phi_2^\phi_2 = -\phi_2 \phi_2^- \\
\phi_1 = \phi_1 \left(\phi_1 - \phi_1^- \right) \\
\phi_2 = \phi_2 \left(\phi_2 - \phi_2^- \right) \]
Case 1:

\[ u_1 = u_1 \quad u_2 = u_2 \]
\[ e_1 = e_1 \quad e_2 = e_2 \]

Then
\[ -\varphi_1 = x_1 \]
\[ -\varphi_2 = x_2 - x_1 \]
\[ x_1 - x_2 = x \quad x_1 - x_2 = x \]
\[ x = x \]
\[ x = x \]

According to the abbreviations:
\[ \varphi_1 \rightarrow x \]
\[ \varphi_2 \rightarrow y \]
\[ \varphi_3 \rightarrow z \]
\[ \varphi_4 \rightarrow t \]

Then
\[ \frac{M_1 \cdot 2m}{4\pi e^2} = \bar{u}_2 \left[ (x - z)S(t - z)x + x(t - z)S(x - z)S \right] u, \]
\[ = \bar{u}_2 \left[ tC + zS - xS + tS + tzS - zC - zS \right] u, \]
\[ = \bar{u}_2 \left[ 2tC - xS - z (1 + C) \right] u, \]

where repeated use has been made of the commutation rules for the \( S \)s:
\[ + t + t = 0 \]
\[ t^2 = -1 \] etc.

To evaluate \( [2tC - xS - z (1 + C)]u \), we make use of the table of \( S \)-matrices on page 7.

Obtain
\[ M_1 \frac{2m}{4\pi e^2} = \bar{u}_2 \begin{pmatrix} 2C \\ 0 \\ 1 + C \\ S \end{pmatrix} \]
\[ M_1 \frac{2m}{4\pi e^2} = 2C + C \Gamma + V \left( C - (S + C) \right) + V \Gamma - m \Gamma/2, \]
\[ = 2C + \cos(\theta \cos^2 \theta_0 - \frac{m}{m} \cos \theta \cos^2 \theta_0) - V \Gamma - m \Gamma/2, \]

or,
\[ M_1 = \frac{4\pi e^2}{V \Gamma} \left( 2C \cos \theta \cos^2 \theta_0 - \frac{m}{m} \cos \theta \cos^2 \theta_0 - V \Gamma - m \Gamma/2 \right) \]
Table of Independent $\gamma$-matrices:

We write this out in a particular representation defined by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $\gamma = \beta \alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\gamma_5 = \beta \gamma \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\gamma_x \gamma_y = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma_x \gamma_y = \begin{pmatrix} -i e_x & 0 \\ 0 & i e_x \end{pmatrix}$$

$$\gamma_y \gamma_x = \begin{pmatrix} 0 & i e_y \\ i e_y & 0 \end{pmatrix}$$

$$\gamma_x \gamma_y = \begin{pmatrix} 0 & e_x \\ e_x & 0 \end{pmatrix}$$

$$\gamma_5 \gamma_x = \begin{pmatrix} -i e_x & 0 \\ 0 & i e_x \end{pmatrix} = \gamma_x \gamma_y \gamma_x$$

$$\gamma_5 \gamma_y = \begin{pmatrix} 0 & -i e_y \\ i e_y & 0 \end{pmatrix} = \gamma_y \gamma_x \gamma_y$$

$$\gamma_5 \gamma_z = \begin{pmatrix} -i e_y & 0 \\ 0 & i e_y \end{pmatrix} = \gamma_z \gamma_y \gamma_z$$

$$\gamma_5 \gamma_t = \begin{pmatrix} i & -i \\ i & 0 \end{pmatrix} = \gamma_x \gamma_y \gamma_z$$

$$\epsilon_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \epsilon_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
\[2. \quad U_{1+} \rightarrow U_{2+}, \quad E_{1a} \rightarrow E_{2a} \quad -\theta_2 = \theta_1 \]

\[M_{2x2m} \frac{\text{meV}}{4\pi e^2} = \bar{u}_2 \left( \begin{array}{c} \frac{1}{2} (t-z) x + x(t-zc-xz) y \frac{1}{2} u_1 \\ \frac{1}{2} \left( -x y + t y - t y + x y + y z^2 + y S \right) u_1 \\ i \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 0 \\ i(c-i) \end{pmatrix} \end{array} \right) = i \sqrt{\text{E-m}} \left( \begin{array}{c} 1 & \frac{1}{2} (t-zc-xz) \frac{1}{2} u_1 \\ \frac{1}{2} (t y - x z - t y - x z + x y + x y + y z^2 + y s^2) \end{array} \right) \]

\[\frac{M_3}{4\pi e^2} = \bar{u}_2 \left( \begin{array}{c} x(c-z) x + x(t-zc-xz)(x-c-zx) \frac{1}{2} u_1 \\ \frac{1}{2} \left( -x y + t y - t y + x y + y z^2 + y s^2 \right) u_1 \\ \left( 2 t x - y z - 3 z x - x S \right) u_1 \end{array} \right) \]

\[\frac{M_3}{4\pi e^2} = \bar{u}_2 \left( \begin{array}{c} \left( \frac{2c}{1+c} \right) \frac{1}{2} \left( 2 \sqrt{\text{E+m}} \cos \theta \sin \frac{\theta}{2} + \sqrt{\text{E-m}} \sin \left( \theta + \frac{\theta}{2} \right) \right) \frac{1}{2} \left( 2 \sqrt{\text{E+m}} \cos \theta \sin \frac{\theta}{2} + \sqrt{\text{E-m}} \sin \left( \theta + \frac{\theta}{2} \right) \right) \right) \]

\[4. \quad U_{1+} \rightarrow U_{2+}, \quad E_{1a} \rightarrow E_{2a} \quad -\theta_2 = \theta_1 \]

\[M_{2x2m} \frac{\text{meV}}{4\pi e^2} = \bar{u}_2 \left( \begin{array}{c} x(c-z) x + x(t-zc-xz)(x-c-zx) \frac{1}{2} u_1 \\ \frac{1}{2} \left( -x y + t y - t y + x y + y z^2 + y s^2 \right) u_1 \\ \left( 2 t x - y z - 3 z x - x S \right) u_1 \end{array} \right) \]

\[\frac{M_3}{4\pi e^2} = -i \sqrt{\text{E-m}} \left( \begin{array}{c} \frac{1}{2} (t-zc-xz) \frac{1}{2} u_1 \\ \frac{1}{2} (t y - x z - t y - x z + x y + x y + y z^2 + y s^2) \end{array} \right) \]

\[5. \quad U_{1+} \rightarrow U_{2+}, \quad E_{1a} \rightarrow E_{2a} \]

\[\frac{M_3}{4\pi e^2} = \frac{1}{2} \left( 2 \sqrt{\text{E+m}} \cos \theta \sin \frac{\theta}{2} + \sqrt{\text{E-m}} \sin \left( \theta + \frac{\theta}{2} \right) \right) \]

\[\frac{M_3}{4\pi e^2} = \sqrt{\text{E-m}} \left( \begin{array}{c} \cos \theta \cos \frac{\theta}{2} - \sqrt{\text{E-m}} \cos \left( \theta + \frac{\theta}{2} \right) \right) \]

\[\frac{M_3}{4\pi e^2} = \sqrt{\text{E-m}} \left( \begin{array}{c} \cos \theta \cos \frac{\theta}{2} - \sqrt{\text{E-m}} \cos \left( \theta + \frac{\theta}{2} \right) \right) \]

\[\frac{M_3}{4\pi e^2} = \sqrt{\text{E-m}} \left( \begin{array}{c} \cos \theta \cos \frac{\theta}{2} - \sqrt{\text{E-m}} \cos \left( \theta + \frac{\theta}{2} \right) \right) \]

\[\frac{M_3}{4\pi e^2} = \sqrt{\text{E-m}} \left( \begin{array}{c} \cos \theta \cos \frac{\theta}{2} - \sqrt{\text{E-m}} \cos \left( \theta + \frac{\theta}{2} \right) \right) \]
Notice that $M_5 = M_1$. This can be seen to be a consequence of the invariance of the coupling $g$ under parity transformation $P$.

