BUMPING IN THE DEFORMATION SPACES OF HYPERBOLIC 3-MANIFOLDS WITH COMPRESSIBLE BOUNDARY

by

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This dissertation is dedicated to my grandmothers Akiko and Maria, my grandfather Sadayuki, and in loving memory of my grandfather Heinz. Thank you for teaching me to enjoy the little moments, to work hard, and to always keep a sense of humor.

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CHAPTER I

Introduction

Let M be a compact, hyperbolizable 3-manifold. The space $AH(M)$ of marked hyperbolic 3-manifolds homotopy equivalent to M gives a natural setting for studying and relating different possible hyperbolic structures on M. Points in $AH(M)$ can be viewed either as pairs (N, h) where $h : M \to N$ is a homotopy equivalence and $N = \mathbb{H}^3/h_*(\pi_1(M))$ is a hyperbolic 3-manifold, or as discrete representations $h_* = \rho : \pi_1(M) \to PSL_2(\mathbb{C})$. The latter point of view identifies $AH(M)$ with a subspace of the character variety in the following way.

$$AH(M) = \{\rho : \pi_1(M) \to PSL_2(\mathbb{C}) | \rho \text{ is discrete, faithful}\}/PSL_2(\mathbb{C})$$

Closely related to $AH(M)$ is the space $\mathcal{A}(M)$ of marked compact hyperbolizable 3-manifolds homotopy equivalent to M. The points of $\mathcal{A}(M)$ can be thought of as equivalence classes $[(M', h')]$ where M' is a compact hyperbolizable 3-manifold and $h : M' \to M'$ is a homotopy equivalence and where two points $[(M_1, h_1)]$ and $[(M_2, h_2)]$ are equivalent if there is an orientation-preserving homeomorphism $f : M_1 \to M_2$ such that h_2 is isotopic to $f \circ h_1$.

The topology of the interior of $AH(M)$ is well understood from work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston. Essentially, the components of the interior are enumerated by marked homeomorphism types $[(M', h')]$ in $\mathcal{A}(M)$

with each component parametrized by the Teichmüller data of the boundary surfaces of M' (a precise statement of this fact can be found in Chapter II, Theorem II.8).

We consider the following question: given two components of the interior of $AH(M)$, under what conditions do these two components "bump," that is, when do their closures intersect in $AH(M)$? When M has incompressible boundary, this question is answered in full by work of Anderson, Canary and McCullough [4, 3, 19]. However, when M has compressible boundary, the global topology of $AH(M)$ is still quite mysterious.

This thesis describes an adaptation of the bumping construction of Anderson-Canary-McCullough to a broad class of compressible boundary hyperbolic 3-manifolds. Our construction stems from the following observation: suppose that M is a hyperbolizable 3-manifold obtained from a hyperbolizable 3-manifold M_0 by attaching a one-handle to the boundary of M along attaching disks D_0 and D_1. Then, under certain conditions, we can "shuffle" the end of the attached one-handle to another boundary component of M_0 to produce a new 3-manifold. Under suitable conditions, this 3-manifold will be homotopy equivalent to M.

As in previous bumping constructions, a key feature in our construction is the presence of primitive essential annuli. An embedded annulus A in M is called **essential** if it is properly embedded (i.e. the boundary of A is sent to the boundary of M), the embedding is π_1-injective, and A is not properly homotopic into the boundary of M. Such an annulus is called **primitive** if $\pi_1(A)$ maps onto a maximal abelian subgroup of $\pi_1(M)$.

Suppose that M_0 is a compact hyperbolizable 3-manifold containing a primitive essential annulus A, of which a regular neighborhood W intersects the boundary of M_0 in annuli V_1 and V_2. Let D_1, respectively D_2 be disks contained in V_1, respectively

V_2. Let D_0 be some other disk in the boundary of M_0. Suppose that M is a hyperbolizable 3-manifold obtained from M_0 by attaching a one-handle along D_0 and D_1 and let M' be a hyperbolizable 3-manifold obtained from M_0 by attaching a one-handle along D_0 and D_2.

We introduce the notion of a *primitive handle shuffle associated to* A, a homotopy equivalence $h_A : M \to M'$ that "shuffles" the position of the attached one-handle around W. We say that equivalence classes $[(M_1, h_1)]$ and $[(M_2, h_2)]$ in $\mathcal{A}(M)$ *differ by a primitive handle shuffle associated to* A if there is a primitive handle shuffle $h_A : M_1 \to M_2$ such that $h_A \circ h_1$ is homotopic to h_2. Our main result is the following:

Theorem I.1. *If two equivalence classes* $[(M_1, h_1)]$ *and* $[(M_2, h_2)]$ *in* $\mathcal{A}(M)$ *differ by a primitive handle shuffle, then the corresponding components of* $int(AH(M))$ *have intersecting closures.*

There are two key ingredients that our construction shares in common with known bumping constructions: the presence of primitive, essential annuli, and the Hyperbolic Dehn Filling Theorem. Let M be a hyperbolic 3-manifold with a toroidal boundary component T. Pick suitable meridian and longitude curves $\{m, l\}$ on T and a meridian curve α on the boundary of a solid torus V. Then the (p, q) Dehn filling $M(p, q)$ of M is obtained by gluing V along T so that α gets sent to a (p, q) curve on T. Thurston discovered that such "fillings" produce hyperbolic 3-manifolds except for finitely many choices of p and q. We make use of the Hyperbolic Dehn Filling Theorem (see Theorem V.2) to control the limit behavior of sequences in $AH(M)$.

The following is a loose sketch of the proof of Theorem I.1. For simplicity of notation, we reduce to the case where $(M_1, h_1) = (M', h_A)$ and $(M_2, h_2) = (M, id)$

where $id : M \to M$ is the identity map. One can remove a neighborhood of the core curve of A from M' to obtain $\hat{M}'$, now containing a toroidal component in its boundary. We use hyperbolic Dehn filling to construct 3-manifolds $M'_n = \hat{M}'(1, n)$ obtained by $(1, n)$ Dehn filling on $\hat{M}'$. Let $i_n : \hat{M}' \to M'_n$ be the natural inclusion map. The Hyperbolic Dehn Filling Theorem gives the existence of (nonfaithful) representations $\beta_n : \pi_1(\hat{M}') \to PSL_2(\mathbb{C})$ and maps $\psi : \hat{M}' \to \hat{M}'$, $\psi_n : int(M'_n) \to \mathbb{H}^3/\beta(\pi_1(\hat{M}'))$, such that $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$

We then construct a cover $\pi : \tilde{M} \to \hat{M}$, and an embedding $\tilde{f} : M \to \tilde{M}$ realizing M as a compact core of $\tilde{M}$. We define representations $\rho_n = \beta_n \circ \psi_* \circ \pi_* \circ \tilde{f}_*$, with algebraic limit ρ. Results of Comar [21][Thm 6.2] and an additional topological argument show that ρ_n all lie in the component of $int(AH(M))$ corresponding to $[(M', h_A)]$.

We use the uniqueness of compact cores, due to McCullough-Miller-Swarup [34], to identify the compact core of (N_ρ, h_ρ) with M. We further show that (N_ρ, h_ρ) has the same marked homeomorphism type as (M, id). It is a consequence of a theorem of Ohshika [42] that (N_ρ, h_ρ) lies in the closure of the component of $int(AH(M))$ corresponding to $[(M, id)]$, thus exhibiting bumping.

The following paragraphs give a roadmap of this thesis. In Chapter II, we survey some background material concerning the known structure of $AH(M)$ including the parametrization of the interior. We also address three major contributions to the understanding of $AH(M)$ in the past twenty years: the Density Theorem, which states that $int(AH(M))$ is dense in $AH(M)$, the Tameness Theorem, which guarantees that all ρ in $AH(M)$ yield hyperbolic 3-manifolds homeomorphic to the interior of a compact 3-manifold, and the Ending Lamination Theorem, which describes how one can understand the boundary of $AH(M)$ by geometric invariants on the boundary surfaces (by Teichmüller data or ending laminations).

In Chapter III, we give a brief survey and some examples of known bumping constructions. We start with the first known examples of bumping in the case where M is a "book of I-bundles", and then proceed to bumping results for incompressible boundary 3-manifolds. We also give some other indications as to the complexity of the global topology of $AH(M)$ such as "self-bumping" and the failure of $AH(S \times [0,1])$ to be locally connected.

In Chapter IV, we build the notation and definitions needed to state our theorem, including the notion of primtive handle shuffles. We also introduce a motivating example where M is formed by attaching a one-handle to the boundary of $M_0 = S \times [0,1]$ with S a closed surface of genus g. For this particular example, Canary-McCullough proved that $int(AH(M))$ consists of only two components. A corollary of our result is that these two components bump, and so $AH(M)$ is connected.

In Chapter V, we give a proof of Theorem I.1. This chapter also contains some important results that are integral to our proof such as the Hyperbolic Dehn Filling Theorem, and a result due to Ohshika which controls the limiting behavior of the sequence of representations.

CHAPTER II

The Topology of $AH(M)$

In this chapter, we give a survey of the known structure of $AH(M)$. We will introduce some important foundational work on the understanding of the interior of $AH(M)$ as well as some other results that give rough outlines of the global topology of $AH(M)$. We will leave results on bumping between components of $int(AH(M))$ until the next chapter, where those constructions will be spelled out in detail.

2.1 Definitions

Definition II.1. A **Kleinian group** Γ is a discrete subgroup of $PSL_2(\mathbb{C})$, viewed as acting on hyperbolic 3-space $\mathbb{H}^3$ by orientation-preserving isometries, and on the boundary Riemann sphere $\bar{\mathbb{C}}$ by Möbius transformations.

