EPISTEMOLOGICAL FOUNDATIONS

OF

FORMAL LOGIC

by

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INTRODUCTION

1. Statement of Main Problem.

This dissertation, as its title suggests, is concerned with the relation of formal logic to knowledge. Traditionally viewed as a science embodying the principles of valid inference, arising out of the critical analysis of reflective thinking, logic is now regarded as "the science of pure form...the general science of order" (Stebbing: A Modern Introduction to Logic, p. 476); in particular, owing to the fact that logical principles were found to be expressible in mathematical symbols, and that the formulae thus obtained could be handled like mathematical formulae for the solution of logical problems, the mathematization of logic has progressed to such an extent that pure mathematics is considered by many to be a branch of pure logic, and logic in its purest form is conceived as the science of abstract deductive systems.

A non-mathematician, already painfully aware of the difficulties attendant on the problem of knowledge and the many controversial issues connected with the relation between formal logic and truth, may well be pardoned for supposing that these developments in logic merely add to his difficulties instead of solving them, and it is by no means reassuring to be told that formal logic has nothing to do with reality. Even a mathematician whose familiarity with postulational technique and the intricacies of abstract deductive systems enables him to grasp the signigicance of these recent developments and even to carry them further may feel an occasional twinge of anxiety regarding the status of logic: seeing that the more mathematical the science of logic appears to be, the more does it appear to stand in need of the same kind of validation as is required for mathematics itself, and the more vitally does it seem to be affected by the current disputes about the foundations of mathematics.

It is obvious that a reinvestigation of the problem of knowledge, to say nothing of the problem of the foundations of mathematics, cannot be attempted in a single dissertation. The following pages represent a very modest endeavor to investigate some of the main principles involved in the construction of those abstract deductive systems which engage the attention of modern formal logicians; and though we venture to hope thus to shed some light on "the bearing of exact methods upon the simple problems of logic," which is still "a more pressing matter, at the present juncture, than the mere manipulation of the mathematical machinery" just as it was when Professor Lewis wrote these words in 1932 (Symbolic Logic, pp. 69-70), we may be pardoned for suggesting that the "business of assessing their precise significance for logic" is not so simple as his comment might lead one to suppose.

2. Method of Treatment.

The following discussion is deliberately conceived and carried on in the simplest terms at our disposal, with a minimum of technical language and on a basis of ordinary commonsense knowledge. This attempt to ensure clarity and intelligibility has led to the avoidance of terminology with which a non-mathematician may be unfamiliar; but it has also entailed a somewhat freer use of certain words than a mathematician would countenance. In particular, we must caution the reader that the word "system" is used in a non-mathematical sense, and also (except in a few easily identifiable passages) the word "set". If the almost complete absence of explicit reference to the works of other writers seems astonishing, it will appear less so in the light of the following observations: first, most of the matters selected for comment are so generally accepted as a part of modern logic that they receive more or less detailed treatment in the standard manuals, and it seemed more advisable to confine ourselves to such an accepted body of doctrine than to discuss variant opinions of individual writers, especially since the business of keeping abreast of contemporary changes of viewpoint is practically impossible; second, if the reader has any doubt about the accuracy of certain comments, and if he finds on reflection that these still persist, let such inaccuracies be attributed to the present writer rather than to any supposed source-material to which no reference has been given.

3. Bibliographical Note:

Below are listed the books which have been found most useful in connection with the preparation of this dissertation. Those marked with an asterisk were studied less carefully than the others, or not so extensively used.

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CHAPTER ONE

FORMAL LOGIC AND ABSTRACT SYSTEMS

1. Uninterpreted Abstract Systems.

The fact that formal logic is an abstract system has an important bearing, as will be seen, on its relations to epistemology. Hence it will be well to examine in what sense it is said to be abstract, and how it is related to other abstract systems. A system may be described, quite generally, as a collection or aggregate of elements arranged in a definite order; and since the notion of order entails the notion of relation, distinction is made between (a) the elements of a system and (b) the relations between these elements. Without attempting to analyze the notion of system more fully, we may remark that the above distinction always holds, even when the elements are themselves relations: as elements of a system, they are not the same as the relations which connect them with other elements, i.e. with one another. In an abstract system, it is usually if not always the case that either the elements or the relations or both are in some sense abstract entities; but apart from any consideration of the components of such a system, the word "abstract" in connection with it expresses a characteristic of the symbols employed to represent the system. A symbol, or group of symbols, is more or less abstract in proportion as it is susceptible of less specific or more specific interpretation. Thus, the symbol "3", inasmuch as it is interpreted as the symbol for a single positive integer, is less abstract than the algebraic symbol "a", which is interpretable as symbolizing indeterminately any positive integer, not to mention a still wider range of possible interpretations. Similarly, the symbol-group "a x b = c", as the expression of an algebraic equation, is more abstract than the symbol-group "2 x 3 = 6"; for the latter symbol-group, according to its ordinary interpretation, represents only one of the many different though similar interpretations of the former.

Just as the word "abstract" refers to symbols which are susceptible of different interpretations, so the phrase "abstract system" often refers, in current usage, to such groups of symbols as, when variously interpreted, represent different systems. Since it is possible to consider, and even to construct, an abstract symbolic system without reference to any particular interpretation of the symbols employed in its construction, attempts have been made to construct such a system without any reference whatever to its possible interpretations. Inasmuch as the most abstract system would be the system which is susceptible of the greatest possible number of interpretations, those who aim at the construction of a completely abstract system cannot entirely lose sight of the question of possible interpretations. In fact, it is precisely in order to ensure unlimited possibility of interpretation that they refuse to consider this question until the work of construction is finished, and meanwhile regard the symbols which they employ as meaningless but recognizable marks.

A system of this sort, which may be described quite nominally as an uninterpreted system, would presumably be constructed along lines suggested by the work of Carnap and others. Certain specifiable collocations of recognizable marks, selected and arranged according to arbitrarily-formulated "formation rules", would be known as "well-formed expressions", and only such expressions would be admitted into the system. Some of these would then be taken as "primitive", and further "transformation rules" would specify the conditions under which, by various permutations, these initially-chosen well-formed expressions could give rise to others equally well-formed. The system as a whole would thus be made up of the initially-chosen well-formed expressions and of all other well-formed expressions derived from them in accordance with the transformation rules.

As we shall see, there are grave reasons for supposing that a completely uninterpreted system, in the above sense, cannot be constructed from entirely meaningless marks; and it would be unfair to suggest, on the basis of an occasional reference to "a system which has no interpretation", that even the most extreme formalists are directing their efforts towards this ideal. We merely wish to note here certain consequences which would follow if the ideal abstract system were to be regarded as a completely uninterpreted set of meaningless marks put together in the fashion suggested above.

The utility of such a system, supposing its construction possible, could not be questioned on the ground that no actual interpretation for it had as yet been discovered: we should have to show that no interpretation could possibly be discovered. Again, the validity of such a system cannot be settled by the usual tests of consistency and coherence. That is, we could not say of any expression in the system that it was incompatible with or contradictory of any other expression, nor could we say of any two expressions that one did or did not follow from the other. On the purely syntactical level, the consistency of any expression would consist merely in its being well-formed, and the consistency of the system as a whole would consist in its being made up of none but well-formed expressions. Similarly, to say that such a system is coherent would merely mean that all expressions except those chosen as primitive were obtained from previous well-formed expressions by legitimate permutations. Hence any such system would be both consistent and coherent by the mere fact of having been constructed according to the arbitrary rules of formation and transformation governing its construction. Finally, we must notice that if formal logic be regarded as a completely uninterpreted system in this sense, the question of the validity of formal logic, or its "formal truth", is an entirely meaningless question; since, as we have seen, validity entails such considerations as mutual compatibility and strict deducibility, which cannot be settled on a basis of mere syntax or symbolism.

2. Analysis of the Notion of System.

If we are to inquire, then, into the validity of formal logic, it must be shown that formal logic cannot be regarded as a completely uninterpreted system such as we have described. An analysis of the notion of system may bring to light certain facts which have a bearing on this problem, and which may help to determine the precise sense in which formal logic is an abstract system. The preliminary description of a system as a collection of elements arranged in a definite order gives rise, as we have noted, to the conclusion that a system is made up of two different kinds of entities: (a) mutually-related elements, and (b) relations between these elements. Although, as was also noted, this distinction always holds in the case of any given system, it does not entail an absolute difference in kind between the entities involved. That is to say, not only may entities which are elements in one system be relations in another, but they may be the kind of entities which, apart from any systemic function, belong to the category of relations. Absolutely speaking, any kind of entities whatever may function as elements in a system; but not every kind of entities may function as relations.

What has been said so far of the notion of system is also applicable to the notion of class: for the members of a class are distinct elements, mutually related because of some likeness between them. The elements of a system, however, as distinct from the elements or members of a class, must not merely be somehow alike, in virtue of some property possessed by the elements either individually or as a group; it is further required that they be somehow ordered. Hence the relation between elements of a system must be what is called an ordering relation. Now, it is plain that mere likeness between two or more entities cannot be a basis for ordering them: that is, for determining the position of each element with respect to some other element. The fact that several entities are alike is no sufficient ground for determining which of them is before or after or next to any other in space or time or thought.

The somewhat loose usage of the word "order" in ordinary speech, to mean "any sort of arrangement or relative position of objects", makes it advisable to insist that the order characteristic of a system must be what is called "regular" order: that is, not only must each element in a system have a definite place, with respect to at least one another element, but the place of each element must be determined by some fixed principle or rule. In other words, the ordering relation of the elements of a system must be a constant relation, because any variation in it will involve a change in the structure of the system as a whole. Such a change need not mean that the elements no longer form a system, but it means that they form a different system.

The fact that a definite order of elements is essential to the notion of system may easily be lost sight of, for two reasons. First, as has just been remarked, elements may still form a system even though their order be changed. This consideration presents no serious difficulty if it be noted that their changed order is no less definite than their original order. Second, in the case of certain systems, it is by no means obvious what the order of the elements is, and hence we may readily suppose that they need not have a definite order at all. This is the case especially of complex organizations which are called systems in virtue of the fact that their elements all function somehow as means to a common end; for example, the post office system, every element of which functions as a means of securing the delivery of mail; or a railway system, each element of which contributes to a certain kind of transportation; or the circulatory system, in which each element plays a definite part in the circulation of the blood through the living organism. In systems such as these, possible variations in (for instance) the spatial order of the elements may not destroy the systematic character of the whole organization; but if the organization as a whole is really a system, not only must each element somehow be ordered as a means to the common end, but each must be definitely subordinated to, or coordinated with, at least one other element of the system.

The fact that the elements of a system must be ordered in a definite way enables us to understand in what sense the notion of determinism or necessity is essential to the notion of system. It cannot be maintained that some one particular or given order is necessary, in the sense that a group of elements cannot form a system unless they are ordered in this particular way and not possibly any other; this cannot be said in general, even though it might be true of some particular group or groups. Neither can we say, in general, that some one particular number of elements is necessary, in the sense that no new elements could be introduced into the system nor any taken away or replaced by others. But it must be maintained that the connection between the members of any group of elements which form a system is necessary in this sense: each element in the group must be connected to at least one other element in the group by that relation which is the ordering relation of the system. Otherwise there would be no warrant for the assertion that the position of every element in a system is determined by at least one other element.

The above analysis could hardly be carried farther without introducing characteristics which apply to systems of a definite kind rather than to the notion of system in general. At any rate, it will suffice to show that a system is not merely an aggregate of elements, but an ordered aggregate, in which the ordering relation is so fixed and definite that the position of each element is determined, with a kind of necessity, by at least one other element.

4. The Notion of Abstract System.

The phrase "abstract system", as we have already remarked, is currently used to mean not a system made up of abstract elements and abstract relations, but an ordered array of abstract symbols: the symbols being more or less abstract in proportion as they are susceptible of more or fewer different interpretations. We must now attempt to state more precisely what this means.

The fact that a system is composed of a number of elements, each of which is definitely related to at least one other element, i.e., is connected therewith in a definite way so that the group as a whole is regularly ordered, gives rise to the notion that a system as a whole has a definite form or shape or structure. Whenever we have to do with anything that has a definite structure, we can state more or less exactly what its structure is, by saying what are its component elements or parts, and how they are connected with one another, in such a way as to indicate the relative position of each element with respect to at least one other element. But it is also possible to represent structure more directly, in a graphic or pictorial fashion, by means of a model or plan or map. The representative force of such devices consists in this: that they themselves possess, and hence directly symbolize, certain characteristics of some other entity, which is at least numerically (or "individually") different from themselves. Their actual use as symbols is a matter of convention; but because of the characteristics which they possess, they are, as it were, "natural" symbols of whatever possesses those same characteristics. The most completely and directly representative symbol will accordingly be one which has all the characteristics of that which it symbolizes, save only those characteristics which constitute individual or numerical difference. It is not easy to think of a symbol which exactly fulfils this condition; but we can readily see that a map, for example, symbolizes all copies of itself which are made of the same material and have the same color, more directly and fully than it symbolizes copies of itself which are differently colored and made of different material. In the case of direct and completely representative symbols, a minimum of interpretation is required, because the connection between the symbols and what they symbolize depends on actual resemblance and hardly at all upon convention.

Whenever a directly representative symbol does not possess characteristics exactly similar to what it symbolizes, its symbolic force is a matter of convention, and the question of interpretation arises. It may even be that certain actual resemblances between that which is a symbol and that of which it is a symbol have no symbolic or representative significance, because of conventions governing the use of such objects as symbols. Thus, for instance, if a map is represented or symbolized by another map of exactly the same size and shape and color, the fact that they are made of the same material may be irrelevant, unless the material of one is being used symbolically: i.e. unless it is intended to represent the material of the other. In all cases of direct representation, therefore, at least this much interpretation is necessary: (a) we must determine what characteristics of the "symbol-object" are symbolic, or representative, or significant; and (b) we must notice what characteristics of the "symbolized object" are thus represented.

In the vast majority of cases where a system is directly represented or graphically symbolized, no attempt is made at complete representation: that is, we do not, and very often cannot, effect such representation by constructing another system which would be an exact replica or instance of the original. The aim is rather to represent the <u>structure</u> of a system, by means of an ordered array of objects, usually marks on paper, which has the same structure as the system in question. It is customary to speak of any ordered array of elements as "a system"; and in particular, an ordered array of symbolic marks is commonly called "a symbol-system", or "a system of symbols". To depart from this usage seems inadvisable, lest confusion arise: but it is far from clear that definite order, which is a necessary condition for a system, is also a sufficient condition. However this may be, it is clear that every ordered array has a definite structure; and hence there is no difficulty about representing the structure of a system by using, as a direct symbol thereof, an ordered array of elements which is similar in structure to the system which we wish to symbolize. For in such a case there is fulfilled that condition which we have noted as requisite for direct pictorial representation: namely, the possession, by the symbol, of the same characteristic as is possessed by what is symbolized.

It must, of course, be remembered that the ordered array is a direct symbol of the structure of the system, rather than of the system itself.

A comparatively simple instance of direct though not completely direct symbolization is the symbolization of a country map drawn on a plane surface. Here we have a case of two concrete objects, one of which is used as a symbol of the other in the following way: certain physical features or characteristics possessed by the one object, i.e. by the map, are intended to represent certain physical features of the other object, i.e. of the country. The map is a direct or "natural" symbol of all and only those characteristics of the country which it has itself: generally speaking, there will be only one such characteristic, namely, shape; hence any other features of the map which are intended to represent certain features of the country have symbolic force only by convention. It is true that the relative spatial position of dots on the map may accurately represent the relative spatial position of cities in the country; but it cannot be said that the dots and the cities possess the same spatial characteristics even relatively. Not only is the distance between one dot and another much less than that between the cities for which the dots stand, but the direction from one dot to another may be entirely different from the relative direction of the corresponding cities: for instance, a dot to the right of another dot usually stands for a city east of another city. To be quite accurate, we should remark that the relative distance between two dots is actually the same as the relative distance between the corresponding cities; for even if the scale of the map be unknown, so that we cannot infer the actual distance between two cities from the actual distance between the two corresponding dots, we know that if the map is drawn to scale,-- as indeed it must be, in order to be an accurate map,-- the actual distance between any two points on the map is a fixed or definite, even though unknown, fraction of the actual distance between the two corresponding points of the country. The actual sameness here is one of proportion; hence it may be said that the distances between any two points on the map are proportionately the same as the distances between two corresponding points of the country.

The fact that direct or graphic representation is possible only if and because a symbol-object possesses the characteristic which it symbolizes may easily be overlooked. For in the first place, even when physical features of a concrete symbol-object are used to symbolize directly the physical features of a concrete symbolized object, the dissimilarities between the symbol-object and the object symbolized are much more noticeable than their similarities, since the former are more numerous. Hence reflection and careful analysis is necessary in order to recognize <u>which</u> features of the symbol-object are directly symbolic. It is obviously easier to recognize as symbolic a feature which is naturally like to that feature which it symbolizes; thus, in the example given above, the shape of the map is naturally,-- i. e., apart from any convention,-- the same as the shape of the country. The comparative ease of recognition here depends on the fact that the feature in question, namely, shape, is readily perceptible in both cases, and hence their natural similarity is also readily perceptible. But when a directly symbolic feature is less readily perceptible owing to its complexity, the likeness between it and what it directly represents is not so easy to recognize. In the second place, when a physical or concrete object is used as the direct symbol of a non-physical or abstract object, we may be inclined to say that the likeness between symbol and what is symbolized is metaphorical rather than real; for here the two objects are so very different that they do not obviously have any characteristic in common. The real state of affairs is somewhat obscured by inexact use of language, as when it is said that a series of dots represents a series of numbers. On reflection, however, it is not difficult to perceive that every characteristic which is <u>directly</u> symbolized is actually possessed by the concrete object that is used as a symbol-object. In the case just mentioned, the symbol object, i.e., the set of dots, possessed the same characteristic as the symbolized object, i.e., the set of numbers: namely, the characteristic of "being arranged in such-and such a definite order"; and it is this characteristic which is directly symbolic. If we wish to be accurate in speaking of such symbolism, we should not say, for instance, that each dot directly represents one and only one number, but rather that the relative position of each dot directly represents the relative position of one and only one number. Again, a dot in the series of dots can be a direct symbol of a number in the series of numbers, because each dot and each number agree in having the characteristic of "being an element in a series".

The main points which we have been suggesting with regard to objects used as direct symbols are: (1) insofar as any object, such as a mark or set of marks, possesses some definite characteristic, either physical or non-physical, it can be used as a direct symbol of some other (i.e. at least numerically distinct) object which possesses that same characteristic. (2) The actual use of such an object as a direct symbol is entirely a matter of convention. There is a third point, concerning the abstractness of direct symbols, which may be noted in passing. Since the abstractness of any symbol consists in its capacity of symbolizing different objects, or of having many different interpretations, we see that a direct symbol can be abstract to this extent: it can symbolize an infinite number of numerically distinct entities, i.e., all those which can be thought of as possessing the same characteristics as itself. For example, a series of dots, inasmuch as it has the characteristic of "being a set of elements arranged in a definite order", can be a direct symbol of anything whatever which has this same characteristic,-- including, be it noted, the same definite order of arrangement; and there is no logical limit to the number of numerically-distinct objects which may have this characteristic.

It is hardly necessary to remark that not all symbolization is directly representative or graphic, as above described. Very often, if not in most cases, concrete objects such as marks or noises are used as symbols of other objects which have little or nothing in common with them. In this way the words of a language or the letters of an alphabet are used as symbols of objects which are neither words nor letters: thus, the word "horse" is often used to symbolize, or represent, or stand for, an animal of a certain kind; and a letter such as "a" is often used to symbolize a positive integer. This sort of symbolization, being non-pictorial,-- i.e., not directly representative or graphic,-- is not subject to the restriction concerning pictorial symbolization: namely, that the symbol-object must itself possess the characteristic or characteristics which it symbolizes. Non-pictorial symbols, therefore, can be much more abstract than pictorial ones; they can stand not only for all those entities which, while differing at least numerically from themselves and from one another, can be thought of as possessing the same characteristics as they themselves possess, but also for all those entities which do not possess those characteristics. In actual usage, of course, any such symbol will be subject to more or less definite restrictions; but the point is that these are entirely a matter of convention, whereas the restriction laid down for directly-representative symbolization is imposed by the very nature of such symbolization: for as we have seen, it presents itself on analysis as a necessary condition of pictorial or direct representation.

Now, when we come to consider any one of the various ordered arrays of recognizable marks which are actually used as abstract symbol-systems by formal logicians, it would seem at first sight as though the symbolization employed in these systems is not entirely direct or pictorial. The use of certain marks to stand for <u>elements</u> in the represented systems presents no difficulty; any mark which is an element in the ordered array can be used as a direct symbol of anything which possesses the same characteristic as itself, i.e., the characteristic of "being an element in an ordered array". The difficulty is rather this: relations between the elements of the represented systems, instead of being symbolized by relations between the marks of the ordered array, are symbolized by other marks in that same array; and since these marks, which stand for relations, have not the same characteristic which they represent, i.e., the characteristic of "being a relation between elements", it would seem that the relations between the elements of represented systems are not directly or pictorially symbolized.

On analysis, however, it becomes clear that the apparent force of this difficulty is due to a misunderstanding. Since none of the marks in any abstract system of symbols are themselves relations, we must indeed admit that no mark in such a system which is used to stand for a relation is a direct symbol of the relation for which it stands. Nevertheless, it would be a mistake to conclude that because a mark has not the characteristic of "being a relation" and hence cannot directly symbolize a relation, it therefore does not possess and hence cannot directly symbolize any characteristic of a relation. As a matter of fact, every mark in an ordered array of marks possesses the characteristic of "having a definite relative position", with respect to the other marks in the array; and since the same characteristic of "having a definite relative position" is possessed by every relation in a system,-- because a system is an ordered array of relations as well as an ordered array of elements,-- it is clear that the marks which are indirect symbols of relations can directly symbolize the relative position of the relations for which they stand, just as the marks which are direct symbols of elements can directly symbolize the relative position of the elements for which they stand.

The above considerations suggest a very important point in connection with the use of concrete objects as direct symbols. It must be remembered that such objects as marks have two kinds of characteristics: those which they possess when considered by themselves, i.e., as individual marks, and those which they possess when considered with reference to other objects, particularly such objects as are regarded to be only numerically distinct from, or other than, themselves, i.e., other individual marks. In order to avoid the many problems which are raised by the words "absolute" and "relative", we shall call the first of these kinds of characteristics "non-relational", and the second kind "relational". Since both these kinds of characteristics can be directly symbolic, both must be taken into account in determining whether or not a mark is being used as a direct symbol.

This point enables us to explain how it is that a set of marks, such as an ordered array, can be directly symbolic of characteristics which the marks in the set, considered individually, cannot directly symbolize. We have seen that each mark in an ordered array can directly symbolize the relative position of something else, inasmuch as it possesses the characteristic of "having a definite relative position". But no individual mark can directly symbolize a definite order, because it does not possess that characteristic,-- either in the sense of "having a definite order", or of "being a definite order". However, an ordered set or array of individual marks does possess this characteristic, and hence can directly symbolize a definite order,-- provided, of course, that the order thus symbolized is the same definite order. It will be observed that "having a definite order" is a non-relational characteristic of the set which possesses it, although the definite order of any set is necessarily connected with a relational characteristic of each of its elements: namely, the relative position of each individual element which is to some extent at least either the logical ground or the logical consequence of the order of the set as a whole.

We are now in a position to say more precisely what is meant by an abstract symbol-system. It is an ordered array of sensibly-perceptible objects, usually marks on paper. It is called a system not only because the marks are arranged in a definite order, in the sense that each mark has a definite spatial position and therefore definite relational characteristics of a spatial sort with respect to the other marks, but because some of the marks are arbitrarily considered as "relation-marks" and others are considered as "element-marks": hence the array is not only ordered, but is composed of elements and relations, after the manner of a system. When it is said that such an array is a symbol-system, or a system of symbols, this means that either the array as a whole, or its individual marks,-- either by themselves or in combination with other marks of the array,-- can be used symbolically, or used as a symbol. The alternatives here are _not_ mutually exclusive of one another. There is of course no reason why such an array, or any mark or set of marks within it, _must_ be used to symbolize anything _directly_ or pictorially, rather than _indirectly_; but since the aim of those who construct such systems is primarily pictorial representation, the somewhat lengthy analysis which we have made of direct or pictorial symbolization is especially relevant. Apart from _a priori_ considerations, these systems are constructed with the express intent of representing directly the structure of other systems which, notwithstanding their many mutual differences, agree with one another and with the symbol-system in having the same structure.

With regard to the abstractness of a symbol-system, we have already noted that a system is more or less abstract in proportion as it has a greater or smaller number of different interpretations. We have also seen that a symbol or system of symbols which is indirectly symbolic has greater abstractness than one which is directly symbolic. Instead of going further into the question of the abstractness and possible interpretations of symbol-systems in general, we shall consider in some detail the special case of an abstract system which is used as a direct symbol of the structure of at least one other system. Not only is this by far the most common sort of symbolization in formal logic, but also the principles governing abstractness and interpretation which reveal themselves as operative in this usage can easily be seen to hold in other cases of direct symbolization, either with no change at all or else with changes so slight and obvious as to need no comment here. Moreover, if our previous analysis is correct, the use of a symbol-system to symbolize directly the structure of other systems depends on the possession, by the symbol-system, of the same structure as is possessed by the systems which it is used to symbolize. In consequence, much of what is said in the following discussion of this sort of symbolization will have a direct bearing on the very important question of similarity of structure, or isomorphism, in general.

5. Abstract Systems as Direct Symbols of Structure.

In order to understand what is meant by saying that two or more systems have the same structure, it will be well to begin by analyzing the notion of structure. The structure of a physical object,-- i.e., of an object which can be thought of as part of the physical universe,-- is a complex characteristic which may be described as "having a definite spatial configuration". Note that in order to be called "physical", such an object need not actually exist. If I think of a house, for example, I am thinking of a physical object, and the structure of the house of which I am thinking is the physical characteristic of "having a definite spatial configuration". If the house of which I am thinking does not actually exist, then neither does that particular structure exist which the house would have if the house itself existed. Unless such objects as houses, and such characteristics as "having a definite spatial configuration," be called physical independently of whether they actually exist or not, we are apt to overlook the difference between those entities which can,-- under certain conditions,-- form part of the actual physical universe, and those which cannot do so under any conditions, e.g., the square root of minus one. Of course, we are not insisting that the word "physical" be used here to the exclusion of other possible words; the point is that some word is needed to cover this situation, and "physical" is perhaps less liable to be misunderstood in this sense.

Analogically, the word "structure" is used to mean a complex characteristic of non-physical objects, which may be described as "having a definite non-spatial configuration". The same sort of analogy is involved in the use of the word "configuration" to mean something non-spatial; and in order to see exactly what this analogy is and on what it is based, we must analyze the notion of physical structure more fully. Before doing so, we may remark that when there is question of non-physical objects or non-physical characteristics, the problem of their actual existence is quite different from the problem of the actual existence of physical objects and physical characteristics. In order to exist, non-physical entities need only to be *thought of*; for existence in thought is the only sort of existence which they can possibly have. It may be advisable also to notice that the word "structure", whether applied to physical or to non-physical objects, may mean either (a) something which *has* a structure, or (b) something which *is* a structure. Thus, a building such as a house is called "a structure"; and it is also said that a house *has* "structure", meaning that it has *a definite* structure. Whenever we have to do with such words, reference to the context in which they are used will ordinarily enable us to decide which of these two possible meanings is intended in a given case. In the present discussion, it should be clear that the word "structure" means the *characteristic* which *is* structure, and *not* something which *has* structure. Strictly speaking, the description we gave of structure as "the characteristic of *having* a definite configuration" is not quite accurate; for structure *is* a definite configuration,-- a characteristic which is *had by* something else, i.e. by something other than, or distinct from, itself, at least numerically. Hence it should rather be said that structure is the characteristic of "*being* a definite configuration".

When we come to reflect upon the notion of physical structure, or definite spatial configuration, we observe that whatever has structure must have parts: for spatial configuration is a matter of spatial arrangement of parts. Hence an object which has physical structure must be composite, not simple. We see, moreover, that physical structure involves (a) extension, or extendedness in space, and (b) a definite shape. The shape of any object which has physical inite; when an object is said to be "shapeless" or "amorphous", this does not mean that it has no shape at all, or that it has not a definite shape, but merely that its shape is too irregular to be readily perceived and exactly defined. The shape of an object as a whole obviously depends upon the spatial characteristics of its parts; hence if an object has a definite spatial configuration, this means that each part occupies a definite spatial position relative to the other parts, and that the parts, collectively considered, are arranged in a definite spatial order. What may be called the dimensional characteristics of the object, i.e., the distance and direction in which its parts extend, appears to affect structure only insofar as these characteristics affect the relative spatial position of the parts. In any case, two or more physical objects may have the same structure or shape even though they differ in size.

When it is used to mean a non-physical characteristic, the word "structure" means "a definite non-spatial configuration". In order that an object have structure in this sense, it must indeed have parts, but not spatially-extended parts; and it can be said to have a definite configuration only because of the following analogy. Just as definite spatial configuration depends upon the relative spatial position of each part and hence upon the spatial arrangement or order of all the parts collectively, so the words "definite configuration" may be used to mean that non-spatial characteristic which arises from definite but non-spatial relative position of individual parts, and definite but non-spatial order of the parts collectively considered. Once we realize that the characteristics of definite relati[illegible] order can be had by entities which are incapable of existing in either space or time, we can readily understand in what sense there is said to be a resemblance between physical structure and non-physical structure; and this resemblance is of course the basis of the analogical use of the word "structure" to mean a non-physical characteristic.

We are now in a position to make clear what is meant by similarity of structure, with special reference to the question of direct symbolism. It will be remembered that we have described non-physical structure as non-physical characteristic, rather than as a characteristic of non-physical objects. The reason for this is that although a non-physical object cannot have any physical characteristics, a physical object can have non-physical characteristics. What we have called "non-physical structure" might more accurately be spoken of as "a-physical structure"; and though we shall continue to make use of the former expression if only to avoid needless introduction of new terminology, the prefix "non-" is to be taken as a mere negative, with no positive opposite connotation. "Non-physical structure", then, means simply structure which does <u>not</u> depend upon physical (i.e. spatial) characteristics, and no reference is intended to those other characteristics on which it does depend: i.e., to the <u>positive</u> qualities of those characteristics, such as their being temporal rather than spatial. In other words, non-physical structure is not the contrary of physical structure, but a more general kind of structure, which arises whenever we have the following conditions fulfilled: (a) a composite object, either physical or non-physical (i.e. incapable of actual existence in the physical universe); (b) ...

relative position of each part, and hence a definite order of all the parts collectively, no matter whether this position and order be in space or in time or in thought. Thus, non-physical structure, notionally considered, is a generic notion, of which spatial structure and temporal structure and also structure which is <u>not</u> spatial and structure which is <u>not</u> temporal are <u>species</u>.

Although the above account of structure may need to be somewhat modified if it is to be accepted as an accurate and adequate explanation of the notion of structure in general, it should suffice to show that there is a sense in which the word "structure" may be applied in the same sense to a characteristic of <u>both</u> physical <u>and</u> non-physical objects,-- the latter being objects which cannot actually exist as part of the physical universe. In this sense, any object has a definite structure if the parts of which it is composed each occupy a definite position relative to the other parts, so that all the parts are arranged in a definite order. Two or more objects, such as systems, or ordered arrays, have the same structure if each of their parts occupies the same relative position with reference to other parts of the system in question, and if the order of parts in one system is the same as the order of parts in the other system.

Similarity of structure, or isomorphism, between two systems, is usually defined in such terms as the following. Remembering that a system is an array of elements standing in a definite order, and that a system is made up of (a) elements and (b) the relations between those elements, we may say that any two systems are isomorphic if for every element in one system there is one and only one corresponding element in the other system, and if the relation which holds between any two elements in one system has the same formal properties as the relation which holds between the corresponding elements in the other system. Because isomorphism is so frequently made use of in formal logic as well as in mathematics, it will be worth while to see as clearly as possible what this notion involves.

In the first place, the one-to-one correspondence of elements in isomorphic systems means that such systems must have the same number of elements. Considering each system as a class of elements, without any reference to definite order, we may say that these two classes of elements have the same cardinal number, or are cardinally similar. Secondly, since the definite order of the systems depends upon the definite relative position of each element, we see that in isomorphic systems, corresponding elements must occupy the same relative position, each in its own system. Moreover, insofar as relations between elements can be regarded as something distinct from the elements between which they hold, it is to be noted that there is a one-to-one correspondence between the relations of isomorphic systems as well as between the elements of such systems; hence the relations, as classes, are cardinally similar: and furthermore, corresponding relations must occupy, each in its own system, the same relative position.

We have already seen that the distinction, within a system or an ordered array, between (a) elements and (b) relations is emphasized by a distinction, on the symbolic level, between (a) element-marks and (b) relation-marks. The further stress laid upon this distinction in the above defininition of isomorphism, which contains an explicit reference to the formal properties of relations, may suggest that the distinction between elements and relations within a system is greater than it actually is. If our analysis of structure is correct, the structure of a system depends upon the definite relative position of each element, and hence upon the definite order of all the elements collectively. This suggests that what are called "relations" are rather relational characteristics of the elements; relative position being a relational characteristic of each element individually, and order being a relational characteristic of the elements collectively, and hence of the system as a whole. The same suggestion emerges from a consideration of the so-called formal properties of relations. It will be observed that every one of these is defined with reference not merely to the relations which are said to have these properties, but also to the terms between which the relations hold, i.e., the referents and the relata, respectively, of the relations. To mention a few examples: a relation is said to be "one-many" because it has one referent and many relata; a relation is said to be "symmetrical" because if it holds between a given referent and a given relatum, it also holds between that relatum and that referent; a relation is said to be "transitive" because if it holds between one term and another and between that other term and a third, it also holds between the first and the third of these terms. Thus it would seem that the formal properties of relations depend upon the relational characteristics of the terms between which these relations hold: that is, upon the relational characteristics of the elements of a system or ordered array. This view is quite compatible with the theory that not all relations are internal; for even if it be granted that the relational characteristics of elements are not all of them due to the nature of those elements, or to their non-relational characteristics, but are some of them due to the fact that an element, without undergoing any internal change, is brought into relation with some other element, it seems true nevertheless that at least one non-relational characteristic of such elements is presupposed: namely, an element must be such that it can be brought into relation with other elements, and so acquire relational characteristics, even though these latter do not affect it internally.