Under $P$, $S$ (spin), like $L$ (orbital angular momentum) is invariant. $P$, however, changes sign, so $S \cdot P$ also changes sign, and we have the result that the helicity of an electron changes sign under parity.

From the form of the amplitude, it is obvious that reversing the directions of polarization of both photons leaves the lowest order amplitude unchanged.

Putting these together, we see that

$$M_1 \left( u_1 \rightarrow u_2 \right) = M_1 \left( -e_2 \rightarrow -e_1 \right) = M_5 \left( u_1 \rightarrow u_2 \right) = M_5 \left( e_1 \rightarrow -e_2 \right)$$

Similarly, $M_6 \left( u_1 \rightarrow u_2 \right) = M_2 \left( u_1 \rightarrow u_2 \right)$

$$M_7 \left( u_1 \rightarrow u_2 \right) = M_5 \left( u_1 \rightarrow e_2 \right) = M_5 \left( e_1 \rightarrow e_2 \right)$$

$$M_8 \left( u_1 \rightarrow u_2 \right) = M_4 \left( u_1 \rightarrow u_2 \right) = M_4 \left( e_1 \rightarrow e_2 \right)$$

9. $u_{1a} \rightarrow u_{2a}$

$$\delta l = \delta l$$

$-\delta l = \delta l$

$$M_9 \frac{\sqrt{2m}}{4\pi^2} = \left\{ 2 \sqrt{E + \frac{m}{2} - \sqrt{E - \frac{m}{2}} - \sqrt{E - \frac{m}{2} + d_2}} \right\}$$

10. $u_{1b} \rightarrow u_{2b}$

$$\delta l = \delta l$$

$-\delta l = \delta l$

$$M_10 \frac{\sqrt{2m}}{4\pi^2} = \frac{1}{\sqrt{2m}} \frac{u_2 (xyz - C_y S - xz) u_1}{i \sqrt{E}} = \sqrt{2m} \left( \cos \frac{2}{2} - \cos \left( \theta + \phi \right) \right)$$
11. \( u_1^- \rightarrow u_2^- \)
\( E_1b \rightarrow E_2a \)
By parity \( M_{11} = M_9 \)

12. \( u_1^- \rightarrow u_2^- \)
\( E_1b \rightarrow E_2a \)
By parity \( M_{12} = M_{10} \)

13. \( u_1^+ \rightarrow u_2^- \)
\( E_1b \rightarrow E_2b \)

\( M_{13} = M_{14} \)

\[ \frac{\sqrt{2m}}{4\pi e^2} M_{13} = 2 \frac{\sqrt{E+m} \sin \theta}{\sqrt{E} \sin \phi} + \frac{\sqrt{E-m} \sin \theta}{\sqrt{E} \sin \phi} + \frac{\sqrt{E-m} \sin(\theta + \phi)}{\sqrt{E} \sin \phi} \]

15. \( u_1^+ \rightarrow u_2^- \)
\( E_1b \rightarrow E_2a \)

\( M_{15} = M_{16} \)

\[ \frac{\sqrt{2m}}{4\pi e^2} M_{15} = \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \left( x y z \bar{y} - y z \bar{x} - y z \bar{x} \right) u_1 \quad \text{(Same as 10.)} \]

\[ = \frac{1}{\sqrt{2m}} \left( \sqrt{E+m} \cos \theta + \sqrt{E-m} \cos(\theta + \phi) \right) \]

\[ = \frac{1}{\sqrt{2m}} \left( \sqrt{E-m} \sin(\theta + \phi) - \sqrt{E-m} \sin \theta \right) \]

The various polarized cross sections are obtained by the replacement \( M \rightarrow M_k \), \( k = 1, 2, \ldots, 16 \) in Equation (1).
Pair production in matter

A photon may create an e⁻-e⁺ pair in the presence of a proton according to the following diagram:

The piece of the matrix element that is represented by this diagram is

\[
(\frac{1}{\sqrt{2\pi c}})^3 J_\mu \frac{1}{q^2} \bar{u}_2 \gamma_\mu \frac{1}{q^2+m} u_1
\]

where \( J_\mu \) = matrix element of the electromagnetic current operator taken between the initial and final free proton states.

From general arguments of relativity and charge and parity conservation, it can be shown that if spinor solutions of the free particle Dirac Equation are used to describe the free proton, then

\[
J_\mu = e \bar{u}_p \left\{ \gamma_\mu F_1(q^2) + \frac{1}{4M} [\gamma_\mu, \gamma_\nu] F_2(q^2) \right\} u_p
\]

\[
= e \bar{u}_p \left\{ \gamma_\mu F_1 + \frac{1}{4M} [\gamma_\mu, \gamma_\nu] F_2 \right\} u_p
\]

The threshold behavior of \( F_1 \) and \( F_2 \), that is \( F_1(0) \) and \( F_2(0) \), may be determined in terms of the charge and magnetic moment of the proton by considering the
non-relativistic limit of the coupling of this $J\mu$ to an external static field $A_\mu = (\phi, \vec{A})$, and identifying terms in the classical formula

$$H_{NR}^I = e\phi + \vec{\mu} \cdot \vec{B}$$

In this manner, one obtains

$$F_1(0) = 1$$

$$\mu = \frac{e}{2M} \left( F_1(0) + F_2(0) \right)$$

$$\Rightarrow F_2(0) \text{ is the anomalous magnetic moment of the proton, in nuclear magnitons.}$$

In lowest order there is another diagram that must be considered for the process: $\gamma + p \rightarrow \gamma + p + e^+ + e^-$, and that is

If we neglect the recoil of the nucleus — that is, if we treat the nucleus as an external, static Coulomb field, we have only two diagrams in lowest order

I.\[ p_2 = p_1 + k + q \]
\[ q = p_2 - p_1 - k \]

II.\[ q' = q = p_2 - p_1 - k \]
The amplitudes associated with these diagrams, $M_I$, $M_{II}$ are

$$M_I = \bar{u}_2 \varphi(q) \frac{1}{p^2 + m^2} \phi u_1 \sqrt{4\pi e^2}$$

and

$$M_{II} = \bar{u}_2 \varphi \frac{1}{p^2 - k^2 - m} \phi(q) u_1 \sqrt{4\pi e^2}$$

where $A_\mu(q) = \int A_\mu(x) e^{iqx} dx$

$$A_\mu = \left( \frac{Ze^2}{R}, 0 \right)$$

$$\varphi(q) = X_t 2\pi \int \frac{QE^2}{R} \delta(q_t) e^{-iQ^2 R^2}$$

$$= 2\pi \delta(q_t) \frac{Q e^2}{Q^2 e^2} \delta_t$$

If we took into consideration the effect of electron shielding of the field of the nucleus by $V(R) = \frac{Ze^2}{R} e^{-br}$ we'd obtain

$$\varphi(q) = \int \delta(q_t) \frac{4\pi e^2 Z Q}{R^2 + Q^2}$$

Substituting $\varphi$ into $M_I + M_{II}$, and squaring and multiplying by the appropriate kinematical factors we obtain the cross-section for pair production in the field of a nucleus. The result is an ugly mess which is discussed in Heitler, Quantum Theory of Radiation 8-26.