In our setting, Kleinian groups will be assumed to be torsion-free. Since the action of a Kleinian group on $\mathbb{H}^3$ is discrete and acting by isometries, the quotient $N_\Gamma = \mathbb{H}^3/\Gamma$ is a 3-manifold that is locally modeled by $\mathbb{H}^3$, hence we call it a *hyperbolic 3-manifold.* Notice that since $\mathbb{H}^3$ is contractible, the fundamental group $\pi_1(N_\Gamma)$ can be naturally identified with Γ and in fact the action of Γ on $\mathbb{H}^3$ is simply the action of $\pi_1(N_\Gamma)$ by covering transformations.

The action of Γ on $\bar{\mathbb{C}}$ partitions $\bar{\mathbb{C}}$ into the *limit set* $\Lambda(\Gamma)$, the smallest closed invariant subset under the action by Γ, and the *domain of discontinuity* $\Omega(\Gamma)$, the largest open subset of $\bar{\mathbb{C}}$ on which Γ acts discontinuously.

A hyperbolic 3-manifold is a geometric object, but it will often be more convenient for us to deal with a purely topological object.

Definition II.2. A compact 3-manifold M is called **hyperbolizable** if its interior admits a complete hyperbolic metric, i.e. there exists a Kleinian group Γ so that $int(M)$ is homeomorphic to $N_\Gamma = \mathbb{H}^3/\Gamma$. In this case we say $int(M)$ is **uniformized** by Γ.

Thurston conjectured in the 1970's that all compact 3-manifolds admit canonical decompositions into geometric pieces, with the geometry on each piece coming from one of eight homogeneous 3-dimensional spaces: Euclidean space $\mathcal{E}^3$, the sphere S^3, hyperbolic space $\mathbb{H}^3$, the fibered geometries $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $P\tilde{S}L_2(\mathbb{R})$, the 3-dimensional solvable group Sol and the 3-dimensional nilpotent Heisenberg group Nil. Of these geometries, hyperbolic geometry is by far the most common and diverse, as demonstrated by the following Theorem, due in part to Thurston, and completed by Perelman.

Theorem II.3. ***Thurston's Hyperbolization Theorem*** *(Thurston [52], Perelman [46, 45, 44])*

M is hyperbolizable if and only if:

1. *$\pi_1(M)$ is infinite or M is the 3-ball,*
2. *M is irreducible, i.e. if every embedded 2-sphere in M bounds a ball in M, and*
3. *M is atoroidal, i.e. if no immersed torus is π_1-injective unless it is homotopic into the boundary of M.*

Thurston proved this in many cases [52], including the particular case where

∂M is nonempty, the case that concerns us in this thesis. Perelman's work on the Geometrization Conjecture [46] [45] [44] proves this in its full generality.

If M has no boundary or only toroidal boundary components, then Mostow/Prasad Rigidity [39] [47] tells us that if Γ and Γ' both uniformize M, then they are conjugate in $Isom(\mathbb{H}^3)$. So in this setting, the potential hyperbolic geometry that can be associated to M is uniquely determined by the topology of M, or equivalently the fundamental group $\pi_1(M) \cong \Gamma$.

Theorem II.4. ***Mostow/Prasad Rigidity*** *(Mostow [39] Prasad [47])*

If M and N are closed or finite volume hyperbolic manifolds of dimension $n \geq 3$ and if there exists an isomorphism $f : \pi_1(M) \to \pi_1(N)$, then this isomorphism is induced by a unique isometry from M to N.

If M has more complicated boundary, then there are typically infinitely many Kleinian groups Γ which uniformize M, and even more that produce hyperbolic 3-manifolds homotopy equivalent to M. We will require a distinction between two types of 3-manifolds with boundary: compressible boundary and incompressible boundary.

Definition II.5. A properly embedded surface S in a 3-manifold M is called **compressible** if either:

1. S bounds a 3-ball in M, i.e. S is a 2-sphere.

2. There is a simple closed curve α in S, bounding an open disk D contained in $M \setminus S$, but which does not bound a disk in $S \setminus \alpha$. The disk D is often referred to as a **compressing disk** for S.

If S is not compressible, then we say that S is **incompressible**. A 3-manifold M has **incompressible boundary** if all of the surfaces that comprise the boundary of M are incompressible. Otherwise, it has **compressible boundary**.

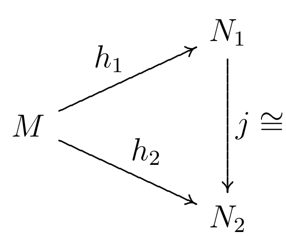
Given a 3-manifold M, a discrete, faithful representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ yields a hyperbolic 3-manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ and a homotopy equivalence $h_\rho : M \to N_\rho$ called the *marking* of N_ρ. We define the space of (marked) hyperbolic 3-manifolds homotopy equivalent to M to be the representation space:

$$AH(M) = \{\rho : \pi_1(M) \to PSL_2(\mathbb{C}) | \rho \text{ is discrete, faithful}\}/PSL_2(\mathbb{C}).$$

The topology on $AH(M)$, called the *topology of algebraic convergence* is the topology inherited from $Hom(\pi_1(M), PSL_2(\mathbb{C}))$ topologized with the compact-open topology. This name is rather suggestive, as we will understand the nature of this space primarily by studying convergent sequences of representations.

Definition II.6. If ρ_n converge to ρ in $AH(M)$, we say that the associated hyperbolic 3-manifolds N_{ρ_n} **converge to** N_ρ **algebraically**.

An equivalent definition of $AH(M)$ mirrors the classical definition of Teichmüller space for surfaces in the following way: we can consider $AH(M)$ to be the collection of pairs (N, h) where $h : M \to N$ is a homotopy equivalence and $N = \mathbb{H}^3/h_*(\pi_1(M))$ is a hyperbolic 3-manifold homotopy equivalent to M. Two points (N_1, h_1) and (N_2, h_2) are equivalent in $AH(M)$ if there is an orientation-preserving isometry $j : N_1 \to N_2$ such that $j \circ h_1$ is homotopic to h_2.



In a similar fashion, we define $\mathcal{A}(M)$ to be the set of marked homeomorphism classes of marked compact hyperbolizable 3-manifolds homotopy equivalent to M. Explicitly, $\mathcal{A}(M)$ consists of equivalence classes (M_1, h_1) where M_1 is a compact hyperbolizable 3-manifold and $h_1 : M \to M_1$ is a homotopy equivalence, and where two points (M_1, h_1), (M_2, h_2) in $AH(M)$ are equivalent if there is an orientation-preserving homeomorphism $f : M_1 \to M_2$ such that h_1 is isotopic to $h_2 \circ f$. For (M', h) in $\mathcal{A}(M)$, let $\mathrm{Mod}_0(M')$ be the group of isotopy classes of orientation-preserving homeomorphisms of M' homotopic to the identity.

2.2 The Interior of $AH(M)$

We will see that the components of the interior of $AH(M)$ are distinguished by their associated marked homeomorphism type. Furthermore, each component of $int(AH(M))$ has a natural manifold structure parametrized by analytic data associated to the boundary surfaces of M.

The first example of this structure was identified by Bers [5] in his Simultaneous Uniformization Theorem. Let S be a closed, oriented surface of genus $g \geq 2$. A point in the Teichmüller space of S is given by a representation $\rho : \pi_1(S) \to PSL_2(\mathbb{R})$. By composing with the inclusion map from $PSL_2(\mathbb{R})$ into $PSL_2(\mathbb{C})$, this representation can be viewed as a representation from $\pi_1(S \times [0,1]) = \pi_1(S)$ into $PSL_2(\mathbb{C})$. There is an orientation-preserving homeomorphism from the quotient manifold $\mathbb{H}^3/\rho(\pi_1(S \times [0,1]))$ to the interior of $S \times [0,1]$ and the limit set $\Lambda(\rho)$ is the circle $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. The domain of discontinuity of this action $\Omega(\rho)$ is the pair of hemispheres in $\hat{\mathbb{C}}$ separated by $\hat{\mathbb{R}}$. The quotient $\Omega(\rho)/\rho(\pi_1(S))$ yields a pair of Riemann surfaces each conformally equivalent to the original surface $\mathbb{H}^2/\rho(\pi_1(S))$ in the Teichmüller space of S. Such a representation is called a *Fuchsian* representation.

Small perturbations in $PSL_2(\mathbb{C})$ of such a representation produce "quasifuchsian" representations. A *quasifuchsian* representation is a discrete, faithful representation of $\pi_1(S \times [0,1]) \cong \pi_1(S)$ where the limit set is a Jordan curve in $\hat{\mathbb{C}}$. In this case, the limit set still bisects $\hat{\mathbb{C}}$ into two topological disks, Ω_+ and Ω_- forming the domain of discontinuity. Both $\Omega_+/\rho(\pi_1(S \times [0,1]))$ and $\Omega_-/\rho(\pi_1(S \times [0,1]))$ are Riemann surfaces homeomorphic to S, and so each represent points in the Teichmüller space of S.

Bers discovered that quasifuchsian Kleinian groups simultaneously uniformize the "top" boundary surface $S \times \{1\}$ and "bottom" boundary surface $S \times \{0\}$ of $S \times [0,1]$. Furthermore, any two hyperbolic structures on the boundary surfaces produce a corresponding quasifuchsian group uniformizing both boundary surfaces.

Theorem II.7. ***Bers' Simultaneous Uniformization Theorem*** *([5])*

For S a surface of genus $g \geq 2$

$$QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$$

where $\mathcal{T}(S)$ denotes the Teichmüller space of S and $QF(S)$ denotes the quasiconformal deformation space of S.