The application of isomorphism to the question of direct symbolism may be explained as follows. To say that two systems are isomorphic is to say that they have the same structure; hence if one of these systems is a symbol-system, it can be used to symbolize directly the structure of the other system, because it possesses the same characteristic that it symbolizes, namely, the same structure. The system ordinarily employed as symbol-systems are ordered arrays of marks on paper; each mark is a sensibly-perceptible (i.e. visible) object, and therefore a physical object, whose definite position relative to other marks is a spatial and therefore a physical characteristic; and the structure of the system as a whole is a physical characteristic, i.e. the definite spatial arrangement of all the marks. Now, even when we take into account the convention whereby some of these marks are arbitrarily used to symbolize relations and not elements, we cannot at once conclude that any physical characteristic of these marks (e.g. their relative spatial position) is directly symbolic. Such a characteristic could be used as a direct symbol of the same physical characteristic, which might be had by another set of marks or of any other physical object; in this usage, the symbol-object would be a map of the object symbolized. But as a matter of fact, even when both the symbol-system and the symbolized system are physical objects, and similar in physical structure, it is their similarity of non-physical structure which is symbolically important. And so we may sum up the situation thus: (a) The physical structure of the set of marks, being a sensibly-perceptible characteristic, furnishes the visibility needed for symbolic representation; (b) the significant characteristic, which is directly representative and which constitutes the <u>directly</u> symbolic force of the set of marks, is non-physical structure.

Because the abstractness of a system of symbols is greater in proportion as it can be used to symbolize directly a greater number of (at least numerically) different systems, the use of the non-physical structure rather than the physical structure of such systems, as their symbolic characteristic, manifestly increases their abstractness; for besides the apparent fact that the number of non-physical objects which have structure is greater than the number of physical objects, we have remarked that non-physical structure is a characteristic of both kinds of objects, physical and non-physical. Any set of elements, each of which occupies a definite relative position with respect to the other elements, so that all the elements together have a definite order, may be isomorphic with a system of symbols, no matter what be the nature of the elements or the nature of the relations between them: provided only that the conditions laid down in the definition of isomorphism be fulfilled.

A clearer understanding of what these conditions mean may be had from considering what changes in the elements and in the relations of a given system involve a change of structure. We have insisted that the characteristic of structure is always a definite characteristic, even though we may not be able to say, in the case of something which we recognize as a system, exactly what its definite structure is. This merely means that "to be a system" entails "to have a definite or particular structure", i.e. some definite structure; it does not mean that "to be a system" entails "to have this definite structure and not possibly any other". When, as in the case of symbol-systems, we are dealing with elements whose characteristics we can alter to some extent, and with systems whose characteristics largely depend upon the way in which we decide to construct them, it is very important to know beforehand, if possible, what changes with respect to the elements and relations which constitute a given system will alter the structure of that system: i.e., will give rise to another system of different structure. The analysis which we have made of isomorphism provides a basis for settling this matter. In the first place, any change in the number of the elements will affect the structure of any system. Two systems which have not the same number of elements cannot be isomorphic, because there will not be a one-to-one correspondence between the elements of these two systems. It should be observed that two systems may have the same number of elements even when both of them have an infinite number of elements; and in such a case, since there is no assignable limit to the number of elements in either system, we cannot say how many elements there are in each, for to do so would be to assign a limit to their number. (One case of this kind deserves special notice, since it is sometimes alleged as an exception to the traditional principle that "a whole is greater than any one of its parts". Consider the following two series: (a) the series of consecutive positive integers, -- 1,2,3,4,5,... and (b) the series of consecutive positive <u>odd</u> integers,-- 1,3.5.7.9,... Inasmuch as (a) and (b) are cardinally similar, they can be said to have the same number of elements; yet since (a) includes not only the odd numbers, i.e. all the elements of (b), but also all the even numbers, it is clear that the number of elements in (b) is only one-half as great as the number of elements in (a). Thus we seem to have a whole, (a), which is no greater than a part of itself, (b). On reflection, it will be seen that the anomaly here arises from what may be called the use of a double standard. Series (a) is said to have the same number of elements as series (b) on the assumption that there is no assignable limit to the number of elements in either series; and on the other hand, series (b) is said to have a lesser number of elements than (a) on the assumption that there <u>is</u> an assignable limit to the number of elements in both series. Unless the number of elements in (b) has an assignable limit, it is plain that the omission of the even numbers,-- i.e. their absence from (b),-- need make no difference. And in general, when it is said that a whole is greater than any one of its parts, the notion of "part" involves the notion of "having assignable limits" at least in principle.) In the second place, any change in the order of the elements will mean a change in the structure of the system or ordered array. Here we must deal with a difficulty which arises from our previous description of structure. We have analyzed structure (e.g., mutually entail each other, we must take care that the characteristic (e.g., y) which we examine to see whether it entails the other (e.g., x) is the same one (y) which we saw was entailed by the other (x). If this principle be kept in mind, the absence of mutual entailment in such cases as the above is not surprising. Insofar as the order "a,b,c,d,e" is entailed by the characteristic "next to", and not by any <u>other</u> relational characteristic of each element, there is absolutely no difference between this order and the order "e,d,c,b,a". (Note that "between" here means "next to" <u>two</u> elements.) The second order is actually different from the first: not because each element is next to at least one other element, but because each element which was "to the left of" some other element in the first order is "to the right of" that same element in the second order. Secondly, it will be noticed that the relative position of any single element in the system or array cannot be <u>uniquely</u> defined, with respect to any other single element, merely in terms of being "next to", or "in immediate contiguity with": for unless such an element is the first in the series,-- i.e., unless its relative position is already partially defined in terms of "ordinal number",-- a unique definition of its relative position involves a reference to <u>two</u> other elements, not just <u>one</u> element.

This point is of importance because it emphasizes the fact that "order", as a characteristic of a system or array of elements, involves a reference to the relative <u>positing</u> of the individual elements, as well as to their relative <u>position</u>. To put the matter in a somewhat clearer way: The definite order of the elements in a system or ordered array involves a reference to the relative position of each element,-- i.e. to cerpp. 24-25, above) in terms of (a) a definite relative position of each element with respect to at least one other element, and (b) a definite order or definite arrangement of all the elements collectively; and we remarked that there is a necessary nexus between the definite order of the elements collectively and the definite relative position of each element individually (p. 18, above). It is clear enough that a change in the relative position of any two elements involves a change in the order or arrangement of the array as a whole; but not every change in the order of elements as a whole involves a change in the relative position of even one individual element. Consider, for example, the first five letters of the alphabet, arranged in the order "a,b,c,d,e". We may describe this arrangement in terms of the relative position of each element, by saying that a is next to b, b is next to c, c is next to d, etc.; and that b is between a and c, c is between b and d, d is between c and e. Now if we consider the same five letters arranged thus: "e,d,c,b,a", although the order of the elements as a whole is different,-- because this latter order is the reverse of the former,-- we notice that no change takes place in the relative position of each element individually; it is still the case that a is next to b, etc., and that b is between a and c, etc., as above said. On reflection, however, it will be seen that this and similar examples do not force us to conclude that a definite relative position of the individual elements merely entails but is not entailed by a definite order of the elements as a whole. For in the first place, the presence or absence of mutual entailment can be determined only if we do not introduce a third characteristic: that is, in order to determine whether two characteristics tain relational characteristics of each element. It is of prime importance that no two elements have the <u>same</u> relative position. If they did, they would not only be indistinguishable, but also, as far as this particular system is concerned, they would be only <u>one element</u>, and not two. For, no matter what characteristics they have apart from the system,-- i.e. no matter what be their individual "nature" and no matter what differences there are between them from other points of view,-- all these characteristics are left out of account, and each of them is considered merely as "being an element in this (i.e. a given) system". Obviously, this characteristic is a characteristic which they have <u>in common with</u> each other (and with every element in the system); therefore it cannot be a basis of <u>difference</u> between them. "Being an element in this system" entails "having a definite relative position in this system"; and this latter may be described in terms of certain relational characteristics, as we have said: but however it be described, "having a definite relative position in this system" can differentiate any element from any other only insofar as the definite relative position in each case is a <u>different</u> relative position. To say that two elements in a given system have the same relative position is to say that there is no difference in the relational characteristics which each of them has in respect of the other elements in that system; hence, that they have exactly the <u>same</u> relational characteristics in respect of those other elements. Since, as we have seen, all other characteristics are left out of account, it becomes clear that "to have the same relational characteristics with respect to all other elements in a given system" entails "to be the same element in that given system"; and thus what appeared to be <u>two</u>

elements is seen to be <u>one and the same</u> element.

The reason why such a relational characteristic as "next to" does not <u>by itself</u> entail that the elements which have it <u>individually</u> possess <u>collectively</u> the characteristic of "a definite order" is therefore this: it does not entail that any two terms which have it occupy a <u>different</u> relative position; on the contrary, it does entail that those two terms have the <u>same</u> relative position, with respect to all other elements in the set or system. Since every term is next to some other term, it cannot be said of any term in the set that the relative position of <u>this</u> term is different from the relative position of every other term. But the very notion of "order" demands that each term have a definite relative position of its own,-- that is, that the relative position of each term be different from the relative position of every other term. If, then, some relational characteristic of the elements in a set is to be a basis for order, it must be such that each element which possesses it will, in consequence, have a definite position of its own relatively to the other elements,-- so that no two elements may occupy the same relative position with respect to the others.

Because a relational characteristic, or "relation", as it is usually called, such as "next to" does not fulfil this condition, it cannot serve to "generate an order", or cannot be "an ordering relation". Relations of this kind are said to be "symmetrical", inasmuch as they (so to speak) work both ways; they are unaffected by a change in the order of the terms between which they hold. Hence it is rightly maintained that only "asymmetrical" relations,-- i.e. those which no longer hold when the order of the terms is changed,-- can generate an order.

In declaring that a change of structure in any system can be effected only by a change in the number of elements or by a change in the order of the elements supposing their number is unchanged, we seem to be omitting all mention of the formal properties of relations, thus ignoring the condition laid down for isomorphism, that the relations between corresponding elements must have the same formal properties, or be similar. It would be easy enough to point out that any change in either the number or the order of elements entails a change in those formal properties. The reason why we have not given prominence to them is this: As has already been noted (pp. 26-27, above), there seems no more ground for saying that they are formal properties of <u>relations</u> than that they are formal properties of <u>relational characteristics</u> of <u>elements</u>. And in attempting an analysis of the notion of structure, we have found that certain characteristics of the <u>elements</u> of a system appear to be more fundamental than are any characteristics of the <u>relations</u>. Thus, a definite structure is seen, on analysis, to depend upon the number and the order of the <u>elements</u> of a system; and a definite order, which is a characteristic of all the elements collectively, is analyzable in terms of the definite and unique relative position of each <u>element</u> individually. Without minimizing the importance of relations in connection with the study of order, we see no reason for obscuring the importance of elements by emphasis upon relations exclusively. And if it be true that relations are really characteristics of elements, the importance of elements cannot be doubted.

If we now apply the results of the above discussion of structure and of isomorphism to the matter of abstract symbol-systems, the relevance of many details of our analysis will be more easily recognized. Any ordered array of recognizably-distinct marks on paper can be regarded as a system having a definite structure. Its definite physical structure will depend upon (a) the number of the marks, which are its elements, and (b) the spatial order or arrangement of those marks,-- that is, the definite relative position of each mark with respect to at least one other mark, in space. Its non-physical structure will depend upon (a) the number of elements, as before, and (b) the non-spatial order or arrangement of those elements. Here it will be observed that because spatial characteristics are left out of account, the definite relative position of each element cannot, in all cases, be uniquely determined without reference to the relative succession of elements: as we have seen, two elements which are "next to" each other will have the same relative position, if only their contiguity be considered. "Being next to in space" ensures a difference of relative position, for it is understood that no two distinct elements can have the same position in space. ("Relative position" here, as always when we have been discussing the elements of a system, means, when used of two elements, not "position relatively to each other", but "position relatively to some other (i.e. third element".) When space is left out of account in such cases, we can ensure a difference of relative position for each element only by taking account of the sequence of adjacent elements,-- i.e. by observing which of the two is before or after the other, in thought at least.

Any ordered array of marks can be used as a direct symbol of either the physical or the non-physical structure of any other system of elements,-- no matter <u>what</u> sort of objects or entities those elements are,-- which has the <u>same</u> structure; inasmuch as such an array fulfils the conditions governing direct symbolism (see, e.g. pp. 13-14, above).

If an ordered array of marks were to be used in this way, it would directly and primarily symbolize the physical structure of all physically-isomorphic systems, or else it would directly and analogically symbolize the non-physical structure of all non-physically-isomorphic systems,-- its use in either case being a matter of convention. Because of the analogy between the physical structure of any system and its non-physical structure (as explained on pp. 19-23, above), its physical structure can serve as a picture, or sensibly-perceptible direct symbol, of its own non-physical structure and of all other instances of that same non-physical structure in other systems.

When we consider the various sets of marks which are actually used as symbol-systems, it seems clear that their physical structure is not directly symbolic. For (a) not every mark is intended to be the symbol of an element, and (b) even when we take account only of such marks as are intended to be symbols of elements, the spatial order of all such marks collectively, which as we have seen is necessarily connected with the relative spatial position of each mark individually, is not intended as a symbol of a similar spatial order of elements. Not only do some of the marks stand for relations, but also the actual spatial relations between element-marks, which are a matter of the spatial position of each element-mark relative to the other element-marks, have no symbolic force. This, of course, is a matter of convention. If we reflect upon it, however, we can see that the adoption of this convention does not involve a departure from the use of physical structure as a direct symbol. It only means that the actual spatial position of each mark in such an array, relatively to the other marks, is not what gives rise to the actual physical structure of the system. To understand this somewhat difficult point, we must understand that the use of relation-marks and other non-element marks, instead of ruling out all reference to the spatial arrangement and order of the set of element-marks as a whole, merely means that this order is other than it appears to be: that is, the non-element marks indicate that the elements between which such marks stand have (in some cases at least) a different relative position than that indicated by their actual spatial position.

The reason for this convention, and the basis of it, is as follows. Theoretically, since the actual structure of an ordered array of marks depends upon their number and upon the relative position of each mark with respect to the others, we could construct as many systems of different structure as there are different ways of spatially ordering any number of recognizably-different marks. We might even use marks which were all of the same size and shape, in which case the only recognizable difference between each would be its different spatial position. In practice, however, this would be extremely difficult and complicated. It would likewise be inconvenient to represent all possible different relative positions by different spatial relative positions of the same marks, or the same number of marks. Consequently, having agreed that marks which have certain recognizable resemblances are to stand for elements in a system, we take other marks, recognizably different from these, to stand for relations between elements. This makes it possible to indicate the relative position of elements without direct reference to the relative spatial position of element-marks. But it should be noted that reference to spatial position is not entirely ruled out. One or two examples will serve to make this clear. In order to indicate that a given element, a, with reference to some other element, b, has the relational characteristic of "being subsequent to", I may assign to <u>a</u> a spatial position to the right of b,— or more accurately, to the right of the position occupied by b. In so doing, I am partially relying on the convention according to which a letter occurring to the right of another letter is understood to be <u>after</u> that letter. If I wish to indicate that same relational characteristic without making use of relative spatial position in this way, I may write these letters in the reverse order, with some symbol between them to indicate this characteristic, thus: a) b. This symbol indicates that although <u>a</u> is actually to the left of <u>b</u> in spatial position, it bears the same relation to <u>a</u> as though it had a spatial position to the right of <u>a</u>. Again, if I wish to indicate the possession by two or more elements of the same relational characteristics, instead of representing this directly by putting each in the same spatial position I may symbolize this relation by inserting between them such a mark as =/

An examination of other non-element marks would reveal the fact that each of them symbolizes indirectly what could be symbolized by the spatial relational characteristics of the element-marks themselves. Hence the introduction of such marks, instead of removing all reference to the physical structure of the system, is merely meant to show that its physical structure is not really what it would seem to be, judging from the actual spatial arrangement of the marks which are its elements.

Considering the way in which these non-element marks are actually employed by those who construct abstract symbol-systems, and the way in which they are actually interpreted by anyone who comes upon them in such a system, we must admit that they are actually used and actually interpreted without explicit reference to spatial relations. Hence it would be simpler to say that when these marks are used as above described, the spatial relations between the marks of the array in which they occur are without symbolic significance. This is true enough. But we have emphasized the idea that the non-spatial relational characteristics which these marks <u>indirectly</u> symbolize are <u>only</u> such as could be <u>directly</u> symbolized by a set composed exclusively of element-marks, which by suitable variations in their number and order,-- i.e. by variations in physical structure,-- could directly represent all possible variations of non-physical structure: because it is important to notice that the use of these non-element marks is rather a matter of convenience than of logical necessity. In a word, the introduction of these marks affects the way in which certain characteristics are symbolized: i.e. the characteristics which could otherwise be directly symbolized are now symbolized indirectly; but they are the very same characteristics in spite of the difference in the method of symbolization.

6. Formal Logic as the Science of System-Structure.

We are now in a position to make some preliminary observations concerning the status of formal logic as the science of system-structure. From this point of view, its aim is to discover the various ways in which any number of entities, no matter what be their non-relational characteristics, can be regularly arranged or ordered, so as to form systems of different structure; and further, to determine the necessary and sufficient conditions for a given structure, with special reference to the relational characteristics of,-- i.e. to the relations between,-- the elements in the system which has that particular structure. Since this inquiry is carried on empirically, by actually ordering certain objects in various ways and seeing under what conditions a given structure arises, the elements in such systems are generally marks, which can be most easily manipulated. For reasons of convenience, some of these marks are selected to stand for elements, while others are taken to stand for relations between elements, or to indicate that a certain set of marks stands for a single element in the system. No reference is intended to the <u>meaning</u> of these marks, apart from the conventions governing their usage in the system as element-marks or non-element-marks respectively. A study of these systems shows that the structure of a system depends on the number and the order of its elements, inasmuch as any change in either of these involves a change of structure; and the order depends on the formal properties of the relations between the elements, for the same reason: i.e. a change in these formal properties involves a change of order. If, having constructed such a system, it is found that another set of objects which are not marks has the same structure, we can at once conclude that this set is a system which has the same number of elements, wherein the relations between the elements have the same formal properties. The significance of this conclusion will be more fully understood when we have discussed the way in which abstract systems are developed. Here we can only say in general that, if our previous analysis is correct, the structural properties of elements in isomorphic systems are the same. These symbol-systems, as we have tried to explain in our discussion of symbolization, are symbols of the structure of all other systems which are isomorphic with them. Considered as direct symbols, according to the view that the systems of formal logic are to be pictorially representative, they symbolize only the characteristics which they have; and this means, as we have seen, that they are symbols only of the structure of the systems which are isomorphic with them.

To what extent such abstract systems can be developed without reference to the "meaning" of the symbols which constitute them, and what is their relation to knowledge, may be more clearly seen by examining the actual methods according to which such a symbol-system is developed or derived. This we shall proceed to do, in the following chapter.

CHAPTER TWO

DEVELOPMENT AND INTERPRETATION OF ABSTRACT SYSTEMS

1. Symbols, Meaning, Interpretation.

We have already referred to certain expressions used by writers on modern logic which, if taken literally, indicate that the marks employed as symbols in abstract symbol-systems are regarded as meaningless but recognizable marks, and that the set of marks which is an abstract system may have no actual interpretation. Precisely what is meant by such expressions, it is the purpose of this chapter to make clear. We may begin with a few reflections on the notion of symbol, which are to some extent suggested by our previous discussion of symbolization.

To say that a sensibly-perceptible object, such as a recognizable mark, is a symbol, is to say that it is used significantly, i.e. used as a sign of something. The word "something" here means, "whatever can be thought, or thought of, as in any sense one": hence it includes whatever can be called "an entity", whether that entity be positive or negative, simple or composite, abstract or concrete, actual or possible. In this sense, even an impossible "entity" is "something". Unless we wish to maintain that "to be understood" is in some cases compatible with "to have no meaning", we ought not to dismiss as meaningless such expressions as "square circle", though it may be difficult to say exactly what their meaning is. The phrase "square circle", for instance, symbolizes "something" quite definite: the combination, in thought, of the characteristics "being square" and "being a circle"; and the fact that such combinations are (to borrow a word from chemistry) too "unstable" to be more than instantaneous in duration should not lead us to overlook them entirely. At first sight, it would appear that to call a square circle "something" is contrary to the general meaning of "something" which we have defined; the characteristics "being square" and "being a circle", it will be said, do not answer to the description, "whatever can be thought as <u>one</u>", because they are obviously incompatible. On reflection, however, it seems clear that we recognize their incompatibility only by bringing them together in thought; hence, unless they can be brought together in thought, or thought as <u>one</u> in some fashion, we have no ground for saying that they are incompatible. This paradox,-- that logical incompatibility presupposes compatibility,-- arises from our psychological limitations, and may therefore be regarded as irrelevant to logic. But anyone who attempts to discuss the relation between logic and knowledge must take our psychological limitations into account to some extent, unless it be assumed that logic has nothing to do with the sort of knowledge which is conditioned by these limitations.

Much of the confusion which is notoriously attendant upon "the meaning of meaning" arises from a failure to distinguish between two quite different questions: (a) To what does a given symbol refer? (b) What <u>is</u> that "something" to which a given symbol refers? The fundamental importance of this distinction will be better understood, and its precise significance more fully realized, in connection with a problem to be dealt with later on: namely, the very similar distinction that should be made between (a) the explicit content and (b) the implicit content of a thought or an assertion. We mention this point at present merely to warn the reader that what is said here about the meaning of symbols is based to some extent on analyses of thought set forth in subsequent chapters, although we shall confine ourselves, as far as possible, to considerations which do not anticipate the results of later analyses.

To understand a symbol is not so much a matter of knowing what the "something" _is_ to which the symbol refers, but rather, of recognizing the symbol as a sign of that "something". The conditions under which certain marks are used as signs,-- that is, the conventions according to which they are employed in a given language,-- enable us not only to recognize these marks as signs, but also to recognize, more or less conjecturally, what they signify. Unless the user of a mark makes plain his intention of departing from these conventions, either by explicit declaration of this intention or by obvious departure from the conventions in his usage, we assume that these conventions are being followed. Often a set of marks is dismissed as meaningless either because (a) it does not obviously accord with these conventions, or because (b) it is used in a way which obviously violates them. Whenever the words "meaningless" or "nonsense" are applied to symbols, it is important to know on what ground they are so applied. An example or two will illustrate this point. We recognize certain marks as "letters", because they are accepted as linguistic units from which other units called "words" are constructed. The possible meanings of such marks are numerous; the English letter "a", for example, is a possible meaning of the mark "a", but this can also mean,-- to mention only a few other possibilities,-- (1) the sound of the English letter "a", which again is something variable, (2) the word "a", which grammarians call "the indefinite article", (3) what is meant by the word "a"; and this, as will be seen, includes more than one possibility. Again, certain combinations of letters are not recognizable as words: e.g., the combination "brif", which is not currently accepted by English-speaking people. Each mark symbolizes a letter, but the set of marks does not symbolize a word. Similarly, certain combinations of words are accepted as grammatical or syntactical units, called "phrases" or "clauses" or "sentences" according to their respective grammatical functions and properties; and a combination of words is often called meaningless because it cannot be recognized as having syntactical unity. Thus the combination "pink of not accordingly", though made up of recognizable words, cannot be understood as a unified phrase.

From what has been said about the possible meanings of a mark or set of marks used symbolically, two important consequences follow. First, when the "something" symbolized is itself an accepted symbol of something (as is the case in every meaning of the mark "a" above mentioned, except that numbered (3), we must ascertain from the context or from previous knowledge whether the initial symbol is used to refer to what is meant by this latter symbol. Second, when there is question of a set or sets of marks, its symbolic use as a grammatical or syntactical unit determines its meaning, and not the symbolic use of individual marks, or combinations of marks, within that set. Unless the combination as a whole is a recognizable grammatical unit (word, phrase, clause, or sentence), it will have no meaning as a whole, and therefore not be a symbol, unless explicit rules indicate how it is being used.

Since the actual meaning of a symbol is entirely a matter of convention (with the single exception already noted regarding the use of direct or pictorial symbols, pp. 13-14, above), it would be much clearer to settle the question of whether a symbol is meaningless on a basis of mere convention. On this basis, a mark or set of marks would be dismissed as meaningless, if and only if (a) it were not the sign of an accepted linguistic unit when considered as a whole, and (b) no conventions governing its usage were either explicitly laid down or discoverable from consideration of its usage. According to condition (a), a mark would be meaningless if it were not a recognized letter. (Since we are confining ourselves to the limits of a given language, we need not take account of symbols such as numbers, which are common to many different languages). Similarly, a combination of letters would be meaningless if it did not form a word; and a combination of words would have meaning only insofar as it formed a recognizable syntactical unit, such as a phrase or a sentence. If this suggestion were to be adopted, we should not be likely to overlook the difference between such completely unintelligible expressions as the examples given above (p. 46), on the one hand, and such intelligible sentences as "Blue is musical", or "Caesar is a prime number", or "paradoxes vote for walls", on the other. These latter expressions are recognizably false, to the point of being absurd or nonsensical; but they do have meaning, whereas the former expressions do not. The latter symbolize a definite thought-complex, even though that complex is dismissed as impossible the moment we consider it attentively; the former are not recognizable symbols, and can be called nonsensical not because they are false to the point of absurdity, but because they have no meaning whatever.

Although it is a matter of convention whether or not any mark or combination of marks be accepted as a recognized instrument for symbolizing "something", and also a matter of convention whether the same mark or marks may be used to symbolize different "somethings" in different usages (e.g., different contexts), it is misleading to assert that the same marks "must be counted as belonging to different symbols". Thus Black (pp. 27-28): "...the same mark, if used with different meanings (e.g. vice, a carpenter's tool, and vice, for which sinners are punished), is said to express different words...the copula in This is green is not the same symbol as that in Green is a colour, and both differ from the is in A man is not a woman" (italics his; see also Stebbing, p. 21). That such a view is likely to lead to confusion may be readily seen without discussing its implications in detail. Once a set of marks is accepted as a recognizable symbolic unit, we may say that this set of marks is the sign of a word, or that it expresses a word. But it is equally true to say that that same set of marks, as an accepted symbolic unit, is a sign of many different meanings: i.e., a sign of many different "somethings" which are meant by words, but which are not themselves words. The view that the same set of marks belongs to different symbols, or expresses different words, when and because it expresses different meanings is likely to suggest that no distinction ought to be made between (a) a word and (b) the meaning of a word. We have already noticed the convention according to which any word (or indeed, any symbol whatever) can be used to mean itself (pp. 45-47, above); and for this reason we can truly assert that every word is the meaning of a word: i.e., it is one of its own possible meanings. But the converse is not true: i.e., we cannot truly assert that every possible meaning of a word is a word. Therefore, it must be maintained that some distinction is to be made between words and meanings, even though we are not yet in a position to say precisely what that distinction is. That it is not entirely a matter of convention seems fairly clear; for in order to be used as a symbol (verbal or non-verbal), an object must be sensibly perceptible, e.g., visible, and it is only in a figurative sense that we are said to "see" the meaning of a symbol.

A more thorough discussion of "meaning" cannot be attempted here without anticipating what must be said later about classification and definition. It may be possible, however, to clarify at least in part some special points arising from the use of abstract symbols. We may say at once that the following comments are intended to cover only those cases wherein symbols are used to signify non-symbols, or "meanings" in the strict sense. Hence no explicit reference is made to such uses as "'Horse' is a word of five letters", or "'Crow means a kind of bird"; although much of what is said might be applied to these uses also.

Since we shall have to say something later about intension and extension, it will be advisable to explain abstract symbols and their meaning with reference to connotation and denotation; for these latter notions are very similar to the two former, as will be seen. It is commonly maintained that "purely demonstrative" symbols, or "logically proper names", such as the word "this", have no connotation: i.e., they merely denote, but do not signify any characteristic of the object denoted. And on the other hand, it is held that some "descriptive" symbols, or descriptive phrases, such as "the present King of France", merely signify characteristics, but do not denote anything: i.e. they have no denotation. Without attempting to discuss this twofold contention directly, we may observe that its truth is far from evident. The word "this", for example, not merely denotes something, but signifies explicitly "something characterized by 'thisness'",--something thought of, or (from the reader's or hearer's point of view) to be thought of, as "being this". The fact that the characteristic signified is unanalyzed, and perhaps unanalyzable, does not indicate that <u>no</u> characteristic is signified. Again, such descriptions as the above have denotation as well as connotation; for, as we have seen (p. 44, above), even the phrase "square circle" denotes "something", although <u>what</u> it denotes has merely "mental" existence of a very transient sort. It will be observed that the phrases used by way of example signify not merely characteristics in the strict sense, i.e. something to be thought of as an attribute or predicate, but also something to be thought of as <u>having</u>, or as characterized by, these signified characteristics. Lest it be supposed that only such phrases have denotation, and that those which signify merely characteristics do not have denotation, we should notice that these latter have denotation also, though in a slightly different sense. Phrases like "square circularity", or "circular squareness", or "the present kingship of France", do not, like the former phrases, denote a combination of characteristics as belonging to something: but they do denote "something", i.e., a combination of characteristics to be thought of; without any reference whatever either to (a) the possibility that this combination might belong to something, or to (b) any possible "something" whereto it might belong.

At this point a difficulty suggests itself: How can these latter phrases have any connotation? We have insisted that they signify nothing but characteristics; and by further insisting that they denote these characteristics, we seem to have left nothing which they can possibly be said to connote. Hence arises the suspicion that we have simply used the word "denotation" to mean what is ordinarily meany by "connotation", and that our main contention, namely that every symbol has both connotation and denotation, is not entirely supported by facts. A somewhat closer consideration of this difficulty will, it is hoped, show that this suspicion is unfounded.

It is a well-known fact that in any well-developed language such as English, certain linguistic forms are accepted and recognized as "concrete", and others as "abstract." We need not pause to discuss which forms are thus distinguished, nor the criteria according to which they are recognized as concrete or abstract, respectively. It is sufficient to note (a) that in the examples used above (pp. 50-51), the phrases "square circle" and "the present king of France" are concrete, whereas "square circularity" and "the present kingship of France" are abstract; (b) that even when the ending of certain words (e.g. "-ity", "-ship") are recognizably abstract endings, and thus suggest that an individual word or the entire phrase containing that word is abstract, we cannot rely entirely on the form of such words, but must consider the context in which they are used before deciding whether a word or phrase is abstract and not concrete, or vice versa; (c) that this distinction is something more than a matter of linguistic convention; for, as will be seen later, it has its roots in our mode of thought, and is not merely verbal. The same distinction could be applied to symbols which are not words, though of course the criteria would be different; but since the phrase "abstract symbol" is currently used with quite another significance, confusion might arise if we extended this distinction to symbols in general.

The difficulty which we are considering can be most easily clarified by making plain the difference between the meaning of abstract words and the meaning of concrete words. What is said of words applies also to groups of words, as we shall see: not only to phrases but also to sentences; but these latter had best be treated when we come to speak of propositions. First let us confine our observations to those individually-intelligible words called "common nouns". When such words are used concretely, they signify, or present to thought, something <u>as having</u> some (more or less complex, and hence more or less analyzable) characteristic. Thus, the word "circle" signifies something having circularity. On the other hand, words which are used abstractly signify something as being some characteristic. Thus, the word "circularity" signifies something as being circularity. Notice that neither of these expressions, when used as words, signifies that something has some characteristic, or that something is some characteristic. If the context, in a given use, shows that they have this latter meaning, then they are not being used as words, but as signs of propositions. Since concrete words signify something as having some characteristic, it is natural to call attention to this twofold aspect of their significance by saying that they denote the "something" which they signify, and that they connote the characteristic which that "something" is signified as having. When we attempt to make a somewhat similar distinction in the case of abstract words, we may say that they denote the "something" which they signify, and that they connote the characteristic which that "something" is signified as being. It is of course quite possible to express this distinction in another way, which would suggest that it is more like the one applied to concrete words. We might maintain that abstract words, like concrete words, denote the "something" which they signify, and that they connote the characteristic which that "something" is signified as having; for it may be quite truly said that abstract words signify something as having some characteristic, inasmuch as their endings indicate (supposing that they are correctly used) that they signify a characteristic: i.e. something as having the characteristic of "being a characteristic". On this view, of course, all abstract words would have the same connotation; all alike would be said to connote the same characteristic, that of "being a characteristic." As we shall see, however, it is misleading to describe "being a characteristic" as a characteristic; hence we shall abide by the distinction as previously expressed, which contains the following implicit admission: The only difference between the denotation and the connotation of an abstract word is a difference, not in what is signified, but in the way in which that "something" is signified. In the case of concrete words, the denotation is never a characteristic, and the connotation always is a characteristic. (To avoid confusion, it must be noted that certain concrete words have an abstract usage: e.g. "color" is often used to mean "coloredness". Hence our insistence on the need of taking usage into account before deciding whether a word is abstract or concrete.) But even in the case of concrete words, it is advisable to distinguish between denotation and connotation according to the way in which words signify, rather than according to a difference in what is signified by words. A full understanding of this last statement cannot be presumed without reference to later discussions; but we shall see at once, when extending the above considerations to groups of words, that the way in which "something" is signified is the decisive factor, regardless of what that "something" is, or what characteristics it has.