One simple result is that if the e- e+ pair are produced with relativistic energies, they are created primarily in the forward direction within the cone defined by the angle $\theta = \frac{m}{p}$. For smaller energies, the effect is less marked.
Another phenomenon of considerable interest is the emission of a photon by an electron in the field of a nucleus ("bremsstrahlung").

The matrix element for this process has the same form as that for pair production by a photon in the field of a nucleus.

γ's create pairs which emit γ's by bremsstrahlung, which in turn create pairs, ... thus creating a "shower."

The "common sense" infinity of quantum electrodynamics

It is not possible for scattering to take place without the emission of photons. That is because the field far away from the charge must change if the particle is deflected. It can change only if photons are emitted.

There is some interest, therefore, in an approximate formula for bremsstrahlung with the emission of low frequency photons.

Consider the emission of one photon. The relevant diagrams are

\[
\text{Rate} = 2\pi \int (E_\gamma + W - E_1) \frac{d^3 p_2}{(2\pi)^3 2E_1 2E_2} \frac{d^3 k}{2\omega (2\pi)^3} |M|^2
\]

\[
M = \left\{ \frac{\bar{u}_2 \not{p} + \not{p} u_1 + \bar{u}_2 \not{p} \not{p} - u_1}{p_2 + k - m} \right\} \frac{\not{u}_1}{\not{p}_1 - k - m} \frac{1}{\sqrt{4\pi \not{p}_1 \not{p}_2}}
\]
\[ M = \left( \frac{k^2 + k + m}{2p_k} \right) \sum \frac{\alpha}{2p_k - k} + \left( \frac{\alpha}{2p_k - k} \right) \sum \frac{u_1}{\sqrt{4\pi e^2}} \]

we are interested only in low-energy photons, so the two terms with arrows above them are small compared to the other terms in the expression, and can be neglected to an accuracy of the order \( \frac{k}{m} \)

Then \( M = \frac{\bar{u}_2}{2p_k} \left( -\frac{p \cdot e + 2p_k \cdot e + \vec{E}_k}{2p_k - k} \right) dA \) \( u_1 \), \( u_2 \) \( \bar{u}_2 \), \( \bar{u}_2 \) \( -2p_k \cdot k \)

(The X's indicate terms that cancel)

\[ = \frac{\bar{u}_2}{2p_k} \left( \frac{p \cdot e}{p_k - k} - \frac{p \cdot e}{p_k - k} \right) \sqrt{4\pi e^2} \]

\[ \text{Rate} = \frac{2\pi}{(2\pi)^3} \delta(E_f - E_i) \frac{d^3 p_k}{d^3 E} \sum \left( u_2 \delta(k) u_1^2 \sqrt{\frac{2}{(2\pi)^3} 2E_2 E_1} \right) \frac{(p \cdot e - p \cdot e)^2}{(p_k - k)^2} \frac{1}{4\pi e^2} \]

\[ \times \frac{d\omega}{\omega} = 2\pi z \frac{d\Omega_z}{(2\pi)^3} \]

\( A \) is the probability of scattering with no photons emitted, in lowest order to the approximation we've made \( \bar{e} \approx p - p_i \)

The term \( \bar{\delta} \bar{\delta} \) may be simplified

\[ \bar{p} \cdot \bar{e} = E_f W - p_i W \cos \theta_1 = W E_1 (1 - \cos \theta_1) \]

\[ \bar{\delta} = \frac{v_1 e - v_2 e}{1 - v_1 \cos \theta_1} - \frac{v_1 e - v_2 e}{1 - v_2 \cos \theta_2} = \frac{v_1 \sin \theta_1}{1 - v_1 \cos \theta_1} - \frac{v_2 \sin \theta_2}{1 - v_2 \cos \theta_2} \]

\[ = 0 \text{ if \( \bar{\delta} \) is perpendicular to the plane of scattering.} \]

... photons polarized in the plane of the collision.
Energy/unit time emitted in range \( dw \)

\[
= \text{same formula (without the factor of } \frac{1}{\nu} \text{)}
\]

which is the same as the classical expression.

**Problem:** Two soft photon emission \((k, l)\)

Show that \( \text{rate} = \text{probability of scattering with emission of no photon} \times \text{probability of emitting one photon } (k) \times \text{probability of emitting other photon } (l) \)

there is also a factor of \( \frac{1}{2} \) from statistics.

(i.e., show that the two photons are emitted with statistically independent probabilities)

**Classical case:** A classical charged particle moves with uniform velocity and is suddenly deflected. What is the radiation like?

Let the trajectory in space time be represented parametrically by the functions \( Z_\mu (x) \)

the current density

\[
\jmath_\mu (x) = e \int \delta^4 (x - Z(x)) \dot{Z}_\mu (x) \, dx
\]

vector potential satisfies the equation

\[
\Box A^{\mu}_\xi = \jmath_\mu (x)
\]

Take the fourier transform of both sides

\[
k^2 \Phi_\mu (k) = \int e^{-ikx} \jmath_\mu (x) \, dx
\]

\[
= \int e^{-ikx} \int Z_\mu (x) \delta^4 (x - Z(x)) \, dx \, dx \, dx
\]

\[
= \int e^{-ikx} Z_\mu (x) Z(x) \, dx = \jmath_\mu (k)
\]
Let the deflection take place at $x=0$, then

\[ \phi_0(x) = \phi_\mu \xi \]

where

\[ \phi_\lambda(x) = \phi_\mu x \]

Then

\[ j_\mu(k) = \int_0^\infty \phi_\mu e^{-ik \rho x} d\rho + \int_0^\infty \rho \phi_\mu e^{-ik \rho x} d\rho \]

\[ = i \left( \frac{\rho \phi_\mu}{k_1 - k} - \frac{\rho \phi_\mu}{k_2 - k} \right) \]

so

\[ \phi_\mu(k) = \frac{i}{k} \left( \frac{\rho \phi_\mu}{k_1 - k} - \frac{\rho \phi_\mu}{k_2 - k} \right) \]