Later, the work of Marden [31] and Sullivan [49] showed that $QF(S)$ is actually the same as the interior of $AH(S \times [0,1])$. In this case, $\mathcal{A}(S \times [0,1])$ only consists of a single point, and we see that the interior of $AH(S \times [0,1])$ forms a ball parametrized by the conformal structures of the "top" and "bottom" boundary surfaces. This behavior is typical of the interior of $AH(M)$: components of the interior are distinguished by marked homeomorphism type where representations in the same component have quasiconformally conjugate actions on $\hat{\mathbb{C}}$. The following result generalizes the Simultaneous Uniformization Theorem, and gives a parametrization of the interior of $AH(M)$.

Theorem II.8. *(Ahlfors-Bers [2], Bers [6], Kra [29], Marden[31], Maskit [33], Sullivan [49], Thurston [52])*

$$int(AH(M)) \cong \bigcup_{(M',h)\in\mathcal{A}(M)} \mathcal{T}(\partial_{NT}M')/\text{Mod}_0(M')$$

where $\partial_{NT}N$ denotes the non-toroidal boundary components of M' and $\mathcal{T}(\partial_{NT}M')$ denotes the Teichmüller space of $\partial_{NT}M'$.

The results of Marden [31] and Sullivan [49] show that if ρ is a representation in $int(AH(M))$, then there exists a compact, atoroidal, irreducible 3-manifold M_ρ and an orientation-preserving homeomorphism $j_\rho : N_\rho \to int(M_\rho) \cup \partial_{NT}M_\rho$, yielding a well-defined marked homeomorphism type $[M_\rho, j_\rho \circ h_\rho]$ in $\mathcal{A}(M)$. Note that j_ρ is well-defined up to post-composition by elements of $\text{Mod}_0(M_\rho)$, and the boundary of N_ρ is a collection of Riemann surfaces, so we get a well-defined element of $\mathcal{T}(\partial_{NT}M_\rho)/\text{Mod}_0(M_\rho)$. The fact that this identification is injective comes from the fact that quasiconformal maps which are conformal on the limit set are globally conformal. The fact that this identification is surjective relies on the Measurable Riemann Mapping Theorem and Thurston's Geometrization Theorem.

2.3 The Boundary of $AH(M)$ and End Invariants

In the last section of this chapter we will survey 3 major results in the understanding of $AH(M)$: the Density Theorem, which shows that $int(AH(M))$ is dense in M, the Tameness Theorem, which shows that all representations ρ in $AH(M)$ give rise to N_ρ that are homeomorphic to the interior of compact 3-manifolds, and the Ending Lamination Theorem, which pins down the structure of hyperbolic 3-manifolds lying on the boundary of $AH(M)$. Together, these results provide a framework by which to understand the global topology of $AH(M)$, namely through sequences of representations in the interior.

In order to understand points on the boundary of $AH(M)$, we will need to discuss ends of 3-manifolds and end invariants. Let ρ be a point in $AH(M)$ and let $N = N_\rho$ be the associated quotient hyperbolic 3-manifold. We can understand the coarse geometry of N by its decomposition into a "compact core" and "ends". The following definition is due to Peter Scott [48]

Definition II.9. Let N be a 3-manifold with finitely generated fundamental group. If N' is a compact, connected 3-manifold in N whose inclusion is a homotopy equivalence, then N' is called a **compact core** of N.

Scott gave the following existence theorem for compact cores.

Theorem II.10. *(Scott [48]) If N is a 3-manifold with finitely generated fundamental group, then N has a compact core.*

Let K be a compact core of N. The components of $N \setminus K$ are called the *ends* of N. This is equivalent to the classical definition of ends, as shown by Bonahon [7]. For a hyperbolic 3-manifold N, it is useful the think of the compact core as carrying the topological information of N and the ends carrying the geometry. This is made explicit by analyzing the intersection of the ends of N with the "convex core" of N (not to be confused with the compact core).

Definition II.11. For a Kleinian group Γ, the **convex core** of N_Γ is $C(N_\Gamma) = Hull(\Lambda_\Gamma)/\Gamma$, where $Hull(\Lambda_\Gamma)$ is the convex hull of the limit set Λ_Γ. The convex core $C(N_\Gamma)$ is the smallest convex subset of N_Γ whose inclusion carries the fundamental group.

Definition II.12. A hyperbolic 3-manifold is **minimally parabolic** if the only maximal parabolic subgroups of $\pi_1(N)$ are rank 2 free-abelian.

The structure of the ends of N is dictated by their intersection with the convex core. If N is minimally parabolic then each end E of N falls into one of the following two categories:

- E is a rank 2 cusp. An end that is cut off by a torus will be a rank two cusp, and so will be entirely contained in the convex core.

- E is geometrically finite. In this case E has a neighborhood that does not intersect the convex core. Each geometrically finite end cut off by a higher genus surface intersects the convex core in a compact set. In this case, since E borders a boundary surface S of K, this neighborhood can be foliated by surfaces S_t isotopic to S where each S_t is convex and the size of S_t grows exponentially with its distance from K. The metrics on these surfaces (after rescaling) converge to hyperbolic metric on S, yielding a point in the Teichmüller space of S.

- E is geometrically infinite. In this case, E has a neighborhood completely contained in the convex core of N.

We see that points in the interior of $AH(M)$ contain only geometrically finite ends or rank 2 cusps. This property is equivalent to the following definition, as shown by Bowditch [9].

Definition II.13. A hyperbolic 3-manifold N with finitely generated fundamental group is called **geometrically finite** if the convex core of N has finite volume.

The interior points of $AH(M)$ are characterized by being geometrically finite and minimally parabolic. Marden [31] showed that the minimally parabolic, geometricallly finite elements of $AH(M)$ lie in the interior, and Sullivan [49] provided the reverse inclusion.

For geometrically finite ends of a hyperbolic 3-manifold N, the geometry of the end is well understood by the Teichmüller data of the boundary surface to that end.

Geometrically infinite ends are much more mysterious. Thurston discovered a way to "tame" the behavior of ends by considering the placement of closed geodesics within them. In the case where M has incompressible boundary, Thurston defined ends to be *simply degenerate* if there is a sequence of simple closed curves α_i on the associated boundary surface S whose geodesic representatives α_i^* in N are eventually contained in any neighborhood of E. In this case we say that the curves α_i^* *exit* N. Canary [20] generalized this notion to compressible boundary: an end E is simply degenerate if it has a neighborhood homeomorphic to $S \times [0, \infty)$ (where S is a compact surface) and there exists a sequence of "hyperbolic surfaces" $\{f_n : S \to U\}$ leaving every compact set for each n, with $f_n(S)$ homotopic to $S \times \{0\}$ in U. One can choose these "hyperbolic surfaces" to be pleated surfaces as in the work of Thurston [51], the simplicial hyperbolic surfaces of Bonahon [7], or images of harmonic maps of surfaces as in Minsky [36].

Thurston [51] and Bonahon [7] proved that when the boundary of the compact core of N is incompressible and without cusps, any sequence of α_i exiting a simply degenerate end E associated to the boundary surface S, converge to a unique unmeasured lamination v_E on S that is *filling*, that is, every nontrivial simple closed curve on S intersects v_E non-trivially. The lamination v_E is called an *ending lamination* of E. Additionally, Thurston showed that such ends E are *topologically tame*, i.e. E has a neighborhood homeomorphic to $S \times (0, \infty)$. There are thematically similar statements for when N has cusps, that are a little more difficult to state, see for example [37, Sec. 5].

Definition II.14. A representation ρ in $AH(M)$ is called **topologically tame** if $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of a compact manifold.

Bonahon [7] proved that if ∂M is incompressible and without cusps and N is a hyperbolic 3-manifold homeomorphic to the interior of M, then every end of N is either simply degenerate or geometrically finite, such ends are called *geometrically tame*. Canary [20] further showed that topological tameness implies geometric tameness. For geometrically tame ends, we have the following description of their "end invariants".

Definition II.15. The **end invariant** of an end facing the component S in ∂M is the associated geometric structure $v_S \in \mathcal{T}(S)$ if the end is geometrically finite, and the associated ending lamination if the end is simply degenerate.

The following important result, originally conjectured by Thurston, and later proved by Brock-Canary-Minsky [11] gives a complete classification of points in $AH(M)$.

Theorem II.16. ***The Ending Lamination Theorem*** *(Brock-Canary-Minsky [11])*

A hyperbolic 3-manifold with finitely generated fundamental group is uniquely determined by its marked homeomorphism type and its end invariants

In our description of ends, we never relied on the fact that N is homeomorphic to M. It is conceivable that some points in $AH(M)$ may not yield manifolds homeomorphic to the interior of any compact manifold.

Marden [31] showed that geometrically finite 3-manifolds are topologically tame and posed the following conjecture in 1974, later proved by Agol in 2004[1] and independently by Calegari-Gabai [16].

Theorem II.17. ***The Tameness Theorem*** *(Agol [1], Calegari-Gabai [16])*

A hyperbolic 3-manifold with finitely generated fundamental group is topologically tame.

Based on a similar conjecture of Bers, Thurston and Sullivan posed a density conjecture for $AH(M)$: that every finitely generated Kleinian group is an algebraic limit of minimally parabolic, geometrically finite groups. This result was proven by Brock and Bromberg [10] for freely indecomposable Kleinian groups without parabolics, and for all freely indecomposable Kleinian groups by Brock, Canary and Minsky ([11]). A full proof of the density conjecture was completed by Namazi and Souto [40] and Ohshika [43] using the Tameness theorem and the Ending Lamination Theorem.

