N. E. 551: NUCLEAR REACTOR INSTRUMENTATION AND CONTROL
(REACTOR DYNAMICS)

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REFERENCES*

1.) A. Z. Akcasu, N.E. 551 course notes (1968)

2.) G. Bell and S. Glasstone, *Advanced Nuclear Reactor Theory* (in press)

3.) M. Ash, *Nuclear Reactor Kinetics* (McGraw-Hill, 1965)

4.) R. Keepin, *Physics of Nuclear Kinetics* (Addison-Wesley, 1965)

5.) M. A. Shultz, *Control of Nuclear Reactors and Power Plants* (McGraw-Hill, 1961)

6.) L. E. Weaver, *Reactor Kinetics and Control*, A.E.C. Sym. Series, No. 2, 1963

* All on reserve in Phoenix library.

TENTATIVE OUTLINE

1.) Elementary Introduction to Reactor Kinetics

2.) Derivation of Point Reactor Kinetics Equations
    Transport equation, delayed neutrons
    Discussion of feedback mechanisms
    Special forms of kinetic equations
    Mathematical preliminaries
    Derivation of point kinetic equations

3.) Solution of Point Kinetics Equations without Feedback
    Standard forms
    Inverse method
    exact solutions
    Approximation schemes
    Numerical solutions

4.) Solution of Point Kinetics Equations with Feedback
    Mathematical description of feedback
    stability theory
    large power excursions
    determination and use of transfer functions

5.) Spatially-dependent Kinetics

I. ELEMENTARY INTRODUCTION TO REACTOR DYNAMICS

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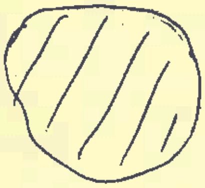
References: Lamarsh, Chapter 12
Glasstone & Edlund, Chapter 10

1.1. REACTOR KINETICS IN ABSENCE OF DELAYED NEUTRONS

Consider a bare, homogeneous reactor as described by one-speed diffusion theory



$$\frac{1}{v}\frac{\partial\phi}{\partial t} = D\nabla^2\phi - \Sigma_a\phi(\underset{\sim}{r},t) + S(\underset{\sim}{r},t)$$

Assume:
i.) constant cross sections (in $\underset{\sim}{r}$ and t)
ii) only "prompt" neutrons (neutron emission occurs instantaneously with fissions)
iii.) only sources due to fissions

$$S(\underset{\sim}{r},t) = P_{NR}P_{re}\nu\Sigma_f^{th}\phi(\underset{\sim}{r},t) = k_\infty P_{NFL}\Sigma_a\phi(\underset{\sim}{r},t)$$

$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a}\epsilon P_{re}$
$\eta f\epsilon P_{re}$
$\frac{\nu\Sigma_f}{\Sigma_a^F}\frac{\Sigma_a^F}{\Sigma_a^T}$ P_{re}

[This assumes instantaneous slowing down.
As example, $P_{NFL} = e^{-B_g^2\tau}$]

Hence the time dependence of the flux is described by

$$\frac{1}{v}\frac{\partial\phi}{\partial t} = D\nabla^2\phi + \Sigma_a\left[k_\infty P_{NFL} - 1\right]\phi(\underset{\sim}{r},t)$$

subject to some initial condition

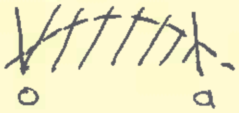
i.c.: $\phi(\underset{\sim}{r},0) = \phi_0(\underset{\sim}{r})$ all $\underset{\sim}{r}\in$ reactor

and the usual extrapolated boundary condition

b.c.: $\phi(\underset{\sim}{\tilde{R}},t) = 0$, $t \geq 0$

Hence we can study the dynamics of this very simple reactor model by merely solving this initial value problem.

EXAMPLE: Slab reactor of width $a = \tilde{a}$



As you will find in the first problem set

$$\phi(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\, e^{-vD\left[\frac{n^2\pi^2}{a^2} - B_m^2\right]t}$$

where

$$B_m^2 \equiv \frac{k_\infty P_{NFL} - 1}{L^2} \qquad \text{"material buckling"}$$

and

$$c_n = \frac{2}{a}\int_0^a dx \sin\frac{n\pi x}{a}\, \phi_0(x)$$

Notice that for long times, the mode with the largest exponential

$$\max\left\{ vD\left[B_m^2 - \frac{n^2\pi^2}{a^2}\right]\right\} = vD\left[B_m^2 - \frac{\pi^2}{a^2}\right] = vD\left[B_m^2 - B_g^2\right]$$

dominates such that

$$\phi(x,t) \sim c_1\left(\sin\frac{\pi x}{a}\right) e^{-vD\left[B_g^2 - B_m^2\right]t}$$

where we define

$$B_g^2 \equiv \frac{\pi^2}{a^2} \qquad \text{"geometric buckling"}$$

Notice:

i.) If $B_g^2 = B_m^2$, $\phi(x,t)$ approaches a steady-state distribution

Hence

$B_g^2 > B_m^2$ subcritical
$B_g^2 = B_m^2$ critical
$B_g^2 < B_m^2$ supercritical

ii.) For large times, the flux assumes a separable form

$$\phi(x,t) = c \sin\frac{\pi x}{a}\, e^{-vD\left[B_g^2 - B_m^2\right]t} = \psi(x)\, T(t)$$

-- that is, every point in the reactor experiences the same time behavior

Let's return now and apply some of these ideas to the more general reactor problem we were considering. As we have seen under certain circumstances (e.g. long times), the time-dependent flux will assume a separable form in space and time.

Assumption: We can <u>approximate</u> the flux as a separable function of space and time

$$\phi(\underset{\sim}{r},t) = v\, n(t)\, \Psi(\underset{\sim}{r})$$

where $\Psi(\underset{\sim}{r})$ is the everywhere positive solution of the Helmholtz equation

$$\nabla^2\Psi(\underset{\sim}{r}) + B_g^2\,\Psi(\underset{\sim}{r}) = 0$$

subject to b.c. $\Psi(\tilde{\underset{\sim}{R}}) = 0$, and B_g^2 is the geometric buckling for the reactor geometry of interest.

Now let's substitute this form into the diffusion equation

$$\frac{1}{v}\, v\,\Psi(\underset{\sim}{r})\frac{dn}{dt} = v D\, n(t)\underbrace{\nabla^2\Psi(\underset{\sim}{r})}_{-B_g^2\Psi(\underset{\sim}{r})} + v\Sigma_a\left[k_\infty P_{NFL} - 1\right] n(t)\,\Psi(\underset{\sim}{r})$$

or

$$\frac{dn}{dt} = \left\{-vDB_g^2 + v\Sigma_a\left[k_\infty P_{NFL} - 1\right]\right\} n(t)$$

$$= v\Sigma_a\left[k_\infty P_{NFL} - (1+L^2B_g^2)\right] n(t) \qquad L^2 = D/\Sigma_a$$

$$= \left[v\Sigma_a(1+L^2B_g^2)\right]\left[\frac{k_\infty P_{NFL}}{1+L^2B_g^2} - 1\right] n(t)$$

But note:

$$\frac{k_\infty P_{NFL}}{1+L^2B_g^2} \equiv k_{eff}$$ "effective multiplication factor"

$$\frac{1}{v\Sigma_a(1+L^2B_g^2)} \equiv \ell$$ "mean lifetime of thermal neutrons in finite reactor", or "prompt neutron lifetime"

[lifetime from moment thermalized to moment absorbed or lost.]

Thus, under the assumption of space-time separability, the time behavior of the total neutron population in the reactor is governed by

$$\frac{dn}{dt} = \left[\frac{k_{eff}-1}{l}\right] n(t)$$

[sometimes one defines
$\delta k \equiv k_{eff} - 1 \equiv k_{ex}$ excess multiplication]

This equation is an example of the so-called "point reactor kinetics equations" [since we have separated out the spatial dependence of the flux--treating the reactor as a point].

We can easily solve this equation, subject to an initial condition

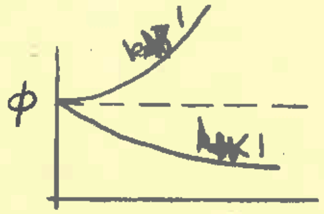
$$\text{i.c.:} \quad \phi(\underset{\sim}{r},0) = \phi_0(\underset{\sim}{r}) = v n_0 \psi(\underset{\sim}{r})$$

to find

$$n(t) = n_0 e^{\left[\frac{k_{eff}-1}{l}\right]t}$$

Hence the flux behaves as

$$\phi(\underset{\sim}{r},t) = v n_0 \psi(\underset{\sim}{r}) e^{\left[\frac{k_{eff}-1}{l}\right]t}$$



Comments:

i.) Note, $k_{eff} = 1 \Rightarrow B_g^2 = B_m^2$ as before

ii.) The time-constant or e-folding time of this exponential behavior is

$$T = \frac{l}{k_{eff}-1} \qquad \text{"reactor period"}$$

But what is l?

Thermal reactors: $l \sim 10^{-4}$ sec [thermal diffusion time]
Hence $k_{eff} - 1 = 0.1\%$, $T = 10^{-4}/10^{-3} = 0.1$ sec
[In one second, power increases by $e^{10} = 2.2 \times 10^4$]

Fast reactors: $l \sim 10^{-7}$ sec
Hence $k_{eff} - 1 = 0.1\% \Rightarrow T = 10^{-4}$ sec !

Hence this analysis would suggest a reactor is uncontrollable.

But, remember two main approximations

i.) fixed spatial shape -- $\phi(\underset{\sim}{r},t) = vn(t)\,\psi(\underset{\sim}{r})$

ii.) all prompt neutrons [assumed to appear promptly after fissions]

1.2 INFLUENCE OF DELAYED NEUTRONS UPON REACTOR DYNAMICS

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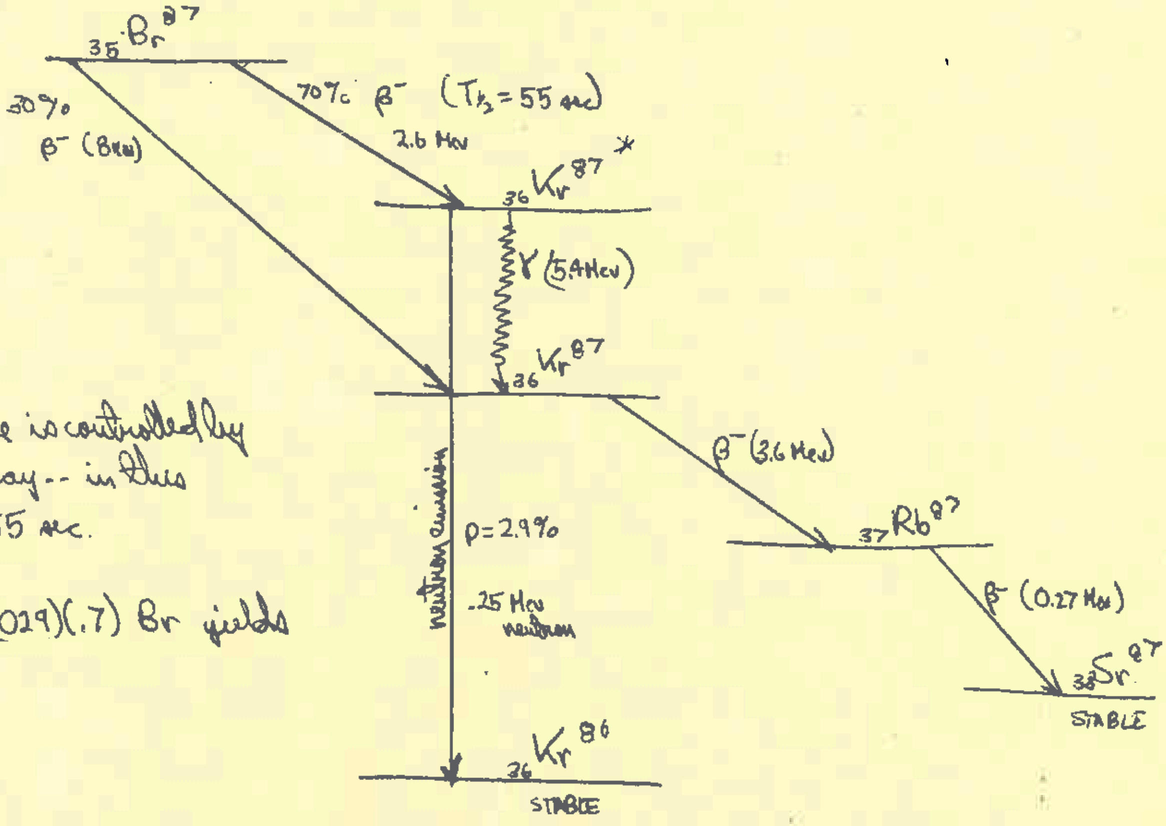
1.2.1. Delayed Neutron Physics

Thus far we have only considered those neutrons which are emitted spontaneously with fissions. However, we must also must consider those neutrons which are emitted in subsequent fission product decays -- so-called delayed neutrons.

EXAMPLE:



i.) Effective lifetime is controlled by preceding β-decay -- in this case, $T_{1/2} = 55$ sec.

ii.) Note only (029)(.7) Br yields a neutron!

Definitions:

delayed neutron precursor ≡ fission fragment whose β-decay yields a daughter nucleus which subsequently decays yielding a delayed neutron [e.g. ${}_{35}Br^{87} \xrightarrow[\beta]{} {}_{36}Kr^{87} \rightarrow n + {}_{36}Kr^{86}$]

λ_i ≡ decay constant (β-decay) of ith kind of precursor

β_i ≡ fraction of fission neutrons due to decay of ith kind of precursor

$\beta \equiv \sum_{i=1}^{6} \beta_i$ total fraction of fission neutrons which are delayed

To date, physicists have been able to classify 6 such groups of delayed neutrons

EXAMPLE: U^{235} [$\beta = .0064$]

$\tau_i = 1/\lambda_i$ sec	β_i
0.321	.000061
0.788	.000194
3.30	.000845
7.65	.000604
29.9	.00073
79.5	.000024

1.2.2. Mean Generation Times

We have seen that the reactor period is essentially proportional to the average lifetime of the neutrons. However in calculating this lifetime we have assumed the neutrons are born simultaneously with fissions. Actually, only (1-β) of the fission neutrons are "prompt". To include the delayed neutron effects on the lifetime we can heuristically calculate:

$$\langle l \rangle = (1-\beta)(l_{so}+l_{dif}) + \sum_{i=1}^{6} \beta_i \left[\frac{1}{\lambda_i} + l_{so_i} + l_{dif}\right]$$

$$\sim (1-\beta)(l_{dif}) + \sum_{i=1}^{6} \beta_i/\lambda_i$$

For thermal reactors, $\sum_{i=1}^{6} \beta_i/\lambda_i \sim 0.1$ sec. Hence

$$\langle l \rangle \sim 0.1 \text{ sec} \Rightarrow T = \frac{0.1}{.001} = 100 \text{ sec} \quad \text{for } k_{eff}-1 = .1\%$$

Hence delayed neutrons increase the time constant of the reactor such that effective control is possible.

[Can imagine reactor being held subcritical on prompt neutrons, with β delayed neutrons providing criticality. However, from this you might suspect that for $k_{eff}-1 > \beta$, the reactor will be critical on prompt neutrons alone, and hence have a very short period.]

1.2.3. Point Reactor Kinetic Equations

In actual fact, we cannot proceed so heuristically in the treatment of reactor dynamics with delayed neutrons. We must first set up a set of equations describing the time dependence of the delayed neutrons. Define

$C_i(\underset{\sim}{r},t)\, d^3\underset{\sim}{r} \equiv$ ~~expected number of fissions~~ precursors of ith kind in d^3r about $\underset{\sim}{r}$ at time t which always decay by emitting a delayed neutron.

[Note C_i is only some fraction of the actual isotope concentration, since only a certain fraction of the ~~isotope~~ nuclei decay by delayed neutron emission -- e.g.

$$C_{Br^{87}}(\underset{\sim}{r},t) = .029\ Br^{87}(\underset{\sim}{r},t) \quad]$$

We can immediately write ~~down~~ a balance relation for these precursor concentrations

$$\begin{pmatrix} \#\text{ of precursors} \\ \text{Decaying in} \\ d^3\underset{\sim}{r}\ /\text{sec} \end{pmatrix} = +\lambda_i\, C_i(\underset{\sim}{r},t)\, d^3r$$

$$\begin{pmatrix} \#\text{ of precursors} \\ \text{being produced} \\ \text{in } d^3r/\text{sec} \end{pmatrix} = \beta_i\, \nu\Sigma_f\, \phi(\underset{\sim}{r},t)\, d^3\underset{\sim}{r}$$

[assumes precursors don't migrate]

Hence equating the sum of gains and losses to the time rate of change of the concentration (and cancelling $d^3\underline{r}$)

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \beta_i \nu \Sigma_f \phi(\underline{r},t) \qquad i=1,\cdots 6$$

We can use the one-speed diffusion equation to describe the flux -- provided we include a term accounting for delayed neutrons

$$\begin{pmatrix}\text{Rate of}\\ \text{production of}\\ \text{delayed neutrons}\end{pmatrix} = \sum_{i=1}^{6} \lambda_i C_i(\underline{r},t)$$

But remember, the one-speed diffusion equation is for thermal neutrons. Hence we must adjust this to account for resonance absorption and non-thermal leakage

$$\begin{pmatrix}\text{rate of}\\ \text{production of}\\ \text{delayed thermal}\\ \text{neutrons}\end{pmatrix} = \sum_{i=1}^{6} P_{re}^{i} P_{NFL}^{i} \lambda_i C_i(\underline{r},t)$$

Note that we must use a different P_{re}^i and P_{NFL}^i in treating the delayed neutrons than we did in treating the fission neutrons, since the energies of delayed neutrons are quite a bit lower [this turns out to be fairly important to account for]. Hence the thermal group diffusion equation becomes

$$\frac{1}{v}\frac{\partial \phi}{\partial t} = D\nabla^2\phi - \Sigma_a \phi(\underline{r},t) + (1-\beta)\nu\Sigma_f P_{re} P_{NFL}\phi(\underline{r},t) + \sum_{i=1}^{6} P_{re}^{i} P_{NFL}^{i} \lambda_i C_i(\underline{r},t)$$

Now we will again separate out the time dependence by

Assumption: $\phi(\underline{r},t) = v\,n(t)\,\psi(\underline{r})$

$$C_i(\underline{r},t) = C_i(t)\,\psi(\underline{r})$$

where $\Psi(\underset{\sim}{r})$ satisfies $\nabla^2\Psi(\underset{\sim}{r}) + B_g^2\Psi(\underset{\sim}{r}) = 0$, B_g^2 being the geometric buckling.

If we substitute in these forms, we find

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$$\frac{dn}{dt} = -vDB_g^2 n - v\Sigma_a n + P_{re}P_{NFL}(1-\beta)\nu v\Sigma_f n + \sum_{i=1}^{6} P_{re}^i P_{NFL}^i \lambda_i C_i$$

$$\frac{dC_i}{dt} = -\lambda_i C_i(t) + \nu\beta_i\Sigma_f v n(t), \qquad i = 1, \cdots 6$$

Now it is customary to make a few definitions:

$$\overline{\beta}_i \equiv \frac{\beta_i P_{re}^i P_{NFL}^i}{(1-\beta)P^p P_{NFL} + \sum_i \beta_i P_{re}^i P_{NFL}^i} \equiv$$ effective delayed neutron fraction which accounts for fact that $P_{re}^i P_{NFL}^i$ differ due to energy difference of each delayed neutron group

$$= \frac{\beta_i P_{re}^i P_{NFL}^i}{\langle P P_{NFL}\rangle}$$

[Note, $\overline{\beta}_i$ may differ from β_i by as much as 25%.]

$$\overline{\beta} \equiv \sum_{i=1}^{6} \overline{\beta}_i$$

[Example: In age theory,

$$\overline{\beta}/\beta = \frac{1}{\beta + (1-\beta)e^{-B_g^2(\tau_p - \tau_d)}}$$]

Now do a little algebra

$$1 - \overline{\beta} = 1 - \frac{\sum_i \beta_i P_{re}^i P_{NFL}^i}{\langle P P_{NFL}\rangle} = (1-\beta)\frac{P_{re}P_{NFL}}{\langle P_{re}P_{NFL}\rangle}$$

Hence can rewrite equations as

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$$\frac{\quad}{dt} = -vDB_g^2 n - v\Sigma_a n + P_{re}P_{NFL}(1-\beta)\nu v\Sigma_f n + \sum_{i=1}^{6} P_{re}^i P_{NFL}^i \lambda_i C_i$$

$$\frac{dC_i}{dt} = -\lambda_i C_i(t) + \nu\beta_i\Sigma_f v n(t), \qquad i=1,\cdots 6$$

Now it is customary to make a few definitions:

$$\bar{\beta}_i \equiv \frac{\beta_i P_{re}^i P_{NFL}^i}{(1-\beta)P^p P_{NFL} + \sum_i \beta_i P_{re}^i P_{NFL}^i} \equiv$$ effective delayed neutron fraction which accounts for fact that $P_{re}^i P_{NFL}^i$ differ due to energy difference of each delayed neutron group

$$= \frac{\beta_i P_{re}^i P_{NFL}^i}{\langle P P_{NFL}\rangle}$$

[Note, $\bar{\beta}_i$ may differ from β_i by as much as 25%.]

$$\bar{\beta} \equiv \sum_{i=1}^{6} \bar{\beta}_i$$

[Example: In age theory,

$$\bar{\beta}/\beta = \frac{1}{\beta + (1-\beta)e^{-B_g^2(\tau_p - \tau_d)}}$$
]

Now do a little algebra

$$1-\bar{\beta} = 1 - \frac{\sum_i \beta_i P_{re}^i P_{NFL}^i}{\langle P P_{NFL}\rangle} = \frac{(1-\beta)P_{re}P_{NFL}}{\langle P_{re}P_{NFL}\rangle}$$

Hence can rewrite equations as

Hence our point reactor kinetics equations (including delayed neutrons) becomes

$$\frac{dn}{dt} = \frac{k_{eff}(1-\bar{\beta}) - 1}{\ell} n(t) + \sum_{i=1}^{6} \lambda_i \bar{C}_i(t)$$

$$\frac{d\bar{C}_i}{dt} = \bar{\beta}_i \frac{k_{eff}}{\ell} n(t) - \lambda_i \bar{C}_i(t) \qquad i=1,\cdots 6$$

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An alternative form of the point reactor kinetics equations is frequently more popular. Define

$$\Lambda \equiv \frac{1}{v\nu\Sigma_f\langle P_{FNL}P_{TNL}\rangle}$$ mean generation time between birth of neutron and subsequent fission (in infinite reactor) $= \frac{1}{\text{fission rate per neutron}}$

Note

$$\Lambda = \ell/k_{eff}$$

Also define

$$\rho(t) \equiv \frac{k_{eff}-1}{k_{eff}}$$ reactivity

Then we can rewrite our equations as

$$\frac{dn}{dt} = \left[\frac{\rho(t)-\bar{\beta}}{\Lambda}\right]n(t) + \sum_{i=1}^{6}\lambda_i \bar{C}_i(t)$$

$$\frac{d\bar{C}_i}{dt} = \frac{\bar{\beta}_i}{\Lambda}n(t) - \lambda_i\bar{C}_i(t) \qquad i=1,\dots,6$$

Comments:

i.) These 7 coupled ordinary differential equations in time will occupy most of our attention in this course

ii.) The point reactor kinetics equations as derived here rest upon a large number of hazy approximations
 (i.) space-time separability (most severe assumption)
 (ii.) one-group diffusion theory
Next week, we shall re-derive these same equations from a much more rigorous standpoint -- transport theory.

iii.) Note three parameters characterizing the reactor enter -- ρ, $\bar{\beta}$, and Λ. Note they only appear in ratios -- ρ/Λ or $\bar{\beta}/\Lambda$. In fact, only these ratios can be measured (or calculated) unambiguously.

iv.) Look again at the "reactivity"

$$\rho = \frac{k_{eff} - 1}{k_{eff}}$$

Recall, keff -- and hence ρ, depends on the size and composition of the reactor. In this case

$$k_{eff} = \frac{\nu \Sigma_f \langle P_{TE} P_{NFL} \rangle}{\Sigma_a (1 + L^2 B_g^2)}$$

We control a reactor by changing ρ -- that is, by changing a control rod position or poison concentration or such. Hence ρ will in general be a function of time -- partly under the control of the reactor operator.

But for any reactor operating at power, ρ will also depend on the flux itself due to temperature feedback effects

i.) geometry changes
ii.) density changes (expansion, moderator boiling)
iii) neutron spectrum
iv.) microscopic cross sections

Hence in general

$$\rho = \rho[n(t), t]$$

This is, we are talking about a set of nonlinear differential equations to solve -- very hard. Actually we will treat

i.) Point reactor equations without feedback -- linear -- ρ is a given function of time

ii.) Point reactor equations with feedback -- nonlinear -- ρ is a functional of $n(t)$.

EXAMPLE: Consider a reactor operating at a steady-state level n_0 for $t<0$. At $t=0^+$, we introduce a reactivity ρ and consider the dynamics of the reactor to be described by the point reactor kinetics equations with <u>one delayed group</u>

$$\frac{dn}{dt} = \left[\frac{\rho-\beta}{\Lambda}\right] n(t) + \lambda C(t)$$

$$\frac{dC}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda C(t) \qquad [\text{Here } \beta = \bar{\beta},\ C = \bar{C}]$$

How do we solve this coupled set of ODE's with constant coefficients? Try Laplace transforms

$$\tilde{f}(s) \equiv \int_0^\infty dt\, e^{-st} f(t) \equiv \mathcal{L}\{f(t)\}$$

and recall

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = s\tilde{f}(s) - f(0)$$

Transforming the kinetics equations

$$s\tilde{n}(s) - n(0) = \left[\frac{\rho-\beta}{\Lambda}\right]\tilde{n}(s) + \lambda \tilde{C}(s)$$

$$s\tilde{C}(s) - c(0) = \beta/\Lambda\, \tilde{n}(s) - \lambda \tilde{C}(s)$$

First solve for

$$\tilde{C}(s) = \frac{C(0)}{s+\lambda} + \frac{\beta}{\Lambda}\frac{\tilde{n}(s)}{s+\lambda}$$

and plug into (*) to find

$$s\tilde{n}(s) - n(0) = \left[\frac{\rho-\beta}{\Lambda}\right]\tilde{n}(s) + \frac{\lambda\beta}{\Lambda}\frac{\tilde{n}(s)}{s+\lambda} + \frac{\lambda C(0)}{s+\lambda}$$

or

$$\tilde{n}(s) = \frac{n(0) + \dfrac{\lambda C(0)}{s+\lambda}}{\left[s - \dfrac{\rho-\beta}{\Lambda} - \dfrac{\beta}{\Lambda}\dfrac{\lambda}{s+\lambda}\right]}$$

But we can manipulate

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$$\frac{1}{s-\frac{\rho-\beta}{\Lambda}-\frac{\beta}{\Lambda}\frac{\lambda}{s+\lambda}} = \frac{\Lambda}{s\Lambda-\rho+\beta-\beta\left(\frac{\lambda}{s+\lambda}\right)}$$

$$= \frac{\Lambda}{s\Lambda-\rho+\frac{\beta s}{s+\lambda}}$$

Furthermore, we can utilize our information that the reactor has been operating at constant n_0 for $t<0$ to find

$$\therefore C(0) = \frac{\beta}{\Lambda}n(0) = \frac{\beta}{\Lambda}n_0$$

Then

$$\tilde{n}(s) = \frac{\Lambda}{s\Lambda-\rho+\frac{\beta s}{s+\lambda}}\left[n_0+\frac{\frac{\beta\lambda}{\Lambda}n_0}{s+\lambda}\right]$$

$$= \frac{\Lambda(s+\lambda)+\lambda\beta}{\Lambda s^2+(\lambda\Lambda+\beta-\rho)s-\rho\lambda}\,n_0$$

Now we must invert to find $\mathcal{L}^{-1}\{\tilde{n}(s)\} = n(t)$. Since we have a ratio of two polynomials, can simply find poles -- i.e. zeros of denominator given by

$$s_{0,1} = \frac{1}{2\Lambda}\left[-(\beta-\rho+\lambda\Lambda) \pm \sqrt{(\beta-\rho+\Lambda)^2+4\Lambda\lambda\rho}\right]$$

This is still too complicated for our purposes. Consider

Special case: $|\rho| << \beta$. Then

$$s_0 \cong \frac{\lambda\rho}{\beta-\rho} \qquad s_1 \cong -\left(\frac{\beta-\rho}{\Lambda}\right)$$

Hence from

$$\tilde{n}(s) = \frac{\Lambda(s+\lambda)+\lambda\beta}{\Lambda(s-s_0)(s-s_1)}\,n_0$$

we can invert to find

$$n(t) = \frac{\Lambda(s+s_0)+\lambda\beta}{\Lambda(s_0-s_1)} n_0 e^{s_0 t} + \frac{\Lambda(s+s_1)+\lambda\beta}{\Lambda(s_1-s_0)} n_0 e^{s_1 t}$$

$$\sim \frac{\beta}{\beta-\rho} e^{\left[\frac{\lambda\rho}{\beta-\rho}\right]t} - \frac{\rho}{\beta-\rho} e^{-\left[\frac{\beta-\rho}{\Lambda}\right]t} \qquad |\rho| << \beta$$

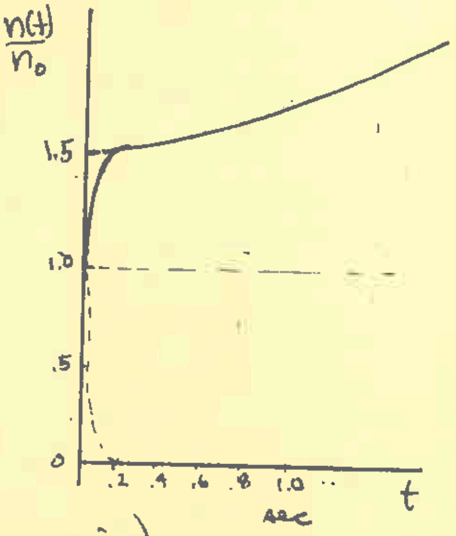
[Notice that regardless of the sign of ρ, one of the exponentials is always decaying. Furthermore, for $\rho=\beta$, we get garbage -- approximations fail.]

Consider:

$\rho = 0.0025$
$\beta = 0.0075$
$\lambda = 0.08\ s^{-1}$
$\Lambda = 10^{-3}\ s$

Then

$$\frac{n(t)}{n_0} = 1.5e^{0.04t} - 0.5e^{-5t}$$



COMMENTS:

i.) Even though delayed neutrons are (percentage-wise) a very small fraction [0.0075] of the neutron population, they effectively control the true behavior -- provided $\rho < \beta$

$$T = \frac{\beta-\rho}{\rho\lambda} = \frac{\overset{2}{\cancel{.0050}}}{(\cancel{.0025})(.08)} = 25 \text{ sec.}$$

1.2.4 The Inhour Equation

As we have seen from our example with one delayed group, the reactor period resulting from a step change in reactivity is determined as the reciprocal of the largest pole s_0 of $\tilde{n}(s)$ -- that is, as the reciprocal of the largest root of the denominator

$$s\Lambda - \rho + \frac{\beta s}{s+\lambda} = 0$$

In practice, one rewrites this in an alternative form. Recall

$$\rho = \frac{k_{eff}-1}{k_{eff}} \Rightarrow k_{eff} = \frac{1}{1-\rho}$$

$$\Lambda = \frac{\ell}{k_{eff}} = \ell(1-\rho)$$

Hence

$$\rho = s\Lambda + \frac{s\beta}{s+\lambda} = s\ell(1-\rho) + \frac{s\beta}{s+\lambda}$$

or

$$\rho = \frac{s\ell}{s\ell+1} + \frac{1}{s\ell+1}\,\frac{s\beta}{s+\lambda}$$

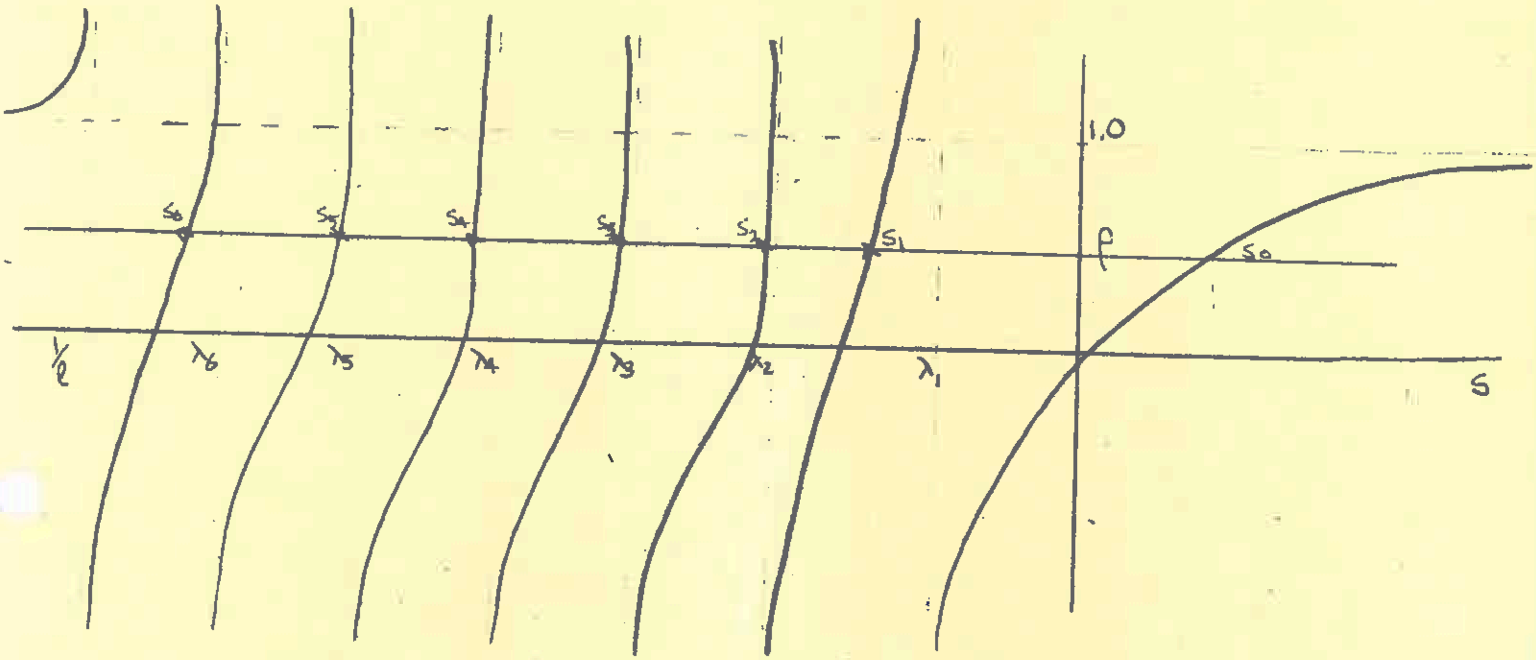
This equation determines the poles of $\tilde{n}(s)$ [and hence the exponents in the time solution] for any constant reactivity ρ. In the more general case of 6 delayed groups, this relation becomes

$$\rho = \frac{s\ell}{s\ell+1} + \frac{1}{s\ell+1}\sum_{i=1}^{6}\frac{s\beta_i}{s+\lambda_i}$$

Inhour Equation

[Reactivity is sometimes expressed in terms of "inverse hour" or inhour ≡ reactivity which will make reactor period equal to one hour -- no longer, however. Use percentage or dollar.]

To determine the roots of this expression, plot the RHS vs. s



COMMENTS:

i.) $$-\infty < \rho = \frac{k_{eff}-1}{k_{eff}} < 1$$

ii.) The reactor period, by definition, is

$$T = 1/s_0 \ , \quad n(t) = \sum_{i=0}^{6} n_i e^{s_i t}$$

iii.) Limits: i.) $\rho = 0 \Rightarrow s_0 = 0 \Rightarrow$ criticality

ii.) $\rho \rightarrow 1 \Rightarrow s_0 \rightarrow \infty$

iii.) $\rho \rightarrow -\infty \Rightarrow s_0 \rightarrow -\lambda_1$

iv.) Again note, that regardless of the sign of ρ, only one root, s_0, can be positive.

Special Cases:

1.) ρ small $\Rightarrow$ s_0 is small, say $|s_0| << \lambda_1 < \lambda_2 < \cdots < 1/\ell$. Hence in the inhour equation

$$\rho = \frac{s_0}{s_0 + 1/\ell} + \frac{1/\ell}{s_0 + 1/\ell} \sum_{i=1}^{6} \frac{s_0 \beta_i}{s_0 + \lambda_i} \sim s_0 \ell + s_0 \sum_{i=1}^{6} \frac{\beta_i}{\lambda_i}$$

or

$$T = \frac{1}{s_0} = \frac{1}{\rho}\left[\ell + \sum_{i=1}^{6} \frac{\beta_i}{\lambda_i}\right] = \frac{1}{\rho}\left[\text{ave. neutron lifetime}\right]$$

2.) ρ large $(\rho >> \beta)$

$$\rho \sim \frac{s_0}{s_0 + 1/\ell} + \frac{1/\ell}{s_0 + 1/\ell} \sum_{i=1}^{6} \beta_i$$

or

$$\rho(s_0 + 1/\ell) = s_0 + \beta/\ell$$

or

$$T = \frac{1}{s_0} = \frac{\ell}{k_{eff}(\rho - \beta)} \sim \frac{\ell}{k_{eff} - 1}$$

-- which is just the result we obtained ignoring delayed neutrons.

3) $\rho = \beta$: This is essentially the break-even point. Recall

$$\frac{dn}{dt} = \left[\frac{k_{eff}(1-\beta) - 1}{\ell}\right] n + \sum_{i=1}^{6} \lambda_i C_i(t)$$

Now for reactor to be critical on prompt neutrons alone, we require

$$k_{eff}(1-\beta) - 1 = 0$$

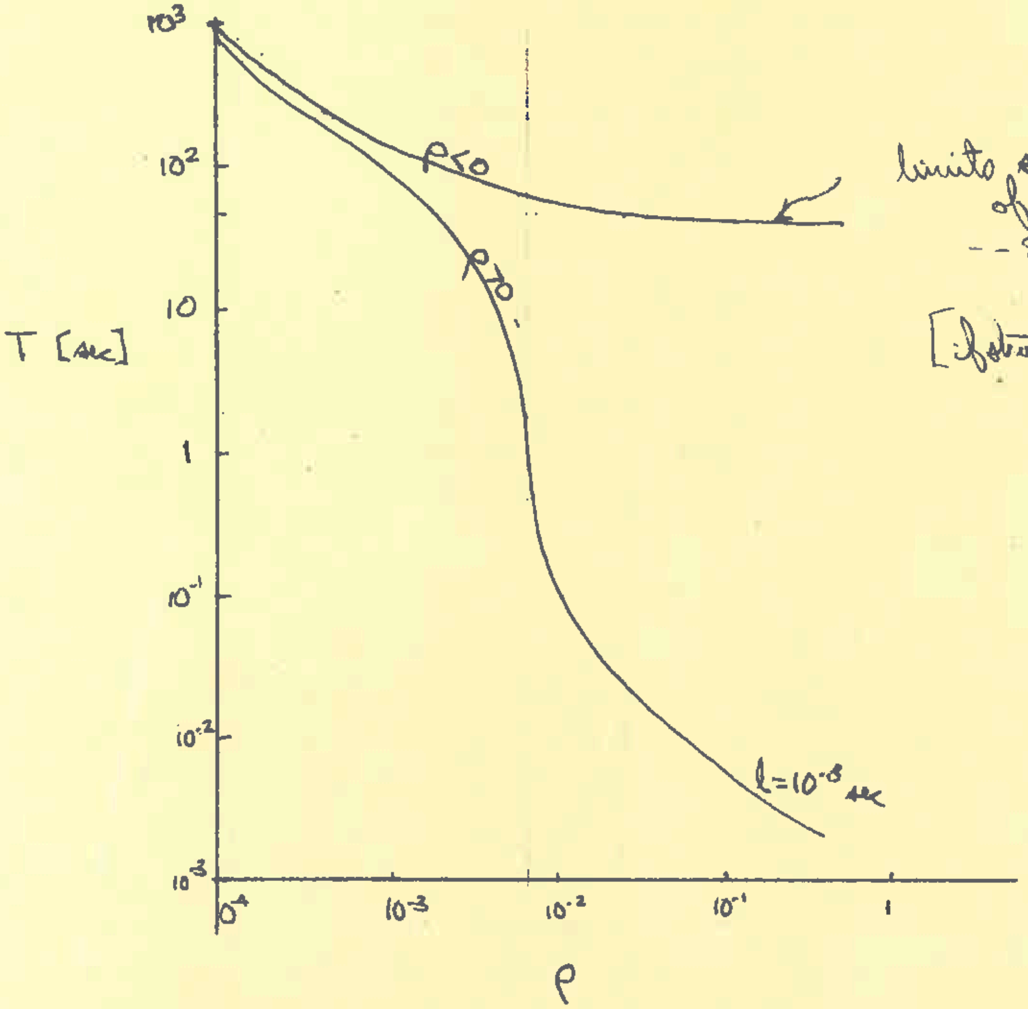
$$\Rightarrow \quad k_{eff} = \frac{1}{1-\beta} = \frac{1}{1-\rho}$$

$$\Rightarrow \quad \rho = \beta$$

Hence for 2/19/70

$\rho < \beta$, reactor is subcritical on prompt neutrons alone
$\rho = \beta$, " " critical " " " "
$\rho > \beta$, " " supercritical " " " "

Terminology: $0 < \rho < \beta$ "delayed critical" [delayed neutrons control T]
$\beta < \rho$ "prompt critical" [prompt neutrons control T]



limits shutdown rate to that of slowest precursor -- 80 sec in thermal reactors

[if shutdown factor 10^{10}, $t_s \sim 30$

[This is motivation for measuring ρ in units of β -- dollars].

Review of Introductory Concepts

prompt neutrons and prompt neutron lifetime l
effective multiplication factor k_{eff}
excess multiplication $k_{ex} = k_{eff} - 1$
space-time separability assumption
point reactor kinetics equations
reactor period T
delayed neutrons
precursor and precursor decay constants λ_i
delayed neutron fraction β_i, β
effective delayed neutron fraction $\bar{\beta}_i, \bar{\beta}$
effective precursor concentration $\bar{C}(t)$
mean generation time Λ
reactivity ρ (in units of inhours, %, \$)
feedback
delayed & prompt critical
inhour equation.

Program

1.) Rederive point kinetics equations rigorously

2.) Solve PRKE for a given reactivity $\rho(t)$

3.) Consider feedback $\rho[n]$

4.) Consider cases in which point kinetics is not valid.

II. DERIVATION OF THE POINT REACTOR KINETICS EQUATIONS

2.1. THE NEUTRON TRANSPORT EQUATION

References: Bell & Glasstone, Chapter 1 (advanced)
Keepin, Chapter 6 (advanced)
Akcasu, Chapter 1
Weston & Edlund, Chapter

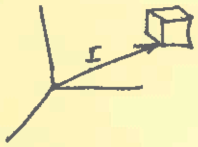
Reactor dynamics is concerned with the time behavior of the neutron population in a reactor. But how do we determine this quantity? Three steps

i.) Define the variables which describe the neutron population (and characterize the material in the reactor if necessary).

ii.) Determine the "kinetic equations" which relate these variables and determine the time behavior of the reactor.

iii.) Try to solve these equations (either rigorously or approximately)

How do we characterize the neutron population in a reactor? Recall that in diffusion theory we introduced the concept of the neutron density or distribution function



$N(\underline{r},t)d^3r \equiv$ expected number of neutrons in d^3r about $\underline{r}$ at time t

[Note we are ignoring fluctuations here.] But we must be careful. Neutrons are characterized by more than just their position. Also

i.) velocity [speed & direction -- e.g. 1 Mev, ssw and down]

ii.) spin [polarization]

iii.) internal states? [excited states of neutron are different baryons]

In essentially all cases, it is found to be sufficient to characterize a neutron by $(\underline{r},\underline{v})$. Hence we must generalize our description of the neutron density

$n(\underline{r},\underline{v},t)d^3r\,d^3v \equiv$ expected number of neutrons in $d^3\underline{r}$ about $\underline{r}$ with velocities in d^3v about $\underline{v}$ at time t

$\equiv$ "angular neutron density"

Next we turn our attention to a description of the interactions of the neutrons with the material (nuclei) in the reactor. We will define the macroscopic cross section for a neutron interaction of type i as a reciprocal mean free path.

$\Sigma_i(\underline{r},v,t) \equiv$ probability of neutron interaction of type i per unit path length traveled by neutron of v at $\underline{r},t$

We can distinguish among three types of interactions

Σ_a absorption
Σ_f fission
Σ_s scattering

In particular, to describe scattering we must introduce a more general quantity -- the "scattering kernel" or differential scattering cross section

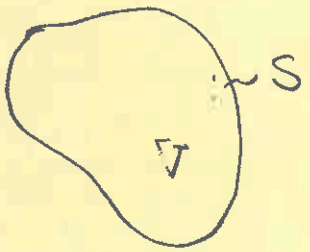
$\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\,d^3v \equiv$ expected number of neutrons of velocity $\underline{v}'$ at $\underline{r},t$ scattered into d^3v about $\underline{v}$ per unit path length traveled.

Note that from this latter definition we find

$$\int d^3v'\, \Sigma_s(\underline{v}\to\underline{v}',\underline{r},t) = \Sigma_s(\underline{r},v,t)$$

Having defined the neutron distribution function and the interaction cross sections, we can now proceed to derive the equation describing the time behavior of $n(\underline{r},\underline{v},t)$. For the present we shall neglect fission events [preferring to include them as an effective source term later in the analysis].

Consider then the neutron balance for an arbitrary volume V of surface area S:



$$\underset{①}{\begin{pmatrix}\text{time rate of}\\ \text{change of}\\ \text{neutrons in}\\ V,\ d^3v\end{pmatrix}} = -\underset{②}{\begin{pmatrix}\text{loss due to}\\ \text{leakage}\\ \text{through}\\ \text{surface } S\end{pmatrix}} - \underset{③}{\begin{pmatrix}\text{loss due}\\ \text{to}\\ \text{absorption}\\ \text{in}\\ V,\ d^3v\end{pmatrix}} + \underset{④}{\begin{pmatrix}\text{change due}\\ \text{to}\\ \text{scattering}\\ \text{in}\\ V,\ d^3v\end{pmatrix}} + \underset{⑤}{\begin{pmatrix}\text{sources}\\ \text{in}\\ V,\ d^3v\end{pmatrix}}$$

We can easily find the mathematical representation for each of these terms. Three of the terms are easy

$$① = \frac{\partial}{\partial t}\left[\int_V d^3r\, n(\underline{r},\underline{v},t)\right] d^3v = \left[\int_V d^3r\, \frac{\partial n}{\partial t}\right] d^3v$$

$$③ = \left[\int_V d^3r\, v\Sigma_a(\underline{r},v,t)\, n(\underline{r},v,t)\right] d^3v$$

$$④ = \left[\int_V d^3r\, Q(\underline{r},\underline{v},t)\right] d^3v$$

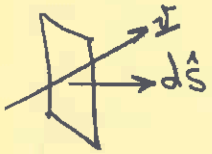
The scattering term is due to scattering in or out of d^3v

$$③ = \left\{\int_V d^3r \left[\int d^3v'\, v'\, \Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\, n(\underline{r},\underline{v}',t) - v\Sigma_s(\underline{r},v,t)\, n(\underline{r},\underline{v},t)\right]\right\} d^3v$$

1/21/70

To calculate the leakage term, first note the number of neutrons passing through a differential surface element $d\hat{S}$ is

$$\#\text{ crossing } dS = n(\underline{r},\underline{v},t)\, \underline{v}\cdot d\hat{S}$$



Hence the total leakage is

$$② = \left[\int_S d\hat{S}\cdot\underline{v}\, n(\underline{r},\underline{v},t)\right] d^3v = \left[\int_V d^3r\, \nabla\cdot(\underline{v}\, n(\underline{r},\underline{v},t))\right] d^3v$$

$$= \left[\int_V d^3r\, \underline{v}\cdot\nabla n(\underline{r},\underline{v},t)\right] d^3v$$

where we have used the divergence theorem $\int_S d\hat{S}\cdot\underline{J} = \int_V d^3r\, \nabla\cdot\underline{J}$

Hence regrouping all of our terms

$$d^3v \int_V d^3r \left[\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v(\Sigma_a+\Sigma_s)\, n(\underline{r},\underline{v},t) - \int d^3v'\, v'\,\Sigma_s(v'\to v)\, n(\underline{r},\underline{v}',t) - Q(\underline{r},\underline{v},t) \right] = 0$$

But recall our volume was chosen arbitrarily. Hence the only way the integral over any arbitrary range can be zero is for the integrand itself to be zero --

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v\Sigma_t(\underline{r},v,t)\, n(\underline{r},\underline{v},t) = \int d^3v'\, v'\,\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\, n(\underline{r},\underline{v}',t) + Q(\underline{r},\underline{v},t)$$

This is the famous [--rather, infamous] <u>neutron transport equation</u>

Now how do we include fissions. We then expect some absorptions will lead to more neutrons. Hence we can include the fission neutrons by modifying the source term $Q(\underline{r},\underline{v},t)$. But we must be careful to distinguish between "prompt" and "delayed" fission neutrons.