This gives the classical expected energy in dw = $e^2 C(w) dw$

mean energy liberated in dw = quantum mechanically

mean no. of photons emitted = $e^2 C(w) dw$

**Emission of many soft photons:**

Rate for one photon emission = $A \left[ \omega \left( \frac{\omega \sin \theta_1 - \omega \sin \theta_2}{1 - \omega \cos \theta_1 \omega \cos \theta_2} \right) \right]^2 \frac{d\omega}{d\omega} \frac{4 \pi \omega}{(2 \pi)^3}$

Now to the same order in $\omega^2$, the cross-section for emitting no photons is

\[ \sigma = \sigma_A (C + 1) \]

where $C$ is a correction factor that comes from interference between the two diagrams.

which is of $O(e^{-A})$

$1- e^{-A}$, $C = O(e^2)$

6 for 1 photon emission = $\sigma_A (1 + C) \left( \frac{d\omega}{d\omega} \right) f(\theta_1) d\Omega_1$

6 for 2 photon emission = $\sigma_A (1 + C) \frac{1}{2} \left( \frac{d\omega}{d\omega} \right) f(\theta_1) d\Omega_1 \left( \frac{d\omega}{d\omega} \right) f(\theta_2) d\Omega_2$

6 for 3 photon emission = $\sigma_A (1 + C) \frac{1}{3!} \left( \frac{d\omega}{d\omega} \right) f(\theta_1) d\Omega_1 \left( \frac{d\omega}{d\omega} \right) f(\theta_2) d\Omega_2 \left( \frac{d\omega}{d\omega} \right) f(\theta_3) d\Omega_3$

etc.
\[ \text{Scattering with energy } E \text{ of } \alpha \text{ is} \]

\[ = 6A C' e^{\frac{1}{2} \int \frac{d\omega}{\omega} f(\omega) \omega^2} \]

The divergence ("infra-red catastrophe") comes from the fact that the integral in the exponential doesn't converge.

However \( C' = e^{\frac{1}{2} \int \frac{d\omega}{\omega} f(\omega) \omega^2} \) is the contribution of the low-energy virtual photons, so \( 6 A C' \) is finite.

The analysis implies that \( 6 A C' = 0 \).

Experimentally, there is an energy \( \Delta \) such that photons with total energy less than \( \Delta \) are not observed, so we measure \( 6 \) of emitting no photon with energy greater than \( \Delta \).

\[ 6(\epsilon(\omega) < \Delta) = 6(\text{no photon}) + 6(\text{one photon}) + 6(\text{two photons}) + \cdots \]

\[ = 6A C' \left( 1 + \int_0^\Delta \frac{d\omega}{\omega} f + \frac{1}{2} \int_0^\Delta \left( \frac{d\omega}{\omega} \int_0^\omega \frac{d\omega'}{\omega'} f(\omega') f(\omega) + \cdots \right) \right) \]

\[ = 6A e^{\int_0^\Delta \frac{d\omega}{\omega} f} \left( 1 + \frac{2d}{\pi} \log \frac{\Delta}{\epsilon} + \frac{1}{2} \rho_\Delta \right)^2 (\log \rho_\Delta)^2 + \cdots \]

\[ = 6A e^{-\frac{2d}{\Delta} \log \frac{\rho_\Delta}{\epsilon}} \]

\[ = 6A e^{-\frac{2d}{\Delta} \log \frac{m_D}{\epsilon}} = 6A \left( \frac{m_D}{\Delta} \right)^{-2d/\Delta} \]

We will show later how to calculate \( C' \).
Additional rules for calculating diagrams: $\text{e}^{-}\text{e}^{-}$ scattering

\[ M_A = \left(\frac{g_H}{4\pi F_2}\right)^2 \frac{\bar{\nu}_2 \gamma_{\mu_2} \nu_4 \gamma_{\mu_4}}{(p_1 - p_3)^2} \]

$A$ is the full amplitude for $\text{e}^-\mu^-$ scattering but not for $\text{e}^-\text{e}^-$ scattering because there is the exchange possibility:

\[ M_B = -\left(\frac{g_H}{4\pi F_2}\right)^2 \frac{\bar{\nu}_2 \gamma_{\mu_2} \nu_4 \gamma_{\mu_4}}{(p_1 - p_3)^2} \]

The minus sign in this contribution to the amplitude comes from the rule of Fermi statistics for electrons, because $B$ differs from $A$ only in that the two outgoing electrons are interchanged.

$\text{e}^-\text{e}^+$ scattering

The matrix element is the same as $A$ above.

What corresponds to the exchange diagram for $\text{e}^-\text{e}^-$ scattering?

The analogy is the annihilation diagram

This is the same as the first diagram rotated $90^\circ$ together with the exchange of two electron exit lines (3 and 4) $\Rightarrow$ relative minus sign in the amplitude.
That this additional term in the $\theta^-\theta^+$ amplitude is present has been checked experimentally to better than 10% by observations of the energy level spectrum of positronium. The "annihilation force" causes a shift in the energy levels from those expected on the basis of pure coulomb attraction.

A technique for calculating unpolarized cross-sections:

\[
\text{Rate} = \frac{1}{2} |\bar{u}_2 N u_1|^2
\]

Suppose the incoming particle is unpolarized and the spin of the final particle is not measured.

Then rate = \( \sum_{\text{spins}} \left| \frac{1}{2} \bar{u}_2 N u_1 \right|^2 = \sum \bar{u}_2 N u_1 (\bar{u}_2 N u_1)^* \)

Let \( \overline{N} \) be defined by \( \left( \overline{\vec{N}} \right)^* = \left( \vec{N} \overline{g} \right) \) (it is easy to see \( \overline{N} = \gamma_5 N \gamma_0 \) )

Then rate = \( \sum \left| \frac{1}{2} \bar{u}_2 N u_1 \right|^2 \)

\[ X = \sum_{\text{spins}} \frac{1}{2} \bar{u}_2 N \frac{p_f + m}{2m} u_1 \overline{u}_1 \overline{N} u_2 \]

\[ = \sum_{\text{spins}} \frac{1}{2} \bar{u}_2 N \frac{p_f + m}{2m} u_1 \overline{u}_1 \overline{N} u_2 \]

Since \( (p_f + m) u_3 = 0 \), \( (p_f + m) u_4 \)

Now \( \frac{1}{2} u_1 \overline{u}_1 = 2m I \), \( I = \text{identity matrix} \)

So \( X = \sum_{\text{spins}} \frac{1}{2} \bar{u}_2 N \frac{p_f + m}{2m} \overline{N} u_2 \frac{2}{r_1} \bar{u}_1 N \frac{p_f + m}{2m} u_1 \overline{N} \]

\[ = \text{trace} \left( N (p_f + m) \overline{N} (p_f + m) \right) \]

Add: \( Tr I = 4, \ Tr \gamma_\mu = 0, \ Tr \gamma_\lambda \gamma_5 = 0, \ Tr (\text{product of any } \gamma_i) = 0, \ Tr \gamma_5 = 0 \)
INTENSITY DISTRIBUTION OF BREMSSTRAHLUNG RADIATION

Consider the factor $\frac{P'c}{P'h} - \frac{P''c}{P''h}$, which may be interpreted as the probability that an electron emit a soft photon of momentum $k$ in going from momentum $P'$ to momentum $P''$.

$$P'c = k \left[ E_1 - P_{1\omega} \theta_1 \right]$$

Hence we obtain a function of shape

$$\frac{\nu \sin \theta_1}{1 - \nu \omega \theta_1}$$

A non-relativistic interpretation of this is to examine the electron before and after.