We have explained the sense in which every symbol, when used significantly, can be said to mean "something". The question "what does this symbol mean?" is one to which we cannot give an unequivocal and complete answer, as it stands; for it contains within itself three questions, each of which must be answered from a different point of view: (a) "How many 'somethings' does this symbol mean,-- only one, or more than one?" (b) "Precisely which 'something', or 'somethings', does it mean?" (c) "What characteristic or characteristics does it signify, either as had by or as being this 'something', or these 'somethings'"? (The "either...or" in this third question is necessary because of what we have said about the difference between concrete words and abstract words, p. 53 above.) These three questions obviously cover the matter of denotation and connotation; for the first two ask "what does this symbol denote?", and the third asks, "what does this symbol connote?"

While insisting, as always, that a conclusive answer to any of these three questions cannot be given without reference to the context in which a symbol occurs, and to the way in which it is used, we wish to remark that in those word-groups known as "descriptions", or "descriptive phrases," the phrase as a whole is a symbolic unit; and within this unit, anyone sufficiently acquainted with the language in which the phrase occurs can distinguish between (a) certain "modifying" words or phrases and (b) certain other words, or at least one word, "modified" thereby,--although words to be classed as either (a) or (b) may have to be "understood" from the context, and the entire phrase may have to be re-worded if the context requires it. Some of these modifiers, e.g., adjectives and adjectival phrases, add to the connotation of the word which they modify; others, e.g., demonstrative pronouns, the definite and the indefinite article, and in general the words known as "quantifying words", indicate more or less determinately the denotation of the word they modify. Since the symbolic function of these modifying words, in such phrases, is to signify the denotation or connotation of some other word, they are not being used as complete symbols, and hence have no denotation or connotation of their own; though they might loosely be said to "connote" the denotation or connotation of the word which they modify, inasmuch as they signify it in part. When so used, they have what may be called "incomplete" meaning, and therefore cannot be called meaningless symbols on the ground of not having denotation and connotation in the same way that complete symbols do.

In saying that the modifying words used in descriptive phrases are incomplete symbols, we are not suggesting that the modified words in these phrases are complete symbols, except in a very relative sense. It is the phrase as a whole which is being used as a complete symbol: and the modified words can be called complete symbols only because (a) they would still have intelligible significance even if the modifying words were omitted,-- whereas the modified words would not, apart from what they modify,-- and (b) they are thus of greater importance in determining the significance of the whole phrase in which they occur. Again, in remarking that some of these modifying words are used to indicate the connotation of the words which they modify (and hence of the entire phrase), while others are used to indicate the denotation thereof, we are not attempting to divide modifiers into two mutually-exclusive classes, the one containing connotational modifiers (adjectival words and phrases) and the other containing denotational modifiers (demonstratives, articles, quantifying words). Just as the distinction between connotation and denotation, in general, depends not so much on the fact that <u>different</u> "somethings" are signified, but rather on the fact that "somethings" are signified <u>in a different way</u>, so here, the distinction between connotational modifiers and denotational modifiers depends not so much on a difference in the <u>kind</u> of words used (e.g., adjectival words on the one hand, and quantifying words on the other), but rather on the <u>way in which</u> these words are used. The second of the three questions concerning the meaning of a given symbol,-- "Precisely <u>which</u> 'something', or 'somethings', does this symbol mean?" (marked (b) on pp. 54-55 above),-- is a question about the denotation of a symbol rather than about its connotation; yet it cannot be answered by considering denotational modifiers alone and refusing to consider connotational modifiers. Among denotational modifiers, the most precisely denotative is admittedly the word "this"; nevertheless, even in a given usage, such as its usage in the phrase "this man", the word "this" <u>alone</u> does not indicate precisely <u>which</u> man is meant by the phrase. And when we discover an answer to our question by referring to the context in which the phrase occurs, the answer involves not merely denotational modifiers, but some word or words signifying a uniquely-determining <u>characteristic</u>: that is to say, some <u>connotational</u> modifier, as well; e.g., "the man whom I have just mentioned", or "this man here and now present."

We are not here relying on the fact that the word "this", by itself, does not <u>denote</u> precisely which man is meant; hence it might be supposed that "this" can, and does, <u>connote</u> precisely which: on the ground that "this" is a shorthand substitute for the phrase, supplied from the context, which signifies a uniquely-determining characteristic of the man referred to by "this man". However plausible such a view may seem, there can be no doubt that its adoption would give rise to more difficulties than it can solve: for it clearly involves extending the notion of connotation so as to include not merely the characteristic, or characteristics, explicitly signified by a symbol in a given usage, but also all characteristics which a symbol <u>implicitly</u> signifies, either (a) by entailing them itself, apart from reference to the context, or (b) by entailing them in the light of a given context. The full force and importance of the distinction here suggested, between what a symbol explicitly signifies and what a symbol implicitly signifies, or entails (either with or without reference to its context), will be more easily appreciated, it is hoped, as a result of later discussions. At present we merely wish to point out its bearing in connection with the above example. Inasmuch as the word "this" explicitly signifies "something having 'thisness'", it explicitly <u>connotes</u> the characteristic meant by the word "thisness", <u>as belonging to</u> (or <u>as had by</u>) the "something" which it <u>denotes</u>. It does <u>not</u> explicitly connote either (a) the characteristic meant by the words "whom I have just mentioned", or (b) the characteristic meant by the words "here and now present"; but it <u>does</u> connote at least one of them <u>implicitly</u>. Whether it does so by itself, or only in relation to its context, makes no difference to the point at issue: though we incline to suggest that in most cases, if not in all, reference to the context is involved. However this may be, in the example under consideration, it is easy to see that "this" does not explicitly connote either of the two characteristics marked "(a)" and "(b)", above. For neither of them is quite the same as the characteristic meant by the word "thisness", which the word "this" explicitly connotes. Whatever be the ultimate analysis of the characteristic "thisness", it can be intelligibly described as "the characteristic of 'being <u>this individual</u> "something"'", in contradistinction to "the characteristic of 'being that individual "something"'"; and the characteristic of "being this individual 'something'" is not quite the same as the characteristic meant either by the words "whom I have just mentioned" or by the words "here and now present". It will be noticed that in order to establish the fact that the characteristics meant by these phrases are not really the same characteristic, we should have to show that, no matter how these phrases be analyzed, the characteristic meant by "thisness" is really a different characteristic from those marked "(a)" and "(b)" (p. 58, above): for it is quite possible that what is meant by these different expressions might be seen, on analysis, to be the very same characteristic. Instead of ruling out this possibility, and thus proving that the characteristics meant are really different characteristics, we merely wish to remark that a clear distinction between the explicit connotation of "this" (i.e. the characteristic meant by "thisness") and its implicit connotation (i.e. either (a) or (b) above, can be made without reference to whether the connoted characteristics are really the same or really different. Even though analysis might show that the same characteristic is meant in both cases,-- i.e. that what is meant is really the same characteristic,-- there can be no doubt that "thisness", on the one hand, and both phrases "(a)" and "(b)", on the other, signify this same characteristic under a different connotation: i.e., symbolize it, or present it to thought, as two different characteristics,-- or rather, as three different characteristics, since the explicit connotation of "(a)" is not the same as the explicit connotation of "(b)".

In thus calling the attention of the reader to a problem which we at once dismiss as irrelevant, we are pointing out, by way of an example, the importance of a distinction already mentioned (pp. 44-45, above): the distinction between the two questions, "To what does a symbol refer?", and "What is that 'something' to which a symbol refers?" So long as one is discussing the first of these questions, he is talking about "what a symbol means", and this is a matter of both logic and language. But when one begins to discuss the second question, he is talking about the "somethings" which are meant by symbols, and the discussion is no longer merely logical, still less is it merely a matter of linguistics or of logical syntax.

Before applying the results of this general discussion of the meaning of symbols to our initial question, "In what sense are abstract symbol-systems meaningless?", we must add a note or two on the notion of interpretation. For it will be necessary to inquire into the meaning, or meaninglessness, not only of systems which are said to have "no actual interpretation", but also of systems in general, prior to their being actually interpreted: and this may well include systems which have an actual interpretation. Now, the word "interpretation", as used in such phrases as "the interpretation of a symbol", may be understood in one of the following three senses. It may mean (a) the business of recognizing what a symbol means, or how it is being used significantly in a given context. The details of this apparently complex and at least partly "mental" process are of interest primarily to students of psychology; hence the word "interpretation", as used in works on logic, is not to be understood in this sense apart from explicit indications to the contrary. The other two senses of the word are these: (b) the meaning which a symbol has in a given context,-- that is, that one of its possible meanings (cf. the example of "possible meanings" given on pp. 45-46 above) which attaches to it in this given context; and (c) the "something" which is meant by a symbol in a given context. Although we have just had occasion to notice the advisability, in general, of distinguishing between (b) and (c) when discussing the problem of meaning, we mention it here merely for the sake of completeness, and shall not insist upon it until we come to explain it more fully.

When it is said that an abstract symbol-system has, or may have, no actual interpretation, the word "interpretation" had best be understood in sense (c); for even those who seem not to distinguish this sense from (b), as we have done, clearly mean to say that there is not, or may possibly not be, anything corresponding to the "meaning" of such a symbol, if "meaning" be understood as in sense (b). In other words, their statement that an abstract symbol-system has no actual interpretation is equivalent to a statement that there is nothing which is meant by such a system.

In order to understand this quite clearly, we must remember that an abstract symbol-system, as has been explained in the preceding chapter, is used to symbolize directly the structure of any set of entities which is similar in structure to it. As soon as an abstract symbol-system is constructed, it will of course have a definite structure; and if exact copies of such a system be made, each of these will have the same structure as the original. But apart from such copies, which would presumably be called "trivial cases of isomorphism", it is quite possible that (a) no other set of entities, whether physical or non-physical (see pp. 19-21, above), is similar to the symbol-system in structure, or that (b) we do not know whether there is such another set or not. Until such a "similarly-structured" set be either discovered or constructed, the symbol-system cannot be said to "have an actual interpretation"; although we should need further evidence in order to assert that it cannot possibly have one, for such a set might be in existence without having as yet been discovered, or such a set might eventually be constructed by someone.

2. Relation between Meaning and Interpretation of Abstract Symbol-Systems.

Having explained the sense in which an abstract symbol-system can be called "meaningless" if it has no actual interpretation, we must now inquire whether such a system is meaningless on any other grounds: in other words, can it be called "meaningless" in a wider sense than that of "having no actual interpretation"? One way to arrive at an answer to this question is, to consider what is involved in the actual interpretation of an abstract system; for in this way we may expect to discover whether or not such a system has "meaning" prior to its being actually interpreted, and if so, in what sense.

A completely uninterpreted abstract system (of the kind mentioned on p. 3, above) is composed, it will be remembered, of recognizably-distinct marks selected and arranged according to certain purely conventional rules. Some of these rules, called "rules of formation", indicate which marks are to be selected and how they are to be arranged so as to be "well-formed expressions" of the system. From certain initial sets constructed according to the formation-rules, all subsequent sets are derived: either from the initial sets alone, or from other sets derived from the initial ones; and this derivation proceeds according to "rules of transformation", so that all such derived sets are "well-formed expressions" of the system. In order to find an actual interpretation of an abstract system, we need not discover or construct an entire system which is similar to it in structure; we need only discover or construct a set of entities which will "satisfy" the initial well-formed expressions of the system. The necessary and sufficient condition for "satisfaction" may be stated as follows. The elements of the set, and the relations between those elements, must be such that if the marks in the initial well-formed expressions be replaced by symbols for those elements and those relations, the set of symbols obtained by this "translation" of each initial set will in every case express a true proposition about those elements and those relations. Once the initial sets are satisfied in this way, the consistency of the abstract system is assured; and we know that all subsequent well-formed expressions in the system, because they are derived according to the transformation-rules, can be similarly translated by appropriate substitutions, into expressions of true propositions about these elements and their relations, without any need of subsequent verification.

It is of course quite possible that an abstract system may have an actual interpretation, even though no one has yet recognized (a) that it has an interpretation, or (b) what its interpretation is; and it is even possible that because of our psychological limitations, some at least of the actual interpretations of some abstract systems may never be recognized. The fact that we are obliged to follow some such procedure as the one above described, in order to recognize any actual interpretation of any abstract system, is doubtless a matter of those same psychological limitations. But the point to note is, that it has important consequences for logic. Whenever we know that someone has verified a proposition, we also know (a) that someone was able to verify it, and (b) that the proposition itself was verifiable. It is clear that in such propositions as we are discussing, the element of time may be ignored; for we are not speaking of propositions whose verifiability is conditioned by temporal factors. Their truth-value may change in time, but their verifiability does not change. Now, to say that a proposition is verifiable involves saying that it has meaning, in some sense, no matter what criteria of verifiability be adopted. This will be admitted even by those who maintain that the meaning of a proposition is its verifiability, i.e. the conditions required for its verification. We are therefore justified in drawing the following conclusions. First, in the case of all abstract systems which have been actually interpreted, the propositions obtained from the initial sets of such systems and verified, in the manner explained above, must have had some meaning prior to their verification, else they could not have been verified; and this meaning is not the same as the actual interpretation of such systems. Second, in the case of abstract systems for which no interpretation has as yet been found, we cannot use this argument to indicate that they have meaning. We do know, however, that if and when they are actually interpreted, the fact that they must have had some meaning can be established in the same way; hence unless other considerations suggest a different view of the matter, it seems advisable to say merely this: we do not know whether they have meaning or not, but it is likely that they have, in proportion as some actual interpretation appears more or less probable.

It will be observed that the above argument does not lead to the conclusion that abstract systems themselves have meaning, independently of interpretation, but only that those expressions have meaning which are derived from the initial sets of such systems, by substituting for the marks therein element-symbols and relation-symbols in the manner outlined above (p.63). If these expressions have meaning, it follows that expressions derived from all subsequent sets of marks in the abstract system which has those same initial sets will also have meaning; for they too, as we have seen (pp. 63-64, above) are expressions of true propositions,--provided, of course, that they are derived by appropriate substitutions. Whether the initial sets themselves, and all derived sets of an abstract system, have meaning apart from, or prior to, such substitution, will be considered in the next section of this chapter. Here we wish to note that the expressions derived by substitution from the initial set of abstract systems, and shown by our previous argument to have some meaning, are of very great importance. For in this class of expressions are included the so-called postulates of all postulational systems, or of all systems which can be developed according to the postulational method: including such interesting isomorphs as Boolean algebra, linear associative algebra, and the various systems derived from the primitive propositions of Principia Mathematica or from some equivalent "reduction" thereof.

The conclusion that such expressions have some meaning immediately raises the further question, "What is this meaning which they have?" It is doubtful whether an adequate answer to this question could be attempted within the limits of a single dissertation; and though much of what will be said later on may suggest the lines along which at least a partial answer could be worked out, or at any rate may indicate some of the problems involved in working out an answer, we must confine ourselves at the moment to the following general comments. First, the meaning of these expressions will vary according to the kind of entities symbolized by the element-symbols and relation-symbols which are substituted for the marks in the initial set of the uninterpreted abstract system whence these expressions are derived. The elements may be any entities whatever, and the relations may be any relations whatever, provided only that these relations have the same formal properties: i.e. provided that the particular relations (e.g. "to the right of", "is greater than") which hold between the particular entities (e.g., points on a line, positive integers) symbolized by the element-symbols according to some particular interpretation have the same formal properties as the particular relations holding between the particular elements symbolized in all other particular interpretations. Secondly, if we try to say quite generally what any such expression will mean, we find that each such expression is interpretable as a statement that "If certain elements stand in a certain relation to one another, then certain other elements stand in certain other relations, either to one another or to the former elements". It should be remarked that this includes the special case in which two or mor elements, or sets of elements, are said to be somehow equivalent: for a statement of equivalence between elements can always be reduced to the above form. Systemic or nominal definitions are not statements of equivalence between elements, but between symbols; hence they are merely transformation rules, or applications thereof, having no direct reference to either elements or relations between elements.

Lest what we have said about postulates be misunderstood, it should be added that in actual practice the postulates of various logics and various postulational systems of mathematics have not been derived, by substitution, from the initial sets of marks of a previously-constructed abstract system. Rather, these sets of postulates have been constructed, for the most part, without reference to a "higher" or more abstract system from which they might be derived. This is clear from the fact that their symbols are not just meaningless marks, but are specifically used to symbolize (a) elements, or classes of elements, on the one hand, and (b) relations, or classes of relations, between these elements, on the other hand. The point we wish to make is simply this: they are on the same level of abstraction, and have therefore the same sort of meaning, as is had by expressions derived, in the fashion we have explained, from the initial sets of a completely-uninterpreted abstract system.

3. Relation between Meaning and Development of Abstract Symbol-Systems.

We have thus far indicated a reason for the view that any abstract system which has actually been interpreted must have had some meaning prior to its actual interpretation, at least at the stage where the marks in the initial sets of such a system are replaced by element-symbols and relation-symbols, thus transforming each initial set into the expression of a postulate or primitive proposition. In asking the further question, "Are the initial sets themselves meaningless, prior to being thus 'transformed' or translated?" we should most probably find that a line of reasoning similar to the one just followed (pp. 62-65) would lead to a similar conclusion: i.e., that they could not have been translated into significant expressions unless they already had some meaning, however general, prior to translation. This is particularly likely in view of the fact that such translation is often a matter, not of replacing the original "meaningless" marks by other marks which are symbols (i.e. which have significance in some sense), but of interpreting, or simply "reading", the original marks themselves as symbols,-- element-symbols on the one hand, relation-symbols on the other. That is to say, the question whether a system is entirely abstract, in the sense of being a set of meaningless but recognizable marks, rather than a set of symbols with some sort of general meaning or significance, is a question not of which marks are substituted for other given marks, but of interpretation: a question of what the given marks mean, or of how they are to be "read".

Since abstract systems are ordinarily constructed with a view to their possible interpretation, it is doubtful whether anything conclusive can be said about their meaning apart from a study of some actual interpretation. But we may here consider whether certain features of their development suggest in what sense they can be called "meaningless" while they are still in the "completely-uninterpreted" stage. In other words, does the construction of a system out of recognizable marks, and the study of the formal or structural properties of marks in connection with system-structure, involve reference to meaning, and if so, in what sense?

A detailed commentary on the formalist view, and on the technique of the postulational or axiomatic method, would manifestly be impossible here, even if these topics had not been thoroughly treated by other and more competent writers. It is hoped that the following reflections may help to show the significance of this view and this method for epistemology, or at least to indicate certain epistemological problems connected therewith.

The general aim of the formalists, whether in mathematics or in logical syntax, is to investigate the formal or structural properties of signs, by selecting and ordering them in various ways; in other words, by constructing abstract systems: and from a study of these systems they hope to arrive at the general principles of system-structure. Moreover, by constructing abstract systems which differ in structure from one another, they will provide means of representing, in pictorial fashion, the structure of the many systems whose elements are not signs, with which the various sciences are concerned. Insofar as these are interpretations of abstract systems, they will be isomorphic with these abstract systems, and their elements will have the same structural or formal properties as do the signs that are elements in the abstract systems.

It appears to be theoretically possible to construct abstract systems, in the sense of "ordered sets of marks", without devising any rules beforehand for the selecting and ordering of marks. If the material for such construction were limited, by a stipulation that a determined kind or a determined number of marks be exclusively employed, these marks could be jotted down at random, and at least some of them might possibly be recognized later as forming one or more ordered set. Since such sets would have structure, they could then be used as direct symbols of any isomorphic set of any elements whatever. But the chances of obtaining abstract symbols of structure in this random fashion are so slight as to be negligible; and if no limit were placed on the number of the kind of marks which might be jotted down at random, the construction of even one ordered set becomes so highly improbable that its theoretical possibility may be questioned: at any rate, the ordinary laws of probability could hardly find application in such a case.

If the above procedure were followed, the marks employed would be entirely meaningless, and no question about their meaning could be raised until certain sets of them, which might have been jotted down so as to be ordered sets, would be selected as symbols. The procedure actually in vogue, however, is rather different, and hence we may expect to find that the marks may not be entirely without significance. We have already referred to the way in which abstract systems are constructed (pp. 3-4, above), according to previously-stated formation-rules which state the conditions under which a set of marks constitutes a well-formed expression, and the previously-stated transformation-rules which state how other well-formed expressions are to be derived from those already formed. Considering this procedure, and confining ourselves principally to the marks in the initial sets, it would seem that they have meaning even before the marks in these sets are replaced by element-symbols and relation-symbols (as described above, p. 63). In the first place, each of these initial sets, because it has been constructed according to the formation-rules, is a well-formed expression; and it is clear that all well-formed expressions are elements in the system which is under construction. According to the transformation rules, any well-formed expression may be replaced, under certain conditions, by another well-formed expression; hence it is quite accurate to say that each of these initial sets signifies, or means, at least one characteristic of at least one of the well-formed expressions which may replace it. In order to say more exactly, though still in a fashion sufficiently general to be true, what that characteristic is, we must note that when the initial sets are composite, as is usually the case, they are composed of two kinds of marks, distinguished by the formation-rules: (a) those which cannot by themselves be well-formed expressions, and (b) those which can be well-formed expressions by themselves. The number and the order of those marks, within a composite, which can by themselves be well-formed expressions, determine the form, or the structure, of the composite well-formed expression: i.e., they give the composite a definitely-recognizable form or structure. If a well-formed expression is not composite, but a single mark, it too has form, in a sense; but its form is not significant in every case. According to the transformation-rules, well-formed expressions may be replaced by other well-formed expressions which are not of the same form. But the added condition, namely, that such replacement must be made in every place in which the original expression is found in a given set, indicates clearly that the universally-significant characteristic of any well-formed expression is, the <u>position</u> which it occupies in a given set. As for those marks which, by themselves, cannot be well-formed expressions, their position in a given set is also significant; but it is understood that they cannot be replaced by any other mark of a different shape, unless the transformation-rules explicitly permit such replacement.

The suggestion we are making, with regard to the rules governing the construction of an abstract symbol-system, is not that these rules themselves have meaning; for it is sufficiently clear that they must have meaning, in the sense of being intelligible and even precise in reference. A consideration of these rules, which admittedly insure the consistency and coherence of the systems constructed according to them (although "consistency" and "coherence", as we noted on p. 4 above, are here used in a peculiar sense), points to the further suggestion that (a) the formation-rules ascribe meaning to the marks out of which initial sets are constructed, and (b) the transformation-rules take this meaning into account in the conditions laid down by them for the derivation of subsequent sets. This statement will perhaps seem less strange if we reflect on each of these classes of rules in turn.

It is customary to speak of the marks selected by the formation-rules as "undefined symbols". But a very little reflection suffices to show that the word "undefined" is somewhat misleading. For according to the formation-rules, certain single marks are definitely indicated, or "characterized as", well-formed expressions, whereas others are definitely characterized as <u>not</u> well-formed expressions. Examples of the former kind are the marks "p,q,r...", which are read as symbols of unanalyzed propositions according to the <u>Principia</u> conventions.

Examples of the latter kind are the marks "+, x, =", and the marks used as symbols for negation and material implication, respectively: "-,)". In order to realize that these have meaning in their most abstract condition, we need only observe that there is a perceptible difference between these marks by themselves and the same marks as referred to by the formation-rules of a system. By itself, for instance, the mark "p" is a physical object with certain definite physical characteristics, e.g., color, size, shape, spatial position. Anyone acquainted with the alphabet of occidental languages will at once tend to interpret it as a symbol: as the letter "p"; but this is a matter of convention, due to its acceptance as a linguistic unit. The formation-rules *denote* certain marks (usually letters) as possible "well-formed expressions of the system"; they also *connote* these same marks, although they do not do so explicitly. Instead of describing the characteristics of the marks which they denote as "well-formed expressions", the formation-rules merely mention a more or less complete list of such marks; on the assumption that the characteristics of the marks, though not explicitly connoted by the rules, will be intuitively recognized: and thus both the connotation and the denotation of the words "well-formed expression" will be sufficiently clear to anyone constructing or studying the system to which the rules apply. Now, the difference between any mark which is thus defined as "a member of the class of well-formed expressions", and that same mark considered apart from such classification, is not merely that the mark receives, in virtue of being thus "defined" by the formation-rules, a characteristic which it did not have before,-- namely, the characteristic of "being a well-formed expression of a given system". The point to notice is that, because the system in question is a symbol-system, every such mark becomes a symbol, and so has a meaning of its own, which is part of the meaning of the symbol-system as a whole. When incorporated into an abstract system, as a well-formed expression or element thereof, it acquires the characteristic of "occupying a definite position with respect to some other element of the system"; and this characteristic is directly symbolic of, or directly signifies, or <u>means</u>, the position of any other well-formed expression which may replace it, according to the transformation-rules.

As they occur in the initial sets of an abstract system, those other marks which are defined by the formation-rules as <u>not</u> well-formed expressions also signify, by their position, the position of any other mark which may replace them, according to the transformation-rules. But in addition to this significance, their special function is to signify the form of the composite well-formed expressions in which they occur. The mark "-", occurring to the left of a well-formed expression, is to be read in conjunction with that expression; and the composite signifies a well-formed expression which has a fixed relation to the original <u>unmarked</u> well-formed expression. Other marks, such as "=, +,), x", occurring between two well-formed expressions, signify that the two expressions between which they occur are to be regarded as <u>one</u> well-formed expression. Speaking in terms of denotation and connotation, we may say that such marks denote the well-formed expressions between which they occur, and that they connote them as <u>one element</u> of the system. For instance, whereas "p", "q", by themselves, would each signify a well-formed expression, and hence would signify two well-formed expressions if set down side by side, the combinations "p + q", "p x q", "p (q", each signify one well-formed expression, and the difference in form is indicated by the different mark in each case. The combination "p = q" has this same significance, with the further connotation that the well-formed expressions between which the mark "=" occurs are <u>one and the same element</u>, in the given system.

These comments on the formation-rules will perhaps be sufficient to show that they are really definitions of the marks to which they refer. Not only do they indicate that certain marks are to be regarded as well-formed expressions (either by themselves or in conjunction with certain other marks in a definite order), and that certain marks are not well-formed expressions, but they also provide that these marks may be used as symbols. Any mark occurring in the initial sets constructed according to the formation-rules is therefore more than a "meaningless mark", recognizably distinct from all other marks because of its physical characteristics (e.g., shape, spatial position). It has meaning, by the very fact that at least some of its characteristics are accepted as significant, or symbolic, in the sense explained above.

Turning now to the transformation-rules, according to which other sets of marks are derived from the initial sets, we notice that the significant characteristics of the marks in the initial sets are constantly taken into account. The rules for altering an initial set, or for "transforming" it into another set which shall be part of the system under construction, provide that only well-formed expressions may be replaced; hence marks which are non-well-formed expressions must be left unaltered. Moreover, none but well-formed expressions may be used for such replacements; and if an expression in any initial set (or previously-transformed set) be replaced by another expression, that other expression must be substituted for the former in every place occupied by the former expression in the initial set. In consequence of this proviso, the well-formed expressions which are elements in the transformed set may be <u>different</u> from those which are elements in the original set; in fact, they <u>must</u> be somehow different, else the set would not really be transformed: but since they occupy the same position as the original elements, the transformed set will be isomorphic with the original set. There is an apparent exception to this rule of "substitution throughout", or replacement each time, in the case of any well-formed expression in an initial set which is separated from another well-formed expression in the same set only by the marks "=". In such cases, either of these two expressions may be replaced by the other in any set in which one of them occurs, but the one need not be replaced by the other everywhere in that set. This exception does not destroy isomorphism between a set thus transformed and the original set, because, as we have remarked (P. 75) the mark "=" connotes that the two expressions between which it occurs are not only each an element of the system, but one and the same element of the system.

It seems fairly clear that the transformation-rules must be understood in a somewhat different sense, if the system whose development they govern is to be useful for knowledge. If they merely insure that from a set of well-formed expressions, taken as "initial strings",-- each string being composed of marks which are recognizably well-formed and non-well-formed expressions, so ordered that every part of well-formed expressions is separated by a non-well-formed expression,-- other sets of marks can be derived which are isomorphic with a given initial string, it would seem that the resulting "system" is merely a group of sets, some of which (the derived strings) are isomorphic with some others (the initial strings from which they have respectively been derived). Even if it be possible to regard such a group as a system, and to find some interpretation for it, it would hardly serve as a symbol for any known system of logic or of mathematics. For, as we have seen (pp. 62-63 above) the whole point of constructing symbol-systems is this: that if the initial strings, when read as propositions about elements and relations, are found to be true of the elements and relations of some system, then the subsequent strings, when read as propositions about the elements and relations of that same system, will also be true without need of further verification. Now, the transformation-rules, if they take no more account of meaning than has been suggested above (p. 76) do indeed guarantee that any set of symbols will be isomorphic with the initial set from which it is derived; but it does not therefore follow that the complex of elements and relations symbolized by the initial set will be isomorphic with the complex symbolized by the derived set. This may be clearly seen from a single example. Let the initial set be "(p v p) .). p". It is permissible to replace the well-formed expression "p" by any other well-formed expression, e.g. "q x r", and so obtain the derived set, "((q x r) v (q x r)) .). (q x r)". Using "W" to indicate "well-formed expression", and using the marks which are non-well-formed expressions, we may show the form common to both these sets thus: "(W v W) .). W". Having a common form, both sets can be called isomorphic. But if these sets are read as symbols of the propositional calculus, standing for propositions and relations between propositions, the initial set and the derived set no longer have the same form.

Even if, on such a reading, the sets in question did have the same form, it would not follow that the truth of the first set would entail the truth of the second set, supposing that each were interpreted as a proposition about elements and relations. Thus, if "q" were substituted for "p" in the initial set, we should have "(q v q) .). q"; and this <u>would</u> be isomorphic with the initial set when both sets are read as symbols of the propositional calculus. The truth of the second is entailed by the truth of the first, not because the two sets of <u>symbols</u> are isomorphic, but because the symbols <u>have the same meaning</u> in each set. In both sets, the symbols "p", "q", mean what is meant by the words "a proposition", and the marks "v" and ")" mean what is meant by the words "or" and "implies", respectively; with the further proviso that "p" and "q" respectively mean the <u>same</u> proposition each time they occur in the same set.

4. Symbolic Force of a Completely Uninterpreted System.

Though the difficulties which we have emphasized as obstacles to the construction of an abstract symbol-system out of meaningless marks can hardly be ignored, it may seem that we are exaggerating their importance. Those who maintain that logic is system-structure, and that the business of logic is to provide a series of maps which shall directly represent the structure of the various systems which we call sciences, may be inclined to dismiss these difficulties as irrelevant, on the ground that the meaning which the symbols have prior to their actual interpretation is entirely left out of account when we come to interpret the system as a whole. It is the form of the abstract system as a whole which is used, or to be used, as a symbol; and although this form depends on certain characteristics of the individual symbols which together make up the abstract system, it does not depend on their meaning.

The analogy between an abstract symbol-system and a map renders such a view as this extremely plausible. Of course, the word "form", as applied to a symbol-system, will not refer to its spatial configuration or shape, as it does in the case of a map. But a symbol-system, considered as a set of meaningless marks arranged in a definite order, appears to have the sort of form or structure which is had by a set of elements other than marks. If we agree that a set of elements can be called a system on condition that the elements be ordered in a regular way, not only can a set of marks be such a system, quite apart from what they may mean, but they will have a definite form or structure, which can represent the structure of all similarly-ordered sets of any elements whatever, provided that there exists a one-to-one correspondence between the elements of each such set and the elements of the set of marks. The analysis which we have made of structure, and our discussion of direct representation of structure by means of symbol-sets which have the same structure as that which they symbolize (pp. 19-40 above) not only recognize the possibility of such diagrammatic symbolization, but indicate the principles involved in it. Hence it may seem that the stress we have laid on the meaning of symbols is not only excessive but inconsistent.

In order to clarify this matter, we must try to see exactly what is the symbolic force of an ordered set of symbols, in which the elements are recognizably-distinct marks having no meaning apart from the ordered set in which they occur. Setting aside for the moment the question of whether such an ordered set can be constructed, we may ask what would be involved in its use as a symbol. The first thing to notice is, that the set as a whole would have form, or structure; it would consist of a number of elements arranged in a regular order. If the set as a whole be taken as a symbol, this means that its form is intended to signify something: i.e., to call attention to something, to present something to awareness or consciousness. Inasmuch as the set is a direct symbol, it must have the structure which it signifies; what we wish to emphasize here is, that if it is used as a symbol, it signifies the structure which it has. When we regard the set merely as an ordered collocation of marks, we observe that it has structure, or is characterized by structure; but when we regard this same ordered collocation of marks as a symbol, it not only <u>has</u> structure but also <u>means</u> structure. In other words, whenever we select some characteristic of an object as significant or symbolic, we are obliged to use <u>that object</u> itself as a symbol: for even if we could effect some sort of separation between the object and the particular characteristic which we select as significant, the characteristic <u>by itself</u> would not have that concrete visible existence which a symbol must have. Thus it is that the object itself becomes a symbol of any characteristic of its own which is selected as significant or symbolic; and therefore the significant characteristic of the symbol-object not only belongs to that object but also <u>is meant by</u> it. There is, in consequence, no ground for an absolute distinction between the structure of an abstract-symbol-set and its meaning; and any attempt to create a sharp distinction is bound to lead to confusion.

With regard to the individual marks in the ordered set, two things are to be noted: (a) they have no meaning in isolation because, although they have certain characteristics apart from being elements in this ordered set, such characteristics are left out of consideration, i.e., not selected as significant; (b) considered as elements in this (or any <u>given</u>) ordered set, they have certain characteristics which are significant: each has a definite position in the set, and all collectively have a definite order. (Characteristics of this kind are what we have called "relational characteristics"; (pp. 16-17, above) Since the structure of the set as a whole is determined by the significant (relational) characteristics of the individual marks, and since the structure of the set is its meaning, each individual mark contributes something to the meaning of the set, and each may therefore be said to have incomplete meaning, when considered as an element in a given set.