First, for <u>prompt neutrons</u> emitted directly in the fission event (or with 10^{-14} seconds thereafter), the fission rate is

$$v\Sigma_f(\underline{r},v,t)\, n(\underline{r},\underline{v},t)$$

Hence, if

$\nu(v) \equiv$ mean number of neutrons released in fission induced by neutron of speed v [in U^{235}, $\nu \in 2.43-2.57$]

$\chi_p(\underline{v}) \equiv$ spectrum of prompt fission neutrons then the effective source of prompt neutrons is

$$Q_p(\underline{r},\underline{v},t) = \chi_p(\underline{v})\int d^3v'\, \nu(v')(1-\beta)\, v'\Sigma_f(\underline{r},\underline{v}',t)\, n(\underline{r},\underline{v}',t)$$

Next, to account for delayed neutrons, we can use the concepts introduced earlier. Let

$C_i(\underline{r},t) \equiv$ precursor concentration for ith type of delayed neutron
$\lambda_i \equiv$ decay constant for precursor
$\beta_i \equiv$ fraction of fission neutrons due to decay of ith precursor
$\chi_i(\underline{v}) \equiv$ speed distribution of delayed neutrons in the ith group $\left(\int dv\, \chi_i(v) = 1\right.$

Then the effective source of delayed neutrons is

$$Q_d(\underline{r},\underline{v},t) = \sum_{i=1}^{6} \lambda_i \chi_i(v)\, C_i(\underline{r},t)$$

where the precursor concentrations satisfy

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \int d^3v'\, \beta_i \nu(v')\, v'\Sigma_f(\underline{r},\underline{v}',t)\, n(\underline{r},\underline{v}',t).$$

[Since there are no spatial operators in this equation, it is evident that we have assumed the fission fragments do not diffuse appreciably from the initial point of fission.]

In summary, then, the complete set of equations describing the neutronic behavior of the reactor (i.e. the reactor kinetics)

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v\Sigma_t(\underline{r},\underline{v},t)\, n(\underline{r},\underline{v},t) = \int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\, n(\underline{r},\underline{v}',t) + \chi(v)\int d^3v'\, \nu(v')(1-\beta)\, v'\Sigma_f(\underline{r},\underline{v}',t)\, n(\underline{r},\underline{v}',t)$$
$$+ \sum_{i=1}^{6} \lambda_i \chi_i(v)\, C_i(\underline{r},t) + Q(\underline{r},\underline{v},t)$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \int d^3v'\, \beta_i \nu(v')\, v'\Sigma_f(\underline{r},\underline{v}',t)\, n(\underline{r},\underline{v}',t) \qquad i = 1,\cdots,6$$

2.2. FEEDBACK MECHANISMS

We now have a set of equations which will yield the neutron distribution $n(\underline{r},\underline{v},t)$ – provided the macroscopic cross-sections are known [and provided, of course, we can somehow solve these very complicated integro-differential equations]. Hence we now turn our attention to a study of the macroscopic cross-sections themselves. To be more specific, write the macroscopic cross section for interactions of type i as

$$\underline{\Sigma_i(\underline{r},v,t) = N(\underline{r},t)\,\sigma_i(\underline{r},v,t)}$$

where

$N(\underline{r},t)\,d^3r \equiv$ number of atomic nuclei in d^3r about $\underline{r}$ at time t

$\sigma_i(\underline{r},v,t) \equiv$ microscopic cross section for interaction of type i

Now it is easily understandable how the atomic density $N(\underline{r},t)$ can depend upon time since

i.) material densities depend upon temperature T which in turn depends on the power distribution and hence the flux.

ii.) the concentrations of certain nuclei is constantly changing due to neutron interactions (buildup of poison or burnup of fuel).

But it should be noted that we have explicitly written the microscopic cross sections as explicit functions of r and t. This requires a more detailed explanation (but is essentially a feature of the thermal motions of the atomic nuclei).

2.2.1. Temperature Dependence of Atomic Densities $N(\underline{r},t)$

1/23/70

The temperature distribution in the reactor, $T(\underline{r},t)$, is described by the conventional heat transfer equation

$$\mu\left[\frac{\partial T}{\partial t} + \underline{u}(\underline{r},t)\cdot\nabla T(\underline{r},t)\right] - \nabla\cdot\kappa\nabla T(\underline{r},t) = H(\underline{r},t)$$

where

μ heat content / unit volume of material

$\underline{u}(\underline{r},t)$ hydrodynamic velocity [usually zero, unless we consider fluid flow]

κ heat conductivity

$H(\underline{r},t)$ distributed heat source

The heat source is due, of course, to nuclear processes -- predominantly fission, but radiative capture as well. [$\Sigma_c/\Sigma_f \sim 1$, but $\epsilon_c \sim 2$ Mev while $\epsilon_f \sim 200$ Mev]. We can further subdivide the <u>fission energy release</u> as

$$H = H_f + H_n + H_\beta + H_\gamma + H_\alpha$$

where

H_f	kinetic energy of fission fragments	165 Mev
H_n	" " " neutrons	5
H_γ	γ's	11
H_β	β's	5
H_α	α's	11
H	Total	197 Mev

Since the fission fragments travel only short distances before releasing their energy in the form of heat, we usually characterize <u>H_f</u> as a <u>localized heat source</u> proportional to the fission rate

$$H_f(\underline{r},t) = w_f \int d^3v'\, v'\Sigma_f(\underline{r},v',t)\, n(\underline{r},\underline{v}',t)$$

The latter components are not so easily treated since they correspond to distributed heat sources (neutrons thermalize over appreciable ranges, as do γ's and β's). One usually neglects them entirely or attempts to readjust w_f to account for this energy source. In our analysis, we shall usually assume a localized heat source.

Equation of State

or

Expansion coefficients

2.2.2. Production and Burnup of Nuclear Species

Several fission fragments have <u>very large capture cross sections</u> as well as <u>relatively high yields</u>, and hence strongly influence the dynamical behavior of the reactor. The three most important isotopes resulting in such "fission product poisoning" are I^{135}, Xe^{135}, and Sm^{149}. To describe the concentrations of these isotopes, we must set up the equations describing the corresponding radioactive chains.

EXAMPLE: The I^{135} chain

$$ {}_{52}Te^{135} \xrightarrow[\beta^-]{(2m)} {}_{35}I^{135} \xrightarrow[\beta^-]{(6.7\,hr)} {}_{54}Xe^{135} \xrightarrow[\beta^-]{(9.2\,hr)} {}_{55}Cs^{135} $$

The balance equations are

$$ \frac{\partial Te}{\partial t} = y_T \int d^3v\, v\, \Sigma_f(v)\, n(\underline{r},\underline{v},t) - \lambda_T Te(\underline{r},t) - Te(\underline{r},t) \int d^3v\, \sigma_{a_T}(v)\, v\, n(\underline{r},\underline{v},t) $$

$$ \frac{\partial I}{\partial t} = \lambda_T Te(\underline{r},t) - I(\underline{r},t)\left[\lambda_I + \int d^3v\, \sigma_{aI}(v)\, v\, n(\underline{r},\underline{v},t)\right] $$

$$ \frac{\partial X}{\partial t} = \lambda_I I(\underline{r},t) - X(\underline{r},t)\left[\lambda_X + \int d^3v\, \sigma_{aX}(v)\, v\, n(\underline{r},\underline{v},t)\right] + y_X \int d^3v\, \Sigma_f(v)\, v\, n(\underline{r},\underline{v},t) $$

$y_i \equiv$ atoms of i per fission
$\lambda_i \equiv$ decay constant of i
$\sigma_{a_i} \equiv$ absorption cross section of i

Similar balances can (and will) be set up for Xe^{135} and Sm^{149}. We will later study the solutions of these equations in some detail.

$$ {}_{60}Nd^{149} \xrightarrow[\beta^-]{(1.7\,hr)} {}_{61}Pm^{149} \xrightarrow[\beta^-]{(47\,hr)} {}_{62}Sm^{149} $$

2.2.3. Variations in the Microscopic Cross Sections

We have alluded to the fact that the microscopic cross sections $\sigma_i(\underline{r},v,t)$ can depend upon position and time. To explain this, we recall that $\sigma_i(\underline{r},v,t)$ is not the theoretical cross section for an individual nucleus, but rather an average of this theoretical cross section over the distribution of atomic velocities. To analyze this, remember that the theoretical microscopic cross sections, σ_i^{th}, depend only upon the relative speed between the neutron and the nucleus

$$\sigma_i^{th}(|\underline{v}-\underline{V}|)$$

But we are interested in calculating the average of this cross section over the velocity distribution of the nuclei:

$$\mathcal{M}(\underline{r},\underline{V},t)\,d^3r\,d^3V \equiv \text{fraction of nuclei having } d^3r \text{ in } \underline{r},\ d^3V \text{ in } \underline{V} \text{ at time } t$$

The average rate of neutron interactions is then

$$\text{Rate} = \int d^3V\ |\underline{v}-\underline{V}|\ \sigma_i^{th}(|\underline{v}-\underline{V}|)\ N(\underline{r},t)\mathcal{M}(\underline{r},\underline{V},t)$$

where recall that $N(\underline{r},t)$ is the density of nuclei. Hence we can find the microscopic cross section presented to the neutrons as

$$\sigma_i(\underline{r},v,t) \equiv \int d^3V\ \frac{|\underline{v}-\underline{V}|}{v}\ \sigma_i^{th}(|\underline{v}-\underline{V}|)\mathcal{M}(\underline{r},\underline{V},t) = \frac{\text{Rate}}{vN(\underline{r},t)}$$

We can similarly define

$$\nu(\underline{r},v,t) \equiv \frac{\int d^3V\ |\underline{v}-\underline{V}|\ \nu(|\underline{v}-\underline{V}|)\ \sigma_f^{th}(|\underline{v}-\underline{V}|)\ \mathcal{M}(\underline{r},\underline{V},t)}{v\int d^3V\ |\underline{v}-\underline{V}|\ \sigma_f^{th}(|\underline{v}-\underline{V}|)\ \mathcal{M}(\underline{r},\underline{V},t)}$$

and

$$\sigma_s(\underline{v}'\rightarrow\underline{v},\underline{r},t) = \int d^3V\ \frac{|\underline{v}'-\underline{V}|}{v'}\ \sigma_s^{th}(\underline{v}'\rightarrow\underline{v},\underline{V})\,\mathcal{M}_i(\underline{r},\underline{V},t)$$

Hence we can see that the time dependence in the microscopic cross section arises in the time dependence of the velocity distribution of the nuclei. But how do we study $\mathcal{M}$?

One usually assumes the atomic nuclei are in local thermodynamic equilibrium

$$M(r,\underline{v},t) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mV^2}{2kT}}$$

where the temperature $T(r,t)$ is allowed to be a function of space and time. Hence the microscopic cross sections depend on the temperature of the reactor

$$\sigma_i = \sigma_i(v,T)$$

-- and thereby upon the flux itself. [As an example, discuss Doppler broadening.]

Also need: i.) initial conditions
ii.) boundary conditions

EQUATIONS OF NUCLEAR REACTOR DYNAMICS

NEUTRON KINETICS EQUATIONS:

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v\Sigma_t(\underline{r},v,t)\,n(\underline{r},\underline{v},t) = \int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\,n(\underline{r},\underline{v}',t) + \chi_p(v)\int d^3v'\,\nu(v')(1-\beta)\,v'\Sigma_f(\underline{r},\underline{v}',t)\,n(\underline{r},\underline{v}',t)$$

$$+ \sum_{i=1}^{6}\lambda_i\chi_i(v)\,C_i(\underline{r},t) + Q(\underline{r},\underline{v},t)$$

Neutron Transport Equation

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \int d^3v'\,\beta_i\,\nu(v')\,v'\Sigma_f(\underline{r},\underline{v}',t)\,n(\underline{r},\underline{v}',t) \qquad i=1,\cdots 6$$

Precusor Equations

EQUATIONS DESCRIBING FEEDBACK MECHANISMS:

$$\Sigma_j^X(\underline{r},v,t) = N_X(\underline{r},t,T)\,\sigma_j^X(v,T)$$

macroscopic cross section

Burnup and Conversion of fissionable nuclei:

$$\frac{\partial N_X}{\partial t} = -\lambda_X N_X(\underline{r},t) + \int d^3v\, v\left[\sigma_Y(v,T)\,N_Y(\underline{r},t) - \sigma_{aX}(v,T)\,N_X(\underline{r},t)\right]n(\underline{r},\underline{v},t)$$

where the Yth element is being burned up and converted into the Xth element.

Fission Product Poison Concentrations:

$$\frac{\partial I}{\partial t} = \gamma_I\int d^3v\, v\Sigma_f(v,T)\,n(\underline{r},\underline{v},t) - I(\underline{r},t)\left[\lambda_I + \int d^3v\, v\,\sigma_{aI}(v,T)\,n(\underline{r},\underline{v},t)\right]$$

$$\frac{\partial X}{\partial t} = \lambda_I\, I(\underline{r},t) - X(\underline{r},t)\left[\lambda_X + \int d^3v\, v\,\sigma_{aX}(v,T)\,n(\underline{r},\underline{v},t)\right] + \gamma_X\int d^3v\, v\Sigma_f(v,T)\,n(\underline{r},\underline{v},t)$$

for I^{135} and Xe^{135}

$$\frac{\partial P_m}{\partial t} = \gamma_p\int d^3v\, v\Sigma_f(v,T)\,n(\underline{r},\underline{v},t) - P_m(\underline{r},t)\left[\lambda_p + \int d^3v\, v\,\sigma_{ap}(v,T)\,n(\underline{r},\underline{v},t)\right]$$

$$\frac{\partial S_m}{\partial t} = \lambda_p P_m(\underline{r},t) + \gamma_S\int d^3v\, v\Sigma_f(v,T)\,n(\underline{r},\underline{v},t) - S_m(\underline{r},t)\int d^3v\, v\,\sigma_{aS}(v,T)\,n(\underline{r},\underline{v},t)$$

for Sm^{149}

$$N_X = N_X(T)$$

equation of state of hydrodynamics or gas dynamics

$$\sigma_j^X(v,T) = \int d^3V\,\frac{|\underline{v}-\underline{V}|}{v}\,\sigma_j^{th}(|\underline{v}-\underline{V}|)\left[\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mV^2}{2kT}}\right]$$

averaged microscopic cross sections

Heat transfer equation:

$$\mu\left[\frac{\partial T}{\partial t} + \underline{u}(\underline{r},t)\cdot\nabla T(\underline{r},t)\right] - \nabla\cdot\kappa\nabla T(\underline{r},t) = w_f\int d^3v\, v\Sigma_f(\underline{r},v,T)\,n(\underline{r},\underline{v},t)$$

2.3. THE REACTOR DYNAMICS EQUATIONS

1/26/70

Now for the first -- and only -- time we will attempt to summarize by writing in one place the equations describing the dynamics of a nuclear reactor:

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v\Sigma_t(\underline{r},v,t)\,n(\underline{r},\underline{v},t) = \int d^3v'\,v'\,\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\,n(\underline{r},\underline{v}',t)$$

$$+\,\chi(\underline{v})\int d^3v'\,\nu(v')(1-\beta)\,v'\,\Sigma_f(\underline{r},v',t)\,n(\underline{r},\underline{v}',t)$$

$$+\,\sum_{i=1}^{6}\lambda_i\,\chi_i(\underline{v})\,C_i(\underline{r},t) + Q(\underline{r},\underline{v},t)$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \int d^3v'\,\beta_i\,\nu(v')\,v'\,\Sigma_f(\underline{r},v',t)\,n(\underline{r},\underline{v}',t) \qquad i=1,\cdots 6$$

$$\Sigma_j(\underline{r},v,t) = N_j(\underline{r},t,T)\,\sigma_j(v,T)$$

A representative equation for the burnup and conversion of fissionable nuclei

$$\frac{\partial N_i}{\partial t} = -\lambda_j N_j(\underline{r},t) + \int d^3v\,v\left[\sigma_i(v,T)N_i(\underline{r},t) - \sigma_{a_j}(v,T)N_j(\underline{r},t)\right]n(\underline{r},\underline{v},t)$$

where ith element is being burned up and converted into jth element. Such equations are written for

i.) fuel burnup
ii.) breeding and conversion

For fission-product poisoning: I^{135} and Xe^{135}

$$\frac{\partial I}{\partial t} = \gamma_I\int d^3v\,v\,\Sigma_f(v,T)\,n(\underline{r},\underline{v},t) - I(\underline{r},t)\left[\lambda_I + \int d^3v\,v\,\sigma_{aI}(v,T)\,n(\underline{r},\underline{v},t)\right]$$

$$\frac{\partial X}{\partial t} = \lambda_I\,I(\underline{r},t) - X(\underline{r},t)\left[\lambda_X + \int d^3v\,v\,\sigma_{aX}(v,T)\,n(\underline{r},\underline{v},t)\right]$$

$$+\,\gamma_X\int d^3v\,v\,\Sigma_f(v,T)\,n(\underline{r},\underline{v},t)$$

$$\frac{\partial P_m}{\partial t} = \gamma_p \int d^3v\, v\, \Sigma_f(v,T)\, n(\underline{r},\underline{v},t) - P_m(\underline{r},t)\left[\lambda_p + \int d^3v\, v\, \sigma_{ap}(v,T)\, n(\underline{r},\underline{v},t)\right]$$

$$\frac{\partial S_m}{\partial t} = \lambda_p P_m(\underline{r},t) + \gamma_s \int d^3v\, v\, \Sigma_f(v,T)\, n(\underline{r},\underline{v},t) - S_m(\underline{r},t) \int d^3v\, v\, \sigma_{as}(v,T)\, n(\underline{r},\underline{v},t)$$

$$\mu\left[\frac{\partial T}{\partial t} + \underline{u}(\underline{r},t)\cdot\nabla T(\underline{r},t)\right] - \nabla\cdot\kappa\nabla T(\underline{r},t) = w_f \int d^3v\, v\,\Sigma_f(\underline{r},v,T)\, n(\underline{r},\underline{v},t)$$

$N_j(\underline{r},t,T)$: equation of state
hydrodynamics or gas dynamics

$$\sigma_j(v,T) = \int d^3V\, \frac{|\underline{v}-\underline{V}|}{v}\, \sigma_j^{th}(|\underline{v}-\underline{V}|) \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mV^2}{2kT}}$$

2.3.1. Approximate Forms of the Kinetics Equations

The kinetic equations are a set of coupled, nonlinear, integro-differential equations of mammoth complexity. The set contains much more information than we are interested in, and is far too complex even for machine calculations. Hence we must simplify this set and develop various analytical and numerical techniques to extract relevant information. Several standard approximations of the transport equation are available.

2.3.1.1. The Diffusion Approximation

Suppose we integrate the transport equation over angle $\hat{\Omega}$ to find

$$\frac{1}{v}\frac{\partial\phi}{\partial t} + \nabla\cdot J + \Sigma_t(\underline{r},v,t)\,\phi(\underline{r},v,t) = \int_0^\infty dv'\, \Sigma_s(v'\to v,\underline{r},t)\,\phi(\underline{r},v',t) + Q(\underline{r},v,t)$$

where

$$\phi(\underline{r},v,t) = \int d\hat{\Omega}\, \Phi(\underline{r},v,\hat{\Omega},t)$$

$$J(\underline{r},v,t) = \int d\hat{\Omega}\, \hat{\Omega}\, \Phi(\underline{r},v,\hat{\Omega},t)$$

and $\Sigma_s(v' \to v, \underline{r}, t) \equiv \int d\hat{\Omega}\, \Sigma_s(v' \to v, \hat{\Omega} \cdot \hat{\Omega}', \underline{r}, t)$.

Now this is still an exact equation -- however it is not closed, since there are now two unknowns, $\phi(\underline{r}, v, t)$ and $\underline{J}(\underline{r}, v, t)$. An approximate relation between these two is given by the "diffusion approximation" [i.e. Fick's law]

$$\underline{J}(\underline{r}, v, t) = -D(\underline{r}, v, t) \nabla \phi(\underline{r}, v, t).$$

[This assumes the angular flux is of the form $\Phi(\underline{r}, v, \hat{\Omega}, t) = \phi(\underline{r}, v, t) + \hat{\Omega} \cdot \underline{J}(\underline{r}, v, \hat{\Omega}, t)$]
Then we find

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D(\underline{r}, v, t) \nabla \phi + \Sigma_t(\underline{r}, v, t)\, \phi(\underline{r}, v, t) = \int_0^\infty dv' \Sigma_s(v' \to v, \underline{r}, t)\, \phi(\underline{r}, v', t) + Q(\underline{r}, v, t)$$

2.3.1.2. *Fermi Age Theory*

But this is still too complicated. We must somehow approximate the energy dependence. One scheme for doing this is age theory. Recall the idea is to define the lethargy variable

$$u = \ln \frac{E_0}{E} = 2 \ln \frac{v_0}{v}$$

Then if the average energy lost per collision is small, we find

$$\int du\, \Sigma_s(u' \to u)\, \phi(u) \to -\frac{\partial q}{\partial u}$$ where q is the slowing down density.

We can approximately express

$$q(\underline{r}, u, t) \sim \xi \Sigma_t(\underline{r}, u, t)\, \phi(\underline{r}, u, t)$$

But actually, the complexities remaining in this model are not worth the approximations used in obtaining it.

2.3.1.3. *Multigroup Diffusion Theory*

The more useful and common approximation in energy involves the use of multigroup theory. The essential idea is to first integrate the diffusion equation from v_j to v_{j-1} to find

$$\frac{1}{v_j}\frac{\partial\phi^j}{\partial t} = \nabla\cdot D^j\cdot\nabla\phi^j - \Sigma_t^j\phi^j(\underline{r},t) + \sum_{k=1}^{N}\left[\chi_p^j \nu^k \Sigma_f^k(1-\beta) + \Sigma_{kj}\right]\phi^k(\underline{r},t)$$
$$+ \sum_{i=1}^{6}\chi_i^k\lambda_i C_i(\underline{r},t)$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \sum_{j=1}^{N}\beta_i \nu^j \Sigma_f^j \phi^j(\underline{r},t)$$

where we define

$$\phi^j(\underline{r},t) = \int_{v_j}^{v_{j-1}} dv\, \phi(\underline{r},v,t)$$

$$\Sigma_t^j(\underline{r},t) = \frac{1}{\phi^j}\int_{v_j}^{v_{j-1}} dv\, \Sigma_t(\underline{r},v,t)\,\phi(\underline{r},v,t) \qquad \text{and so on.}$$

These are the most popular set of equations used to describe general reactor dynamics. They are still much too complicated to solve (nonlinearities). We must somehow simplify things further. We could do this by making a further drastic approximation (e.g., one-speed diffusion theory). We shall instead proceed on a more rigorous course.

2.4. DERIVATION OF THE POINT REACTOR KINETICS EQUATIONS

Frequently we are interested only in rather gross features of the behavior of reactors — e.g. the total neutron population or power level as a function of time. One never needs the angular neutron dependence, and rarely needs the energy or space dependence. Here we will develop a set of equations describing $n(t)$ and $p(t)$. But in contrast to our earlier derivation (a la the one-speed diffusion equation), we will derive the point reactor kinetics equations directly from the exact transport equation.

Needless to say, this requires a few mathematical preliminaries. *Don't let these scare you!* Nothing else we do in the course will be nearly this abstract, but we do need a little bit of the theory of linear operators here.

2.4.1. Mathematical Preliminaries

2.4.1.1. Scalar Products

We define the *scalar (or inner) product* of two functions $f(\underline{r},\underline{v})$ and $g(\underline{r},\underline{v})$ as

$$(f,g) \equiv \int d^3\underline{r} \int d^3\underline{v}\; f^*(\underline{r},\underline{v})\, g(\underline{r},\underline{v}) \equiv \langle f \,|\, g \rangle$$

This is *just a complex number*. [It's very similar to a dot product

$$(\underline{A},\underline{B}) = \underline{A}\cdot\underline{B} = A_1B_1 + A_2B_2 + A_3B_3 \quad]$$

Notice: i.) $(f,g) = (g,f)^*$

ii.) $(f, ag_1 + bg_2) = a(f,g_1) + b(f,g_2)$

iii.) $(f,f) > 0$ if $f(x) \neq 0$

Two functions are *orthogonal* if their scalar product vanishes

$$(f,g) = 0$$

[Like perpendicular: $\underline{A}\cdot\underline{B} = 0 \Rightarrow \underline{A} \perp \underline{B}$.]

2.4.12 Linear Operators

An operator refers to a mathematical operation by which we convert a function $f(x)$ into another function $g(x)$ ... i.e.

$$A\,f(x) = g(x)$$

examples:

i.) differential operator $\quad A\cdot = \frac{d}{dx}\cdot \quad , \quad Af = df/dx$

ii.) integral operator $\quad A\cdot = \int_a^b dx'\,k(x,x')\cdot \quad , \quad Af = \int_a^b dx'\,k(x,x')f(x')$

iii.) unit operator $\quad A\cdot = 1\cdot \quad , \quad Af = f(x)$

iv.) null operator $\quad A\cdot = 0\cdot \quad , \quad Af = 0$

v.) $A\cdot = (\cdot)^2 \quad , \quad Af = [f(x)]^2$

If $A(af+bg) = aAf + bAg$, then A is a linear operator.
Can manipulate algebraically -- except for noncommutativity

$$AB\,f(x) \neq BA\,f(x)$$

A related animal -- functional -- which converts a function into a scalar

$$\mathcal{F}\{f(x)\} = a = \text{number}$$

Thus equipped with the concept of an operator and a scalar product, we can now proceed to define the very important concept of an "adjoint operator". An operator $A^\dagger$ is defined to be the adjoint of A if

$$(A^\dagger f, g) = (f, Ag) \quad \text{for every function } f \text{ and } g.$$

Actually for the case of differential operators we will frequently have to consider also the boundary conditions on the function $g(\underline{r},\underline{v})$, and analogously define "adjoint" boundary conditions to be satisfied by the function $f(\underline{r},\underline{v})$.

EXAMPLE: The Adjoint Transport Equation. 1/28/70

Suppose we write the transport equation in the form

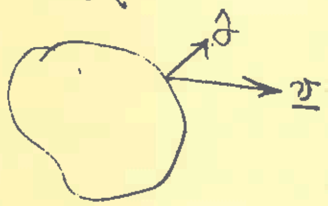
$$\frac{\partial n}{\partial t} = -\underline{v}\cdot\nabla n - v\Sigma_t(\underline{r},v,t)\,n(\underline{r},\underline{v},t) + \int d^3v'\,v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\,n(\underline{r},\underline{v}',t) + Q$$

$$\equiv Ln + Q$$

where we have defined the "transport operator"

$$L\circ \equiv \underbrace{-\underline{v}\cdot\nabla\circ}_{L_1} \underbrace{- v\Sigma_t(\underline{r},v,t)\circ}_{L_2} + \underbrace{\int d^3v'\,v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\circ}_{L_3}$$

Associated with this equation are the boundary conditions that on the surface of a reactor there can be no incoming neutrons



$$n(\underline{r}_s,\underline{v},t) = 0 \qquad \hat{\sigma}\cdot\underline{v} < 0$$

We will now compute the adjoint of L, piece by piece, using the definition:

$$(L^\dagger f, g) = (f, Lg)$$

i.) For $L_2 = v\Sigma_t(\underline{r},\underline{v},t)\circ$ (a simple multiplicative operator)

$$(f, L_2 g) = \int d^3r \int d^3v\; f^*(\underline{r},\underline{v})\, v\Sigma_t(\underline{r},v,t)\, g(\underline{r},\underline{v})$$

$$= \int d^3r \int d^3v\, \left[v\Sigma_t(\underline{r},v,t) f(\underline{r},\underline{v})\right]^* g(\underline{r},\underline{v}) = (L_2^\dagger f, g)\;;$$

hence

$$L_2^\dagger = v\Sigma_t(\underline{r},v,t)\circ = L_2$$ [so-called "self-adjoint"]

ii.) Next $L_3 = \int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\circ$ (an integral operator)

$$(f,L_3g) = \int d^3r\int d^3v\, f^*(\underline{r},\underline{v})\int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\, g(\underline{r},\underline{v}')$$

$$= \int d^3r\int d^3\underline{v}' \left[\int d^3v\, v'\Sigma_s(\underline{v}'\to\underline{v},\underline{r},t)\, f(\underline{r},\underline{v})\right]^* g(\underline{r},\underline{v}')$$

or letting $\underline{v}'\leftrightarrow\underline{v}$

$$= \int d^3r\int d^3\underline{v}\left[\int d^3v'\, v\Sigma_s(\underline{v}\to\underline{v}',\underline{r},t)\, f(\underline{r},\underline{v}')\right]^* g(\underline{r},\underline{v}) = (L_3^\dagger f,g)$$

hence

$$L_3^\dagger\circ = \int d^3v'\, v\Sigma_s(\underline{v}\to\underline{v}',\underline{r},t)\circ \qquad [\underline{\text{not}}\ \text{self-adjoint}]$$

iii.) Finally $L_1 = \underline{v}\cdot\nabla\circ$ (a differential operator)

The trick here (and with all differential operators) is to use integration by parts -- in this case

$$(f,L_1g) = \int d^3v\int d^3r\, f^*(\underline{r},\underline{v})\ \underline{v}\cdot\nabla g(\underline{r},\underline{v})$$

$$= -\int d^3v\int d^3r\,[\underline{v}\cdot\nabla f(\underline{r},\underline{v})]^* g(\underline{r},\underline{v}) + \int d^3v\int_S d^2r\, \hat{n}\cdot\underline{v}\, f^*(\underline{r},\underline{v})\, g(\underline{r},\underline{v})$$

where we have used a vector identity

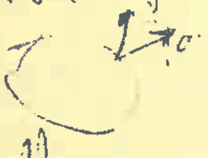
$$\int_V d^3\underline{r}\,\phi\,\nabla\psi = -\int_V d^3r\,(\nabla\phi)\,\psi + \int_S d^2r\,\hat{n}\,\phi\,\psi$$

Suppose we could demonstrate the surface term vanishes. Then we could identify

$$L_1^\dagger\circ = -\underline{v}\cdot\nabla\circ$$

But how would we do this? By restricting the type of functions on which the L operates on. In particular, suppose we only allow $g(\underline{r},\underline{v})$ to be functions which satisfy the b.c. of no incoming neutrons on the surface

$$g(\underline{r}_s,\underline{v}) = 0 \qquad \hat{n}\cdot\underline{v} < 0$$



Notice that this implies the surface integral vanishes over these functions. The only way we can for the remainder of the integral to vanish is to place a restriction on the type of functions $f(\underline{r},\underline{v})$

-- in particular, we require that $f(\underline{r},\underline{v})$ satisfy

$$f(\underline{r}_s,\underline{v}) = 0 \qquad \hat{\nu}\cdot\underline{v} > 0$$

This is [illegible] "adjoint" boundary condition, i.e. functions which satisfy [illegible] "adjoint functions"

In conclusion then, the adjoint transport operator is

$$L^{\dagger}\circ = +\underline{v}\cdot\nabla - v\Sigma(\underline{r},v,t) + \int d^3v'\, v\Sigma_s(v\to\underline{v}',\underline{r},t)\,\circ$$

with corresponding adjoint b.c.

$$f^{\dagger}(\underline{r}_s,\underline{v}) = 0 \qquad \hat{\nu}\cdot\underline{v} > 0$$

2.4.1.3: The Adjoint Flux and Neutron Importance

Consider the steady state transport equation for a subcritical system containing an independent source

$$\underline{v}\cdot\nabla n + v\Sigma\, n(\underline{r},\underline{v}) = \int d^3v'\,\Sigma_s(\underline{v}'\to\underline{v},\underline{r})\, n(\underline{r},\underline{v}') + Q(\underline{r},\underline{v}) \qquad (1)$$

with b.c: $\quad n(\underline{r}_s,\underline{v}) = 0 \qquad \hat{\nu}\cdot\underline{v} < 0$

Then consider the adjoint equation -- also with a source, Q^{+}

$$-\underline{v}\cdot\nabla n^{+} + \Sigma_t\, n^{+}(\underline{r},\underline{v}) = \int d^3v'\,\Sigma_s(v\to\underline{v}',\underline{r})\, n^{+}(\underline{r},\underline{v}') + Q^{+}(\underline{r},\underline{v}) \qquad (2)$$

with adjoint b.c: $\quad n^{+}(\underline{r}_s,\underline{v}) = 0 \qquad \hat{\nu}\cdot\underline{v} > 0$

Suppose we multiply (1) by n^{+} and integrate over $(\underline{r},\underline{v})$, then multiply (2) by n and integrate over $(\underline{r},\underline{v})$ and subtract to find

$$\int d^3r \int d^3v\; Q(\underline{r},\underline{v})\, n^{+}(\underline{r},\underline{v}) = \int d^3r \int d^3v\; Q^{+}(\underline{r},\underline{v})\, n(\underline{r},\underline{v})$$

This relation holds for arbitrary sources $Q(\underline{r},\underline{v})$ and $Q^{+}(\underline{r},\underline{v})$. In particular, it holds for the case where we choose

$$Q(\underline{r},\underline{v}) = \delta(\underline{r}-\underline{r}_0)\,\delta(\underline{v}-\underline{v}_0) \qquad (*)$$

$$Q^{+}(\underline{r},\underline{v}) = v\,\Sigma_d(\underline{r},\underline{v})$$

where $\Sigma_d(\underline{r},\underline{v})$ is the cross section for a neutron detector. Then

$$n^{+}(\underline{r}_0,\underline{v}_0) = \int d^3r \int d^3v\, v\,\Sigma_d(\underline{r},\underline{v})\, n(\underline{r},\underline{v})$$

Notice that the right hand side of this equation is proportional to the detector response created by the unit source (*). Hence $n^{+}(\underline{r}_0,\underline{v}_0)$ is proportional to the detector response due to a unit point source at $(\underline{r}_0,\underline{v}_0)$. In this sense, $n^{+}(\underline{r}_0,\underline{v}_0)$ is a measure of the "importance" of a neutron at $(\underline{r}_0,\underline{v}_0)$ in contributing to the response of a detector with cross section $\Sigma_d(\underline{r},\underline{v})$.

Notice that this physical interpretation is compatible with the adjoint boundary condition which demands that the importance of a neutron leaving the system is zero

$$n^{+}(\underline{r}_s,\underline{v}) = 0 \qquad \hat{\sigma}\cdot\underline{v} > 0$$

2.4.1.4. Eigenvalue Problems

The problem of finding the non-trivial solutions of a homogeneous equation of the form

$$H\,\Psi_n = \lambda_n \Psi_n$$

with certain boundary and regularity conditions on Ψ_n is called an eigenvalue problem. The set of complex numbers λ_n for which (*) has non-trivial solutions Ψ_n are called eigenvalues, and the corresponding solutions Ψ_n are called eigenfunctions of the operator H. Generally there will be many such eigenvalues and eigenfunctions (usually an infinite number for the problems we will deal with).

If we recall the definition of the adjoint operator:

$$(H^\dagger f, g) = (f, Hg)$$

then we can similarly consider eigenvalues and eigenfunctions of the adjoint operator

$$H^\dagger \psi_n^\dagger = \lambda_n^\dagger \psi_n^\dagger$$

One can prove:

i.) $\{\lambda_n\}$ and $\{\lambda_n^\dagger\}$ are complex conjugates of each other -- i.e. for a given λ_n there is a $\lambda_n^\dagger$ such that $\lambda_n^\dagger = \lambda_n^*$.

ii.) $\{\psi_n\}$ and $\{\psi_n^\dagger\}$ form a biorthogonal set

$$(\psi_m^\dagger, \psi_n) = 0 \qquad m \neq n.$$

iii.) $\{\psi_n\}$ are $\overset{\text{usually}}{\text{complete}}$ -- i.e. any function can be expanded in $\{\psi_n\}$ as

$$f(\underline{r}, \underline{v}) = \sum_n c_n \psi_n(\underline{r}, \underline{v})$$

where

$$c_n = \frac{(\psi_n^\dagger, f)}{(\psi_n^\dagger, \psi_n)}$$

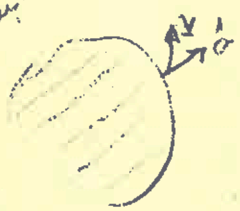
iv.) The eigenvalues of a self-adjoint operator $[H^\dagger = H]$ are real and the eigenfunctions form a complete, orthogonal set.

v.) The eigenfunctions satisfy the "closure" relation

$$\sum_n \psi_n^\dagger(\underline{r}', \underline{v}')\, \psi_n(\underline{r}, \underline{v}) = \delta(\underline{r} - \underline{r}')\, \delta(\underline{v} - \underline{v}')$$

[illegible]
[illegible]

Actually, such eigenvalue problems are rather intimately connected with separation of variables methods in partial differential equations. Consider an initial value problem involving the transport operator



$$\frac{\partial n}{\partial t} = Ln \qquad \text{i.c:}\ n(\underline{r},\underline{v},0) = n_0(\underline{r},\underline{v})$$

$$\text{bc:}\ n(\underline{r}_s,\underline{v},t) = 0 \quad \hat{n}\cdot\underline{v}<0$$

Assume a separable solution

$$n(\underline{r},\underline{v},t) = T(t)\,\psi(\underline{r},\underline{v})$$

Then find

$$\psi\frac{dT}{dt} = T\,L\psi$$

or

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\psi}L\psi = \lambda \qquad \text{separation constant}$$

Hence

$$\frac{dT}{dt} - \lambda T = 0 \quad\Rightarrow\quad T(t) \sim e^{\lambda t}$$

The remainder of the problem takes the form

$$L\psi = \lambda\psi$$

or since there will in general be many λ for which there are nontrivial solutions ψ, rewrite this as

$$L\psi_\lambda = \lambda\psi_\lambda \qquad \text{-- an eigenvalue problem}$$

$$\psi_\lambda(\underline{r}_s,\underline{v}) = 0 \quad \hat{n}\cdot\underline{v}<0$$

Where to now? Expand general solution in a complete set of the eigenfunctions

$$n(\underline{r},\underline{v},t) = \sum_n c_n\,\psi_n(\underline{r},\underline{v})\,e^{\lambda_n t}$$

How do we get c_n? Use initial condition

$$n_0(\underline{r},\underline{v}) = \sum_n c_n\,\psi_n(\underline{r},\underline{v})$$

Now multiply by ψ_m^+, integrate, and use biorthogonality to find

$$c_n = \frac{(\psi_n^+, n_0)}{(\psi_n^+, \psi_n)}$$

Hence the solution to the original problem is just

$$n(\underline{r},\underline{v},t) = \sum_n \frac{(\psi_n^+, n_0)}{(\psi_n^+, \psi_n)} \psi_n(\underline{r},\underline{v}) e^{\lambda_n t}$$

-- very easy, <u>provided</u> we know the eigenfunctions ψ_n and eigenvalues λ_n [which we usually don't].

2.4.2. AMPLITUDE AND SHAPE FACTORS

Consider once again the neutron kinetics equations

$$\frac{\partial n}{\partial t} + \underline{v}\cdot\nabla n + v\Sigma_t n(\underline{r},\underline{v},t) = \int d^3v' \left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu(v')(1-\beta)\chi_p(v)v'\Sigma_f(v')\right] n(\underline{r},\underline{v}',t) + \sum_{i=1}^{6}\lambda_i\chi_i(v)C_i(\underline{r},t) + Q(\underline{r},\underline{v},t) \qquad (1)$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + \int d^3v\, \beta_i\nu(v)v\Sigma_f(v)\, n(\underline{r},\underline{v},t) \qquad i = 1,\dots 6$$

Here we have left as implicit the dependence of the cross sections on r and t [and, perhaps, upon $n(\underline{r},\underline{v},t)$ as well.]

For very simple problems (e.g. simple geometry, [illegible] feedback, etc.) we can [illegible] these equations [illegible] numerically (e.g. [illegible]) [illegible] "reactor".

[illegible] equations (1) by integrating over r and v. But this [illegible] produce [illegible] would [illegible] This approach [illegible] It has been found to be [illegible] critical (time-independent) reference system. Such a critical system as described by the [illegible] equations (1)

$$\underline{v}\cdot\nabla n_0 + v\Sigma_t^0 n_0(\underline{r},\underline{v}) = \int d^3v' \left[v'\Sigma_s^0(\underline{v}'\to\underline{v}) + \nu(v')\chi(v)v'\Sigma_f^0(v')\right] n_0(\underline{r},\underline{v}') \qquad (2)$$

where we have eliminated C_i using

$$\lambda_i C_i = \int d^3v'\, \beta_i\nu(v')v'\Sigma_f(v')\, n_0(\underline{r},\underline{v}')$$

and defined the total emergent fission spectrum

$$\chi(v) \equiv (1-\beta)\chi_p(v) + \sum_{i=1}^{6}\beta_i\chi_i(v)$$

while the zero superscript denotes cross section values for which the reactor will be critical.

Actually, it will be of more use to consider the adjoint equation for this reference reactor

$$-\underline{v}\cdot\nabla n_0^{+} + v\Sigma_t^0\, n_0^{+}(\underline{r},\underline{v}) = \int d^3v' \left[v\Sigma_s^0(\underline{v}\to\underline{v}') + \nu(v)\chi(v')v\Sigma_f^0(v) \right] n_0^{+}(\underline{r},\underline{v}') \qquad (3)$$

In practice, $n_0^{+}(\underline{r},\underline{v})$ can be determined by a standard criticality search in the same manner that one determines $n_0(\underline{r},\underline{v})$. Recall the essential idea in this latter scheme is to introduce an artificial scaling parameter, k_{eff}, which is adjusted such that

$$\underline{v}\cdot\nabla n_0 + v\Sigma_t\, n_0(\underline{r},\underline{v}) = \int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v})\, n_0(\underline{r},\underline{v}') + \frac{1}{k_{eff}} \int d^3v'\, \nu(v')\chi(v)v'\Sigma_f(v')\, n_0(\underline{r},\underline{v}')$$

has a nontrivial solution for any choice of cross sections or geometry. The idea is to adjust the cross sections and geometry until $k_{eff} = 1$ -- which is just the criticality condition

2/4/70

We will now use this critical reference reactor to derive the P.R.K.E. This has been accomplished by a number of authors [Hurwitz (1949), Ussachoff (1955), Henry (1955), Gyftopoulos (1964), Becker (1968)]. We will follow the derivation due to Henry, since this is perhaps the most popular treatment in current literature references.

The essential idea is to formally write $n(\underline{r},\underline{v},t)$ as the product of an "amplitude factor" $P(t)$ and a "shape factor" $\psi(\underline{r},\underline{v},t)$

$$n(\underline{r},\underline{v},t) = P(t)\,\psi(\underline{r},\underline{v},t) \qquad (4)$$

where the shape factor $\psi(\underline{r},\underline{v},t)$ is hopefully slowly varying in time. Of course this separation is rather arbitrary. We shall further specify (actually, "normalize") the shape factor such that

$$\frac{\partial}{\partial t}\left[\int d^3r \int d^3v \; n_0^+(\underline{r},\underline{v})\,\Psi(\underline{r},\underline{v},t)\right] \equiv \frac{\partial}{\partial t}\,(n_0^+,\Psi) \qquad (5)$$

Comments:

i.) Notice that this condition does not imply that $\Psi(\underline{r},\underline{v},t)$ is independent of time -- only that its "importance-weighted" integral is.

Such a normalization is always possible, although Ψ is sometimes very difficult to determine -- indeed, this is actually only a useful scheme for small perturbations from criticality.

ii.) Note that (n_0^+,Ψ) is proportional to the total importance of neutrons in the reference reactor distributed with a shape Ψ. We are demanding in (5) that this total importance be time-independent

iii.) Suppose we take the scalar product of n_0^+ with (4) to find

$$P(t) = \frac{(n_0^+,n)}{(n_0^+,\Psi)}$$

But since (5) implies that (n_0^+,Ψ) is independent of time, we find that $P(t)$ is proportional to the importance of neutrons placed in the critical reference reactor with distribution $n(\underline{r},\underline{v},t)$. We are still free to normalize $P(t)$. There are two popular choices

a.) Set $P(t_0)$ equal to the reactor power at time t_0: Then since

$$\text{Power} = w_f\int d^3r\int d^3v\, v\Sigma_f(v)\,n(\underline{r},\underline{v},t) = P(t)\, w_f\int d^3r\int d^3v\, v\Sigma_f(v)\,\Psi(\underline{r},\underline{v},t)$$

this implies that at time t_0

$$w_f \int d^3r \int d^3v \, v \Sigma_f(v) \psi(\underline{r},\underline{v},t_0) = 1$$

Since $\psi(\underline{r},\underline{v},t)$ is a function of time, this normalization will only hold at t_0, and hence $P(t)$ can only be truly identified as the power at this time. However if the shape factor is slowly varying in time, the $P(t)$ will be, to a good approximation, a measure of the reactor power.

b.) We could have also normalized $P(t_0) \to n(t_0)$, the total neutron population in the reactor at t_0. Usually, however, power is the more convenient normalization (particularly when considering feedback).

2.4.3. THE "EXACT" POINT REACTOR KINETICS EQUATIONS

We will now utilize the neutron importance $n_0^{+}(\underline{r},\underline{v})$ for the critical reference reactor to derive the point reactor kinetics equations (P.R.K.E.). First substitute the forms $n(\underline{r},\underline{v},t) = P(t)\,\psi(\underline{r},\underline{v},t)$ into the kinetics equations (1)

$$\psi \frac{dP}{dt} + P\frac{\partial \psi}{\partial t} + P\underline{v}\cdot\nabla\psi + P v \Sigma_t \psi = P\int d^3v' \left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu(v')(1-\beta)\chi_p(v) v'\Sigma_f(v') \right] \psi(\underline{r},\underline{v}',$$
$$+ \sum_{i=1}^{6} \lambda_i \chi_i(v) C_i(\underline{r},t) + Q(\underline{r},\underline{v},t) \qquad (6).$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i(\underline{r},t) + P\int d^3v \, \beta_i \nu(v) v \Sigma_f(v) \psi(\underline{r},\underline{v},t) \qquad i = 1, \cdots 6 \qquad (7)$$

Now multiply by $n_0^{+}(\underline{r},\underline{v})$ and integrate over $\underline{r}$ & $\underline{v}$, using the scalar product notation for convenience

$$(f,g) \equiv \int d^3r \int d^3v \; f^*(\underline{r},\underline{v})\, g(\underline{r},\underline{v})$$

$$\frac{dP}{dt}(n_0^+,\psi) + P\cancelto{0}{\frac{\partial}{\partial t}(n_0^+,\psi)} + P(n_0^+,\underline{v}\cdot\nabla\psi) + P(n_0^+,v\Sigma_t\psi)$$
$$= P\left(n_0^+,\int d^3v'\left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu(v')(1-\beta)\chi_p(v)v'\Sigma_f(v')\right]\psi(\underline{r},\underline{v}',t)\right.$$
$$+\sum_{i=1}^{6}\lambda_i(n_0^+,\chi_i C_i) + (n_0^+,Q) \qquad (8)$$

where we have noted explicitly the normalization condition ⑤. Further, take the scalar product of the precursor concentration equation with $\chi_i n_0^+$

$$\frac{\partial}{\partial t}(n_0^+,\chi_i C_i) = -\lambda_i(n_0^+,\chi_i C_i) + P\left(n_0^+,\chi_i(v)\int d^3v\,\beta_i(v)\nu v\Sigma_f(v)\psi(\underline{r},\underline{v},t)\right) \qquad i=1,\cdots 6 \qquad (9)$$

Also take the scalar product of the adjoint criticality equation ③ with $P(t)\psi(\underline{r},\underline{v},t)$ (using the definition of the adjoint operator) to find

$$-P(\underline{v}\cdot\nabla n_0^+,\psi) + P(n_0^+,v\Sigma_t\psi) = P\left(n_0^+,\int d^3v'\left[v'\Sigma_s^0(v'\to v) + \nu(v')\chi(v)v'\Sigma_f^0(v')\right]\psi(\underline{r},\underline{v}',t)\right) \qquad (10)$$

Next, subtract equation (9) from (8) [using the adjoint boundary conditions to eliminate the divergence terms], and divide by (n_0^+,ψ) to find

$$\frac{dP}{dt} + P\frac{(n_0^+,v\,\delta\Sigma_t\psi)}{(n_0^+,\psi)} = P\frac{\left(n_0^+,\int d^3v'\,\delta\left[v'\Sigma_s(v'\to v)+\nu(v')\chi(v)v'\Sigma_f(v')\right]\psi(\underline{r},\underline{v}',t)\right)}{(n_0^+,\psi)}$$
$$-P\frac{\left(n_0^+,\int d^3v'\,\nu(v')\sum_i\beta_i\chi_i v\Sigma_f\psi(\underline{r},\underline{v}',t)\right)}{(n_0^+,\psi)} + \sum_{i=1}^{6}\lambda_i\frac{(n_0^+,\chi_i C_i)}{(n_0^+,\psi)} + \frac{(n_0^+,Q)}{(n_0^+,\psi)}$$

where we define the variations of the cross sections from those of the reference reactor by

$$\delta\Sigma_t = \Sigma_t - \Sigma_t^0$$
$$\delta\left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu(v')\chi(v)v\Sigma_f(v')\right] = v'\Sigma_s(v'\to v) - v'\Sigma_s^0(v'\to v)$$
$$+\left[\nu(v')\chi(v)v'\Sigma_f(v')\right]$$
$$-\left[\nu(v')\chi(v)v'\Sigma_f^0(v')\right] \qquad (11)$$

Suppose we give some of these adjoint weighted averages names. In particular, define

$$\rho(t) \equiv \frac{1}{F}\left(n_0^+, \int d^3v'\, S\left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu\chi(v)v'\Sigma_f(v')\right]\psi(\underline{r},\underline{v}',t)\right) - \frac{1}{F}\left(n_0^+, v\, S\, \Sigma_t \psi\right) \qquad (12)$$

$$\bar{\beta}_i(t) \equiv \frac{1}{F}\left(n_0^+, \int d^3v'\, \nu(v')\beta_i\chi_i(v)\, v'\Sigma_f(v')\psi(\underline{r},\underline{v}',t)\right), \quad \bar{\beta} \equiv \sum_{i=1}^{6}\bar{\beta}_i(t) \qquad (13)$$

$$\Lambda(t) \equiv \frac{1}{F}\left(n_0^+, \psi\right) \qquad (14)$$

$$\bar{C}_i(t) \equiv \frac{1}{\Lambda F}\left(n_0^+, \chi_i C_i\right) \qquad (15)$$

$$Q(t) = \frac{1}{\Lambda F}\left(n_0^+, Q\right) \qquad (16)$$

[The factor F in these definitions is just an arbitrary scaling factor (note it cancels out in the equations themselves), and is inserted as a normalization to facilitate an easier interpretation of these parameters. The most common choice is

$$F(t) = \left(n_0^+, \int d^3v'\, \nu(v')\chi(v)\, v'\Sigma_f(v')\, \psi(\underline{r},\underline{v}',t)\right) \qquad (17)$$
]

We then can rewrite (9) and (11) in the more familiar form

$$\begin{aligned} \frac{dP}{dt} &= \left[\frac{\rho(t) - \bar{\beta}(t)}{\Lambda(t)}\right] P(t) + \sum_{i=1}^{6}\lambda_i \bar{C}_i(t) + Q(t) \\ \frac{d\bar{C}_i}{dt} &= \left[\frac{\bar{\beta}_i(t)}{\Lambda(t)}\right] P(t) - \lambda_i \bar{C}_i(t) \qquad i = 1,\dots,6 \end{aligned} \qquad (18)$$

These, then, are the general form of the "POINT-REACTOR KINETICS EQUATIONS".