The electron is accelerated in the direction $\Delta \nu$ by the scattering, and hence must emit light. Detailed comparison of the above result with classical results (see Jackson, around p. 472) is left as an exercise for the student.

In the highly relativistic region, where $\nu \sim c$, $P = \sqrt{E^2 - m^2} \rightarrow E - \frac{m^2}{2E}$

Thus

$$\frac{P \sin \theta}{E - \nu c} \rightarrow \frac{E \theta}{E \left[ \theta^2 + \frac{m^2}{E^2} \right]} \sim \frac{\theta}{\theta^2 + \left( \frac{mc}{E} \right)^2}$$

This takes on the shape

When the deflection angle due to scattering is big compared to $\nu \gamma$, then either one or the other of the above terms is big, and the radiation emitted takes on the shape plotted above.

For small angle scattering the shape gets more complicated.

Notice that if you exit two identical particles, some care must be taken in obtaining a final answer.

Suppose you wish to measure the differential cross section and total rate for a process in which 2 photons are emitted, along with some other stuff. Assume you have one photon counter.
Every time any photon hits the counter, it records a count. For each such count, the other photon could have gone anywhere. Thus the rate measured by the counter when it is placed at angle $\mathcal{J}_i$ is $\frac{d\sigma}{d\mathcal{J}_i}$ - the differential cross section for particle 1, where we define particle 1 as the one that hit the counter.

That is, the answer got by integrating the familiar expression over $\mathcal{J}_2$ gives the cross section to measure the emission of some photon into angle $d\mathcal{J}_i$. We have no way of telling whether this is the "red" photon or the "blue" one in the Feynman diagram, and we don't care.

If we now integrate over angles $\mathcal{J}_i$, we are summing cases like

$$\begin{array}{c}
\begin{array}{c}
\int d\mathcal{J}_i, \quad d\mathcal{J}_2
\end{array}
\end{array}$$

where 1 and 2 have been assigned simply on the basis of the counter position.

These cases are indistinguishable.

To get a total rate, all we want is the probability that the reaction went.

$$\int d\mathcal{J}_i \int d\mathcal{J}_2 = \frac{d\sigma}{d\mathcal{J}_2}$$

gives twice this probability. Hence, in obtaining a total rate, we must divide the final integral by $m_i!$, where $n$ is the number of indistinguishable particles in the final state.

MORE ABOUT THE DIRAC EQUATION

Velocity in Dirac language

$$\hat{H} = \beta m + \frac{e}{\hbar} (\hat{\mathbf{p}} - \hat{\mathbf{A}}) t \nu \quad \Rightarrow \quad \hat{x} = i \left[ \hat{H}, \hat{x} \right] = \alpha \hat{x}$$

Hence one would like to find some interpretation for $\alpha$ in terms of particle velocity.

However, $\alpha = 1$; thus the eigenvalues are $\pm 1$, and this seems to imply that a measurement of the velocity in the x direction would yield $\pm c$.

This has caused some concern.

Dirac's explanation:

To measure the velocity, you must measure the position twice. But the first accurate measurement of position would make the momentum totally uncertain, and thus you would measure $c$. 

A measurement of average velocity, given by
\[ v^2 = \frac{\left(\frac{x'_t + T - x_t}{T}\right)^2}{\frac{p}{\sqrt{p^2 + m^2}}} \]
is roughly
\[ \frac{p}{\sqrt{p^2 + m^2}} \]
(can you show this?) but measurement of the instantaneous value would
come out wrong.

Feynman's explanation

Dirac's logic is wrong. For \( [\alpha, \beta \gamma] = 0 \) implies they can be measured
simultaneously. If one fumbles around, one finds that it is indeed possible
to construct a solution of the Dirac equation which has a definite eigenvalue of
\( \alpha_x \) and \( \beta \gamma \); but such a solution does not have a definite energy. In order to
be a solution with definite momentum \( p \), it must have energy \( \pm \sqrt{p^2 + m^2} \). Hence
the solution in question must be some mixture of electron and positron.

Given that the system must have a certain net charge, the Dirac equation can
describe systems with one electron or one electron and pair(s); thus \( \alpha \gamma \) is not
necessarily a single particle operator and might better be thought of as
a sort of current density.

The commutators of such operators as \( \alpha_x \beta \gamma \), \( \beta \alpha_x \beta \gamma \) etc. with the Hamiltonian
have not been completely explored.

RELATIVISTIC INVARIENCE OF DIRAC EQUATION

So far we have calculated all answers with a given set of \( \gamma \) matrices. But we have
assigned a Lorentz index to these matrices and treated them like a 4 vector. Now,
then, do we know we are using the right \( \gamma \)'s? Why don't we use some \( \gamma'_\mu = a_\gamma \gamma \mu \)
with the transformation \( a_\gamma \gamma \mu \) depending on the frame of the problem?

A clever answer to this is to say that if you read your Dirac equation off some
moving system with a telescope, it wouldn't change anything. This argument, however,
doesn't get to the core of the problem. The explanation of this apparent paradox
is that \( \gamma \mu \) and \( a_\gamma \gamma \mu \) are related by an equivalence transformation! Provided they are
used with solutions of the Dirac equation that are transformed in the same way,
all answers will be independent of the representation used.

To discover in general the conditions that we have the same physics with different
matrices
\[
\left[ \gamma_{\mu} \left( i \nabla_{\mu} - A_{\mu} \right) + m \right] \gamma = 0
\]

Define \( \gamma'' = S \gamma \), where \( S \) is a matrix of constants

Then \( \left[ \gamma'' \left( i \nabla_{\mu} - A_{\mu} \right) + m \right] \gamma'' = 0 \) will be equivalent to the above

if \( S \gamma'' S^{-1} = \gamma'' \)

\( \gamma_{\mu} = S^{-1} \gamma'' S \)

This sort of transformation preserves all algebraic relations between the matrices (in particular their commutation relations)

However, we must demand one more thing in order that the physics be unchanged.

Matrix elements must also be preserved. \( \gamma A \gamma = \gamma'' A'' \gamma'' \)

Hence \( S = \gamma_{o} S^{t} \gamma_{o} \) must equal \( S^{-1} \)

Then an equivalence transformation by any \( S \) such that \( S = S^{-1} \) leaves everything unchanged.