In declaring that there is no ground for an absolute distinction between the structure of an abstract symbol-set and its meaning (par.1, above) we do not mean to suggest that no distinction whatever should be made between them. The structure of such a set, considered as a characteristic of the set, is a complex of actually-existing properties, no less real than the set to which it belongs; hence it is something concrete and physical. Considered as something <u>meant by</u> the set which has it, that same complex of actually-existing properties is thought of in the abstract: i.e., we leave out of account any peculiarities it might have as a characteristic of this particular set, and consider it as a possible characteristic of any set whatever, including this one. As a characteristic of this particular set, for example, it is subject to spatio-temporal limitations of the set to which it belongs; hence we leave these limitations out of account, and think of it as possibly belonging to some other set at some other time and in some other place. A fuller discussion of this point will be necessary in connection with the question of universals; but attention to it here enables us to make plain the difference between (a) a characteristic and (b) a significant characteristic: i.e., between a characteristic as such, and a characteristic as significant. A characteristic as such, i.e., as it actually is, has no concrete existence in isolation from the object to which it belongs. When we think of a characteristic in isolation, we recognize it as an abstraction; and even though we may call it "an object", or refer to it as "something", we should admit that these words are being used ambiguously: that it is not "something" in exactly the same sense as the object which has it is "something". So long as we are thinking of a characteristic in isolation from any and all objects in which it actually exists, the fact that it is an abstraction can be readily appreciated on reflection. But when we think of a characteristic of a given object as belonging to another given object, we are apt to overlook the fact that we are dealing with abstractions.

These general considerations have a direct bearing on the question of the symbolic use of the structure of ordered sets, as follows. The structure of a given ordered set, as found in, or existing in, that set, is something quite as concrete as the set to which it belongs. This is true of any given set. In the case of two or more isomorphic sets, we should not say without reservation that one of these sets <u>is</u> another set; and it would be inaccurate to say that the structure of any one of these sets <u>is</u> the structure of any other set, if this means that they all have one and the same concrete characteristic. We can and do say that all these sets have the same structure; but this is correct only inasmuch as each set possesses a different concrete instance of the <u>universal</u> "structure" which is common to them all. This universal structure is an abstraction, a product of our thinking. It is verified in, and has concrete existence in, the individual sets which possess it. But it is not in all respects identical with any concrete instance of itself. Applying this to the question of direct symbolization, we may say: (1) Any ordered set of marks has a given structure; (2) this given structure, thought of as an abstraction, can signify, and hence be a symbol of, all concrete instances of itself, wherever it may exist; (3) therefore a given ordered set can be used as a symbol of any concrete instance of structure which is the same as, or similar to, its own concrete structure; (4) it can also, less directly, be a symbol of any object, other than itself, which has another concrete instance of that same abstract structure, whereof its own concrete structure is an instance. The matter may be put more briefly, in terms of connotation and denotation, thus: A given ordered set, because it has concrete structure, <u>connotes</u> that same structure in the abstract, or as an abstraction; and it <u>denotes</u> (a) directly, all concrete instances of that same abstract structure, as well as (b) indirectly, all objects which severally possess such instances.

What we have said about the distinction between structure as a characteristic and structure as a significant characteristic applies also to the individual symbols which make up a given ordered set, as well as to the set as a whole. Without working this out in detail, we can readily see that just as the characteristic "structure" which belongs to the set is analyzable into the structural or formal characteristics which belong to the elements and the relations of that set, so the meaning of the set, which is "structure" regarded as a significant characteristic, depends on those same structural or formal characteristics regarded as significant or meaningful. Hence, although the individual marks have no meaning in isolation, they each have meaning as members of a given set; the meaning of each is part of the meaning of the set as a whole. Having explained the symbolic force of an abstract symbol-system whose structure is taken as symbolic or significant, in order to show how an object which has structure also means the structure which it has, we must examine more closely this characteristic of "structure": for its precise significance, as a direct symbol, will depend upon what it <u>is</u>. The standard definition of isomorphism, according to which two ordered sets or systems are isomorphic if and only if there exists a one-to-one correspondence between the elements of these sets, <u>and</u> if the relations between the elements of one set have the same formal properties as the relations between the corresponding elements of the other set, calls attention to certain conditions which must be fulfilled in order that two structures be recognized as somehow the same, or similar. But it does not tell us precisely what structure is. We have already analyzed the notion of structure (pp. 19-40 above) with special reference to this definition; and if our analysis is correct, the structure of an abstract system depends upon the number and the order of its elements. Since the formal properties of relations involve a reference to the number and the order of the elements between which they hold (p. 35 above)and since the relations themselves may be regarded as relational characteristics of their elements, the number and the order of elements appear to be fundamental. In the light of this analysis, we may be able to see whether the view that an abstract symbol-system is a map of the structure of the systems which it symbolizes applies to such abstract systems as those of logic and mathematics, and whether such map-like symbolization is adequate for the purpose of logic.

5. Structure and System-Structure.

As was noted in the analysis of structure already referred to, spatial structure, or shape, depends upon the number and spatial arrangement of the elements whereof an object which has structure is composed (p. 22 above). Each element must occupy a definite spatial position relatively to at least one other element: that is, only such changes in the distance and direction from one element to another are permitted as will not alter the shape of the whole configuration. In non-spatial structure, the arrangement of the elements is not a matter of the relative spatial position of each with reference to some other; for spatial considerations are left out of account, and the arrangement of the elements becomes a matter of their order in thought. No matter what meaning be attached to the words "order" and "arrangement", it is evident that in any ordered set of elements, each element will have a definite relative position with respect to at least one other element of the set, and further, that no two elements in such a set will have one and the same position. (This point has already been discussed at some length on pp. 32-34.

It seems fairly clear that every system is an ordered set of elements; hence every system will have structure, in the same sense as every ordered set is said to have structure. It is not so clear that every ordered set is a system, in spite of the use of the word "system" to describe, or to refer to, any set of elements arranged in a recognizable order. Without attempting to discuss this question further, we may note that all those systems which are called "deductive systems" are not merely sets of elements arranged in a given order. They are called "deductive" because the order which their elements manifest is not simply an order of sequence but an order of consequence. Each element in a deductive system occupies a definite position, with respect to at least one other element; hence the elements of such a system are an ordered set of elements. What distinguishes them from all non-deductive ordered sets is this: the definite position of each element in the system, and in fact its status as an element in a given deductive system, depends on its being a consequence of at least one other element.

Since a deductive system is an ordered set, it will not only have structure in the same sense as an ordered set has structure, but may also have the same structure as a given ordered set. If we grant that a set of marks can be arranged so as to form an ordered set without any reference to their meaning, such a set may be isomorphic with some deductive system or other, and so may be used to symbolize the structure of that system. But we cannot say of such a set (as we said on p. 83, above) that it also, though indirectly, symbolizes a deductive system. The structure which it directly symbolizes belongs to the deductive system in question, but it belongs to it, or characterizes it, as an ordered set, not as a <u>deductive</u> ordered set.

6. Main Problem Raised by Extreme Formalist View of Logic.

The current disagreement about the nature of mathematics and the relations between mathematics and formal logic has thus far led to no dispute regarding the accepted view that both formal logic and mathematics are deductive systems. If, as the extreme formalists maintain, the various systems of logical syntax and formal logic and mathematics are abstract symbol-systems which are constructed without reference to the meaning of the marks of which they are composed, and which have no meaning apart from their actual interpretation, the least that can be expected is, that these systems should be <u>deductive</u> systems. As a matter of fact, they are said to be deductive systems; but it is not clear that they are really deductive: i.e., it is not clear that (a) any element-mark is a consequence of any other element-mark, or (b) that any one of the subsequent sets of marks is a consequence of any initial set of marks, even when it has been derived from a given initial set according to the transformation-rules.

For this derivation is supposed to be a mere matter of replacing some mark or set of marks,-- arbitrarily chosen as a "well-formed expression",-- by some other mark or set of marks selected from the class of well-formed expressions, without reference to the <u>meaning</u> of the mark or marks in question. As we have seen, there is reason to suppose that some reference to meaning is always involved in the actual process of derivation. But we are here assuming that, as the formalists maintain, an abstract system can be constructed without any reference to the meaning of the marks of which it is composed; hence the question to be discussed is, whether a system so constructed is really a deductive system. An answer to this question should enable us to arrive at some definite conclusion about the relation between formal logic and abstract symbol-systems, and help us to decide the grounds on which the validity of formal logic ought to be settled.

CHAPTER THREE

ABSTRACT SYMBOL-SYSTEMS AND DEDUCTIVE SYSTEMS

1. Derivation from Completely-Uninterpreted Sets.

In order to ascertain whether and in what sense an abstract symbol-system is a deductive system, it will be well to consider a particular group of initial sets of symbols, and to notice what is involved in deriving other sets from each of these. The particular group chosen is quite similar to one of Huntington's sets of postulates for Boolean algebra (see Mind, 1933, pp. 203 ff.), but it will be observed that certain alterations have been made not only in his notation but in the method of presenting them. First we shall present this group as a group of completely-uninterpreted initial sets, each set being made up of a "meaningless but recognizable" collocation of symbols, using Roman numerals to number the sets for convenience of reference.

I. $a, b * ab$

II. $a \circ a'$

III. $ab = ba$

IV. $(ab)c = a(bc)$

V. $-(a'b').-(a'b) = a$

VI. $(a + b) = -(a'b')$

Two remarks need to be made about this series of sets, in explanation of the symbolism. First, wherever two letters occur together, as do the letters ab, this is an abbreviation for a.b; thus, a'b' is a mere abbreviation for a'.b', and (ab)c is an abbreviation for (a.b).c. Second, the mark - to the left of any letter, or any letters enclosed by the marks (), is merely a substitute for the mark ' to the right of that letter or enclosed letters; thus, for a' in set II we might have written -a, and in set V -(a'b') could have been written (a'b')'.

Without assigning any meaning to any one of these sets as a whole, or to any mark in any one of these sets, we may now formulate the following rules whereby any set may be transformed:

Rule A. Any set may be transformed by substituting, for one or more than one of the single letters it contains (e.g., a, b, c in sets I to VI) some other single letter of the English alphabet, subject to the following conditions:

1. Every letter so replaced must be replaced each time it occurs in the original set, by some one and the same single letter.

2. All non-literal symbols must be retained, without any change of relative position.

Rule B. In any set containing the mark °, any collocation of symbols to the right of the mark ° may be used for purposes of replacement according to Rule A, exactly as if it were a single letter.

Rule C. Rule B applies also to any set containing the mark *.

Rule D. In any set containing the mark =, a collocation of symbols to the left of the mark = may replace, or be replaced by, the collocation of symbols to the right of that same mark;

and such replacement may be made in any other set whatever wherein one of these collocations occurs, without observing condition 1 of Rule A.

By applying these rules to set V alone, we can transform that set into the following:

T1. $aa = a$

And by applying these rules to this new set, as also to sets I to V, we can transform V, successively, into each of the following:

T2. $-(a') = a$

T3. $aa' = bb'$ (It is convenient to use the symbol O for aa'.)

If we next apply the above transformation-rules to all the sets which we have thus far been considering, we can derive another group of sets whose resemblance to those which we already have may be seen from the following table:

I. $a,b \circ ab$	Ia. $a,b \circ a + b$
II. $a * a'$	IIa. same as II.
III. $ab = ba$	IIIa. $a + b = b + a$
IV. $(ab)c = a(bc)$	IVa. $(a + b) + c = a + (b + c)$
V. $-(a'b').-(a'b) = a$	Va. $-(a' + b')+-(a' + b) = a$
VI. $(a + b) = -(a'b')$	VIa. $ab = -(a' + b')$
T1. $aa = a$	T1a. $a + a = a$
T2. $-(a') = a$	T2a. same as T2.
T3. $aa' = bb'$	T3a. $a + a' = b + b'$
Vb . $ab + ab' = a$	Vb1. $(a + b).(a + b') = a$

Note that Vb1 is derived from V by means of sets I to T3a. Vb is derived by means of these and also of Vb1.

On comparing any set in either of these two groups with the set directly opposite it in the other group, we observe that the only difference between any two such sets is this: where one set has the mark +, the other has a dot, and vice versa (although the dot is often omitted, as we have said, and is to be supplied according to the convention stated on p. 90, above). Since this resemblance appears to be the result of having begun with certain definite symbol-sets and of having transformed these according to certain definite rules, it is reasonable to conclude that if we adhere to these same rules of transformation and apply them to no other symbol-sets but the above sets and others derived from them according to the rules above mentioned (A to D, pp. 90-91 above) a similar resemblance will hold between pairs of subsequent sets. Accepting this conclusion as a general rule of derivation, we may state it as follows: From any symbol-set containing no symbols but those above indicated, another set may be derived containing the same symbols in the same order, except that every dot in the first set must be replaced by the mark + in the second set, and every + mark in the first must be replaced by a dot in the second. Here, it will be observed, we have the well-known "principle of duality" stated as a transformation-rule.

The proviso that no symbols may be employed except those already indicated does not rule out the introduction of mere "shorthand substitutes" for unwieldy collocations of those same symbols. We have already noted that the mark 0 is used to stand for aa'; similarly, the mark 1 stands for a + a', and the combination a (b stands for the combination ab = a. Nevertheless, as we shall see later, it would be a mistake to treat such shorthand substitutes as though they were subject to the same transformation-rules as are the expressions which they replace.

To mention only two of the uses which have been made of these sets, we may observe in the first place that from them alone and their derivatives according to rules A to D, it is possible to derive all the sets employed in the class-calculus of Boolean algebra; and secondly, if we add one other set,—namely, -a = (a = aa'), whence is obtainable the derivative set, a = (a =(a + a'),— it is possible to derive all the sets of the so-called "two valued" algebra, which differs from the calculus of elementary propositions in <u>Principia Mathematica</u> only because the letters p,q,r... are used instead of the letters a,b,c..., and because the mark (between two letters is replaced by the mark); thus, instead of the combination a (b, we have the combination p) q. It would be superfluous to mention these two well-known uses, except by way of calling attention to a point of some importance: namely, the reason why our transformation-rules A to D do not include any form of the familiar "principle of inference".

In view of the fact that derivatives of the above sets can be interpreted as theorems of the calculus of elementary propositions, the mark (may be expected to occur in such derivatives just as frequently as the mark) occurs in the elementary propositional calculus of <u>Principia</u>. Sets containing the mark (will therefore be very numerous, and it may seem advisable to have a transformation-rule which refers to them explicitly. This rule could be stated quite abstractly as follows.

Whenever the mark (appears in any set of the system, in such wise that the collocation of marks to the left thereof, when considered in isolation, is recognizable as an initial or previously derived set, then the collocation of marks to the right of the mark (may be used by itself as a set of the system.

It would undoubtedly be convenient to formulate such a transformation rule governing sets containing the mark (, just as it is convenient to employ the principle of duality as a transformation rule governing sets containing the dot or the mark +. But the transformation rules A to D, already given, suffice by themselves for the derivation of all sets used in the two valued algebra, provided we apply them to the set a' = (a =aa') and its derivatives, as well as to the other sets above mentioned and derivatives thereof. Before proceeding to establish this latter contention, we must observe that the initial a' in the set a' = (a = aa') is introduced as a mere shorthand substitute for the bracketed expression to the right of the subsequent mark =. When we speak of the set a = (a = a + a') as a derivative of the above, we do not mean that the initial a in this latter set appears in virtue of applying rules A to D to the above a'; for as has been noted, such shorthand substitutes are not subject to these transformation rules directly. In the cases previously cited, e.g., the use of 0 for aa', confusion is not likely to arise, because the substitute sign is not the same as any previously-admitted sign of the system. Here, however, the introduced substitutes a' and a are indistinguishable from the signs a' and a which occur in sets I to VI and their derivatives according to rules A to D. In spite of this resemblance, we have no right to say that they come under these transformation-rules, unless it can be shown that each of them, respectively, may anywhere replace some one and the same single letter (or collocation which may be regarded as a single letter) that does come under the rules in question. Instead of saying, then, that the set $a = (a = a + a')$ is a derivative of the set $a' = (a = aa')$,-- thus suggesting that the initial signs a' and a are subject to rules A to D, which in fact they are not,-- we should rather say that if a' be used as a substitute for the set $(a = aa')$, the use of a as a substitute for the set $(a = a + a')$ enables us to develop a system in which these new signs can be manipulated according to rules A to D. In order to show that this latter statement is true, we must anticipate a point to be mentioned later: namely, that the mark = is to be read as "is anywhere interchangeable with"; hence we are no longer regarding the mark = as meaningless. If we begin with the initial set VI, and substitute a' for b, and then apply T3 and III to the result, we get: T4. $a + a' = -(aa')$. Next, according to T2, $-(a') = a$. Now if we agree to use a' as a substitute for "a is anywhere interchangeable with aa'", we shall be following the rules suggested by the above two sets if we use a,-- i.e. $-(a')$,-- as a substitute for "a is anywhere interchangeable with $a + a'$,-- i.e., $-(aa')$". The new signs are thus brought under our original rules of transformation; but it must be remembered that they are so only on condition that every a for which a' is a substitute be everywhere interchangeable with aa', and similarly, that every a for which a is a substitute be everywhere interchangeable with $a + a'$. It is of course questionable whether we can introduce into the system, as it stands, both (1) an a everywhere interchangeable with aa', and (2) an a everywhere interchangeable with $a + a'$, so that a' and a may themselves be used as substitutes, respectively, for the former and for the latter. The difficulty is that by T2, -(a') = a; and by T4 (p. 95, above), a + a' = -(aa'): hence if we introduce both (1) and (2) above mentioned, either T2 or T4 will have to be dropped from the system. This difficulty cannot be solved so long as we regard our symbols as meaningless but recognizable marks. In fact, it would hardly have arisen at all if we had not already allowed the mark = to have some meaning: to mean what is meant by the words "is everywhere interchangeable with". What we wish to point out at present is that <u>if</u> we can and do introduce <u>both</u> the set a = a + a', for which a is to be a substitute, <u>and</u> the set a = aa', for which a' is to be a substitute, it will then be possible to derive all the sets used as theorems in the two-valued algebra merely by using rules A to D, without having to employ some form of the principle of inference as a transformation-rule. We have already suggested how such a transformation-rule might be worded (93-94, above) its effect would be, that if we have a set a (b, then if that same a, by itself, is a set of the system, b may be used by itself as a set of the system. The contention here is that such a transformation-rule is not strictly necessary, because if the a in question is merely a substitute for an a which is everywhere interchangeable with a + a', our rules A to D, when applied to the sets a and a(b, will produce the very same result as would such a transformation-rule. For, once we have introduced the set a = a + a', we may, according to rule A, substitute b for a therein and so obtain the set b = b + b'. Since, by T3a, a + a' = b + b', the application of rule D gives us the set a = b. (It may be noted, incidentally, that if we had not already introduced the set a(b, we should now be in a position to derive it: for a(b is merely shorthand for ab = a; and given a = b, we can obtain the set ab = a from T1, aa = a, by substituting b for the second a according to rule D.) Now, the set a = b means that a and b are everywhere interchangeable within the system. Under the above conditions, then, given a(b, if a occurs by itself as a set, then b may be used by itself as a set. Therefore we do not need any form of the principle of inference among our transformation-rules, since the same result can be obtained without it. It is hardly necessary to show, by a similar line of reasoning, that the appearance within the system of the two sets a' and a'(b', warrants the use of b' as a separate set; for this conclusion is easily arrived at if it be recalled (as was noted on pp. 94-95, above) that a' is merely a shorthand substitute for the set a = aa', and if reference is then made to T3 instead of to T3a.

2. Logical Significance of Purely Formalist Derivation.

It is generally admitted that, as we have said above (p. 93) all the sets used as theorems in Boolean algebra and (provided suitable changes in notation be adopted) in the elementary calculus of the Principia can be derived from sets I to VI according to rules A to D; and further, that such derivation can be effected by a kind of mechanical application of these rules to these sets, without reference to the logical significance of the operations thus performed. Once certain derived sets have actually been selected as theorems, to the exclusion of an indefinite number of other sets (which, though similarly derived, have been set aside as mere steps in the process), the matter of recognizing any given derived set as a theorem is a mere matter of observing similarity of physical characteristics, which of course involves no reference to logical significance. But since we wish to inquire what is meant by the statement, "These initial sets and the theorems derived from them according to the above rules constitute a deductive system" and since we are further interested in the reasons why this statement is true (or alternatively, the reasons why it is false), an investigation of the logical significance of the operations above described assumes importance. If the words "deductive system" be taken in their ordinary sense, these initial sets and derivatives thereof will constitute a deductive system only on condition that the derived sets are consequences of the initial sets. In view of the fact that derivatives are obtained from initial sets by performing the operations permitted by the rules, a closer study of these rules, and of their effects when applied, may show us more clearly the relation between any set and its derivatives.

If we consider sets I and II, together with rules A, B and C, the following points come to light. First, by means of these sets and these rules, certain marks are specified as usable in this system. Secondly, among the marks so specified, a twofold division is indicated: (a) marks which may be replaced by other marks, and (b) marks which may not thus be replaced. Thirdly, in the light of the rules, sets I and II are readable as propositions. A closer study of these three points will contribute to an understanding of the logical significance of the operations under discussion. In what follows, for convenience of reference, the phrase "well-formed expression" will be used as an abbreviation of "any single letter, or any collocation of marks which may be substituted for a single letter according to rules A,B,C."

To begin with the third of the above points: anyone who understands the meaning of rules A to C will recognize that set I may be read as follows. "Any two well-formed expressions, a,b, <u>may be joined by a dot and taken together as a single</u> well-formed expression, (a.b)." Similarly, set II may be read: "Any well-formed expression, a, <u>may be followed by the mark ' and taken as a marked</u> well-formed expression, a'." (As was noted on p. 90 above, the dot is often omitted; hence any two well-formed expressions between which no mark appears are understood to be joined by a dot. As will be seen later, the omission of brackets causes no confusion; and finally, it will be recalled that a bracketed expression, if marked, is preceded by the mark - instead of being followed by the mark ': merely because this notation seems clearer).

The reading of these sets as propositions focuses attention on the other two points under discussion. The marks usable in this system are thus far seen to include letters of the alphabet, and the non-literal marks *, °, -,(),', and the dot. Any letter may be replaced by (a) a single letter, or (b) any collocation of letters <u>and</u> non-literal marks which appear to the right of the marks *,°, in sets I and II, or which will appear there as a result of substitution according to the rules A,B,C. The non-literal marks are not subject to replacement. Of these latter, the first two mean what is meant by the words which are underlined, in the reading of their respective sets, namely: (a) the operation of "joining together by a dot and taking together as a unit (or single expression)"; (b) the operation of "marking, and taking as a marked expression". The others, in conjunction with the letters accompanying them on the right hand side of sets I and II, signify the results obtained by performing these operations upon these same letters which, as will be observed, appear on the left hand side of sets I and II respectively. Note further that the various collocations which occur in sets III, IV and V on either side of the mark = are each obtainable from sets I and II. For instance, the complex expression -(a'b').-(a'b), in V, can be thus obtained; take a' as in II, and make in order the following substitutions: b for a in II; a',b', for a,b, in I; (a'b') for a in II; a' for a in I; (a'b) for a in II; -(a'b'), -(a'b) for a,b in I. Since each complex expression obtainable in this fashion from set I or set II or both is (according to rules B and C) a well-formed expression, we may take it as sufficiently clear, without need of further examples, that the class of well-formed expressions includes not only single letters, both marked and unmarked, but also "dot-complexes" of the same (i.e., two or more successive letters each joined by a dot to its immediate successor may be bracketed together and taken as a unit), and the latter may be marked or unmarked. It is of course understood that there is no upper limit either to the number of single letters which may be taken together as a dot-complex, or to the number of marks which may follow a given letter or complex. The reason for this is that both marked and complex expressions which are obtained from sets I and II may themselves be substituted for a,b, and a' in those sets to produce indefinitely many expressions of increasing complexity.

In order that the logical significance of this seemingly mechanical procedure may be realized, we must observe that rules A, B and C, in addition to having a clear meaning of their own, render sets I and II just as clearly meaningful as they themselves are. We have already indicated in a general way that this is so, by showing how these two sets may be read as propositions. It will be worth while to examine in fuller detail the precise meaning given by these same rules to every mark in the two sets. What the marks ° and * mean has been verbally stated above (p. 99) besides looking into their meaning more closely, we must compare this with the meaning of the other literal and non-literal marks which appear in these sets,-- the so-called variable symbols and constant symbols which are well-formed expressions of this system.

3. Truth and Meaning of the Formation Rules.

Since a detailed analysis of the meaning of the formation rules is likely to involve us sooner or later in a discussion of their truth, it will be well to point out at once in what sense, if any, they can be called true, and why they can be so called. As has been suggested (p. 99, above), both set I and set II are readable as statements to the effect that certain specified operations may be performed upon any well-formed expressions, or any pair of such expressions. On this reading, the only significance of these sets is what may be called "permissive": they inform us that certain operations are permissible on any well-formed expression; and hence they are rather rules of procedure than statements whose truth or falsity can be questioned. Nevertheless, as mere rules of procedure, they rest on certain assumptions whose truth can be directly investigated. It would be pointless, for instance, to frame rules permitting certain operations upon certain specified entities, unless the operations were such, and unless the entities were such, that these operations could be performed upon these entities. We are justified in concluding, therefore, that these sets, which explicitly contain and expressly state a mere permission to perform certain operations upon certain entities, also contain implicitly,—— that is, entail,—— the following statements whose truth can be tested: (1) These operations are such that they can be performed upon these entities. (2) These entities are such that they can be subjected to these operations. The truth of these statements becomes manifest when we observe that the entities referred to are all marks, and that the entities operations involved are such operations as "separating by a dot","bracketing together in a given order", and "placing a given mark to the right of".

Besides explicitly permitting the performance of certain operations upon certain marks, and implicitly asserting the possibility of these operations, sets I and II should be recognized as explicitly asserting that there is a necessary connection between the actual performance of the operations in question and the results of these same operations. Set I asserts a proposition (or expresses a proposition) which may be worded as follows: "If any two well-formed expressions, taken in a given order, are joined by a dot and bracketed together, the result will be a composite well-formed expression, enclosed in brackets, consisting of one of these same well-formed expressions joined by a dot to the other in that same order." And set II: "If any well-formed expression is followed by the mark ' and bracketed with it, the result will be a marked well-formed expression consisting of the originally-selected well-formed expression and that same mark."

The truth of these propositions is so manifest that one may be inclined to dismiss them as trivial. But their blatant obviousness need not prevent us from inquiring why they are true. It is tempting to suppose that they are mere tautologies, whose "truth" is entirely a matter of verbal or symbolic (i.e. purely nominal) definition, and has nothing to do with "what is the case". For we see, in the first place, that these particular marks and these particular operations have been quite arbitrarily selected and specified as "usable in this system", out of an indefinite number of differently-shaped marks and an indefinite number of different operations. And there are other systems which have been constructed by selecting different marks and performing different operations upon them; e.g., the stroke-system of Nicod and Sheffer. In the second place, the meaning of the phrase "a well-formed expression of this system" appears to be a matter of nominal and hence of arbitrary definition, inasmuch as anyone who wishes to construct a particular system can quite arbitrarily stipulate, beforehand, the conditions which any expression whatever must fulfil if it is to be regarded as well-formed, in this system. With reference to the particular system which we have selected for discussion, such a stipulation is contained in the nominal definition of "well-formed expression" stated above (p. 99) : "Any single letter, or any collocation of marks which can be substituted therefor according to rules A,B,C, in sets I and II, or in sets III to VI, or in any derivative set."

On closer consideration, however, we find that the truth of sets I and II, even when they are read as abstractly as possible,-- that is, as statements about marks and about operations performable upon marks,-- cannot be fully accounted for on a basis of arbitrary decision or nominal definition. After due allowance has been made for the arbitrary factors just mentioned, it yet remains undeniable that the very possibility of obtaining these results by performing these operations upon these marks fundamentally depends to some extent on the nature of the marks selected and of the operations specified as performable upon them. Unless the marks selected are marks of a given kind (here, letters of the alphabet), and unless the permitted operations are precisely those mentioned above, the collocations of marks which appear on the right side of sets I and II above, simply cannot be obtained. Hence the statement that these results not only will be obtained, but inevitably must be obtained, through the performance of these operations upon these marks is fundamentally a statement whose truth depends, at least in some measure, on "the way things <u>are</u>", and not entirely on our arbitrary arrangements and decisions. In other words, the truth of these statements is not a mere matter of nominal definition.

<u>4. Significance of Substitution in Sets I and II.</u>

We have already pointed out (p. 99 above) that the collocations of marks which appear as well-formed expressions in sets III to V are obtained from sets I and II by substitutions in these latter sets according to rules A,B,C. It is furthermore apparent, from a consideration of these three rules, that any collocation of marks appearing on the right-hand side of any such derived set whatever will be a well-formed expression usable in this system. We now wish to inquire whether every set derived from either I or II by substitution according to the rules is actually a consequence of the set from which it is so derived. If this question happens to be answered in the affirmative, we shall be led to the conclusion that there is a necessary connection between such derivatives and the original, in each case, and that the truth of each derivative is guaranteed by the truth of the original: for we have already indicated that the original sets in question are both true, by analysis of their terms. From this it will follow that the statement, "Any collocation of marks appearing on the right-hand side of such derivatives is a well-formed expression" must also be true. If, on the other hand, our inquiry should result in a negative answer to the question proposed, we may at least see more clearly how this kind of substitution is related to ordinary deductive reasoning.

Since ordinary deductive reasoning is carried on by taking account of the meaning of words and of sentences, it will be useful to ask whether the substitution-process whereby derivatives are obtained effects a difference of meaning in the derived set as compared with the original set. In order to answer this question as simply as possible, we must state more definitely the meaning of each symbol in sets I and II; and the easiest method of doing this seems to be as follows. Remembering that the meaning of set I is thus expressed in words: "If any pair of well-formed expressions of whatever form is selected in any order and joined by a dot and bracketed together, the result will be a well-formed expression composed of one of these two expressions joined by a dot to the other in that same order and bracketed with that other", we observe that the task of conveying this same meaning is performed by non-verbal symbols in the following fashion. (1) The symbols a,b mean what is meant by "any pair of well-formed expressions of whatever form selected in a given order". (2) The operation-mark ° means what is meant by "joined by a dot and bracketed together". (3) The collocation (a.b), symbolizing the result of these operations on these two selected expressions, means what is meant by "a well-formed expression composed of one...that other" (as above stated). Similarly, with regard to set II: (1) The symbol a means what is meant by the words "any well-formed expression of whatever form". (2) The operation-mark * means what is meant by the words "followed by the mark ' and bracketed therewith". (3) The resultant collocation of symbols, (a'), means what is meant by "a well-formed expression made up of that same well-formed expression bracketed together with the mark ' to the right of itself". It is important to observe that we are here confronted with those two very different methods of symbolization which we discussed at considerable length in an earlier chapter: direct or pictorial symbolization, and non-pictorial symbolization (see pp. 8-19, above) All the verbal symbols used to express the meaning of sets I and II, whether individual words or significant groups of words, are non-pictorial symbols. They do not directly picture that for which they stand; this is plain from the fact that they do not themselves possess the characteristics which they signify: thus, for example, the phrase "well-formed expression" is not a well-formed expression, for although it is an expression (as indeed every significant word is), it has not the characteristics summed up in the phrase "well-formed for use in this system". What has just been said of all verbal symbols is equally true of those non-verbal symbols which we have called operation-marks; these, as is evident, are simply abbreviations, arbitrarily chosen, for the group of words we have used to indicate their meaning. On the other hand, all other symbols, both literal and non-literal, are direct or pictorial symbols. The non-literal symbols, which appear together with literal ones in the collocations on the right-hand side of the operation-marks, directly picture what they stand for: namely, a mark of certain shape occupying a definite position with respect to some well-formed expression as a result of an operation on this same expression. The literal symbols, which are the same on both sides of the operation-marks, likewise directly picture what they stand for: each is, and each directly represents, a letter of a definite shape occupying a definite position with respect to some other letter or mark. Now, in all derivatives of I and II, the marks which appear on the right-hand side as a result of the operation signified by the operation-mark are the same as in the original set; or, to be quite precise: although they are entitatively or individually other than the marks in the original, each is a mere instance of the original mark which it replaces, differing from the original not at all in physical characteristics such as shape, size, and relative position. Inasmuch as each such mark has the same characteristics as the mark which it replaces, it will have the same pictorial symbolic force as its original, apart from some explicit statement to the contrary; for as we have seen, any change in the meaning of a pictorial symbol involves a change in the characteristics of such a symbol (see pp. 9-10, above). No matter what these non-literal marks mean in the original, therefore, they will have the same meaning in all derivatives; in a word, their meaning is as unaffected by the process of substitution as they themselves are. With regard to the literal symbols, we observe that these are not the same in derivative sets as in the original set; for the process of substitution consists in replacing a given literal symbol by a different one. If, for instance, we wish to obtain a derivative of set I, we must substitute some other letter for a or for b or for both; this holds true even when, as a result of substitution, we obtain the same pair of letters in reverse order. It appears, then, that if the substitution-process effects a difference of meaning in any derived set as compared with an original set, such difference will be found in the meaning of the literal symbols. In any event, the meaning of the literal symbols is worth examining more closely; for it naturally involves a discussion of the phrase "well-formed expression", and leads to a suggested distinction between the content and the form of such expressions, as well as to some further observations about the meaning of non-literal symbols, all of which have to do with the analysis of the substitution-process in sets I and II which we are now attempting to make explicit.