Note: i.) These equations are still quite exact and hence equivalent in every respect to the original neutron kinetics equations (including their complexity).

ii.) Since F is an arbitrary scale factor, it is clear that the definitions of ρ, Λ, and β are meaningless in themselves.

only their ratios can be interpreted unambiguously [e.g. ρ/Λ, or β/Λ.]

ii) Of course we still haven't accomplished anything. These equations contain lots of undetermined parameters -- in particular the shape function $\psi(\underline{r},\underline{v},t)$ -- which are still too difficult to determine. Their importance, however, lies in the fact that they provide a basis for introducing approximations which will allow the practical study of nuclear reactor dynamics.

Let's now turn our attention to the quantities defined in the P.R.K.E.

Reactivity: The quantity $\rho(t)$ as defined in equation (12) is referred to as the "reactivity". As expected, it is proportional to an average of the variation of the cross sections from their values in the critical reference reactor. Actually there has been quite a bit of controversy‡ over the interpretation of $\rho(t)$ as a reactivity. In the event that the shape function should have the same dependence on $(\underline{r},\underline{v})$ as the fundamental mode for the critical reference reactor, $n_0(\underline{r},\underline{v})$, then $\rho(t)$ reduces to the more conventional "static reactivity" defined in elementary treatments of reactor kinetics. We will return momentarily to discuss the generalization, the "dynamic reactivity".

Mean Neutron Generation Time

$$\Lambda(t) = \frac{(n_0^+, \psi)}{(n_0^+, \int d^3v'\, \nu(v')\chi(v)\nu\Sigma_f(v')\psi(\underline{r},\underline{v}',t))} = \frac{\text{importance of all neutrons in reactor at } t}{\text{importance of all fission neutrons produced per second}}$$

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‡ See, for example, T. Gozani, Nukleonik 5, 55 (1963) and N. Corngold, Trans. Am. Nuc. Soc. 7, 211 (1964)

Effective Delayed Neutron Fraction

$$\overline{\beta_i}(t) = \frac{\left(n_0^+, \int d^3v'\, \nu(v')\, \beta_i\, \chi_i(v)\, v'\, \Sigma_f(v')\, \psi(\underline{r},\underline{v}',t)\right)}{\left(n_0^+, \int d^3v'\, \nu(v')\, \chi(v)\, v'\, \Sigma_f(v')\, \psi(\underline{r},\underline{v}',t)\right)}$$

$$\equiv \frac{\text{importance of all delayed neutrons of } i\text{th group emitted/sec}}{\text{'' \quad '' \quad '' fission neutrons (delayed \& prompt) '' \quad '' \quad ''}}$$

Effective Number of Delayed Neutron Precursors

$$\overline{C}_i(t) = \frac{\left(n_0^+, \chi_i\, C_i\right)}{\Lambda\left(n_0^+, \int d^3v'\, \nu(v')\, \chi(v)\, v'\, \Sigma_f(v')\, \psi(\underline{r},\underline{v}',t)\right)}$$

Be a little careful here since $\overline{C}_i$ is the effective number of delayed neutron precursors of the ith group -- provided $P(t)$ is normalized as the total neutron population. However, should $P(t)$ be normalized to the reactor power, then $\overline{C}_i(t)$ must be interpreted as the number of precursors multiplied by the rate at which energy is produced per neutron.

Effective Source

$$Q(t) = \frac{\left(n_0^+, Q\right)}{\Lambda\left(n_0^+, \int d^3v'\, \nu(v')\, \chi(v)\, v'\, \Sigma_f(v')\, \psi(\underline{r},\underline{v}',t)\right)}$$

which is proportional to the importance of source neutrons introduced into reactor per second.

2.4.4. APPROXIMATION OF THE SHAPE FACTOR

We still haven't accomplished anything, all of the quantities in the P.R.K.E. [e.g. $\rho, \bar{\beta}_i, \Lambda$] depend on a knowledge of $\psi(\underline{r},\underline{v},t)$ -- which we could only determine rigorously by solving the reactor kinetics equations themselves (precisely what we are trying to avoid). We will now approximate the form of the shape factor to allow an explicit calculation of $\rho, \bar{\beta}$, and Λ.

a.) Constant Shape Approximation

Frequently we can neglect the time dependence of $\psi(\underline{r},\underline{v},t)$. [In particular, if the reactor is critical (steady-state) or on an asymptotic period, the angular density becomes rigorously separable in $(\underline{r},\underline{v})$ and t.] This approximation is of particular interest since it implies Λ and $\bar{\beta}$ are independent of time [$\rho(t)$ may still depend on time through the time dependence of the cross sections], and this is the only situation allowing an analytical study of the P.R.K.E.

The common assumption is to approximate $\psi(\underline{r},\underline{v},t)$ by the critical distribution $n_0(\underline{r},\underline{v})$ -- i.e.

$$n(\underline{r},\underline{v},t) \sim P(t)\, n_0(\underline{r},\underline{v})$$

The normalization condition (5) is satisfied automatically (since ψ is independent of time). [Note this was essentially the assumption we used in our early derivation of the P.R.K.E.]

b.) Adiabatic Approximation -- λ-mode expansions

An improvement upon the constant shape approximation is to allow $\psi(\underline{r},\underline{v},t)$ to depend "parametrically" upon time. To motivate this, recall the equation for the critical distribution

$$\underline{v}\cdot\nabla n_0 + v\Sigma_t^0\, n_0(\underline{r},\underline{v}) - \int d^3v'\, v'\,\Sigma_s^0(\underline{v}'\to\underline{v})\, n_0(\underline{r},\underline{v}) = \int d^3v'\, \nu(v')\chi(v)\, v'\,\Sigma_f^0(v')\, n_0(\underline{r}$$

or more symbolically

$$L^\circ n_0 = M^\circ n_0$$

where L° is the transport operator (with critical cross sections), while M° is the multiplication operator. Now to perform a criticality calculation, it was mentioned that the actual scheme was to insert a factor keff

$$\underline{v}\cdot\nabla n_{keff} + v\Sigma_t n_{keff}(\underline{r},\underline{v}) - \int d^3v'\, v'\Sigma_s(\underline{v}'\to\underline{v})\, n_{keff}(\underline{r},\underline{v}) = \frac{1}{k_{eff}}\int d^3v'\, \nu(v')f(v)\, v'\Sigma_f(v')\, n_{keff} \qquad (*)$$

or symbolically

$$L\, n_{keff} = \frac{1}{k_{eff}} M\, n_{keff} \qquad (**)$$

Again the idea is to adjust keff to that value for which (*) has a non-trivial solution. Then by adjusting the cross sections and geometry such that keff = 1, we achieve criticality $[\,k_{eff} = 1 \Rightarrow n_{keff} \to n_0\,]$

Now in general the cross sections will depend upon time. The adiabatic approximation will make to to calculate n_{keff} for a given instant using

$$L(t)\, n_{keff} = \frac{1}{k_{eff}} M(t)\, n_{keff}$$

and then use this as our shape factor. Since L and M depend parametrically on time through the cross sections, so do keff and n_{keff}. Such an approximation is termed the 'adiabatic approximation'

$$n(\underline{r},\underline{v},t) \sim P(t)\, n_{keff}(\underline{r},\underline{v},t)$$

and would be expected to be valid for slow power changes [actually this approximation works well for the major portion of the startup effects in reactor kinetics even for fairly rapid power transients.] Note that we no longer have any guarantee that the normalization is satisfied:

$$\frac{d}{dt}(n_0^{\dagger}, n_{keff}) \neq 0$$

— but, hopefully, the time dependence here will be small

Incidentally, the equation

$$L n_{k_{eff}} = \frac{1}{k_{eff}} M n_{k_{eff}}$$

may have many values of keff for which nontrivial solutions exist. It is just an eigenvalue problem

$$L n_\lambda = \frac{1}{\lambda} M n_\lambda$$

The eigenfunctions n_λ corresponding to a given value of λ are called "reactivity modes", or more commonly, "λ-modes". The lowest eigenvalue λ_0 is what we have been calling keff, while $n_0 \rightarrow n_{k_{eff}}$. Hence we are replacing ψ by the lowest λ-mode at each instant of time.

One can actually demonstrate that in this approximation,

$$\rho(t) = \frac{k_{eff}(t) - 1}{k_{eff}(t)}$$

where keff(t) is the lowest eigenvalue λ determined at each instant of time. This quantity is referred to as the "static reactivity".

◆

There is an alternative form of the adiabatic approximation based upon so-called "prompt" or "ω-modes". Suppose we write out the neutron kinetics equations (in operator notation)

$$\frac{\partial n}{\partial t} = Ln + M_p n + \sum_{i=1}^{6} \lambda_i \chi_i C_i + S$$

$$\frac{\partial C_i}{\partial t} = -\lambda_i C_i + \frac{1}{\chi_i} M_i n$$

where, again, L is the transport operator and M_i is the multiplication operator

$$M_i \circ \equiv \int d^3v' \, \nu(v') \, \beta_i \, \chi_i(v) \, v' \Sigma_f(v') \circ$$

Now recall that for problems of the form

$$\frac{\partial n}{\partial t} = Ln$$

the natural scheme suggested by separation of variables was to consider the eigenvalue problem

$$L\Psi_\lambda = \lambda\Psi_\lambda$$

and then expand $n(t)$ in the eigenfunctions Ψ_λ.

But the only way we can write (*) in this form is to use matrix notation:

$$\frac{\partial}{\partial t}\begin{pmatrix} n \\ C_1 \\ \vdots \\ C_6 \end{pmatrix} = \begin{pmatrix} L+M_p & \lambda_1 & \lambda_2 & \cdots & \lambda_6 \\ \frac{1}{\chi_1}M_1 & -\lambda_1 & 0 & & \\ \vdots & 0 & & & \\ \frac{1}{\chi_6}M_6 & 0 & & & -\lambda_6 \end{pmatrix}\begin{pmatrix} n \\ C_1 \\ \vdots \\ C_6 \end{pmatrix}$$

or

$$\frac{\partial \underline{n}}{\partial t} = \underline{\underline{H}}\,\underline{n}$$

We now consider the matrix operator eigenvalue problem

$$\underline{\underline{H}}\,\underline{\Psi_n} = \omega_n \underline{\Psi_n}$$

Note that the general solution will then be expressed as

$$\underline{n}(t) = \sum_n a_n \underline{\Psi_n} e^{\omega_n t}$$

One refers to the $\underline{\Psi_n}$ as "period-modes" or "ω-modes". In particular the ω-mode corresponding to the largest value, ω_0, of ω will be the persisting mode, while the reactor period is just $T = 1/\omega_0$.

Now again, in general the cross sections in the neutron kinetics equations will depend upon time t. The adiabatic approximation using ω-modes takes this into account by taking

$\Psi(\underline{r},\underline{v},t)$ to the the first component of the lowest ω-mode satisfying

$$\underline{\underline{H}}(t)\,\underline{\Psi_n} = \omega_n \underline{\Psi_n}$$

More specifically, $\Psi(\underline{r},\underline{v},t)$ is taken as the lowest mode satisfying

$$\left[L(t) + M_p(t) + \sum_{i=1}^{6} M_i(t)\right] N_n(\underline{r},\underline{v}) = \omega_n \left[1 + \sum_{i=1}^{6} \frac{M_i}{\lambda_i + \omega_n}\right] N_n(\underline{r},\underline{v})$$

One can show that the reactivity then becomes

$$\rho(t) = \omega_0(t)\left[\Lambda(t) + \sum_{i=1}^{6} \frac{\overline{\beta}_i(t)}{\lambda_i + \omega_0(t)}\right]$$

-- the so-called "dynamic reactivity" [see A. Henry, Nuc. Sci. Eng. $\underline{20}$, 338 (1964)]

[Note that once again the normalization $\frac{\partial}{\partial t}(N_0^{\dagger}, N_0) \neq 0$ unless the time variation is very small.]

c) Further Comments

It should be remarked once again that only the constant shape approximation allows an analytical study of the P.R.K.E. The adiabatic approximations necessitate numerical calculation of either N_{adj} or N_0 at several times, which are then used to compute $\rho(t)$, $\overline{\beta}_i(t)$, and $\Lambda(t)$. However even this adiabatic approximation gives a very considerable savings over a direct numerical solution of the neutron kinetics equations themselves, since the shape function need only be calculated a few times, using the P.R.K.E. to determine $F(t)$ for intermediate times.

2.4.5: ALTERNATIVE DERIVATIONS OF THE P.R.K.E.

Recall the key steps involved in Henry's derivation

i.) average the neutron kinetics equations over the importance function $n_0^+(\underline{r},\underline{v})$ for a critical reference reactor
[this expresses $\rho(t)$ as a functional of the variations of the cross sections from their critical values]

ii.) write $n(\underline{r},\underline{v},t) = P(t)\,\psi(\underline{r},\underline{v},t)$ subject to the restriction

$$\frac{\partial}{\partial t}(n_0^+,\psi)=0$$

[this yields $P(t)$ proportional to the time variation of the fundamental mode in the reactor]

We must recognize that $P(t)$ is rigorously neither the power nor total neutron population. There have been attempts to modify the derivation of the P.R.K.E. to bypass this difficulty -- in particular by averaging the neutron kinetics equations over a time dependent quantity $n(\underline{r},\underline{v},t)$ rather than $n_0^+(\underline{r},\underline{v})$. For more details, see

E.P. Gyftopoulos, in "Technology of Nuclear Reactor Safety"
Vol. I, pp. 175-204, ed. by Thompson and Beckerley
[MIT Press, 1964]

M. Becker, Nuc. Sci. Eng. 31, 458 (1968)

The derivation of the P.R.K.E. has also been extended to include the case in which the cross sections are functionals of $n(\underline{r},\underline{v},t)$ [see Akcasu]. Rather complicated so I won't get involved here. Merely note this results in

$$\rho(t) \rightarrow \rho[P(t),t]$$

III. SOLUTION OF THE POINT-REACTOR KINETICS EQUATIONS WITHOUT FEEDBACK

We will now direct our attention to the study of the point reactor kinetics equations

$$\begin{aligned} \frac{dP}{dt} &= \left[\frac{\rho(t)-\bar{\beta}}{\Lambda}\right]P(t) + \sum_{i=1}^{6}\lambda_i\bar{C}_i(t) + Q(t) \\ \frac{d\bar{C}_i}{dt} &= \frac{\bar{\beta}_i}{\Lambda}P(t) - \lambda_i\bar{C}_i(t) \qquad i=1,\ldots,6 \end{aligned} \tag{1}$$

Here we have assumed that $\bar{\beta}_i$ and Λ are time-independent [which implies that we have employed the constant shape approximation for $\psi(\underline{r},\underline{v},t)$]. Note that $\rho(t)$ is still time-dependent due to

i.) externally controlled changes in the cross sections (e.g., control rod motion)

ii.) feedback effects which imply that $\rho(t)$ is actually a functional $\rho(t,P(t))$ of the power $P(t)$.

These latter feedback effects imply that the P.R.K.E. are actually nonlinear and hence very complicated to study.

There are certain instances in which we can neglect the feedback effects and consider $\rho(t)$ to be a known function of time [hence converting (1) into a linear set of 7 ordinary differential equations — with a variable coefficient, $\rho(t)$, of course].

i.) Zero-power reactors (ZPR): In this case, the power level is so low that there is no appreciable feedback. This situation is of particular concern in reactor startup and critical mockup experiments.

ii.) Knowledge of the time behavior of $P(t)$ for a given $\rho(t)$ can be compared with experimental measurements to determine feedback.

3.1. ALTERNATIVE FORMS OF THE P.R.K.E. 2/11/70

3.1.1 The Standard Form

First we will rewrite the P.R.K.E. in a slightly more convenient form (the so-called "standard" form)

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t) - 1]P(t) + \sum_{i=1}^{6}\lambda_i C_i(t) + \frac{\Lambda}{\beta}Q(t) \qquad (3a)$$

$$\frac{dC_i}{dt} = a_i P(t) - \lambda_i C_i(t) \qquad i = 1, \cdots 6 \qquad (3b)$$

where we define

$$a_i \equiv \bar{\beta}_i/\bar{\beta} \qquad \left[\sum_{i=1}^{6} a_i = 1\right]$$

$$\$(t) = \rho(t)/\beta$$

$$C_i(t) = \bar{C}_i(t)\,\Lambda/\beta$$

(where we will occasionally omit the bar on $\bar{\beta}_i$ and $\bar{\beta}$ for convenience]

Note:

i.) We have introduced the relative delayed neutron fractions a_i since these quantities are rather insensitive to reactor type and hence more useful than $\bar{\beta}_i$

ii.) $\$(t)$ is a measure of reactivity in dollars -- i.e.

$$\rho = \bar{\beta} \Rightarrow \$ = 1 \text{ dollar}$$

[$\$(t)$ is sometimes denoted as $k(t)$, but this latter notation can be confused with the multiplication constant.]

3.1.2. The Integro-differential Form

There is an alternative and occasionally more useful form of (3). First solve (2b) for

$$C_i(t) = C_i(t_0)\, e^{-\lambda_i(t-t_0)} + \int_{t_0}^{t} dt'\, a_i P(t')\, e^{-\lambda_i(t-t')}$$

Now if we assume

$$\lim_{t_0 \to -\infty} C_i(t_0)\, e^{\lambda_i t_0} = 0$$

then we find

$$C_i(t) = \int_{-\infty}^{t} dt'\, a_i P(t')\, e^{-\lambda_i(t-t')}$$

or letting $\tau = t - t'$

$$C_i(t) = \int_0^{\infty} d\tau\, a_i e^{-\lambda_i \tau}\, P(t-\tau).$$

Now substitute this into (2a) to find

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = (\$ - 1)P + \int_0^{\infty} d\tau \left[\sum_{i=1}^{6} a_i e^{-\lambda_i \tau}\right] P(t-\tau) + \frac{\Lambda}{\beta} Q(t)$$

or defining the "delayed neutron kernel"

$$D(\tau) = \sum_{i=1}^{6} \lambda_i a_i e^{-\lambda_i \tau}$$

(note $D(\tau)d\tau \equiv$ probability that a delayed neutron will be emitted in $d\tau$ about τ following a fission event at $\tau = 0$), we find the "integro-differential form" of the P.R.K.E.

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t) - 1]P(t) + \int_0^{\infty} d\tau\, D(\tau) P(t-\tau) + \frac{\Lambda}{\beta} Q(t) \qquad (3)$$

3.2. EXACT SOLUTIONS OF THE P.R.K.E. FOR A KNOWN REACTIVITY INSERTION

We will now attempt to solve the P.R.K.E. for a specified reactivity insertion $\$(t)$. It must be admitted at the outset that there are very few forms of $\$(t)$ for which exact solutions to the P.R.K.E. are known.

3.2.1. Step Reactivity Insertion

The response of reactor power to a step reactivity insertion $\$(t) = k_0,\ t > 0$, can be obtained by taking a Laplace transform in time

$$\tilde{P}(s) \equiv \int_0^\infty dt\, e^{-st} P(t)$$

to find

$$\tilde{P}(s) = \frac{\frac{\Lambda}{\beta} P(0) + \sum_{i=1}^{6} \frac{\lambda_i C_i(0)}{\lambda_i + s} + \frac{\Lambda}{\beta} \tilde{Q}(s)}{Y(s) - k_0}$$

where

$$Y(s) \equiv s\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \frac{a_i}{\lambda_i + s}\right] \equiv \frac{1}{Z(s)} \tag{4}$$

[The inverse of $Y(s)$ is sometimes referred to as the "zero power transfer function" $Z(s)$. More on this later.]

Suppose we assume that prior to the reactivity insertion, the reactor was operating at a constant power P_0. Then

$$C_i(0) = a_i P(0)/\lambda_i = a_i P_0/\lambda_i$$

Hence

$$\tilde{P}(s) = \frac{Y(s)\frac{P_0}{s} + \frac{\Lambda}{\beta}\tilde{Q}(s)}{Y(s) - k_0} \tag{5}$$

To invert, we merely find the zeros of the denominator

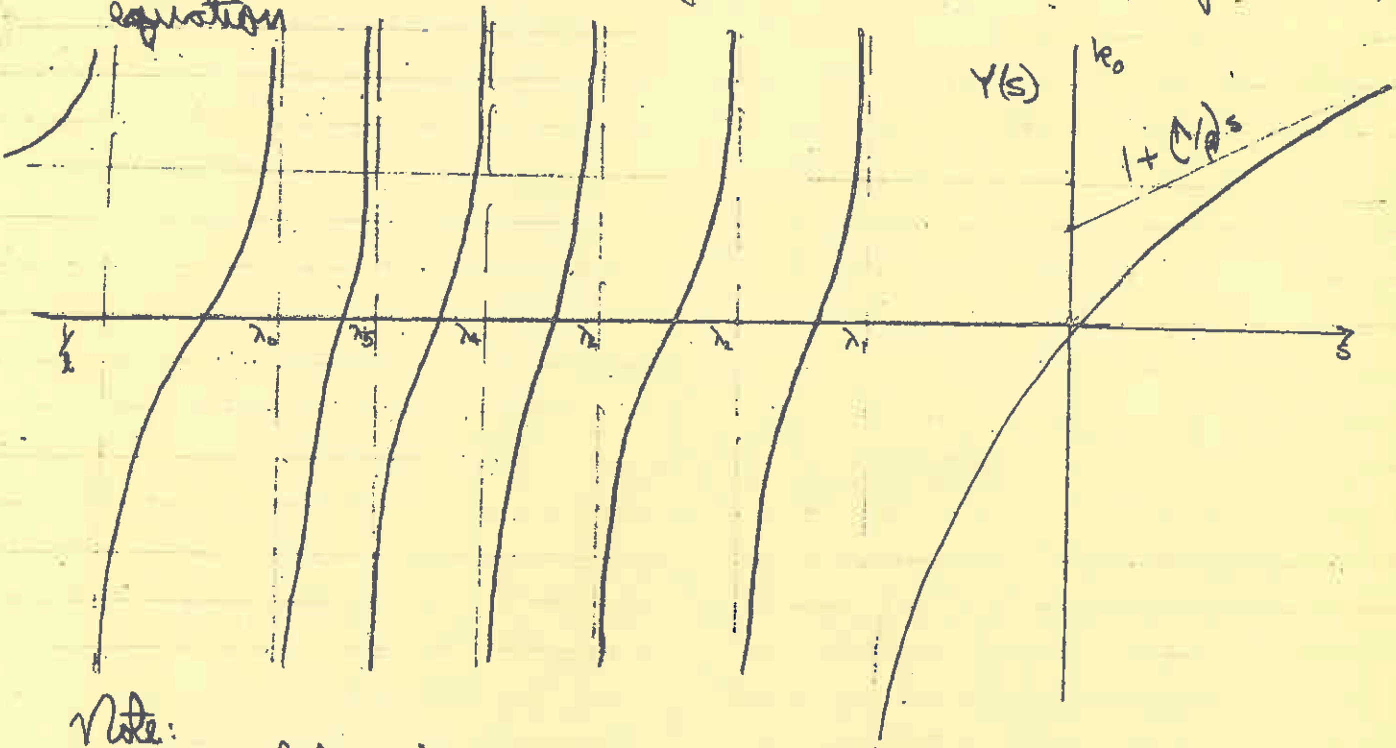
$$k_o = s\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \frac{a_i}{\lambda_i + s}\right]. \qquad \text{(the inhour equation)}$$

There will be seven such zeros, ω_j, $j = 0, 1, \cdots 6$. Hence we can write the general solution as

$$P(t) = P_o \sum_{j=0}^{6} \frac{k_o e^{\omega_j t}}{\omega_j \left.\frac{dY}{ds}\right|_{\omega_j}} + \frac{\Lambda}{\beta} \sum_{j=0}^{6} \frac{\tilde{Q}(\omega_j) e^{\omega_j t}}{\left.\frac{dY}{ds}\right|_{\omega_j}} + \frac{\Lambda}{\beta} \sum_{i} \frac{e^{\mu_i t}}{Y(\mu_i) - k_o} \operatorname{Res}\left[\tilde{Q}(s)\right]_{\mu_i} \qquad (6)$$

where μ_i are the poles of the source term $\tilde{Q}(s)$.

Recall that we know a good deal about the roots of the inhour equation



Note:

i.) Six of the roots, ω_j, $j = 1, \cdots, 6$ are less than zero.

ii.) Asymptotic or stable reactor period

$$T = 1/\omega_0.$$

iii.) For large, negative k_o, $T \Rightarrow -\lambda_1 = -80$ sec.

Notice that we can determine the reactivity of a reactor by measuring its stable period T and then relating it to reactivity through the inhour equation

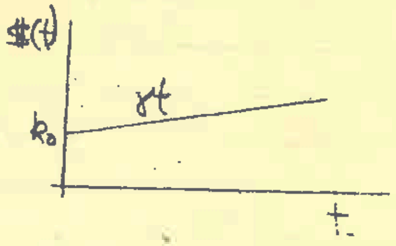
$$k_0 = \frac{1}{T}\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i T}{1+\lambda_i T}\right]$$

[One must be very careful to insure that the reactor is actually on its asymptotic period. Otherwise an appreciable error in k_0 can be made.]

5.2.2. Ramp Reactivity Insertion

Consider now a ramp reactivity insertion

$$\$(t) = k_0 + \gamma t$$



[This would model reactor startup.] The P.R.K.E. now become

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [k_0 + \gamma t - 1]P(t) + \sum_{i=1}^{6}\lambda_i C_i(t) + \frac{\Lambda}{\beta}Q_0$$

$$\frac{dC_i}{dt} = a_i P(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, 6$$

Unfortunately, we can no longer use Laplace transforms to solve these equations [since we will find $P(t) \sim e^{t^2}$ and hence the Laplace transform $\tilde{P}(s)$ does not exist].

Suppose we instead try to represent the solutions as definite integrals

$$P(t) = \int_{x_1}^{x_2} ds\, \hat{P}(s)\, e^{st}$$

$$C_i(t) = \int_{x_1}^{x_2} ds\, \hat{C}_i(s)\, e^{st}$$

If we note

$$\frac{dP}{dt} = \int_{x_1}^{x_2} ds\, s\hat{P}(s)e^{st}$$

and

$$tP(t) = \hat{P}(s)e^{st}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} ds\, \frac{d\hat{P}}{ds}e^{st}$$

then we can rewrite (7) as

$$\int_{x_1}^{x_2} ds\, e^{st}\left\{[Y(s)-k_0]\hat{P}(s) + \gamma\frac{d\hat{P}}{ds}\right\} = \frac{\Lambda}{\beta}Q_0 + \gamma\hat{P}(s)e^{st}\Big|_{x_1}^{x_2}$$

Since x_1 and x_2 are arbitrary, we will choose them such that the RHS is identically zero:

$$\frac{\Lambda}{\beta}Q_0 + \gamma\hat{P}(s)e^{st}\Big|_{x_1}^{x_2} = 0 \qquad (8)$$

Then we find

$$\gamma\frac{d\hat{P}}{ds} + [Y(s)-k_0]\hat{P}(s) = 0 \qquad (9)$$

-- but we can easily integrate this first order O.D.E. in s to find

$$\hat{P}(s) = B\prod_{i=1}^{6}\left(1+\frac{s}{\lambda_i}\right)^{\frac{a_i\lambda_i}{\gamma}} e^{-\frac{s^2\Lambda}{2\beta\gamma} + \frac{k_0-1}{\gamma}s} \qquad (10)$$

Now to determine x_1 and x_2, we will use this solution. But remember, (7) is an inhomogeneous O.D.E. Hence we need both the particular and homogeneous solutions. If we define

$$F(s,t) \equiv \prod_{i=1}^{6}\left(1+\frac{s}{\lambda_i}\right)^{\frac{a_i\lambda_i}{\gamma}} e^{-\frac{s^2\Lambda}{2\gamma\beta} + \left(\frac{k_0+\gamma t-1}{\gamma}\right)s} \qquad (11)$$

then

$$\hat{P}(s)e^{st}\Big|_{x_1}^{x_2} = F(x_2,t) - F(x_1,t) = 0$$

One can verify that this is satisfied for pairs

$$(x_1, x_2) = (-\infty, -\lambda_6), (-\lambda_6, -\lambda_5), \ldots, (-\lambda_1, \infty) \rightarrow (x_j, x_{j+1})$$

If we now superimpose the homogeneous and particular solutions, then we find (upon integration)

$$P(t) = S_0 \frac{\Delta}{\beta} \int_0^\infty ds\, F(s,t) + \sum_{j=1}^{7} B_j \int_{x_j}^{x_{j+1}} ds\, F(s,t)$$

where the B_j are to be determined from the seven initial conditions on $P(0)$, $C_1(0) \cdots C_6(0)$

For $S_0 = 0$, the asymptotic behavior is given by

$$P(t) \sim B_7 \int_{-\lambda_1}^{\infty} F(s,t)\, ds$$

However one must use either computers or asymptotic expansions to get anything useful at this point. Refer to

Garabedian, Varga, Bilodeau, Nucl. Sci. & Eng. 3, 548 (1958)
J. E. Wilkins, Nucl. Sci. & Eng. 5, 207 (1959)

3.2.3. *Other Reactivity Insertions*

One can also obtain exact solutions for

$$\$(t) = k_0 - k_2 e^{-\gamma t} \quad \text{(exponential)}$$

$$\$(t) = k_0 - 1/\gamma t \quad \text{(reciprocal)}$$

See

H. B. Smets, Acad. Roy. Bel., Bull. Cl. Sc., XLV-3 (1959)

3.3. THE INVERSE METHOD

As we have seen, exact solutions of the P.R.K.E. are known only for a few types of reactivity insertions. However, it is relatively easy to solve the "inverse problem" of determining exactly the reactivity $\$(t)$ required to yield a known variation of P(t). Just solve (3) for

$$\$(t) = 1 + \frac{\Lambda}{\beta}\frac{d}{dt}\left[\ln P(t)\right] - \int_0^\infty d\tau\, D(\tau)\frac{P(t-\tau)}{P(t)} - \frac{\Lambda}{\beta}\frac{Q(t)}{P(t)} \qquad (8)$$

Such information is important for several reasons:

i.) In reactor operation, the time dependence of the applied reactivity required to yield a specified power variation must be known in order to program the control rod motion.

ii.) The interpretation of measured power responses in transient analyses of reactivity changes provides information about the feedback mechanism in the reactor.

3.3.1. Periodic Power Variation

Suppose $\quad P(t) = P_0 + P_1 \sin\omega t \quad , \quad Q(t) = 0$

Then one finds from (8)

$$\$(t) = \frac{P_1}{P_0} Y(i\omega)\frac{\sin(\omega t - \phi)}{1 + \frac{P_1}{P_0}\sin\omega t} \quad , \quad \phi = \arg\left[\frac{1}{Y(i\omega)}\right]$$

Notice in particular that the reactivity insertion which gives rise to a pure sinusoidal power variation is periodic — but not sinusoidal (at least for large power variations).

One can in fact show that $\$(t)$ has a negative bias:

$$\$_{av} \equiv \frac{1}{T}\int_0^T \$(t)\,dt = -\,\mathrm{Re}\{Y(i\omega)\}\left[\sqrt{1-(P_1/P_0)^2} - 1\right]$$

$$\sim -\left(\frac{P_1}{P_0}\right)^2 \mathrm{Re}\{Y(i\omega)\}$$

It is interesting to note that these features carry through for more general periodic power variations. That is, suppose

$$P(t) = P(t+nT) \qquad n=0,\pm 1,\cdots$$

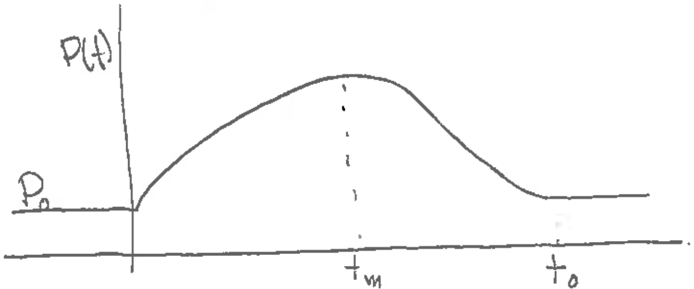
Then i.) $\$(t)$ must also be periodic with the same period

ii.) $\$_{av} < 0$

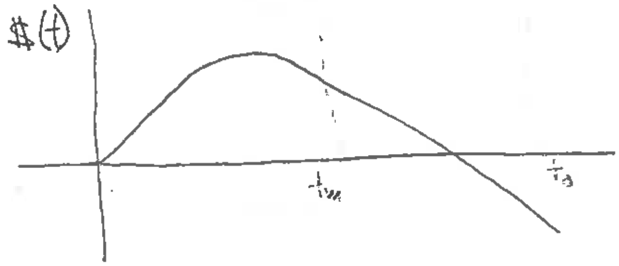
[For proof, see either Akcasu or Smets, Nukleonik 5, 181 (1961)]

3.3.2. Reactivity After a Positive Power Excursion

As a second example of the inverse method, consider a positive power excursion of the form:



Then in fact we can demonstrate that the reactivity is negative at the time t_0 when the power returns to its initial value P_0



The idea is to use

$$\$(t_0) = \frac{\Lambda}{\beta}\frac{1}{P_0}\frac{dP}{dt}\Big|_{t_0} - \int_0^{t_0} du\, D(u)\left[\frac{P(t_0-u)}{P_0}-1\right] \qquad (*)$$

and note that the integrand must be positive $[P(t_0-u) > P(t_0)]$ while the slope $dP/dt|_{t_0} \lesssim 0$

We can in fact, compute the time dependence of the reactivity if we assume an excursion of very short duration in comparison to delayed neutron rates. For $t > t_0$, assume $P(t) = P_0 = \text{const.}$ Then from (*)

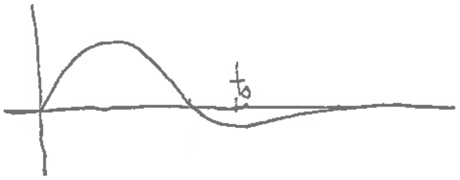
since $P(\tau)=P_0,\ \tau>t_0$

$$\$(t) = -\int_0^t du\, D(u)\left[\frac{P(t-u)-P_0}{P_0}\right] = -\int_0^{t_0} d\tau\, D(t-\tau)\left[\frac{P(\tau)-P_0}{P_0}\right]$$

$$\cong -D(t-t_0)\int_0^{t_0} d\tau\left[\frac{P(\tau)-P_0}{P_0}\right] = -\frac{I}{P_0}\sum_{i=1}^{6} a_i\lambda_i e^{-\lambda(t-t_0)}, \quad t \geq t_0$$

where

$$I \equiv \int_0^{t_0} du\,[P(u)-P_0]$$ is gross energy release in excursion.



i.) Note as $t \to \infty$, $\$(t) \to 0$

ii.) $\$(t_0) = -\frac{I}{P_0}\left[\sum_i a_i\lambda_i\right]$

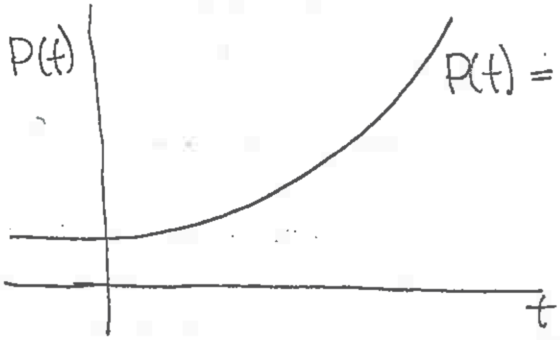
3.3.3. Reactivity for $P(t) = P_0\, e^{\alpha t^2}$

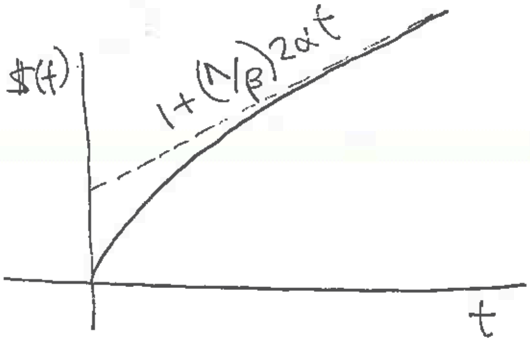
Assume $P(t) = P_0 \quad t<0$
$\qquad = P_0 e^{\alpha t^2} \quad t \geq 0$

Then $$\$(t) = 1 + \frac{\Lambda}{\beta}2\alpha t - e^{-\alpha t^2}\sum_{i=1}^{6} a_i e^{-\lambda_i t} - \sum_{i=1}^{6} a_i\lambda_i \underbrace{\int_0^t d\tau\, e^{-\alpha\tau^2 - 2\alpha t\tau - \lambda_i\tau}}_{\frac{1}{2\alpha(t+\lambda_i/2\alpha)}}$$

$$\cong 1 + \frac{\Lambda}{\beta}2\alpha t - \sum_{i=1}^{6}\frac{a_i\lambda_i}{\lambda_i + 2\alpha t} + O(e^{-\alpha t^2})$$

Note \$(t) approaches a linear function which implies that the response of the reactor power P(t) to a ramp insertion $\$(t) = \gamma t$ should behave as $e^{\beta\gamma t^2/2\Lambda}$ for large times





3.4. APPROXIMATE SOLUTIONS

As we have seen, exact solutions of the P.R.K.E. are known for only a few special reactivity insertions. Hence we now turn our attention to approximate schemes for solving these equations in the absence of feedback.

3.4.1. Constant Delayed Neutron Production Rate Approximation

In certain problems, such as when the reactor is shut down by rapid insertion of the safety rods we are interested in the response of the reactor power to a given reactivity insertion in short time intervals following a time t_0. During such short time intervals we can ignore the change in the rate of production of delayed neutrons, replacing $C_i(t)$ by $C_i(0)$. Hence in this approximation, the P.R.K.E. becomes

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \left[\$(t)-1\right]P(t) + \frac{\Lambda}{\beta}Q^*(t)$$

where the "effective source" $Q^*(t)$ includes the delayed neutrons

$$Q^*(t) = \frac{\beta}{\Lambda}\sum_{i=1}^{6}\lambda_i C_i(0) + Q(t)$$

Since this is just a first order ODE, it can be integrated to find

$$P(t) = e^{A(t)}\left[P(0) + \int_0^t dt'\, e^{-A(t')}\, Q^*(t')\right]$$

$$\text{where} \quad A(t) = \frac{\beta}{\Lambda}\int_0^t dt'\left[\$(t')-1\right]$$

EXAMPLE: Fast Ramp Reactivity Insertion and Reactor Shut-Down

In the case of an emergency, such as the loss of coolant flow, the reactor is shut down by a rapid insertion of the safety rods. Since the rod insertion takes a finite time, we cannot really treat this as a step reactivity change. A more reasonable model is to assume a negative ramp insertion, i.e. $\$(t) = -\gamma t$. Then, in our earlier notation,

$$A(t) = \frac{\beta}{\Lambda}\int_0^t [-\gamma t' - 1]\, dt' = -\frac{\beta}{\Lambda}\gamma t^2/_2 - \frac{\beta}{\Lambda} t$$

If we further assume $Q(t) = 0$, we find

$$P(t) = e^{-\frac{\beta}{\Lambda}\left[\frac{\gamma t^2}{2} + t\right]} P(0) + \int_0^t dt' \, e^{-\frac{\beta}{\Lambda}\left[\frac{\gamma (t-t')^2}{2} - (t-t')\right]} \frac{\beta}{\Lambda}\sum_{i=1}^{6} \lambda_i C_i(0)$$

Further noting that the initial conditions imply

$$\sum_i \lambda_i C_i(0) = P(0) \sum_i a_i = P(0)$$

we can simplify to find

$$\frac{P(t)}{P(0)} = e^{-(T^2 - T_0^2)/2}\left[1 - \frac{T_0}{\sqrt{2}} F(T_0)\right] + \frac{T_0}{\sqrt{2}} F(T)$$

where

$$T = (1+\gamma t) T_0, \quad T_0 = \sqrt{\frac{\beta}{\Lambda\gamma}}$$

$$F(T) = \int_0^T e^{(T'^2 - T^2)} \, dT'$$

At the end of the ramp insertion, the reactivity becomes a constant k_0 and we can find $P(t)$ by solving the P.R.K.E. for a constant reactivity as we already have done.

3.4.2. The Prompt Jump Approximation

If the relative rate of change of reactor power in a mean prompt generation time is sufficiently small, i.e.

$$\left|\frac{\Lambda}{\beta}\,\frac{\dot{P}(t)}{P(t)}\right| << \left|1 - \$(t)\right]$$

then we can neglect the $\frac{\Lambda}{\beta}\frac{dP}{dt}$ in the P.R.K.E. and consider

$$0 = [\$(t) - 1]P(t) + \sum_{i=1}^{6} \lambda_i C_i(t) + \frac{\Lambda}{\beta} Q(t)$$

$$\frac{dC_i}{dt} = a_i P(t) - \lambda_i C_i(t) \quad , \quad i = 1, \cdots, 6$$

This is called the "prompt jump approximation" since it predicts a sudden change in the power $P(t)$ following a sudden change in reactivity. Note that in the PJA, a reactivity jump from $\$_1$ to $\$_2$ causes a change from P_1 to P_2 given by

$$\frac{P_2}{P_1} = \frac{\$_1 - 1}{\$_2 - 1}$$

EXAMPLE: This approximation is particularly useful in the case of one delayed group: Then we can eliminate C in terms of P to find

$$[1 - \$(t)]\frac{dP}{dt} = [\dot{\$} + \lambda \$]P(t) + \frac{\Lambda}{\beta}[\dot{Q} + \lambda Q]$$

Hence if $\$(t)$ and $Q(t)$ are given, we can solve for $P(t)$:

$$P(t) = e^{A(t)}\left[P(0) + \int_0^t dt'\, e^{-A(t')} q(t')\right]$$

where

$$A(t) = \int_0^t d\tau \left[\frac{\dot{\$}(\tau) + \lambda \$(\tau)}{1 - \$(\tau)}\right]$$

$$q(t) = \frac{\Lambda}{\beta} \frac{\dot{Q}(t) + \lambda Q}{1 - \$(t)}$$

[For example, if $\$(t) = \gamma t$, $Q \equiv 0$, then

$$P(t) = P(0) e^{-\lambda t} [1 - \gamma t]^{-(1 + \lambda/\gamma)}$$]

Notice from (*) that in the absence of external sources

$$\frac{\dot{P}}{P} = \frac{\dot{\$} + \lambda \$}{1 - \$}$$

Hence our condition for the validity of the PJA becomes

$$(1-\$)^2 >> \frac{\Lambda}{\beta}(\dot{\$} + \lambda \$) \sim \frac{\Lambda \$ \lambda}{\beta}$$

Using $\lambda \sim .1\ \text{sec}^{-1}$, $\Lambda = 10^{-5}$ sec, $\beta = 10^{-2}$, this implies that the PJA will be valid until reactivity reaches roughly 80% of prompt criticality. Numerical solutions have demonstrated the PJA to be within 2% to .008% of the true solution after a $k_0 = .5\ \$$ step reactivity insertion.

There are a number of improved modifications of the PJA. See Goldstein & Shotkin, Nuc. Sci. & Eng. (1969) for more details.

3.4.3. Gradual Reactivity Changes

a.) *Hurwitz's Method:* Nucleonics 5, 61 (1949)
Nucl. Sci. & Eng. 6, 11 (1959)

b.) *The W.K.B. Method:* S. Tan, Nukleonik 8, 480 (1966)

3.4.4. Small Amplitude Approximation (Linearization)

Suppose that we assume small reactivity variations will produce only small changes in the reactor power from its equilibrium value P_0. We already know that this assumption is not true for a critical reactor since even a slight positive step in reactivity gives rise to an exponentially increasing power response which eventually grows beyond any bound. However the assumption will still be true if we consider only short times following the step insertion. Furthermore for certain classes of reactivity changes such as a periodic reactivity insertion with an appropriate negative bias, the resulting power variations remain small for all times. In this case, the P.R.K.E. reduce from a set of linear O.D.E.'s with variable coefficients to a set of linear O.D.E.'s with constant coefficients.

Consider again the integro-differential form of the P.R.K.E. [setting the source term equal to zero for convenience]

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t)-1]P(t) + \int_0^t d\tau\, D(\tau)P(t-\tau) + \sum_{i=1}^{\infty}\lambda_i C_i(0)e^{-\lambda_i t} \qquad (1)$$

It is convenient (and conventional) to assume the reactor is operating at a fixed power level P_0 prior to $t=0$. Then

$$\sum_{i=1}^{\infty}\lambda_i C_i(0)e^{-\lambda_i t} = P_0\sum_{i=1}^{\infty}a_i\lambda_i e^{-\lambda_i t} = P_0\int_0^{t} d\tau\, D(\tau) \qquad (2)$$

If we now let $p(t)$ denote the power variations about the reference level P_0

$$P(t) = P_0 + p(t) \qquad (3)$$

then (1) becomes

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = \$(t)P_0 + \$(t)p(t) - p(t) + \cancel{\int_0^t d\tau\, D(\tau)P_0} + \int_0^t d\tau\, D(\tau)p(t-\tau)$$

$$\cancel{-P_0\int_0^t d\tau\, D(\tau}$$

$$= \$(t)P_0 + \$(t)p(t) + \int_0^t d\tau\, D(\tau)p(t-\tau) - p(t) \qquad (4)$$

Our approximation will be to assume that $k(t)$ and $p(t)$ are sufficiently small that we can neglect $\$(t)p(t)$ to obtain

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = P_0\$(t) + \int_0^t d\tau\, D(\tau)\, p(t-\tau) - p(t) \qquad (5)$$

This approximation is sometimes (incorrectly) referred to as the "linearization approximation". [Note that both (4) and (5) are linear. Actually, what this approximation does is to linearize the functional relation between $\$(t)$ and $p(t)$, which, as we have found earlier, is not linear in general.]

Since (5) is now just an integro-differential equation with constant coefficients, we can easily solve it by Laplace transforms to find

$$\tilde{p}(s) = \left[s\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i}{s+\lambda_i}\right)\right]^{-1} P_0\,\tilde{\$}(s)$$

or

$$\frac{\tilde{p}(s)}{P_0} = Z(s)\,\tilde{\$}(s) \qquad (6)$$

where we have defined the "zero power transfer function" $Z(s)$

$$Z(s) \equiv \left[s\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i}{s+\lambda_i}\right)\right]^{-1} \equiv 1/Y(s) \qquad (7)$$

Hence to compute $\tilde{p}(s)$, we need only study the poles of $Z(s)$ and $\tilde{\$}(s)$. However there is a great deal more we can do by employing the very powerful methods of linear systems analysis. In particular, we can study the stability of the reactor when it is operating at power (when we introduce feedback).

3.4.4.1. Some Elements of Linear System Theory

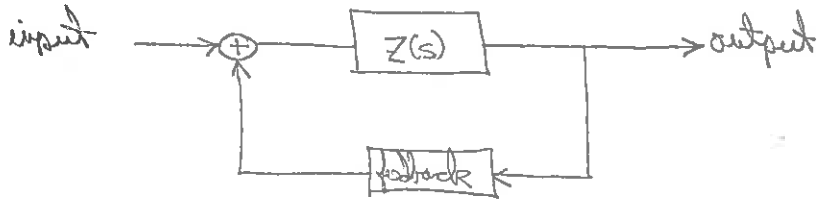
The response (or output) of any physical system to a signal (or input) applied to it can be expressed in terms of a quantity called the "transfer function" of the system. More precisely, we define

$$\text{Transfer function} = \frac{\text{Laplace transform of response}}{\text{Laplace transform of input}} \equiv Z(s)$$

This can be conveniently represented by a "block diagram"

In our case, the input is the reactivity $\$(t)$, while the output is the fractional power change $p(t)/P_0$. [Of course other choices are possible, such as the inlet coolant temperature and pressure and so on.]

The transfer function defined for a linear system in this way is sometimes called the "open-loop" transfer function, since we have assumed that the output (power level) does not affect the input (reactivity) in any way. Later we will consider the case in which we allow feedback effects:



[that is, we "close" the feedback loop.]

Comments:

i) As we have defined it, the transfer function obeys the principle of superposition -- that is the sum of the outputs corresponding to two inputs, is equal to the output resulting from the sum of the two inputs.

ii.) Notice that $\mathcal{L}\{\delta(t)\} = 1$. Hence the response due to a δ-function reactivity input is

$$\frac{\tilde{p}(s)}{P_0} = Z(s)\cdot 1 = Z(s)$$

or inverting

$$\frac{\tilde{p}(t)}{P_0} = \mathcal{L}^{-1}\{Z(s)\} = \frac{\beta}{\Lambda} + \sum_{j=0}^{6} \frac{e^{\omega_j t}}{\omega_j\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]} \equiv z(t)$$

Here, $z(t)$ is the so-called "unit impulse response" -- that is, the power response to a unit impulse reactivity insertion.

If we return now to the case of more general inputs $\$(t)$, and in particular reexpress

$$\frac{\tilde{p}(s)}{P_0} = Z(s)\,\tilde{\$}(s)$$

in the "time domain" using the convolution theorem

$$p(t) = P_0\int_0^t dt'\, z(t-t')\,\$(t')$$

we can see that $z(t)$ is just the Green's function for the linearized point reactor kinetics equation. That is, $z(t-t')$ is the solution to

$$\frac{\Lambda}{\beta}\frac{dz}{dt} = P_0\delta(t-t') + \int_0^t d\tau\, D(\tau)\, z(t-\tau) - z(t) .$$

Hence tis not surprising that $z(t)$ plays an extremely important role in the study of the linearized P.R.K.E. A couple of interesting points:

a.) $\lim_{t\to\infty} z(t) = \frac{\beta}{\Lambda}$ This implies that the critical reactor has "infinite memory".

b.) $\int_0^\infty |z(t)|dt = \infty$

This implies that a critical reactor without feedback is <u>unstable</u> with respect to bounded inputs.