**Relativistic Adjoint**

\[ (EMq)^{t} = \overline{S} M \overline{f} \]

defines \( \overline{M} = \beta M^{t} \beta \)

For matrices \( A, B, C \) and constants \( \alpha \),

\[ \begin{vmatrix} \alpha & A & \cdots & B \\ C & \cdots & \cdots & \cdots \end{vmatrix} = \alpha^{t} C \cdots B^{t} \]

\( \overline{s_{\mu}} = s_{\mu} \)

Hence the relativistic adjoint of any number of \( \gamma \) matrices is got simply by reversing their order

**Examples of Useful Equivalence Transformations**

1) Define \( \gamma \) by \( \gamma_{\mu} = \gamma_{\mu} m h \)

Then the velocity transformations assume a form similar to that for rotations

For the velocity transform in \( z \) direction,

\[ \begin{align*}
\gamma_{z} & \rightarrow \left( \cosh w \right) \gamma_{z} - \gamma_{t} \sinh w \\
\gamma_{x} & \rightarrow \gamma_{x} \\
\gamma_{y} & \rightarrow \gamma_{y} \\
\gamma_{t} & \rightarrow \left( \cosh w \right) \gamma_{t} - \left( \sinh w \right) \gamma_{z}
\end{align*} \]

If \( S = e^{w_{z}} \gamma_{z} \gamma_{t} \), then \( S^{-1} = e^{-w_{z}} \gamma_{t}^{t} \gamma_{z} = \overline{S} \)

\[ \overline{s} = \gamma_{0} S^{t} \gamma_{0} = \gamma_{0} e^{-w_{z}} \gamma_{t}^{t} \gamma_{z} = \gamma_{0} e^{-w_{z}} \gamma_{t}^{t} \gamma_{z}^{t} \gamma_{t}^{t} \gamma_{z}^{t} = e^{w_{z}} \gamma_{t}^{t} \gamma_{z}^{t} \gamma_{t}^{t} \gamma_{z}^{t} = e^{-w_{z}} \gamma_{t}^{t} \gamma_{z}^{t} \gamma_{t}^{t} \gamma_{z}^{t} = e^{-w_{z}} \gamma_{t}^{t} \gamma_{z}^{t} \gamma_{t}^{t} \gamma_{z}^{t} = e^{w_{z}} \gamma_{t}^{t} \gamma_{z}^{t} \gamma_{t}^{t} \gamma_{z}^{t} = \overline{S} \]

Expansion shows that

\[ S = \cosh w_{z} \gamma_{z} + \gamma_{t} \gamma_{z} s_{z} \sinh w_{z} \]

\[ S^{-1} = \cosh w_{z} - \gamma_{t} \gamma_{z} s_{z} \sinh w_{z} \]

2) Show that the corresponding rotation operator is \( S = e^{w_{z} \gamma_{z} \times \gamma_{y}} \)

Then
\[ \psi' (x, y, z, t) = e^{\frac{i}{2} \int \mathcal{J} \cdot \mathbf{D} \, d^4 \tau} \psi (x, y, z, t) \]

(work through the algebra to get some familiarity)

Using this definition of \( \mathcal{J} \), one can show for the Dirac equation that if there is no vector potential, and if \( V(\tau) = \nu(\nu) \), then angular momentum is a constant of the motion.

**PROBLEM:** See how much you can get from the non-relativistic point of view of the formula

\[ 2 m A m = (\text{const}) \ e^2 \frac{4 \pi}{\hbar} \int \frac{[\mathbf{u} \cdot \nabla (m^2 - m^2 \mathbf{u} \cdot \mathbf{u}) \mathbf{u}]}{[m^2 - m^2]} d^4 \mathbf{u} \]

for the correction in mass of an electron of momentum \( p \) due to second order interactions with the electromagnetic field.

**Hint:** If you start with transverse waves you will get only the \( \gamma_\tau \) piece. There is also a longitudinal contribution to the self energy, which looks like \( \frac{e^2}{r_{\tau}} \).

**Computational Aids**

Spin summations for electrons and positrons:

\[ \sum_{\text{spins}} \mathbf{u}_2 \mathbf{A} \mathbf{u}_1 \mathbf{B} \mathbf{u}_2 = \sum_{\text{spins}} \mathbf{u}_3 \mathbf{A} (\mathbf{u}_3 \mathbf{u}_1 \mathbf{u}_2) \]

Thus

\[ \sum_{\text{spins}} |\mathbf{u}_2 M \mathbf{u}_2|^2 = \mathbf{u}_2 M (\mathbf{u}_2 \mathbf{u}_1 \mathbf{u}_2) \mathbf{u}_2 \]

To calculate the traces:

\[ T_3 \mathbf{1} = 4 \]

\[ T_3 \gamma_\tau = 0 \]

Trace of any odd number of \( \gamma \)'s is 0.

**Terminology**

\[ \text{Trace} (X) = \text{Spur} (X) \]

\[ \text{Tr} (\mathbf{AB}) = \text{Tr} (\mathbf{BA}) \]

\[ \text{Tr} (\mathbf{ABC}) = \text{Tr} (\mathbf{BCA}) \]

\[ \text{Sp} (\mathbf{A}) = 4 \mathbf{a} \cdot \mathbf{b} \]

\[ \text{Sp} (\mathbf{A} \mathbf{B} \mathbf{C}) = 4 (a \cdot b c + a \cdot b c d + a \cdot b c d \mathbf{e}) \]

Always try to use the Dirac equation \( \mathbf{p} \mathbf{u} = m \mathbf{u} \)

\[ \mathbf{p} \gamma = -m \mathbf{p} \gamma \]

to reduce the number of \( \gamma \) matrices in the matrix element before taking traces.

There are tricks for taking traces of any number of \( \gamma \) matrices, but in practice even

6 \( \gamma \) matrices in a row lead to headaches. If you find you have to take traces of more than 4 \( \gamma \)'s, it is worth trying to find some simplification. One helpful thing in problems involving photons is
Summing over Polarizations

Suppose you get something like
\[ \sum_{\text{light polarizations}} S \rho \left[ \begin{array}{c} \rho_i \rightarrow \rho_i \\
\rho \end{array} \right] \]

Provided you have not used special properties of \( e_i \) in some gauge (like assuming it has no time component) then the summation over transverse directions of the light can be replaced by
\[ \begin{array}{c} \rho_i \rightarrow \rho_i \\
\rho \end{array} \rightarrow \begin{array}{c} \gamma_m \rightarrow \gamma_m \\
\rho \end{array} \]

where summation is implied (also multiply the mess by -1)

Reason:

Gauge invariance implies that replacing \( e_i \) by \( h_i \) should give 0. If \( e_i \leftrightarrow (\rho_i, e_\rho, \rho) \) then \( h_i = (k_x, k_\rho, \rho, k) \)

and gauge invariance says
\[ k \left[ \begin{array}{c} (\gamma_x \rightarrow \gamma_x) \\
(\gamma_\rho \rightarrow \gamma_\rho) \end{array} \right] = 0 \]

Hence \( \gamma_m \rightarrow \gamma_m \) boils down to
\[ -\frac{1}{2} \left[ (\gamma_x \rightarrow \gamma_x) + (\gamma_\rho \rightarrow \gamma_\rho) \right] \]

which is what you expect from the property \( e^j_i * e^j_k = \delta^j_i \delta^i_k \) (i and k are spacelike)

Once the quantity is in the form
\[ \begin{array}{c} \gamma_m \\
\rho \end{array} \rightarrow \begin{array}{c} \gamma_m \\
\rho \end{array} \]

one then uses
\[ \gamma_m \gamma_m = 4 \]
\[ \gamma_m \gamma_{\rho} = -2 \rho \]
\[ \gamma_{\rho} \gamma_{\rho} = 4 \rho \cdot \rho \]