We have seen (p. 105, above) that the literal symbols, a,b, in set I, mean what is meant by the words "any pair of well-formed expressions selected in a given order, of whatever form". While it is true that the order in which a,b occur furnishes a pictorial representation of the "given order" here mentioned, there can be no doubt that the pair a,b do not pictorially symbolize "<u>any</u> pair of well-formed expressions of <u>whatever</u> form". For they are just one definite pair, i.e. the pair made up of the letters a,b; and each of these letters has just one definite form of its own. At this point, however, a difficulty presents itself. The word "form", as applied to well-formed expressions, is manifestly ambiguous; for it may be said on the one hand that the form of a differs from the form of b, inasmuch as the actual shape of these marks is different, and on the other hand we may truly say that a and b are of the same form, inasmuch as each is a single letter. The removal of this ambiguity is very necessary if we wish to understand fully the meaning of any well-formed expression which is a direct or pictorial symbol. All such symbols must possess the form which they directly signify; hence unless we know precisely what form such a symbol has, we cannot begin to find out precisely what form it means. Besides attempting to make more precise the notion of form by removing the above ambiguity, we must also further clarify it by considering its relations to the cognate notions of content and of structure.

5. Form and Content of Well-Formed Expressions.

It will be obvious to anyone who realizes what a variety of meanings have been given to the words "form" and "content" by past and present philosophers that we cannot here embark upon a full discussion of these various meanings. Hence, however interesting and fruitful might be the attempt to establish some general principle according to which the seemingly divergent meanings of these words could be coordinated and unified, we must limit ourselves to a study of their meaning when used of well-formed expressions. We have noted that the two well-formed expressions a and b can truly be said to be of the same form, and with equal truth, from a different point of view, can be said to be of different forms; and the situation is still further complicated when we recall (as was indicated on pp. 77-78, above) that from still another point of view any two collocations, each of which may be regarded as a unit and is also a well-formed expression, may be described as isomorphic with each other: thus it may be said that a and (b.c.d) are of the same form, on the ground that each is a single well-formed expression. These different viewpoints are easy enough to identify; the question is, what is their basis, and what difference, if any, do they bring about in the meaning of "form" as applied to well-formed expressions. An answer to this question appears necessary before we can see whether, and why, a change in the form of a well-formed expression involves a change in the meaning of that expression.

To begin with the simplest and clearest kind of case: it is true to say that a and a are well-formed expressions of the same form. Here we may be inclined to say that they are of the same form because they are instances of the same letter; but on reflection this reason is seen not to be fundamental. For it must be admitted that many instances of the letter a are very different from a in form. To say nothing of the various forms used in different languages, nor of the difference within the same language between capital letters and small letters, it is plain that many instances of the letter a have not the same form as a: consider, for example, the various forms of a available to the modern printer, even when we take no account of the use of italics, or of script. The ultimate factual basis for the statement that a is of the same form as a seems rather to be this: that a and a are instances of the same mark. Without relying unduly on linguistic considerations, we may note that in ordinary usage the words "mark" and "sign" express much the same idea. A physical object is called a mark inasmuch as it is a sign of some other object: that is to say, once a given physical object is known, and once we perceive a connection between it and some other object, it is for us a means whereby that other object is presented to thought. In all cases of direct symbolism, as we have seen, the connection between symbol and referend depends on the fact that they have in common some characteristic, notably shape, (form, outline); and it may be said quite generally that, apart from any convention or understanding, whether private or public, any physical object whatever is naturally such that it directly pictures or represents all instances of itself, that is, all other objects which have the same characteristics as itself. On the other hand, in all cases of non-pictorial symbolism the connection between symbol and referend is either entirely a matter of convention and hence entirely arbitrary, or at any rate is entirely independent of physical similarity. In particular, the connection between the physical object a and the linguistic unit a of which that object is the accepted mark or sign or symbol is entirely a matter of convention; and the same is true of all other letters. That is why, although the shape of marks can be perceived, letters have to be learned. This fact, as also the fact that such conventional connections must be known before any object or mark can be actually recognized as the symbol of a letter, is of interest to the psychologist rather than to the logician. What we are emphasizing here is, that this kind of connection simply does not <u>exist</u>, apart from arbitrary convention, and hence it is entirely independent of any characteristic of the "symbol-object", or mark, itself.

To insist on saying "the mark a is a symbol of the letter a (or, "symbolizes the letter a"), instead of "the mark a is the letter a", would in most contexts be mere pedantry; particularly because the letter a, as an accepted linguistic unit, is itself a symbol, with meaning of its own, whether by itself or as part of those collocations of letters which are accepted words of some language. But anything like a careful analysis forces us to recognize that the latter of these two descriptions is less accurate than the former: that a letter is not precisely and adequately describable as "a mark of a given shape", but rather as "a linguistic unit, which is a single integral part of those larger linguistic units called 'words', and of which a mark of a given shape is the accepted symbol".

Two very interesting consequences follow from this view. If by "letter" we mean always and only a single integral part of a word, then such an isolated mark as "a" in such a collocation as "a dog" does not symbolize a letter; instead, it is the symbol of a word, namely, of the part of speech known as "the indefinite article". We are thus led to the general observation that a single mark may symbolize either a letter or a word; and as regards collocations of marks, it should be noted that these do not always symbolize words even when each individual mark is the accepted symbol of a letter. A word is not merely a group of letters arranged in a definite order; it is a definitely-ordered group of letters accepted as a linguistic unit: that is, accepted for use as a symbolic unit in a given language. A letter in isolation is something that can be part of a word; but it is not, and will not be, actually part of a word until and unless it occupies a definite position in some group of letters which, as a whole, is conventionally accepted as a word in a given language.

The second consequence which we wish to point out is as follows. Strictly speaking, the marks a,b,c..., specified as usable in this system, do not symbolize letters of the English alphabet when used in this system; for according to the rules of the system, no collocation of these marks can ever symbolize an English word, and therefore no individual mark can stand for something that can be part of such a word. Taking both these consequences together, we may say: (a) A single mark may be used to symbolize either a letter or a word, of a given language. (b) A collocation of marks may be used to symbolize either a part of a word, e.g., a diphthong, or a word, provided that each mark in the collocation is the accepted symbol for a letter of that same language.

Perhaps the most exact way of stating what ought to be meant by "mark", "letter", and "word", in order to have a clear and consistent view of the interrelation of these entities with reference to symbolic usage, is as follows. First, a symbol is a sensibly-perceptible object (and therefore an existing, material object), used as a sign of something. Any sensibly-perceptible object, inasmuch as it has certain perceptible physical characteristics, is a natural sign of any other object which has those same physical characteristics. In particular, a mark is a natural sign of all other marks which have the same non-relational physical characteristics as itself; i.e., of all "instances" of itself. Not to depart unnecessarily from the ordinary way of speaking about words and letters, we may say that a mark of a given shape is a letter, if and only if it is accepted, by convention, as one of the basic objects which serve as signs in one of those complex symbol-structures called "written languages". Further, a collocation of letters is a word, if and only if it is accepted, by convention, as one of those linguistic symbols called "parts of speech", the precise functions of which, in those larger symbolic units called "sentences", are defined by the grammatical rules of the language. Finally, by way of clarifying what is meant by speaking of a mark as a letter if it is accepted as a "basic" part of a language, we may note that a mark is not considered as a letter unless it can form part of a word.

The preceding considerations suggest that the ambiguities attaching to the words "form" and "content" as applied to well-formed expressions may be at least partially removed by attending to the different points of view from which a well-formed expression may be regarded. Considered as a physical entity, a well-formed expression is either a single mark, or else a collocation of marks. Two single marks, such as a and a, are said to be "instances of the same mark" if they are of the same form, i.e. of the same shape. The different content of each mark, which makes each a different instance, does not affect their sameness of form; though it would if the component parts of each mark were differently shaped or differently arranged, with reference to corresponding component parts of the other. Thus, for example, the difference of form between the marks p and b, d and q, is a mere matter of different arrangement of the same component parts, i.e. of similarly-shaped component parts. So long as we have to do with individual marks, difference of content is ignored, unless it be so shaped and ordered as to make the marks not only individually different but specifically different, i.e. different in form or shape, so that we no longer have two instances of the same kind of mark, but two instances, each of which is an instance of a different kind of mark. When we wish to consider merely as physical entities those well-formed expressions which are collocations of individual marks, the distinction between form and content is more clear-cut, and content itself provides a basis of difference in kind. Two collocations of marks will be of the same form if they are composed of the same number of component parts, i.e. of individual marks, arranged in the same order. If the corresponding parts of two such collocations are marks which differ in kind from each other, the collocations will differ in content, and in that case they can not be called "instances of the same collocation" even though both collocations are of the same form. In order to be instances of the same collocation, both must have not only the same form but also the same content, i.e. be made up of the same component parts, the same number of individual marks of the same kind.

It will be seen that the statement which we have just been examining, namely, "a is of the same form as a", holds true of a and a considered as marks. If we look at the statement "a is of the same form as b", the viewpoint from which this is seen to be true is slightly different. The marks a and b are obviously not of the same form; hence if this statement is true, a and b must be viewed not as mere marks, but as well-formed expressions, that is, as marks more or less arbitrarily selected for use as the basic element-symbols of a definite symbol-system. Quite apart from any reference to interpretation, or to any system of entities which the symbol-system may symbolize, the basic or elementary well-formed expressions, a,b,c...are element-symbols inasmuch as each is a member of that arbitrarily specified class of marks which can become well-formed expressions of this system merely by selection. They are, accordingly, the least complex of all the well-formed expressions of the system, and may be called "simple" or elementary well-formed expressions; out of them the more complex well-formed expressions are constructed, by subjecting them to the operations permitted by the formation-rules of the system. Such simple well-formed expressions as a,b,c... are all of the same form precisely because each is simple: a mere element-symbol selected for use in this system, and not operated upon in any way. They cannot, however, be regarded as well-formed expressions of the same content, unless they are not only of the same form but also instances of the same mark. To say truly that a and b are of the same content, or that they have the same content, one would have to mean either (a) that each is made up of the same number of elementary well-formed expressions, viz. one, or (b) that a and b are identical in reference, or have the same range of values. In the former sense, the statement appears to be trivial, and in the latter sense it is a statement not about the well-formed expressions themselves, but about the meaning of these expressions. Hence it seems more advisable to say that a and b are of the same form but not the same in content.

A discussion of the third statement, "a is of the same form as (a.b.c.d)", will enable us to show how the form and the content of well-formed expressions are affected by the operations permitted according to sets I and II, and also how one well-formed expression may be used as a variable, to symbolize other well-formed expressions. We cannot say, as above, that these two expressions are of the same form because they are marks of the same shape, nor yet because each is a simply or elementary well-formed expression; for obviously neither of these reasons holds good here. Considered as well-formed expressions of this system, these two expressions have two features in common: each is a single well-formed expression, and each is unmarked. This latter point of resemblance evidently arises from the fact that neither has been subjected to the operation of marking, permitted by set II. It is, however, a point of less importance than the former, since even a marked expression might be said to be of the same form as a, on the gound that it, like a, is a single well-formed expression: thus, a and (b') are of the same form, in this sense.

We can now see the basis of the connection that exists between a simple well-formed expression such as a, when a is used as a variable, and all those well-formed expressions which are indeterminately symbolized by a, and are called "values of a". Some of these values, namely, all other instances of the mark a, will be strictly the same as a both in form and in content, and the symbolization in all such cases will be most direct, independent of convention. Other values, namely, all other elementary or simple well-formed expressions, will be of the same form as a, because independently of any systematic considerations, each is a single letter; but each will differ from a in content because each is a different letter from a. The variable a will in such cases directly symbolize the form of all these values of itself; independently of any systematic considerations, it has the form which it symbolizes, that of being a single letter. All other well-formed expressions not included in the two above-mentioned classes will really differ from a not only in content but also in form; and such expressions can be said to be of the same form as a only because of the systematic conventions whereby each of them is regardable as a single well-formed expression of this system. The variable a, in such cases, does not and cannot directly symbolize the form of its values, precisely because they are not really of the same form as a; but it can directly symbolize the relative position of these values in a set or in a more complex expression, because each of these values must occupy the same position as a occupies, when any one of them is substituted for a in any set or expression.

It has already been remarked (pp. 100, 104, above) that the well-formed expressions appearing in sets III to V are all derivatives of sets I and II. Before passing on to consider the special problems presented by these other sets, and the somewhat different problem connected with set VI, there is one further point of particular interest to be mentioned as regards I and II, namely: the operations permitted by these two sets affect only the form, but not the content, of all well-formed expressions derived from them by rules A,B,C.

That is to say, if we take a given number of instances of the same or of different initial well-formed expressions (i.e. letters, a,b,c...) and subject them to the operations permitted by sets I, II, and by rules A,B,C. the form of the resultant well-formed expressions will vary according to the order and number of operations to which the initially-selected expressions are subjected in each case; but the content of the resultant expressions will be the same: each will contain all and only those letters which were initially selected for operation upon, no matter what be the order or the number of times of the operations performed on each, once they have been selected. For example, take the five letters a, b, a, c, b as those to be operated upon, and subject them to two series of operations, as follows. Series A: (1) Combine a, b, by I, into (a.b); (2) Mark this, according to II, -(a.b); (3) Mark a,c separately, by II, and combine the results, by I, into (a'.c'); (4) Combine the remaining b with (a.b) above, by I, into ((b).(a.b)); (5) Mark this, by II, and (6) Combine the result, by I, with (a'.c') above, so as to get (-((b).(a.b)).(a'.c')). Series B: (1) Combine a,a, by I, into (a.a); (2) Mark b, by II, b'; (3) Combine the remaining b,c, by I, into (b.c); (4) Mark (a.a) above, by II, -(a.a); (5) Combine b' and (b.c) above, by I, into ((b').(b.c)); (6) Mark this last, by II, and (7) Combine the result with -(a.a) above, so as to get (-((b').(b.c)).-(a.a)). Although the letters initially chosen to be operated upon, and also the operations performed upon them, were selected at random, it may be unsafe to generalize from this single example. Nevertheless, we have here some ground for the general statements already made (p. 118 above): (a) that the form of well-formed expressions in this system is determined by the number and order of operations I and II performed on a given number of instances of initial letters; and (b) that the content of the resultant expressions, i.e. the number and kind of initial letters which appear in the resultant expressions, remains the same, no matter how many times, or in what order, these operations are performed on a given number of instances of initial letters.

6. Significance of Sets III to V.

Sets I and II and their derivatives, together with rules A, B and C, may be said to define the class of well-formed expressions usable in this system, inasmuch as they provide for the construction of composite expressions out of simple ones, and indicate the marks to be used as simple expressions. Sets III to V and their derivatives, as well as all subsequent sets of the system together with the derivatives thereof, contain only such expressions as appear in sets I and II or are derivable therefrom according to the rules; but they differ from the first two sets in the following important respect: each contains the conventional sign of equality, which is one of the marks indicated as a non-well-formed expression of this system.

A consideration of the significance of these sets, when they are read as abstractly as possible without reference to any of the possible interpretations of the system whereof they form a part, must accordingly include some discussion of the significance of this sign of equality. We may begin by noting that sets III and IV each contain, on either side of this sign, two well-formed expressions which differ in form but consist of the same elementary or simple components: a,b in set III and a,b,c in set IV. Hence they are the same in content, taking "content" to mean "number of instances of the same simple expressions (i.e. letters) included in each". This is not true of the two well-formed expressions which occur in set V on either side of the sign of equality; for obviously the expression a differs both in content and in form from the expression -(a'.b').-(a'.b).

As has already been suggested (p. 95, above) , the sign of equality, insofar as it has any meaning at all,-- and it <u>must</u> have meaning if we are to apply Rule D in the development of this abstract system,-- means at least what is meant by the words "is everywhere interchangeable with". On this minimal interpretation (using the word "interpretation" in a wide sense), all sets containing the sign of equality have the same kind of permissive significance which we mentioned as attaching to sets I and II, and especially to the operation-marks in those sets. Taken in conjunction with rule D, all these sets declare that the two expressions which appear on either side of the "equals" sign are everywhere interchangeable within this system. Thus interpreted, they are mere rules for the development of the system by means of substitution. But the point we wish to stress is that they, like the sets we have already discussed, rest upon certain assumptions whose truth is taken for granted; and unless these assumptions are true, these sets are useless even as rules of procedure.

It must always be borne in mind that what we are here considering is not a mere jumble of marks set down more or less at random, but an abstract symbol-system intended for use as a direct or pictorial representation. Hence the utility of this system will depend on its conforming to the primary requirement of direct symbolism; namely, that the objects (here, the marks) chosen as direct symbols of this system be so chosen and so arranged as to have the characteristics which they are intended to symbolize. If an expression is to be a direct symbol of content, it must have the content which it is meant to symbolize; and the same holds true of form and of structure. From what has already been said about the meaning of the words "form" and "content" as applied to well-formed expressions of this system, similarity of structure between any two such expressions consists in this: that each of the two must be made up of the same number of simple well-formed expressions, and that corresponding simple constituents occupy, in their respective expressions, the same relative position with respect to the other constituents of the expression in which they occur. The kind of substitution permitted by Rule A (p. 90, above) insures the preservation of the structure of all expressions in which such substitution takes place; for although the content of an expression is altered by the substitution of one kind of simple constituent (e.g. a) for another kind of simple constituent (e.g. b), this alteration of content is determined beforehand by the proviso that whenever a simple constituent is replaced, it must be replaced by the same substitute throughout the expression in which it occurs; and consequently the respective position of substitute and original constituents remains entirely unaltered. The modified expression as a whole will thus be isomorphic with the original expression, and can therefore be used to symbolize directly the structure possessed by and symbolizable by the original. But when we extend Rule A, so as to permit not merely the replacement of simple constituents by simple constituents (i.e. single letters by single letters), but "each time" substitution, for any well-formed expression, of any other well-formed expression of whatever structure, it is evident that such substitute expressions as differ in structure from the originals whose position they occupy can not directly symbolize the structure possessed by and symbolized by the originals.

It may indeed be objected that there is a sense in which every substitute expression can be regarded as isomorphic with any original expression which it replaces: namely, that each is a single well-formed expression of this system. The fact remains, however, that there is a real difference in structure between those well-formed expressions which are simple, and those which are formed by operating upon simple expressions according to sets I and II, as well as between any two well-formed expressions whose simple constituents have been subjected to the above operations in a different order or a different number of times. The singleness or oneness attributable to non-simple expressions is purely systemic, whereas the singleness of simple expressions is extra-systemic, based on the fact that simple expressions are single letters, and hence have a oneness that is independent of their status as expressions in this system. We may, it is true, agree to treat all marked or composite expressions as though they were simple expressions; but if we do, we are agreeing to ignore their structural differences, and hence not to make use of them as direct symbols of structure; for as has been said, the only structure which they can directly symbolize is the structure which they have, and when we ignore this structure we cannot consistently make use of it for purposes of direct symbolism. If we wish to maintain that (a.b.c.d) and a are both single expressions and can therefore be regarded as isomorphic, consistency obliges us to maintain this position when using these two expressions as direct symbols of structure, and therefore not to employ as directly symbolic the actual structural features of the composite expression (a.b.c.d). To employ these actual structural features entails, on the other hand, an admission that (a.b.c.d) is not isomorphic with a.

Rule D, it will be observed, permits even greater latitude in the matter of substitution than does the above-mentioned extension of Rule A, for according to Rule D, two expressions occurring on either side of the sign of equality are everywhere interchangeable, and this means that when one is substituted for the other, such substitution need not be made each time, throughout the whole expression in which that other occurs. To understand the implications of this rule, we shall examine in detail the sets in which the sign of equality first makes its appearance.

Set III, if written in full, would be as follows: (a.b) = (b.a). Taking the bracketed expressions as mere resultants of an operation permitted within this system, namely, the operation signified by set I, we may read set III thus: "The expression obtained by operating on the letters a,b according to set I is everywhere interchangeable with the expression obtained by similarly operating on the letters b,a." If set III be regarded as a mere rule of procedure, the above statement is simply an arbitrary declaration, or permission to the effect that these two expressions may replace each other anywhere within this system. And the same may be said when the set is given a general interpretation, by taking a,b and b, a as variables meaning any pair of well-formed expressions in a given order, and that same pair of well-formed expressions in reverse order. If the sets of this system were indeed meaningless marks, or if the symbolic force of the marks occurring in this system were to be settled entirely by convention, there would be little point in raising further questions. But as a matter of fact, the very readability of these sets as sentences shows that they have at least as much meaning as sentences have; and because they are to be used as direct symbols of structure, all rules of procedure touching their development must conform to the requirements of direct symbolism already noted. It is therefore important to examine the implications of these rules of procedure from an extra-systemic point of view, and to see what other statements there are, if any, which must at least be assumed as true in order that such rules of procedure may be useful for the development of a directly symbolic system of abstract symbols.

Since (a.b) and (b.a) differ in the order of their component parts, set III would be false if read as a statement that these two expressions were exactly the same, i.e. mere different instances of the same expression, such as a and a are. Clearly, "a joined by a dot to b and bracketed together with b as a single expression" is not exactly the same as "b joined by a dot to a and bracketed together with a as a single expression." This observation applies no less clearly when a and b are taken as variables, standing for "any pair of well-formed expressions in a given order". Hence we see that if this set is read as a statement about marks of a given kind, or about the results of an operation upon marks (i.e. the operation of joining by a dot in a given order, and bracketing together), it would be a false statement were we to read the sign of equality as meaning "is identical with". The difficulty is apparently solved if we read this sign as above, meaning "is everywhere interchangeable with"; but only apparently: for this last implies "for purposes of direct symbolism within this system," and as we have seen, such interchangeable expressions must be identical at least in structure. Though (a.b) and (b.a) can be regarded as identical in structure in spite of the different order of their components, no such identity is attributable to the many other expressions obtainable by substituting various values of a and b when these latter are taken as variables. Hence another solution of the difficulty must be sought for.

The simplest solution would seem to be this. Take a,b as meaning two distinct elements, or individual members, of an aggregate, and take the dot as meaning a relation which holds between these two elements in consequence of some operation whereby the two are combined without losing their individual distinctness; finally, let the brackets mean the unity which attaches to the combination of these elements so related. Set III may then be read as follows: "Any pair of elements bound together into a composite unit in a given order by some relation or other is, when considered as a unit, identical with that same pair of elements bound together into a composite unit in the reverse order by that same relation, when this differently-ordered pair is considered as a unit." Lest the word "same" in the above statement be misleading, it should perhaps be remarked that it is mere shorthand for "a pair of instances of the same elements" (i.e. "another pair..") and "another instance of the same relation". For in strict accuracy, two things which are individually distinct cannot be called "identical with" each other; though they may be identical in kind: i.e. they may be two different instances of the same kind of thing.

Whether the above statement is true or false, it is at least not evidently self-contradictory, and cannot be dismissed as meaningless nonsense. Moreover, it will be a true statement if the following condition is fulfilled: if there already exist at least one class of elements and at least one relation, such that if this relation holds between any pair of elements in a given order, it will also hold between that same pair of elements in reverse order. We need not pause at this point to inquire whether such a class of elements and such a relation does in fact exist, nor attempt to say anything further about the nature of these entities if they do exist. It is enough to note here that the existence of such entities is at least possible, for we can conceive of them as existing, either as actual entities in the physical universe (under clearly definable conditions), or as mental constructs whose existence is a mere matter of their being thought. We may remark, further, that the letters a,b,c... themselves constitute just such a class of elements; but the relation signified by the dots and brackets cannot be the relation which holds between a,b as a result of placing a dot between these two letters in a given order and bracketing them as a single expression. For although the pair a,b has the same components as the pair b,a, it cannot be the case that "a joined by a dot to b in this order and bracketed together" is the same as "b joined by a dot to a in this order and bracketed together". The reason is, obviously, that as a result of the first operation we have "a to the left of b", and as a result of the second operation we have "a to the right of b". If, however, the relation signified by the dot is the relation signified by the words "next to", set III can be read as a statement about the letters a,b, c..., or any well-formed expression constructible therefrom according to the rules of this system, and the truth of this statement will be evident on reflection. For no matter what letters or well-formed expressions the marks a,b stand for, ~~it~~ is undeniable that "a next to b" and "b next to a" are everywhere interchangeable.

A very similar line of thought suggests itself in connection with Set IV, which if written in full would be as follows: ((a.b).c)=(a.(b.c)). In the two well-formed expressions here indicated as everywhere interchangeable, we do not find, as above, the same pair of simple expressions dotted and bracketed together in two different orders, one the reverse of the other. Each expression is indeed made up of the same simple components a,b,c; but the manner in which these have been operated on according to the operations mentioned in set I is in each case different. In the first, the pair a,b has been dotted and bracketed together in that order, and the composite expression thus formed has been dotted and bracketed together with c in that order. In the second, the pair b,c has similarly been dotted and bracketed together, and then a has been dotted and bracketed together with this composite expression. If, as in the case of set III, we take a, b,c as standing for what they are, i.e. instances of letters,-- whether of the same or of different letters does not matter, though actually they are instances of different letters,-- set IV may be read as a significant statement to the effect that certain operations performed upon the same three instances of letters yield the same results; but the operations signified cannot be those of dotting and bracketing together, or else the statement will not be true. On the face of it, the performance of these operations on these three letters in the two different fashions above described yields results which are different in each case; hence to say that these results are the same would be to make a false statement. To take the dot as standing for the relation meant by the words "next to" does not give us an obviously true statement as it did in the case of set III, for it is not easy to see what ought to be meant by the brackets here. The simplest reading of this set would seem to be a statement to the effect that the operations of dotting and bracketing together, performed in the two different manners above mentioned on the same three well-formed expressions will not change the expressions thus operated upon. Such a statement is no doubt trivial; but it has the advantage of being analytically true, apart from any conventions, and it is a statement about well-formed expressions of this system, so that the problem of existence does not arise.

This suggests a way of avoiding the problem of existence which was raised by the reading of set III previously suggested

(p. 124, above). We may read set III as a statement about well-formed expressions of this system, to the effect that the relation brought about by the operation of dotting and combining any pair of such expressions remains the same, no matter in what order these expressions are subjected to these operations; and further, that the expressions thus operated upon undergo no change in consequence of the operations to which they are subjected. This, again, however trivial, is a statement which is analytically true, apart from any conventional considerations.

Finally, sets III and IV may be read conjointly as a statement about the results of subjecting two or more well-formed expressions to the operations permitted by set I: a statement, namely, that the results of performing these operations on any number of well-formed expressions will be the same, no matter in what order these expressions are combined, nor how many of them are bracketed together, provided the same number of instances of the same expressions be subjected to these two operations.

If we content ourselves with remarking that the sameness of these resultants is merely a sameness of content in the sense of "consisting of the same simple well-formed expressions", sets III and IV add nothing to the information already contained in sets I and II regarding the effect of operations on simple well-formed expressions of this system; for we have already noted that if the same number of instances of the same simple expressions be subjected, in any order, and any number of times, to the operations permitted by sets I and II, the content of the resultant expressions will be the same, although their form will be different according to the different number of times each has been subjected to the same, or to different, operations. From this point of view, for instance, the expressions-(a.b) and (a'.b') are the same in content though different in form. Yet we cannot assume that these two are interchangeable merely because (a.b).c and a.(b.c) are interchangeable. In brief, it seems that interchangeability within this system presupposes something more than sameness of content, where content means only "same number of instances of same letters".

7. Significance of Set V.

When we come to examine the fifth of our initial sets of marks, the implications of interchangeability are seen to be more far-reaching than we have thus far had reason to believe, and it is no longer possible to maintain that the initial sets can be read as analytically true statements about the results of subjecting well-formed expressions to the operations of joining by a dot and bracketing together and placing the mark ' to the right of. Set V, it will be recalled, is as follows: -(a'.b').-(a'.b) = a. Here, as is evident, the well-formed expressions on either side of the sign of equality differ not only in form but also in content: on the left hand side we have two instances of the mark a and two of the mark b, while on the right hand side there is but a single instance of the mark a and no instance of b. And it is certainly not true to say that if two instances of a and two instances of b are subjected to the operations of dotting and bracketing together and marking, in the manner indicated by the expression to the left of the sign of equality, this resultant is somehow the same as a single instance of a not operated upon at all. Furthermore, it is very difficult, if not impossible, to imagine what other operations, which, like those of dotting and bracketing and marking, are operations performable upon letters, could effect any kind of sameness between -(a'.b').-(a'.b) on the one hand and a on the other. The conclusion therefore seems justified that this set cannot be read as a true statement about the results of subjecting letters, or well-formed expressions, to any operations on letters which might be symbolized indirectly by the marks . and () and '.

Such a conclusion, it may be said, suggests that any attempt to read these sets as true statements about anything whatever ought to be abandoned: that they should be taken merely as statements about the way certain marks are going to be used within this system, and hence that the question of their implications is entirely irrelevant. Thus Set V merely says that the two expressions -(a'.b').-(a'b) and a may be used interchangeably throughout the present system, one being a permissible substitute for the other wherever that other occurs. This way of looking at the matter is not very satisfactory, however, for it merely evades, without settling, the main problem which we have been considering. Every expression in this system is intended for use as a direct symbol of structure, to symbolize the structure of any entities in any existing or constructible system which happen to be isomorphic with expressions in this system. Hence if two expressions in this system are to be designated as everywhere interchangeable, they must either have the same structure or else have some one and the same characteristic in common, so that they may thus be both equally susceptible of use as a direct symbol of the same entity. Now, taking a and -(a'.b').-(a'.b) simply as possibly interchangeable well-formed expressions, it is plain that the only characteristic they have in common is that each is a single well-formed expression of this system. They are not isomorphic with each other, unless we wish to remove all definiteness from the notion of isomorphism by insisting that any two or more single well-formed expressions of this system are similar in structure.

The problem of significance becomes more acute when we realize the capital importance of this particular Set V in the present system. Upon it, as we have noted (p. 91, above) depend the three sets designated as T1, T2, and T3, whereby the following interchanges of expression are permitted anywhere in the system: aa for a, -(a') for a, aa' for bb'; and vice versa, in each case. Without these three sets, this system could not well be used, as it is in fact used, to symbolize the structure of logically-related classes, propositions, and relations; and therefore any discussion of the validity of this system must include an examination of the basic set, i.e. set V, from which these three sets are derived.

Having thus far arrived at the negative conclusion that this set cannot be read as a true statement about the results of operations on well-formed expressions, because, as has been said, a and (a'.b').-(a'.b) are certainly not everywhere interchangeable as direct symbols of the same structure, so long as the literal symbols are taken to stand for well-formed expressions and the non-literal symbols are taken to stand for relations or modifications brought about by operations on well-formed expressions, we must see whether some positive position can be taken, at least tentatively, regarding the reading of this set as a true proposition. A suggestion previously made seems to be of some help in this connection. The literal symbols may be taken as signifying, not symbols or letters, but some non-symbolic entities which are sufficiently homogeneous to be regarded as elements of some aggregate; and the non-literal symbols may be taken as signifying relations between these elements, or modifications of them, which are the result of performing specified operations on these elements. This view entails two important limitations. Not only must the elements be sufficiently homogeneous to be regardable as elements of some one and the same aggregate,-- which means that they must all have at least one characteristic in common,-- but they must be such, and also the relations or modifications referred to must be such, that any element so modified or so related remains an element of that same aggregate in spite of the relations or modifications it thus acquires.

Considering these two conditions, it becomes necessary to alter a statement contained in the beginning of the preceding paragraph. We need not insist that the literal symbols be taken to mean non-symbolic entities: because such symbolic entities as letters and well-formed expressions, constructed out of marks according to a definite pattern, are sufficiently homogeneous to be regarded as elements of one and the same aggregate. But we must insist that the non-literal symbols be taken to stand for some other modifications of well-formed expressions, or some other relations between them, than is indicated directly by these symbols themselves. That is, the dot and brackets must mean something other than "the result of dotting and bracketing together", and the mark ' must mean something other than "the result of placing the mark ' to the right of". For if the non-literal symbols in these sets are taken to stand for the results of these operations upon literal symbols or well-formed expressions, the sets will, when read in this sense as propositions, be manifestly false.

The fundamental reason why the dot and the brackets and the mark ' cannot be read as meaning the results of the operations of dotting and bracketing in a given order and of placing the mark ' to the right of, seems to be this: these operations do not actually effect modifications of well-formed expressions, nor do they bring about relations between them. If a pair of simple expressions, such as a,b, are subjected to the dot-operation, we may indeed read the bracketed result as "a joined by a dot to b"; but it is no less true that the result is "a separated by a dot from b". And the result of bracketing is not to bring a and b together into unity, but merely to leave the mark (at the left of a and the mark) at the right of b. The brackets can indeed serve as a sign of the togetherness of a and b as a composite unit; but in that event, the operation whose result they symbolize is not the operation of enclosing in brackets, but the mental operation of "thinking a and b together", or of "regarding a,b as a unit consisting of two expressions mutually related". Finally, a' does not mean simply "a with the mark ' to its right", because the mark ' is taken together with a, even when no enclosing brackets are used to indicate this togetherness: for a' is to be regarded as a single well-formed expression in this system. The modification signified by "the mark I to the right of" cannot be one which actually effects a change in a itself; otherwise it would not be appropriately symbolized by "the mark ' to the right of", which obviously leaves a unaltered. Hence we may say that it must stand either for some mental operation,-- or rather, for the result of some mental operation,-- or else for the result of some physical operation whereby a', as a whole, is rendered different from a although a itself undergoes no change. Even in the latter case, the brackets which visibly enclose a', or are "understood" to enclose a' even if omitted, must be taken to stand for the result of the mental operation of "taking together with"; for the marked letter is a single well-formed expression, a symbolic unit, though composite and not simple.