Definition: A linear system is said to be <u>stable</u> if its response to any bounded input is also bounded.

Theorem: A necessary and sufficient condition for stability is

$$\int_0^\infty |z(t)|dt < \infty \qquad (*)$$

Proof: To prove sufficiency, note

$$|p(t)| < P_0 \int_0^\infty dt' \, |\$(t-t')| |z(t')| < MP_0 \int_0^\infty |z(t)|dt$$

where M is the bound of the input, i.e. $\$(t) < M$.

To prove necessity, we merely construct a bounded input for which the output is unbounded if (*) does not hold. Consider

$$\$(-t) = z(t)/|z(t)|$$

Then the response at $t = 0$ is

$$p(0) = P_0 \int_0^\infty dt' \, \$(-t') \, z(t') = P_0 \int_0^\infty dt' \, \frac{z^2(t')}{|z(t')|} = P_0 \int_0^\infty |z(t)|dt$$

which is unbounded if (*) does not hold.

3.4.4.2 Response to a Small Sinusoidal Input

Consider now a sinusoidal reactivity input

$$\$(t) = \delta k \sin\omega t$$

Then

$$\tilde{\$}(s) = \frac{\omega\delta k}{s^2+\omega^2} = \frac{\omega\delta k}{(s-i\omega)(s+i\omega)}$$

We now can solve for

$$p(t)/P_0 = \mathcal{L}^{-1}\left\{\frac{Z(s)\,\omega\delta k}{(s^2+\omega^2)}\right\}$$

$$= \delta k\left[\frac{Z(i\omega)}{i\omega}e^{i\omega t} - \frac{Z(-i\omega)}{i\omega}e^{-i\omega t}\right] + \omega\delta k\sum_{j=0}^{6}\frac{e^{\omega_j t}}{(\omega^2+\omega_j^2)\left.\frac{dY}{ds}\right|_{\omega_j}}$$

where the first two terms arise from the poles of $\tilde{\$}(s)$ on the imaginary axis at $s=i\omega$ and $s=-i\omega$, and the remaining terms are due to the poles of $Z(s)$ which are the roots ω_j of the inhour equation $Y(\omega_j)=0$. Note that for the critical system we are considering

$$\omega_6 < \omega_5 < \cdots < \omega_0 = 0$$

Hence as $t\to\infty$ only the oscillating terms and the $\omega_0=0$ term remain and we find the asymptotic behavior of the power oscillations as

$$p(t)/P_0 = \delta k\, G(\omega)\sin(\omega t+\phi) + \frac{\delta k}{\omega}\frac{\beta}{\Lambda}$$

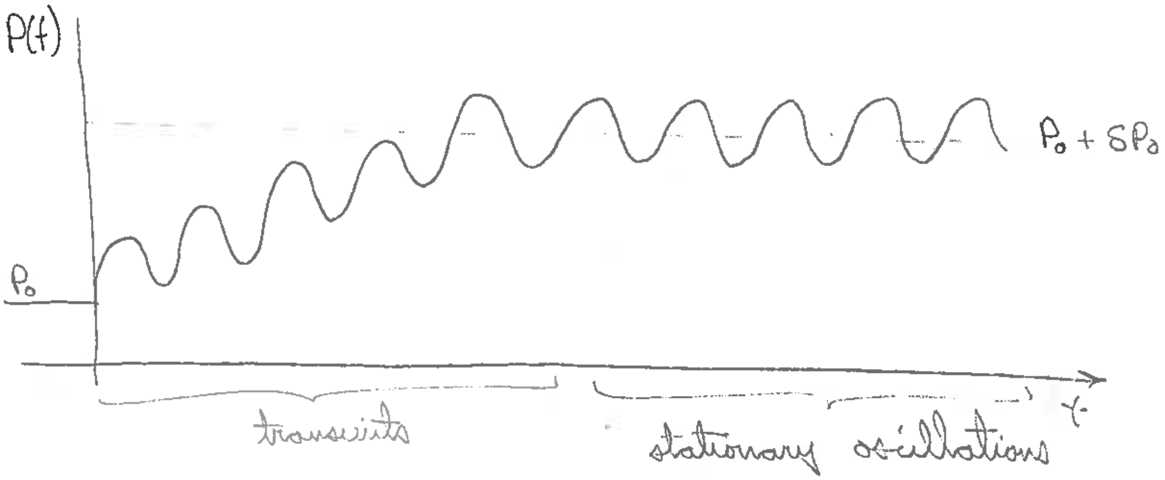
where

$$G(\omega) = |Z(i\omega)| \qquad \text{"gain"}$$

$$\phi(\omega) = \tan^{-1}\left[\operatorname{Im}\{Z(i\omega)\}/\operatorname{Re}\{Z(i\omega)\}\right] \qquad \text{"phase"}$$

Notice we find a shift in the average power level

$$\delta P_0 = P_0\lim_{s\to 0} s\,Z(s)\,\tilde{\$}(s) = P_0\frac{\delta k}{\omega}\frac{\beta}{\Lambda}$$



Now it is customary to define

$$\text{gain} \equiv \frac{\text{relative amplitude of power oscillations}}{\text{amplitude of reactivity oscillations}} = \frac{G(\omega)}{1 + \frac{\delta k \beta}{\omega \Lambda}}$$

To circumvent this difficulty, one can choose an alternative reference power level in the original linearization

$$P(t) = P_r + p(t)$$

where

$$P_r = P_0 \left(1 - \frac{\delta k \beta}{\omega \Lambda}\right)$$

Then we find that for long times

$$\frac{p(t)}{P_r} = \delta k \, G(\omega) \sin(\omega t + \phi)$$

That is, if a reactor operating at steady state power level is subjected to a sinusoidal reactivity insertion, the power will oscillate with the same frequency, but will be shifted in phase $\Theta \equiv \arg\{Z\}$ (actually a phase lag) and an amplitude proportional to $G(\omega) \equiv |Z(i\omega)|$. Hence we can obtain the values of $Z(s)$ on the imaginary axis in the complex s-plane by measuring experimentally the amplitude and relative phase of power oscillations as a function of frequency, induced by a sinusoidal reactivity variation with constant amplitude. This is the basis of the zero-power pile-oscillator experiment.

* * *

Suppose we have determined $Z(i\omega) = G(\omega)e^{i\Theta(\omega)}$ for all $\omega \in [0,\infty)$. Can we infer $Z(s)$ for arbitrary s from this data? Yes. For the particular transfer functions of interest to us one can show

$$Z(s) = \frac{\beta}{\Lambda s} + \frac{2}{\pi} s \int_0^\infty d\omega \frac{G(\omega)\cos\Theta(\omega)}{s^2+\omega^2} \qquad \mathrm{Re}\{s\} > 0$$

The derivation of this relationship is based upon the theory of functions of complex variables (and in particular, upon so-called "dispersion relations")

Actually, however, we can obtain all the information we need from $G(\omega)$ and $\Theta(\omega)$. Hence we will confine our attention to the study of these quantities

3.4.4.3. Bode Diagrams

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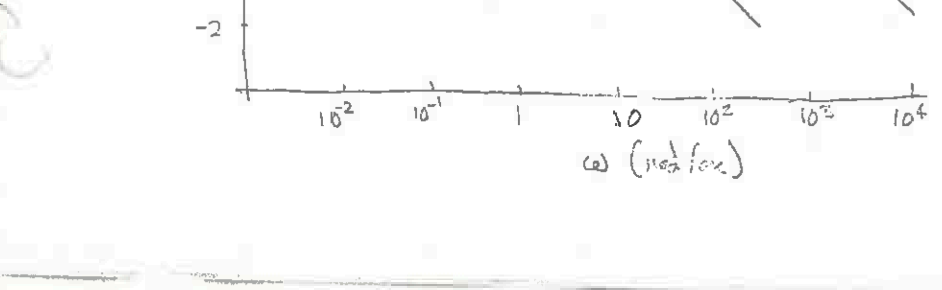
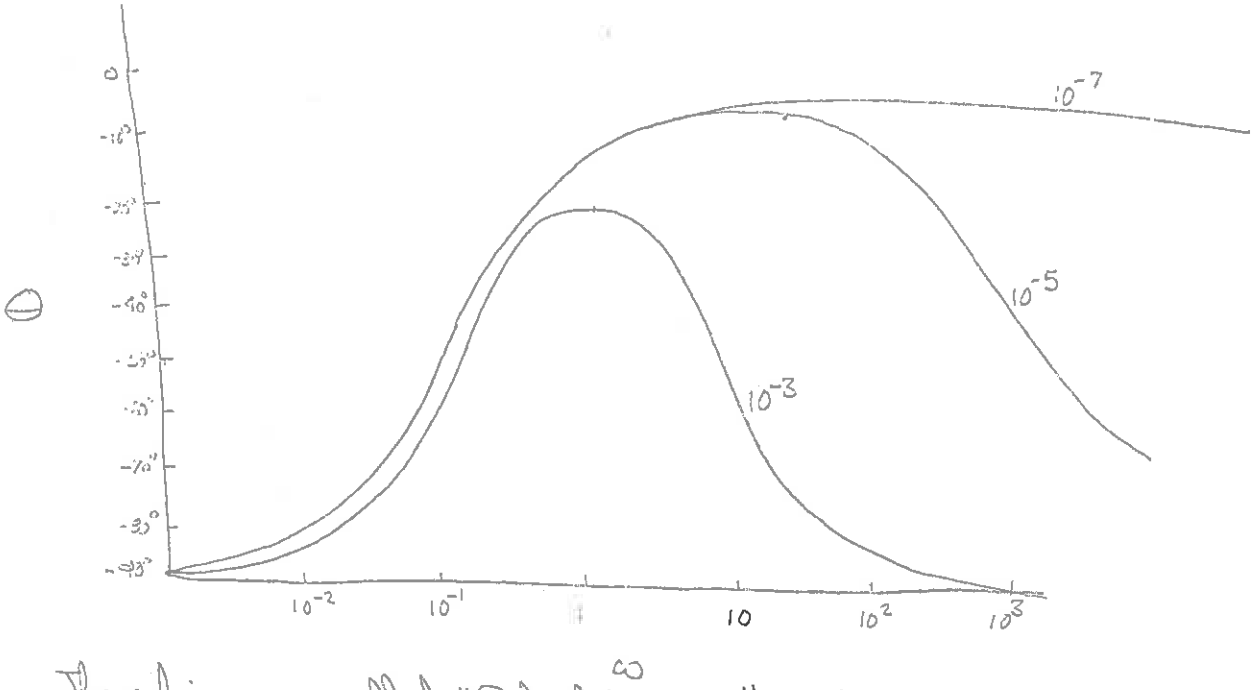
Consider, as an example, the amplitude and phase characteristics for a U^{235} fueled reactor with $\Lambda = 10^{-3}, 10^{-5}$, and 10^{-7}





These figures are called "Bode diagrams". Note in particular that we have plotted the amplitude in decibels

$$G_{db}(\omega) \equiv 20 \log_{10} G(\omega)$$

To understand these curves in more detail, note we can write

$$Z(i\omega) = \left[i\omega\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \frac{a_i}{i\omega+\lambda_i}\right)\right]^{-1} = K\,\frac{(1+i\omega\tau_1)\dots(1+i\omega\tau_6)}{i\omega(1+i\omega t_1)\dots\dots(1+i\omega t_6)}$$

where

$$\tau_j \equiv 1/\lambda_j$$

$$t_j \equiv 1/\omega_j \qquad (\omega_j \text{ are roots of inhour equation})$$

$$K \equiv \frac{\beta}{\Lambda}\,\frac{\lambda_1\lambda_2\cdots\lambda_6}{\omega_1\omega_2\cdots\omega_6}$$

In this form we write

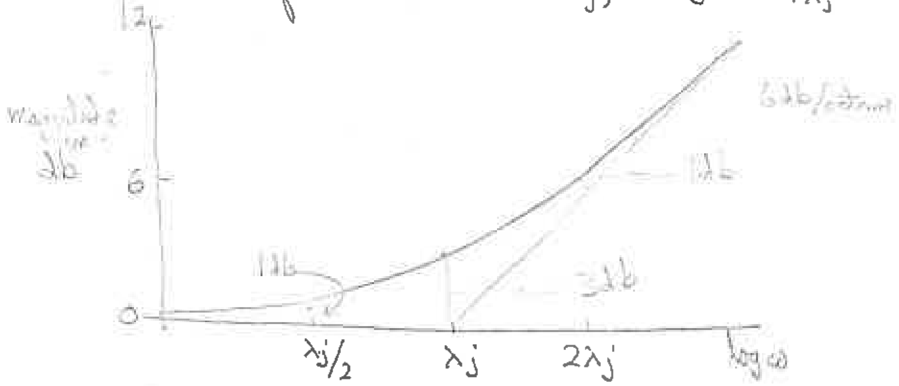
$$G_{db}(\omega) = 20\log_{10}K + 10\Big[\log_{10}(1+\omega^2\tau_1^2) + \cdots + \log_{10}(1+\omega^2\tau_6^2) - \log_{10}\omega - \log_{10}(1+\omega^2 t_1^2) - \cdots - \log_{10}(1+\omega^2 t_6^2)\Big]$$

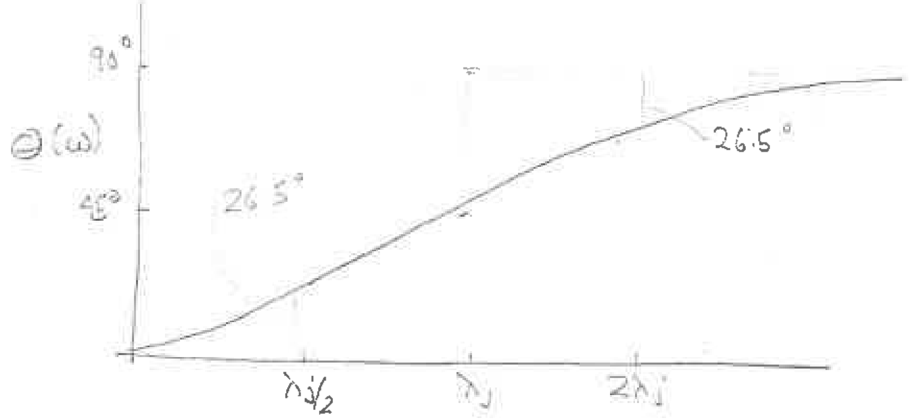
and

$$\Theta(\omega) = \tan^{-1}\omega\tau_1 + \cdots + \tan^{-1}\omega\tau_6 - 90° - \tan^{-1}\omega t_1 - \dots - \tan^{-1}\omega t_6$$

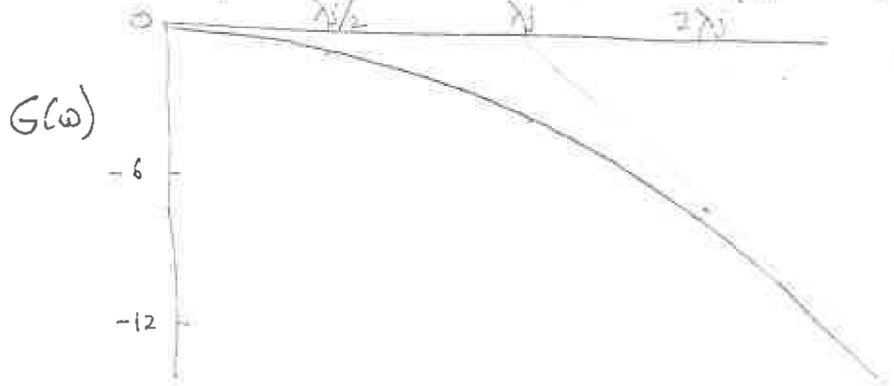
Hence since these expressions represent frequency in a natural fashion, we need only plot the magnitude and phase of each [illegible] as a function of frequency.

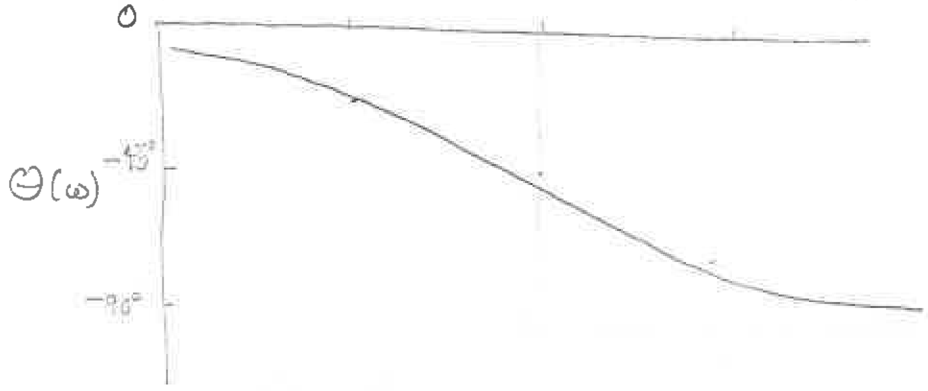
Consider first $(1 + i\omega\tau_j) = (1 + i\omega/\lambda_j)$ $= \sqrt{1 + \omega^2/\lambda_j^2}$





In a similar fashion one can show that for $(1 + i\omega\tau_j)^{-1}$





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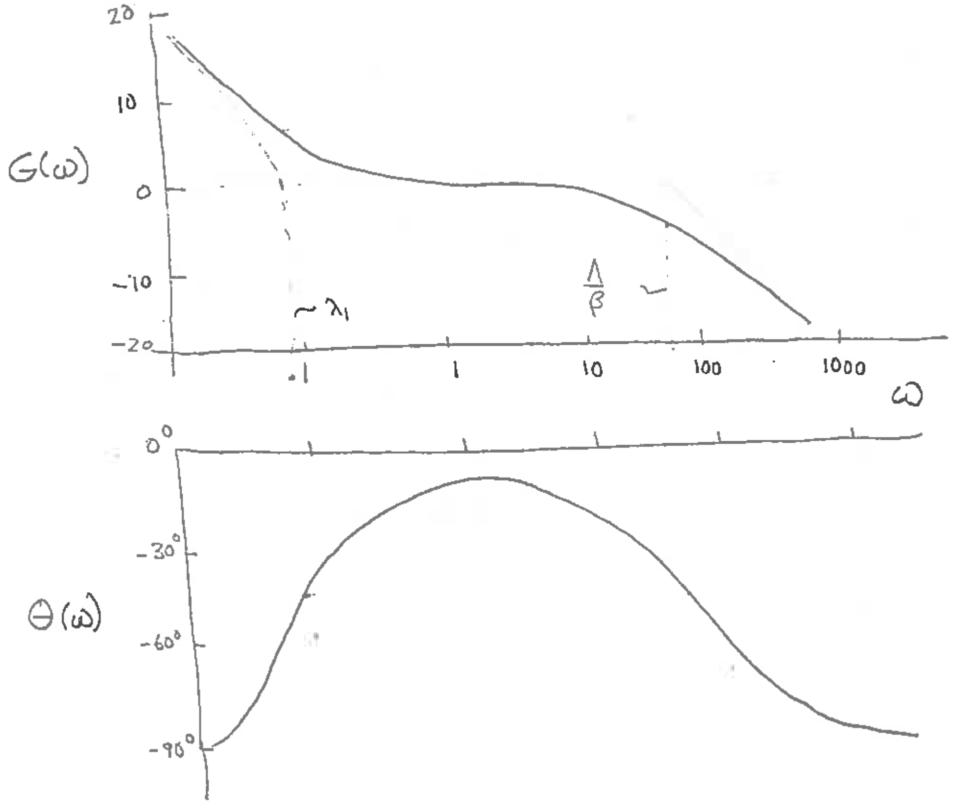
EXAMPLE: One-delayed neutron group

$$Z(i\omega) = K\frac{(1+i\omega\tau_1)}{i\omega(1+i\omega t_6)} = \left[i\omega\left(\frac{\Delta}{\beta} + \frac{1}{i\omega+\lambda_1}\right)\right]^{-1}$$

where

$$K = \frac{\beta}{\Lambda}\frac{\lambda_1}{\omega_6}, \quad \tau_1 = 1/\lambda_1, \quad t_6 = 1/\omega_6 \sim \frac{\Delta}{\beta}$$

If we plot these for $\Lambda = 10^{-4}$ sec, $\lambda_1 = 0.08$ sec^{-1}, $\beta = 0.0075$



$1/\lambda_1 \quad \frac{\beta}{\Lambda}\frac{\lambda_1}{\omega_6} \quad \frac{\tau_1}{t_6}$

Note: i.) $\Theta(\omega) < 0$ (true in general)

ii.) The "break frequencies" $1/\lambda_1$ and $1/t_1$ could be determined experimentally by fitting straight lines to the measured amplitude and phase characteristics. In this way we could measure λ_1 and Λ/β.

iii.) For very low frequencies $\omega \ll \lambda_1$,

$$Z(s) \sim \frac{\beta}{\Lambda^* s} \qquad \Lambda^* = \Lambda + \sum_i \frac{\beta_i}{\lambda_i}$$

For intermediate frequencies $\lambda_1 < \omega < \beta/\Lambda$

$$Z(s) = 1$$

For high frequencies

$$Z(s) = \frac{\beta}{\Lambda s}$$

We will return later to this topic when we discuss feedback and control theory.

3.4.5. Logarithmic Linearization

The small amplitude or "linearization" approximation discussed in the previous section is only valid for small power variations. We can relax this restriction somewhat by linearizing the functional relationship between the logarithm of the power and reactivity $\$(t)$. Again we start from the P.R.K.E.

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \left[\$(t) - 1\right]P(t) + \int_0^\infty d\tau\, D(\tau)\, P(t-\tau) \qquad (1)$$

Now define

$$y(t) \equiv \ln P(t)/P_0$$

Then we can rewrite (1) as

$$\frac{\Lambda}{\beta}\frac{dy}{dt} - \int_0^\infty d\tau\, D(\tau)\left[e^{y(t-\tau)-y(t)} - 1\right] = \$(t) \qquad (2)$$

Our basic approximation will be to assume

$$|y(t-\tau) - y(t)| << 1 \qquad \text{or} \qquad \left|\ln\frac{P(t-\tau)}{P(t)}\right| << 1$$

We can then use

$$e^{y(t-\tau)-y(t)} \sim 1 + y(t-\tau) - y(t)$$

to re-write (2) as

$$\frac{\Lambda}{\beta}\frac{dy}{dt} - \int_0^\infty d\tau\, D(\tau)\left[y(t-\tau) - y(t)\right] = \$(t) \tag{3}$$

If we assume the reactor is operating at a power level P_0 prior to $t = 0$, then (3) becomes

$$\frac{\Lambda}{\beta}\frac{dy}{dt} + y(t) - \int_0^t d\tau\, D(\tau)\, y(t-\tau) = \$(t) \tag{4}$$

This is again a linear integro-differential equation which can be easily solved using Laplace transforms to find

$$\tilde{y}(s) = Z(s)\,\tilde{\$}(s)$$

— hence the linear system obtained by logarithmic linearization is also described by the zero-power transfer function $Z(s)$.

EXAMPLE: The response to a step input k_0 is.

$$y(t) = k_0 \frac{\beta}{\Lambda^*}\left(t - \frac{\beta}{\Lambda^*}\overline{\tau^2}\right) + \sum_{j=1}^{6} \frac{k_0 e^{\omega_j t}}{\omega_j Y'(\omega_j)}$$

where $$\overline{\tau^2} = \sum_{i=1}^{6} a_i/\lambda_i^2$$

It is important to note that for long times this implies

$$P(t) = P_0 e^{-k_0\left(\frac{\beta}{\Lambda^*}\right)^2\overline{\tau^2} + k_0 \frac{\beta}{\Lambda^*} t}$$

which is identical to the exact solution of the P.R.K.E. as $k_0 \to 0$. By way of contrast, the "linearization" approximation predicts

$$P(t) = P_0 + P_0 k_0 \frac{\beta}{\Lambda^*}\left(t - \frac{\beta}{\Lambda^*}\overline{\tau^2}\right)$$

IV. THE POINT REACTOR KINETICS EQUATIONS WITH FEEDBACK

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4.1. MATHEMATICAL DESCRIPTION OF FEEDBACK

4.1.1. Introduction

In our formal derivation of the P.R.K.E. we made the assumption that the macroscopic cross sections were known functions of time. However we have seen that these cross sections do in fact depend upon the neutron density itself (through temperature or isotope buildup effects). It is possible to carry through the very complicated re-derivation[‡] of the P.R.K.E. allowing the cross sections to be nonlinear functionals of the neutron density itself, i.e.

$$\Sigma(\underline{r},v,t) = \Sigma(\underline{r},v,t;T) = \Sigma(\underline{r},v,t;n)$$

This derivation while interesting, is of a very limited usefulness since we in fact are usually unable to describe in detail the dependence of reactor temperature upon the neutron density. [This dependence is given by a very formidable nonlinear set of integro-differential equations describing heat transfer, fluid flow, spectrum effects, etc.]

We shall instead introduce alternative simpler mathematical "models" of feedback by expressing the reactivity \$ as a simple functional of the power $P(t)$

$$\$ = \$(P)$$

Of course the true test of such feedback models will come only with a direct comparison of the predictions obtained via a given model with actual experimental data.

[‡] See O'Dean.

In this spirit then, we shall write the P.R.K.E. in the presence of feedback as

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \left\{\delta\$_{ext}(t) + \delta\$_f[P] - 1\right\}P(t) + \sum_{i=1}^{6}\lambda_i C_i(t) + \frac{\Lambda}{\beta}Q(t)$$

$$\frac{dC_i}{dt} = a_i P(t) - \lambda_i C_i(t) \qquad i=1,\dots 6 \qquad (1)$$

Here we have written the reactivity as the sum of two terms

$$\$ = \delta\$_{ext}(t) + \delta\$_f[P]$$

The $\delta\$$ notation signifies that the reactivity is measured with respect to the equilibrium power level P_0. Furthermore, $\delta\$_{ext}(t)$ represents the "external" reactivity insertion such as by adjusting a control rod. $\delta\$_f[P]$ denotes the change in reactivity due to feedback.

When the reactor is operating at a steady-state power level P_0, then there will be a certain feedback reactivity $\$_f(P_0)$. To sustain the criticality of the system, we must supply a counteracting external feedback $\$_0$ such that

$$\$_0 + \$_f(P_0) = 0$$

In this sense we are defining the incremental reactivities,

$$\delta\$_{ext}(t) = \$_{ext}(t) - \$_0$$

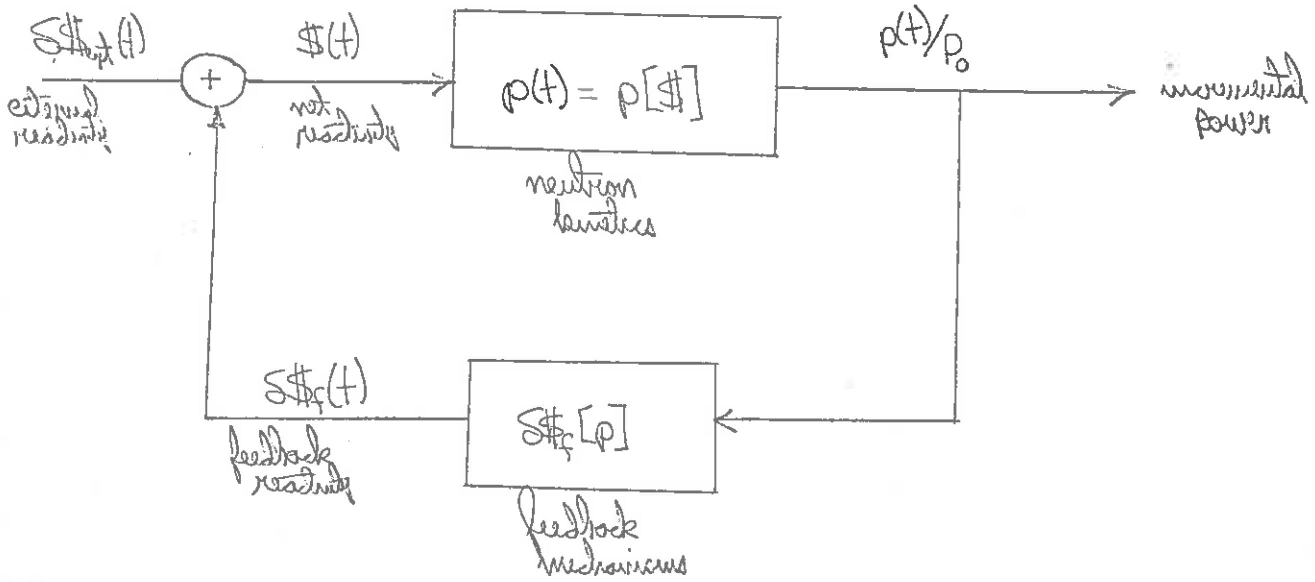
$$\delta\$_f[P] = \$_f[P] - \$_f[P_0]$$

It is also useful to recall our definition of the incremental power

$$p(t) = P(t) - P_0$$

Note then that $\delta\$_f[p\equiv 0] = \delta\$_f[P_0] = 0$

We can now sketch the "block" diagram for the reactor with feedback as



We have already analyzed the "black box" describing the neutron kinetics -- i.e. the functional $p[\$]$ is given by the solution of the P.R.K.E. *without* feedback

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = \$(t)\left[P_0+p(t)\right] + \int_0^\infty d\tau\, D(\tau)\left[p(t-\tau)-p(t)\right]$$

Notice that in general we can only determine $p[\$]$ *approximately*, e.g.

i.) no delayed neutrons
ii.) linearization

$$p[\$] = P_0\int_0^t d\tau\, z(\tau)\,\$(t-\tau)$$

iii.) logarithmic linearization
iv.) prompt jump approximation

and so on.

We now turn our attention to a study of the "black box" describing the feedback functional $\delta\$_f[p]$.

4.1.2. Specific Feedback Models

As an example of what is involved in a typical feedback calculation, consider the description of temperature feedback as given by

$$\delta \$_f(t) = \left[\left(n_0^+, \int d^3v' \delta\left[v'\Sigma_s(v'\to \underline{v}) + v\chi(v)v'\Sigma_f(v')\right]\Psi(\underline{r},v',t)\right) - \left(n_0^+, v\delta\Sigma_t \Psi\right)\right]\frac{1}{F\bar{\beta}}$$

$$\delta\Sigma(\underline{r},v,t) = \Sigma(\underline{r},v;T) - \Sigma^0(\underline{r},v)$$

$$\mu\left[\frac{\partial T}{\partial t} + \underline{u}(\underline{r},t)\cdot\nabla T(\underline{r},t)\right] - \nabla\cdot\kappa\nabla T(\underline{r},t) = H(\underline{r},t)$$

$$H(\underline{r},t) = w_f \int d^3v\, v\Sigma_f(\underline{r},v,t)\, n(\underline{r},\underline{v},t)$$

Needless to say, it is quite a job to determine $\delta\$_f[P(t)]$. In fact even the steady-state calculation of $\$_f[P_0]$ is an incredible mess involving very detailed neutronics calculations coupled with engineering calculations of heat transfer and coolant flow.

Later, when we discuss spatially-dependent reactor kinetics, we will treat temperature feedback as a "distributed parameter" system -- that is, one in which we retain the spatial dependence. In this description, the feedback functional becomes

$$\delta\$_f[p] = \int_0^t d\tau \left[\int d^3r\, \alpha(\underline{r}) \int d^3r'\, G(\underline{r},\underline{r}',\tau)\, H(\underline{r}',\tau)\right] p(t-\tau)$$

where $\alpha(\underline{r})$ is a "local" temperature coefficient of reactivity, while $G(\underline{r},\underline{r}',\tau)$ essentially gives the temperature $T(\underline{r},\tau)$ at a point $\underline{r}$ to a unit heat source at $(\underline{r}',\tau)$.

However, consistent with the neglect of spatial dependence in the P.R.K.E., we will consider in this chapter a "lumped parameter" description of the temperature feedback. That is, we will characterize the fuel, moderator, and coolant by one respective average temperature, T_F, T_m, or T_c.

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EXAMPLE: Consider a "two temperature" feedback model in which

$T_f(t)$ is the average (lumped) temperature of the fuel

$T_m(t)$ " " " " " " moderator (or coolant)

Now the fuel temperature will respond with little delay to changes in reactor power. By way of contrast, there may be an appreciable lag in the response of the coolant temperature because of the delay introduced in the heat transfer from the fuel to the coolant.

Then if Suppose that in steady-state, $P(t) = P_0$, $T_f = T_{f0}$, $T_m = T_{m0}$.

$$\delta T_f = T_f(t) - T_{f0}\ ,\quad \delta T_m = T_m(t) - T_{m0}$$

are small, we can represent

$$\frac{d(\delta T_f)}{dt} = a p - \omega_f \delta T_f$$

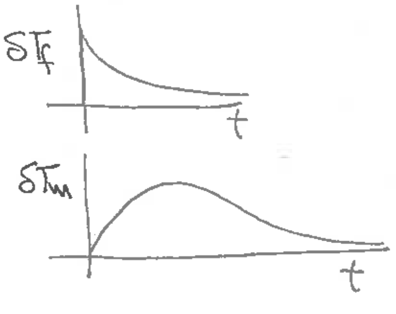
$$\frac{d(\delta T_m)}{dt} = b\delta T_f - \omega_m \delta T_m$$

$\frac{1}{\omega_f}, \frac{1}{\omega_m}$ time constants of fuel and moderator temperatures

Note that for a sharp power pulse, $p = P_0 \delta(t)$,

$$\delta T_f = a P_0 e^{-\omega_f t}$$

$$\delta T_m = \frac{a b P_0}{\omega_m - \omega_f}\left(e^{-\omega_f t} - e^{-\omega_m t}\right)$$



We will introduce alternative models of temperature feedback later in this chapter.

4.1.3. Some General Mathematical Properties of Feedback Functionals

We will now state a few properties of the feedback functional $\delta\$_f[p]$ (without proof)

i.) $\delta\$_f[p]$ is invariant under time translation provided the feedback parameters are not explicit functions of time

$$\delta\$_f(t-t_0) = \delta\$_f[p(t-t_0)] \qquad \text{for arbitrary } t_0$$

ii.) $\delta\$_f[p]$ is uniquely determined at a time t provided $p(t)$ is known for all times prior to t [causality]

iii.) The feedback functional is bounded for any bounded input (stability)

$$|\delta\$_f[p]| < M \quad \text{if} \quad |p(t)| < m$$

Actually we can manipulate this functional much as one would manipulate functions [continuity, derivatives, power series expansions, integration, etc.] As a couple of more concrete examples, we will frequently be concerned with linear feedback functionals

$$\delta\$_f[p] = \int_{-\infty}^{t} d\tau\, G(t-\tau)\, p(\tau)$$

Note here that $G(t)$ can be interpreted as the reactivity at time t due to a unit energy release at $t=0$.

Sometimes it is necessary to consider nonlinear feedback functions (such as in xenon oscillations) which we write as

$$\delta\$_f[p] = \int_{-\infty}^{t} d\tau\, G_1(t-\tau)\, p(\tau) + \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau'\, G_2(t-\tau, t-\tau')\, p(\tau)\, p(\tau')$$

EXAMPLE: The linear feedback functional model is sufficient for the stability analysis of most of reactor types. Hence let us examine

$$\delta\$_f[p] = \int_{-\infty} d\tau\, G(t-\tau)\, p(\tau)$$

in a bit more detail. In particular, if $\delta\$_F$ is to satisfy the above listed properties of feedback functionals, we must require

i) $G(t) = 0 \qquad t<0$ (causality)

ii) $\int_0^\infty |G(t)|\, dt < \infty$ (stability)

Now the P.R.K.E. with linear feedback becomes

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \left\{\delta\$_{ext} + \int_0^\infty d\tau\, G(\tau)\, p(t-\tau)\right\}\left[P_0 + p(t)\right] + \int_0^\infty d\tau\, D(\tau)\left[p(t-\tau) - p(t)\right]$$

Notice that even though the feedback is linear, it contributes a *nonlinear* term into the P.R.K.E.

Now suppose we wish to operate the reactor at a steady-state power level $P_0' \neq P_0$. Then

$$p(t) = P_0' - P_0 = \text{constant}.$$

Hence since $dP/dt = 0$, we must have

$$\delta\$_{ext} + (P_0' - P_0)\int_0^\infty d\tau\, G(\tau) = 0$$

That is, we must supply an external reactivity

$$\delta\$_{ext} = -\gamma\,(P_0' - P_0)$$

where $\gamma \equiv \int_0^\infty d\tau\, G(\tau)$

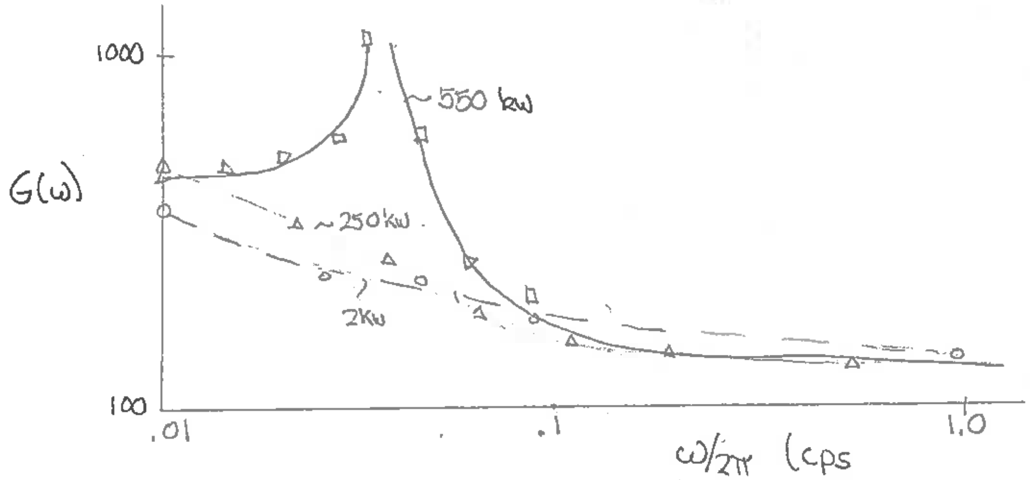
Thus the incremental changes in the steady state power level is proportional to the incremental changes in the external reactivity. The proportionality constant γ is called "power coefficient" of reactivity.

We will later find that stability considerations require γ to be negative. [Indeed, if γ were positive, an increase in the power level would produce a positive feedback reactivity. But this causes the power to increase even more, hence increasing δH_f and so on.]

4.2. THE TRANSFER FUNCTION OF A REACTOR WITH FEEDBACK

Let us now return to consider the dynamic behavior of a reactor with feedback. In particular, we will consider the effects of feedback on the reactor response to an external reactivity insertion. First we will consider the power oscillations resulting from a periodic reactivity insertion of small amplitude, since this in effect measures the reactivity to power transfer function.

A typical series of reactivity - power transfer function gain measurements are shown below. [These measurements were for EBR-I.]



In particular, notice how different the transfer function at power is from the zero power transfer function. The marked resonance behavior in the vicinity of .03 cps at $P_0 = 550$ kw is apparent. Such behavior is due to the presence of feedback. As the power level increases, the resonance peak becomes narrower and higher. As we shall see, this implies that for sufficiently large powers, the reactor is unstable.

Let's go back and develop a mathematical expression for the transfer function of the reactor with feedback. Return to the P.R.K.E. with $\rho(t) = P(t) - P_0$

$$\frac{\Lambda}{\beta}\frac{d\rho}{dt} = \$(t)(\rho + P_0) + \int_0^\infty d\tau\, D(\tau)\left[\rho(t-\tau) - \rho(t)\right] \qquad (1)$$

where we will use

$$\$(t) = \delta\$_{ext}(t) + \delta\$_f(t)$$

We shall now restrict ourselves to small power oscillations about the equilibrium level P_0 so that the feedback functional can be adequately represented as a linear functional

$$\delta\$_f[\rho] = \int_0^\infty d\tau\, G(\tau)\,\rho(t-\tau)$$

[It should be kept in mind that the feedback kernel $G(\tau)$ actually depends on P_0.] Now (1) becomes

$$\frac{\Lambda}{\beta}\frac{d\rho}{dt} = \left[\delta\$_{ext}(t) + \int_0^\infty d\tau\, G(\tau)\rho(t-\tau)\right](\rho + P_0) + \int_0^\infty d\tau\, D(\tau)\left[\rho(t-\tau) - \rho(t)\right] \qquad (2)$$

Of course this equation is still nonlinear. We shall consider only small power variations such that $\rho(t) << P_0$ and we can linearize (2) as

$$\frac{\Lambda}{\beta}\frac{d\rho}{dt} + \rho(t) = P_0\,\delta\$_{ext}(t) + \int_0^\infty d\tau\left[D(\tau) + P_0 G(\tau)\right]\rho(t-\tau)$$

Now as before, we assume $P(t) = P_0$ for $t<0$ and Laplace transform to find

$$\frac{\Lambda}{\beta}\left[s\tilde{\rho} - \rho(0)\right] + \tilde{\rho}(s) = P_0\,\delta\tilde{\$}_{ext}(s) + \tilde{D}(s)\,\tilde{\rho}(s) + P_0\,\tilde{G}(s)\,\tilde{\rho}(s)$$

or

$$\frac{\tilde{p}(s)}{P_0} = \left\{ \frac{Z(s)}{1 - P_0 H(s) Z(s)} \right\} \delta \$_{ext}(s) \equiv L(s)\, \delta \tilde{\$}_{ext}(s) \qquad (3)$$

where $Z(s)$ is the usual zero-power transfer function

$$Z(s) = \left[s\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \frac{a_i}{s + \lambda_i} \right) \right]^{-1} \qquad (4)$$

and $H(s)$ is the "feedback" transfer function

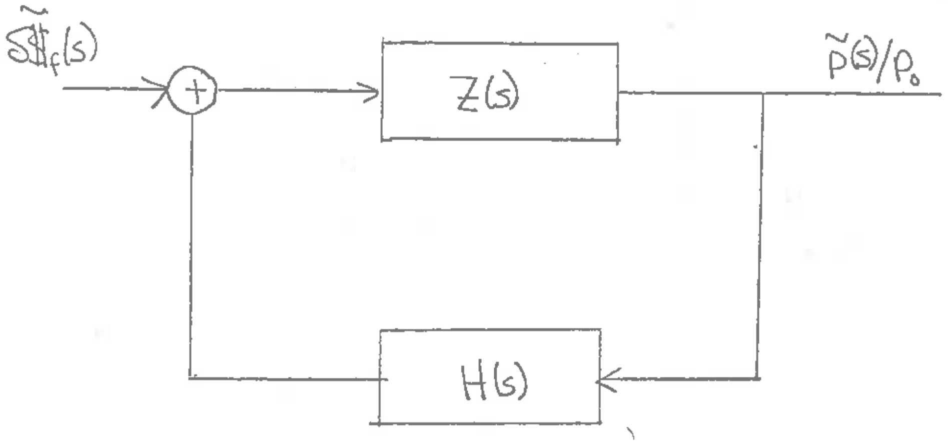
$$H(s) = \tilde{G}(s) \qquad (5)$$

We have further defined the "reactivity-to-power" or "closed loop" transfer function $\tilde{L}(s)$

$$L(s) \equiv \frac{Z(s)}{1 - P_0 H(s) Z(s)} \qquad (6)$$

[Note that as $P_0 \to 0$, $L(s) \to Z(s)$, the "zero-power" transfer function.]

This notation is consistent with our earlier block diagram



Note that if we can compute $l(t) = \mathcal{L}^{-1}\{L(s)\}$, then we find

$$p(t) = P_0 \int_0^t d\tau\, l(t-\tau)\, \delta \$_{ext}(\tau) \qquad (7)$$

Now let's examine $L(s)$ a bit. First note that unlike $Z(s)$, $L(s)$ is analytical $s=0$ with a value

$$L(0) = \frac{1}{-P_0 H(0)} = \int_0^\infty l(t)\,dt \qquad (8)$$

Hence the long time response to a step reactivity insertion is

$$p(t) = P_0\,\delta\$_0 \int_0^t l(t')\,dt' \rightarrow P_0\,\delta\$_0\,L(0)$$

Hence we approach a new equilibrium state

$$P(\infty) = P_0\left[1 + \delta\$_0\,L(0)\right]$$

This in contrast to the zero-power reactor whose power grew exponentially for long times and occurs because the reactivity

$$\delta\$_f(t) = \int_0^t G(t-\tau)\,p(\tau)\,d\tau \rightarrow p(\infty)\,H(0) \qquad (9)$$

such that $\delta\$_f(t) \rightarrow -\delta k_0$ to just compensate the step input

4.2.1. Response to a Sinusoidal Reactivity Insertion

Now for $\delta\$_{ext}(t) = k_0 \sin\omega t$

we find the long time response

$$p(t)/P_0 = |L(i\omega)|\sin(\omega t + \phi)\ , \quad \phi(\omega) = \arg\left[L(i\omega)\right]$$

Now from (6), we can see that a resonance in the gain $G(\omega) = |L(i\omega)|$ will occur when

$$1 - P_0 H(i\omega) Z(i\omega) = 1 + P_0\left|H(i\omega)\,Z(i\omega)\right| e^{i\Phi} \rightarrow 0 \qquad (10)$$

where $\Phi(\omega) = \arg[-Z(i\omega)H(i\omega)]$

For (10) to be satisfied, we require

$$P_0 |H(i\omega) Z(i\omega)| = 1$$

and

$$\Phi(\omega) = 180^\circ$$

to be simultaneously satisfied. This determines a critical power level P_c and a resonance frequency ω_0. For $P_0 < P_c$, there will be a finite resonance peak

$$|L(i\omega_0)| = \frac{|Z(i\omega_0)|}{1-(P_0/P_c)}$$

Note that the average power level $P_{ave} = P_0$ when feedback is included [this is again due to fact that $L(s)$ is analytic at the origin]

4.3 LINEAR STABILITY ANALYSIS

4.3.1. Introduction

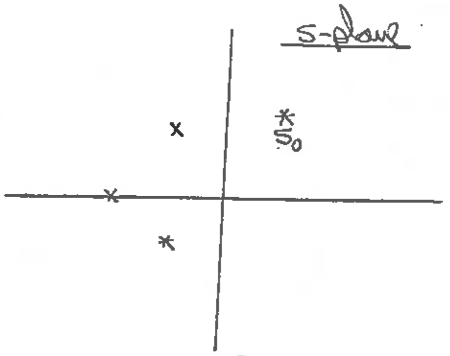
The stability of a reactor with feedback can be investigated by examining the poles of

$$L(s) = \frac{Z(s)}{1 - P_0 H(s) Z(s)} \qquad (1)$$

First note that since Z(s) appears both in the numerator and denominator, its poles ω_j "cancel". Hence the poles of L(s) are merely the zeros of

$$1 - P_0 H(s) Z(s) = 0 \qquad (2)$$

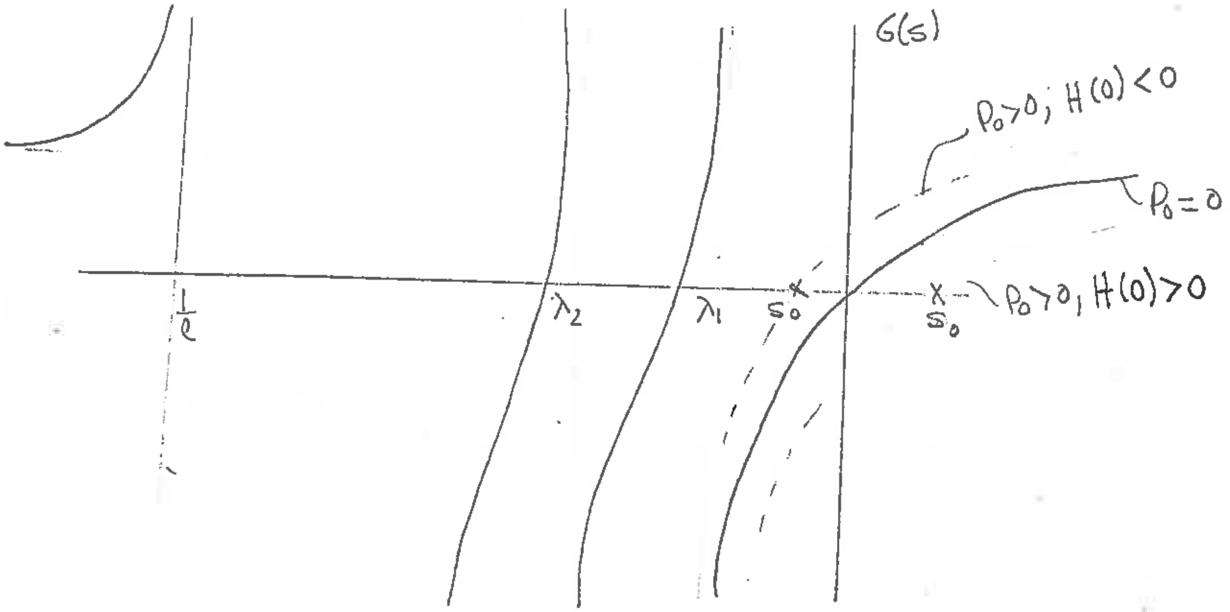
Now suppose (2) has a simple root (a pole of L(s)) at $s = s_0$. Then when we invert the Laplace transform, this will contribute a term in p(t) of the form e^{s_0}. Hence if s_0 is in the RHP, then p(t) will grow exponentially in time, thus indicating an unstable response to an applied reactivity perturbation [within the linear approximation, of course.] If the root s_0 lies in the LHP, then e^{-st} will decay in time. Hence to study reactor stability, it is obviously important to determine if any of the poles of L(s) [zeros of (2)] lie in the LHP.