Thus each summation over photon polarizations reduces the number of \( \gamma \) matrices in the trace by 2

**Summary of Rules**

\[ \text{Prob of Transition} = \frac{2 \pi}{\text{sec}} f(E) \prod_{i=m} \left[ \frac{1}{2 E_i} \right] \prod_{\rho} \left[ \frac{1}{2 \rho_i} \right] | \mathcal{M} |^2 \]

\[ = (2 \pi)^4 k \left[ \prod_{i=m} \left( \frac{1}{2 E_i} \right) \prod_{\rho} \left( \frac{1}{2 \rho_i} \right) \right] \frac{d^4 \rho_i}{(2 \pi)^4} \frac{d^4 \rho_i}{(2 \pi)^4} | \mathcal{M} |^2 \]

where \( \mathcal{M} \) is a relativistically invariant matrix

Using the normalization \( \overline{u} u = 2 m \quad e \mu e \mu = -1 \)

gives as the sum over states
\[ \sum_{\rho} \{ \left( \rho^2 + m^2 \right) \overline{u} u \left( \rho^2 + m^2 \right) \mathcal{M} \}

Calculation of \( \mathcal{M} \)
Electron propagator \[ \frac{i}{\not{p} - m + i\varepsilon} \]
Photon propagator \[ \frac{i}{\not{p} + i\varepsilon} \]
Spin 0 meson propagator \[ \frac{i}{p^2 - m^2 + i\varepsilon} \]

Couplings:

Fermions to a real photon \[ -i \frac{\not{p}}{\sqrt{4\pi\alpha}} \]
Fermions to a potential \[ \Phi(q) = \int e^{-i\not{q}\cdot x} \mathcal{A}(x) d^3x \]
To a virtual photon \[ (-i\sqrt{4\pi\alpha}) \gamma_{\mu} \ldots - \gamma_{\mu} (-i\sqrt{4\pi\alpha}) \]
To pseudoscalar meson \[ \sqrt{4\pi\alpha} \gamma_5 \ldots - \gamma_5 \sqrt{4\pi\alpha} \]
To scalar meson \[ \sqrt{4\pi\alpha} \]

For a closed loop of electrons, the rule is \[ -Sp \]

If there is an indeterminate momentum, sum by
\[ \int \frac{d^4p}{(2\pi)^4} \]
The correction to the energy of a free electron due to the interaction of the electron with its own electromagnetic field is represented in lowest order perturbation theory by the diagram:

\[ \Delta E^2 = 4\pi e^2 \int u \, \delta \mu \left( \frac{1}{p-k-m} \right) \frac{d^4k}{(2\pi)^4} \]

\[ = \frac{4\pi e^2}{(2\pi)^4} \int u \, \delta \mu \left[ \frac{(p-k+m)}{(p-k)^2 - m^2 + i\varepsilon} \right] \frac{d^4k}{k^2 + i\varepsilon} \]

This formula may be obtained from the expression for the second order correction to the energy of a state in ordinary perturbation theory:

\[ \Delta E_i = \sum_n \frac{H_i \text{H}^n_i}{E_i - E_n} \]

with \( H' = \text{perturbation} = e(\xi - A + \phi) \) in our case.

\( \Delta E^2 \) will turn out to be logarithmically divergent.

\[ \Delta E^2 = \frac{4\pi e^2}{(2\pi)^4} \int u \, \frac{(-2p^2 + 2k^2 + 4m)}{k^2 + i\varepsilon} \frac{d^4k}{(p-k)^2 - m^2 + i\varepsilon} \]

\[ = \frac{4\pi e^2}{(2\pi)^4} \int \frac{(-4m^2 + 4p \cdot k + 8m^2)}{(k^2 + i\varepsilon) \left[ (p-k)^2 - m^2 + i\varepsilon \right]} \frac{d^4k}{m} \]
For the correction to the rest energy, put $\beta^2 = 0$

Then $p.k = \omega m$

$$(p - k)^2 = k^2 + m^2 - 2\omega m$$

and

$$\Delta m^2 = \frac{4\pi C^2}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \frac{m(m+\omega)}{(w^2 - k^2 + i\epsilon)(\omega^2 - K^2 - 2\omega m - i\epsilon)}$$

$$= \frac{\epsilon^2}{\pi^2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \int_{-\infty}^{\infty} \frac{dK}{K^2} \frac{m(m+\omega)}{(w^2 - K^2 + i\epsilon)(\omega^2 - K^2 - 2\omega m + i\epsilon)}$$

The integrand has poles as a function of $\omega$ at

$$\omega = K - i\epsilon = \omega_1$$

$$\omega = -K + i\epsilon = \omega_2$$

The $\omega$ plane

$$\omega_1$$

$$\omega_2$$

$$\omega_3$$

$$\omega_4$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{(w - m - \omega_1)(w - m + \omega_1)(w - \omega_1)(w - \omega_2)} = \int_{-\infty}^{\infty} \frac{d\omega}{(w - m + \omega_2)(w - m + \omega_1)(w - \omega_2)(w - \omega_3)}$$

where $C$ is the contour shown.

The contribution from the large semicircle vanishes in the limit of infinite radius.
\[
\int = 2\pi i \left( \text{sum of residues \& poles of integrand inside } C \right)
\]

\[
= 2\pi i \left\{ \frac{m-K}{(-2K) \left[ \sqrt{K^2+m^2}+m-K \right] \left[ -\sqrt{K^2+m^2}-m-K \right]} \right\} \quad \text{res. at } w_C
\]

\[
+ 2\pi i \left\{ \frac{2m - \sqrt{K^2+m^2}}{(-2 \sqrt{K^2+m^2})[m-K-\sqrt{K^2+m^2}][K+m-\sqrt{K^2+m^2}]} \right\} \quad \text{res. at } w_q
\]

\[
= 2\pi \left\{ \frac{K-m}{4K^2m} + \frac{1-\frac{2m}{\sqrt{K^2+m^2}}}{-4m(\sqrt{K^2+m^2}-m)} \right\}
\]

\[
2m \Delta m = \frac{e^2}{\pi} \times 4m \times 2\pi i \int_0^\infty dK \left\{ \frac{K-m-K^2}{4m} \left( 1-\frac{2m}{\sqrt{K^2+m^2}} \right) \right\}
\]

\[
= \frac{2e^2i}{\pi} \int_0^\infty I(K) dK
\]

for small \( K \), \( I(K) \) is regular.

For large \( K \), \( I(K) \approx K - m - K \left( 1 - \frac{2m}{K} + \frac{m^2}{K^2} \right) \)

\[
= K - m - K \left( 1 - \frac{m}{K} + \frac{3m^2}{2K^2} \right) = \frac{3M^2}{2K}
\]

- If we integrate from 0 to \( \Lambda \), then the leading term in \( \Lambda \) is

\[
\frac{2e^2i}{\pi} \cdot \frac{3M^2}{2} \log \Lambda \quad \text{which is, as we asserted, logarithmically infinite.}
\]

Of course, other cutoff procedures are possible, and we consider now one in more detail.
The cutoff procedure we will adopt is relativistically invariant. It consists of modifying the propagator at high energies.