Theorem II.18. ***The Density Theorem*** *(Brock-Bromberg [10], Brock-Canary-Minsky [11], Ohshika [43], Namazi-Souto [40])*

$int(AH(M))$ is dense in $AH(M)$. Therefore all representations in $AH(M)$ can be realized as the algebraic limits of sequences of minimally parabolic, geometrically finite representations.

We see that $AH(M)$ can be completely understood by studying the minimally parabolic geometrically finite representations, for which we already have a nice parametrization, and points on the boundary that can be thought of as algebraic limits of sequences of geometrically finite minimally parabolic representations. One might hope that the parametrization of the interior extends in some nice continuous way to the boundary. The presence of bumping of components of $int(AH(M))$, as we will see in the next chapter, prohibits this. In fact, we will see that the global topology of $AH(M)$ is quite poorly behaved.

CHAPTER III

Known Bumping Results

In the previous chapter, we saw that $int(AH(M))$ admits a nice parametrization, first into components distinguished by marked homeomorphism type, and second by analytic data on the boundary surfaces of M. One might hope that this parametrization of the interior extends to the boundary in some nice way. Anderson and Canary [3] (1996) showed this to not be the case by exhibiting "bumping" of components of $int(AH(M))$ when M is a book of I-bundles.

Definition III.1. Two components of $int(AH(M))$ are said to **bump** in $AH(M)$ if they have intersecting closures.

Later Anderson, Canary and McCullough [4] generalized this construction to describe bumping in the case when M has incompressible boundary. In this chapter, we will give the statements of known bumping results and we will provide some examples that illustrate their constructions. A key feature present in all of these constructions is the presence of primitive essential annuli.

Definition III.2. An embedded annulus A in M is called **essential** if it is properly embedded (i.e. the boundary of A is sent to the boundary of M), the embedding is π_1-injective, and A is not properly homotopic into the boundary of M. Such an annulus is called **primitive** if $\pi_1(A)$ maps onto a maximal abelian subgroup of

$\pi_1(M)$.

We end the chapter with a discussion of other, related results indicating the peculiar global topology of $AH(M)$, including self-bumping of components in $int(AH(M))$ and failure of local connectivity of $AH(S \times [0,1])$.

3.1 Bumping for books of I-bundles

Interestingly, Anderson and Canary [3] stumbled on their bumping construction after years of attempting to show that bumping didn't exist, as it is somewhat unintuitive that a hyperbolic 3-manifold in the boundary of a component of $int(AH(M))$ could have different marked homeomorphism type from manifolds in the interior.

The "books of I-bundles" used in Anderson and Canary's construction are obtained as follows: for $i = 1, .., n$, consider $J_i = S_i \times I$ where S_i is a surface with one boundary disk removed. Let $\{A_i\}_{1,..,n}$ be a collection of disjoint, consecutively ordered, parallel, longitudinal annuli in the boundary of a solid torus V. Define M_n by attaching the J_i to V by gluing ∂J_i along A_i. One may obtain a homotopy equivalent 3-manifold by attaching the J_i in a different order, that is, for every permutation τ in Σ_n, one obtains M_n^τ by gluing J_i to V by gluing ∂J_i along $A_{\tau(i)}$. We can obtain an explicit homotopy equivalence $h_\tau : M_n \to M_n^\tau$ by extending the identity map on $\{J_1, ..., J_n\}$.

Johannson's Deformation Theorem [27] shows that every element (M', h') is equivalent to (M_n^τ, h_τ) in $\mathcal{A}(M_n)$ for some permutation $\tau \in \Sigma_n$, i.e. that re-arranging the pages in the book of I-bundles gives rise to every marked homeomorphism type in $AH(M_n)$ [3]. In general, (M_n, id) and (M_n^τ, h_τ) are not equivalent in $\mathcal{A}(M_n)$, however if τ is any multiplication of the rotation $(12...n)$, then (M_n, id) and (M_n^τ, h_τ) are clearly equivalent. In fact, $\mathcal{A}(M_n)$ can be associated with the cosets of the subgroup generated by $(12...n)$ in Σ_n. As a result, $int(AH(M_n))$ has $(n-1)!$ components. Anderson and Canary gave the following surprising result.

Theorem III.3. *(Anderson-Canary [3]) For M_n as above, if $n \geq 3$ any pair of components of $int(AH(M_n))$ have intersecting closures.*

In light of The Density Theorem, the above result shows that $AH(M_n)$ is connected. The proof of this theorem is constructive: given $[(M_n, id)]$ and $[(M_n^\tau, h_\tau)]$ in $\mathcal{A}(M)$, one can produce a sequence in $int(AH(M))$ lying in the component corresponding to $[(M_n^\tau, h_\tau)]$ whose limit lies on the boundary of the component of $int(AH(M))$ corresponding to $[(M, id)]$. The following example (also found in [18]) illustrates this construction, and serves as a roadmap for later bumping constructions.

Example III.4. Define M_4 to be a book of I-bundles formed by attaching J_i to a solid torus V along longitudinal annuli, where each attached J_i is of genus $i = 1, 2, 3, 4$. Construct M_4 so that the cyclic order of the attached components is J_1, J_2, J_3, J_4. Let τ be the permutation $(2,3)$ and consider M_4^τ obtained by attaching the S_i in the cyclic order J_1, J_3, J_2, J_4. We can extend the identity map on J_1, J_2, J_3, J_4 to a homotopy equivalence $h_\tau : M_4 \to M_4^\tau$. It is clear the (M_4, id) and (M_4^τ, h_τ) are not equivalent in $\mathcal{A}(M_4)$: the boundary components of M_4 have genera 3, 5, 7 and 5 and the boundary components of M_4^τ have genera 4, 5, 6 and 5.

Let $\hat{M}_4^\tau$ be the manifold obtained by removing a small neighborhood of the core curve of V from M_4^τ. We construct an infinite cover $\tilde{M}_4^\tau$ of $\hat{M}_4^\tau$ from an infinite thickened annulus $S^1 \times [0,1] \times \mathbb{R}$ by attaching infinitely many copies of J_i to the outer boundary of $S \times \{0\} \times \mathbb{R}$ so that these copies occur in the cyclic order $...J_1, J_3, J_2, J_4, J_1, J_3, ...$ prescribed by τ. More concretely, one attaches a copy of J_i

to the thickened annulus by identifying, via an orientation-preserving homeomorphism, the boundary of J_i with $S^1 \times \{0\} \times [8n + 2\tau(i) - 2, 8n + 2\tau(i)]$ where n is an integer. Translation by 8 along the last component generates the group of covering transformations of $\tilde{M}_4^\tau$ over M_4^τ. Let $\pi : \tilde{M}_4^\tau \to \hat{M}_4^\tau$ be the covering map. We construct an orientation-preserving embedding $\tilde{f}_\tau : M_4 \to \tilde{M}_4^\tau$ which sends each copy J_i homeomorphically to a copy of J_i in the cover. Explicitly, each J_i is sent to a copy of J_i attached to $S^1 \times [0,1] \times [10i - 2\tau(i) - 1, 10i + 2\tau(i) + 1]$ in the cover. Let $f_\tau = \pi \circ \tilde{f}_\tau$.

The key technical result that drives this construction is the Hyperbolic Dehn Filling Theorem. If M is a hyperbolizable 3-manifold with a toroidal boundary component T, one can "fill" this boundary component by gluing in a solid torus. If (m, l) are a pair of meridian and longitude curves on T, then given a pair of relatively prime integers (p, q), we may define a new 3-manifold $M(p, q)$ by attaching a solid torus V to T so that a meridian c of V is sent to a (p, q) curve on T. In the case where M is hyperbolizable with a single toroidal boundary component, Thurston proved that $M(p, q)$ is hyperbolizable for all but finitely many choices of (p, q). Others (see Bonahon-Otal [8], Comar [21, Thm 12.5] generalized this result to the setting of geometrically finite hyperbolic 3-manifolds.

Let T denote the toroidal boundary component of $\hat{M}_4^\tau$, and choose a meridian curve m that bounds a disk in M_4^τ and a longitude curve l that bounds an essential annulus A in $\hat{M}_4^\tau$. The Dehn fillings $\hat{M}_4^\tau(1, n)$ of $\hat{M}_4^\tau$ are all homeomorphic to M_4^τ, indeed $\hat{M}_4^\tau(1, 0) = \hat{M}_4^\tau$ and $\hat{M}_4^\tau(1, n)$ is obtained from $\hat{M}_4^\tau(1, 0)$ by Dehn twisting n times around A. Let $i_n : \hat{M}_4^\tau \to \hat{M}_4^\tau(1, n)$ denote the inclusion map. We see that $(\hat{M}_4^\tau(1, n), i_n \circ f_\tau)$ is equivalent to (M_4^τ, h_τ) in $\mathcal{A}(M_4)$.

Let $N = \mathbb{H}^3/\Gamma$ be a geometrically finite hyperbolic 3-manifold admitting an orientation preserving homeomorphism $\psi : int(\hat{M}_4^\tau) \to N$. The Hyperbolic Dehn Filling Theorem provides a sequence of (non-faithful) representations $\beta_n : \Gamma \to PSL_2(\mathbb{C})$ corresponding to $\hat{M}_4^\tau(1,n)$. In order to get representations in $AH(M_4)$, we consider $\rho_n = \beta_n \circ \psi_* \circ (f_\tau)_*$. Let ρ denote the algebraic limit of the sequence (ρ_n) in $AH(M_4)$.

It is a consequence of the Hyperbolic Dehn Filling Theorem, together with the fact that $(\hat{M}_4^\tau(1,n), i_n \circ f_\tau)$ is equivalent to (M_4^τ, h_τ) in $\mathcal{A}(M_4)$, that ρ_n all lie in the component of $int(AH(M_4))$ corresponding to (M_4^τ, h_τ). Since f_τ lifts to an embedding, the Tameness Theorem and a result of McCullough-Miller-Swarup [34] (see Theorem V.7 in this thesis) show that N_ρ is homeomorphic to $int(M_4)$.