7. <u>Comparison between Sets I and II, and Sets III to V.</u>

By way of summarizing what has been said about the significance of all the sets of marks which we have thus far considered, it will be useful to compare the first two with the last three. The first two, as we have seen, taken in conjunction with Rules A,B,C, specify the material out of which all well-formed expressions of this system are to be constructed, and provide fixed rules for the construction of complex expressions out of simple ones. The reading of these sets (pp. 102, 105-106, above) indicates that they are not only significant statements, but analytically true statements about the marks specified as well-formed expressions and about the results of subjecting these marks to certain operations (joining by a dot in a given order, placing the mark ' to the right of , and bracketing together, which we shall hereafter refer to as a,b,c, respectively). The symbolic force of the marks used in these sets is as follows. The non-literal marks, none of which are usable as well-formed expressions, are of two kinds: first, the operation-marks ° and *, which indirectly symbolize, respectively the operations a and c (set I) and b and c (set II); second, the dot and brackets, and the mark ' and brackets, which directly symbolize (i.e. picture), respectively, the results of these operations. The literal symbols a,b,c..., considered in the light of Rules A, B,C, are seen to be variable symbols: that is, each is replaceable by, and therefore represents or stands for, any well-formed expression of whatever form. Each directly symbolizes the content and the form of every instance of itself; likewise, each directly symbolizes the structure, that is, the oneness and non-complexity, of all simple well-formed expressions; and finally, each directly symbolizes the position, but not the structure, of all complex well-formed expressions, as well as the oneness of such complexes. With regard to all derivatives of sets I and II, a study of the process of substitution whereby derivatives are obtained indicates that every such derivative is a consequence of the set from which it is derivable and derived; and this without altering the ordinary meaning of the phrase "is a consequence of". The line of thought which leads to this conclusion may be sketched as follows: If the performance of operations a and c, or of operations b and c, produces a given effect upon a, or upon a,b, then the performance of these same operations will produce the same effect upon any value of a, or upon any values of a,b. Now, the collocation of marks to the right of the operation sign in sets I and II makes clear the fact that a given effect is produced, and also shows what that effect is: namely, (a.b), and (a'). Hence the above condition is fulfilled. The necessary connection between this antecedent and its consequent is based on the fact that all values of a, or of a,b, are fundamentally of the same kind as a; that is, either simple marks or collocations of simple marks: the collocations being operated on as a unit, just as though they were themselves simple and single. Since sets I and II are analytically true, and since all derivatives of these sets are consequences of their originals, it follows that all derivative sets are likewise true, even if owing to their complexity their truth may not be at once intuitively evident.

Finally, two points are to be noted regarding the truth of these sets and their derivatives, and regarding the referential force of the symbols employed. First, insofar as truth depends on the existence of the well-formed expressions and of the operations thereon and of the results or effects of these operations,-- all of which the sets and their derivatives are "statements about",-- no difficulty arises in this connection. For the existence of the operations and of their results is guaranteed by actual experience, and we acquire knowledge of their existence by easy and direct reflection. The existence of simple well-formed expressions is assured, because they are all instances of marks in actual use as letters of the English alphabet; and complex well-formed expressions are merely collocations of these marks, together with other specified marks, whereof the order and method of arrangement is clearly prescribed by the rules. If a collocation is very complex, we may indeed find it difficult to know whether this collocation is a well-formed expression; but such knowledge is in any event unnecessary, since no such collocation will be actually substituted for the variables in sets I or II until we know that it is a value of the variable in each case: that is, until we know that it is a well-formed expression. Second, it is of the utmost importance to notice that when these sets or their derivatives are read as statements, in the manner already described at length, such statements are always about what the marks in the sets mean, that is, about the entities meant by these marks, rather than about the marks themselves. In other words, we must never lose sight of the difference between a sensibly-perceptible object (such as a mark) used as a symbol, on the one hand, and on the other hand the entity referred to or meant by such a symbol-object. A symbol is always a significant mark (supposing, of course, that the object used as a symbol is a mark and not some other sensibly-perceptible object); and in all cases, even when the symbol refers to itself, or to some characteristic of itself as found in some other object, or to another instance of the same kind of thing as itself,— as happens in direct symbolism,— what the symbol refers to must be recognized as somehow distinct from the symbol itself, so that we may compare the symbol with its referend and thus perceive whether symbol and referend are in fact identical. A full discussion of the implications of this distinction cannot be attempted here; but we may suggest in passing that anyone who considers it carefully will see that it has something to do with the theory of types, insofar as this theory is more than a mere systemic device. Only by ignoring this distinction, it would seem, can the suggestion arise that a function might be one of its own values, or that a proposition might be about itself. The point to note here, in connection with the sets we are studying, is that these sets are about the values of the symbols contained in them rather than about the symbols which they contain, in spite of the fact that some of these values are themselves symbols.

Our study of the three latter sets, III, IV and V, made clear the necessity of reading them somewhat differently from the first two sets. Taken in conjunction with Rule D as mere rules of procedure, they give rise to further possible transformations of well-formed expressions than are provided for by the "each time" substitutions allowed by Rules A,B,C, inasmuch as they permit unrestricted interchangeability of the well-formed expressions which appear in them on either side of the sign of equality. If these sets are to be read as permitting the interchange of the two well-formed expressions which they contain, then it must be the case that each of the two expressions has the same symbolic force; and because there is question here of direct or pictorial symbolism, the two expressions specified as everywhere interchangeable must be in some sense the same: for if they are to have the same directly symbolic force, they must have in common at least one characteristic which can be used as a directly symbolic characteristic in the case of each. Since the well-formed expressions of this system are intended for use as direct symbols of structure, it ought to be the case that expressions specified as everywhere interchangeable have the same structure; for only on that condition can they be used to symbolize directly the same structure. Set III does indeed fulfil this requirement; and even set IV can be regarded as containing two isomorphic expressions; but if the two expressions in set V be considered isomorphic with each other, the meaning of isomorphism is practically lost, since in this case any two expressions of whatever form can be called isomorphic with each other provided each can be regarded as a single well-formed expression. Having thus concluded that the two expressions in each of these sets have not the same symbolic force, we further conclude that their permitted interchangeability should not be taken to entail the contrary-to-fact proposition that they do have it; and this in turn warrants the further conclusion that there must be some other ground for the interchangeability here allowed, or that there is something else which these sets do entail. In spite of the fact that these expressions have not the same symbolic force as direct symbols of structure, their status as elements in a symbol-system indicates that they are intended to be somehow used as symbols; and there appears to be no other basis for permitting unrestricted interchangeability except that they can somehow be identical in reference, in a pictorial fashion. The only way in which two interchangeable expressions of different form can be regarded as identical in reference would seem to be this: Take each as picturing the result of certain operations upon certain elements, so that by these operations the elements are modified, or are related to one another, or both; and regard two interchangeable expressions as picturing some sort of sameness between the two resultants, in spite of the pictured differences of their structure. Hence it is suggested that these sets are to be read as follows: III. (a.b) = (b.a): "Any pair of elements, joined together in a given order by some relation or other so as to form a single complex element, is identical with that same pair of elements joined together in reverse order by that same relation so as to form a single complex element." IV. ((a.b).c) = (a.(b.c)): "A single complex element made up of any pair of elements joined together in a given order by some relation or other and then joined as a unit to a third element in a given order by that same relation is identical with a single complex element made up of the same three basic component elements, the first of which is joined, in that same order and by that same relation, to the second and third joined together as a unit in the same order as before and by the same relation." V. (-(a'.b'). -(a'.b)) = a: "A single complex element made up of two pairs of elements, each pair being modified in the same given way and both joined together in a given order by some relation or other: the first pair consisting of two elements, both modified in the way above referred to and joined together in a given order by the same relation as above; the second pair consisting of the same two elements, of which the first alone is modified, and both of which are joined together in the same order and by the same relation as before, is identical with the unmodified first element of each pair."

We must now go on to inquire into the implications of the above statements, which can be so much more shortly expressed in symbols than in words by the simple device of using the same letter to stand for the same basic element and letting the dots and brackets stand for the same relation and as a sign of unity, and by letting the marks - and ' represent a modification of the bracketed elements or the single element next to these marks. But it must be remarked that the dot and the brackets and the marks do not stand, as they did in sets I and II, for the results of the operations a,b,c. Hence in order to have the same reading for sets I and II as for sets III to V, we must take these marks to stand for the results of some other operations: namely, the same relation and the same modifications which they symbolize in sets III to V. On this principle, set I should be read as follows: "Any pair of elements whatever may be joined together by some relation or other, and the result will be a complex element consisting of those same two elements joined together by that same relation in a given order." And set II should be read: "Any element whatever may be subjected to a certain modification, and the result will be that same element modified in that same way."

8. Consequences of a Uniform Reading of these Five Sets.

The above translation of sets I to V into words shows that each can be read as a general proposition about elements of some class or other, and about the results of subjecting these elements to certain operations whereby they are modified or joined together as a unit by some relation. All these sets are alike, inasmuch as all of them contain, at least in part, instances of the same symbolic marks; but this surface resemblance would not suffice to establish a real connection between them unless the same marks had the same significance in every set in which they occur. Letters enclosed in brackets signify elements regarded as a single unit; an element followed by the mark ' is understood to be bracketed with that mark, and the whole expression signifies a modified form of the unmarked element represented by the letter which the mark follows; moreover, the mark ' stands for the same kind of modification, wherever it occurs; and what is said of this mark applies also to the mark - preceding a bracketed expression. Finally, the dot between two letters or two bracketed expressions signifies one and the same relation, wherever it appears. With regard to the marks ° and * in sets I and II, which we have called "operation-marks", they must stand for operations which, if performed upon elements, will give rise respectively to the relation signified by the dot and to the modification signified by the mark '; and both of them must stand for the operation of "joining together into unity in a given order", the resultant unity being indicated by the brackets.

The reading initially suggested for sets I and II, according to which the operation-marks and relation-marks and modification-marks and brackets were taken at their face value as meaning the operations a,b,c and the results thereof, had to be abandoned because these same marks could not be given this meaning in sets III to V. The question whether the literal symbols, either individually or when bracketed together as a unit, could be taken at their face value in all five sets, as meaning "a well-formed expression of a given form", was left open: the suggestion being made that it is answerable affirmatively if the non-literal symbols can be taken to stand for such relations between, or such modifications of, well-formed expressions (i.e. letters or collocations of letters permissible in this system) as will satisfy the requirements of direct symbolism regarding those expressions which are allowed by Sets III to V to be everywhere interchanged. In any event, the literal symbols and bracketed collocations must stand for elements of some one and the same class, whether or not this class could be the class of well-formed expressions itself; and this imposes a very definite restriction on the possible meanings of the operation-marks and other non-literal symbols. We saw that the truth of the statements obtained by reading sets I and II at their face value is due at least in part to the nature of the entities which these statements are about: i.e. to their being marks or collocations of marks, and operations performable on marks (see pp. 101-104, above). Manifestly, no change is made in the marks initially selected as well-formed expressions by the operations performable on them; they themselves are not altered by being joined by a dot to some other mark in a given order, or by being followed by the mark ', or by being enclosed in brackets. Hence there is a factual basis for the seemingly quite arbitrary statements: "If a,b are well-formed expressions of this system, (a.b) will also be a well-formed expression", and "If a is a well-formed expression of this system, a' will also be a well-formed expression". In like manner, when these five sets are read as statements about elements of some class, and about the results of operations performable on these elements, the operations must be such that elements which are subjected to them remain fundamentally unaltered, so as to be still elements of that same class in spite of receiving new characteristics. It ought to be true that "if a,b are elements of a given class, then (a.b) will be an element of that same class, no matter what values be assigned to a,b". Clearly, this will not be a true statement unless (1) the relation meant by the dot is such that it can hold between a and b without effecting a fundamental change in either a or b, and (2) the combination "a joined to b by this relation in a given order" is so like to a,b in isolation that it can be regarded as a complex element of the same class whereof a,b are simple elements. Similarly, it ought to be true that "if a is an element of a given class, then a' will be an element of that same class, no matter what value be assigned to a." And this will not be a true statement unless (1) the modification signified by the mark ' is such as to effect no fundamental change in a itself, and consequently (2) the modified element a' can be regarded as of the same class as a in spite of this modification.

If all five sets are to be read as connected statements, the above restrictions concerning the meaning of the symbols they contain must apply to the last three as well as to the first two. But sets III to V impose further restrictions, which affect the symbols of all the sets. The well-formed expressions occurring on either side of the mark = in sets containing this mark must be identical in reference, because, as we have seen, they are everywhere interchangeable and are intended to have the same directly symbolic force. The precise restriction imposed by each set can be gathered from a consideration of each in turn. It must be remembered that the literal symbols a,b,c... are each identical in reference throughout, whether they occur in the same set or in different sets, and also that the marks - and ' and ., as well as the brackets, have the same significance in every set and in any given set.

Set III, in permitting the unrestricted interchangeability of (a.b) and (b.a), not only limits the reference of a,b to such elements or members of an aggregate as can be combined into complex unity without alteration of themselves and in such wise that the resultant complex is the same kind of entity as a,b individually, but also limits the reference of the dot, so that it can mean only such a relation as will hold between two elements in reverse order if it holds between them in a given order. So, too, set IV imposes the further condition that the relation meant by the dot must be associative: i.e. if it holds between three elements, it will hold between any two successive elements taken together, of the three, and the third of the three. Taking this condition in conjunction with the one laid down in set III, we find that the word "successive" may be omitted from the preceding sentence. Finally, the conditions imposed by set V, which permits unrestricted interchange of the elements meant by a and (-(a'.b').-(a'.b)), respectively, limit the reference of the mark ' and its defined equivalent mark -. This mark, it will be remembered, stands for the modification effected in any element by performing upon it the operation signified by the mark * in set II; it being understood that this modification leaves unaltered the element thus operated on. From the reading already given of set V (p. 142, above) it is plain that this set not only limits the meaning of the mark ' but also the meaning of the dot; for the complex element which is declared interchangeable with any single element a is obtained by successively applying to two instances of this element and two instances of another element the operations whereby the modification meant by the mark and the relation meant by the dot and brackets are produced. Exactly what are the limitations thus imposed could be put into words only with great difficulty; but the rules already laid down enable us to deduce from this set, with the help of the preceding ones, two other sets which make these limitations clearer. The first set thus derivable, already given as T1. (a.a) = a, informs us that any element joined to another instance of itself by the dot-relation so as to form a single complex element is identical with that initial element. And the second set, already given as T2.-(a') = a, informs us that if any element is twice subjected to the operation mentioned in set II, the modifications thus effected cancel out, so that the result is the same as the original element unoperated on. The dot-relation, then, must be such that when it holds between two instances of the same element, the complex element thus formed is really the same as a single instance of the initial element; and the modification meant by the mark ', or -, must be such that if it is effected twice in succession on any given element, the result will be really the same as though this modification had never been effected.

Another limitation of the meaning of the dot and mark is implicit in set V and the preceding sets; for from these sets is derivable the set given earlier as T3. (a.a') = (b.b'); and if these two complex expressions are to be used interchangeably, the meaning of the dot and mark must be such that if any element whatever is joined by the dot-relation to a modified instance of itself, the complex element thus formed will be really one and the same element, no matter what be the element whereof two instances are thus operated on.

9. Derivability and Deducibility.

When speaking of derivatives of sets I and II (pp. 137-138, above) we indicated why derivatives of these sets were to be regarded as actual consequences of their originals, in the sense of being strictly deducible from them. Having established this, we cannot at once conclude, on the same basis, that a derivative of any set whatever is a consequence of the set wherefrom it is derivable according to the transformation rules of this system. For, in the first place, the conclusion arrived at regarding derivatives of sets I and II relied partly on the fact that these derivatives are obtained by rules A, B,C, and not by rule D; hence their derivation is a matter of "each time" substitution, or of substitution throughout a given expression. In obtaining derivatives of all subsequent sets, however, the above limitation is removed; and derivation may involve the use of rule D, with its permission of unrestricted interchangeability. In the second place, the contention that derivatives of sets I and II are actually consequences of their originals was based in part on the possibility of reading these sets as statements about well-formed expressions and about the results of operations performable on well-formed expressions considered as marks or collocations of marks. Now that we have shown that sets III to V cannot be read as statements about these same operations and their results, and now that we have, in consequence, suggested a different reading of sets I and II in order to be able to attach the same meaning to the non-literal symbols in all five sets, we must see whether this difference affects the assertion that derivatives of I and II are consequences of their originals.

With regard to sets I and II, the conclusion that derivability is the same as deducibility except for the difference of method employed, and that therefore this mechanical method is as valid as ordinary deduction, is fairly easy to establish. The single point which needs to be proved is this: if these sets are readable as true statements, then their derivatives are readable as true statements, and (which is the crux of the whole matter) the truth of the latter is entailed by the truth of the former. We have already seen how sets I and II and their derivatives can be read as statements whose truth is at least conceivable. What needs to be made clear, if the above conclusion is to be established, is that their actual truth guarantees the actual truth of their derivatives. According to set I, any pair of elements, a,b, may be operated on in such wise as to form a combination of these same two elements connected by a certain relation in a given order; and this combination, or complex, taken as a unit, is itself regardable as an element of the same kind as a,b: that is to say, it possesses the same fundamental characteristics as a,b, in spite of having other characteristics of its own. Very little reflection is needed to see that if this statement is true, all derivatives of set I will therefore be true; for what is true of any pair of elements must be true of a given pair, and derivatives are obtained by replacing the variable symbols a,b in set I by symbols for some given pair of elements, it being understood that the variables be replaced by the same symbol throughout, in the case of any single derivative set. According to set II, any element, a, may be operated on in such wise as to form another element, (a'), which is but a modified form of the original element and is of the same kind as the original, in the sense just explained. And it is easy to see how the comment made above regarding derivatives of set I applies to derivatives of this set also.

The derivatives of sets III to V are evidently divisible into two groups: (1) those obtained by each time substitution according to rules A,B,C, and (2) those obtained by the unrestricted interchangeability permitted according to Rule D. Derivatives belonging to the first group present no special difficulty. The identity of reference entailed as regards the symbols occurring on either side of the mark = is safeguarded by the rules governing the formation of derivatives from originals (or in other words, the transformation of originals into derivatives). For none of the original relation-marks or modification-marks are altered in any derivative of this first group, and the only change is the replacement of one or more element-marks of the original by symbols for an element of equal or of greater complexity. Seeing that all elements are basically of the same kind, and since furthermore every element-mark thus replaced is replaced by some one and the same symbol wherever it occurs in the original set, any identity existing between referends of the original symbols must also exist between referends of the substituted symbols. All this does not apply, however, to derivatives belonging to the second group above mentioned, which involve the use of Rule D. In many cases, sets derived by the use of this rule have nothing in common with the original sets, except that every derivative, like every original, will contain on either side of the sign of equality some collocation of marks which is a single well-formed expression. However, since all sets in this system which contain the sign of equality have in common just this same characteristic, it will hardly serve as a means of determining whether a given set is a derivative of any other given set.

Perhaps the simplest way of showing how derivatives obtained by the use of Rule D are really consequences of the sets, or formulae, from which they are derived, is to begin with two observations already made. The first of these observations is as follows: Although sets containing the mark = explicitly declare that two well-formed expressions are everywhere interchangeable within this system, and hence such sets may be read as permissive statements regarding the use of well-formed expressions, they nevertheless entail something further, not about the well-formed expressions which they contain, but about the referends of those expressions: that is, about the elements of the system or systems of entities which the well-formed expressions themselves are intended to symbolize pictorially. And the second observation is that the referends of well-formed expressions, about which something is entailed, must either be some other kind of entities than well-formed expressions, or else the dot and the brackets and the mark ' cannot be taken as direct symbols; that is, they cannot be taken quite literally, as meaning "joined by a dot to, and bracketed with, in a given order", and "followed by the mark '", respectively.

To make clear the force of these two observations, it must be remarked that every set containing the mark = entails a statement to the effect that the referends of the "everywhere interchangeable" well-formed expressions on either side of this mark are actually the same: hence, that they are either two different instances of the same element, or else actually one and the same individual element thought of and symbolized in two different ways. This latter distinction appears to be called for in order to cover all possibilities. In any system of related entities, the elements will be either concrete particulars, or else abstractions, i.e. logical constructions or purely mental entities. If they are concrete particulars, then what is referred to by two different but everywhere interchangeable well-formed expressions of a directly-symbolic symbol system will be actually one and the same individual element, denoted under two different connotations, or (which comes to the same thing) described by two different descriptions. And if they are abstractions, what is referred to will be two different instances of the same element. Without insisting further on this point, which obviously has to do with the problem of universals, we may note that whenever two well-formed expressions are declared to be everywhere interchangeable within a symbol system, the entailed sameness of reference amounts at least to this, that the performance of certain specified operations on certain specified instances of elements is said to lead to the same results. If the specified instances of elements are the same, as in set III where we have on each side of the mark = one instance of a and one instance of b, and if the operations performed are the same, performed the same number of times on each instance, as is also the case in set III, then at least the order in which the same instances are subjected to the same operations will be different. In most cases, however, the difference indicated by two everywhere interchangeable well-formed expressions is greater than this, even when only one of the two permissible operations (i.e. dotting and bracketing, or marking and bracketing) is involved. Thus, in set IV, where we have on either side of the mark = the results of subjecting instances of the same three elements a,b,c to the operation of dotting and bracketing, this operation is performed a different number of times on the instances on the one side and on the other. In ((a.(b.c)), b and c are dotted and bracketed with each other and again with a, while a is dotted and bracketed only once; whereas on the other side, in ((a.b).c)), a and b are dotted and bracketed twice, and c only once. In set V, evidently, two further differences are manifest in the permissibly interchangeable well-formed expressions: not only does the expression -(a'.b').-(a'.b) differ from the expression a in the number of operations indicated, but also in the number of elements indicated on each side of the mark =. The information entailed by this set is very compactly conveyed to us in a brief formula, because the formula is to be understood in the light of the four preceding sets and the rules A to D which make their significance intelligible. If put into words, it would be something like the following:

Select an instance of the element referred to by a, and subject it to the operation allowed by set II; do the same with an instance of the element referred to by b; subject the resultant pair of modified elements, a',b', to the operation allowed by set I, and operate on this result according to set II. Then take another instance of a, operate on it by II, and subject this modified element a' along with another instance of b to operation I, operating on the resultant (a'.b) according to II. Finally, combine this result with the result obtained by operating as above described on the first instances of a and b, according to the operation allowed by set I. The result, -(a'.b').-(a'.b), will be the same as a single instance of a, not operated on at all.

Taking into account the total information similarly entailed by sets I to IV, we see that these five sets convey some very definite information about something other than the marks which they contain. They give us some definite characteristics which collectively serve as a description, though not necessarily a complete description, of the operations for which the operation-marks in sets I and II may be taken to stand; and since they convey this information in terms of the results obtained by performing these operations on members of a class of definitely-specified elements, they give us indirectly some of the characteristics which all members of this class of elements must possess. We know, for instance, that the operation permitted by set I must be an operation of combining two elements into one; further, that it must be such, and the elements to be combined must be such, that the complex element formed by combining two or more simple elements, or two or more complex elements, is basically of the same kind as the simple elements themselves, and that moreover the component elements thus combined into unity remain the same as before being combined, save that each acquires a new relation with respect to all other components. The nature of this relation is not fully defined; but two of its properties, namely, symmetry and associativity, are entailed by sets III and IV. Set V is our only source of information regarding the operation permitted by set II; but from it we can derive the formula -(a') = a, which tells us, in effect, that two successive performances of this operation on any element leave that element entirely unaltered; or in other words, that the modification, or additional characteristic, conferred on an element by the performance of this operation is entirely removed by another performance of the same operation on the element thus modified. Availing ourselves of this information, we can derive from set V another formula, (a.a) = a, which tells us that if two instances of the same element are subjected to operation I, the result will be the same as if we merely selected but did not operate on one single instance of that element; or in other words, that if a pair of instances of the same element are subjected to operation I, one member of the pair will be entirely removed or eliminated. These two latter pieces of information, obtained from derivatives of set V, can be called consequences of the information conveyed by set V itself, only if the derivation-process is a valid kind of deduction, or a valid substitute therefor. After what has been said, we are perhaps in a position to show that this condition is fulfilled, and to explain how the semi-mechanical device of symbol-substitution yields results no less reliable than those of ordinary deductive reasoning.

Whenever any other set is derived from a set or formula containing the sign of equality, such derivation is managed by substitution of different symbols in the place of certain symbols occurring in the original set. Now, whether such substitution is made "throughout", according to rules A,B,C, or whether it is made quite unrestrictedly, according to rule D, two things are always observed: the only symbols subject to replacement are element-symbols, and all replaceable or variable element-symbols are symbols of unoperated-on elements, that is, of simple elements, not complex, and not modified. If our analysis of the entailments of sets containing the sign of equality is correct, each such set conveys the information that the results of subjecting a given number of instances of elements to a given number of permissible operations are the same, in spite of certain specified differences in the number of instances chosen, or the kind of elements chosen, or the number and order of the operations performed. The simple element-symbols occurring in such sets can be, and are, regarded as variables,-- that is, they can and do stand for any element whatever, whether simple, or complex, or modified, or both complex and modified,-- and may be replaced by any symbol of any element whatever, for the simple reason that the result of performing one or both of the permitted operations of this system is entirely the same, no matter what be the element subjected to the operation or operations in question. That is to say, the relation signified by the dot, and the modification signified by the mark ', is always precisely the same, no matter what be the element subjected to the operation whereby this relation or this modification (as the case may be) is produced. Hence, with respect to the results effected by the performance upon them of a given number of these operations in a given order, all elements of this system are on the same footing, and what is true of a given simple element, a, is true of any element whatever. This being so, it is easy to see that any stated equivalence regarding the results of a given number of operations on a given number of elements will hold good when the element-symbols of the original are replaced by symbols for quite different elements: provided that every replaced symbol of the original be replaced throughout by the symbol for some one and the same element, and that the marks indicating the results of the performed operations be retained, exactly as they appeared in the original set or formula. When the above restriction about "replacement throughout" is removed, as happens by the use of rule D, the stated equivalence of the original formula is not disturbed thereby; for as our analysis indicates, the element symbolized by one of two everywhere interchangeable expressions is actually the same as the element symbolized by the other expression, and hence the unrestricted interchange of symbols permitted according to rule D does not entail a difference in what is symbolized. If, then, the statement entailed by the original set is true, the statement entailed by the derived set will also be true, in consequence of the fact that any expression in the derived set which replaces an expression in the original according to rule D is identical in reference with the original expression which it thus replaces.

The preceding considerations appear to afford sufficient warrant for the conclusion that the substitution process whereby other sets or formulae are derived from these first five sets according to the transformation rules of the system is equivalently a kind of deduction, depending for its validity on the same sort of principles as ordinary deduction; and further, that the information conveyed by all derivatives is a logical consequence of the information conveyed by the sets from which these are derived. It is important to remember that the information here spoken of is more than a series of statements about how a given set of marks are to be used as symbols of a given system; it is also, and especially, a series of statements about the results obtained by the performance of given operations on an aggregate of simple elements; and in particular, it is a series of statements to the effect that a given number of elements, combined and modified by the performance of a given number of operations, is the same as a given number of different elements, or of the same elements, combined and modified in a given way by the same operations. From such statements and from their consequences we learn, more or less directly, something about the nature of the operations symbolizable by the operation-marks of the symbol system, and also the nature of the results effected by these operations upon the elements symbolizable by the simple or complex element-symbols, these results being characteristics of the elements, either relations (i.e. relational characteristics) or non-relational modifications; and finally, though less directly, we learn something about the nature of the elements themselves, which the element-symbols can directly represent by way of picturing their form, or their structure.

The detailed study which we have made of the first five sets of the abstract system under discussion makes it possible to attempt an answer to the question raised at the beginning of the present chapter: whether, and in what sense, an abstract system of this kind is a deductive system. We say, then, first of all, that the collocations of marks which constitute the sets of an abstract symbol-system do not themselves make up a deductive system; for the sets derived from the initial sets of an abstract system by means of the transformation rules are not logical consequences of the sets from which they are thus derived, but are rather effects or results produced by applying the transformation rules to the initial sets. Secondly, as we have seen, derivatives may be regarded as logical consequences of original sets, if we consider not simply the collocations of marks whereof both originals and derivatives are composed, but the meaning of these collocations, according to which originals and derivatives alike are readable as propositions, stating equivalence between the results of given operations performed upon given elements. From this point of view, the propositions symbolized by derivatives are seen to be strict consequences of the propositions symbolized by originals, and the process of substitution permitted by the transformation rules is seen to conform to the ordinary principles of deduction.

The conclusion thus arrived at might have been reached more simply and directly, without examining the way in which an abstract symbol-system is constructed according to the accepted postulational method. For strictly speaking, no system ought to be called a deductive system unless (a) the related elements which make up the system are all of them propositions, and (b) the relation between these elements is a relation of deducibility, in every instance. The reason for this second condition becomes clear merely by reflecting on the words "deductive" and deducibility. And the reason for the first condition is this: propositions are the only known kinds of entities between which a relation of deducibility can hold. On the other hand, the elements of abstract symbol-systems are not propositions; therefore such systems cannot possibly be deductive systems. It is by no means a waste of time, however, to have arrived at this conclusion by a detailed study of the initial sets of a given abstract system and the connection between them and their derivatives; for the analysis we have been making of an abstract system of symbols will enable us to understand better how this sort of symbol-system is related to those deductive systems which it symbolizes. Before taking up this latter problem, we must round out the present discussion by examining the sixth of the initial sets listed at the beginning of this chapter (pp. 89-91, above).

10. Set VI and the Principle of Duality.

The importance of set VI, as we have already noted, consists in this: that by means of it we can derive from the first five sets and from set VI itself, as well as from the sets marked T1, T2, T3, another series of sets, each of which is exactly like one of those in our original series, except that all sets in the derived series will contain the mark +

wherever the mark . occurs in the corresponding original set (see table, p. 91 above). Since the use of set VI enables us to derive a similarly corresponding set (that is, one containing the mark + wherever the mark . occurs in the original), not only from each set in our original series, but from all sets derived from those in this series according to rules A to D, we are hereby provided with a time-saving transformation rule applicable to any set in the system containing only the marks . and ' and (), and the operation-marks ° and *, as well as the literal symbols a,b,c... (This rule has already been stated on p. 92, above.) We now wish to inquire into the significance of this transformation rule, and the significance of set VI which makes the rule possible.

Set VI, it will be recalled, reads as follows:
VI. (a + b) = -(a'.b'). Here again, as in the case of sets III to V, we must ask what is entailed by the permission, conveyed by the sign of equality, to employ the two expressions (a + b) and - (a'.b') interchangeably in this system. The fact that this set is often introduced as a definition might give rise to the notion that the expressions on either side of the sign of equality are merely intended as interchangeable symbols; that the marks (a + b) are here set down merely as a more convenient way of writing the marks -(a'.b'), as the marks 0 and 1 are introduced as a convenient alternative symbol instead of aa' and a + a', respectively. Considering the use made of this set, however, we see that it is intended not merely to introduce a more convenient symbolic equivalent of the expression -(a'.b'), but especially to introduce and to define a new relation-mark, +. It therefore serves as a partial

description of the relation meant by the mark +; or rather, of all the possible relations for which the mark + may stand, since this mark, like the dot, may represent any relation whatever which has the properties stated in the propositions signified by the sets in which this mark occurs. From sets Ia, it would appear that the relation meant by + is, like the dot-relation, one which holds between any pair of elements, a,b, when these are combined together by a given operation, the effect whereof is to produce a complex element (a + b); this latter, as the symbols indicate, being composed of that same pair of elements in the order of selection, with the relation meant by + holding between them. Sets IIIa and IVa similarly inform us that this relation, like the dot-relation, is symmetrical and associative; but from set VI itself we see that this relation is not the same as the dot-relation, in spite of having the above properties in common therewith. The same line of thought which guided us to the significance of sets III to V leads to the conclusion that because the complex expressions (a + b) and -(a'.b') are everywhere interchangeable within this system, they must therefore be identical in reference; that is to say, the complex elements to which they refer must be really the same. Now, each of these two complex elements is manifestly made up of the same simple elements: each contains one instance of the element meant by a, and one instance of the element meant by b. We know that the complex element -(a'.b') is the result of the following operations: Perform operation II on one instance of a, and also on one instance of b, to get the pair of a', b'; then combine this pair according to operation I, to get (a'.b');

and finally subject this combination to operation II. If this is indeed equivalent to the element meant by (a + b), it must be the case that the single operation which results in the complex (a + b) has the same effect on a pair of simple elements a,b, as is produced by the series of operations whereby the complex element -(a'.b') is formed out of another pair of instances of the same elements a,b. Taking + and . to stand for some unspecified pair of relations, we see that the relation meant by + is much more complicated than the relation meant by ., or at any rate is by no means the same as the latter relation, though the two have in common the formal properties already spoken of. Since these two relations are different, the operations which give rise to them should, for the sake of clearness, be represented by a different operation-mark; hence it would be better to employ, in set Ia, a mark other than the mark ° which occurs in set I. We have used the same mark in both these sets merely because when the sets are read as propositions about elements, this mark is translated by the same words in each case. Set I informs us that any pair of simple elements, a,b, may be combined to form a complex element (a.b); and similarly, set Ia informs us that any pair of simple elements, a,b, may be combined to form a complex element (a + b). If it be remembered that the operations whereby these different elements are formed are different in each case, as has been already explained, no confusion will arise.

A consideration of set VI in connection with the Principle of Duality enables us to emphasize from a slightly different point of view the necessity of attaching a meaning to all initial and derived sets of such an abstract system as the one under discussion; the necessity, that is, of being able to read all these formulae as propositions, apart from any specific interpretation. Set VI, as is well known, appears in the class-calculus as one form of De Morgan's theorem; and by means of some of the preceding sets, we can derive from it according to the transformation rules of the present system a series of derivative sets, such as: $-(a + b) = (a'.b')$, $+(a.b) = (a' + b')$, and also set **VIa** itself, $(a.b) = -(a' + b')$, as well as the important sets $(a . a' = -(a + a')$, already mentioned (on p. 95, above) as T4, and the converse of this, $-(a.a') = (a + a')$. These last two are important in the light of sets T3 and T3a, whereby we are informed that the elements symbolized by $(a.a')$ and by $(a + a')$, respectively, are unique elements of the system symbolized by this abstract system; and as has been remarked, they are often represented by the shorthand symbols 0 and 1. Now, in order to obtain these derivatives of set VI, it is necessary merely to transform set VI according to the rules of substitution, without regard to any possible meaning, or any reading of the set as a proposition. But if this set and its derivatives is to give us any information regarding the operations, or the relations, or the elements, which the symbols may possibly stand for, it is necessary to regard each of these sets as a statement to the effect that two different complexes, constructed out of the same pair of simple elements, are actually the same element: i.e., that the collocation of symbols on one side of the mark = is identical in reference with the collocation on the other side. Regarding set VI and its derivatives from this point of view, and thus getting all the information to be had from all forms of De Morgan's theorem, we find ourselves in possession of the following data regarding any system symbolizable by the abstract symbol-system in which set VI and its derivatives occur. The elements symbolizable by this system are such, and the modifications and relations produced by operations permissible upon them are such, that (1) for every complex element formed by combining any two elements whatever according to operation Ia, there is a corresponding complex element made up of two other instances of the same elements, but in such fashion that the whole complex has the modification resulting from operation II, and each component has that same modification, and the relation between the components thus modified is the relation resulting from operation I; (2) for every complex element formed in exactly the same way according to operation I, there is a corresponding element which fulfills all the above conditions, save that the relation between the components is the relation resulting from operation Ia; (3) the two complex elements referred to in each of the above statements are actually two instances of one and the same element, in spite of the indicated differences in their modifications and in the relations and modifications of their component parts.