To gain some insight into the onset of instability, write (2) as

$$D(s) = \frac{1}{Z(s)} - P_0 H(s) = s\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \frac{a_i}{s+\lambda_i}\right] - P_0 H(s) = 0$$

Now suppose P_0 is <u>very small</u>. Then the roots of D(s) are near those of $1/Z(s) = Y(s)$ --- as indicated in the diagram

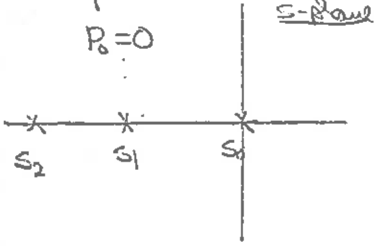


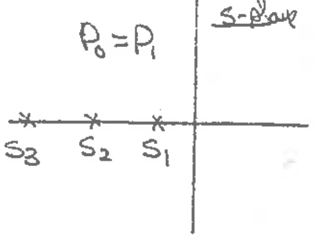
Note now that for a small, nonzero value of P_0, the curve shifts up or down depending upon the sign of the steady-state power coefficient

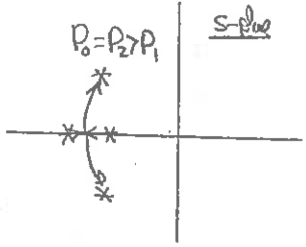
$$H(0) = \int_0^\infty d\tau\, G(\tau)$$

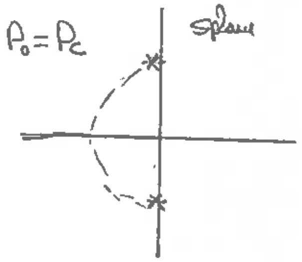
In particular, if $H(0) > 0$, s_0 is in the RHP and the system will <u>be unstable</u>
$H(0) < 0$, LHP " " " stable

For larger powers P_0, the situation is something as shown. For $P_0 = 0$, we can identify the poles of $L(s)$ as just those of $Z(s)$. Then if $H(0)<0$, as we increase P_0, the root s_0 moves to the left. But other roots will move toward the right.









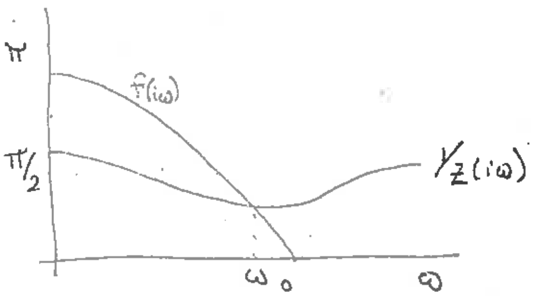
For <u>some</u> sufficiently large P_0, s_0 will encounter s_1 and form a complex conjugate pair. For still larger P_0 this pair moves back to the right, until for some critical power $P_0 = P_c$, the roots move into the RHP and the reactor becomes unstable. When the poles cross the imaginary axis, we observe a "resonance" in the transfer function gain. (as we mentioned earlier).

Hence to investigate the onset of instability, we can merely determine when

$$D(i\omega) = i\omega\left[\frac{\Delta}{\beta} + \sum_{i=1}^{6}\frac{a_i}{i\omega+\lambda}\right] - P_0 H(i\omega) = 0 = \frac{1}{Z(i\omega)} - P_0 H(i\omega)$$

If for some value of ω, say ω_0, $1/Z(i\omega)$ and $H(i\omega)$ have the same phase, there will be a value of P_0 for $D(i\omega) = 0$ -- and hence there will be an instability.

For example, consider the phase variation as shown. For sufficiently large power P_0, instability will occur ω_0.



We can examine in more detail $1/Z(i\omega)$. In fact one can show $1/Z(i\omega)$ lies always in upper right quadrant of complex plane. Thus for instability to be possible, the feedback function $H(i\omega)$ must lie in the same quadrant. Hence necessary conditions for stability are

$$\mathrm{Re}\{H(i\omega)\} > 0 \quad \mathrm{Im}\{H(i\omega)\} > 0$$

In this sense then, a sufficient condition for stability is that for no value of ω are both $\mathrm{Re}\{H(i\omega)\} > 0$ and $\mathrm{Im}\{H(i\omega)\} > 0$

We can occasionally study a given feedback model directly.

EXAMPLE: Consider the two-temperature model we were studying before

$$\frac{d\,\delta T_f}{dt} = a\,\delta P - \omega_f\,\delta T_f$$

$$\frac{d(\delta T_m)}{dt} = b\,\delta T_f - \omega_m\,\delta T_m$$

Laplace transforming, we find

$$H(s) = \frac{H_f(0)}{1 + s/\omega_f} + \frac{H_m(0)}{(1+s/\omega_f)(1+s/\omega_m)}$$

where

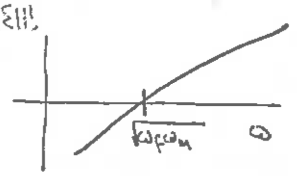
$$H_f(0) = \frac{a r_c}{\omega_f} \qquad H_m(0) = \frac{a b r_m}{\omega_f \omega_m}$$

Now for stability we require $H_m(0) + H_f(0) = H(0) < 0$. But if this is achieved by having a large negative $H_m(0)$ [large delayed feedback], then instability may occur.

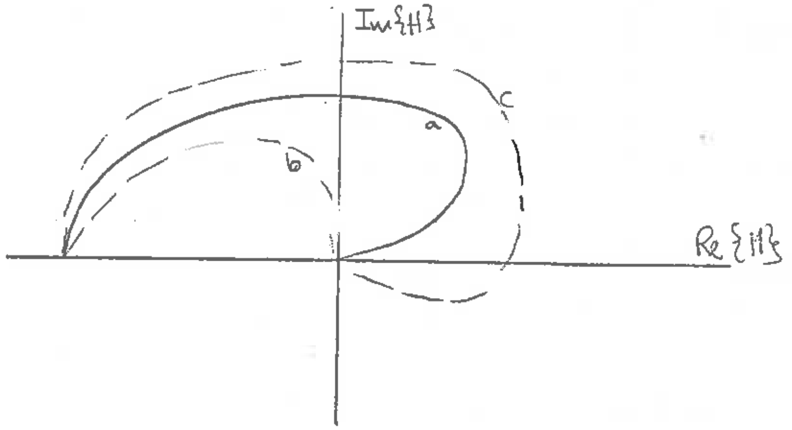
Consider first the case in which $H_f(0) = 0$. Then

$$H(i\omega) = \frac{H_m(0)\left[1 - \frac{\omega^2}{\omega_f \omega_m} - i\left(\frac{\omega}{\omega_f} + \frac{\omega}{\omega_m}\right)\right]}{(1+\omega^2/\omega_f^2)(1+\omega^2/\omega_m^2)}$$

Now if $H_m(0) < 0$, $\operatorname{Im}\{H(i\omega)\} > 0$ all $\omega > 0$. But $\operatorname{Re}\{H(i\omega)\}$ changes sign at $\omega = \sqrt{\omega_f \omega_m}$



A more detailed plot indicates



Here, curve a is for the case $H_f(0) = 0$ (possible instability)
curve b " " " $H_f(0) < 0$, $|H_f| >> |H_m|$ (stable)
curve c " " " " $H_f(0) > 0$ but $H(0) < 0$ (possible instability)

Hence we find that negative feedback systems with appreciable delay (or phase lag) may in fact be unstable.

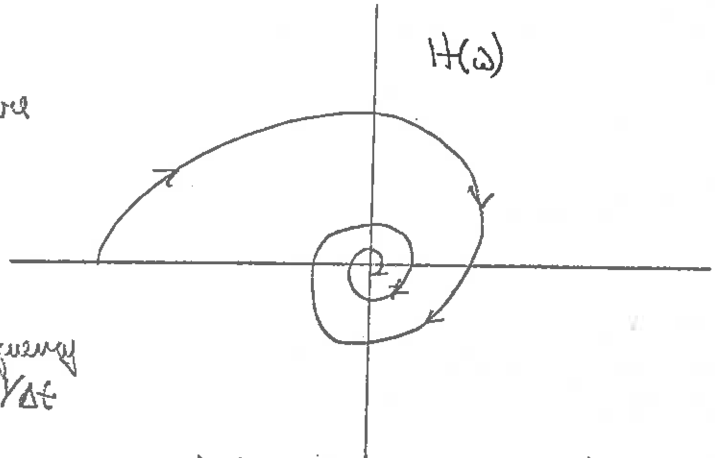
Yet another source of instability may be that caused by the time lag for fluid flow. In this case the equation for the moderator temperature becomes

$$\frac{d}{dt}[\delta T_m] = b\,\delta T_F(t-\Delta t) - \omega_m\,\delta T_m(t)$$

where Δt is the time delay. Then the feedback due to the moderator becomes

$$H_m(i\omega) = \frac{H_m(0)\,e^{-i\omega\Delta t}}{\left(1+\frac{i\omega}{\omega_F}\right)\left(1+\frac{i\omega}{\omega_m}\right)}$$

Now if $H_F(0)$ is positive, then we find the plot as shown. There will then be several frequencies at which enhanced responses (or resonances) to reactivity disturbances will occur.
The fundamental resonant frequency for instability is of the order of $1/\Delta t$



Still another instability mechanism is that in which acoustical modes of mechanical oscillations may be amplified to cause instability.

4.3.2. More Formal Considerations of Reactor Stability

We are interested in the response of a reactor to small power perturbations. The possibility of an unstable response to a small disturbance in the power is characteristic of systems with feedback. We will be concerned with "autonomous" systems in which none of the parameters characterizing the system depend on time. This implies that the external reactivity $\delta\$_{ext}$ and external source S_0 are constant. Then the P.R.K.E. becomes

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \{\$_{ext} + \$_f[P]\}P(t) + \int_0^\infty d\tau\, D(\tau)\left[P(t-\tau) - P(t)\right] + S_0 \qquad (1)$$

For an equilibrium state P_0 we require

$$\{\$_{ext} + \$_f[P_0]\}P_0 + S_0 = 0 \qquad (2)$$

For general $\$_f[P_0]$, there might be many equilibrium states. However if $-\$_f[P_0]$ is a non-decreasing function of P_0 (as we expect it to be) there will be only one such nontrivial state P_0. [E.g. for a linear functional, we have $\$_f[P_0] = H(0)P_0$ and we have only one equilibrium state]

Now (2) indicates that $P_0 = 0$ is always an equilibrium state if $S_0 = 0$. But note that in the shutdown state, the net reactivity

$$\$ = \$_{ext} + \$_f[0] = \$_{ext} > 0$$

and hence the power level will increase exponentially if it is perturbed from zero -- that is, the shutdown state $P_0 = 0$ is unstable.

We will instead direct our attention to the non-zero solution of (2) in the absence of sources, i.e.

$$\$_{ext} + \$_f[P_0] = 0 \qquad (3)$$

We will again introduce the incremental feedback reactivity and power

$$\delta\$_f[p] \equiv \$_f[P] - \$_f[P_0]$$
$$p(t) = P(t) - P_0$$

in which case (1) becomes

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = \delta \$_f[p](p+P_0) + \int_0^\infty d\tau\, D(\tau)[p(t-\tau)-p(t)]$$

or linearizing and using a linear feedback functional

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = P_0\int_0^t d\tau\, G(\tau)p(t-\tau) + \int_0^t d\tau\, D(\tau)p(t-\tau) - p(t)$$

where we have assumed that the reactor is operated at P_0 prior to $t=0$. Further, we will introduce an initial perturbation in the power $p(0)$ at $t=0$. If we now Laplace transform

$$\tilde{p}(s) = \frac{p(0)}{\frac{1}{Z(s)} - P_0 H(s)} = L(s)\,p(0)$$

The behavior of $p(t)$ is determined by the singularities of $L(s)$ which occur at the zeros of the "characteristic equation"

$$\mathcal{G}(s) = 1 - P_0 H(s) Z(s)$$

Hence the problem of the linear stability of our equilibrium state reduces to the problem of determining the sign of the real parts of the roots of the characteristic equation. In particular we have found

CLAIM: A reactor is linearly "strictly stable" when the roots of the characteristic equation all have negative real parts. The response of the power in the critical case, when any of these roots have real parts equal to zero, is not correctly described by linear analysis and depends upon nonlinearities of the P.R.K.E.

There are lots of tricks to determine the signs of the roots of $\mathcal{G}(s)$ without actually solving the characteristic equation. We will now outline several of these "linear stability criteria".

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4.3.3 Routh-Hurwitz Stability Criterion

Whenever the system is characterized by a set of linear equations with constant coefficients (a lumped parameter system), then the characteristic equation can be written as a polynomial in (s)

$$G(s) \equiv a_0 s^n + a_1 s^{n-1} + \cdot \quad + a_{n-1} s + a_n = 0 \qquad (4)$$

[Here we will assume $a_0 > 0$ without loss of generality.]

Preliminary Comments:

i.) A necessary condition for all of the roots to have negative real parts is for all the coefficients a_n to be non-zero and positive.
ii.) This is also a sufficient condition for first or second order polynomials.
iii.) For $n \geq 3$, $a_n > 0$ only insures negativeness of real roots

A necessary and sufficient condition on the $\{a_n\}$ such that $G(s)$ has only roots with negative real parts is given by

THEOREM: The roots of $G(s) = 0$ all have negative real parts if and only if

$$a_0 > 0$$

$$\Delta_1 = a_1 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0$$

$$\vdots$$

$\Delta_n > 0$ where

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & & \\ 0 & 0 & 0 & 0 & \cdot & a_{n-2} & a_n & 0 \\ 0 & 0 & 0 & 0 & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{vmatrix}$$

Comments: These conditions are not independent of each other, and hence for a given system may be written in more simplified forms. Such criteria are useful for defining stable operating regions which are based on the physical parameters of reactor models.

4.3.4. Nyquist Stability Criterion

There are several very useful graphical stability tests used in linear system analysis such as "root-locus method", Bode diagrams, and the "Nyquist criterion".

Ref: E. A. Guillemin, The Mathematics of Circuit Analysis, (John Wiley, New York, 1949)
I. M. Horowitz, Synthesis of Feedback Systems (Academic Press, New York, 1963)

The Nyquist criterion is a graphical method based on Cauchy's residue theorem.

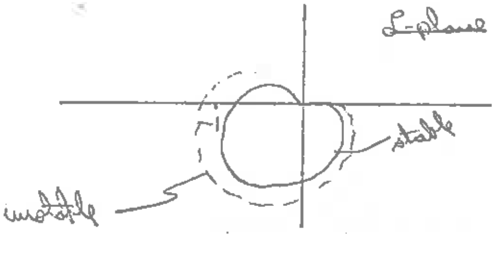
NYQUIST STABILITY CRITERION: Consider the "open-loop frequency response"

$$\mathcal{L}(s) = -P_0 H(s) Z(s)$$

for $s = i\omega$. Now vary ω from $-\infty$ to $+\infty$ and observe the behavior of $\mathcal{L}(i\omega)$. If $\mathcal{L}(i\omega)$ does not encircle the -1 point in the complex $\mathcal{L}$-plane, then $\mathcal{D}(s)$

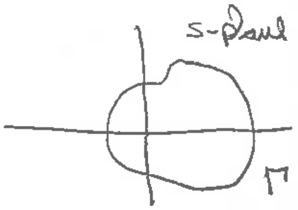
$$\mathcal{D}(s) = 1 - P_0 H(s) Z(s) = 0$$

cannot have any roots with real positive parts, and the reactor is stable.

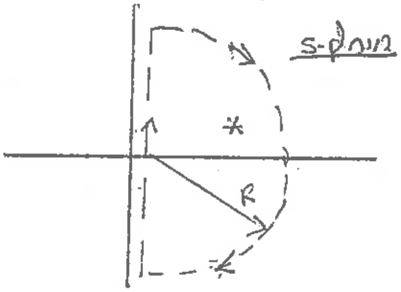


Proof: Consider the function $G(s) = 1 - P_0 H(s) Z(s)$. It is a well known theorem in complex variable theory that the change of the argument of $G(s)$ along a closed contour Γ is 2π times the number of zeros of $G(s)$ minus the number of poles of $G(s)$

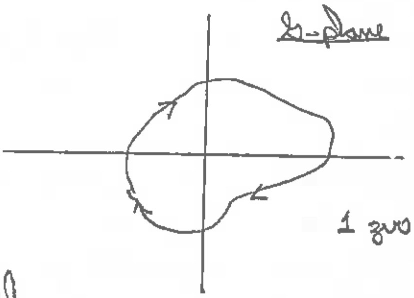
$$\Delta \arg G(s)\Big|_{\Gamma} = 2\pi\left(\#_{zeros} - \#_{poles}\right) = 2\pi(Z-P)$$



Since we wish to determine if arg G(s) has any zeros in the RHP, we need only apply this theorem along the indicated contour in the s-plane [as $R \to \infty$]



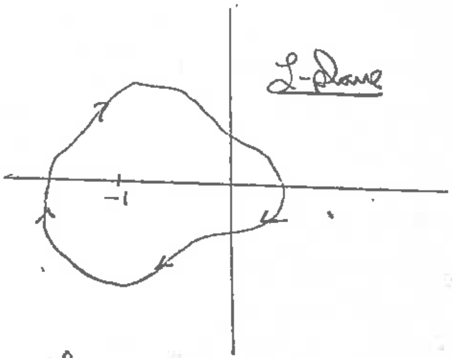
Notice that an alternative interpretation of this theorem states that the point $G(s)$ in the G-plane encircles the origin $(Z-P)$ times in the same direction as the original motion of s along Γ. (We have indicated this schematically for one zero).



Actually it is most convenient to study the motion of

$$\mathcal{L}(s) = -P_0 Z(s) H(s) = G(s) - 1$$

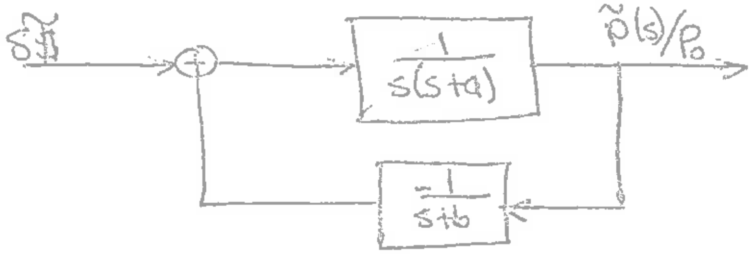
which translates the origin of the G-plane to the point -1 of the $\mathcal{L}$-plane. This is because one can show that $\mathcal{L}(s) \sim 1/s^n$ for large s, hence we can ignore the contribution from the semi-circular contour in the s-plane. [Note $\mathcal{L}(s)$ has no poles in RHP since feedback system is stable.]



Actually, we need only plot $\mathcal{L}(s)$ for $s = i\omega$, $0 < \omega < \infty$ since the plot for $-i\omega$ is the mirror image.

Hence our "sketch" of the proof is complete.

EXAMPLE



Here

$$Z(s) = \frac{1}{s(s+a)} \qquad H(s) = \frac{-1}{s+b}$$

$$L(s) = \frac{Z(s)}{1 - P_0 H(s) Z(s)} = \frac{1/s(s+a)}{1 + \frac{P_0}{s(s+a)(s+b)}}$$

Root Locus Method:

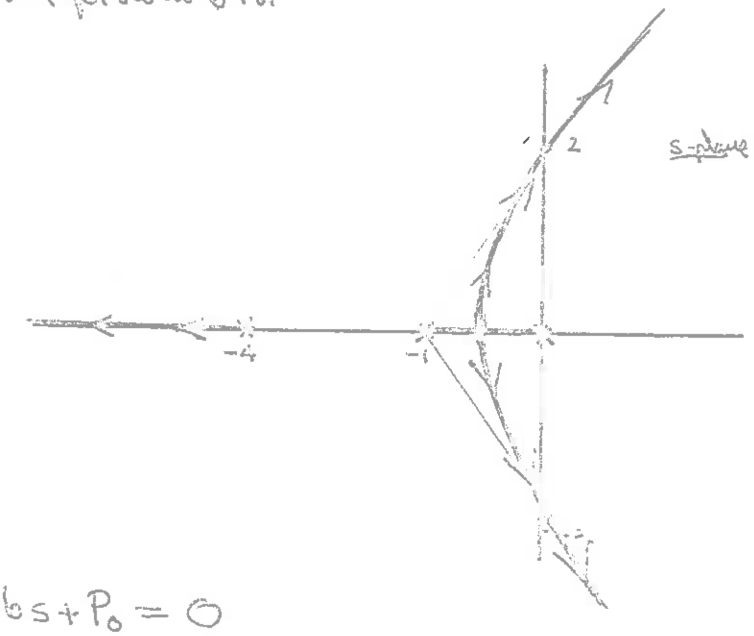
The zeros of $\mathcal{L}(s) = 1 - P_0 H(s) Z(s)$ are given by the roots of

$$s^3 + (a+b)s^2 + abs + P_0 = 0$$

We will now plot the zeros of this polynomial for various P_0.
To make life easy, take $a = 1$, $b = 4$.
Then note

i.) $P_0 = 0 \Rightarrow$ roots at $s = 0, -1, -4$

ii.) For $P_0 = 20$, roots cross into RHP. [Occurs at $\omega = 2$]
Hence $P_0 < 20$ for stability



Routh-Hurwitz

Again consider $s^3 + (a+b)s^2 + abs + P_0 = 0$
Then

$$a_0 = 1 > 0, \quad \Delta_1 = a_1 = (a+b) > 0, \quad \Delta_2 = \begin{vmatrix} (a+b) & P_0 \\ 1 & ab \end{vmatrix} > 0 \Rightarrow P_0 < ab(a+b)$$

For the case $a = 1$, $b = 4$, this demands again

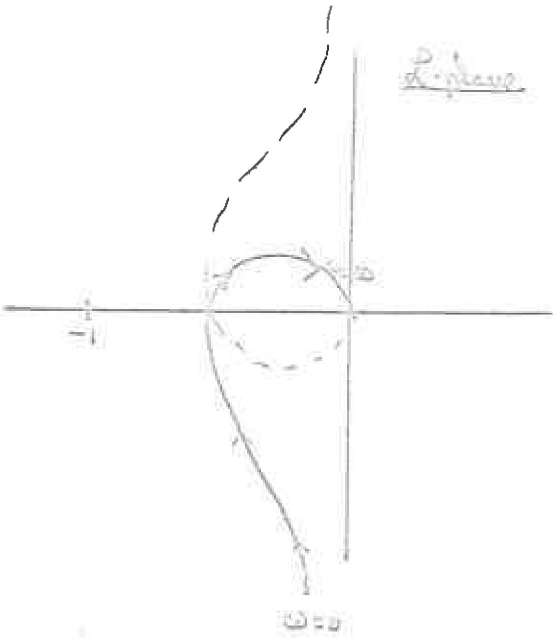
$$P_0 < 20 \quad \text{for stability}$$

Nyquist Diagram

We now want to plot

$$\mathcal{L}(s) = - P_0 H(s) Z(s)$$

$$= \frac{P_0}{s(s+a)(s+b)}$$



for $\omega \in (-\infty, \infty)$, Note

$\omega = 0, \ |\mathcal{L}(s)| = \infty, \ \arg\{\mathcal{L}(s)\} = -90°$

$\omega = \infty, \ |\mathcal{L}(s)| = 0, \ \arg\{\mathcal{L}(s)\} = -270°$

Note also that for $\omega = \sqrt{ab}$

$$\mathcal{L}(s) = \frac{P_0}{i\sqrt{ab}\,(i\sqrt{ab}+a)(i\sqrt{ab}+b)} = \frac{-P_0}{ab(a+b)}$$

Hence for

$P_0 < ab(a+b)$ the system is stable.

4.3.5. Stability Margins

Suppose we have determined that the reactor with feedback is stable for a given set of parameters. We now want to introduce the concepts of "gain" and "phase" stability margins.

We have seen that there will usually be a critical frequency ω_c and a critical power level P_c for which

$$P_c Z(i\omega_c) H(i\omega_c) = 1 \qquad (1)$$

— the reactor becomes unstable. In particular, we have seen that if

$$\mathrm{Re}\{H(\omega)\} \le 0 \qquad \text{for all } \omega$$

then the reactor can never be linearly unstable.

[The one-temperature feedback model for which

$$H(s) = \frac{\alpha n}{s+d}$$

is a system of this type since

$$\mathrm{Re}\{H(\omega)\} = \frac{\alpha n d}{\omega^2+d^2} < 0 \quad]$$

We can rewrite (1) as

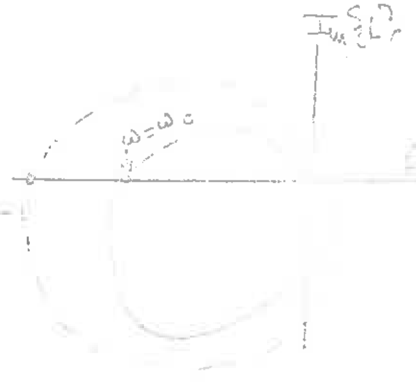
$$-\mathcal{L}(i\omega) = 1$$

or:

$$|\mathcal{L}(i\omega)| = 1 \quad \text{and} \quad \arg\{\mathcal{L}(i\omega)\} = 180^\circ$$

Now if we say then the "gain margin" is defined to be the factor by which the open loop gain must be increased to cause instability at a frequency ω_c at which the phase angle is 180°

$$\arg\{\mathcal{L}(i\omega)\} = 180^\circ$$

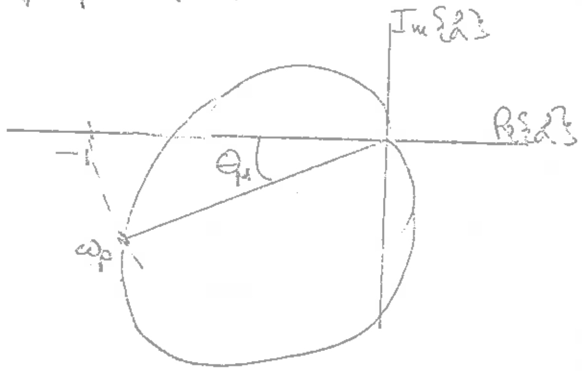


More explicitly the gain margin in decibels is

$$G_M = 20 \log \frac{P_c}{P_0}$$ where P_c is the critical power level

The "phase margin" is similarly defined as the amount of additional phase lag required to cause instability at a frequency ω_p at which the gain is unity

$$|L(i\omega_p)| = 1$$



Such phase and gain margins are measured experimentally. Feedback control mechanisms may well be inserted into the feedback loop if it is desirable to increase G_M or Θ_M.

Frequently one analyzes the stability of a system [illegible] the [illegible] of the system. [illegible] But one must [illegible] the stability of the system.

4.4. NONLINEAR POINT REACTOR KINETICS

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Thus far our study of the P.R.K.E. with feedback has been restricted to situations in which the reactivity changes, and corresponding power changes are sufficiently small that these equations can be linearized. In particular, our study of the stability of the reactor has been restricted to the consideration of "stability in the small" -- that is, to the study of perturbations and responses sufficiently small for a linear analysis. However for larger perturbations, nonlinear effects must be taken into account. In these cases, the resulting conclusions about stability may be quite different.

For example, we have seen that the linearized P.R.K.E. predicts the reactor will be unstable if the power exceeds some critical value P_c. However even though the reactor is linearly unstable, it may be stable in the nonlinear description. Hence it is of considerable importance to study the significance of linear stability theory within the more general framework of nonlinear stability theory.

* * *

Before we set off into this venture, it should be mentioned that there are many different approaches to nonlinear point reactor kinetics -- none of which is completely satisfactory. Most theories are limited to very simple forms of feedback functions. Further, such theories as do exist provide only sufficient conditions for stability (as opposed to necessary conditions). Sufficient conditions are usually much too restrictive for practical applications.

"From a practical standpoint, nonlinear stability analysis has proved to be much less important than linear analysis. The nonlinear analysis does not lend itself to experimental verification. Nonlinear studies have usually indicated that results of linear analysis are not likely to be misleading." -- Bell & Glasstone

* * *

4.4.1. A Simple Example of Nonlinear Reactor Analysis (Ash, p.109)

Consider a simple model in which we neglect delayed neutrons and take a very simple temperature feedback

$$\frac{dP}{dt} = \frac{\rho}{\Lambda} P(t) \qquad \text{where} \quad P(t) = \frac{P(t)}{P_0}, \quad P(0) = 1$$

$$C \frac{dT}{dt} = P(t) - 1 \qquad T(t) = T(t) - T_0, \quad T(0) = 0$$

We also assume a simple reactivity dependence

$$\rho(t) = \rho_0 - aT$$

Here C is the heat capacity of the reactor and a is the temperature coefficient of reactivity.

Rather than attempt to solve these equations directly, suppose we take their ratio to find

$$\frac{dP}{dT} = \frac{C}{\Lambda}\left[\frac{(\rho_0 - aT)P}{P-1}\right] \qquad \text{with } P[T(0)] = P(0) = 1$$

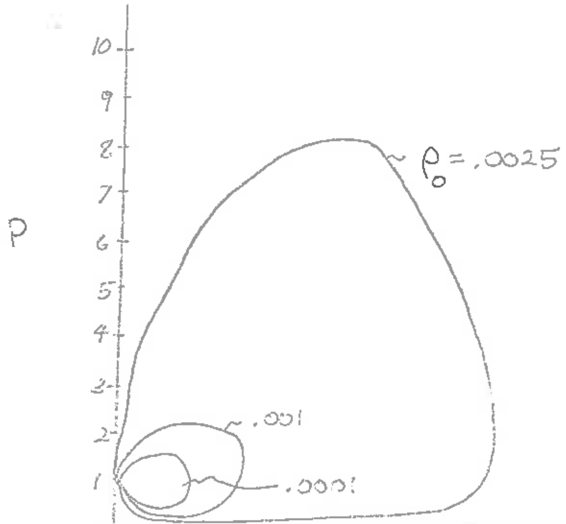
But we can easily integrate this to find

$$P - \ln P = \frac{C}{\Lambda}\left[\rho_0 T - \frac{a}{2}T^2\right] + 1$$

We can plot this dependence of P upon the temperature T as shown for

$C = 0.1$ sec/deg

$a = 10^{-4} \left(\frac{\rho_0}{\beta}\right)$/degree



We can see that the system will circle around these closed paths in a bounded, stable oscillation. Such a parametric plot is called a "phase plane", while such paths are called "limit cycles".

To determine P(t) and T(t), we must still solve the nonlinear ordinary differential equations. This usually can only be done approximately.

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What would a linear analysis of this system have predicted?

Let

$$\rho(t) = P(t) - 1$$

We can then linearize

$$\frac{d\rho}{dt} = \left[\frac{\rho_0 - aT}{\Lambda}\right]\left[\rho(t)+1\right] \sim \frac{\rho_0\,\rho(t) + \rho_0 - a\,T(t)}{\Lambda}$$

$$c\,\frac{dT}{dt} = \rho(t)$$

Now Laplace transform to find

$$\tilde{\rho}(s) = \frac{\rho_0}{s^2 - \frac{\rho_0}{\Lambda}s + \frac{a}{c}}$$

The poles are at

$$s = \frac{+\frac{\rho_0}{\Lambda} \pm \sqrt{\left(\frac{\rho_0}{\Lambda}\right)^2 - 4a/c}}{2} \sim \frac{\rho_0}{\Lambda}\,;\ \frac{a}{c}\left(\frac{\Lambda}{\rho_0}\right)^2$$

Since the poles are in the RHP, the linear analysis predicts instability for any $\rho_0 > 0$.

4.4.2. Phase Plane Analysis

The study of such a phase plane provides a great deal of information about the stability and actual time behavior of nonlinear systems. Indeed this method is regarded as the "classical" approach to the analysis of nonlinear O.D.E.'s.

To illustrate such an approach, let's consider a slightly more complicated example. We will again describe the neutronics by

$$\frac{dP}{dt} = \left[\frac{\rho_o - aT}{\Lambda}\right] P(t) \quad , \qquad P(0) = 0$$

But we will now add a heat loss term to the temperature balance equation

$$\frac{dT}{dt} = \frac{P(t) - 1}{c} - \mu T(t) \, , \qquad T(0) = 0$$

To determine the phase plane structure, we again take the ratio

$$\frac{dP}{dT} = \frac{m(\rho_o - aT)P}{P - 1 - \mu c T}$$

where we will take

$a = 10^{-4}$ (ρ_o/β)/deg (reactivity temperature coefficient)
$c = .10$ sec/deg (reactor heat capacity)
$m = c/\Lambda = 8 \times 10^{-3}$/deg
$\mu = 0.01$ sec^{-1} (heat transfer coefficient)

Note we have again written the system as

$$\frac{dP}{dT} = \frac{N(P,T)}{D(P,T)}$$

Now we first locate those points in the P-T phase plane which are singular in the sense that

$$N(P,T) = D(P,T) = 0 \quad \text{simultaneously}$$

G. Birkhoff & G. Rota, ODEs, (Ginn & Co., 1962) Chapter 6.
Stoker, "Nonlinear Vibrations" (Interscience)

Now, without loss of generality, suppose origin is the singular point. Then expand

$$N(P,T) = aP + bT + O(P^2+T^2)$$

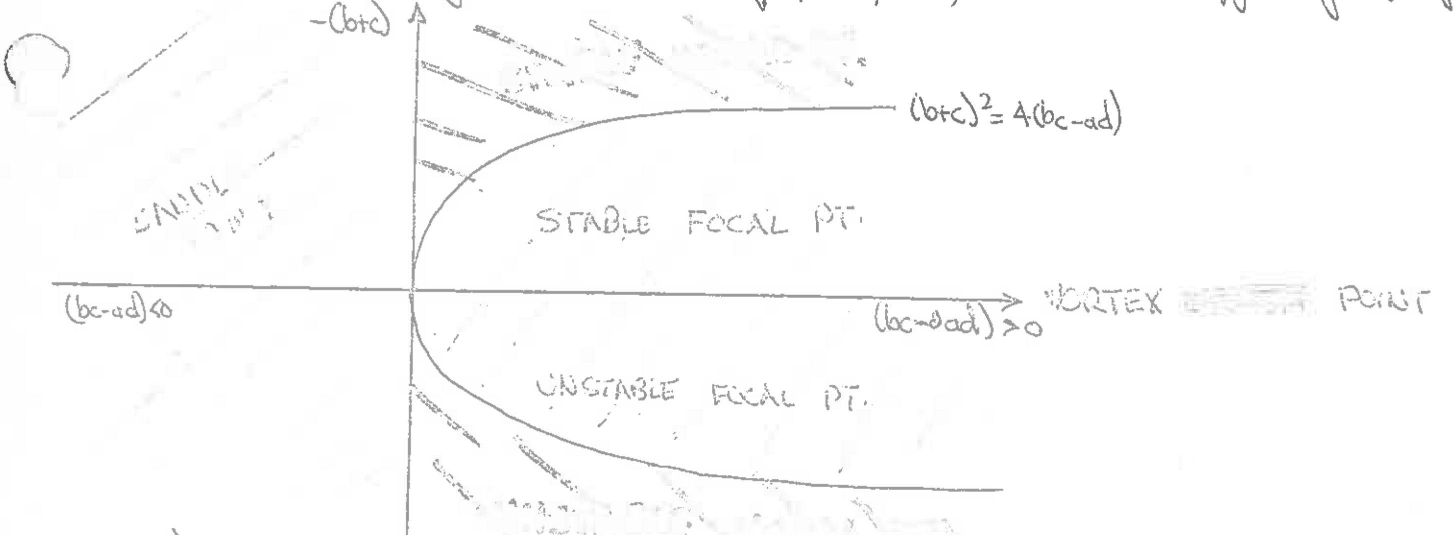
$$D(P,T) = cP + dT + O(P^2+T^2)$$

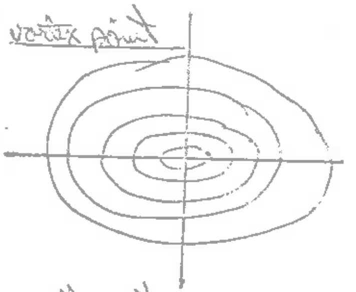
Hence we can now use

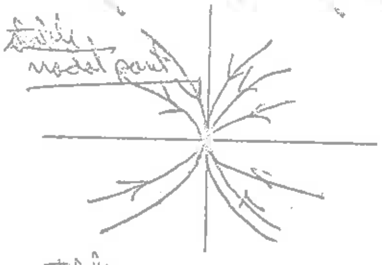
Poincaré's Theorem: The singularities of the nonlinear equations have the same nature as those of the linear, autonomous system

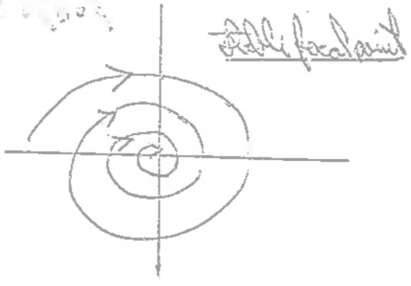
$$\frac{dP}{dT} = \frac{aP+bT}{cP+dT}$$

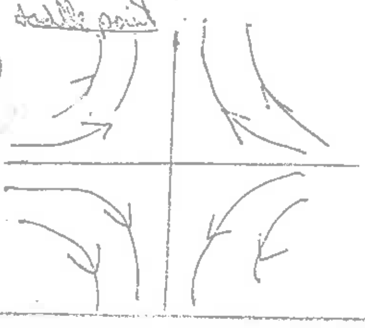
Now, depending on the values of a, b, c, & d, we can identify singularity

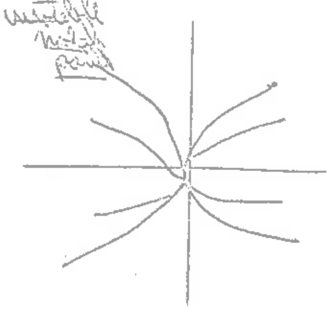


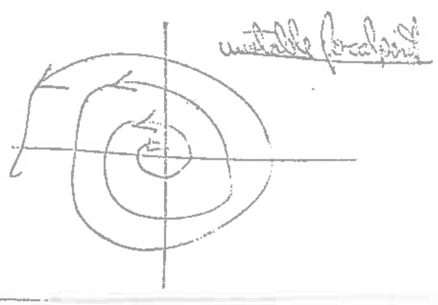












As you can verify, there are two singular points:

① $P=0, \ T=-1/\mu C$

② $P=1+\mu C\rho_0/a, \ T=\rho_0/a$

Our next job is to determine the behavior of the system in the vicinity of each of these points. We do this by linearizing about each point. Consider point ①

$$\frac{dP}{dT} \cong \frac{m(\rho_0+a/\mu C)P}{P-1-\mu CT}$$

But this is equivalent to the linear system

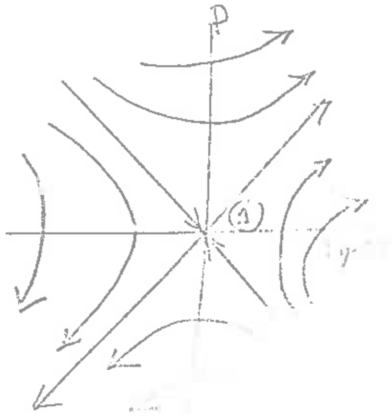
$$\frac{dP}{dt} = m\left(\rho_0+\frac{a}{\mu C}\right)P(t)$$

$$\frac{d}{dt}(1+\mu CT) = P(t)-(1+\mu CT)$$

We can solve this system using Laplace transforms to find

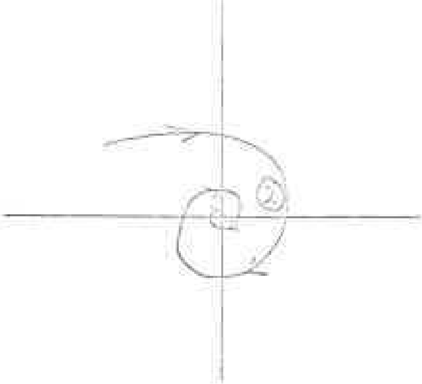
$$P(t) = A\exp\left[m\left(\rho_0+\frac{a}{\mu C}\right)t\right]+Be^{-t}$$

$$1+\mu CT(t) = C\exp\left[m\left(\rho_0+\frac{a}{\mu C}\right)t\right]+De^{-t}$$

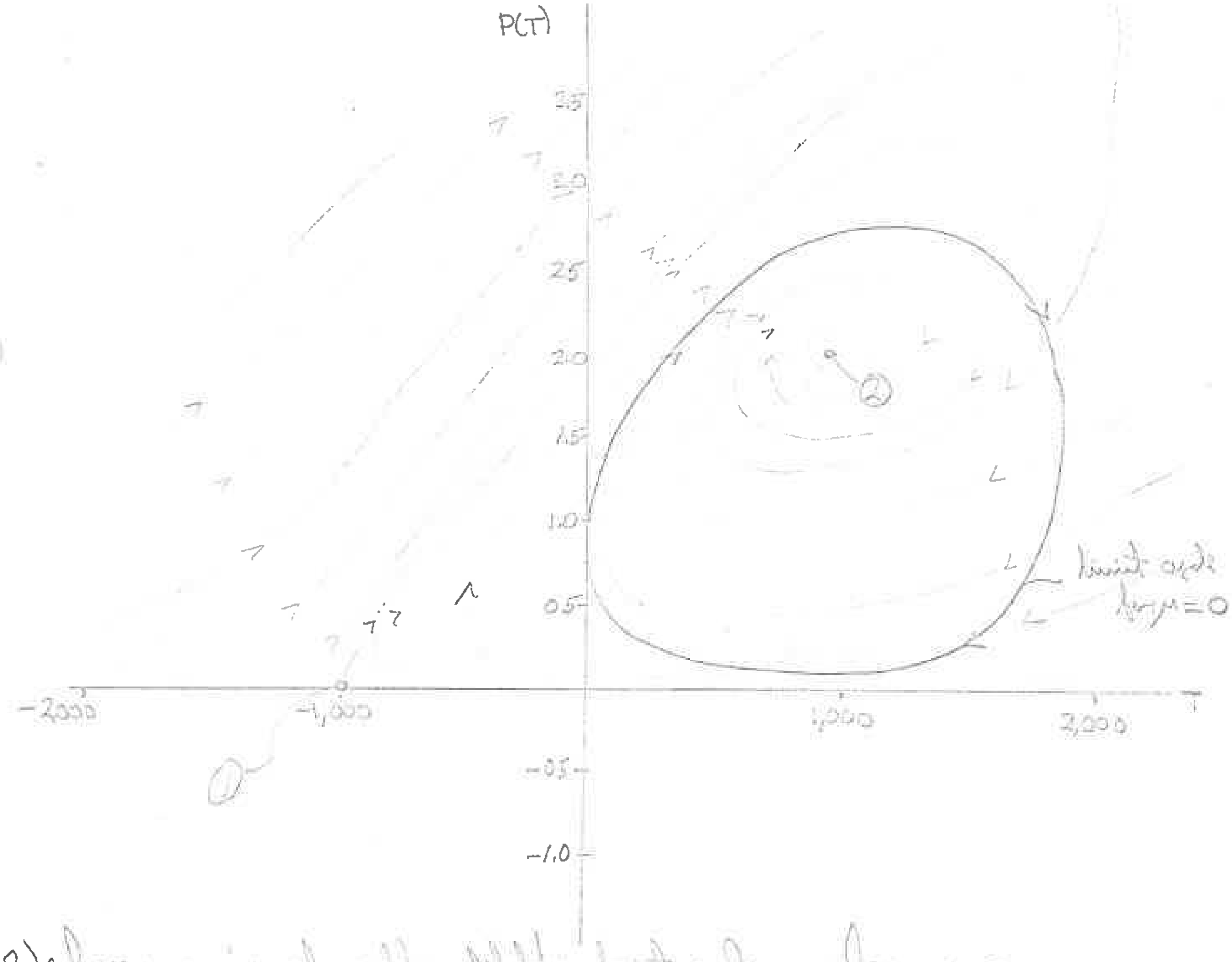


Here, A, B, C & D are determined by the initial conditions on P(0) and T(0). In particular, note that the system diverges unless A = C = 0. The singular point (1) is an example of an "unstable saddle point" since the system always moves initially away from such a point.

A similar linearized analysis indicates that the singular point ② is a "stable spiral point". [illegible] spiral in on such a point.



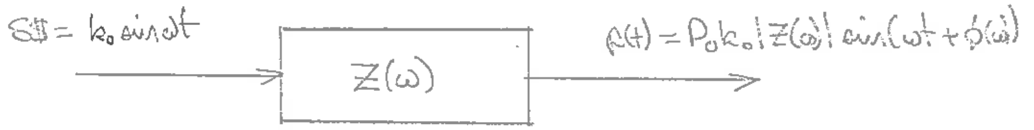
A more detailed plot in the phase plane then follows.



We have superimposed on this plot the "limit cycle" curve for $\mu=0$

4.4.3 Describing Functions

Before we cont[illegible] on to a more [illegible] treatment of nonlinear problems (stability, etc.) [illegible] if we can generalize the concept of a transfer function to nonlinear systems. [illegible] for linear systems the transfer function was defined by assuming a sinusoidal signal and measuring the output.



Unfortunately, in a nonlinear system it is possible to excite higher harmonics of the excitation frequency ω such that the general output is

$$p(t) = P_0 \sum_{n=-\infty}^{\infty} P_n(k_0,\omega) \sin n\omega t$$

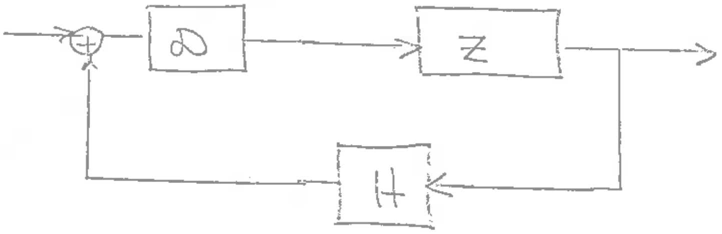
In the event that we can neglect all but the fundamental component, i.e.

$$p(t) \sim P_0 \, |P_1(k_0,\omega)| \sin(\omega t + \phi_0)$$

one can define the nonlinear analogue to a transfer function, the so-called "describing function"

$$\mathcal{D}(k_0,\omega) \equiv \frac{|P_1(k_0,\omega)|}{k_0} e^{i\phi(k_0,\omega)}$$

One can then insert this into the feedback loop and continue with the analysis



For more details, see Ott, pp. 102-107
Weaver, pp. 147-154

4.4.4. Nonlinear Stability Theory

[illegible] — a very complicated [illegible] (as there are, for example, in linear systems theory). For that reason, each particular [illegible] considered usually requires a unique approach (usually far from obvious).

To provide a flavor of just what is involved in such analyses, I'll just run through a sampling of concepts in nonlinear stability theory.

We have already seen that linear stability analysis often gives information about the more general nonlinear problem. This feature can be stated as a theorem

THEOREM: Suppose we linearize the nonlinear ODE's about a certain reference point. Then if this linearized system is asymptotically stable with respect to small perturbations about the reference point, the nonlinear system is also asymptotically stable within small [illegible] about this point.

This very important result is known as Liapunov's First or Indirect Method, and usually serves as the first step in the analysis of the system. In the example we considered on p. 112 & 113, the linearization suggested an instability. Hence the above theorem did not apply [

What can we say about the stability of the reactor system for arbitrarily large perturbations in the power level? This involves the study of so-called "global stability", and general results are usually quite hard to come by. As an example of a global stability criterion, we will discuss Liapunov's second method which is based on the following theorem:

THEOREM: Consider a system of nonlinear ODE's

$$\frac{d\underline{f}}{dt} = \underline{A}(\underline{f}(t))$$

where we will assume that $\underline{A}(\underline{0}) = \underline{0}$. Then if there exists a function $V(\underline{f}(t))$ of the solutions $\underline{f}(t)$ which possesses the properties:

i.) $V(\underline{0}) = 0$,

ii.) $V(\underline{f}) \geqslant 0$

iii.) $V(\infty) = \infty$

iv.) $\frac{dV}{dt} < 0$

$\Big\{$ $V(\underline{f})$ is known as the Liapunov function for the system.

then the system is absolutely stable. [The motion following an initial perturbation $\underline{f}(0)$ will be such that $\underline{f}(t) \to 0$ as $t \to \infty$, regardless of the value of $\underline{f}(0)$]

If the condition iv.) is replaced by $\frac{dV}{dt} \leq 0$, then we can only say the system is "uniformly stable" -- i.e. the perturbations remain bounded in time.

Proof: See, for example, Ergen, Lipkin, & Nohel, J. Math & Phys. 36, 36 (1957)

EXAMPLE: Consider a critical reactor with a simple temperature feedback

$$\frac{dP}{dt} = \frac{\rho}{\Lambda} P(t) \qquad\qquad P(t) \to P(t)/P_0$$

$$\rho = -\alpha T(t)$$

with the temperature equation containing a term describing a constant power removal

$$c\frac{dT}{dt} = P(t) - 1$$

It is convenient to work with a new variable $Q(t) = \ln P(t)$. Then our set of equations becomes

$$\frac{dQ}{dt} = -\frac{\alpha}{\Lambda} T(t) \qquad\qquad \frac{dQ}{dt}(0) = 0$$

$$c\frac{dT}{dt} = e^{Q(t)} - 1 \qquad\qquad T(0) = 0$$

Consider now the function

$$V(Q,T;t) = e^{Q(t)} - Q(t) - 1 + \frac{\alpha c}{2\Lambda} T^2(t)$$

Note:
i.) $V(0,0;t) = 0$
ii.) $V(Q,T;t) > 0$
iii.) $V(\infty) = \infty$
iv.) $\frac{dV}{dt} = \dot{Q}e^{Q} - \dot{Q} + \frac{\alpha c}{\Lambda} T\dot{T} = c\dot{T}\dot{Q} - c\dot{T}\dot{Q} = 0$

Hence $V(Q,T;t)$ is indeed a Liapunov function, but it only guarantees the uniform stability of the system.

EXAMPLE: We'll consider a slightly more complicated model in which we consider a "Newton's law of cooling" heat extraction

$$\frac{dP}{dt} = -\frac{\alpha}{\Lambda} T(t)\, P(t)$$

$$c\frac{dT}{dt} = P(t) - \sigma(T(t) - T_0)$$

Again defining $Q(t) = \ln P(t)$, the Lyapunov function becomes

$$V_1(Q,T;t) = e^{Q} + \sigma T_0 Q(t) - 1 + \frac{\alpha C}{2\kappa}(T^2 - T_0^2)$$

Again $V(0)=0$, $V(\infty)=\infty$, $V(t)>0$.
But now

$$\frac{dV_1}{dt} = -\frac{\alpha\sigma}{\kappa}T^2 < 0$$

Hence this system is asymptotically stable.

Of course the question you probably want to ask [illegible] just how one goes about finding a Lyapunov function for a given system. Unfortunately, [illegible] be done by [illegible]. There are [illegible]. This is one of the principal disadvantages of [illegible] stability criterion. Needless to say, such methods are not yet sufficiently well developed to be of a truly practical significance.