It is a consequence of a result due to Ohshika [42, Cor. 6] (stated later as Thm V.9) that ρ must lie in the closure of the component of $int(AH(M))$ corresponding to (M_4, id). Therefore, we see that the component of $int(AH(M_4))$ containing (M_4, id) and the component containing (M_4^τ, h_τ) have intersecting closures in $AH(M_4)$.

Holt [24] (2003) later showed that one can choose a single point in $AH(M_n)$ that lies in the boundary of every component of $int(AH(M_n))$. Holt further showed that the set of of such points contains a complex co-dimension 2 subvariety of the character variety.

3.2 The Incompressible Case

Anderson, Canary and McCullough [4] (2000) utilized the characteristic submanifold theory of Johansson to generalize their original bumping construction and totally characterize bumping in the case when M has incompressible boundary.

If M has nonempty incompressible boundary, separate works of Jaco-Shalen [26] and Johannson [27] showed the existence of a *characteristic submanifold* $\Sigma(M)$ of M, consisting of a disjoint collection of I-bundles and Seifert-fibered submanifolds whose frontiers are essential annuli. If M is hyperbolizable, then each Seifert-fibered component of $\Sigma(M)$ is homeomorphic to a solid torus or a thickened torus. A solid torus component V of $\Sigma(M)$ is called *primitive* if $\partial M \cap V$ consists of annuli whose inclusions into V are homotopy equivalences.

Given two irreducible 3-manifolds M_1 and M_2, with incompressible boundary, a *primitive shuffle* between M_1 and M_2 is a homotopy equivalence $h : M_1 \to M_2$ if there is a finite collection $\mathcal{V}_1$ of primitive solid torus components of $\Sigma(M_1)$ and a finite collection $\mathcal{V}_2$ of solid torus components of $\Sigma(M_2)$ such that $h^{-1}(\mathcal{V}_2) = \mathcal{V}_1$ and so that h is an orientation preserving homeomorphism from $\overline{M_1 \setminus \mathcal{V}_1}$ to $\overline{M_2 \setminus \mathcal{V}_2}$. Two elements (M_1, h_1) and (M_2, h_2) of $\mathcal{A}(M)$ are called *primitive shuffle equivalent* if there is a primitive shuffle $h : M_1 \to M_2$ such that (M_2, h_2) is equivalent to $(M_2, h \circ h_1)$ in $\mathcal{A}$.

Intuitively, these primitive shuffles are homotopy equivalences obtained by "shuffling" the way in which the manifold is glued around the solid torus components of its characteristic submanifold. Anderson-Canary-McCullough [4] showed that primitive shuffle equivalence gives an equivalence relation on $\mathcal{A}$ and that the quotient is finite to one. The following result of Anderson-Canary-McCullough provides the existence of primitive shuffles.

Theorem III.5. *(Anderson-Canary-McCullough [4]) Let M be a compact, hyperbolizable 3-manifold with non-empty incompressible boundary. Let ρ_n be a sequence in $AH(M)$ that converges to ρ. Then for sufficiently large n, the element of $\mathcal{A}(M)$ corresponding to ρ_n is primitive shuffle equivalent to the element corresponding to ρ.*

Anderson-Canary-McCullough then generalized the book of I-bundles bumping construction to the setting of incompressible boundary hyperbolizable 3-manifolds.

Theorem III.6. *(Anderson-Canary-McCullough [4]) Let M be a compact, hyperbolizable 3-manifold with non-empty incompressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. If $[(M_2, h_2)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$, then the associated components of $int(AH(M))$ have intersecting closures.*

Combining these two results, we see that components in $int(AH(M))$ bump if and only if their corresponding marked homeomorphism types differ by a primitive shuffle.

Corollary III.7. *(Anderson-Canary-McCullough [19]) Let M be a compact, hyperbolizable 3-manifold with nonempty incompressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be two elements of $\mathcal{A}(M)$. The associated components of $int(AH(M))$ have intersecting closures if and only if $[(M_2, h_2)]$ is primitive shuffle equivalent to $[(M_1, h_1)]$.*

We see that the components of $AH(M)$ are enumerated by primitive shuffle equivalence classes in $\mathcal{A}(M)$. The work of Canary and McCullough [19] gives a complete enumeration of $\mathcal{A}(M)$ when M has incompressible boundary. This space is typically finite, except in the case when M has "double trouble".

Definition III.8. A 3-manifold M has **double trouble** if there exist simple closed curves α and β in $\partial_{NT} M$ which are both homotopic to a curve γ in a toroidal boundary component of M but are not homotopic in ∂M.

Theorem III.9. *(Canary-McCullough[19]) If M has incompressible boundary, then $int(AH(M))$ is homeomorphic to a collection of disjoint balls. This collection is infinite if and only if M has "double trouble".*

3.3 Known Examples with Compressible Boundary

When M has compressible boundary, Canary and McCullough showed that the interior of $AH(M)$ contains infinitely many components outside of a few specific examples they referred to as "small". The following is a summary of Theorem 6.2.1 in [19].

Theorem III.10. *(Canary-McCullough [19]) The interior of $AH(M)$ has finitely many components if and only if M has incompressible boundary (and no double trouble), M is a compression body, or M is obtained from one or two I-bundles (over closed surfaces) by adding a 1-handle. These examples with compressible boundary are called **small**.*

Canary-McCullough give explicit enumerations of the components of $int(AH(M))$ for these small examples. Of particular interest to our own construction, they showed that when M is obtained by attaching a one-handle to $S \times [0, 1]$ where S is a closed surface of genus $g \geq 2$, $int(AH(M))$ contains only two components. This example motivates our own construction in Chapter V, where we will see that these two components bump in $AH(M)$. As a corollary, we see that $AH(M)$ is connected in this case.

3.4 Self-Bumping and Other Strange Behavior

Soon after the emergence of these bumping construction, other constructions exhibiting the peculiar nature of the global topology of $AH(M)$ began to appear.

Definition III.11. A component B of $int(AH(M))$ **self-bumps** if there is a representation ρ in the closure of B such that for any sufficiently small neighborhood X of ρ in $AH(M)$, the set $X \cap B$ is disconnected.

McMullen [35] (1998) exhibited self-bumping in the only component of $int(AH(S\times [0,1]))$.

Theorem III.12. *(McMullen [35]) If S is a closed surface, then $int(AH(S\times[0,1]))$ self-bumps.*

In particular, this shows that $AH(S\times[0,1])$ is not a manifold. McMullen's proof utilized the theory of projective structures on surfaces, and so his techniques did not immediately generalize to manifolds that are not I-bundles. Bromberg and Holt generalized McMullen's result in 2001 to give the following characterization of self bumping:

Theorem III.13. *(Bromberg-Holt [15]) Let M be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that M contains an essential, primitive annulus, then every component of $int(AH(M))$ self-bumps.*

As a consequence, we see that $AH(M)$ is not a manifold if M contains a primitive essential annulus. Of note, the above result applies when M has compressible boundary. As in Anderson and Canary's bumping example, Bromberg and Holt prove the existence of self-bumping representations by constructing them explicitly. The following example (also found in [18]) illustrates how one can augment the construction in Example III.1 to find self-bumping representations.

Example III.14. Let M_2 be the book of I-bundles obtained by attaching J_1 and J_2 of genus 1 and 2 respectively to a solid torus V along longitudinal annuli. Observe that $M_2 = S_3\times[0,1]$ where S_3 is a closed surface of genus 3. As in Example III.1, let $\hat{M}_2$ be the 3-manifold obtained by removing an open neighborhood of the core curve of V from M_2. Let $\tilde{M}_2$ be the infinite cyclic cover of $\hat{M}_2$ obtained by gluing copies of J_1 and J_2 to $S^1\times[0,1]\times R$. Index these copies by the cyclic order in which they appear in the cover, e.g. ..., $J_1^{-1}, J_2^{-1}, J_1^0, J_2^0, J_1^1, J_2^1, ...$, and construct an embedding $\tilde{f} : M_2 \to \tilde{M}_2$ sending J_1 to J_1^0 and J_2 to J_2^1 (so that there are copies of J_2 and J_1 lying between their images in the cover). This embedding serves as the analogue of $\tilde{f}_\tau$ in Examples III.4.

We now proceed as in Examples III.4 and use the Hyperbolic Dehn Filling Theorem to produce a sequence $\{\rho_n\}$ in $int(AH(M_2)) = QF(S_3)$ that converges to a self-bumping point ρ. The fact that ρ is a self-bumping point is not obvious, see Bromberg-Holt [15] or McMullen [35].

Self-bumping leads one to believe that the topology of $AH(M)$ may be quite bad. The following result, due to Bromberg for the space of Kleinian punctured torus groups, and to Magid for higher genus surfaces, confirms this:

Theorem III.15. *(Bromberg [14] Magid [30]) The space $AH(S \times [0,1])$ is not locally connected where S is a closed surface of genus $g \geq 2$.*

Results of Evans-Holt [22] show that the set of self-bumping points, and hence the set of points on the boundary where the deformation space is not locally connected is not dense for the space of Kleinian punctured torus groups. Ito [25] has given a complete description of the self-bumping points in the punctured torus case. Ohshika has been able to generalize many of these results to the quasifuchsian case. In addition, Ohshika [41] and Bromberg-Brock-Canary-Minsky [12] are able to demonstrate many points in the boundary of quasifuchsian space where there is no self bumping.

Bromberg conjectures that $AH(M)$ is not locally connected as long as M has a non-toroidal boundary component. This conjecture remains open.