This information may be summed up by saying that De Morgan's theorem defines operation I, or the relation resulting therefrom, in terms of the results of successively performing operations II, Ia, and II upon the same pair of initially-chosen elements; also, that it defines the relation resulting from operation Ia in terms of the results of successively performing operations II, I, and II upon the same pair of elements. This kind of definition, it should be noted, can hardly serve as an adequate account of the two entities involved: i.e., of the result of the single operation I (or Ia) and the result of the series of operations II, Ia (or I), and II. Hence in strict accuracy it ought to be said that we have here only a description, or partial definition, of the operations in question, and of the relations or modifications arising from them.

Whereas De Morgan's theorem thus informs us that two complex elements, formed by different operations on two instances of the same pair of elements, are really the same in spite of the different modifications and relations of their components, the Principle of Duality informs us about a relation, not between complex elements, but between statements about elements. It tells us, in effect, that for every stated equivalence between two elements, at least one of which is a complex element formed by operation I or Ia, there is also an equivalence between two other elements which have the same components, respectively, as the first pair, but a different relation between these components, so that wherever a + relation occurs in the first of these equivalences, a dot-relation is to be found in the other, and vice versa; and moreover, that the second of these equivalences is deducible from the first, and the first is deducible from the second. We may, of course, consider this principle merely as a transformation rule, just as we may regard as transformation rules every form of De Morgan's theorem, or indeed any formula in the system which, because it contains the mark =, permits the unrestricted interchange of the two well-formed expressions between which this mark occurs. From this point of view, the Principle of Duality is seen to be merely a general statement about common characteristics of sets which can be derived from other given sets by using all forms of De Morgan's theorem as transformation rules; that is, by using, as everywhere interchangeable, the pair of expressions on either side of the mark = in the various forms of this theorem. By confining ourselves to this viewpoint, however, we might fail to notice this very important fact: that the abstract symbol-system into which various forms of De Morgan's theorem, and therefore also the Principle of Duality, are introduced as transformation rules, can symbolize only those existing or constructible systems whose elements, with their modifications and relations, fulfill the conditions laid down in the readings already given of this principle (p. 167, above) and of De Morgan's theorem (pp. 166-167).

11. Results of Inquiry into Nature of Abstract Symbol-Systems.

In the light of the analysis which we made in the first two chapters, of such notions as "system" and "structure" and "similarity of structure", and also the study which we attempted of the principles involved in symbolism, we came to the conclusion at the end of Chapter Two (pp. 85-87, above) that an abstract symbol-system is fundamentally a collocation of definitely-specified marks, some of which are initially chosen and so ordered as to form initial sets of a recognizable pattern; these initial sets being subject, according to rule, to certain transformations, whereby other ordered sets can be derived from them. Some of these marks, either by themselves or in specified combination with other marks, are indicated as "well-formed expressions"; all other marks or combinations are understood to be "non-well-formed expressions". The formation rules specify which marks by themselves shall be simple well-formed expressions, and indicate how these can be arranged in conjunction with other marks to constitute complex well-formed expressions. The formation rules also indicate how, by means of suitably juxtaposed non-well-formed expressions, any given well-formed expression can receive an external modification by itself, or else acquire a relational characteristic whereby it is combined with another well-formed expression so that the two constitute a single but complex well-formed expression. In the particular abstract symbol system studied in the present chapter, the formation rules are summarily expressed by sets I, II, and Ia. These sets, when read as statements about the marks they contain and about operations performable on those marks and also about the results of those operations (see on pp. 101-102, above) and when read in the light of the transformation rules A,B,C which permit substitution of other specified marks for the marks they contain (as explained on pp. 98 ff., above) serve to define in general all well-formed expressions of this system, by indicating how any well-formed expression is to be constructed.

The other initial sets, which are not formation rules, contain two well-formed expressions separated by a non-well-formed expression. At first sight (as was suggested on above) we might be inclined to regard each such set as simply a complex well-formed expression, seeing that there are many complex well-formed expressions made up of two well-formed expressions separated by a non-well-formed expression. But two considerations militate against this view. First, the formation-rules nowhere provide for the use of this particular non-well-formed expression as a medium of combination for well-formed expressions; and second, no matter how "meaningless" or uninterpreted all other marks usable in this system may be, this particular mark must have meaning: for it is understood that any two well-formed expressions between which it occurs are thereby indicated as everywhere interchangeable within this system, rather than being joined to each other as components of a more complex well-formed expression. From this it follows that every set wherein the mark in question occurs as above described is, in effect, a transformation rule of the system; inasmuch as each such set provided for the transformation of any other set containing one of the expressions which it contains itself, into a set containing the other of the two expressions which it contains. In the abstract symbol-system chosen for detailed examination in this chapter, the mark in question is the mark *, the sign of equality; and it is to be found in every initial set other than the formation-rule sets. Other systems which apparently dispense with the use of this mark will, it is true, contain sets in which this mark does not appear; but such sets either contain another mark which has precisely the same meaning, or else are translatable exactly into a set containing such a mark.

We are therefore justified in concluding that every initial set in an abstract symbol-system, apart from the sets readable as formation rules, is not just a complex well-formed expression, serving as an element of the system, but an intelligible statement to the effect that two well-formed expressions are everywhere interchangeable. The formation-rules, namely, the first two sets and the rules A,B,C which provide for transformations of these sets, afford an indefinite number of well-formed expressions usable as elements of this system; and, once certain expressions have been used in the other initial sets, the transformation rules A and D provide for the formation of other sets which are each recognizable as derivatives, i.e. as transformations, of some initial set; there being no limit to the number of such derivatives, since the operations permitted by the transformation rules may be repeated any number of times. Which of the many possible derivatives of initial sets will actually be used as theorems in the system, there is no rule to determine; but on grounds of ordinary common sense we may expect to find only those derivatives used as theorems which are (a) considerably unlike the sets from which they are derived, and (b) likely to give rise to a large number of markedly different derivatives. When it comes to settling what well-formed expressions will be chosen for use in the initial sets, not only are we left without any rule to guide us, but also we find little help from such common sense considerations as the above. The second of the two mentioned above is hardly helpful, except in a general way, and the first is obviously not to the point at all.

We have tried to show in the present chapter that the sets employed as formation rules are readable as analytic (i.e. intuitively true) propositions about the marks they contain, and that derivatives of these sets obtained by applying rules A,B,C thereto are likewise readable in a similar fashion, and are recognizably consequences of the original propositions. To this extent, that is, so far as the existence and status of well-formed expressions are concerned, reliance is placed on ordinary deductive procedure in the development of this system, and the apparently mechanical transformation rules are seen to depend on ordinary deductive principles and to produce results which are intelligible in the light of ordinary logic. Strictly speaking, no derivative of these sets is a consequence of the original set whence it was derived; but the proposition meant by any derivative is a consequence of the proposition meant by its original set, and we can see why this must be the case when derivatives are formed according to the transformation rules referred to. The same observation may be made regarding derivatives obtained by applying rule A to initial sets which are not formation rules; and thus the mechanical process of "substitution throughout" is seen to be valid as a substitute for ordinary inferential thinking. Be we can not read any of these other initial sets, nor in consequence any derivative thereof, as analytically true propositions about the well-formed expressions which they contain; and therefore we have no guarantee of their truth so long as they are taken to express equivalence between these well-formed expressions. In fact, on this reading they seem to be positively false propositions. Thus their validity as transformation rules, in connection with rule D, is seriously called into question.

A way out of this difficulty was suggested, by explaining at some length how sets containing the mark = are readable as statements to the effect that the symbols which occur as well-formed expressions on either side of this mark are identical in reference. We have yet to inquire by what means, if any, the truth of such statements can be ascertained; but it is clear that the validity of these sets, as transformation rules, depends on their being readable as true statements, for otherwise the element-symbols which they indicate as everywhere interchangeable may not be identical in reference, and thus confusion will arise in the final interpretation of the system.

With regard to the symbolic force of an abstract symbol-system, a distinction must be made between its use as a direct symbol, and its indirect symbolic force. As a direct symbol, any well-formed expression of such a system can represent the structure of some element in any system which the abstract symbol-system symbolizes, provided that the symbol-element has the same structure as the symbolized element. This means, as we have explained at considerable length in the first two chapters, that the symbol-element must contain the same number of components as the element whose structure it pictures, and that corresponding components must occupy the same relative position in symbol-element and symbolized element, respectively. Non-well-formed expressions, which serve as symbols of non-relational or else relational characteristics (always excepting the mark =, which has the special significance stated above on p. 170 directly symbolize, in virtue of their juxtaposition to some well-formed expression, the possession by the element corresponding to that expression, of the characteristic which they signify. If an abstract symbol-system is to be taken as a direct or pictorial symbol in this way, the isomorphism, or similarity of structure, which obtains between its element-symbols and the elements of another such ordered set, is basically a matter of one-to-one correspondence of component elements, and of sameness in order of these component elements. All that an abstract symbol-system can picture in this direct fashion is therefore the number and order of elements in any system isomorphic with it.

In actual practice, however, more is involved in the use of an abstract symbol-system than this purely pictorial representation of structure. Besides the direct symbolic force of the number and order of its component element-marks and relation-marks or modification-marks, a certain amount of indirect symbolism enters in; that is to say, the various sets of the system are readable as propositions about the elements which their element-marks can symbolize. The elements and relations and modifications symbolizable by a given abstract symbol-system must be such that when the symbols of any set in the system are read as referring to them, the set in question is a true proposition about the elements and relations and modifications to which the symbols of the set refer. If every set in the abstract symbol-system can be read in this way as a true proposition about the elements of some existing or constructible system or systems of entities together with their relations and modifications, then that system, or those systems, are said to be isomorphic with the abstract symbol-system and with each other. We have not attempted to justify this seemingly odd extension of the notion of isomorphism; but it is not at all obvious that some justification for this use of the term might not be worked out: by showing, for instance, that any elements which, together with their relations and modifications, happen to satisfy the various symbol-sets of a given abstract symbol-system can on this ground be said to have the same structural or formal properties as any other similarly "satisfying" class of elements with <u>their</u> relations and modifications. If this could be done, it would show merely that the systems of elements representable by the sets of the abstract symbol-system in question were isomorphic with each other, but it would not indicate that they were any of them isomorphic with the symbol-system which thus indirectly symbolized each of them. For the elements of this symbol-system are the symbols which occur as well-formed expressions in the various sets; and we have given reason for the statement that these sets cannot be read as true propositions about the well-formed expressions which they contain; whence it follows that the class of well-formed expressions of this system is not one of the classes of elements which satisfies the sets of the system itself. There is, accordingly, no obvious objection to the view that all systems symbolizable by a given abstract symbol-system in this fashion are isomorphic with each other; but it is very difficult to see how any of these systems can be called, in the same sense, isomorphic with the symbol-system itself. The only kind of isomorphism present in this latter case seems to be the kind described in our detailed analysis of this notion: whereby the elements of the symbol-system, insofar as they are similar in number and order of components to the elements of some other system, can and do pictorially represent the structure which is possessed by themselves and those other elements.

From what has been said in the present chapter (especially on pages 160-161, above) it should be clear that an abstract symbol-system is not a deductive system, in the ordinary sense of the word: that the sets obtainable by transforming initial sets according to rule are not logical consequences of the sets from which, respectively, they are thus derived. At the same time, we have tried to show in some detail that the general propositions meant by these initial sets do strictly entail the general propositions meant by their respective derivative sets, and hence that this body of propositions does constitute a deductive system. Before proceeding to examine, in our next and final chapter, what happens when such an abstract symbol-system as we have been studying is interpreted as a system of formal logic, we must briefly discuss a question raised by our inquiry into the nature of an abstract symbol-system: the question (mentioned on p. 173, above) whether it is possible, and if so, by what means, to settle the truth of the statements obtainable by reading as propositions those sets which contain the mark = and which serve as transformation rules of this system.

12. Validity of Transformation-Rule Sets.

We have already noted (e.g., pp. 100-101, above) that the well-formed expressions in all sets containing the mark

= are either borrowed from or derived from the formation-rule sets. The formation-rule sets, as we have seen, can be read as intuitively-true statements about the well-formed expressions which they contain; but when the attempt to read any other sets in this same fashion proved abortive, we noticed that the formation-rule sets and all other sets are readable as statements about the elements and modifications and relations for which their well-formed and non-well-formed expressions may perhaps stand. At first sight, it would seem that these so-called statements can hardly be called statements at all; and since they are in fact spoken of as propositional functions rather than as propositions, it appears premature to raise any question of their truth or falsity. The point we wish to make, however, is that these sets, when read as previously suggested, serve to convey some information, however vague and general this may be, about some entities, however difficult may be the task of identifying them. Initially, in the case of any set containing well-formed and non-well-formed expressions, all we may be able to do is to discriminate between element-marks and relation-marks and modification-marks; and when we inquire what these marks can possibly refer to, the only clue to an answer may seem to be such phrases as "some element or other", "some relation or other", "some non-relational modification or other". If we consider each set more closely, however, the problem of reading these sets intelligibly is seen to be somewhat less insoluble. Whatever the individual marks may mean, we may take for granted that each of them will mean the same thing on every occasion of its occurrence, whether in the same set or in different sets. Assuming that the marks specified as simple well-formed expressions refer to a class of homogenous elements, and without supposing that different well-formed expressions necessarily refer to different members of this class,-- since we have no reason to make any assumptions about the number of members thereof,-- we learn from the formation-rules some further characteristics of these elements. What these further characteristics are may be gathered from previous pages of the present chapter (notably pp. 143, 146 above) but it is advantageous to list them here in perhaps a clearer way. The formation-rule sets are those marked I, Ia, and II, in the list at the beginning of this chapter (p. 91, above) ; and the information each conveys is as follows:

I. For every pair of elements in this class, whether simple or complex, modified or unmodified, there is a complex element made up of this same pair of elements in a given order, these components being joined together by some relation, as yet undefined, so as to form a single element which is itself a member of the same class as the original pair.

II. For every single element in this class, whether simple or complex, modified or unmodified, there is a modified element which is in all respects the same as this single element, save that it possesses an as yet undefined characteristic or modification not possessed by the single element itself.

Ia. For every pair of elements...(etc., as in I above)... by some relation, as yet undefined, but not the same in all respects as the relation mentioned in I, so as to form...

(etc., as in I above).

Evidently, the above statements are not categorical assertions. They do not purport to tell us that there exists a class of homogeneous elements, the members of which possess the properties here mentioned. They do not tell us that such a class, even if it existed, would contain a pair of elements; and, even supposing such a pair of elements to exist, or supposing a single element to exist, they do not tell us that there is in existence a complex element made up of that pair, or a modified element corresponding to that single element. They do tell us categorically that if such a pair of elements exists, any such pair can be combined in the fashion described; and that if such a single element exists, a corresponding modified element is constructible: hence, any existing pair must be such as to be combinable, and any existing element must be such as to be modifiable, in a way that fulfills the conditions above stated. What is being presented to us by these sets is a kind of cumulative description of the class of elements symbolizable by this system: a statement of the characteristics which such a class of elements must possess if it is to satisfy the requirements of this symbol-system.

These characteristics, at first thus indefinitely stated, are progressively made more definite in the light of similar descriptive details added by subsequent initial sets. Set III informs us that whatever else be said about the undefined relation mentioned in I, it must be such that if it holds between any pair of elements in a given order, it will also hold between that same pair in reverse order; and set IV adds the further stipulation that this same relation must be associative. Taking these two items of information together, they tell us that if this relation holds between every pair of components of a complex element, it will continue to hold within a complex element made up of those same components, no matter in what order they are taken and no matter which consecutive components are taken as a single unit. Set V states an equivalence between any single element and a complex element made up of two pairs, both pairs being modified as described in II above: the first consisting of the modified element corresponding to the original single element, and some (other) modified element, between which holds the relation mentioned in I; the second pair having the same components similarly interrelated, save that the second component is unmodified; both these pairs being each taken as a single complex element and joined together into unity by the relation mentioned in I. From the information thus given, we can arrive at the information conveyed by the set marked T1, $-(a') = a$, which tells us that the modification mentioned in II must be such as to be removed by two consecutive applications of itself to the same element (i.e. to any given element); and this bit of data added to the previous information enables us to infer what is stated in T2, $a.a = a$, about the relation mentioned in I: that when this relation holds between two instances of the same element, the apparent complex thus formed is actually the same as one instance of that element. It is easy to see that both set V and T2 also serve to define the elements meant by the simple well-formed expressions they contain. No class of elements will be symbolizable by this system, in virtue of set V, unless, when two modified instances of one element and one modified and one unmodified instance of another element are related and modified and related according to the process whose result is pictured in V, all instances of element disappear together with their modifications and relations, except a single unmodified instance of the first element selected. And in virtue of T2, the elements of any class symbolizable by this system must be such that when two instances of the same element are joined by the relation mentioned in I, one of these instances disappears.

Admittedly, the information thus far given is not sufficient to serve as an adequate description of the elements or relations or modifications symbolizable by the marks of this system; but its implications may well be more far-reaching than one might think. For instance, the reference just made to a disappearance of instances of elements as a result of certain successively-performed operations, and the cognate notion that an element joined to another instance of itself by the relation mentioned in I, seems to necessitate the conclusion that no class whose elements are concretely-existing physical objects can possibly be symbolized by this abstract system. For it is difficult to see by what conceivable relation two such objects might be combined as a unit, in such wise that this combination of the two objects should be identical with only one of the two in question. If such a relation between physically-existing objects is indeed inconceivable, this particular symbol-system can symbolize only abstractions and other mental entities. We mention this instance in passing, as an illustration of the importance of attending to the information conveyed by the symbol sets of an abstract system apart from any specific interpretation of that system.

Set VI, by means of the formula $a + b = -(a'.b')$, describes the relation mentioned in Ia by making clear its connection with the relation mentioned in I and the modification mentioned in II. We are here succinctly informed that a complex element consisting of any pair of elements joined by the relation mentioned in Ia is actually the same as a modified complex element consisting of the pair of modified elements which corresponds to that same pair, joined by the relation mentioned in I. The importance of this information, as has been seen, consists in the fact that because of it we can arrive at the information conveyed by Set Ia and all the other sets marked with the letters a or b (p. 90, above). In this way we are assured, for instance, that any class of elements whose members fulfill the conditions stated by sets I to VI must therefore fulfill the conditions stated by Sets Ia to VIa, including Vb and all derivatives of all these sets. The set marked T3 is of special interest because of the information it gives concerning all complex elements which consist of any element whatever and the modified element corresponding thereto joined together by the relation mentioned in I. All such complex elements, we are told, are actually one and the same element, irrespective of the different components which are found in each complex. A similar piece of information is conveyed by set T3a regarding a complex composed in like manner with the relation mentioned in Ia joining its components; and from T4 (p. 95, above) which reads (a + a') = -(a.a'), together with another previously-unmentioned set derivable from this, (a.a') = -(a + a'), we learn that any complex element of this sort which contains the relation mentioned in Ia is actually the same as the modified element corresponding to a similarly-constructed complex containing the relation mentioned in I; and conversely. Knowing this, we are in a position to say with certainty that if we can find a relation which fulfills the conditions laid down for either of the two mentioned in I and Ia, we shall find a relation which fulfills the conditions laid down for the other, without having to institute an independent inquiry into the existence of that other.

We have already attempted to show (especially on pp. 157-160, above) how the mechanical process of substitution whereby derivatives are obtained from initial sets affords a guarantee that the propositions, or statements, obtained by reading these derivatives are logical consequences of the statements obtained by reading the originals. Insofar as this attempt may have been successful, it is true to say that the information conveyed by the original sets entails the information conveyed by their derivatives, even when the connection between any two such pieces of information is very difficult to see because of the complexities involved in thinking about verbally-expressed propositions. If this is indeed true, we should find that once we have grasped the information conveyed by the initial sets and their main derivatives,--meaning by "main derivatives" those which add to our store of information items not easily perceptible as consequences of previously-acquired data,-- we have a sufficiently complete list of those characteristics which must be possessed, or of those conditions which must be fulfilled, by the members of any class of elements which our abstract symbol-system can symbolize. We have as yet, however, no positive reason for asserting that even one such class of elements exists; and of course, if no such class exists, our store of information will be a description which describes nothing, and the statements obtained from our initial sets, as well as the consequences obtained from the derivatives of these sets, will in fact be false. If, on the other hand, only one such class exists, the information in hand will in fact be a unique description, and the symbol-system will hardly deserve to be called abstract, in spite of the apparent generality of its symbols and the truth of the seemingly general statements obtained from its sets of symbols.

We have suggested (p. 181, above) that the particular abstract symbol-system here selected for study will most likely not serve as a symbol for any class of concretely-existing objects in the physical universe, for some of the conditions above mentioned appear unfulfillable by any such class. If this be so, we need not expect to find an interpretation of our system in the world of physical existents. We might accordingly attempt to conceive, i.e. to construct, one or more class of abstract elements which would fulfill these conditions, knowing that a successful attempt in this direction would furnish us with at least one interpretation of the system and thus verify the statements obtained from its sets. As a matter of fact, the various classes of elements which this particular system is used to symbolize are classes of abstract elements, as will be seen in the next chapter, wherein we shall touch upon a few of the problems arising from interpreting this system as a system of formal logic. The question now under consideration is, what, if anything, can be said about the truth or falsity of those statements derived by reading the initial sets in the general fashion above described, apart from finding or constructing any specific interpretation of the system.

It seems clear enough that we cannot know these statements to be actually true prior to finding a specific interpretation for them. This consideration, coupled with the relatively general character of the information conveyed by the sets,-- especially when we consider only initial sets without working out their derivatives,-- may incline us to accept the view that it is premature to raise any question of truth or falsity until a specific interpretation has been found, or until failure to find one makes the existence of one extremely improbable. In actual practice, it may be advisable to postpone the question of truth or falsity until we have obtained as much information from the initial sets as we can, by working out their main derivatives and reading them as above described. Yet it is even more advisable to keep in mind from the very beginning of our inquiry into an abstract symbol-system the conditions which it must fulfill in order to be at least possibly true, or possibly interpretable. And there are certain negative criteria which can be applied, in this connection, as soon as we are in possession of the information conveyed by the initial sets, without reference to the further information obtainable from their derivatives. We know, for instance, that if a statement obtained by reading any set is self-contradictory, as would be the case if it imposed, as a condition, the possession by some one and the same element of characteristics that were mutually incompatible, such a statement could not possibly be true, and no entity fitting such a description could ever be discovered or constructed. Similarly, if any two statements obtained by reading any two sets are mutually contradictory, the abstract-symbol-system in which both these sets occur is certainly uninterpretable, since no entity could be found which would simultaneously fulfill two contradictory conditions. This means, of course, that the initial sets of any abstract symbol-system must be such that the statements obtained by reading these sets are each free from inner contradiction and that no one statement is the contradictory of any other.

A positive criterion also is at hand, as a guarantee of the possibility of interpretation, or of the possible truth of these statements; but it can be applied only when the statements convey such information as amounts to a fairly adequate description of the elements and relations and modifications to which they apparently refer. In such a case, we can sometimes see, not merely that no inner contradiction or mutual contradiction is present, but also that the statements in question are each self-consistent and consistent with one another. Whenever this is so, we can say at once that any statements entailed by these initial ones, individually or collectively, will likewise be self-consistent and consistent with one another.

The negative criteria are evidently based on the principle of contradiction, in the sense that they have no value whatever unless this principle is true. That is one reason why this principle must be regarded as fundamental, in spite of the fact that no symbolic expression of it is to be found among the primitive propositions of such systems of logic as Principia Mathematica, though some form of it appears as a theorem, or derivative of the primitive propositions. The positive criterion, besides relying on the same principle, depends for its value on the truth of another principle, which may be stated as follows: Whatever is entailed by a self-consistent proposition must be self-consistent, and all propositions entailed by mutually consistent propositions must be consistent with one another. A direct demonstration of the truth of this latter principle would be rather an explicitation of its terms than a strict demonstration, and the analysis involved would very likely raise more philosophic problems than could be discussed with profit in the space of a single dissertation. Suffice is to say here that the principle is evidently analogous to the somewhat clearer principle, "A proposition which is entailed by a true proposition cannot be false"; and further, that from a denial of the principle above mentioned, it would follow that all knowledge obtained by inference is worthless apart from separate verification of every conclusion arrived at.

The distinction here made between positive and negative criteria for testing the possible truth of statements obtainable from the initial sets of an abstract symbol-system is one which must be made if we wish to avoid confusion. It is true to say about each of these statements, and indeed about any statement whatever, that it must be either self-consistent or not; or about any series of statements collectively considered, that they must be mutually consistent or else mutually contradictory. But unless we have some assurance that a statement is self-consistent, or that two statements are mutually consistent, we cannot arrive at this conclusion simply on the ground that we see no inconsistency in a statement or between two statements. For the information we obtain from such statements may be so very vague and indefinite as to contain, at least implicitly, a contradiction which we do not see; though such a contradiction, if implicitly contained in the statements, will presumably manifest itself on further reflection and analysis.

The considerations just presented, it will be observed, merely indicate that we can, within certain limits, assure ourselves that the statements obtainable from reading the initial sets are themselves both individually and mutually consistent, and will therefore not lead to any contradictory consequences. The importance of this fact becomes clear when we reflect that many of the interpretations which fit abstract symbol-systems are themselves abstract systems: i.e., systems in which the elements are abstractions. If attention is paid, along the lines we have been suggesting, to the information conveyed by initial sets and derivatives of a symbol-system, it may well be that this information will contain such a complete description of the elements symbolizable by this system that we can at once conceive or construct such elements mentally; and once this stage has been reached, their actual existence becomes a mere matter of whether or not we take the trouble to conceive or construct them. For this reason the information conveyed by sets and derivatives of an abstract symbol-system is well worth taking into account. By attending to it, we will surely notice obvious self-contradictions and obvious mutual inconsistencies, the presence of which will assure us that no interpretation of a symbol-system can be found unless these are removed; and it is not unlikely that we may obtain sufficient data to assure ourselves of the constructibility of an abstract interpretation, if we do not actually obtain a complete set of rules for its construction.

Finally, once we construct or discover an interpretation which satisfies the initial sets, i.e. which fulfills the conditions therein laid down concerning what characteristics any elements must possess in order to be symbolizable by this system, we are thus assured not only of the actual truth of the initial sets, but also of the actual truth of statements obtainable from their derivatives. At any rate, this last assertion is true of all abstract interpretations of the system. For if the initial statements are true, all their consequences must be true; and though we might have room for doubt if there were question of concretely-existent particulars, which might conceivably possess the properties stated in the initial sets and yet happen somehow not to possess all the properties theoretically entailed by these, no such doubt can arise regarding abstract elements, whose existence depends on their constructibility.

Further consequences of finding an interpretation for an abstract symbol-system will be examined in what follows. For we shall now attempt to make more concrete the discussion of the relations between abstract symbol-systems and formal logic, by considering one or two of the main problems raised when the particular system we chose for comment is interpreted as a system of formal logic.

CHAPTER FOUR

AN ABSTRACT SYMBOL-SYSTEM INTERPRETED AS A SYSTEM OF LOGIC

1. Main Problem Connected with Interpretation.

Thus far, by studying an abstract symbol-system independently of any specific interpretation, we have attempted to discover and explain the principles involved in its formation and development, as well as those governing its utility as a symbol of structure. We have suggested how the sets of such a system are readable as general propositions, and how on this reading they purport to convey information, not about the symbolic marks which they contain, but about the possible entities,-- whether elements or modifications of elements or relations between elements,-- for which these marks may stand, supposing such entities to be constructible or discoverable. From this point of view, an abstract symbol-system is seen to be a kind of descriptive and cumulative definition: each set states a condition which must be fulfilled by any entities in terms of which the system is interpretable, and the condition thus laid down by any derived set is guaranteed to be a consequence of the conditions laid down by the previous set or sets whence the set in question is derived, inasmuch as the process of substitution whereby such derivation is effected appears on analysis to rely on ordinary principles of deduction.

It might be supposed that once we have discovered or constructed an interpretation, or even several different interpretations, of a symbol-system devised with such meticulous accuracy, no further problems would remain to be solved. For we are now in a position to read each set as a true proposition about the entities which constitute the system we have found, and it would seem that we may at once proceed to avail ourselves of the information thus placed at our disposal about these entities without the slightest hesitation. As a matter of fact, however, one very important question remains to be settled. Granted that the information afforded us by this system about these entities is reliable so far as it goes, we must now ascertain precisely how far it goes; and the question is, does this system provide us with an adequate description of the entities to which its symbols refer?

It is easy to dismiss this question with a negative answer. For, apart from the fact that no abstract symbol-system which has as yet been devised lays any claim to being representative of every existent or constructible system of entities, the very abstractness of each extant symbol-system makes each in principle, as many are in practice, susceptible of more than one interpretation; and since each can thus represent several systems, the constituent elements of any one of which have (at least presumably) some characteristics not possessed by the constituent elements of every other, it seems clear that no system of entities which serves as one interpretation of an abstract-symbol-system can be adequately described thereby, in the sense that all the characteristics of these entities are thus made manifest. Nevertheless, it is far from clear that the question we have raised is an idle one; and in this concluding chapter, which might well be expanded into a dissertation, we shall try to say in general why it is important, and consider in detail one or two of the problems with which it is connected.

The importance of this question may be seen in general by reflecting on one of the dangers inseparable from the construction and development and use of abstract symbol-systems. Initially, the marks specified as usable in such systems are taken as standing for quite undefined entities, whose characteristics are then set forth in a series of recursive definitions; or rather, in a series of statements which collectively serve as a recursive definition. The consequences of these primitive propositions are next made explicit, until we have a description of these entities which is sufficient for the purposes of this system. When we find an interpretation of any such system, by discovering entities which fulfill these conditions, or answer to this description, it will of course be the case that all the statements obtainable by reading the various sets of the system in terms of these entities will be true propositions about those same entities. The danger is, at this point, that the accuracy and purity of the method employed in arriving at these true propositions may give rise to the idea that they collectively afford a real definition of the entities to which they refer. To speak of such an idea as a danger is apparently to under-estimate the mental acumen of those who make it their business to study abstract methods. For the fallacy to which attention is here being called is one which the veriest tyro in logic would recognize and avoid almost immediately: the fallacy of supposing that because the statements made by a given system all apply to a particular system of entities, these statements afford a complete description of these entities. A little reflection on the situation we are considering, however, will show that this fallacy may very easily intrude itself without being detected so readily: not for lack of mental acumen, or because of carelessness, but because those who are most concerned with abstract symbol-systems are entirely immune from this fallacy, and realize that it has no occasion of arising. If a mathematician, for example, having developed an abstract symbol-system, fails to find an interpretation for his system among the familiar entities of his science, he may easily manage to invent an interpretation, by constructing unfamiliar but clearly conceivable and describable abstractions. These, being constructed specifically to fit his system, will accordingly have all those characteristics stated or entailed by the propositions obtainable from the reading of the system's formulae. If they have any other characteristics,-- and in some cases at least it is difficult to see how they could have any others,-- he has no reason whatever to be concerned with those, seeing that they do not follow from his initial postulates. The postulates, in other words, state and entail conditions which are not only necessary but also sufficient, regarding the characteristics to be possessed by any entities in terms of which the system is interpretable. Even when the mathematician finds an interpretation for his system among concretely existing objects in the physical universe, or among such relatively familiar abstractions as points on a line and areas in a plane and ordinary integers, he need not raise the question whether these entities have any other characteristics than those stated and entailed by the postulates of his system. But when an abstract symbol-system is interpreted as a system of formal logic, the situation is rather different. If the initial and derivative sets of such a system are to be read as propositions about classes and propositions and relations, describing (whether pictorially or non-pictorially) the structural properties of these entities and of the relations between them, the question whether the description thus given is not only true but also complete is by no means unimportant. In particular, one would like to be sure that the definitions and descriptions thus "systemically" formulated include not merely some of the characteristics possessed by these logical entities, nor yet just those which afford a uniquely-descriptive account of these same entities individually, but also and especially those characteristics which are most fundamental quite apart from such systemic considerations as their order of derivation.

Unlike the extreme formalists, who maintain that the formation-rules and transformation-rules of an abstract symbol-system are merely rules for the manipulation of symbols, and hence involve no analysis of the entities symbolizable by these symbols, the so-called logistic school of mathematical philosophers and logicians insist upon the necessity of such analysis. Though they may introduce primitive ideas as undefined, and subscribe to the general view that definitions are always of symbols, many would agree that analysis of symbols entails analysis of concepts; and in practice the latter kind of analysis is understood as at least a useful preliminary to the construction of a system by the logistic method. The work done by Russell and others in this field is too well known to leave room for doubt about the painstaking accuracy and thoroughness with which such analysis has been carried out, and it would indeed be unfortunate if the suggestion that much remains to be done were construed as an unfavorable comment on what has already been achieved. Even more unfortunate, however, both for philosophy and for mathematics itself, would be the consequences of taking for granted that the concepts thus carefully analyzed are actually the concepts which they seem to be, particularly when the words used to express them are the same as those of ordinary discourse.