4.5. LARGE POWER EXCURSIONS

We now turn our attention to a study of large reactor transients which are of interest in the following situations:

i.) pulsed reactors (Godiva, TRIGA)
ii.) intentional large insertions of reactivity (SPERT)
iii.) analysis of postulated reactor accidents

In all of these situations one is usually concerned with a fast insertion of a large reactivity in excess of prompt critical such that delayed neutrons are unimportant, and the excursion is terminated by temperature feedback or mechanical mechanisms. [In this regard, one must be very careful since the point reactor model itself is of questionable validity for such transients.]

4.5.1. The Fuchs-Hansen Model

We will consider a step injection of reactivity $\rho > \beta$. Since the reactor is super-prompt critical, we can then neglect delayed neutrons to write:

$$\frac{dP}{dt} = \frac{\rho(t)-\beta}{\Lambda} P(t)$$

We will assume the temperature feedback reactivity is proportional to the energy generated such that

$$\rho(t) = \rho_0 - \gamma E(t) = \rho_0 - \gamma \int_0^t dt' P(t')$$

Then

$$\frac{dP}{dt} = P(t)\left[\alpha_0 - b\int_0^t dt' P(t')\right]$$

where $\alpha_0 = \frac{\rho_0-\beta}{\Lambda}$, $b = \frac{\gamma}{\Lambda}$

Amazingly enough, we can solve this equation exactly to find

$$P(t) = \frac{2c^2Ae^{-ct}}{b[Ae^{-ct}+1]^2}$$

and

$$E(t) = \frac{\alpha_0 + c}{b}\left[\frac{1 - e^{-ct}}{Ae^{-ct} + 1}\right]$$

where $c = \sqrt{\alpha_0^2 + 2bP_0}$, $A = \frac{c + \alpha_0}{c - \alpha_0}$

To interpret this more easily, assume the initial power level is low -- i.e.

$$c \sim \alpha_0 \Rightarrow A \sim \frac{2\alpha_0^2}{bP_0} >> 1$$

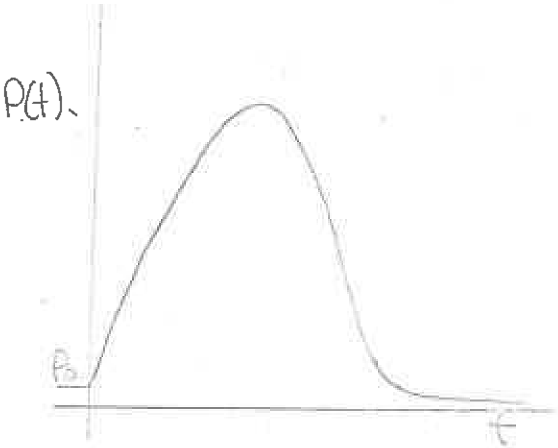
Then at early times

$$E(t) \sim P(t) \sim e^{\alpha_0 t}$$

The power eventually reaches a maximum when

$$\frac{dP}{dt} = \frac{2c^3 A}{b} \frac{e^{-ct}[Ae^{-ct} - 1]}{[Ae^{-ct} + 1]^3} = 0$$

$$\Rightarrow A e^{-ct} = 1$$



Hence

$$t_{P_{max}} = \frac{\ln A}{c} \sim \frac{\ln A}{\alpha_0}$$

Then

$$P_{max} \sim \frac{\alpha_0^2}{2b} = \frac{(\rho_0 - \beta)^2}{2\Lambda\gamma}$$

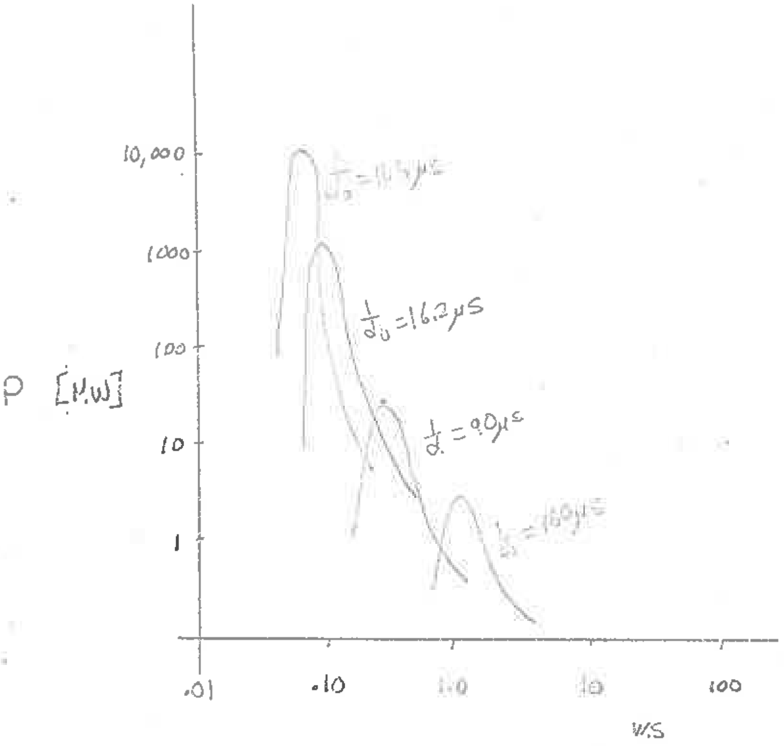
For long times, $P(t) \sim e^{-\alpha_0 t}$ which suggests a symmetric pulse shape. Actually the power tails off more slowly because of delayed neutrons (can calculate these separately).

The total energy generated is

$$E(t) \xrightarrow[t \to \infty]{} \frac{2\alpha_0}{b} = \frac{2(\rho_0 - \beta)}{\gamma}$$

Note that the essential parameters are the excess reactivity $\rho_0 - \beta$ and the feedback coefficient γ. But the peak power also depends on the prompt neutron lifetime Λ. This is important, since the peak pressure, etc., will depend on P_{max}. Since $\Lambda \sim 10^{-4}$ sec in thermal reactors while $\Lambda \sim 10^{-8}$ in fast systems, we can see that the potential for a damaging power excursion is far more serious in the latter. This analysis suggests that the reactor should have a large negative feedback coefficient to minimize this effect.

As an example, we have sketched the excursions of a Godiva system.



4.5.2. Analysis of Fast Reactor Accidents

As we have seen, peak powers and pressures accompanying a reactivity excursion in a fast reactor will be very much greater than in a thermal reactor. Further there is much more fissile material in a fast reactor core. Hence the MCA for such a system involves a complete loss of coolant. The reactor will be shutdown, but [illegible] fission product decay heat leads to core meltdown and reassembly in a prompt critical mass. The parameters necessary for the analysis of such an accident include the gross core geometry, initial power level, and the rate of reactivity increase during the supercritical phase.

We will essentially outline the so-called Bethe-Tait analysis of such an accident. Assume that the [illegible] are such as to achieve prompt criticality at $t=0$. Since delayed neutrons are insignificant, we write

$$\frac{dP}{dt} = \frac{\rho-\beta}{\Lambda} P(t)$$

Now suppose the reactivity is increasing in a linear manner with time

$$\rho(t) = \beta + \dot{\rho}_0 t$$

where $\dot{\rho}_0$ must be estimated from the model of core collapse.

Then

$$P(t) = P(0)\, e^{\dot{\rho}_0 t^2/2\Lambda}$$

Now suppose the feedback is negligible until the total energy generated is E_1. If this energy is generated at time t_1, then

$$\rho(t_1) = \rho + \dot{\rho}_0 t_1 \Rightarrow \rho(t_1) - \beta = \dot{\rho}_0 t_1$$

where t_1 is defined by

$$E_1 = \int_0^{t_1} P(t)\,dt = P(0)\int_0^{t_1} dt\, e^{\dot{\rho}_0 t^2/2\Lambda}$$

If we approximate

$$E_1 \sim \frac{P(0)\Lambda}{\dot{\rho}_0 t_1} e^{\dot{\rho}_0 t_1^2/2\Lambda}$$

we find

$$t_1^2 \sim \frac{\Lambda}{\dot{\rho}_0}\left[\ln\left(\frac{E_1^2\dot{\rho}_0}{P^2(0)\Lambda}\right)\right]$$

which yields

$$\rho(t_1) - \beta = \sqrt{\Lambda\dot{\rho}_0 \ln\left(\frac{E_1^2\dot{\rho}_0}{P^2(0)\Lambda}\right)}$$

This then is the excess reactivity over prompt critical which is reached when feedback becomes significant. In some cases, $\rho(t_1)-\beta$ may range up to β.

We now must introduce feedback. The Doppler coefficient is the only mechanism which can be relied on (one major reason for its significance in fast reactor design).

In the Bethe-Tait analysis, it is postulated that there is no feedback mechanism until an energy $E = E^*$ at which point the core material vaporizes. The vaporizing core builds up a pressure which tends to expand the core -- hence decreasing reactivity and terminating the excursion (the reactor blows up). To analyze this one must really use a coupled neutronics-hydrodynamics calculation. Typical results are

$$\left(\frac{E}{E^*} - 1\right) \sim \left[\frac{(\Delta\rho)^3 R^2}{\Lambda^2}\right]^{2/9}$$

But
$$\frac{(\Delta\rho)^3}{\Lambda^2} \sim \frac{\dot{\rho}_0^{3/2}}{\Lambda}$$

This $\dot{\rho}_0$, the reassembly reactivity rate, is the single most important factor in determining how serious a fast reactor accident might be.

V. SPACE DEPENDENT REACTOR KINETICS

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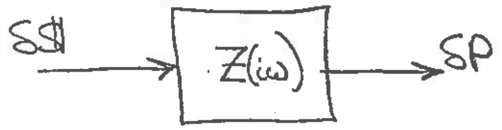
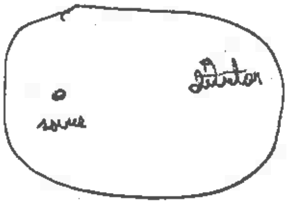
5.1. INTRODUCTION

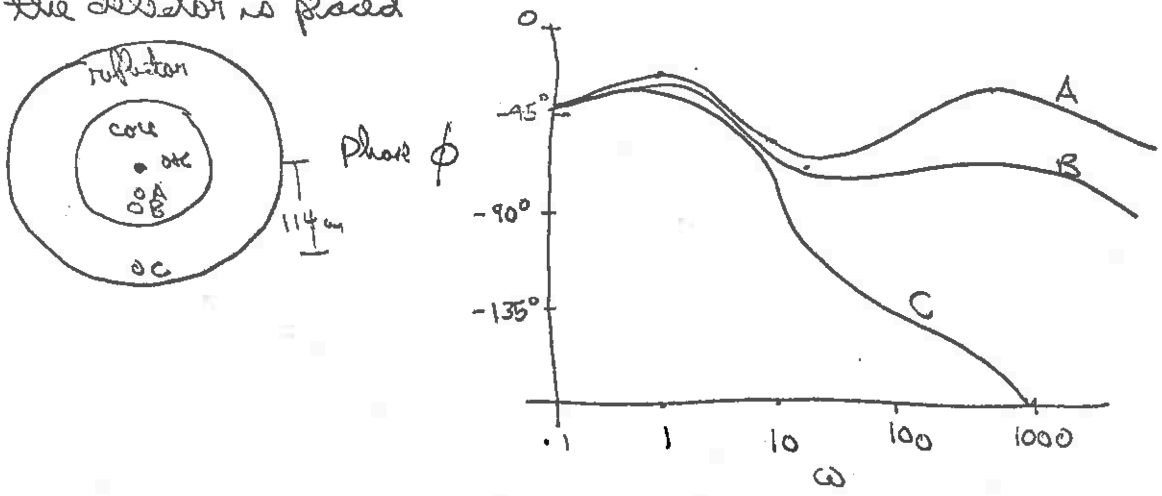
Suppose we return for a moment to reconsider the measurement of the transfer function of a reactor [at zero power, for convenience]



Recall this is accomplished by applying sinusoidal reactivity variation and measuring the amplitude and phase of resulting power oscillations for various reactivity frequencies ω. Let's look a little more closely at this measurement. First, the reactivity perturbation will be localized at the point of the oscillating absorber rod. Furthermore, the neutron detector itself is localized at a point in the reactor. Hence the measurement actually measures a transfer function between two points in the reactor.



Suppose we perform the measurement at various different points in the reactor. If the reactor is truly described by the P.R.K.E., then these measurements should give the same results. But, in fact, actual measurements yield different results, depending upon where the detector is placed



These measurements reveal the spatial dependence of the transfer function -- and, in fact, the breakdown of the P.R.K.E. Recall that in our consideration of these equations, we have been assuming that the shape function $\Psi(\underline{r},\underline{v},t)$ in

$$n(\underline{r},\underline{v},t) = P(t)\,\Psi(\underline{r},\underline{v},t)$$

was time-independent and equal to the distribution $n_0(\underline{r},\underline{v})$ in the critical reference reactor.

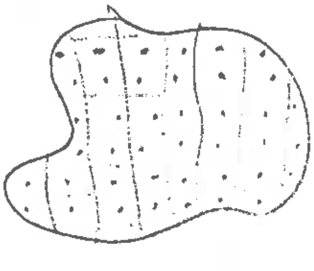
However this is clearly a very severe approximation, as measurements such as those above emphasize. Let us turn now to a discussion of those situations in which spatial effects in reactor kinetics are of importance, and in particular, describe analytical methods used to analyze such problems in "space dependent reactor kinetics".

4.2. CALCULATIONAL METHODS

The workhorse of modern reactor analysis are the multigroup diffusion equations. A "<u>brute force</u>" calculation of spatially-dependent reactor kinetics would involve a direct numerical solution of these equations. Such calculations are prohibitively expensive for any realistic reactor configuration. Hence we are forced to turn to either of two alternative approaches: a "<u>nodal</u>" analysis or a "<u>modal</u>" approach.

4.2.1. Nodal Methods:

In the nodal scheme, one divides the reactor into a number of regions or nodes. Each node constitutes a space point in the problem, and the parameters that couple the flux at various nodes must be specified



4.2.2. Modal Expansion Methods

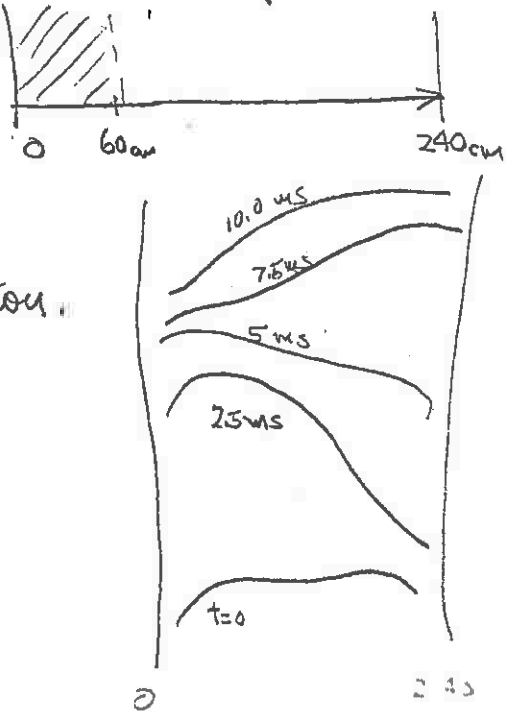
An alternative approach is to expand the flux in the normal modes of the system

$$\phi(\underline{r},t) = \sum_j T_j(t)\,\psi_j(\underline{r})$$

and then consider the resultant set of equations for the time-dependent coefficients $T_j(t)$. The problem with this approach is that one usually does not know these modes $\psi_j(\underline{r})$ -- and, in fact, frequently does not know much about them. Various schemes that bypass these difficulties have been suggested [such as "synthesis methods"], but we shall defer their discussion in order to turn to a more qualitative discussion of various phenomena in space time kinetics.

4.3. EXTREME FLUX TILTING IN LARGE THERMAL REACTORS

As a illustrative example of spatial effects in reactor kinetics, Bell & Glasstone‡ have considered spatial transients by changing the reactivity in one section of a large, critical thermal 2-group reactor. The subsequent time behavior was then analyzed using various approximate methods [point reactor kinetics, adiabatic and quasistatic methods] and comparing these results with an "exact" two-group kinetic calculation. [Reactivity was varied by changing ν in this first section -- by 9.5% [> prompt critical] The "exact" calculations are shown revealing a strong flux tilting



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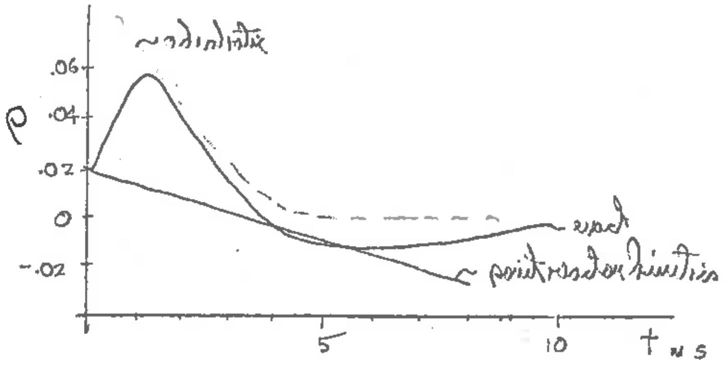
‡ Chapter 8, pp. 103-11

As one method of comparison the reactivity ρ was calculated for each of the various models.



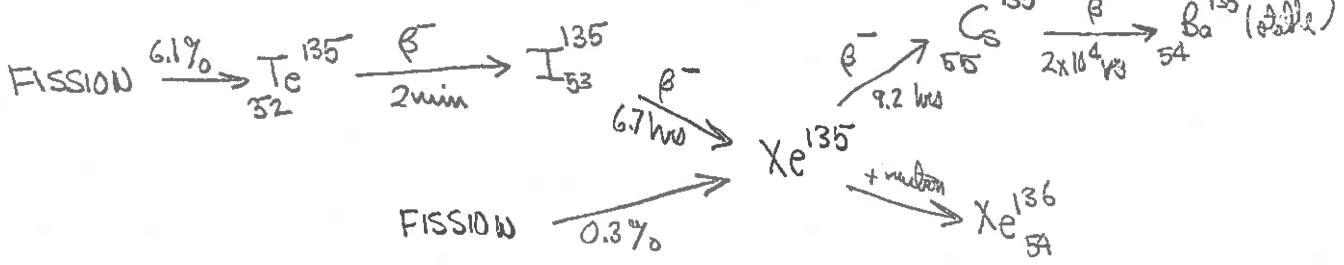
Conventional P.R.K. underestimated the peak flux by a factor of ~ 10^4. Even the adiabatic approximation failed to yield decent agreement. By way of comparison, modal synthesis (3 modes) yielded very good agreement.

These calculations indicate that when applied to transients involving marked changes in the shape function, the P.R.K.E. may be grossly misleading. The adiabatic approximation tends to overcorrect. The modal synthesis or quasistatic approximations are the most accurate

4.4. XENON-INDUCED POWER OSCILLATIONS

The significance of fission product poisoning [such as Xe^{135} with $\sigma_a \sim 3\times10^6$ b] in thermal reactor operation is well known. Buildup of such fission products can greatly decrease the excess reactivity of such reactors. However Xe^{135} buildup can actually lead to rather severe spatial oscillations in large thermal reactors, and perhaps even instability

The xenon decay chain is as follows



We will approximate this chain by

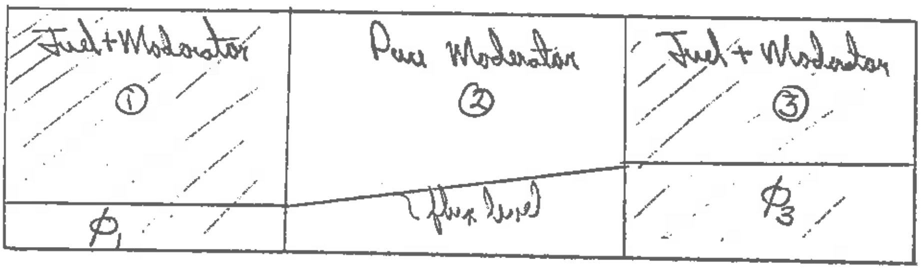
$$\begin{tikzcd}
 & & & C_s^{135} \\
\text{FISSION} \arrow[r] & I^{135} \arrow[r, "6.7\text{ hrs}"] & Xe^{135} \arrow[ur, "9.2\text{ hrs}"] \arrow[dr, "\text{neutron}\rightarrow"'] & \\
 & \text{FISSION} \arrow[ur] & & Xe^{136}
\end{tikzcd}$$

Point Reactor Model: To examine the effect of this decay chain, first consider a point reactor model in which the reactor has been operating at a steady flux level for some time. This has resulted in a buildup of a steady-state Xe^{136} concentration which results from the balance of Xe^{135} production (via direct fission and I^{135} decay) and loss due to decay to Cs^{135} and transmutation (via neutron absorption) to Xe^{126}. Now suppose a perturbation increase in the flux occurs. Then Xe^{135} will transmute more rapidly to Xe^{136} (instantaneously) depleting the Xe^{135} concentration, hence decreasing the absorption and increasing the flux. But the increased flux transmutes even more Xe^{135}, and hence the initial flux perturbation grows with time (unstable).

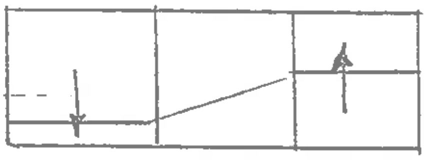
Actually, such an instability can only exist for power levels higher than a certain threshold value. For U^{235} fueled reactors this threshold is $\phi \sim 3 \times 10^{11}$ neutrons/cm^2-sec. Below this threshold the stabilizing effect of the direct xenon yield from fission is more important than the destabilizing effect of the xenon decaying from I^{135}

Actually this type of instability is relatively unimportant in practical reactor operations, since it is easily controlled by normal control rod movement. A much more serious spatial xenon instability can arise, however, which requires a more complex control rod program.

Spacial Xenon Oscillations: An instructive example of a space dependent xenon buildup and decay instability can be given by considering the following very simple model -- two coupled xenon unstable point reactors, separated by a moderating material.



Suppose there is a control system keeping the total power of all three regions a constant (although the flux or power in an individual region is not constant).

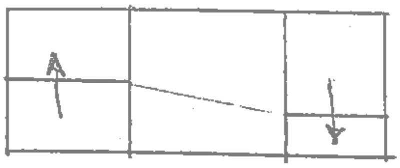


A slight increase in the power level on one side gives rise to the unstable xenon process described for a point reactor. Since the control systems keeps the total power constant, the flux on the other side decreases. This process continues with a steeper and steeper tilt in the flux occurs. Two effects will limit this tilt

i.) burnup of most of the xenon on the high flux side

ii.) the steep flux tilt creates a flux gradient and hence a current which carries all the excess neutrons being produced to the other side.

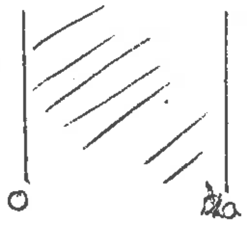
The flux will remain tilted for several hours. Eventually the high flux side will have created an I^{135} concentration much greater than that originally present. Since the decay constant of I^{135} is 6.7 hours, more xenon will be created after this delay period. Similarly on the low side less xenon will be created. This reverses the flux tilt eventually and produces a side to side oscillation with a period of from 15 to 30 hours. Thus the xenon process tends to be self limiting and produces the effect of a moving "hot spot" to the reactor operator.



The control of such oscillations is of paramount importance, since the flux in the hot spot may be quite large, leading to fuel element damage.

Suppose we consider a more detailed model of this phenomenon using one-speed diffusion equation. We will treat the delayed neutrons as if they appeared promptly, as assume reactivity feedback is represented by $f\phi(x,t)$, where f is the power coefficient of reactivity. Then the reactor kinetics equations become:

$$\frac{1}{v}\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2} + (k_\infty - 1 + f\phi)\Sigma_a\phi(x,t) - \sigma_X X(x,t)\phi(x,t)$$



We will consider the geometry to be that of a well-reflected slab reactor such that the steady-state flux is spatially uniform

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = \left.\frac{\partial\phi}{\partial x}\right|_{x=a} = 0$$

The equations for iodine and xenon concentration are

$$\frac{\partial I}{\partial t} = \overset{\text{prompt fission}}{\gamma_I}\Sigma_f\phi(x,t) - \overset{\text{decay const}}{\lambda_I} I(x,t)$$

$$\frac{\partial X}{\partial t} = \lambda_I I(x,t) - \lambda_X X(x,t) - \sigma_X X(x,t)\phi(x,t)$$

Now if the distributions in the steady state system ϕ_0, I_0, X_0 are spatially uniform, we find these equations become

$$(k_\infty - 1 + f\phi_0)\Sigma_a\phi_0 - \sigma_X X_0\phi_0 = 0$$

$$I_0 = \frac{\gamma_I\Sigma_f\phi_0}{\lambda_I}$$

$$X_0 = \frac{\lambda_I I_0}{\lambda_X + \sigma_X\phi_0} = \frac{\gamma_I\Sigma_f\phi_0}{\lambda_X + \sigma_X\phi_0}$$

Here k_∞ is the multiplication factor the reactor would have in the absence of Xe^{135} and power feedback. We can solve for

$$\phi_0 = \frac{k_\infty - 1}{\dfrac{\sigma_X\gamma_I\Sigma_f}{\Sigma_a(\lambda_X + \sigma_X\phi_0)} - f}$$

which has a solution if $k_\infty > 1$ and $f < 0$

We shall now linearize our system of equations by assuming the perturbations φ, $\mathcal{I}$, χ about these steady state values are small. Then if we substitute

$$\phi(x,t) = \phi_0 + \varphi(x,t)\ , \quad I(x,t) = I_0 + \mathcal{I}(x,t)\ , \quad X(x,t) = X_0 + \chi(x,t)$$

into these equations and neglect second order terms we find

$$\frac{1}{v}\frac{\partial \varphi}{\partial t} = D\frac{\partial^2 \varphi}{\partial x^2} + (k_\infty - 1 + 2f\phi_0)\Sigma_a \varphi(x,t) - \sigma_X\left[X_0\varphi(x,t) + \phi_0\chi(x,t)\right]$$

$$\frac{\partial \mathcal{I}}{\partial t} = \gamma_I \Sigma_f \varphi(x,t) - \lambda_I \mathcal{I}(x,t) \qquad (*$$

$$\frac{\partial \chi}{\partial t} = \lambda_I \mathcal{I}(x,t) - \lambda_X \chi(x,t) - \sigma_X\left[X_0\varphi(x,t) + \phi_0\chi(x,t)\right]$$

To solve this set of coupled P.D.E.'s, we will use Laplace transforms in time coupled with a modal expansion in space. In this case the appropriate "modes" are $\sin n\pi x/2a$, $n = \cancel{0}, 1, \ldots \infty$ such that we expand

$$\varphi(x,t) = \sum_{n=\cancel{0}}^{\infty} A_n(t) \overset{\sin}{\cancel{\cos}} \frac{n\pi x}{2a}$$

$$\mathcal{I}(x,t) = \sum_{n=\cancel{0}}^{\infty} I_n(t) \overset{\sin}{\cancel{\cos}} \frac{n\pi x}{2a}$$

$$\chi(x,t) = \sum_{n=\cancel{0}}^{\infty} X_n(t) \overset{\sin}{\cancel{\cos}} \frac{n\pi x}{2a}$$

In the usual manner, we can substitute these expansions into the linearized set (*), multiply by $\sin \frac{n\pi x}{2a}$, integrate over x and use orthogonality to find a set of equations for the coefficients A_n, I_n, X_n. If we also Laplace transform in time, this set becomes

$$\frac{s\tilde{A}_n(s)}{v} = -D\left(\frac{n\pi}{2a}\right)^2 A_n + (k_\infty - 1 + 2f\phi_0)\Sigma_a \tilde{A}_n - \sigma_X\left[X_0\tilde{A}_n + \phi_0\tilde{X}_n\right]$$

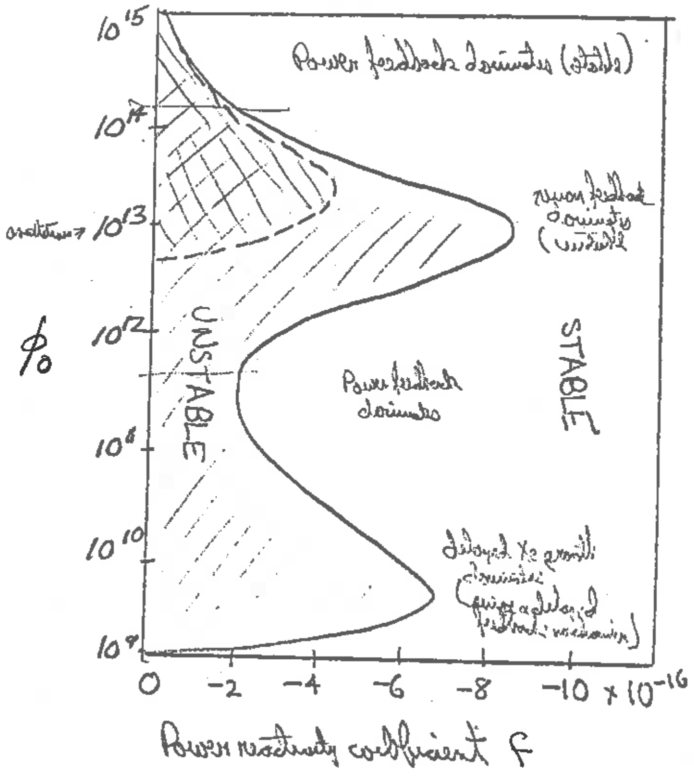
$$s\tilde{I}_n(s) = \gamma_I \Sigma_f \tilde{A}_n - \lambda_I \tilde{I}_n$$

$$s\tilde{X}_n(s) = \lambda_I \tilde{I}_n - \lambda_X \tilde{X}_n - \sigma_X\left[X_0\tilde{A}_n + \phi_0\tilde{X}_n\right]$$

After a bit of algebra, one can then solve this set for $\tilde{A}_n(s)$. In particular, the poles of $A_n(s)$ are determined by the roots of the characteristic equation

$$\frac{s}{v} = -D\left(\frac{n\pi}{2a}\right)^2 + f\Sigma_a\phi_0 - \frac{\sigma_X\gamma_I\lambda_I\Sigma_f\phi_0}{(s+\lambda_I)(s+\lambda_X+\sigma_X\phi_0)} + \frac{\sigma_X^2\gamma_I\sigma_f\phi_0^2}{(\lambda_X+\sigma_X\phi_0)(s+\lambda_X+\sigma_X\phi_0)}$$

There are three roots to this equation. The threshold of instability occurs when one of the roots becomes imaginary. Suppose we fix D, 2a, Σ_f, Σ_a, ... and vary f and ϕ_0. Then one can actually sketch an instability parameter region. We have indicated the stable region for the fundamental mode $A_0(t)$. [The dashed region is for the first harmonic $A_1(t)$. Note that it is harder to excite the higher harmonic oscillations.]

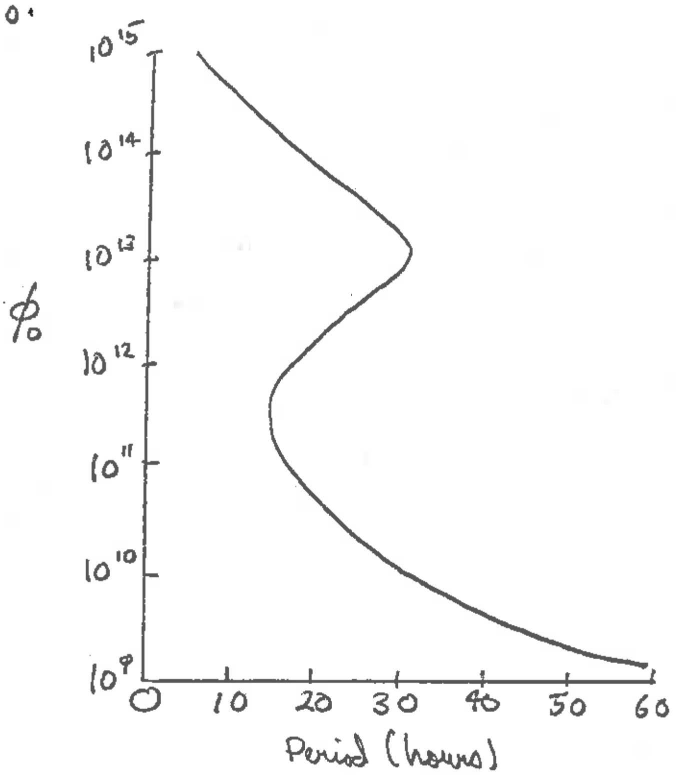


These features hold in general. When the flux ϕ_0 is less than 10^9 /cm² sec, the system is stable against Xe^{135} oscillations -- regardless of f. [Burnup rate of Xe^{135} is small.]

At higher ϕ_0, the fundamental mode becomes unstable due to delayed growth of Xe^{135}. For still higher flux, the power feedback will dominate and stabilize -- but about 10^{-13} the xenon instability takes over again [until Φ_0 exceeds 10^{16}].

One must watch out for the higher harmonics, since sometimes normal control motions might tend to aggravate things.

One can also plot the period of these oscillations vs. the power level ϕ_0.



This model, as crude as it is, contains most of the physics of these oscillations, and has proved useful in setting up effective automatic control systems.

Mention stability vs. control philosophy.

VI. NOISE ANALYSIS IN NUCLEAR REACTORS ‡

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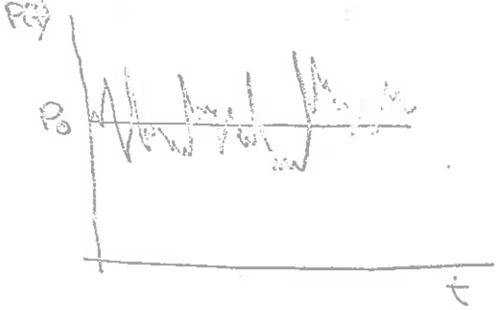
6.1. INTRODUCTION

Thus far we have been treating the reactor as essentially a deterministic system. Actually, the reactor operates on a statistical basis. That is, all of the dynamic variables describing the reactor (power level, flux, temperature, etc.) actually fluctuate in a random fashion about some mean value. We are unable to predict with certainty the future values of these variables but rather can only specify the probability that they will assume a certain value.



The statistical nature of neutron diffusion is well known to you. Moreover the concept of a cross section -- and indeed even the quantum mechanical description of the neutron interactions must be interpreted in a probabilistic sense. There are numerous other sources of such statistical fluctuations (or "noise") in a reactor, such as fluid flow, boiling, mechanical vibrations, etc. In fact, statistics enters even into the measurement of the reactor state via detector noise.

At low power levels, the statistical fluctuations associated with the fundamental nuclear processes will be dominant. However, at higher power levels, the reactor noise will be predominantly due to disturbances of a non-nuclear nature.

Regardless of its origin, reactor noise is important in reactor

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‡ Ref: J. A. Thie, "Reactor Noise", (ANS Monograph, 1963)
A.Z. Akcasu, N.E. 551 course notes, 1969

dynamics for two reasons:

i.) It often interferes with the precision to which a desired quantity can be measured in a reactor. That is, one must extract the signal of interest out of the background noise.

ii.) However since the noise arises from various processes occurring in the reactor, it can actually serve as a source of information about the system.

The latter of these properties is the one of most interest to us here. We shall study how noise analysis can be used to measure the transfer function of the reactor.

6.2. CROSS-CORRELATION METHODS

Suppose we wish to measure the reactivity to power transfer function of a reactor

$$\delta\tilde{\$}_{ext} \longrightarrow \boxed{\tilde{Z}(s)} \longrightarrow \delta\tilde{P}$$

We can write the variation in the power $\delta P(t)$ in terms of the impulse response $z(t) = \mathcal{L}^{-1}\{\tilde{Z}(s)\}$ and the reactivity insertion $\delta\$_{ex}(t)$ as a convolution (here we are describing the reactor for all $-\infty < t' < t$)

$$\delta P(t) = \int_{-\infty}^{t} d\tau\, \delta\$_{ex}(\tau)\, z(t-\tau) = \int_{0}^{\infty} d\tau\, \delta\$_{ex}(t-\tau)\, z(\tau)$$

This will form the basis of the cross correlation measurement

Now for a couple of definitions: We will define the autocorrelation of a function x(t) by

$$\varphi_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, x(t+\tau)\, d\tau \qquad (2$$

and the cross-correlation of two functions x(t) and y(t) by

$$\varphi_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\, y(t+\tau)\, d\tau \qquad (3)$$

[If the functions are periodic, then the limit process can be omitted provided T is chosen as the period. In general, however, x(t) and y(t) will not be periodic (in fact we shall later consider them to be "random variables" in the sense that only their probability distributions can be specified).]

In particular, we can set up the cross correlation between reactivity and power [where the limit process T→∞ will be understood]

$$\varphi_{kP}(\tau) = \frac{1}{2T}\int_{-T}^{T} \delta k_{ext}(t)\, \delta P(t+\tau)\, dt = \frac{1}{2T}\int_{-T}^{T} \delta k_{ext}(t-\tau)\, \delta P(t)\, dt$$

Now suppose we substitute our expression for δP(t) from (1) into (4)

$$\varphi_{kP}(\tau) = \frac{1}{2T}\int_{-T}^{T} \delta k_{ext}(t-\tau)\left[\int_0^\infty du\, \delta k_{ext}(t-u)\, Z(u)\right] dt$$

$$= \int_0^\infty Z(u)\left[\frac{1}{2T}\int_{-T}^{T} \delta k_{ext}(t-\tau)\, \delta k_{ext}(t-u)\, dt\right] du$$

$$= \int_0^\infty Z(u)\, \varphi_{kk}(\tau-u)\, du$$

where we have identified the autocorrelation function of reactivity as

$$\phi_{\$\$}(\tau) = \frac{1}{2T}\int_{-T}^{T} \delta\$_{ext}(t)\,\delta\$_{ext}(t+\tau)\,dt \qquad ($$

Now finally take the Fourier transform of (5) as defined by

$$\tilde{\phi}_{\$P}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\phi_{\$P}(\tau) \equiv \mathcal{F}\{\phi_{\$P}\}$$

to find

$$\tilde{\phi}_{\$P}(\omega) = \mathcal{F}\{z(\omega)\}\,\tilde{\phi}_{\$\$}(\omega)$$

But $\mathcal{F}\{z(\omega)\}$ is just the transfer function $Z(i\omega)$. Hence we find

$$Z(i\omega) = \frac{\mathcal{F}\{\phi_{\$P}\}}{\mathcal{F}\{\phi_{\$\$}\}} \qquad (7$$

Here, $\tilde{\phi}_{\$P} = \mathcal{F}\{\phi_{\$P}\}$ is referred to as the "cross spectral density" while $\tilde{\phi}_{\$\$} = \mathcal{F}\{\phi_{\$\$}\}$ is referred to as the "reactivity (or input) spectral density".

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How is this expression used in practice? One varies the reactivity of the reactor in a random manner (much as one would vary the reactivity in an oscillator measurement) and then measures the corresponding variations in the power (or flux). To obtain $\phi_{\$\$}$ at a time t, one measures $\delta\$_{ext}(t)$ at the time t and at $t+\tau$ for a series of delay intervals τ increasing in discrete steps of $\Delta\tau$ about 0.01 sec. The cross correlation $\phi_{\$P}(t)$ is obtained in a similar manner from measurements of $\delta\$_{ext}(t)$ and $\delta P(t+\tau)$. The integrals in the definitions (4) and (6) are then evaluated numerically over the period of observation (usually about [illegible].)

One then takes the Fourier transforms of $\phi_{\#p}$ and $\phi_{\#\#}$ numerically -- i.e.

$$\mathcal{F}\{\phi_{\#p}(\tau)\} \cong \sum_n \phi_{\#p}(n\Delta\tau)\,[\cos(n\omega\Delta\tau) + i\sin(n\omega\Delta\tau)]\,\Delta\tau$$

and then computes

$$Z(i\omega) = \frac{\mathcal{F}\{\phi_{\#p}\}}{\mathcal{F}\{\phi_{\#\#}\}}$$

Note that this measurement yields both the amplitude and phase of $Z(i\omega)$. [Also note it is highly dependent on the availability of a high speed computer.]

* * *

This experiment becomes particularly simple if one can use a "white" noise source as input. [A white noise source is one whose spectral density is constant in frequency.] -- i.e.

$$\mathcal{F}\{\phi_{\#\#}\} = \text{constant} = A$$

Then

$$\mathcal{F}\{\phi_{\#p}(\tau)\} = A\,Z(i\omega) \tag{8}$$

-- hence by merely measuring the cross correlation one can determine both the amplitude and phase of the transfer function.

* * *

This experiment serves, then, as an alternative to a reactor oscillator transfer function measurement. Unlike the latter, it does not suffer from background noise (rather taking
(1) advantage of such random fluctuations) -- and hence does not require nearly so large an input signal.

However both of these experiments suffer from the fact that one must perturb the reactor by introducing an externally controlled reactivity signal in order to perform the measurement. It is possible to bypass this difficulty and measure the amplitude of $Z(i\omega)$ directly from the inherent noise naturally present in the reactor.

6.3. AUTOCORRELATION MEASUREMENTS

Consider the autocorrelation of the fluctuations in the reactor power

$$\varphi_{PP}(\tau) = \frac{1}{2T}\int_{-T}^{T} \delta P(t)\, \delta P(t+\tau)\, dt \tag{10}$$

Let's now use the convolution expression (1) in (10) to find

$$\varphi_{PP}(\tau) = \frac{1}{2T}\int_{-T}^{T} dt \left[\int_0^\infty z(u)\, \delta\$_{in}(t-u)\, du\right]\left[\int_0^\infty z(v)\, \delta\$_{in}(t+\tau-v)\, dv\right]$$

$$= \int_0^\infty z(u)\, du \int_0^\infty z(v)\, dv \underbrace{\left[\frac{1}{2T}\int_{-T}^{T} \delta\$_{in}(t-\tau)\, \delta\$_{in}(t+\tau-v)\, dt\right]}_{\varphi_{\$\$}(\tau+u-v)}$$

Fourier transformation then gives

$$\mathcal{F}\{\varphi_{PP}(\tau)\} = Z(-i\omega)\, Z(i\omega)\, \mathcal{F}\{\varphi_{\$\$}(\tau)\}$$

or $$|Z(i\omega)|^2 = \frac{\mathcal{F}\{\varphi_{PP}(\tau)\}}{\mathcal{F}\{\varphi_{\rho\rho}(\tau)\}}$$

Hence the square of the magnitude of the transfer function is thus equal to the ratio of the power and reactivity spectral densities.

Of course the autocorrelation function of the power fluctuations can easily be measured as before. The fluctuations in the reactivity being of internal origin, cannot be measured. If these fluctuations are random ("white noise"), then again

$$\mathcal{F}\{\varphi_{\rho\rho}(\tau)\} = \text{constant} = A$$

and the measurement of the power autocorrelation function by itself will yield the amplitude of the transfer function. Note, however, that all phase information has been lost.

This measurement of $|Z(i\omega)|$ is extraordinarily simple and cheap (requiring only a detector, a recorder, and a computer). In fact it is so cheap that panoramic frequency instrumentation can be used to obtain a visual real-time display of the transfer function which can be monitored along with other instruments on the operators console. Furthermore it does not disturb the reactor (which is of great importance in power reactors). It suffers in accuracy however.

Noise measurements have been used to determine a number of other quantities characterizing reactor kinetics. However for these details I will refer you to Thie.

VII. THE KINETICS OF SPECIFIC REACTOR TYPES

4/13/70

7.1. WATER MODERATED REACTORS

Ref: J.A. Thie, Chpt. 8, in "The Technology of Nuclear Reactor Safety", edited by Thompson & Beckerley.

Here we are referring to thermal reactors which are moderated and cooled by either H_2O or D_2O. In particular, we will direct most of our attention to PWR and BWR reactor types. As we will see, the <u>primary differences</u> between such reactors and those with solid moderators occurs due to heating, evaporation, and motion of the coolant.

7.1.1. Reactivity Control

1.) Control rods: principal control mechanism. Must be careful in worth calculations [usually from 10 to 25% worth] [shadowing]

2.) Liquid poisons: boric acid or cadmium sulfate [usually as a backup shutdown device] -- be careful -- decreases negative void coefficient.

* 3.) Moderator and [illegible] control -- in PWR, changes in moderator density (temperature feedback) [35° F → \$]. Things a little trickier with BWR's. But void volume can be changed by pump rates

7.1.2. Dynamics

The dynamics analysis of a power plant is [illegible]lly just that of thermal and hydraulic analysis -- with the a[illegible] in the kinetics equations. Of prime concern are various reactivity effects:

Principal Causes of Reactivity Effects in Short-Term (minutes or less) Transients in Water Reactors

Fuel Effects: - Doppler

Fuel motion (including bowing) [hazard during fuel loading]

Core expansion (level changes) [power or volume changes]

Fuel phase changes (melting in dry reactors)

Moderator effects:‡ - moderator temperature (non-1/v absorption; larger neutron T)

± moderator volume (including steam voids) [‡

moderator isotopic constitution (including spectral shift)

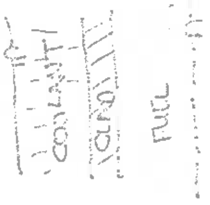
radiolytic gas

gas microbubbles

Other effects: poison changes

neutron shifting [moderator forced out by carrying neutrons or precursors with it]

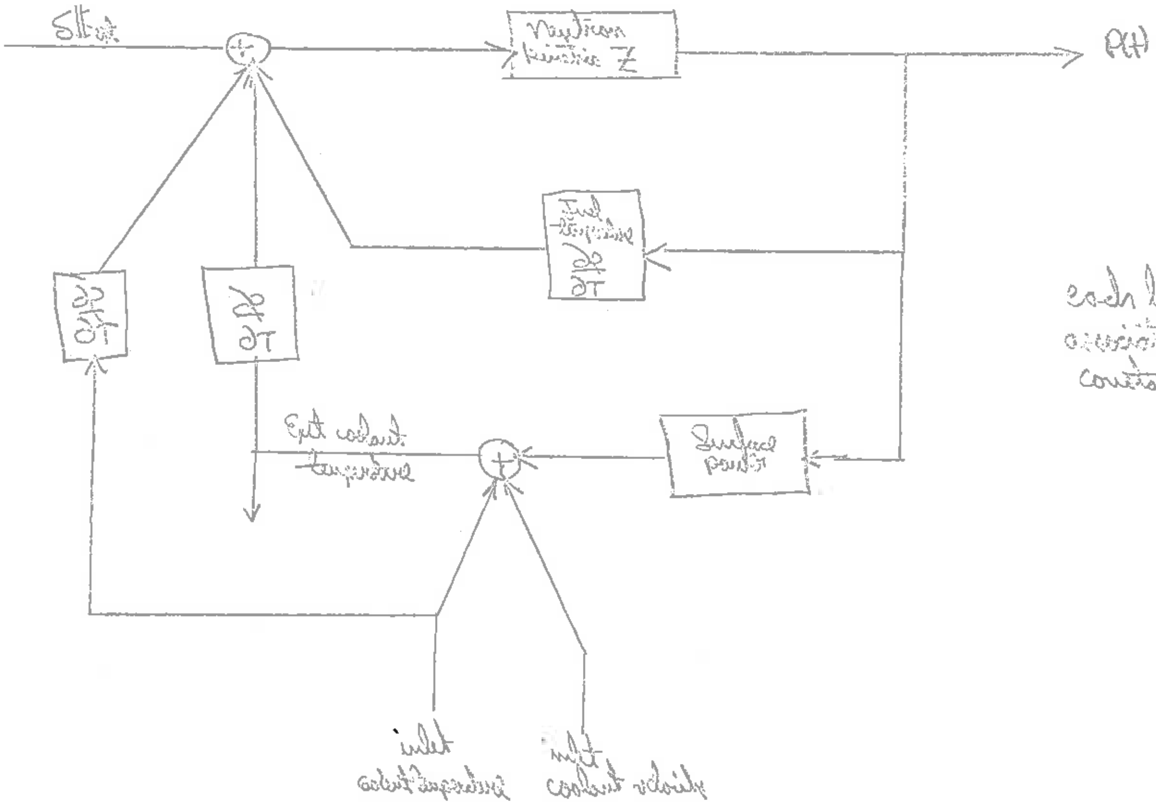
The next step is to calculate the various feedback transfer functions. For example, one might study heat transfer in the reactor. Each physical process occurring in the reactor will have a characteristic time constant. Typical time constants



- i.) fuel rod
- ii.) cladding
- iii.) coolant transport in core
- iv.) coolant transport through heat exchanger

- i.) fuel rod
- ii.) cladding
- iii.) coolant transport in non-boiling zone
- iv.) steam bubble formation in boiling zone
- v.) recirculation
- vi.) subcooled water transport

Example. [PWR]



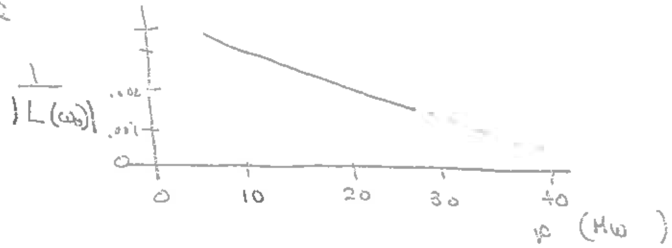
each loop has an associated time constant

The dynamics of a BWR is more complex [strong influence of system pressure on reactor], since one must worry about the dynamic interaction between the steam & liquid phases.

7.1.3. Measurement of Transfer Functions

Use a mixture of experiments & theory. For example, in BWR's theory predicted a resonance due to phase lag of steam void formation. Rod oscillator measurements confirmed this.

Can use gain & phase margins in theoretical studies. But can also just measure magnitude to measure peak



Can oscillate feedwater flow.

Also can use noise measurements (usually cross correlation). But flow or temperature fluctuations in feedwater serve as a noise source.