CHAPTER IV

Primitive Handle Shuffles

Let M be a connected, orientable, hyperbolizable, compact 3-manifold obtained by attaching a one-handle to the boundary of M_0, a compact, orientable, hyperbolizable 3-manifold. Our bumping construction stems from the following simple idea: by attaching a one-handle along different boundary components of M_0, we can obtain hyperbolizable 3-manifolds homotopy equivalent to M, but not homeomorphic to M.

For the purpose of our construction we fix one attaching disk D_0 in the boundary of M_0. The other attaching disk, D_1 or D_2 lies in a boundary component of M_0 that lies in a regular neighborhood of a primitive essential annulus A (recall the importance of primitive essential annuli in previous bumping constructions, see Chapter III). Let M be the 3-manifold resulting from attaching a one-handle to D_0 and D_1 and let M' be the 3-manifold obtained from attaching one-handle to D_0 and D_2.

In this chapter, we introduce the concept of a "primitive handle shuffle associated to A", establishing a homotopy equivalence $h_A : M \to M'$, allowing us to relate the two points (M, id) and (M', h_A) in $\mathcal{A}(M)$. More generally, we will establish the notion that two points (M_1, h_1), (M_2, h_2) in $\mathcal{A}(M)$ differ by a primitive handle shuffle associated to A. In Chapter 5, we will see that the components of $int(AH(M))$ associated to points in $\mathcal{A}$ differing by primitive handle shuffles have intersecting closures in $AH(M)$.

4.1 An Illustrative Example

Throughout this chapter and the next, we will return again and again to an illustrative example that served as an inspiration for our final result.

Example IV.1. Consider $M_0 = S \times [0, 1]$ where S is a surface of genus $g \geq 2$. M_0 is clearly compact and hyperbolizable, it is uniformized by a quasifuchsian group. We wish to attach a one-handle to the boundary of M_0. Topologically, there are two choices for the locations of D_0 and D_1 up to homeomorphism: they can either be contained in the same boundary component of M_0 or they can each be contained in one of the two different boundary components. Attaching one-handles in these two configurations clearly yields two 3-manifolds with different homeomorphism type. In the first case, which we'll refer to as M, the resulting 3-manifold has two boundary components, one of genus g and the other of genus $g + 1$. In the second case, M', the resulting 3-manifold has only one boundary component of genus $2g$.

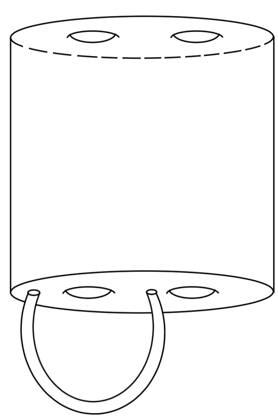


Figure 4.1: A schematic of the 3-manifold M

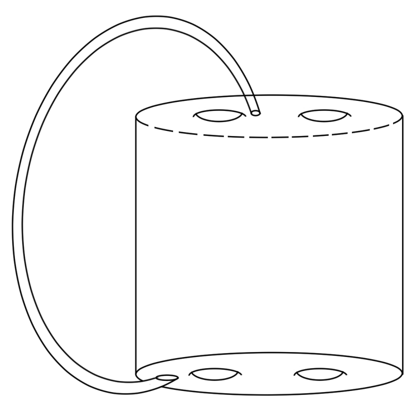


Figure 4.2: A similar schematic for M'

In fact, these two homeomorphism types enumerate $\mathcal{A}(M)$. The manifold M is an example of the "small" compressible boundary 3-manifolds discussed in Canary-McCullough [19] (see Theorem III.7 in the previous chapter). Our bumping construction will show bumping between the two associated components of the interior of $AH(M)$.

4.2 Attaching a One-Handle

Let M be a compact hyperbolizable 3-manifold obtained from M_0, a compact, orientable hyperbolizable 3-manifold containing a primitive essential annulus A such that a regular neighborhood W of A intersects the boundary of M_0 in two annuli V_1 and V_2. Let D_1 and D_2 be disks contained in V_1 and V_2 respectively, and let D_0 be a disk in the boundary of M_0 not contained in V_1 or V_2. Suppose that M is obtained from M_0 by attaching a one-handle along D_0 and D_1.

By "shuffling" one end of the attached one-handle over the annulus A, we construct a new 3-manifold. Let M' be the 3-manifold obtained by attaching a one-handle to M_0 along D_0 and D_2. If V_1 and V_2 lie in the same boundary component of M_0, then M and M' are clearly homeomorphic. So for the purpose of our bumping construction, we will assume that V_1 and V_2 lie in distinct components of the boundary of M_0. If V_1 and V_2 lie in distinct boundary components of M_0, then M and M' may have different homeomorphism type (the boundary surfaces of M and M' may have different genera, as in Example IV.1).

Example IV.2. We return to Example IV.1 where $M_0 = S \times [0,1]$. We obtain a primitive, essential annulus A in M_0 by taking nontrivial curve c in S and considering $c \times [0,1]$. Let W be small neighborhood of A, and let D_1 be contained in the intersection of W and $S \times \{0\}$. Let D_2 similarly be be contained in the intersection of W and $S \times \{1\}$. Let D_0 be a disk in $S \times \{0\}$ that does not intersect W. Let M

be the 3-manifold obtained by attaching a one-handle to disks D_0 and D_1, and let M' be obtained by attaching a one handle along the disks D_0 and D_2.

4.3 Primitive Handle Shuffles

We need to pin down the marked homeomorphism types we would like to compare for M and M'. For M, we can clearly take the point (M, id) where $id : M \to M$ is just the identity map. For M', we will construct a homotopy equivalence $h_A : M \to M'$ which effectively moves the attaching disk of the one-handle from D_1 to D_2. This "handle shuffle" will be used to relate the marked homeomorphism types (M, id) and (M', h_A) in $\mathcal{A}(M)$.

Recall that W, a regular neighborhood of the incompressible annulus A inside of M_0, intersects the boundary of M_0 in two annuli containing the two potential attaching disks D_1 and D_2. Therefore, W contains a solid cylinder C containing neighborhoods of D_1 and D_2 in its boundary. We can parametrize C so that $C = \mathbb{D}^2 \times [0,1]$ where $\mathbb{D}^2$ is the unit disk and with $D_1 = \{(x,1)||x| < 1/2\}$, and similarly $D_2 = \{(x,0)||x| < 1/2\}$. Define $h_A : M \to M'$ as follows: for $y \in M \setminus C$, $h_A(y) = y$ is the identity and for $(x,t) \in C$, define

$$h_A(x,t) = \begin{cases} (x,0) & \text{if } |x| < 1/2 \\ (x, t(2|x|-1) & \text{if } |x| \geq 1/2 \end{cases}$$

Intuitively, h_A corresponds to "pushing" D_1 down into D_2 where the deformation takes place entirely within W. We wish to show that h_A is a homotopy equivalence. By reversing the roles of D_1 and D_2 above, we get a candidate homotopy inverse, j_A, for h_A on C:

$$j_A(x,t) = \begin{cases} (x,1) & \text{if } |x| < 1/2 \\ (x, 2-2|x| + t(1-(2-2|x|))) & \text{if } |x| \geq 1/2 \end{cases}$$

The following figure shows the subsequent image of C under h_A and j_A.

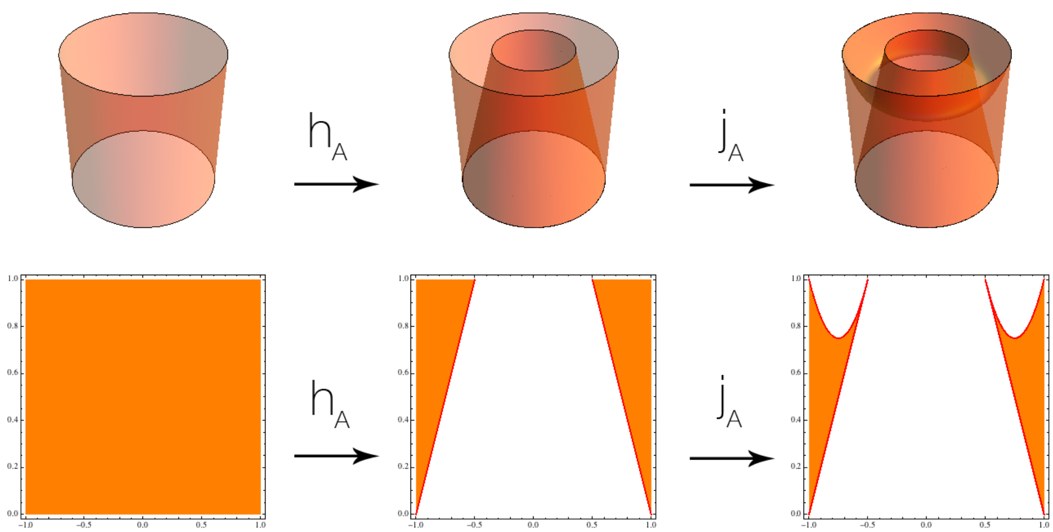


Figure 4.3: The image of C and a vertical cross section of C under h_A and then j_A

Note the image of C under h_A is the union of D_2 and the following subset of C, $\{(x,t)|0 \leq t \leq 2|x|-1\}$. The image of D_1 under j_A is D_2 and the image of $\{(x,t)|0 \leq t \leq 2|x|-1\}$ under j_A is $\{(x,t)2-2|x| \leq t \leq 3-6|x|+4|x|^2\}$, a contractible subset of C.