I.e., replace \( \frac{1}{k^2} \) by \( \frac{C(k^2)}{k^2} \)

where \( C(k^2) \rightarrow 1 \) when \( k^2 < \Lambda^2 \)

\( \Lambda = \text{cutoff} \)

we choose \( C(k^2) = \frac{-\Lambda^2}{k^2 - \Lambda^2} \rightarrow \frac{1}{k^2} \) as \( k^2 \rightarrow \infty \).

Suppose now that with this cutoff all integrals are convergent (they are not). If the answer for a problem depends on \( \Lambda \), we consider that we don’t know the answer. But if as \( \Lambda \rightarrow \infty \), a finite limit ensues, then we consider the answer known in quantum electrodynamics.

**Discussion of the infinite correction to the electron mass**

\( M_0 \), the mass that appears in the Dirac Equation is not the physical mass of the electron, because there are corrections to \( M_0 \) due to the interaction of the electron with the electromagnetic field.

Theoretically, \( m^{\text{exp}} = \frac{m_0^2 + \frac{3e^2}{4\pi} \frac{m_0^2}{\log \frac{\Lambda}{m_0}}}{\frac{1}{\sqrt{2}}} + O(e^4) \)

The problem is of course that \( \Delta m^2 = \infty \)

\( m^{\text{exp}} \) is the experimental, measured mass.

**Aside:** If you look at the mass difference between the neutral and charged pion in the same way, you will obtain a quadratically divergent infinity.

The electromagnetic structure of the pion is simply not known.
Let's look at a problem that we can solve in higher order.

Calculate the shift in a particular energy level of the hydrogen atom due to the possibility of virtual emission and reabsorption of a photon by the electron.

Obtain
\[ \Delta E_n = E_1 + i \frac{\Gamma}{2} \]

\( E_1, \Gamma \) are real — \( \Gamma \) gives the lifetime of the state (rate of disintegration).

Ground state:
\[ \text{Re } \Delta E_0 = E_1 = \Delta M_0 + \frac{m e^4}{8 \pi^2} \]

If you re-express the energy in terms of the physical mass \( M_{\text{exp}} \) rather than \( M_0 \), then the correction is finite as \( \lambda \to \infty \).

There are ways of by-passing the infinities such as dispersion theory, but at the present time there is no method that is uniformly better than any other method for dealing with higher order corrections in gad.
Theory of $\beta$ Decay:

Ref. PR 109, 193 (1958), Feynman & Gell-Mann

There are 2 different kinds of neutrinos: $\nu_e, \nu_m$

Muon decay occurs as: $\mu \rightarrow e + \overline{\nu}_e + \nu_m$

The neutrinos are massless and obey Dirac's equation. $p\nu = 0$

$\nu^2 = 0$

Muon decay, $\beta$ decay, and other process are described by the weak coupling:

$$\text{Rate} = \frac{G}{m^2}$$

where $m$ is derived from a 4 particle point interaction

$$\begin{align*}
  a+c &\rightarrow b+d \\
  \overline{a} d &\rightarrow \frac{1}{2} \overline{b} c
\end{align*}$$

$$M = G\sqrt{8} (\overline{u}_d (\bar{\nu}_e a u_c)) (\overline{u}_b (\bar{\nu}_m a u_a)) = G\sqrt{8} (\overline{d c}) (\bar{b} a)$$

Abbreviated notation

$\sqrt{8}$ is there for historical reasons.

$$a = \frac{1 + i\gamma_5}{2}$$

$G$ is a constant with dimensions: $6M_p^2 = 1.01 \pm 0.1 \times 10^{-5}$

$M_p = \text{mass of proton}$

For particular weak processes, one must pick $a, b, c, d$ suitably.

For $\mu$ decay: $(\overline{e} \nu_e) (\overline{\nu}_\mu \mu)$

and that is the same as for $\nu_e + \mu \rightarrow \nu_\mu + e$
Problem: Calculate the shape of the energy spectrum in $\mu$ decay.

Aside: There is the IWB (intermediate vector boson) theory which says that $\mu + \nu_e \rightarrow e + \nu_\mu$, for example, is

\[ e \quad \mu \] 
rather than \[ e \quad \nu_\mu \]

i.e., that the interaction is mediated by a vector boson, but that the point interaction description is a good approximation because the IWB (called the $W$ particle) is massive, and therefore corresponds to a short range force

The interaction is written $G \sqrt{8} \, J^\mu \, J^\mu$

where $J^\mu = (\bar{\nu}_e \, \gamma^\mu \, \nu_\mu) + (\bar{u} \, \gamma^\mu \, d) + J^\mu$ hadrons

$J^\mu$ hadron \[ \quad \text{weak current of strongly interacting particles (e.g., } u, p, \pi, K, \lambda, \ldots \) \]

Problem: Calculate the rate of any weak process that interests you.

Aside: One can also do electromagnetic corrections to weak processes. For example, in $\mu$ decay, neutron $\beta$ decay

\[ \bar{\nu}_e \quad \mu \] 
there is the diagram \[ \nu \]
Returning now to the problem of infinities in qed, we write

\[ C(k^2) = \frac{-\lambda^2}{k^2 - (k^2 - \Lambda^2)} = \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \]

The first term is the one obtained from a massless photon, the second is obtained from a photon of mass M that couples to an imaginary charge. The charge is imaginary because \( \omega = 4\pi e^2 \) is now \(-4\pi e^2\). The imaginary charge implies that the Hamiltonian is non-Hermitian, and so unitarity is violated.

In the limit \( \Lambda \to \infty \), things seem to be OK, but that has never been proved.

I do not believe that everything has been straightened out. The writers always have a weak point in their arguments where they cannot prove anything.

**Question:** Is the final answer consistent with unitarity?

The procedures \( \{ M_0 \to M_{\text{exp}} \} \) in all calculations does do not straighten \( \frac{1}{k^2} - \frac{C(k^2)}{k^2} \) not straighten out all infinities.

**Ex.: e-e scattering in 4th order**

\[ M = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \]

The 4th-order diagrams give a contribution \( e^2 (1 + e^2 \log \frac{\Lambda}{M_0}) \delta \mu - \delta \mu \).
If then we identify

\[ e_{\text{exp}}^2 = e_{\text{th}}^2 (1 + e_{\text{th}}^2 \log \frac{\Lambda'}{\Lambda}) \]

and say that \( e_{\text{exp}} \) = observed charge

\( e_{\text{th}} \) = unrenormalized ("theoretical") charge

we once again have an infinite renormalization.

If now we find the rate, in any order, of a process

\[ \Gamma (m_0, e_{\text{th}}, \Lambda, \Lambda') = G (m_{\text{exp}}, e_{\text{exp}}, \Lambda, \Lambda') \]

then

\[ \lim_{N \to \infty} \lim_{N' \to \infty} G (m_{\text{exp}}, e_{\text{exp}}, \Lambda, \Lambda') \]

exists.

The process of finding

\( M_{\text{exp}} = M_{\text{exp}} (m_0, \Lambda) \)

and

\( e_{\exp} = e_{\exp} (e_{\text{th}}, \Lambda') \)

is known as mass and charge renormalization.