Pih

7.1.4. Power Excursions [periods < 1 sec]

Possible causes

1.) control rod withdrawal
2.) absorber (core) withdrawn
3.) fuel added too rapidly
4.) sudden rearrangement of core
5.) voids are removed [collapse of voids as pressure increases, sweeping out of voids or void redistribution)
6.) instability

Experiments: — Spert tests — Major shutdown feedback is due to water heating and fuel expansion effects. — Except for oxide fuels in which doppler effect is dominant. In thin metallic plate cores, boiling is dominant shutdown mechanism.

7.1.5. Spatially Dependent Dynamics

Necessary to include spatial effects for large cores, ① find xenon instabilities ② also local coolant temperature transients (boiling) ③ localized control rods [can actually excite xenon oscillations, e.g. in PWR's].

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7.2. KINETICS OF SOLID-MODERATOR REACTORS

Ref: H.B. Stewart & M.H. Merrill, in "Technology of Nuclear Reactor Safety", ed. by Thompson and Beckerley.

We recall that the kinetics of water-moderated reactors were strongly affected by the relatively large change in density with temperature. By way of contrast, for solid moderator reactors:

a) spectra predominantly thermal or intermediate

b) density changes have a minor effect

[moderated by graphite, ZH_2, Be, BeO and cooled by gas, liquid metal, or water]

Now T_m, T_f, and structural expansion become more important.

7.2.1. Control Mechanisms

A variety has been used: control rods, control cylinders, poison cruciform blades. Use burnable poisons to minimize reactivity swing due to fuel depletion.

7.2.2 Reactivity Effects

By way of comparison, the various overall temperature coefficients of reactivity are

$10^{-4}/°C$	water-moderated
$10^{-5}/°C$	solid-moderated
$10^{-6}/°C$	fast reactors

Dominant moderator contribution is due to changes in neutron spectrum -- not thermal expansion (requires a fairly careful thermalization calculation). [Spectrum hardens and changes thermal cross sections.] Need to include chemical binding.

Prompt neutron lifetimes range from 10^{-3} to 10^{-5} sec [for intermediate spectrum systems]. Delayed photoneutron problem.

Doppler effect can be quite important.

Find xenon flux oscillations in Calder Hall type reactors.

Energy storage in graphite [lattice displacement in irradiation]. Subsequent release leading to oxidation (fire) -- e.g. Windscale. Occurs in reactors with graphite temperature below 300°C [~570°F].

7.3. FAST REACTORS

7.3.1. Introduction

Ref: W.J. McCarthy & D. Okrent, in "Technology of Nuclear Reactor Safety", ed. by Thompson & Beckerley

① Fast reactors are characterized by the absence of moderator. [large mfp
② No large capture resonances (hence no special reactivity effects such as xenon or samarium poisoning). Also no strong absorber to use as a control.
③ Cross sections are small and mfp long, hence no steep flux gradients (as in a thermal reactor). ④ Spectral effects dominate diffusion effects.

Only direct temperature effect (other than density change) is Doppler shift. Typical reactivity mechanisms
i.) Doppler effect (+ or −)
ii.) sodium loss -- spectral hardening near center of core (+)
-- increased leakage near boundary (−)
iii.) sodium and fuel expansion (−)

For small reactors (large $k_\infty \sim 2$), large leakage, hence can use reflector control. For larger cores, one uses fuel movement or absorbers such as B^{10}.

7.3.2. Reactivity Mechanisms

7.3.2.1. Doppler Effect

Possible to have a positive Doppler coefficient in fast reactors (competition between nonproductive capture and fission resonances). As core size increases, fuel becomes more dilute, lower spectrum energy lowers leading to an increase in Doppler effect (+ or −). In large oxide reactors, breeder calculations a negative Doppler coefficient.

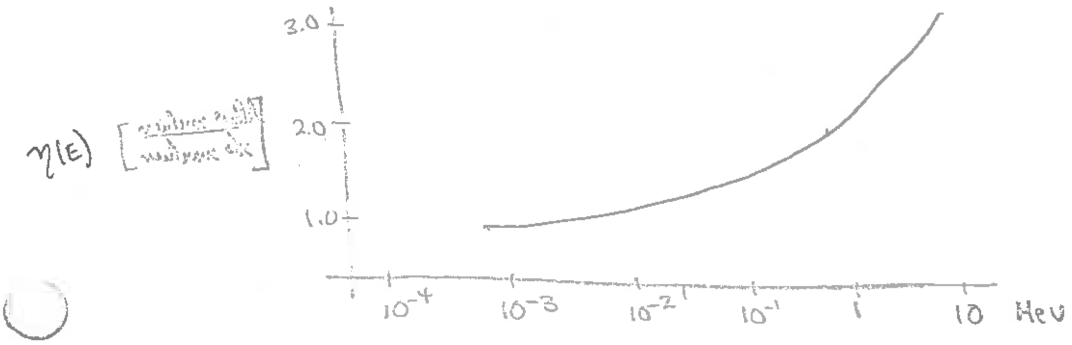
A reactor with $E_{av} \sim 100$ keV will have zero D.C. if equal amounts of U^{235} and U^{238} are in core. In Fermi, $U^{238}/U^{235} \sim$

For metal fuel systems, D.C. [illegible] than for oxide systems. Very sensitive to design, however. [SEFOR as a test]

7.3.2.2. Sodium Void Coefficient

Due to several effects

i.) neutron capture in sodium (small - effect)
ii.) leakage (−, depends on core size)
iii.) spectral shift (+, since as spectrum hardens, η(E)↑)



Lots of work on designing reactors with negative sodium void coefficients, e.g.

Very tricky calculation. (particularly for large reactors)

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7.3.2.3. Temperature and Power Coefficients (main feedback effects)

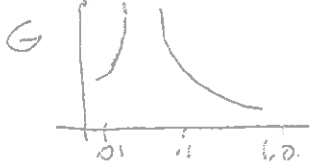
Primarily due to thermal expansion which changes the geometry of the core and density of material in it.

EXAMPLE (Fermi):

δP↑ → δφ↑ → T↑ → radial outward fuel exp → reduce reactivity

These are very difficult calculations to perform.

A problem with fuel rod bowing (EBR-I meltdown). Put in expansion of fuel.



7.3.2.4. Miscellaneous Effects

i.) entrained argon bubbles and bubble collapse

ii.) fuel and jacket growth under irradiation

iii.) fission product poisoning (incomplete cross section measurements -- but not much of problem ~ Δk/k ~ .01 - .03

iv.) fuel phase changes [~ 0.14% Δk/k]

7.3.3. Control Design

For small reactors, any of three standard methods seem OK --
i.) poison
ii.) moving fuel
iii.) reflector
For larger reactors, reflector control becomes impractical. In Fermi, $B^{10}C$ is used for both control and safety.

No problem with hot spots or rod perturbations because of long mfp's.

Control limited to 2-3% as opposed to 10-20% [OK since no Xe poisoning]

7.3.4. Reactor Dynamics

Value of Λ may range between 5×10^{-7} sec (ceramic) to 3×10^{-9} sec (bare plutonium sphere) [as compared to $10^{-3} - 10^{-5}$ for thermal reactors]. [Due to $\Lambda \sim 1/v\Sigma_a$ [illegible]

Values of β and γ depend on fuel type, not energy spectrum. However $\bar{\beta}$ depends on spectrum. [For Pu, $\beta \sim \frac{1}{3}$ that of U^{235}]

Main differences between thermal & fast reactor kinetics
i.) small Λ
(ii) core compaction due to meltdown can lead to large reactivity increase

Stability Studies -- spurred by EBR-I instability [prompt positive and delayed negative]

"At present it is believed that no purely nuclear oscillatory instabilities can occur in fast reactors." Nothing akin to xenon oscillations.

Lots of models in McCarthy & Okrent.

Transfer Function measurement -- Rod oscillator is still most popular, although noise has been used.

7.3.5. Accident Analysis

At first worry about nuclear explosion (kiloton range). But eventually it became clear that proper design eliminated this as MCA.

Now replaced by meltdown and recriticality. This is a major key problem studied. Very complicated (and [illegible]) -- literature [illegible] Fermi meltdown. Bethe-Tait is still preferred design tool.

PROBLEM SET #1

1.) Consider a slab reactor of extrapolated width a in which delayed neutrons are neglected. Assuming one-speed diffusion theory is valid, determine the time dependent flux distribution in the reactor subject to initial condition

$$\phi(x,0) = \phi_0(x) \quad , \quad 0 \le x \le a$$

and boundary conditions

$$\phi(0,t) = \phi(a,t) = 0 \, , \quad t > 0 \, .$$

Utilize separation of variables to solve for $\phi(x,t)$.

2.) Repeat the solution of problem (1) using Laplace transforms in time—i.e.

$$\tilde{\phi}(x,s) = \int_0^\infty dt \, e^{-st} \phi(x,t) \, .$$

Verify that the answer you obtain is identical to that obtained using separation of variables. (Useful Reference: H. S. Carslaw and J. C. Jaeger, Operational Methods in Applied Mathematics (Dover, 1941), Chapter V, VI)

3.) Using the solution obtained in (1) or (2), determine under what conditions (e.g. on t, a, k_{eff}, etc.) one could approximate the time dependent flux to be separable in space and time—i.e.

$$\phi(x,t) = P(t) \sin \frac{\pi x}{a} \, .$$

PROBLEM 1:

Diffusion equation: $\frac{1}{v}\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2} + (\nu\Sigma_f - \Sigma_a)\phi(x,t)$

or $\frac{1}{vD}\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + B_m^2\phi(x,t)$, $B_m^2 \equiv \frac{\nu\Sigma_f - \Sigma_a}{D}$ (material buckling

with

i.c: $\phi(x,0) = \phi_0(x)$; b.c. $\phi(0,t) = \phi(a,t)$

Seek a solution of the form $\phi(x,t) = T(t)\,\psi(x)$, or plugging into (2) and dividing by T

$$\frac{1}{vD}\frac{1}{T}\frac{dT}{dt} = \frac{1}{\psi}\frac{d^2\psi}{dx^2} + B_m^2 \equiv -\gamma^2 \quad \text{(separation constant)}$$

Suppose γ^2 is known; then we can solve

$$\frac{dT}{dt} + vD\gamma^2 T(t) = 0 \Rightarrow T(t) = T(0)e^{-vD\gamma^2 t}$$

To determine γ, consider $\frac{d^2\psi}{dx^2} + (B_m^2+\gamma^2)\psi(x) = 0$

subject to b.c: $\psi(0) = \psi(a) = 0$

General solution:

$$\psi(x) = C_1\cos\kappa x + C_2\sin\kappa x, \quad \kappa^2 \equiv B_m^2 - \gamma^2$$

Applying b.c:

$$\psi(0) = 0 \Rightarrow C_1 = 0$$

$$\psi(a) = 0 = C_2\sin\kappa a \Rightarrow \kappa a = n\pi,\ n = 1, 2, \cdots$$

Hence for any value γ_n such that

$$\kappa_n^2 = B_m^2 + \gamma_n^2 = n^2\pi^2/a^2, \quad n = 1, 2, 3$$

or

$$\gamma_n^2 = -B_m^2 + n^2\pi^2/a^2$$

we find a solution

$$\psi_n(x) = C_n\sin(n\pi x/a)$$

We can now return to solve the original initial value problem (2) & (3) by seeking the solution as a superposition of these solutions

$$\phi(x,t) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{a}\right)e^{-vD\gamma_n^2 t}$$

Note that by construction, $\phi(0,t) = \phi(a,t)$, $t > 0$. Hence to determine the C_n, we apply the initial condition

$$\phi(x,0) = \phi_0(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}$$

Now multiply by $\sin\frac{l\pi x}{a}$ and integrate from $x=0$ to $x=a$

$$\int_0^a dx \sin\frac{l\pi x}{a}\, \phi_0(x) = \sum_{n=1}^{\infty} C_n \underbrace{\int_0^a dx \left(\sin\frac{l\pi x}{a}\right)\left(\sin\frac{n\pi x}{a}\right)}_{\frac{a}{2}\delta_{nl}} = \frac{a}{2} C_l$$

Hence we can solve for

$$C_n = \frac{2}{a}\int_0^a dx \sin\frac{l\pi x}{a}\, \phi_0(x)$$

to find our final solution

$$\phi(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{a}\int_0^a dx' \sin\frac{l\pi x'}{a}\, \phi_0(x')\right] \sin\frac{n\pi x}{a}\; e^{-vD\left[\frac{n^2\pi^2}{a^2} - B_m^2\right]t}$$

Comments:

i.) $B_m^2 = \frac{k_\infty - 1}{L^2}$. Hence for $k_\infty < 1$, $B_m^2 < 0$ and all modes are decaying in time. For long times, the mode with the smallest exponential $\left\{\frac{n^2\pi^2}{a^2} + |B_m^2|\right\}t$, i.e. the $n=1$ mode, dominates the time behavior.

$$\phi(x,t) \sim C_1 \sin\frac{\pi x}{a}\, e^{-vD\left[\frac{\pi^2}{a^2} + |B_m^2|\right]t}, \quad t \to \infty$$

This persisting mode is sometimes referred to as the "fundamental" mode. It should be noted that the form (11) is separable in x and t.

ii.) The fundamental mode also dominates the long time behavior for $k_\infty \geq 1$. Note for $B_m^2 = \frac{k_\infty - 1}{L^2} = \pi^2/a^2$, the flux approaches a steady-state value

$$\phi(x,t) \sim C_1 \sin\frac{\pi x}{a}$$

Of course, $B_m^2 = \pi^2/a^2 = B_g^2$ is just the criticality condition

PROBLEM 2:

An alternative approach is to first Laplace transform (2) in time, noting

$$\mathcal{L}\left\{\frac{\partial\phi}{\partial t}\right\} = s\phi(x,s) - \phi_0(x)$$

to find

$$\frac{s}{vD}\tilde{\phi}(x,s) - \frac{1}{vD}\phi_0(x) = \frac{\partial^2\tilde{\phi}}{\partial x^2} + B_m^2\tilde{\phi}(x,s)$$

or

$$\frac{d^2\tilde{\phi}}{dx^2} + \left(B_m^2 - \frac{s}{vD}\right)\tilde{\phi}(x,s) = -\frac{1}{vD}\phi_0(x) \qquad (11)$$

subject to transformed b.c.:

$$\tilde{\phi}(0,s) = \tilde{\phi}(a,s) = 0$$

To proceed further, we must solve this inhomogeneous O.D.E. Several ways to do this. One way is to use Green's functions. Try a short cut however. Seek the solution to (11) in the form of a Fourier series

$$\tilde{\phi}(x,s) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{a} + \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{a}$$

First note the b.c. ⇒ $b_n = 0,\ n = 0, 1, \cdots \infty$. Hence if we expand

$$\phi_0(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a}\ ; \qquad c_n = \frac{2}{a}\int_0^a dx\, \sin\frac{n\pi x}{a}\,\phi_0(x)$$

we find

$$\sum_{n=1}^{\infty}\left\{\left(-\frac{n^2\pi^2}{a^2}\right)a_n + \left(B_m^2 - \frac{s}{vD}\right)a_n + \frac{1}{vD}c_n\right\}\sin\frac{n\pi x}{a} = 0$$

Multiply by $\sin\frac{k\pi x}{a}$, integrate from 0 to a, and use orthogonality to find

$$a_n\left[B_m^2 - \frac{n^2\pi^2}{a^2} - \frac{s}{vD}\right] = -\frac{1}{vD}c_n$$

or

$$a_n = \frac{c_n}{s - vD\left[B_m^2 - \frac{n^2\pi^2}{a^2}\right]}$$

Hence we find

$$\tilde{\phi}(x,s) = \sum_{n=1}^{\infty} \frac{c_n}{s - vD\left[B_m^2 - \frac{n^2\pi^2}{a^2}\right]} \sin\frac{n\pi x}{a}$$

We can now invert term by term, noting

$$\mathcal{L}^{-1}\left\{\frac{1}{s-vD\left[B_m^2-\frac{n^2\pi^2}{a^2}\right]}\right\} = e^{-vD\left[\frac{n^2\pi^2}{a^2}-B_m^2\right]t}$$

to find

$$\phi(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\, e^{-vD\left[\frac{n^2\pi^2}{a^2}-B_m^2\right]t}$$

which is identical to our earlier answer (10).

<u>PROBLEM 3</u>:

If we examine (10) more carefully, we find that the higher modes become negligible for long times. In particular, for sufficiently large t we are left with the two lowest modes

$$\phi(x,t) \sim C_1 \sin\frac{\pi x}{a}\, e^{-vD\left[\frac{\pi^2}{a^2}-B_m^2\right]t} + C_2 \sin\frac{2\pi x}{a}\, e^{-vD\left[\frac{4\pi^2}{a^2}-B_m^2\right]t}$$

Hence we can easily formulate the criterion such that the fundamental mode dominates [and hence yields space-time separability]. Assume $C_1 \sim C_2$. Then we require

$$e^{-vD\frac{\pi^2}{a^2}t} \gg e^{-vD\frac{4\pi^2}{a^2}t}$$

or

$$e^{vD\frac{3\pi^2}{a^2}t} \gg 1$$

or

$$vD\frac{\pi^2}{a^2}t \gg 1$$

or

$$t \gg \frac{a^2}{vD\pi^2}$$

Hence the higher modes become negligible more rapidly in small systems, or systems characterized by fast neutrons (large v).

PROBLEM SET # 2

4.) (Lamarsh 12-16) The production of neutrons from the (γ,n) reaction can have an important effect on the kinetic behavior of D_2O and Be moderated reactors. These neutrons can be traced to certain fission products which decay with the emission of energetic γ-rays. It is possible, therefore, to treat photoneutrons as delayed neutrons having these fission products as precursors. The table below gives data for U^{235} fissions (saturation fission product activity) in D_2O.

i	$\lambda_i(\text{sec}^{-1})$	$\beta_i \times 10^5$	i	$\lambda_i(\text{sec}^{-1})$	$\beta_i \times 10^5$
1	6.26×10^{-7}	0.05	6	1.50×10^{-3}	3.36
2	3.63×10^{-6}	0.103	7	4.81×10^{-3}	7.00
3	4.37×10^{-5}	0.323	8	1.69×10^{-2}	20.4
4	1.17×10^{-4}	2.34	9	2.77×10^{-1}	65.1
5	4.28×10^{-4}	2.07		$\beta = \sum \beta_i =$	100.75×10^{-5}

a.) Show that the reactivity equation for a D_2O moderated reactor is

$$\rho = \frac{s\ell}{s\ell+1} + \frac{s}{s\ell+1} \sum_{i=1}^{15} \frac{\beta_i}{s+\lambda_i}$$

where i = 1,2,...6 refers to delayed fission neutrons and i = 7,8,...15 refers to delayed photoneutrons.

b.) Compare the stable periods of an infinite, U^{235} fueled, D_2O moderated reactor, following a step reactivity insertion of \$0.10, with and without the delayed photoneutrons taken into consideration.

5.) (Lamarsh 12-18) A uniform, infinite, subcritical assembly of multiplication factor k_∞ contains uniformly distributed sources emitting S fast neutrons/cm^{2}sec. For $t < 0$ the system is in the steady state, but at $t = 0$ the sources are suddenly removed.

a.) Show that the thermal flux quickly drops to the value‡

$$vn(t) \to \frac{\beta k_\infty p_e S}{\Sigma_a(1-k_\infty)[1-(1-\beta)k_\infty]} = \frac{\beta k_\infty v n(0)}{1-(1-\beta)k_\infty}$$

b.) Compute the fractional change in the flux for a natural uranium-water assembly having $k_\infty = 0.98$.

c.) Show that following the prompt drop the assembly goes on a stable period of approximately 80 sec.

d.) Discuss the prompt drop in flux in a finite subcritical assembly when a point source is suddenly removed.

e.) Show how this drop in flux can be used to measure k_∞ of a subcritical assembly.

‡ Use one effective delayed group to simplify algebra.

6.) List all the assumptions or approximations involved in deriving the neutron transport equation. [Most of these were not mentioned in class.]

PROBLEM 4:

a) The mechanism of photoneutron production is

$$\boxed{\text{fission product}} \xrightarrow{\gamma} \boxed{\text{Be or } D_2O} \xrightarrow{n}$$

It is customary to treat these by defining an effective concentration C_i of γ precursors [(γ,n) reaction is effectively instantaneous]

$$\frac{dC_i}{dt} = \beta_i \frac{k_{eff}}{\ell} n(t) - \lambda_i C_i(t) \qquad i = 7, \cdots, 15$$

[Note this assumes the photoneutrons are produced at the point of fission. This is actually a crummy assumption. The γ's can travel an appreciable distance -- as indeed they must if they are to induce a (γ,n) reaction in the moderator]

It is obvious, therefore, that the modification of the inhour equation is trivial

$$\rho = \frac{s\ell}{s\ell+1} + \frac{s}{s\lambda+1} \sum_{i=1}^{15} \frac{\beta_i}{s+\lambda_i}$$

b.) To a good estimate, the stable period <u>without</u> photoneutrons is

$$T = \frac{1}{\rho}\left[\ell + \sum_{i=1}^{6} \frac{\beta_i}{\lambda_i}\right] \cong 340 \text{ sec}$$

Including photoneutrons

$$T = \frac{1}{\rho}\left[\lambda + \sum_{i=1}^{15} \frac{\beta_i}{\lambda_i}\right] \doteq 2200 \text{ sec}$$

This is a rather considerable difference, needless to say.

PROBLEM 5:

a.) For $t<0$, $$\left[\frac{k_\infty(1-\beta)-1}{\ell}\right] n(0) + \lambda C(0) + P_{re} S = 0$$

$$\frac{\beta k_\infty}{\ell} n(0) = \lambda C(0)$$

$$\Longrightarrow n(0) = \frac{-\ell P_{re} S}{1-k_\infty}$$

For $t>0$, consider P.R.K.E.

$$\frac{dn}{dt} = \left[\frac{k_\infty(1-\beta)-1}{\ell}\right] n(t) + \lambda C(t)$$

$$\frac{dC}{dt} = \beta \frac{k_\infty}{\ell} n(t) - \lambda C(t)$$

Since we are only interested in the initial rapid decrease in the flux after the source is withdrawn, it is valid to make the "constant delayed neutron production rate approximation":

$$C(t) = C(0) = \frac{\beta k_\infty}{\lambda \ell} n(0)$$

Hence we must solve

$$\frac{dn}{dt} = \left[\frac{k_\infty(1-\beta)-1}{\ell}\right] n(t) + \frac{\beta k_\infty}{\ell} n(0)$$

But this can be easily integrated to find

$$n(t) = e^{\lambda t}\left[n(0) + \underline{\int_0^t d\tau\, e^{-\lambda t} \frac{\beta k_\infty}{\ell} n(0)}\right] \qquad \lambda = \frac{k_\infty(1-\beta)-1}{\ell}$$

$$= n(0)\left[e^{\lambda t} + e^{\lambda t}\frac{\beta k_\infty}{\ell\lambda}\left(1-e^{-\lambda t}\right)\right]$$

$$\cong -\frac{\beta k_\infty}{\ell\lambda} n(0) \quad \text{for } t >> \ell$$

$$\therefore\ v n(0^+) = \frac{v n(0)\, \beta k_\infty}{1-(1-\beta)k_\infty} = \frac{\beta k_\infty P_{re} S}{\Sigma_a (1-k_\infty)[1-(1-\beta)k_\infty]}$$

b.) For natural uranium – water system

$$\frac{v n(0^+)}{v n(0^-)} = \frac{\beta k_\infty}{1-(1-\beta)k_\infty} = 0.245$$

c.) After the prompt drop, the time behavior is determined by the decay rate of the longest lived precursor – in this case

$$T = \frac{1}{\lambda_1} = 80 \text{ sec}$$

d.) For a point source in a subcritical assembly it is obvious that one must account for leakage. Provided one can neglect the excitation of higher spatial modes, the leakage term can be included by merely modifying $k_\infty \rightarrow k_{eff}$.

e.) Recall we have

$$n(0^+) = \frac{n(0^-)\,\beta k_{eff}}{1-(1-\beta)k_{eff}}$$

or

$$\frac{n(0^-)}{n(0^+)} = \frac{1-k_{eff}+\beta k_{eff}}{\beta k_{eff}} = \frac{1-k_{eff}}{\beta k_{eff}} + 1$$

Hence

$$\frac{n(0^+)}{n(0^+)} - \frac{n(0^-)}{n(0^+)} = \frac{k_{eff}-1}{\beta k_{eff}} = \frac{\rho}{\beta} = \$$$

or

$$\left(\begin{array}{c}\text{fractional}\\ \text{change in}\\ \text{power}\end{array}\right) = \frac{n(0^-)-n(0^+)}{n(0^+)} = -\$$$ reactivity of subcritical system in dollars.

Hence by measuring the fractional change in power level, we directly measure the reactivity of the system.

PROBLEM SET # 3

7.) It has been mentioned in class that the angular neutron density $n(\underline{r},\underline{v},t)$ can be regarded as a function of several sets of independent variables. Derive the expressions relating $n(\underline{r},\underline{v},t)$, $n(\underline{r},v,\hat{\Omega},t)$, and $n(\underline{r},E,\hat{\Omega},t)$, where

$$\hat{\Omega} \equiv \underline{v}/|\underline{v}| \quad , \quad v \equiv |\underline{v}| \quad , \quad E \equiv \tfrac{1}{2}mv^2 \ .$$

8.) Consider the transport equation for the angular neutron flux $\Phi(\underline{r},v,\hat{\Omega},t)$

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} + \hat{\Omega}\cdot\nabla\Phi + \Sigma_t(\underline{r},v,t)\Phi(\underline{r},v,\hat{\Omega},t) = \int_0^\infty dv' \int d\hat{\Omega}' \, \Sigma_s(v'\to v,\hat{\Omega}'\to\hat{\Omega},\underline{r},t)\Phi(\underline{r},v',\hat{\Omega}',t) + Q(\underline{r},v,\hat{\Omega},t)$$

Integrate this equation over angle $\hat{\Omega}$ to obtain an equation forthe angle-integrated neutron flux

$$\phi(\underline{r},v,t) \equiv \int d\hat{\Omega}\ \Phi(\underline{r},v,\hat{\Omega},t) \ .$$

(It will be helpful to note that for essentially all materials of interest in reactor design, the scattering kernel is only a function of the relative angle $\hat{\Omega}\cdot\hat{\Omega}'$, i.e.

$$\Sigma_s(v'\to v,\hat{\Omega}'\to\hat{\Omega},\underline{r},t) = \Sigma_s(v'\to v,\hat{\Omega}'\cdot\hat{\Omega},\underline{r},t)$$

Hence we can define the angle-integrated scattering kernel as

$$\Sigma_s(v'\to v,\underline{r},t) \equiv \int d\hat{\Omega}\ \Sigma_s(v'\to v,\hat{\Omega}'\cdot\hat{\Omega},\underline{r},t) \qquad)$$

9.) Prove that a self-adjoint operator L
   - i.) has only real eigenvalues;
   - ii.) further, the eigenfunctions corresponding to nondegenerate eigenvalues are orthogonal (nondegenerate eigenvalues are ones to which there corresponds only one eigenfunction).

10.) Consider a matrix operator:

$$\underline{\underline{L}} \equiv \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & & & \vdots \\ \vdots & & & \\ L_{n1} & \cdots & \cdots & L_{nn} \end{pmatrix}$$

whose components are themselves operators. Note that $\underline{\underline{L}}$ must be defined as operating on a vector whose components are functions:

$$\underline{f} \equiv \begin{pmatrix} f_1(\underline{r}) \\ \vdots \\ f_n(\underline{r}) \end{pmatrix}$$

   - i.) Define a suitable scalar product for the set of vectors $\underline{f}$.
   - ii.) Prove that the adjoint of $\underline{\underline{L}} = [L_{ji}]$ is just

$$\underline{\underline{L}}^{\dagger} \equiv [L^{\dagger}_{ij}]$$

11.) Prove that the diffusion operator $\nabla\cdot D(\underline{r})\nabla$ is self-adjoint for the class of all functions $\phi(\underline{r})$ which vanish on the surface of a volume V.

PROBLEM 7:

By definition

$$n(\underline{r},\underline{v},t)\,d^3r\,d^3v = n(\underline{r},v,\hat{\Omega},t)\,d^3r\,dv\,d\hat{\Omega} = n(\underline{r},E,\hat{\Omega},t)\,d^3r\,dE\,d\hat{\Omega}$$

Hence

$$n(\underline{r},v,\hat{\Omega},t) = n(\underline{r},\underline{v},t)\frac{\cancel{d^3r}\,d^3v}{\cancel{d^3r}\,dv\,d\hat{\Omega}} = n(\underline{r},\underline{v},t)\frac{v^2\,dv\,d\hat{\Omega}}{dv\,d\hat{\Omega}} = v^2\,n(\underline{r},\underline{v},t)$$

$$n(\underline{r},E,\hat{\Omega},t) = n(\underline{r},\underline{v},t)\frac{d^3r\,d^3v}{d^3r\,dE\,d\hat{\Omega}} = n(\underline{r},\underline{v},t)\frac{v^{\cancel{2}}(dv)\,d\hat{\Omega}}{(dE)\,d\hat{\Omega}} = \frac{v}{m}\,n(\underline{r},\underline{v},t)$$

$$n(\underline{r},v,\hat{\Omega},t) = n(\underline{r},E,\hat{\Omega},t)\frac{d^3r\,dv\,d\hat{\Omega}}{d^3r\,dE\,d\hat{\Omega}} = n(\underline{r},E,\hat{\Omega},t)\frac{dv}{dE} = \frac{1}{mv}\,n(\underline{r},E,\hat{\Omega},t)$$

PROBLEM 8:

Consider

$$\frac{1}{v}\frac{\partial\Phi}{\partial t} + \hat{\Omega}\cdot\nabla\Phi + \Sigma_t(\underline{r},v,t)\Phi(\underline{r},v,\hat{\Omega},t) = \int_0^\infty dv'\int d\hat{\Omega}'\,\Sigma_s(v'\to v,\hat{\Omega}'\to\hat{\Omega})\,\Phi(\underline{r},v',\hat{\Omega}',t) + Q(\underline{r},v,$$

Integrate each term over $\hat{\Omega}$

$$\int d\hat{\Omega}\,\frac{1}{v}\frac{\partial\Phi}{\partial t} = \frac{1}{v}\frac{\partial}{\partial t}\int d\hat{\Omega}\,\Phi = \frac{1}{v}\frac{\partial\phi}{\partial t}$$

$$\int d\hat{\Omega}\,\hat{\Omega}\cdot\nabla\Phi = \nabla\cdot\int d\hat{\Omega}\,\hat{\Omega}\,\Phi = \nabla\cdot J$$

$$\int d\hat{\Omega}\,\Sigma_t\Phi = \Sigma_t\int d\hat{\Omega}\,\Phi = \Sigma_t\,\phi(\underline{r},v,t)$$

$$\int d\hat{\Omega}\int_0^\infty dv'\int d\hat{\Omega}'\,\Sigma_s\Phi = \int_0^\infty dv'\int d\hat{\Omega}'\left[\int d\hat{\Omega}\,\Sigma_s(v'\to v,\hat{\Omega}'\cdot\hat{\Omega})\right]\Phi(\underline{r},v',\hat{\Omega}',t)$$

$$= \int_0^\infty dv'\,\Sigma_s(v'\to v)\int d\hat{\Omega}'\,\Phi(\underline{r},v',\hat{\Omega}',t) = \int_0^\infty dv'\,\Sigma_s(v'\to v)\,\phi(\underline{r},v',t)$$

$$\int d\hat{\Omega}\,Q \equiv q(\underline{r},v,t)$$

Hence we find a "conservation equation"

$$\frac{1}{v}\frac{\partial\phi}{\partial t} + \nabla\cdot J + \Sigma_t(\underline{r},v,t)\,\phi(\underline{r},v,t) = \int_0^\infty dv'\,\Sigma_s(v'\to v,\underline{r},t)\,\phi(\underline{r},v',t) + q(\underline{r},v,t)$$

PROBLEM 9:

By definition, a self-adjoint operator L satisfies

$$(f, Lg) = (Lf, g) \quad , \quad (f,g) = \int d^3r \int d^3v \, f^*(\underline{r},\underline{v})\, g(\underline{r},\underline{v})$$

a) Consider the eigenvalue problem for such an operator

$$L\psi_\lambda = \lambda \psi_\lambda$$

Now multiply by ψ_λ^* and integrate to find

$$(\psi_\lambda, L\psi_\lambda) = \lambda\,(\psi_\lambda, \psi_\lambda)$$

But we also know

$$(L\psi_\lambda, \psi_\lambda) = \lambda^*\,(\psi_\lambda, \psi_\lambda)$$

Hence using (1) we find

$$\lambda(\psi_\lambda, \psi_\lambda) = \lambda^*(\psi_\lambda, \psi_\lambda) \implies \lambda = \lambda^* \implies \lambda \text{ is real.}$$

b.) Consider now

$$L\psi_n = \lambda_n \psi_n$$

and

$$L\psi_m = \lambda_m \psi_m$$

Take the scalar product of (2) with ψ_m^* and 3) with ψ_n to find

$$(\psi_m, L\psi_n) = \lambda_n(\psi_m, \psi_n)$$

$$(L\psi_m, \psi_n) = \lambda_m(\psi_m, \psi_n)$$

Subtracting, using (1), we find

$$(\lambda_n - \lambda_m)(\psi_m, \psi_n) = 0$$

Hence if $\lambda_n \neq \lambda_m$ [i.e., the eigenvalues are nondegenerate]

$$(\psi_m, \psi_n) = 0 \quad , \quad m \neq n \quad \text{(orthogonality).}$$

PROBLEM 10:

i.) For $\underline{f} = \mathrm{col}\,[f_1, \ldots f_n(\underline{r})]$, the appropriate scalar product is

$$\langle \underline{f} | \underline{g} \rangle \equiv \sum_{i=1}^{n} \int d^3r\, f_i^*(\underline{r})\, g_i(\underline{r}) = \sum_{i=1}^{n} (f_i, g_i)$$

ii.)
$$\langle \underline{f} | \underline{\underline{L}}\, \underline{g} \rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} (f_j, L_{ji}\, g_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (f_i, L_{ij}\, g_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (L_{ij}^{+} f_i, g_j) \equiv \langle \underline{\underline{L}}^{+} \underline{f} | \underline{g} \rangle$$

where we have define the adjoint of the <u>components</u> L_{ij} by

$$(L_{ij}^{+} f, g) = (f, L_{ij}\, g)$$

Thus we identify

$$\underline{\underline{L}}^{+} = [L_{ji}]^{+} = [L_{ij}^{+}]$$

PROBLEM 11:

Consider

$$(\psi, \nabla \cdot D(\underline{r}) \nabla \phi) = \int_V d^3r\, \psi^*(\underline{r})\, \nabla \cdot D(\underline{r}) \nabla \phi(\underline{r})$$

$$= \int_S d^2r\, \psi^*(\underline{r})\, D(\underline{r})\, \hat{n} \cdot \nabla \phi(\underline{r}) - \int_V d^3r\, \nabla \psi^* \cdot D(\underline{r}) \nabla \phi$$

$$= \underset{①}{\int_S d^2r\, \psi^*(\underline{r})\, D(\underline{r})\, \hat{n} \cdot \nabla \phi(\underline{r})} - \underset{②}{\int_S d^2r\, \hat{n} \cdot \nabla \psi^*\, D(\underline{r})\, \phi(\underline{r})}$$

$$+ \underset{③}{\int_V d^3r\, \nabla \cdot D(\underline{r}) \nabla \psi^*(\underline{r})\, \phi(}$$

Now we have required $\phi(\underline{r})$ to vanish on the surface. Hence the second term vanishes.

Suppose we also require $\Psi(\underline{r})$ to vanish on the surface such that the first term vanishes. Then we find

$$(\Psi, \nabla\cdot D(r)\nabla\phi) = \int_V d^3r \; \nabla\cdot D(r)\nabla\Psi^*(\underline{r})\, \phi(\underline{r}) = (\nabla\cdot D(r)\nabla\Psi, \phi)$$

which implies that $\nabla\cdot D(r)\nabla$ is self-adjoint [and also that the accompanying boundary condition, $\phi(\underline{r_s}) = 0$, $\underline{r_s}$ on surface, is also self-adjoint since we demand $\psi(\underline{r_s}) = 0$, $\underline{r_s}$ on surface].

PROBLEM SET # 4

12.) Using one-group diffusion theory with one-group of delayed neutrons, show that the ω-modes for an infinite bare slab reactor of thickness are

$$\underline{\Psi}_{nJ} = \begin{pmatrix} 1 \\ \frac{\nu\Sigma_f\beta}{\lambda+\omega_{nJ}} \end{pmatrix} A_{nJ} \sin\frac{n\pi x}{\tilde{a}}$$

where ω_{nJ} are the two roots of

$$\frac{(k_{eff})_n - 1}{(k_{eff})_n} = \omega_{nJ}\left[\frac{1}{v\nu\Sigma_f} + \frac{\beta}{\lambda+\omega_{nJ}}\right] \qquad J=1,2$$

where $(k_{eff})_n = \dfrac{\nu\Sigma_f/\Sigma_a}{1+L^2\left[\frac{\pi}{\tilde{a}}n\right]^2} \qquad n=1,2,\cdots$

and $A_{nJ} = \sqrt{\dfrac{2}{\tilde{a}}}\;\dfrac{1}{1+\left(\frac{\nu\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)}$

(Hint: Use

$$\underline{\underline{H}} \equiv \begin{bmatrix} D\frac{d^2}{dx^2}+(1-\beta)\nu\Sigma_f-\Sigma_a & \lambda \\ \nu\Sigma_f\beta & -\lambda \end{bmatrix} \qquad \underline{\Psi}_{nJ} = \begin{pmatrix} N_{nJ}(x) \\ C_{nJ}(x) \end{pmatrix}$$

and solve the eigenvalue problem

$$\underline{\underline{H}}\,\underline{\Psi}_{nJ} = \omega_{nJ}\underline{\Psi}_{nJ}$$

with the boundary conditions $N_{nJ}(0) = N_{nJ}(\tilde{a}) = 0.$)

Observe that $N_{nJ}(x) = A_{nJ}\sin(n\pi x/\tilde{a})$ has the same spatial distribution for J = 1,2 for a fixed n. Verify the orthogonality

$$(\underline{\Psi}_{nJ}, \underline{\Psi}_{n'J'}) = \delta_{nn'}\delta_{JJ'}$$

13.) Read either:

Bell & Glasstone, Chapter 8, pp. 1-23
Keepin, Chapter 6
Naval Reactors Physics Handbook, Vol. I, Ed. by
A. Radkowsky, pp. 853-872

PROBLEM 12:

Recall that the period eigenvalue problem is

$$\underline{\underline{H}}\,\underline{\Psi}_n = \omega_n \underline{\Psi}_n \qquad (1$$

where in the one-group diffusion, one delayed group approximation $\underline{\underline{H}}$ is just

$$\underline{\underline{H}} = \begin{bmatrix} vD\frac{d^2}{dx^2} + (1-\beta)v\nu\Sigma_f - v\Sigma_a & \lambda \\ v\nu\Sigma_f\beta & -\lambda \end{bmatrix} \qquad (2$$

Hence writing out the eigenvalue problem explicitly

$$vD\frac{d^2N_n}{dx^2} + \left[(1-\beta)v\nu\Sigma_f - v\Sigma_a\right]N_n(x) + \lambda C_n(x) = \omega_n N_n(x) \qquad (3$$

$$v\nu\Sigma_f\beta N_n(x) - \lambda C_n(x) = \omega_n C_n(x) \qquad (4$$

We can solve for

$$C_n(x) = \frac{\nu v\Sigma_f\beta}{\lambda+\omega_n} N_n(x) \qquad (5$$

and substitute into (3) to find

$$vD\frac{d^2N_n}{dx^2} + \left[(1-\beta)v\nu\Sigma_f - v\Sigma_a\right]N_n(x) + \frac{\lambda\nu v\Sigma_f\beta}{\lambda+\omega_n}N_n(x) = \omega_n N_n(x)$$

Now the solutions to this equation which satisfy the boundary conditions $N_n(0) = N_n(\tilde{a}) = 0$ for a slab are

$$N_n(x) \sim \sin\frac{n\pi x}{\tilde{a}} \qquad (6)$$

Hence the eigenvalues ω_n are given as the roots of

$$-\frac{n^2\pi^2}{a^2}vD + (1-\beta)v\nu\Sigma_f - v\Sigma_a + \frac{\lambda\nu\Sigma_f\beta}{\lambda+\omega_n} = \omega_n$$

or

$$-\frac{n^2\pi^2}{a^2}L^2 + (1-\beta)\frac{\nu\Sigma_f}{\Sigma_a} - 1 + \lambda\frac{\nu\,\Sigma_f/\Sigma_a\,\beta}{\lambda+\omega_n} = \frac{\omega_n}{v\Sigma_a}$$

or
$$-\left[1+\frac{n^2\pi^2}{a^2}L^2\right]+\frac{\nu\Sigma_f}{\Sigma_a}+\beta\underbrace{\left[-\frac{\nu\Sigma_f}{\Sigma_a}+\frac{\lambda\nu\Sigma_f/\Sigma_a}{\lambda+\omega_n}\right]}_{-\frac{\nu\Sigma_f}{\Sigma_a}\frac{\omega_n}{\lambda+\omega_n}}=\omega_n/\Sigma_a$$

or
$$1-\frac{\left[1+\frac{n^2\pi^2}{a^2}L^2\right]}{\nu\Sigma_f/\Sigma_a}=\omega_n\left[\frac{1}{\nu v\Sigma_f}+\frac{\beta}{\lambda+\omega_n}\right]$$

or defining $(k_{eff})_n=\dfrac{\nu\Sigma_f/\Sigma_a}{1+\frac{n^2\pi^2}{a^2}L^2}$ and noting $1-\dfrac{1}{(k_{eff})_n}=\dfrac{(k_{eff})_n-1}{(k_{eff})_n}$

we find the equation for the period eigenvalues for a given n is
$$\frac{(k_{eff})_n-1}{(k_{eff})_n}=\omega_{nJ}\left[\frac{1}{\nu v\Sigma_f}+\frac{\beta}{\lambda+\omega_{nJ}}\right]$$

Since this is quadratic in ω_n, we have recognized that there will in general be two roots, and have denoted them by ω_{nJ}, $J=1$ or 2.

Hence
$$N_{nJ}(x)=A_{nJ}\sin\frac{n\pi x}{a}$$
and using ⑤
$$C_{nJ}(x)=\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}A_{nJ}\sin\frac{n\pi x}{a}$$

We can rewrite these in vector form as
$$\underline{\Psi}_{nJ}(x)=\begin{pmatrix}1\\ \frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\end{pmatrix}A_{nJ}\sin\frac{n\pi x}{a}$$

Here A_{nJ} is a normalization factor we can determine by requiring
$$1=\langle\Psi_{nJ}|\Psi_{nJ}\rangle=A_{nJ}^2\Bigg[\underbrace{\int_0^a dx\sin^2\frac{n\pi x}{a}}_{a/2}+\left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)^2\int_0^a dx\sin^2\frac{n\pi x}{a}$$
which implies
$$A_{nJ}^2=\frac{2}{a}\left[1+\left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)^2\right]^{-1}$$

or
$$A_{nJ}=\sqrt{\frac{2}{a}}\frac{1}{\sqrt{1+\left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)^2}}$$

We can verify orthogonality in n immediately

$$\langle \Psi_{nJ} | \Psi_{mJ} \rangle = A_{nJ}A_{mJ}\Big[\underbrace{\int_0^a dx \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a}}_{0} + \left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)\left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{mJ}}\right) \underbrace{\int_0^a dx \sin\frac{n\pi x}{a} \sin}_{0}$$

$$= 0 \quad \text{if } m \neq n.$$

For orthogonality in J, note

$$\langle \Psi_{nJ} | \Psi_{nJ'} \rangle = A_{nJ}A_{nJ'}\left[1 + \left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ}}\right)\left(\frac{\nu v\Sigma_f\beta}{\lambda+\omega_{nJ'}}\right)\right] \int_0^a dx \sin^2\frac{n\pi x}{a}$$

$$\neq 0 \quad \text{apparently.}$$

Hence Ψ_{nJ} is <u>not</u> orthogonal in the index J.

PROBLEM SET # 5

14.) Demonstrate that the "amplitude factor" $P(t)$ in the point reactor kinetics equations can be interpreted as proportional to the time dependence of the fundamental mode $n_o(\underline{r},\underline{v})$ if one expands $n(\underline{r},\underline{v},t)$ in the modes (either reactivity or period-modes) of the critical reference reactor. (Assume that these modes form a complete, bi-orthogonal set.)

15.) a.) Consider the adiabatic approximation in which we replace the shape factor by the lowest "λ-mode" satisfying

$$L(t)\, N_{k_{eff}} = \frac{1}{k_{eff}} M(t)\, N_{k_{eff}}$$

Demonstrate that in this approximation (i.e. $\Psi(\underline{r},\underline{v},t) = N_{k_{eff}}(\underline{r},\underline{v})$) our expression for the reactivity reduces to

$$\rho(t) = \frac{k_{eff}(t) - 1}{k_{eff}(t)}$$

(the "static" reactivity).

b.) In a similar manner, demonstrate that replacing $\Psi(\underline{r},\underline{v},t)$ by the lowest "ω-mode" $N_{\omega_o}(\underline{r},\underline{v})$ satisfying

$$\left[L(t) + N_p(t) + \sum_{i=1}^{6} M_i(t)\right] N_{\omega_o}(\underline{r},\underline{v}) = \omega_o \left[1 + \sum_{i=1}^{6} \frac{M_i(t)}{\lambda_i + \omega_o}\right] N_{\omega_o}(\underline{r},\underline{v})$$

leads to the "dynamic" reactivity:

$$\rho(t) = \omega_o(t)\left[\Lambda(t) + \sum_{i=1}^{6} \frac{\bar{\beta}_i(t)}{\lambda_i + \omega_o(t)}\right].$$

16.) Describe qualitatively the approximations necessary to reduce the general point reactor kinetics equations to the forms we derived via one-speed diffusion theory in the beginning of the course. (i.e. what does one choose for $\Psi(\underline{r},\underline{v},t)$, $P(t)$, $n_o^{\dagger}(\underline{r},\underline{v})$, etc.)

PROBLEM 14:

Just expand $n(\underline{r},\underline{v},t)$ in the "modes" of the critical reference system (assuming these to be complete and biorthogonal)

$$n(\underline{r},\underline{v},t) = \sum_n c_n(t)\, n_n(\underline{r},\underline{v})$$

or multiplying by n_n^+ and integrating, using biorthogonality, yields

$$c_n(t) = \frac{(n_n^+, n)}{(n_n^+, n_n)}$$

But

$$P(t) = \frac{(n_0^+, n)}{(n_0^+, \psi)}$$

Since $\frac{d}{dt}(n_0^+, \psi) = 0$, we find

$$P(t) \propto c_0(t)$$

PROBLEM 15:

a.) First recall

$$\rho(t) = \frac{\left(n_0^+, \int d^3v'\, \delta\left[v'\Sigma_s(\underline{v}'\to\underline{v}) + \nu\chi(v)v'\Sigma_f(v')\right]\psi(\underline{r},\underline{v}',t)\right) - (n_0^+, v\,\delta\Sigma_t\psi)}{\left(n_0^+, \int d^3v'\, \nu(v')\chi(v)v'\Sigma_f(v')\,\psi(\underline{r},\underline{v}',t)\right)}$$

or if we introduce operator notation

$$\rho(t) = \frac{(n_0^+, [L_0 - L + M_0 - M]\,\psi)}{(n_0^+, M\psi)}$$

Now use $\psi = N_{k_{eff}}$ where $L(t)N_{k_{eff}} = \frac{1}{k_{eff}} M(t) N_{k_{eff}}$.
If we note

$$L_0 N_{k_{eff}} = M_0 N_{k_{eff}},$$

then

$$\rho(t) = -\frac{(n_0^+, L N_{keff}) - (n_0^+, M N_{keff})}{(n_0^+, M N_{keff})}$$

$$= -\frac{(n_0^+, L N_{keff}) - k_{eff}(n_0^+, L N_{keff})}{k_{eff}(n_0^+, L N_{keff})} = \frac{k_{eff}(t) - 1}{k_{eff}(t)}$$ Q.E.D.

b.) We now use the lowest "ω-mode" $\psi = N_{\omega_0}$ where

$$\left[L(t) - N_p(t) - \sum_{i=1}^{6} M_i(t)\right] N_{\omega_0} = -\omega_0 \left[1 + \sum_{i=1}^{6} \frac{M_i(t)}{\lambda_i + \omega_0}\right] N_{\omega_0}$$

Again note

$$L_0 N_{\omega_0} = M N_{\omega_0}$$

Hence

$$\rho(t) = -\frac{(n_0^+, (L' - M) N_{\omega_0})}{(n_0^+, M N_{\omega_0})}$$

$$= +\frac{\left(n_0^+, \omega_0 \left[1 + \sum_{i=1}^{6} \frac{M_i(t)}{\lambda_i + \omega_0}\right] N_{\omega_0}\right)}{(n_0^+, M N_{\omega_0})}$$

$$= \omega_0 \left[\frac{(n_0^+, N_{\omega_0})}{(n_0^+, M N_{\omega_0})} + \sum_{i=1}^{6} \frac{1}{\lambda_i + \omega_0} \frac{(n_0^+, M_i N_{\omega_0})}{(n_0^+, M N_{\omega_0})}\right]$$

But recall for $\psi = N_{\omega_0}$

$$\Lambda(t) \equiv \frac{(n_0^+, N_{\omega_0})}{(n_0, M N_{\omega_0})}, \qquad \bar{\beta}_i \equiv \frac{(n_0^+, M_i N_{\omega_0})}{(n_0^+, M N_{\omega_0})}$$

Hence

$$\rho(t) = \omega_0(t) \left[\Lambda(t) + \sum_{i=1}^{6} \frac{\bar{\beta}_i(t)}{\lambda_i + \omega_0(t)}\right]$$ Q.E.D.

PROBLEM SET # 6

17.) According to the point reactor kinetics equations (with no feedback) with one equivalent group of delayed neutrons, how long should a steady source of S_0 neutrons per second be left on, in order to raise the steady neutron density in a critical reactor from n_1 to n_2? Draw a rough sketch of $n(t)$ vs. t.