Since $j_A \circ h_A$ is the identity off of C and $j_A \circ h_A$ is a homotopy equivalence on C that fixes the boundary of C, we see that h_A is a homotopy equivalence with homotopy inverse j_A. The homotopy equivalence h_A is an example of a *primitive handle shuffle associated to A*. This motivates the following definition:

Definition IV.3. Suppose that M_1 and M_2 are compact, orientable, hyperbolizable 3-manifolds obtained from M_0 by attaching a one handle with one attaching disk at D_1, respectively D_2. Suppose that M_0 contains a primitive essential annulus A, of which a regular neighborhood W intersects the boundary of M_0 in annuli V_1 and V_2 containing D_1 and D_2 respectively. A **primitive handle shuffle associated to a primitive essential annulus A in M_1 and M_2** is a homotopy equivalence $h_A : M_1 \to M_2$ such that $h_A^{-1}(W) = W$ and h_A restricts to an orientation-preserving homeomorphism from $\overline{M_1 \setminus W} \to \overline{M_2 \setminus W}$ where W is a neighborhood of A. We say that points (M_1, h_1) and (M_2, h_2) in $\mathcal{A}(M)$ **differ by a primitive handle shuffle associated to A** if there is a primitive handle shuffle $h_A : M_1 \to M_2$ associated to A such that the following diagram commutes up to homotopy,

$$\begin{array}{ccc} & & M_1 \\ & \overset{h_1}{\nearrow} & \\ M & & \Big\downarrow h_A \cong \\ & \underset{h_2}{\searrow} & \\ & & M_2 \end{array}$$

that is $h_A \circ h_1$ is homotopic to h_2.

CHAPTER V

Bumping with Primitive Handle Shuffles

We now have the needed terminology to state our main theorem.

Theorem V.1. *Let M be a compact, orientable hyperbolizable 3-manifold with compressible boundary, and let $[(M_1, h_1)]$ and $[(M_2, h_2)]$ be equivalence classes in $\mathcal{A}(M)$ that differ by a primitive handle shuffle h_A associated to a primitive essential annulus A in M_1. Then the components of $int(AH(M))$ corresponding to $[(M_1, h_1)]$ and $[(M_2, h_2)]$ have intersecting closures in $AH(M)$.*

We may immediately reduce to the case where (M_1, h_1) is (M, id) and (M_2, h_2) is (M', h_A) where h_A is the primitive handle shuffle corresponding to A. Let $M_1 = M$, we may assume this since $AH(M_1) = AH(M)$ by definition (in fact, $AH(M)$ depends only on $\pi_1(M)$ and the choice of base point is just for convention).

Let j_1 be a homotopy inverse of h_1, then observe that by precomposing with j_1, we see that $[(M_1, h_1 \circ j_1)] = [(M_1, id)]$. Let $M' = M_2$ and observe that $h_2 \circ j_1$ is homotopic to h_A.The components of $int(AH(M))$ corresponding to $[(M, id)]$ and $[(M', h_A)]$ are sent to the components corresponding to $[(M_1, h_1)]$ and $[(M_2, h_2)]$ by composition with h_1. As a result, we see that if the components of $int(AH(M)$ corresponding to $[(M, id)]$ and $[(M', h_A)]$ bump in $AH(M)$, then so do the components corresponding to $[(M_1, h_1)]$ and $[(M_2, h_2)]$.

The remainder of this chapter will be spent on the proof of this theorem. The proof can be broken into the following broad steps:

1. Take the primitive essential annulus A in M' associated to the handle shuffle and drill out the core curve to obtain a new hyperbolizable 3-manifold, $\hat{M}'$.

2. Use the Hyperbolic Dehn Filling Theorem to produce $(1, n)$ Dehn fillings of $\hat{M}$ called $M'_n = \hat{M}'(1, n)$ with inclusions $i_n : \hat{M}' \to M'_n$, a sequence of discrete (nonfaithful) representations $\beta_n : \pi_1(\hat{M}) \to PSL_2(\mathbb{C})$, and homeomorphisms $\psi_n : N'_n \to \mathbb{H}^3/\beta(\pi_1(\hat{M}'), \psi : \hat{M}' \to \hat{M}'$ such that $\psi_{n*} \circ i_{n*}$ is conjugate to $\beta_n \circ \psi_*$ and β_n converge to the identity.

3. Construct a cover $\tilde{M}'$ of $\hat{M}'$ obtained by unwinding the core curve of A. Let $\pi : \tilde{M}' \to \hat{M}'$ be the covering map. Construct an embedding $\tilde{f} : M \to \tilde{M}'$ identifying M as a compact core of $\tilde{M}'$ and let $f = \pi \circ \tilde{f}$. We define $\rho_n = \beta_n \circ \psi_* \circ f_*$ to be a sequence of discrete faithful representations in $AH(M)$ with limit $\rho = \psi_* \circ f_*$. We construct f in such a way that the marked homeomorphism type of $[(\mathbb{H}^3/\rho_n(\pi_1(M)), h_{\rho_n})]$ coincides with $[(M', h_A)]$ for all n and the marked homeomorphism type of $[(\mathbb{H}^3/\rho(\pi_1(M)), h_\rho)]$ coincides with $[(M, id)]$.

4. Invoke a theorem of Ohshika to show that ρ lies in the closure of a component of $int(AH(M))$ with corresponding marked homeomorphism type $[(M, id)]$. This shows that the components of $int(AH(M))$ corresponding to $[(M, id)]$ and $[(M', h_A)]$ bump in $AH(M)$.

5.1 Dehn Filling

In order to show bumping, we will need to produce a sequence of representations in the interior of $AH(M)$ that lie in the component corresponding to $[(M', h_A)]$, but whose limit approaches the component corresponding to $[(M, id)]$. As in the constructions used by Anderson, Canary and McCullough that provided bumping criteria for incompressible boundary hyperbolic 3-manifolds and the similar wrapping constructions that Bromberg and Holt used to demonstrate self-bumping, we will use hyperbolic Dehn filling to generate this sequence.

Let M be a compact 3-manifold with toroidal boundary component T. By attaching a solid torus W to M along T, we obtain a new 3-manifold. Specifically, select a meridian m and longitude l on T and a meridian α on the boundary of W. Given a pair of relatively prime integers (p, q), define the (p, q)-Dehn filling of M to be $M(p, q)$ the 3-manifold obtained by attaching W to M along T with attaching map $g : \partial W \to T$ so that $g(\alpha)$ is a (p, q) curve on T. The homeomorphism type of the result depends only on the pair (p, q).

In the case that M is hyperbolizable and $\partial M = W$, Thurston [51] proved that the Dehn fillings $M(p, q)$ are hyperbolizable except for finitely many choices of (p, q). Dehn filling will be our primary tool in constructing sequences of representations in $AH(M)$. The specific formulation of the Dehn filling Theorem we use can be found in Comar [21, Thm. 12.5].

Theorem V.2. ***Hyperbolic Dehn Filling Theorem***

Let M be a compact, hyperbolizable 3-manifold and let T be a toroidal boundary component of M. Let $N = \mathbb{H}^3/\Gamma$ be the geometrically finite hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : int(M) \to N$. Assume that every parabolic element in Γ is contained in a rank-two parabolic subgroup. Let $\{(p_n, q_n)\}$ be a sequence of distinct pairs of relatively prime integers. Then for sufficiently large n, there exist (non-faithful) discrete representations $\beta_n : \Gamma \to PSL_2(\mathbb{C})$ such that:

1. *$\beta_n(\Gamma)$ is geometrically finite and every parabolic element lies in a rank-two subgroup.*

2. *The sequence $\{\beta_n\}$ converges to the identity representation on Γ.*

3. *Let $i_n : M \to M(p_n, q_n)$ be the inclusion map. Then for each n, there is an orientation-preserving homeomorphism $\psi_n : int(M(p_n, q_n)) \to \mathbb{H}^3/\beta_n(\Gamma)$ such that $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$.*

5.2 Drilling and Filling

Consider the 3-manifold $\hat{M}'$ obtained by drilling a regular neighborhood, V, of c, the core curve of the essential primitive annulus A, from M'. The following theorem combines results of Comar [21] to show that $\hat{M}'$ is hyperbolizable.

Theorem V.3. *(Comar [21, Thm 13.3, 13.6]) Let M be a compact hyperbolizable 3-manifold. Let δ be a primitive, homotopically nontrivial, π_1-injective curve in M parallel into the boundary of M. Let V be a regular neighborhood of δ in M. Let $\hat{M} = M \setminus V$. Then $\hat{M}$ is hyperbolizable.*

We wish to construct Dehn fillings of $\hat{M}'$ in order to construct a sequence in $AH(M)$. Let Y denote a collar neighborhood of the boundary component of M' containing the attached one-handle. We can choose meridian and longitude curves (m, l) on the boundary of V so that l is isotopic to c in S, a component of ∂Y, and bounds an essential annulus in $\hat{M}'$, and m a simple closed curve in ∂V so that m bounds a disk D in V with $\partial D \cap \partial V = m$. Define $M'_n = \hat{M}'(1, n)$ to be the $(1, n)$-Dehn filling of $\hat{M}$ with respect to this meridian-longitude pair. The following result, due to Comar [21], allows us to control the homeomorphism type of the Dehn fillings.