Precisely because the logistic analysis of concepts is so thoroughgoing and symbolized with such exactness, one is apt to regard it as much more reliable than any other; and the very idea that it may be open to question seems quite incompatible with a knowledge of the advances in science made possible by the results of this analysis. For this reason the question we have raised may well seem pointless without some specific evidence of its relevance and importance.

The question of determining precisely what is meant by interpreted symbols, whether verbal or non-verbal, is one which arises within the science of mathematics itself. To cite but one example, the word "number", as applied to transfinite numbers, refers to entities whose properties are so different from those of the so-called natural numbers that the same word takes on a very different meaning. No confusion can arise in the mind of anyone who knows the accurately defined properties whereby these two kinds of numbers are respectively described and clearly differentiated from each other. But when a mathematician attempts to define the notion of number in general by saying that it is a class of cardinally similar classes, it is well to make sure that no analogous use of these words is intended, before taking them literally. The notion of cardinal number, as defined by the well-known Frege-Russell definition, may be sufficient for purposes of mathematics and for formal logic; it may be the case that this definition applies to what is meant by the words "cardinal number" apart from exclusively mathematical uses and independently of systemic considerations; but it is at least arguable that this definition not only leaves out of account the property which differentiates numbers from other entities which are not numbers, but that it tends to ignore a distinction which is important for philosophy but not likely to arise within mathematics: the distinction between "having a number" and "being a number". We are not here concerned to argue this point, nor to discuss such cognate problems as the difference between a class and a characteristic, and why this difference raises no special problem for the mathematician. We merely call attention, by way of this single example, to the fact that an analysis satisfactory from a mathematical point of view may well prove unsatisfactory and even quite misleading when transferred to a non-mathematical context.

The same question, of the adequacy of logistic analysis when its results are applied outside mathematics, is brought to the fore in connection with the relation of implication. Anyone acquainted with the literature of modern logic will remember how much insistence was placed on the view that material implication is that relation between propositions in virtue of which one is deducible from the other. The late Mr. W. E. Johnson, in particular, wrote a lengthy exposition of the so-called paradoxes of material implication, by way of pointing out that this relation is really fundamental to all inference, though the inferences based on it are not always useful. The point to notice is that when this mathematical kind of analysis leads to results seemingly at variance with the principles and methods of ordinary reflective thinking, there is a tendency to revise these latter, or at least to judge them in the light of mathematically-obtained results which are themselves assumed to be more reliable. The subsequent work of Professor C. I. Lewis has made manifest the inadequacy of material implication, and indeed of any truth-implication appearing in any truth-value system whatever, to be <u>the</u> relation in virtue of which valid inference is possible; but it is far from clear that his "relation of Strict Implication" is a completely acceptable substitute. It would be unfair to suggest that he himself attempts to prove that it is; for instead of maintaining that Strict Implication is, as we have said above, the relation <u>in virtue of which</u> valid inference is possible, he is content to speak of it (as, e.g., on p. 247 of Lewis and Langford's <u>Symbolic</u>

Logic) as "that relation which holds when valid deduction is possible, and fails to hold when valid deduction is not possible"; that is to say, he holds that it is a necessary condition for valid inference, but not that it is a necessary and sufficient condition, still less a cause or ground of inference, as our phrase "in virtue of which" suggests. Nevertheless, it is somewhat disturbing to note that he too seems inclined to give systemic considerations precedence over non-systemic ones, taking for granted the accuracy of that previous analysis which gave rise to his initial definitions and postulates. For although he avoids the paradoxes of material implication, and makes it clear that they arise because this relation fails to fulfill the conditions laid down for Strict Implication (see end of p. 247, op.cit.), he adopts much the same attitude towards the corresponding paradoxes to which Strict Implication gives rise as was adopted towards the paradoxes of material implication by Johnson. Instead of re-examining the postulates which served as premisses whence these paradoxes follow, he gives a formal proof, based on his own definitions and postulates and developed by the ordinary substitution-process used within his system, in support of these paradoxical conslusions, by way of showing that they "are paradoxical only in the sense of expressing logical truths which are easily overlooked" (p. 248, op. cit.; see also pp. 249-251).

A detailed discussion of the many issues involved in the two instances we have mentioned, of this tendency to rely unduly upon systemic considerations, can hardly be attempted within the limits of a single dissertation. To deal in anything like an adequate fashion with the latter alone, we should have to inquire into the nature of Lewis's system of Strict Implication, by way of discovering whether it is more than technically different from the truth-value systems which, like itself, are developed according to the logistic method. Instead of pursuing these topics further, we may well bring the present dissertation to a close by investigating a problem more directly connected with the validity of any abstract symbol-system interpreted as a system of formal logic.

2. Truth-functions as a Guarantee of Validity of Formal Logic.

In the concluding chapter of his comprehensive three-volume Treatise of Formal Logic, under the title, "The Ideal Presuppositions of Formal Logic", the Danish Logician Jørgen Jørgensen deals with the problem of validity in a way that is particularly interesting. In selecting for comment some of the points developed in the chapter referred to (Chapter XV, pp. 276-293 of Volume III), we are not directly concerned with passing judgment on the merits of his proposed solution, nor even on the adequacy of his statement of the problem. Our main purpose is to emphasize certain features of his attempt to use truth-functions as a basis for the validity of formal logic, in order that we may thus give a concrete illustration of the importance of some of the main points we have been stressing in preceding pages.

His statement of the problem may be summarized as follows. There is an important difference, in principle, between a deductive system of formal logic and all other deductive systems, such as for instance those of mathematics: the latter need only be formally true (i.e. composed of theorems which are validly deduced from consistent postulates), whereas formal logic must be materially true (i.e. its theorems must be validly deduced from postulates which are not merely consistent but also true in the ordinary sense of the word "true"). The exact significance of the phrase "material truth" as he uses it (p. 276: "By material truth is understood a relation between an objective and the fact to which it refers") may seem to demand further elucidation: for although by "objective" he clearly means "propositions" (cf. Sheffer's use of the word "ascript"), many would object to using "fact" with reference to what is meant by a general proposition; hence we suggest that by material truth he means truth in the ordinary sense, and whatever the ordinary sense of the word may be, it admittedly involves more than mere absence of contradiction, or consistency. The reason for this difference between formal logic and all other deductive systems is that formallogic is not simply a deductive science, but also a deductive science of deduction; hence in it the principles of deduction, according to which the theorems of any deductive system are validly deduced, must be formulated as premisses, not tacitly assumed and employed as principles. In any deductive system whatever, "the principles on which the theorems are deduced from the primitive propositions must be materially true, for otherwise the deduction would be invalid" (p. 278, <u>op</u>. <u>cit</u>.). Since these very principles form the subject-matter of the science of deduction, i.e., of formal logic, they must appear as premisses in that science; and if the science of deduction is developed and presented as a deductive science, or deductive system of propositions, the basic principles of deduction must of course appear as primitive propositions in this system. The problem of the validity of formal logic is thus seen to be a problem of finding some guarantee for the material truth of the propositions employed as primitive propositions in a system of formal logic.

Jørgensen's solution of the problem thus stated is based on the second of two considerations (mentioned by him on pp. 278-279). Relying on "the fact that the primitive propositions of formal logic can themselves be employed as principles of deduction", he observes that for this reason (1) "the material truth of the theorems is guaranteed by the fact that the primitive propositions are materially true, and that the theorems are deduced in a manner formally valid from the primitive propositions, solely by means of the primitive propositions themselves," and (2) "that the primitive propositions are materially true is guaranteed by the fact that they deal solely with relations between the truth values of objectives and especially with such relations as subsist irrespective of whether their constituents are true or false objectives, whence it follows that the falsity of the primitive propositions is altogether inconceivable."

In support of this second contention, he proceeds to explain (following Nicod) what is meant by "truth value relations", remarking that such relations are expressed in what Russell and Whitehead call "truth functions", and referring to Wittgenstein's table of the sixteen possible truth functions which together express all the truth value relations that can exist between two elementary (i.e. unanalyzed) propositions, p,q. Special attention is then directed (p. 288) to those truth functions whose truth is independent of the truth values of their constituents, e.g., p) p, p.q.).p, p .). p v q, "which are true regardless of whether p is true or false or q is true or false". Every logical principle, he maintains, is a truth function of this kind; and in particular, such are the propositions in "the elementary theory of deduction in the Principia...all generalized propositions which in reality do not contain real but only apparent variables".

Since the elementary propositional calculus of the Principia is, supposing suitable symbolic changes, one of the interpretations of the abstract symbol-system whose initial sets we have been discussing, an examination of the ideas underlying Jørgensen's views as above indicated will enable us to stress an important point or two regarding the interpretation of this system as a system of formal logic. Granting the force of his argument as above outlined, and assuming that a system of formal logic has been constructed with such primitive propositions as he describes, we wish to inquire whether the material truth of such propositions is indeed "guaranteed by the fact that they deal solely with relations between the truth values of objectives...such relations as subsist irrespective of whether their constituents are true or false objectives".

At first sight there appears no reason for taking this statement as a topic of investigation. For if we observe that these propositions are asserted as primitive propositions in a truth-value system, and agree with Professor Lewis's statement, "Nothing is ever asserted in a truth-value system unless it is a tautology" (Symbolic Logic, p. 240), the obviously tautological character of these same propositions would seem to afford a guarantee of their truth. There still remains, however, the vital question, "In what sense are these propositions tautologies?" The force of this question will be appreciated if we notice the difference between statements which are tautological in virtue of containing symbol-complexes arbitrarily defined as identical in reference, and those which are tautological because the symbol-complexes they contain are really identical in reference: that is to say, because their terms are in fact different descriptions of the same thing. As an instance of the first kind, it may be said that the expression "p) q .=. pp v q" is a tautology because the two symbol-complexes on either side of the mark = are defined equivalents; or that the sentence "Triangles are three-sided plane figures" is a tautology because the word "triangle" is chosen as a more convenient symbol in place of the more cumbersome expression "three-sided plane figure". This same sentence, however, would express a tautology of the second kind (which in pre-Kantian philosophy was spoken of as an analytic proposition), if we understood it to mean that the description conveyed by the word "triangle" and the description conveyed by the phrase "three-sided plane figure" apply to the same object; with the implication that this identity of reference will be manifest to anyone who compares the meaning of this word with the meaning of this phrase.

Special care is needed if we wish to avoid confusing these two different kinds of tautologies, when we are dealing with any interpretation of an abstract symbol-system which is itself an abstract system. For in order to discover whether a given symbol-complex expresses a nominal tautology or a strictly analytic proposition, we must make sure that the relation stated by the symbol-complex is a relation not merely between the symbols but also between the entities meant by the symbols. The task thus incumbent upon us becomes doubly difficult when we have to do with a logistically-developed system based on ititially-undefined concepts: since it may be necessary to investigate whether, once the systemic definition of a given concept has been ascertained, such a systemic definition does full justice to that same concept as it occurs outside the system.

So in the case of these generalized propositions, or truth-functions, which we are discussing, it will be worth while to make sure of their exact meaning, if possible; and in any event, the effort to discover precisely what information they purport to convey will illustrate the importance of our previous cautionary remarks about the interpretation of an abstract symbol-system.

To begin with, a consideration of the way in which any of these truth-functions is constructed is sufficient to raise some doubt about "the fact that they deal solely with relations between the truth-values of objectives"; or at any rate, supposing this to be indeed a fact, it is very difficult to be sure that such literal symbols as occur in the expressions of these functions mean what is meant by the words "any proposition". We have been taking for granted, by the way, that "objective" as used by Jørgensen is synonymous with "proposition" as used in modern logic generally. It would be more accurate to say that by "objective" he means "unasserted proposition"; and in this event some would insist than an objective has no truth-value. To adopt this view, however, would be to risk obscuring the difference between a concept and a proposition. Whether a given proposition is in fact either asserted or denied by anyone, it would not be a proposition at all, but merely a concept, unless it were such that it could be significantly asserted or denied by someone, at least mentally. This being so, even an unasserted proposition has truth-value; it is in fact determinately true, or in fact determinately false, though we may not know which of these two truth-values it possesses. If now we were to ask, with regard to any two given propositions p,q, what relation could exist between them in respect of their truth-values, we should incline to say at once that there are only two relations possible: the truth-value of p would either be the same as the truth-value of q, or else different from (i.e. the opposite of) the truth-value of q. Inasmuch as any proposition may be more or less false, degrees of difference might be indicated according to some agreed standard; but an exact standard would hardly be applicable in every case, even among those relatively few cases (i.e. propositions about measurable characteristics which can be quantitatively determined) where such a standard is available. Some such procedure as this may well have been involved in the traditional account of truth-value relations subsisting between objectives in virtue of their form, which is epitomized in the familiar square of opposition, and the rules accompanying this diagram: according to which, as is well known, the truth, or alternatively the falsity, of a given proposition is determinable by the truth, or in some cases by the falsity, of another proposition which is formally related to it as contradictory, contrary, subcontrary, or subaltern. At any rate, whatever may be thought of the inadequacies of this earlier treatment of truth-value relations, it must be admitted that the relation described in each of these cases is a recognizable truth-value relation, and even those who find difficulty with the doctrine because of considerations connected with the problem of existence will agree that all these relations afford a basis for valid inference once the existence of the subjects of the various propositions is assumed.

The truth-value relations symbolized by Wittgenstein's truth-functions (and other symbolically different but significantly equivalent expressions) seem, however, to be much more complex. They are all based on the fact that in the case of any two given propositions, p, q, there are four possible combinations of these which differ from one another according as the truth-value of the propositions themselves varies, respectively, in each case: that is, (1) we may have p true and q true, or (2) p true and q false, or (3) p false and q true, or (4) p false and q false. As our previous remarks about truth-value relations suggest, it is difficult to see how, in the case of any given pair of propositions, the relation between their respective truth-values in case (1) differs from that in case (4), and the same applies to cases (2) and (3); though the truth-value relation in the two latter cases is clearly not the same as that in the two former. However this may be, it is manifest that with reference to any given pair of propositions, these four possibilities are such that no two of them can be simultaneously fulfilled; that is to say, if (1), then not (2) nor (3) nor (4); if (2), then not (1) nor (3) nor (4); if (3), then...etc. In other words, if the symbols p,q each mean what is meant by the words "some proposition or other", we seem forced to conclude from the above considerations that there is not, and that there cannot be, any values whatever of p,q such that the same pair of values would satisfy any two of these four cases; unless, of course, some one and the same pair, or one member thereof, somehow underwent a change in its truth-value.

A symbolic expression of any one of these four cases, however, would not be said to express, or to be an instance of, a truth-function. The possible truth-functions of p,q are not four but sixteen in number; "according as", in Jørgensen's words (op. cit., Vol. III, p. 286), "all four combinations or only three of them or only two of them or only one of them or none of them exist...Each of these 16 possibilities", he goes on to say (p. 287), "expresses a definite value relation between p and q, and taken all together, they represent all possible value relations between p and q. The objectives which assert that one or another of these value relations subsists between p and q are truth functions, their truth or falsity depending solely on the truth values of p and q." The truth function "p) q", for instance (mentioned as number 5 in the table quoted from Wittgenstein on p. 287), expresses the truth value relation known as material implication; this relation is defined as holding in the first, second and fourth of our original cases, and failing to hold only in the third; or as Jørgensen puts it (p. 286): "If we know for instance, that the combinations I, II, and IV exist, while III is out of the question, then we shall have the relation known as material implication between the truth values of p and q." (It will be seen that his III is our (2) above).

How a statement to the effect that none, or that more than one, of these four possible truth value combinations exist for a given p,q can be a true statement if p,q are a pair of propositions, we have seen reason to wonder. And how such a statement can be said to "express a definite value relation between p and q", in any ordinary sense of the word "relation", is almost as difficult to understand as the matter of saying precisely what that "definite" relation is appears difficult to accomplish. Jørgensen himself leaves room for doubt whether these value relations are relations between p and q (as stated on p. 287), or between the truth values of p and q (as is said of material implication on p. 286).

It may be objected that the difficulties above mentioned with regard to the significance of these truth functions all arise from ignoring the fact that they are intended to be interpreted as extensional functions. Every relation, as is evident, requires two or more terms in order to exist. Remembering that any two terms between which a relation holds are called, respectively, the referent and the relatum of that relation, and that the totality of all possible referents (i.e. the domain of the relation) together with the totality of all possible relatums (the converse domain of the relation) make up a totality of terms which are the field of the relation in question, we may consider any given relation as something having a definite field, or range of terms between which it may hold. From this point of view, the field or extension of a relation is taken to be the totality of all possible terms which can serve either as referents or as relata for that relation; and if any symbol-complex consisting of a symbol for some relation and on either side thereof a variable symbol be interpreted as an extensional symbol, the first variable symbol will refer indeterminately to the totality of possible referents, and the second variable to the totality of possible relatums, of the relation in question. Such an extensional interpretation of the expression "p) q", for instance, would indicate that the symbols p,q do not mean, respectively, what is meant by the words "some proposition or other"; rather, p means what is meant by the words "the first of any pair of propositions whereof the first member is true and the second true, or else the first of any pair whereof the first member is false and the second true, or else the first of any pair whereof both members are false"; and what q means could be described in the same way verbally, as "the second of any pair which fulfills one of the above conditions." Seeing that p,q thus represent indeterminately any pair of propositions which fulfill the conditions just given regarding the truth value of their members, we must try to see in what sense the fulfillment of these conditions, in the case of a given pair, can be said to give rise to a relation between the two members of that pair, or perhaps between the truth-values of each of the two members. Otherwise the assertion that there is a relation between such pairs, or rather between the members of such pairs as fulfill these conditions remains open to question.

To approach this problem as extensionally as possible, we shall endeavor to speak in terms of totalities, or extensions. Consider first the totality of elementary or unanalyzed propositions (or in fact any other totality recognizable as legitimate according to the theory of types). Since every member of this totality has either the character "true" or the character "false", or at any rate is such that it can have one and only one of these two characteristics, the totality will be composed of two different kinds of propositions, those which are true and those which are false. Thus we have two mutually-exclusive totalities within the totality of propositions under consideration. We might use the symbols P,Q,R...to denote any member of the original totality; the symbols p,q,r...to denote a member of the totality of true propositions; and the symbols -p,-q,-r...to denote a member of the totality of false propositions. Next consider the totality of pairs of propositions; this totality will consist of all such pairs as P and Q, P and R, Q and R... etc. If regard is had to the truth value of the members of each such pair, it will be seen that the totality of pairs of propositions contains four mutually-exclusive subtotalities: (1) all such pairs as p and q, q and r...etc.; (2) all such pairs as -p and q, -q and r...etc.; (3) all such pairs as p and -q, q and -r...etc.; (4) all such pairs as -p and -q, -q and -r,...etc. The symbolism made use of is clearly such that by means of it we can univocally refer to any one of these sub-totalities. Now suppose that we wish to refer to more than one of them, indeterminately; and in particular, wish to indicate that the range of values of P and Q includes the range of values of p and q, -p and q, -p and -q, but does not include any of the range of values of p and -q, such an indication might be given by using the symbol-complex P) Q. We have, by so doing, provided ourselves with a means of referring to a new sub-totality contained in the totality of all pairs of propositions; this new one consists of all those pairs of propositions which are values of p and q, all those which are values of -p and q, as well as all those which are values of -p and -q, but it excludes all pairs of propositions which are values of p and -q. Thus the range of values of P,Q in the expression P) Q is made sufficiently definite. At this stage it may seem safe to maintain that there is a relation of some kind between the two members of any pair of propositions within the totality of values of P and Q thus specified; we seem to have no difficulty in determining certain properties of this relation, noting that it has a definite direction (from P to Q), that it is transitive, and that it is not symmetrical. The fact that we cannot so easily say precisely what this relation is may not raise any doubts about whether or not it exists, especially since a fuller account of its nature appears unnecessary for practical purposes. And because the members of all these pairs of propositions have been classified and subdivided according to their truth value, respecitvely, in every pair, we may feel no hesitation in declaring that this relation between every pair of values of P and Q is a truth-value relation.

It is hardly necessary to remark that this relation is not such as to give rise to a uniquely-determined value of Q corresponding to a determinate value of P, when it holds between some pair of values of P and Q. And it certainly cannot be said that whenever this relation holds between any such pair of values, the truth value of the second member of such a pair is uniquely determined by the truth-value of the first member of that same pair. The preceding statement is true only when the values of P and Q between which it holds contain as their first member some propositions which is also a value of p (and therefore not a value of -p). It is only in such cases that the truth-value of every satisfactory value of Q is determined; for only in such cases are these values of Q definitely specified as being also values of q, but not also values of -q. Here, be it noted, we are calling attention to something which (according to the terminology in vogue since Mr. Johnson's time) is not simply an epistemic condition of inference, but a truly constitutive condition thereof; hence it bears directly upon the relation of implication itself, rather than upon one's knowledge of that relation.

Enough has been said, perhaps, to show that if the mark) in the expression P) Q does really stand for a relation, the entity meant by the word "relation" in this context is something much more complex than, or at any rate something very different from, what is meant by the word "relation" in ordinary philosophical usage. We may add that its existence appears to depend on something more than the respective truth value of the members of those pairs of propositions which satisfy the expression P) Q. At least, if we compare the totality of pairs of propositions within which the relation does not hold with the other totalities within which it does hold, the following statements appear to be true. (a) This relation holds between all pairs of propositions whose members, taken pair by pair, have the same truth value. (b) It does not hold between all pairs whose members, pair by pair, differ from each other in truth value; with regard to such pairs, it holds only if the first member is false and the second member true, but not if the first member is true and the second member false. Thus the existence of this relation, and therefore the truth of any proposition asserting its existence between the members of any given pair of propositions, depends not merely on the sameness or the difference of the truth value possessed by the members respectively, but also on the order in which the members stand, in any given pair.

As was noted at the beginning of this section (p. 200, above), we are not here concerned with the correctness of Jørgensen's solution of the problem of validity; and in commenting on his statement that the truth of the primitive propositions of a system of formal logic is guaranteed by the twofold fact mentioned by him,-- namely, (a) that these propositions deal solely with relations between truth values of objectives, and (b) that the relations in question subsist irrespective of whether the constituents of these propositions are true or false,-- our main concern is not to insist that his statement is false. What has been said thus far may serve to cast doubt on the first of his two facts; but the main point we wish to make is, that both these facts must be verified by extra-systemic considerations, if they are to guarantee the truth of the primitive propositions which presumably express them. Hence we have suggested that the nature of the entities whose existence these propositions assert must be settled by other means than by merely systemic definition; and in particular, that the truth-value relations referred to by these propositions should be recognizable as such outside this system; for otherwise we cannot be sure _what_ they are, nor whether statements about them have reference to anything outside the system in which these statements occur. And in that case, we cannot say that they are tautologies in the sense of statements expressing analytic propositions.

Just as it is important to know something, extra-systemically, about the nature of the relations symbolized by the constant symbols occurring in the expression of these primitive propositions if we wish to discover whether they are analytically true, so it is no less important to know something about the nature of the entities which are values of their variable symbols, for the same reason. The special difficulty in this connection is one to which Professor Langford has called attention in his treatment of the logical paradoxes: the word "value", he observes (_Symbolic Logic_, p. 444), has a twofold sense, owing to the two very different kinds of substitution by the use whereof a symbolic expression may be transformed. The example employed by him to illustrate this point is particularly appropriate to our present discussion, for it is not only one of those tautological truth-functions mentioned by Jørgensen, but is admitted to be the simplest of all tautologies; and moreover, as Professor Lewis has clearly shown (op. cit., pp. 249-250), every tautology is expressible as a general proposition of this form. The form in question is (p). p v -p. Professor Langford notes the difference between two derivatives obtainable therefrom by substitution: (1) (q,r). (q v r) v -(q v r), where we replace "the generic expression p by the more specific expression q v r"; this he calls "a case of 'genus-species' substitution", valid because "the original expression implies the derived one", and the ground of validity is "that whatever is true of all propositions of the form p must be true of all of the form q v r"; (2) "Men of either do or do not exist", where we replace p by "one of its values...one of the propositions which it denotes" (i.e. "Men exist"); and this he calls "a case of 'genus-instance' substitution", valid on the ground that whatever holds for all values of -p v p holds for this particular one."

Although we have referred to the above-quoted account in order to show that the word "value" is ambiguous, and hence that caution must be exercised in interpreting statements containing variables, there are one or two points which may be worth noting in this passage, apart from this principal one. We have here a clue to the way in which a generalized truth-function should be read as a general proposition. The original formula given by way of example has a meaning which can be stated thus: "For all values of p, the statement -p v p is true". This statement can be read: "Either not -p is true or p is true". It thus appears as an assertion about any proposition of whatever form, to the effect that, as regards any given proposition, either the negative of that proposition is true or else the proposition itself is true. The assumption underlying this last reading is, of course, that the symbol p, considered in conjunction with the prefix (p) of the original formula, means what is meant by the words "any member of the class of propositions"; and the sameness of the symbol p on both occasions of its occurrence indicates that reference is made both times to one and the same given member of this class, although this may be any member whatever. The actual expression obtained from the original formula by 'genus-instance' substitution is obtained, evidently, by replacing the symbol p, each time, by a sentence which expresses some one and the same proposition.

Another method of reading the original formula is as follows: "Any compound proposition whatever, consisting of a pair of instances of one and the same proposition, the first of the pair being the negative form of the other and both members being united by the relation signified by 'or', is always true, no matter what be the form or the truth-value or the meaning of the proposition whereof these instances are instances." The truth of this lengthy assertion, which can be recognized as an analytic proposition by anyone who understands the meaning of its terms, is a guarantee of the truth of all propositions obtainable by 'genus-instance' substitution in the original formula.

With regard to 'genus-species' substitution, two points may be worth noting. It would seem clearer to differentiate this from the former kind on the ground of a more fundamental difference than is suggested by the contrast between 'genus-species' and 'genus-instance'. Quite apart from any question of terminology, the really important difference between them appears to be this: When p is replaced by some other expression such as q v r, it is clear that one symbol is being replaced by another (more complex) symbol. This means that in such cases we are merely replacing one kind of symbol by another kind of symbol; and that when a replaceable symbol is viewed in relation to such other symbols as may replace it, it is not fundamentally different from any of these "replacement-values" in spite of possibly differing from them in form. The symbol p, in any expression, can refer indeterminately to any of its "replacement-values"; but the question which must be answered before we can say what those values are is the question whether p is a direct symbol of the form of those replacement-values, or merely a direct symbol of the position they must occupy. The symbol p is not a direct symbol of the form of such expressions as q v r; hence it seems that the replacement-values of p may be of any form whatever. In the case of 'genus-instance' substitution, although the substitution itself is a mere replacement of one symbol by another symbol or set of symbols, the change involves a complete alteration of viewpoint. Once we have introduced verbal symbols, reference is made to the meanings of the words, and thus to the content of the propositions expressed in these words. Such symbols as p, or such symbols as q v r, on the other hand, refer not to propositions, but to forms of propositions; thus p may symbolize directly the form of a non-compound proposition, and indirectly it may symbolize the form of any proposition. It is for this latter reason that p is more generic than the expression q v r.

We may say, then, that such symbols as p, or any replacement-values of p in a given system, symbolize forms of propositions, and hence bear a relation to all those propositions whose forms they symbolize. If it be admitted that the notion of meaning is a more useful criterion than the notion of form, in distinguishing one proposition from another, and that, if we regard both meaning and form as characteristics of propositions, meaning is somehow more fundamental than form, then it may also be granted that a symbol of form is not so closely related to a proposition as is a symbol of meaning. The question thus raised is, with regard to all such symbols as p, or q v r, "Are the values of these symbols propositions, or are they rather forms of propositions?" Form may be a mere matter of syntactic arrangement: a question not of logic but of language. In genus-species substitution, if it be true that the replacement-values agree with one another and with the original symbol in being all symbols of form, no fundamental change in reference will be entailed by such substitution. In genus-instance substitution, on the other hand, inasmuch as symbols of form are replaced by symbols of content (i.e. symbols of meaning), there seems to be a more marked change of reference involved. We cannot pursue this topic further at present; but it may well be that along this line of thought a clearer insight could be obtained into the nature of the logical paradoxes;

for, as Professor Langford remarks, these paradoxes never arise in cases of 'genus-species' substitution, but only in cases where the 'genus-instance' kind of substitution occurs.

The status of the primitive propositions obtained by generalizing truth functions which yield true propositions irrespective of the truth value of their constituents, and in particular the question whether they are analytic propositions, is likely to remain doubtful until we have ascertained whether the entities to which their variables refer are propositions, or merely forms of propositions. And as we have said, their truth must be settled by extra-systemic considerations. One further comment suggests itself, in connection with the statement that these primitive propositions are not only premisses from which the theorems of logic are derived, but also principles of inference according to which they are derived. Even supposing that their truth as premisses can be guaranteed in some such fashion as we have suggested, so that they can be understood as analytic propositions: they might then be regarded as principles of inference, in the sense of being true statements about the relations between propositions which, when they hold, make inference valid, but it would not at once follow that they are used as principles of inference in the development of a given system of logic. In other words, because such development proceeds by way of substitution, it would have to be shown (as we have actually tried to show in the preceding chapter) that the method of substitution whereby sets of symbols are derived from other sets involves a reference to those principles of inference whereby propositions are deduced from other propositions.

Although we have touched upon only a few of the problems connected with the interpretation of an abstract symbol-system as a system of formal logic, enough has been said to show that the adequacy of any such system depends upon the care and completeness with which its basic notions have been analyzed, and the extent to which the results of such analyses can stand the test of extra-systemic criteria. In conclusion, we shall try to sum up the main points of the present study, and to suggest a few tentative conclusions based thereon.

CONCLUSION

In the course of the preceding pages, we have been specially concerned with emphasizing the following general considerations:

First, a sharp distinction must be made between those different systems which have been constructed as systems of logic, on the one hand, and the different abstract symbol-systems which have been devised to symbolize systems of logic, on the other hand.

Second, if formal logic itself be looked upon as the science of system-structure, having as its main concern the investigation of the so-called structural properties of symbols and the uses to which these properties can be put in constructing systems of symbols whereby the structure of other systems of entities can be directly represented or pictured, certain definite limitations impose themselves upon such a science if it is to constitute a contribution to human knowledge or even provide a means of putting in order such knowledge as one already possesses. In particular:

(a) If symbol-systems are to be really constructed,-- that is, if the marks whereof they are composed are to be so arranged as to constitute an orderly array, the order of which is not simply an accidental result of their being selected and set down at random,-- account must be taken throughout, not merely of the actual characteristics of the individual marks and collocations of marks, but of the possible significance of these characteristics: that is, the possible meanings of the system.

(b) Even supposing it were possible to construct an abstract symbol-system out of marks without such reference to meaning as the above statement mentions, and supposing such a system to be used as a map-like symbol of the structure of deductive systems (e.g. of mathematics or of logic), no hint of the specifically deductive character of these latter systems could be conveyed by such representation; the order, or sequence, of their elements might indeed be shown, but not the dependence of one element on any other, and hence not any necessary connection, or relation of consequence, such as serves to distinguish deductive systems from all non-deductive systems whose elements have a definite sequence order.

By way of giving point to the analysis attempted, especially in the first two chapters, of such general notions as system, and structure, and isomorphism, and the meaning of symbols, and interpretation, and pictorial representation of structure, an attempt was next made to apply some of the general principles previously considered to an investigation of a given abstract symbol-system. Taking a variation of Huntington's postulates, and regarding them in the most completely abstract fashion, a concrete study of the method of derivation of theorems from abstract postulates led us to conclude that this method involved reliance on the ordinary principles of deduction. Assuming that the system derived from these postulates was intended to be more than a map-like representation of the structure of systems isomorphic with it, we endeavored to point out:

(a) That the strings of marks which made up the sets, or formulae, of the system can be read as propositions, apart from any specific interpretation.

(b) That from this point of view they convey information not about the symbols they contain, but about any entities whatever which may be meant by these symbols.

(c) That they thus afford, collectively, a description of those entities, which sets quite clear, though not narrow, limits to the possible interpretations of the system.

(d) That the individual sets, thus read as propositions, can be tested for consistency, if not for actual truth, on extra-systemic grounds, by analysis of their terms.

(e) That when read in this way, the theorems are seen to be consequences of the postulates and of previously-derived theorems.

In our closing chapter, we have outlined some reasons for being very cautious in accepting any conclusions about the nature of logic or the status of such logical entities as propositions, when these conclusions rest upon systemic analyses of logical concepts or upon results that seem to emerge when some abstract symbol-system is presented as a system of logic. If what we have said in connection with the particular abstract symbol-system whose development we studied is reliable, the semi-mechanical method of symbol substitution gives rise to conclusions no less valid than ordinary methods of deductive thinking; but such thought-saving procedures will surely breed confusion in the realm of thought unless the symbols which express their conclusions be rightly interpreted, and their referenda identified by extra-systemic criteria.

Thus, in an indirect way, we have tried to suggest certain limitations which appear to be inseparable from a formalist approach to logic, and also the necessity of validating any system of logic on extra-systemic grounds, as well as the lines along which such validation might be worked out. To accept any system of mathematical logic as a thoroughly reliable description of logical entities, to say nothing of accepting it as a substitute for epistemology, will perhaps not seem advisable to anyone who reflects upon the following fact. Every one of the postulates and theorems which appear in the familiar two-valued algebra as statements about propositions and the relations between them with respect to their truth or falsity can be read as true statements about numbers and the relations between them with respect to their being even or odd. This seems to suggest that the postulates and theorems of any mathematically-developed system have to do only with such properties of whatever entities their variable symbols can represent, as are possessed by mathematical entities, or are analogous to specifically mathematical properties. At any rate, it would seem unsafe to subscribe to the view that pure mathematics is a branch of pure logic until one has ascertained by careful analysis of both mathematical and logical concepts that the "pure logic" referred to is not merely an abstract mathematical system expressing properties which logical and mathematical entities happen to possess in common.