18.) Determine the reactivity $\$(t)$ necessary to achieve a sinusoidal power variation

$$P(t) = P_0 + P_1 \sin\omega t$$

and then compute

$$\$_{av} = \frac{1}{T}\int_0^T dt\ \$(t)$$

(Hint: First compute $P(t)\$(t)$ noting that

$$\int_0^\infty D(\tau)\sin\omega(t-\tau)\,d\tau = \mathrm{Im}\left\{e^{i\omega t}\tilde{D}(i\omega)\right\}$$

where

$$\tilde{D}(s) = \int_0^\infty d\tau\, e^{-s\tau} D(\tau) = \sum_{i=1}^{6} \frac{\lambda_i a_i}{\lambda_i + s}$$

To compute $\$_{av}$, note that $\displaystyle\int_0^{\pi/2} \frac{dx}{1 - a\sin^2 x} = \frac{\pi}{2\sqrt{1-a}}$.)

19.) Find the response of a critical reactor to the following reactivity insertions, ignoring delayed neutrons:

i.) $\$(t) = k_0 + \gamma t$

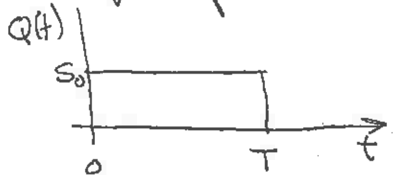
ii.) $\$(t) = k_0 \sin\omega t$

17) The P.R.K.E. with one delayed group are

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t)-1]P(t) + \lambda C(t) + \frac{\Lambda}{\beta}Q(t)$$

$$\frac{dC}{dt} = P(t) - \lambda C(t)$$

Now in a steady state reactor, $\$(t) \equiv 0$. Moreover, we will consider a source of the form



$$Q(t) = S_0[\Theta(t) - \Theta(t-T)]$$

$$\text{where } \Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{(step fu}$$

We can Laplace transform to find

$$\frac{\Lambda}{\beta}s\tilde{P}(s) - \frac{\Lambda}{\beta}P(0) = -\tilde{P}(s) + \lambda\tilde{C}(s) + \frac{\Lambda}{\beta}S_0\frac{(1-e^{-Ts})}{s} \qquad (*$$

$$s\tilde{C}(s) - C(0) = \tilde{P}(s) - \lambda\tilde{C}(s)$$

But for a critical reactor we find $\quad \lambda C(0) = P(0)$

Hence

$$\lambda\tilde{C}(s) = \frac{1}{s+\lambda}\left[\lambda\frac{P(0)}{\lambda} + \lambda\tilde{P}(s)\right]$$

or substituting into (*)

$$\left[\frac{\Lambda}{\beta}s + \overset{+\lambda\ -\lambda}{1} - \frac{\lambda}{s+\lambda}\right]\tilde{P}(s) = P(0)\left[\frac{\Lambda}{\beta} + \frac{1}{s+\lambda}\right] + \frac{\Lambda}{\beta}S_0\frac{(1-e^{-Ts})}{s}$$

or

$$\tilde{P}(s) = \frac{P(0)\left[\frac{\Lambda}{\beta} + \frac{1}{s+\lambda}\right] + \frac{\Lambda}{\beta}\overset{S_0 Ts}{S_0\frac{(1-e^{-Ts})}{s}}}{s\left[\frac{\Lambda}{\beta} + \frac{1}{s+\lambda}\right]}$$

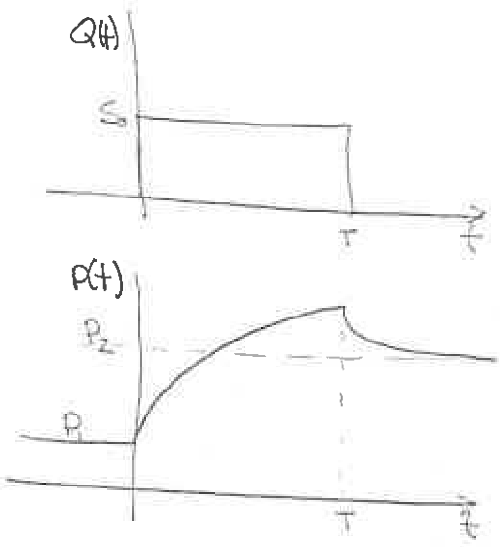
Now we can use the "final value theorem"

$$\lim_{t\to\infty} P(t) = \lim_{s\to 0} s\tilde{P}(s) = P(0) + \frac{S_0 T\frac{\Lambda}{\beta}}{\left(\frac{\Lambda}{\beta} + \frac{1}{\lambda}\right)} = \frac{[\lambda\Lambda+\beta]P(0) + \lambda\Lambda S_0}{\lambda\Lambda + \beta}$$

Hence if we wish to raise the power level from $P(0) = P_1$ to $P(t=\infty) = P_2$, we need to have the source on for

$$T = \frac{(P_2 - P_1)(\lambda\Lambda + \beta)}{\lambda\Lambda S_0} \sim \frac{(P_2 - P_1)\beta}{\lambda\Lambda S_0}$$

Graphically



Note that

$$P(T) - P_2 > 0$$

18.) We want to determine the reactivity \$(t) necessary to achieve a sinusoidal power variation

$$P(t) = P_0 + P_1 \sin\omega t$$

We can compute \$(t) using the inverse method

$$P(t)\$(t) = P(t) + \frac{\Lambda}{\beta}\frac{dP}{dt} - \int_0^\infty d\tau\, D(\tau)\, P(t-\tau)$$

$$= \cancel{P_0} + P_1\sin\omega t + \frac{\Lambda\omega}{\beta}P_1\cos\omega t - \underbrace{\int_0^\infty d\tau\, D(\tau)}_{\overset{\shortparallel}{1}} \cancel{P_0} - P_1 \underbrace{\int_0^\infty d\tau\, D(\tau)\sin\omega(t-\tau)}_{\mathrm{Im}\left\{e^{i\omega t}\sum_{i=1}^{6}\frac{\lambda_i a_i}{\lambda_i + i\omega}\right\}}$$

$$= P_1\,\mathrm{Im}\left\{\left[\underbrace{i\omega\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i}{i\omega+\lambda_i}\right)}_{Y(i\omega)}\right] e^{i\omega t}\right\}$$

$$= P_1\,|Y(i\omega)|\sin(\omega t - \phi) \qquad \phi = \arg\left[\frac{1}{Y(i\omega)}\right]$$

or

$$\$(t) = \frac{P_1}{P_0}|Y(i\omega)|\,\frac{\sin(\omega t - \phi)}{1 + \frac{P_1}{P_0}\sin\omega t}$$

Now we must compute

$$\$_{av} = \frac{1}{T}\int_0^T \$(t)\,dt = \frac{P_1}{P_0}\frac{|Y(i\omega)|}{T}\int_0^{T=\frac{2\pi}{\omega}} \frac{\sin(\omega t-\phi)}{1+\frac{P_1}{P_0}\sin\omega t}\,dt$$

19.) If we ignore delayed neutrons, the P.R.K.E. become (for $Q(t)\equiv 0$)

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t)-1]\,P(t)$$

or

$$P(t) = P(0)\,e^{A(t)} \quad , \quad A(t) = \frac{\beta}{\Lambda}\int_0^t dt'\,[\$(t')-1]$$

i.) $\$(t) = k_0 + \gamma t, \quad A(t) = \frac{\beta}{\Lambda}\int_0^t dt'\,[k_0 - 1 + \gamma t'] = \frac{\beta}{\Lambda}(k_0-1)t + \frac{\beta}{\Lambda}t^2/2$

or

$$P(t) = P(0)\,e^{\frac{\beta}{\Lambda}(k_0-1)t}\,e^{\frac{\beta}{\Lambda}t^2/2}$$

ii.) $\$(t) = k_0 \sin\omega t, \quad A(t) = \frac{\beta}{\Lambda}\int_0^t dt'\,[k_0\sin\omega t' - 1] \quad \frac{\beta}{\Lambda}k_0\left[\frac{1-\cos\omega t}{\omega} - t\right]$

or

$$P(t) = P(0)\,e^{\frac{\beta}{\Lambda}\frac{k_0}{\omega}[1-\cos\omega t - \omega t]}$$

PROBLEM SET # 7

20.) Consider the point reactor kinetics equations in the one delayed neutron group approximation:

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t) - 1]\,P(t) + \lambda C(t)$$

$$\frac{dC}{dt} = P(t) - \lambda C(t)$$

Solve these equations for a ramp reactivity insertion $\$(t) = \gamma t$ for $t > 0$ using

a.) The constant delayed neutron production rate approximation.

b.) The prompt jump approximation.

(Assume the reactor is operating at a constant power level P_0 for $t < 0$ and assume the source term $Q(t)$ is zero.) Discuss the validity of each of these solutions, and sketch the solutions.

21.) Find the response of a critical reactor to a sinusoidal reactivity insertion $\$(t) = k_0 \sin \omega t$ using the prompt jump approximation and one delayed neutron group.

22.) Find the response of a critical reactor to a step reactivity input at $t = 0$ using the small amplitude approximation, and discuss the result, comparing it with the exact solution.

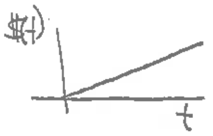
23.) "Linearize" the point reactor kinetic equations about an arbitrary reference power level P_r. Using this result, find the stationary (i.e. long time) response of a critical reactor at P_0 to a sinusoidal reactivity input $\$(t) = k_0 \sin \omega t$. Compare the two results obtained by choosing $P_r = P_0$ and $P_r = P_{av}$.

20.) Consider

$$\frac{\Delta}{\beta}\frac{dP}{dt} = [\$(t)-1]P(t) + \lambda C(t)$$

$$\frac{dC}{dt} = P(t) - \lambda C(t)$$

We want to solve these equations for $\$(t) = \gamma t$



a.) Constant delayed neutron production rate approximation

We assume $C(t) = C(0) = P(0)/\lambda$. Then

$$\frac{\Delta}{\beta}\frac{dP}{dt} = [\$(t)-1]\,P(t) + P(0)$$

Then integrating

$$P(t) = e^{A(t)}\left[P(0) + \int_0^t dt'\,P(0)\,e^{-A(t')}\right]$$

$$\text{where} \quad A(t) = \frac{\beta}{\Lambda}\int_0^t dt'[\gamma t - 1] = \frac{\beta}{\Lambda}\left[\gamma t^2/2 - t\right]$$

Hence

$$P(t) = P_0\,e^{\frac{\beta}{\Lambda}[\gamma t^2/2 - t]} + P_0\int_0^t dt'\,e^{\frac{\beta}{\Lambda}[\gamma t^2/2 - t]}\,e^{-\frac{\beta}{\Lambda}[\gamma t'^2/2 - t]}$$

b.) Prompt jump approximation

$$[1-\$(t)]\frac{dP}{dt} = [\dot{\$}(t) + \lambda\,\$(t)]\,P(t)$$

or

$$P(t) = P(0)\,e^{A(t)}$$

$$A(t) = \int_0^t d\tau\left[\frac{\dot{\$}(\tau) + \lambda\,\$(\tau)}{1-\$(\tau)}\right] = \int_0^t d\tau\left[\frac{\gamma + \lambda\gamma\tau}{1-\gamma\tau}\right]$$

$$= \frac{1}{\gamma}\int_{1-\gamma t}^{1} dt'\,\frac{\gamma + \lambda - \lambda t'}{t'} = -\frac{(\gamma+\lambda)}{\gamma}\ln(1-\gamma t) - \lambda t$$

Hence

$$P(t) = P(0)\, e^{-\lambda t}\left[1-\gamma t\right]^{-(1+\lambda/\gamma)}$$

Now we recall the constant delayed neutron production rate approximation is valid for short time intervals during which $C(t)$ remains essentially equal to $C(0)$ -- i.e. for $t << \frac{1}{\lambda}$. Moreover, the prompt jump approximation is valid for times $t >> \Lambda$.

21.) The P.R.K.E. with one delayed group in the prompt jump approximation becomes

$$\left[1-\$(t)\right]\frac{dP}{dt} = \left[\dot{\$}+\lambda \$\right]P(t)$$

Now for $\$(t) = k_0 \sin\omega t$, we can use

$$P(t) = P(0)\, e^{A(t)}$$

$$A(t) = k_0 \int_0^t d\tau \left[\frac{\omega\cos\omega\tau + \lambda\sin\omega\tau}{1-k_0\sin\omega\tau}\right]$$

Now

$$\int_0^t \frac{k_0\omega\cos\omega\tau\, d\tau}{1-k_0\sin\omega\tau} = -\ln\left(1-k_0\sin\omega\tau\right)\Big|_0^t = -\ln\left(1-k_0\sin\omega t\right)$$

$$\int_0^t \frac{k_0\lambda\sin\omega\tau\, d\tau}{1-k_0\sin\omega\tau} = -\lambda\left[\int_0^t 1\, d\tau \div \int_0^t \frac{d\tau}{1-k_0\sin\omega\tau}\right] = -\lambda t - \frac{\lambda}{\omega}\frac{1}{\sqrt{k_0^2-1}}\left\{\ln\frac{-\tan\frac{\omega t}{2}+k_0-\sqrt{k_0^2-1}}{-\tan\frac{\omega t}{2}+k_0+\sqrt{k_0^2-1}} - \ln\frac{k_0-\sqrt{k_0^2-1}}{k_0+\sqrt{k_0^2-1}}\right\}$$

Hence

$$\frac{P(t)}{P_0} = \frac{e^{-\lambda t}}{1-k_0\sin\omega t}\left\{\frac{\left[\tan\frac{\omega t}{2}-k_0-\sqrt{k_0^2-1}\right]\left[k_0-\sqrt{k_0^2-1}\right]}{\left[\tan\frac{\omega t}{2}-k_0+\sqrt{k_0^2-1}\right]\left[k_0+\sqrt{k_0^2-1}\right]}\right\}^{\frac{\lambda}{\omega\sqrt{k_0^2-1}}}$$

22.) Recall that the Laplace transform of the incremental power, $\tilde{p}(s)/P_0$, for the "linearized" P.R.K.E. for a critical reactor is given by

$$\tilde{p}(s)/P_0 = Z(s)\,\tilde{\$}(s) = \left[s\left(\frac{\Lambda}{\beta}+\sum_{i=1}^{6}\frac{a_i}{s+\lambda_i}\right)\right]^{-1}\tilde{\$}(s)$$

It is more convenient to work with

$$p(t) = P_0\int_0^t d\tau\, z(t-\tau)\,\$(\tau)$$

Now for a step input, $\$(\tau)=\$_0$, $\tau>0$, and we find

$$p(t) = P_0\$_0\int_0^t d\tau\left[\frac{\beta}{\Lambda}+\sum_{j=0}^{6}\frac{e^{\omega_j(t-\tau)}}{\omega_j\left[\frac{\Lambda}{\beta}+\sum_{i=1}^{6}\frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]}\right]$$

$$= P_0\$_0\frac{\beta}{\Lambda}t + P_0\$_0\sum_{j=0}^{6}\frac{(e^{\omega_j t}-1)}{\omega_j\left[\frac{\Lambda}{\beta}+\sum_{i=1}^{6}\frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]}$$

Hence

$$P(t) = P_0+p(t) = P_0+\$_0P_0\left[\frac{\beta}{\Lambda}t+\sum_{j=0}^{6}\frac{(e^{\omega_j t}-1)}{\omega_j\left[\frac{\Lambda}{\beta}+\sum_i\frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]}\right] \qquad Y(\omega_j)=0$$

But the exact solution of the P.R.K.E. for a step yielded

$$P(t) = \$_0P_0\sum_{j=0}^{6}\frac{e^{s_j t}}{\omega_j\left[\frac{\Lambda}{\beta}+\sum_i\frac{a_i\lambda_i}{(\omega_j+\lambda_j)^2}\right]} \qquad \text{where } \$_0-Y(s_j)=0.$$

Note that these solutions are only similar for very small times.

B.) Begin with

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = \$(t)\,P(t) + \int_0^\infty d\tau\, D(\tau)\left[P(t-\tau) - P(t)\right]$$

Now let $P(t) = \rho(t) + P_r$. Then linearizing

$$\frac{\Lambda}{\beta}\frac{d\rho}{dt} = \$(t)P_r + \cancel{\$(t)\rho(t)} + \int_0^\infty d\tau\, D(\tau)\left[\rho(t-\tau) - \rho(t)\right]$$

To solve, we Laplace transform, assume $P(0) = P_0 \Rightarrow \rho(0) = P_0 - P_r$

$$\frac{\Lambda}{\beta}\left[s\tilde{\rho}(s) - \rho(0)\right] = P_r\tilde{\$}(s) + \tilde{\rho}(s)\tilde{D}(s) - \tilde{\rho}(s)$$

or

$$\tilde{\rho}(s) = \frac{\frac{\Lambda}{\beta}\left[P_r - P_0\right] + P_r\tilde{\$}(s)}{Y(s)}$$

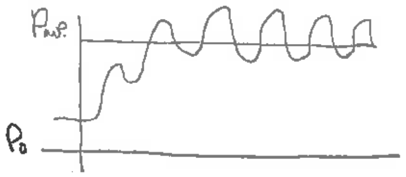
Now if $\$(t) = k_0 \sin\omega t$, $\tilde{\$}(s) = k_0\omega/s^2+\omega^2$. Then

$$\tilde{\rho}(s) = \frac{\frac{\Lambda}{\beta}\left[P_r - P_0\right]}{Y(s)} + \frac{P_r k_0 \omega}{Y(s)(s^2+\omega^2)} \qquad (*)$$

To determine the long time response, use the final value theorem

$$\lim_{t\to\infty} P_{ave}(t) = \lim_{s\to 0} s\rho(s) = \frac{\frac{\Lambda}{\beta}\left[P_r - P_0\right]}{\Lambda^*/\beta} + \frac{P_r\omega k_0}{\omega^2\Lambda^*/\beta} \qquad \frac{\Lambda^*}{\beta} \equiv \left(\frac{\Lambda}{\beta} + \sum_i \frac{a_i}{\lambda_i}\right)$$

$$= \frac{\Lambda}{\Lambda^*}\left[P_r - P_0\right] + \frac{P_r}{\omega}\frac{\beta}{\Lambda^*}k_0$$



Inverting (*) yields

$$\rho(t) = \frac{\Lambda}{\Lambda^*}\left[P_r - P_0\right] + \frac{P_r k_0}{\omega}\frac{\beta}{\Lambda^*} + \left\{\frac{\Lambda}{\beta}\left[P_r - P_0\right] + P_r\omega k_0\right\}\left\{\sum_{j=0}^{6'} \frac{e^{\omega_j t}}{\left[\frac{\Lambda}{\beta} + \sum_i \frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]}\right\}$$

$$+ P_r k_0 |Z(i\omega)| \sin(\omega t + \phi)$$

Now choose $P_r = P_0$

$$\rho(t) = \frac{P_0 k_0 \beta}{\omega\Lambda^*} + P_0\omega k_0\left\{\sum_{j=0}^{6} \frac{e^{\omega_j t}}{\left[\frac{\Lambda}{\beta} + \sum_i \frac{a_i\lambda_i}{(\omega_j+\lambda_i)^2}\right]}\right\} + P_0 k_0 |Z(i\omega)| \sin(\omega t + \phi)$$

Now recall $P_{av} = \frac{\Delta}{\Lambda^*}\left[P_r - P_0\right] + \frac{P_r \beta k_0}{\omega \lambda^*}$

Hence if we choose $P_r = P_{ave} = P_0 + P_{ave}$, we find

$$p(t) = \frac{\Delta}{\Lambda^*} P_{av} + (P_0 + P_{av}) \frac{k_0 \beta}{\omega \Lambda^*} + \left\{ \frac{\Delta}{\beta} P_{av} + (P_0 + P_{av}) \omega k_0 \right\} \left\{ \sum_{j=0}^{6} \frac{e^{\omega_j t}}{\left[\frac{\Delta}{\beta} + \sum_i \frac{a_i \lambda_i}{(\omega_j + \lambda_i)^2}\right]} \right\}$$

$$+ (P_0 + P_{av}) |Z(i\omega)| \sin(\omega t + \phi)$$

Hence for long times

$$p(t) = \frac{\Delta}{\Lambda^*} P_{av} + (P_0 + P_{av}) \frac{k_0 \beta}{\omega \Lambda^*} + (P_0 + P_{av}) |Z(i\omega) \sin(\omega t + \phi)$$

which is more convenient for observing oscillations about P_{ave}.

PROBLEM SET # 8

24.) Determine the transfer function of a subcritical reactor using one group of delayed neutrons, and present its Bode plot using $\$_0 = -10$ \$, $\beta = 7.5 \times 10^{-3}$, $\Lambda = 10^{-4}$ sec, and $\lambda = 0.1$ sec^{-1}. Indicate the low and high frequency break frequencies, and investigate the effect of the subcriticality considering the plot for $\$_0 = 0$. (Remember, to have a steady-state, non-zero power level P_0 in a subcritical reactor, you will need to consider the system sustained by a constant source S_0.)

25.) The transfer function of a reactor when the power is rising on a constant period $T = 1/\omega_0$ is given by

$$Z(s,\omega_0) = \left[s \left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6} \left(\frac{\lambda_i}{\lambda_i + \omega_0} \right) \left(\frac{a_i}{s + \lambda_i + \omega_0} \right) \right) \right]^{-1}$$

Using one group of delayed neutrons (as in problem 24), plot the Bode diagram for $\omega_0 = 0$ and $\omega_0 = 2$ sec^{-1}, and discuss the effect of the period on the mid-frequency gain, low and high frequency response.

24.) Again start from

$$\frac{\Lambda}{\beta}\frac{dP}{dt} = [\$(t)-1]P(t) + \int_0^\infty d\tau\, D(\tau)P(t-\tau) + \frac{\Lambda}{\beta}S_0$$

or if $P(t) = P_0 + p(t)$,

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = \$(t)\left[P_0 + p(t)\right] + \int_0^\infty d\tau\, D(\tau)\left[P(t-\tau) - p(t)\right] + \frac{\Lambda}{\beta}S_0 \qquad (*)$$

Now for a subcritical reactor with $k_0 < 0$, we find the source condition

$$0 = [k_0 - 1]P_0 + P_0 + \frac{\Lambda}{\beta}S_0 \Rightarrow k_0 P_0 = -\frac{\Lambda}{\beta}S_0$$

Then linearizing (*) for $\$(t) = k_0 + \delta\$(t)$, we find

$$\frac{\Lambda}{\beta}\frac{dp}{dt} = \cancel{k_0 P_0} + k_0 p(t) + \delta\$(t)P_0 + \int_0^\infty d\tau\left[p(t-\tau) - p(t)\right]D(\tau) + \cancel{\frac{\Lambda}{\beta}S_0}$$

Laplace transforming

$$\frac{\Lambda}{\beta}s\tilde{p}(s) = \delta\tilde{\$}(s)P_0 + \tilde{p}(s)\tilde{D}(s) - \tilde{p}(s)$$

or

$$\tilde{p}(s) = \frac{\delta\tilde{\$}(s)P_0}{s\frac{\Lambda}{\beta} - \tilde{D}(s) + 1 - k_0} = \frac{\delta\tilde{\$}(s)P_0}{s\left[\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\frac{a_i}{s+\lambda_i}\right] - k_0}$$

Now for one delayed group

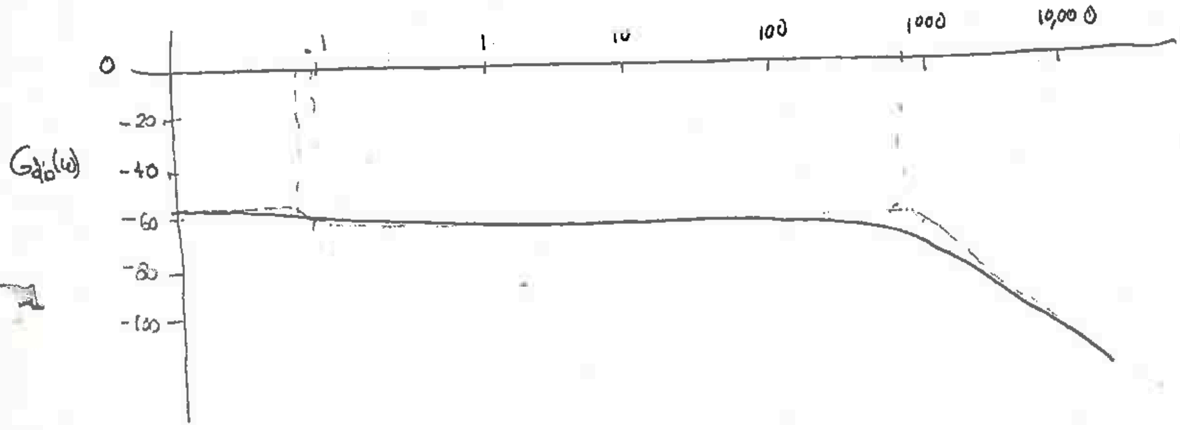
$$\tilde{p}(s)/P_0 = \frac{1}{s\left(\frac{\Lambda}{\beta} + \frac{1}{s+\lambda}\right) - k_0}\,\delta\tilde{\$}(s) \equiv Z(s)\,\delta\tilde{\$}(s)$$

Plugging in numbers we find

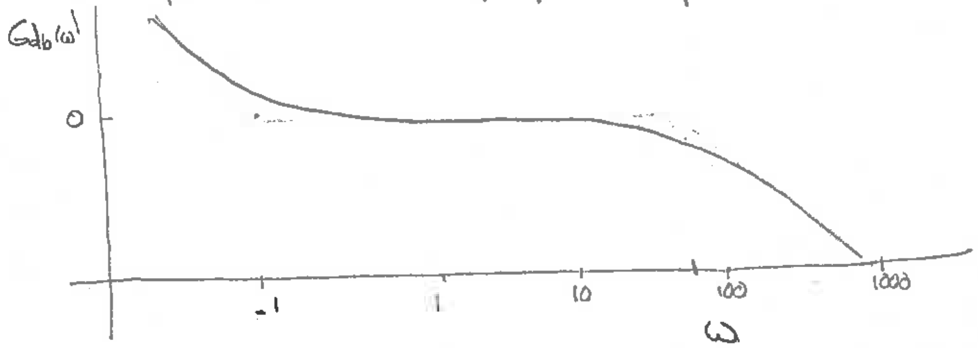
$$Z(s) = \frac{s+\lambda}{\left(\frac{\Lambda}{\beta}s^2 + (1-k_0)s - k_0\lambda\right)} = \frac{s+.1}{(s+825)(s+.09)}$$

Note as $s \to 0$, $Z(s) \to \frac{.1}{(825)(.09)} \Rightarrow G_{db}(0) = -61$ db

$s \to \infty$ $Z(s) \to 1/s$



This compares with the transfer function for a critical reactor [$\$_0 = 0$]



25.) We are given

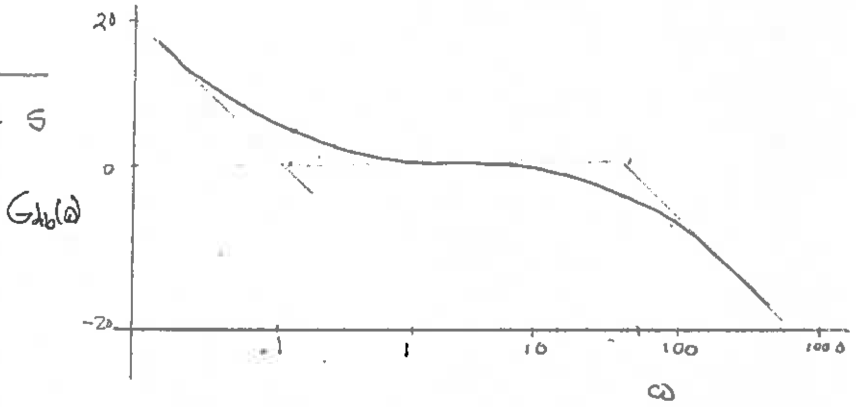
$$Z(s,\omega_0) = \left[s\left(\frac{\Lambda}{\beta} + \sum_{i=1}^{6}\left(\frac{\lambda_i}{\lambda_i+\omega_0}\right)\left(\frac{a_i}{s+\lambda_i+\omega_0}\right)\right)\right]^{-1}$$

$$\sim \left[s\left(\frac{\Lambda}{\beta} + \frac{\lambda}{(\lambda+\omega_0)(s+\lambda+\omega_0)}\right)\right]^{-1} = \frac{(\lambda+\omega_0)(s+\lambda+\omega_0)}{s\left[\frac{\Lambda}{\beta}(\lambda+\omega_0)(s+\lambda+\omega_0)+\lambda\right]}$$

For $\omega_0 = 0$, we just find the zero power transfer function of problem (25)

$$Z(s,\omega_0) = \frac{s+\lambda}{\frac{\Lambda}{\beta}s^2 + \frac{\Lambda}{\beta}\lambda s + s}$$

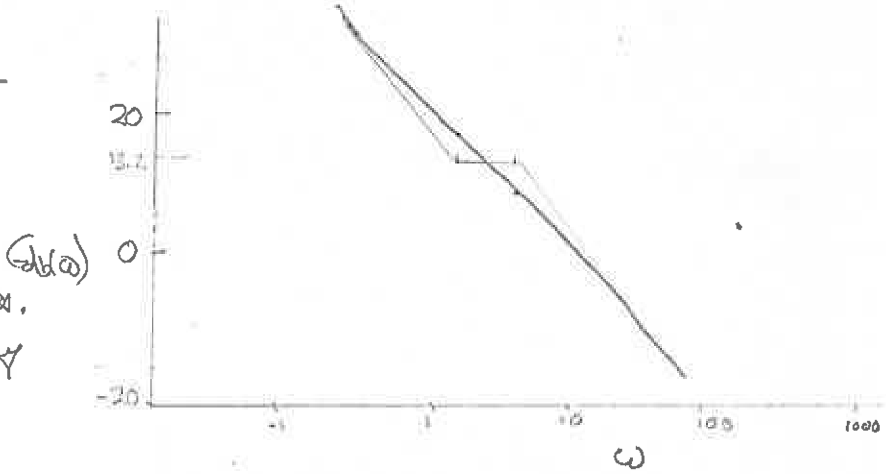
$$= \frac{75\,(s+.1)}{s\,(s+75)}$$



For $\omega_0 = 2\,\text{sec}^{-1}$

$$Z(s) = \frac{75.2\,(s+2.1)}{s\,(s+5.7)}$$

Note that ω_0 increases the gain for low frequencies, but decreases G_{db} for large frequencies. It also shortens the mid frequency plateau very considerably.



PROBLEM SET # 9

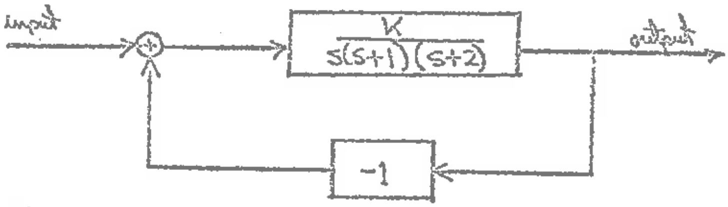
26.) Using the Routh-Hurwitz stability criterion,

i.) determine whether

$$L(s) = \frac{15}{s^3+2s^2+5s+15}$$

is stable.

ii.) Determine the values of K for which the system below is stable.



27.) Plot the Nyquist diagram for

$$\mathcal{L}(s) = \frac{P_0}{(s+1)(s+2)(s+3)} \qquad \text{for } P_o = 10 \text{ and } P_o = 100$$

Comment on the stability of each of these systems. For the case $P_o = 10$, determine the gain and phase stability margins.

28.) (optional) Let C be a simple closed contour in the z-plane, and let f(z) be analytic on and within C, except for poles at $b_1, \ldots, b_n$, none of which lie on C. Suppose further that f(z) has zeros at the points $a_1, \ldots a_m$ inside C, but does not vanish on C. Let

$$g(z) = f'(z) / f(z)$$

a.) If b_k is a pole of order s_k of f(z), show that

$$\operatorname*{Res}_{z=b_k} \{g(z)\} = -s_k$$

b.) If a_k is a zero of order r_k of f(z), show that

$$\operatorname*{Res}_{z=a_k} \{g(z)\} = +r_k$$

c.) Show that

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} r_k - \sum_{k=1}^{n} s_k$$

d.) If $[\arg f(z)]_C$ denotes the change in arg f(z) after one positive circuit around C, show that the result of part (c) can be interpreted as meaning

$$\frac{1}{2\pi} [\arg f(z)]_C = \text{\# of zeros of } f(z) \text{ inside C} - \text{\# of poles of } f(z) \text{ inside C}$$

(Poles and zeros are to be "counted" according to their orders.)

26.) i.) $L(s) = \dfrac{15}{s^3+2s^2+5s+15}$

Examine the roots of

$$s^3+2s^2+5s+15=0$$

Using the Routh-Hurwitz criteria

$$a_0 = 1 > 0, \quad \Delta_1 = a_1 = 2 > 0, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = 2\cdot5-15 \not> 0$$

Hence system is unstable.

ii.) For this system

$$L(s) = \frac{K}{s(s+1)(s+2)}\left[1+\frac{K}{s(s+1)(s+2)}\right]^{-1}$$

Hence we must consider roots of

$$s(s+1)(s+2)+K = s^3+3s^2+2s+K=0$$

Again applying the Routh-Hurwitz criterion

$$a_0=1>0, \quad \Delta_1=a_1=3>0, \quad \Delta_2=\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = 6-K>0$$

Hence we require $K<6$ for stability.

27.) We are to plot the Nyquist diagram for

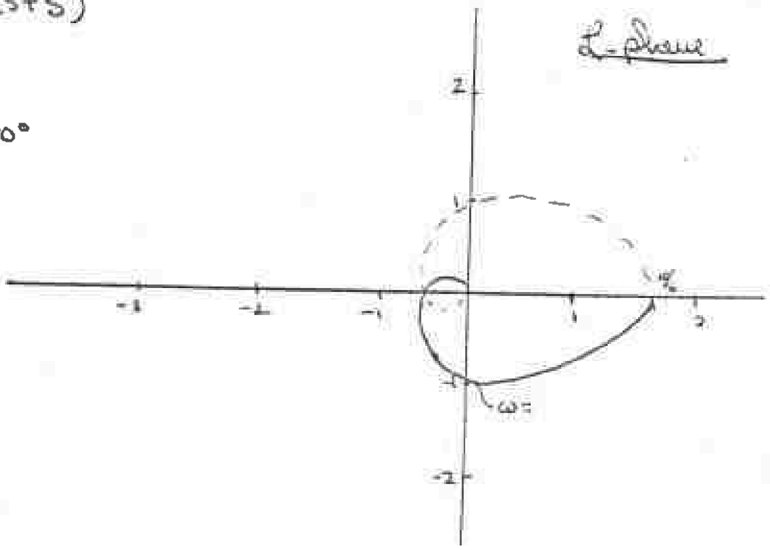
$$\mathcal{L}(s) = \frac{P_0}{(s+1)(s+2)(s+3)}$$

Note for $i\omega=0$, $\mathcal{L}(0) = P_0/6$

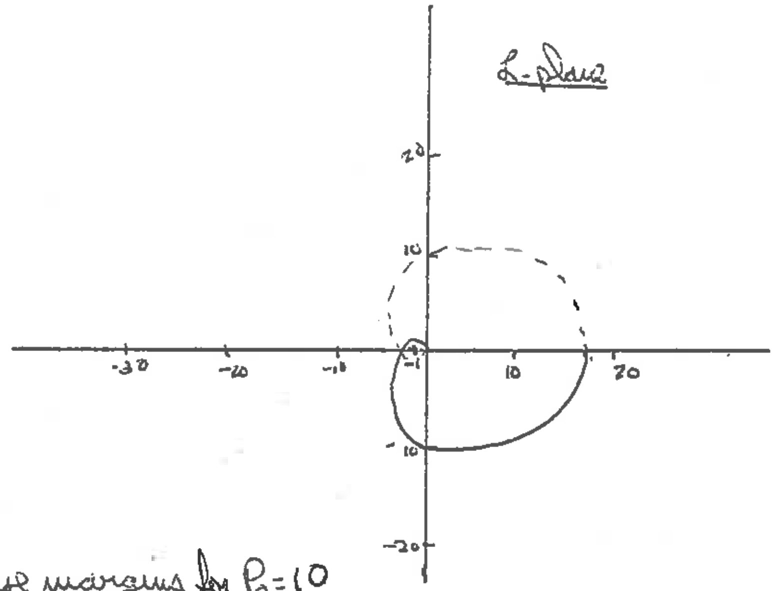
As $\omega\to\infty$, $\mathcal{L}(i\omega) \to 0\,e^{i270°}$

Hence the plot becomes for $P_0=10$

The system is evidently stable.



For $P_c = 100$, we find a scaling up by 10. Note this system is now unstable



One can estimate the gain and phase margins for $P_0 = 10$ either graphically or analytically.

$G_M = P_c - P_0$ But the frequency at which $\mathcal{L}$ crosses the negative real axis is $\omega = \sqrt{11}$. Then $\mathcal{L}(i\sqrt{11}) = P_c/6(1-11) = -1$
$\Rightarrow P_0 = 60$

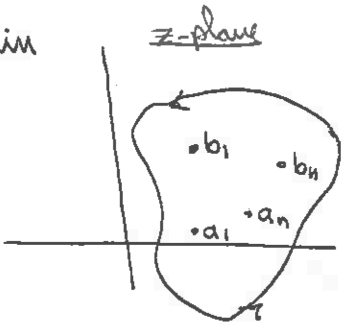
$\therefore \quad G_M = 60 - 10 = 50$

$\theta_M = 180° - 90° = 90°$ [easiest to do this graphically]

8.) Let C be a simple closed curve, $f(z)$ analytic on and within C except for poles $b_1, b_2, \cdots b_n$ and zeros $a_1, a_2, \cdots a_m$ in C, none of which lie on c.

Define

$$g(z) \triangleq f'(z)/f(z)$$



a.) To consider $\operatorname*{Res}_{z=b_k}\{g(z)\}$, expand $g(z)$ in a <u>Laurent</u> series about b_k. We do this first by expanding $f(z)$ in a neighborhood of b_k

$$f(z) = \frac{a_{s_k}}{(z-b_k)^{s_k}} + \frac{a_{(s_k-1)}}{(z-b_k)^{s_k-1}} + \cdots$$ where s_k is the order of the pole b_k.

Then

$$f'(z) = \frac{-s_k a_{s_k}}{(z-b_k)^{s_n+1}} - \frac{(+s_k-1)\, a_{(s_k-1)}}{(z-b_k)^{s_k}} + \cdots .$$

Hence

$$\frac{f'(z)}{f(z)} = \frac{-s_k}{(z-b_k)} + c_0 + c_1(z-b_k) + \cdots$$

But the leading term is just the $n=-1$ power term in a Laurent expansion of $g(z)$, and its coefficient is the residue

$$\operatorname*{Res}_{z=b_k} g(z) = -s_k$$

b.) Now expand $f(z)$ in a <u>Taylor</u> series about a_k

$$f(z) = a_{r_k}(z-a_k)^{r_k} + a_{r_k+1}(z-a_k)^{r_k+1} + \cdots$$

and

$$f'(z) = r_k a_{r_k}(z-a_k)^{r_k-1} + (r_k+1)\, a_{r_k+1}(z-a_k)^{r_k} + \cdots$$

Thus

$$\frac{f'(z)}{f(z)} = \frac{r_k}{(z-a_k)^{r_k}} + c_0' + c_1'(z-a_k) .$$

-- which again isolates the $n=-1$ term in a Laurent expansion of $g(z)$ about a_k. Thus

$$\operatorname*{Res}_{z=a_k} g(z) = r_k$$

c.) Now by the residue theorem

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \sum \text{Res}\{g(z)\} = \sum_{k=1}^{m} r_k - \sum_{k=1}^{n} s_k$$

d.) Now note $\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$

Then

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\log f(z)\Big|_C = \frac{1}{2\pi i}\Big[\underbrace{\log|f(z)|\Big|_C}_{\overset{\shortparallel}{0}} + i\arg f(z)\Big|_C\Big]$$

Hence

$$\frac{1\Delta}{2\pi}\arg f(z)\Big|_C = \sum_{k=1}^{m} r_k - \sum_{k=1}^{n} s_k = \#\text{ of zeros of } f \text{ in } C - \#\text{ of poles of } f \text{ in } C$$

PROBLEM SET # 10

29.) Study the following set of differential equations in the phase-plane

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x + y$$

Determine the singular points in the (x,y) phase-plane, and study the behavior of the system in the vicinity of these points by classifying the singularities as spiral points, saddle points, etc. Sketch in the phase-plane behavior.

30.) Repeat this analysis for

$$\frac{d\theta}{dt} = y$$

$$\frac{dy}{dt} = -k^2 \sin\theta$$

(Incidently, if you eliminate the variable y(t) in each of these equations, you will find that equation (29) is just the problem of a harmonic oscillator with negative damping, while (30) is a mathematical description of a simple pendulum of length ℓ such that $k^2 = g/\ell$.)

31.) Consider the system

$$\frac{dx}{dt} = y - x^3$$

$$\frac{dy}{dt} = -x^3$$

i.) Demonstrate that this system is <u>linearly unstable</u>.

ii.) Demonstrate, however, that it is <u>stable</u> in the nonlinear sense. (Hint: Verify that

$$V(x,y) = x^{[illegible]} + [illegible]\,y^2$$

is a Liapunov function.)

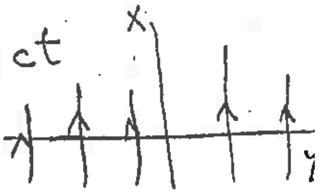
$$x^4 + 2y^2$$

31.) Consider $\frac{dx}{dt} = y - x^3$ $\frac{dy}{dt} = -x^3$

Hence $\frac{dx}{dy} = \frac{y-x^3}{-x^3} \Rightarrow$ $x=0, y=0$ is singular point.

In the linear approximation $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 0 \Rightarrow y = c \Rightarrow x = ct$

This is obviously unstable.



Consider $V(x,y) = x^4 + 2y^2$

i.) $V(0,0) = 0$

ii.) $V(\infty,\infty) \to \infty$

iii.) $V(x,y) > 0$ all x, y

iv.) $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = 4x^3\frac{dx}{dt} + 4y\frac{dy}{dt} = 4x^3(y-x^3) + 4y(-x^3)$

$= -4x^6 < 0$

Hence $V(x,y)$ is indeed a Liapunov function, and the system is absolutely stable.

$$\sin(\phi+2n\pi) = \sin(2n\pi) + \phi\cos 2n\pi + \cdots = \phi$$

Thus the linearized system becomes

$$\frac{d\phi}{dy} \sim \frac{y}{-k^2\phi} \Rightarrow a=0,\ b=1,\ c=-k^2,\ d=0$$

or $\Delta = -4k^2 < 0,\ p=0 \Rightarrow \Theta = 2n\pi,\ n=0,\pm1,\cdots$ are <u>all</u> vortex points

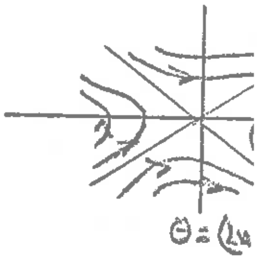
On the other hand, if we consider $\Theta = (2n+1)\pi,\ n=0,\pm1,\pm2\cdots$ and define

$$\varphi = \Theta - (2n+1)\pi,$$

$$\sin(\varphi+(2n+1)\pi) = \sin[(2n+1)\pi] + \varphi\cos(2n+1)\pi + \cdots = -\varphi$$

Hence $\frac{d\varphi}{dy} \sim \frac{y}{+k^2\varphi} \Rightarrow a=0,\ b=1,\ c=+k^2,\ d=0$

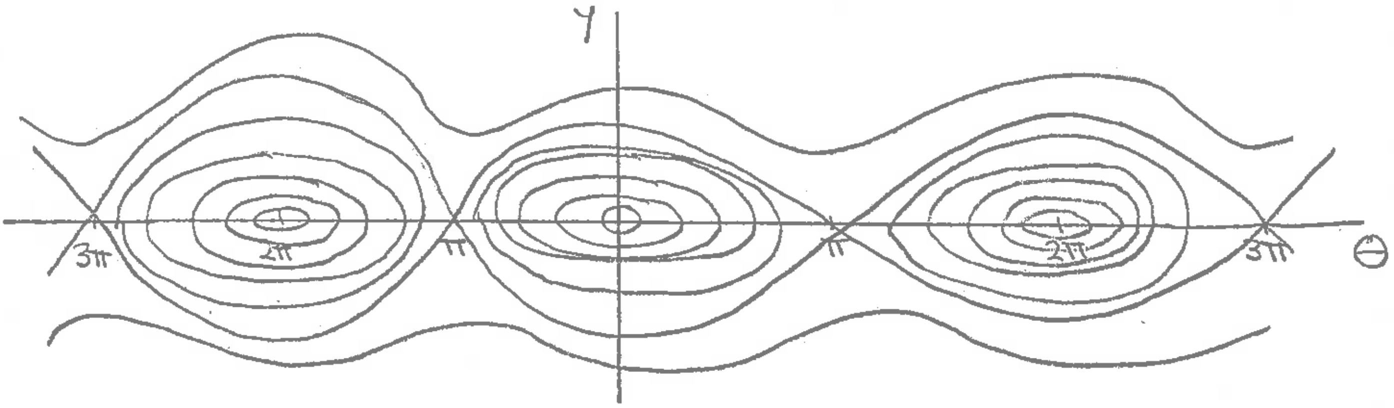
Now $\Delta = +4k^2 > 0,\ q = -k^2 < 0 \Rightarrow$ saddle point.



The asymptotes are given for large $y=\varphi$ by

$$\varphi \sim \frac{1}{k^2}y$$

Hence our phase plane has the indicated structure



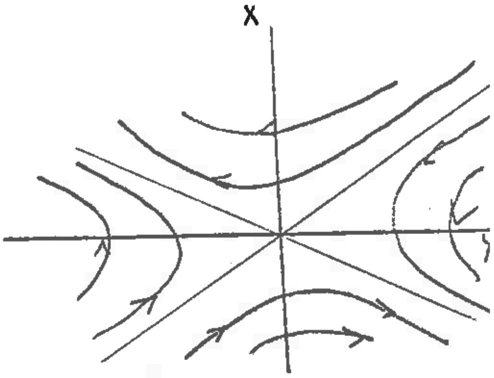
29.) Consider $\frac{dx}{dt}=y \qquad \frac{dy}{dt}=x+y$

Then $$\frac{dx}{dy}=\frac{y}{x+y}=\frac{ax+by}{cx+dy} \Rightarrow a=0, b=1, c=1, d=1.$$

Now $$\Delta=(a-d)^2+4bc = 1+4=5>0$$

$$p=-(a+d)=-1<0$$

$$q=(ad-bc)=-1<0$$



Hence we have a saddle point. To determine the asymptotes, note for large $x=y$

$$\frac{dx}{dy}\sim\frac{1}{2} \Rightarrow x=\tfrac{1}{2}y+c$$

30.) Now consider $\frac{d\theta}{dt}=y \qquad \frac{dy}{dt}=-k^2\sin\theta$

or

$$\frac{d\theta}{dy}=\frac{y}{-k^2\sin\theta}$$

First, note singular points are

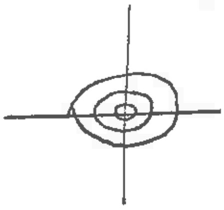
$y=0, \quad \theta=n\pi \qquad n=0,\pm1,\pm2,\ldots$

Consider first the point at the origin $y=0, \theta=0$. Then expanding

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\cdots$$

Hence $$\frac{d\theta}{dy}\simeq\frac{y}{-k^2\theta}=\frac{a\theta+by}{c\theta+dy} \Rightarrow a=0, b=1, c=-k^2, d=0$$

Thus $\Delta=-4k^2<0, \quad p=0 \Rightarrow$ the origin is a vortex point



In fact, consider all $\theta=2n\pi, \quad n=0,\pm1,\pm2.$ Then Make a variable transformation to origin by defining

$$\phi=\theta-2n\pi$$

POSSIBLE TERM PAPER TOPICS FOR N. E. 551

1.) Delayed neutron multiplication in fer-subcritical assemblies: A study of schemes used to determine the amount of fissionable material in reactor and fuel processing wastes (to be used in inspection procedures associated with the Nuclear Non-proliferation Treaty). Ref: Lee, et. al., Nuc. Sci. & Eng. 38, 114 (1969) *

2.) Pulsed neutron and delayed neutron measurements as nondestructive testing techniques. Ref: T. Gozani, Trans. Am. Nuc. Soc. 12, 258 (1969)

3.) Space-Time Synthesis methods and their application to spatially dependent reactor kinetics calculations. Ref: J. B. Yasinsky, Nuc. Sci. & Eng. 34, 158 (1968)

4.) Pulsed neutron experiments in small, highly enriched uranium spheres Ref: T. Gozani, Nuc. Sci. & Eng. 36, 143 (1969)

5.) Numericalsolutions of the point reactor kinetics equations. Ref: K. F. Hansen, et. al., Trans. Am. Nuc. Soc. 12, 616 (1969) , R. Goldstein and L. Shotkin, ibid, p. 622.

6.) Pulsed reactors for use as intense neutron sources. Ref: Asaoka and Larrimore, Trans. Am. Nucl. Soc. 12, 655 (1969)

7.) The kinetics of coupled core reactors. Ref: Umar and Ram, Trans. Am. Nuc. Soc. 12, 706 (1969)

8.) Xenon oscillations in large thermal reactors. Ref: Trans. Am. Nuc. Soc. 12, pp. 760 to 767 (1969)

9.) Sodium voiding and the calculation of sodium void coefficients in fast reactors. Ref: Trans. Am. Nuc. Soc. 12, pp. 904-913 (1969)

10.) Explosive disassemblyof fast reactor cores. Ref: J. C. Lee, Trans. Am. Nuc. Soc. 12, pp. 917-920 (1969)

11.) Pulsed neutron measurements in multiplying media. Ref: E. Garelis, Trans. Am. Nuc. Soc. 11, 40 (1968)

12.) Derivations of the point reactor kinetics equations. Ref: M. Becker, Nuc. Sci. & Eng. 31 , 458 (1968)

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*NOTE: These references are usually to the most recent work available. The references contained in these papers will probably be more useful.