Theorem V.4. *(Comar [21, Thm. 6.2]) Let M be a compact, connected 3-manifold such that $M = X \cup_{id} Y$ where X and Y are 3-submanifolds of M, each of which contains a boundary component S, which is a closed, orientable surface of genus greater than one and $id : S \to S$ is the identity map. Let δ be a simple closed curve on S which is homotopically nontrivial in $\pi_1(M)$. Let V be a regular solid torus neighborhood of δ. Let m, l be a meridian-longitude pair of curves on the boundary of V so that l is isotopic to δ in S and bounds an essential annulus in M, and so that m is a simple closed curve in ∂V so that m bounds a disk D in V with $\partial D \cap \partial V = m$ Then the manifold $M_h = X \cup_h Y$ where $h : S \to S$ is a n-fold Dehn twist along δ, is homeomorphic to the manifold $M(1, n)$ obtained from M by $(1, n)$ Dehn surgery along δ for all n. Furthermore if Y is homeomorphic to $S \times [0, 1]$, then $M(1, n)$ is homeomorphic to M for all n.*

The above theorem guarantees that M'_n is homeomorphic to M' for all n.

5.3 Constructing a Cover

We construct a cover $\tilde{M}'$ of $\hat{M}'$, and an embedding $\tilde{f} : M \to \tilde{M}'$ that identifies $\tilde{f}(M)$ as a compact core of $\tilde{M}$. In contrast with previous bumping constructions (see Example III.4), the cover we construct will be an irregular cover of $\hat{M}$.

We now define $\tilde{M}'$. Recall that M' contains a primitive essential annulus A, of which a regular neighborhood W intersects the boundary of M_0 in annuli V_1 and V_2 containing the candidate attaching disks D_1 and D_2. Let c be the core curve of A and let V be a regular neighborhood of c in M'. The outside boundary component of $W \setminus V$ decomposes into four distinct pieces, V_1, V_2 and two disjoint pieces bordering the interior of M_0. For simplicity, we can refer to these boundary pieces as ∂W_l and ∂W_r (the "left" and "right" pieces of the boundary of W).

Consider the infinite cyclic cover of $W \setminus V$ obtained by unwinding c. This infinite cyclic cover of $W \setminus V$ cyclically repeats copies of V_1, W_l, V_2, W_r along one of its boundary components. Explicitly, consider $S^1 \times [0,1] \times \mathbb{R}$. we identify the segments of the outer boundary component $S^1 \times \{0\} \times [n, n+1]$ with copies of the annuli V_1, W_l, V_2, and W_r in the following cyclic order.

$$..., W_r^{-1}, V_1^0, W_l^0, V_2^0, W_r^0, V_1^1, W_l^1, V_2^1, W_r^1, V_1^2, ...$$

We want M to embed in the cover we are building. The boundary components of $M \setminus W$ include W_l, W_r and the attaching disk (that was attached to D_1 in V_1). Attach $M \setminus W$ to the cyclic cover of $W \setminus V$ above by identifying W_l, D_1 and W_r along W_l^1, D_2^0 in V_2^0 and W_r^{-1} respectively via a lift of the restriction of h_A to $M \setminus W$. The important aspect of this identification is that these attached components of $M \setminus W$ appear in the same cyclic order around the boundary of W as in our cover.

For each other copy V_2^n of V_2 in $S^1 \times \{0\} \times \mathbb{R}$, consider the cover $\tilde{M}'_{V_2}$ of $\hat{M}'$ associated to $\pi_1(V_2)$ and choose a lift of V_2 in this cover. This lift borders a lift of the one-handle and a lift of the interior of $M_0 \setminus W$. Cut $\tilde{M}'_{V_2}$ along this lift of V_2 and identify V_2^n with the boundary of the one-handle-adjacent portion.

Likewise, for each other copy W_l^n and W_r^n of W_l and W_r, consider the cover of $\hat{M}$ associated to $\pi_1(W_l)$ (respectively $\pi_1(W_r)$), choose a lift of W_l (W_r), cut along this lift, and glue the portion of the cover corresponding to the adjacent lift of the interior of $M' \setminus W$ via the lift of the restriction of h_A. Call the final result of these gluings $\tilde{M}'$.

We see that $\tilde{M}'$ forms an irregular cover of $\hat{M}'$. Let $\pi : \tilde{M}' \to \hat{M}'$ be the covering map. We define $\tilde{f} : M \to \tilde{M}'$ in the following way. Decompose M into W and $M \setminus W$. Note that $M \setminus W$ is topologically equivalent to $M' \setminus W$. Define $\tilde{f}$ to map $M \setminus W$ homeomorphically along the embedded copy of $M' \setminus W$ attached to W_l^1, W_r^{-1}

and V_2^0. Embed W to the portion of the cyclic cover of $W \setminus V$ stretching between W_r^{-1} and W_l^1. Note that $\tilde{f}$ is an orientation-preserving embedding of M into $\tilde{M}'$. For convenience with later notation, let $f = \pi \circ \tilde{f} : M \to \hat{M}'$ be the composition down to $\hat{M}'$.

Proposition V.5. *The space $\tilde{M}'$ deformation retracts onto $\tilde{f}(M')$ and so $\tilde{f}(M')$ forms a compact core of $\tilde{M}'$.*

The proof of this fact follows from our construction. Indeed, for each segment V_2^n that is not contained in the image of $\tilde{f}$, we see that the attached portion of $\tilde{M}'_{V_2}$ deformation retracts onto V_2^n. Similarly, for W_l^n, we see that the attached portions of $\tilde{M}'_{W_l}$ deformation retract onto W_l^n (and also for W_r^n). We see that $\tilde{M}$ deformation retracts onto $(S^1 \times [0,1] \times \mathbb{R}) \cup \tilde{f}(M')$. This space clearly deformation retracts onto $\tilde{f}(M')$ by collapsing $S^1 \times [0,1] \times \mathbb{R}$ onto $S^1 \times [0,1] \times \mathbb{R} \cap \tilde{f}(M')$.

5.4 Proof of the main theorem

We will now give a proof of Theorem V.1, our main result. Suppose that $[(M, id)]$ and $[(M', h_A)]$ are two points in $\mathcal{A}(M)$ that are related by a primitive handle shuffle associated to A.

Let $\hat{M}'$ be the hyperbolizable 3-manifold obtained from M' by removing a regular neighborhood V of the core curve c of A. Let $\hat{N}' = \mathbb{H}^3/\Gamma$ be the geometrically finite hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : int(\hat{M}') \to \hat{N}'$ guaranteed by Theorem V.3. In the context of the Hyperbolic Dehn Filling Theorem, choose (p_n, q_n) to be $(1, n)$ and let $M'_n = \hat{M}'(1, n)$ be the filled manifolds. Let $i_n : \hat{M}' \to M'_n$ be the inclusion maps, let $\beta_n : \pi_1(\hat{M}') \to PSL_2(\mathbb{C})$ be the sequence of (nonfaithful) discrete representations guaranteed by the Hyperbolic Dehn Filling Theorem, and let $\psi_n : int(M'_n) \to \mathbb{H}^3/\beta(\pi_1(\hat{M}'))$, and $\psi : \hat{M}' \to \hat{M}'$ be as in the Hyperbolic Dehn Filling Theorem.

We will construct a sequence of geometrically finite, minimally parabolic representations ρ_n that lie in the component of $int(AH(M))$ corresponding to $[(M', h_A)]$, but whose algebraic limit lies on the boundary of the component of $int(AH(M))$ corresponding to $[(M, id)]$. The following commutative diagram may be useful in following our argument.

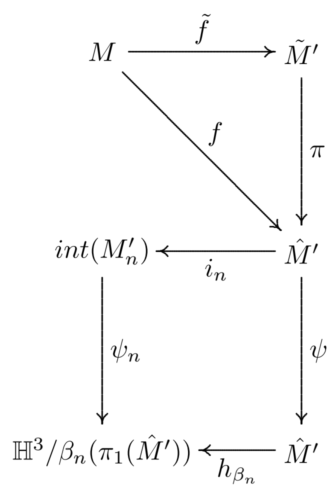


Figure 5.1: The topological picture

Define $\rho_n = \beta_n \circ \psi_* \circ f_*$. Algebraically, ρ_n converges to $\rho = \psi_* \circ f_*$ since β_n converges to the identity. The Hyperbolic Dehn Filling Theorem gives that $\beta_n \circ \psi_*$ is conjugate to $(\psi_n)_* \circ (i_n)_*$. We see that ρ_n is conjugate to $(\psi_n)_* \circ (i_n)_* \circ f_*$.

By construction, the map $\pi|_{\tilde{f}(M \setminus W)} : \tilde{f}(M \setminus W) \to \hat{M}'$ is a homeomorphism onto its image. Moreover, i_n is a homeomorphism outside a neighborhood of V. We appeal to the following technical lemma.

Lemma V.6. *(Anderson-Canary-McCullough [4, Lemma 5.2]) Let X, Y be compact, orientable, irreducible 3-manifolds, and let $\mathcal{V}_X$, (respectively $\mathcal{V}_Y$) be a codimension-zero submanifold of X (resp. Y), whose frontier $Fr(\mathcal{V}_X)$ is nonempty and incompressible. Let $h : X \to Y$ be a map such that:*

1. $h^{-1}(\mathcal{V}_Y) = \mathcal{V}_X$ *and* $h|_{\overline{X \setminus \mathcal{V}_X}} : \overline{X \setminus \mathcal{V}_X} \to \overline{Y \setminus \mathcal{V}_Y}$ *is a homeomorphism, and*

2. $h|_{\mathcal{V}_X} : \mathcal{V}_X \to \mathcal{V}_Y$ *is a homotopy equivalence.*

Then, h *is a homotopy equivalence, and there exists a homotopy inverse* $\bar{h} : Y \to X$ *for* h *so that* $\bar{h}^{-1}(\mathcal{V}_X) = \mathcal{V}_Y$*, so that* $\bar{h}|_{\overline{Y \setminus \mathcal{V}_Y}}$ *is the inverse of* $h|_{\overline{X \setminus \mathcal{V}_X}}$*, and so that* $\bar{h}_{\mathcal{V}_2} : \mathcal{V}_Y \to \mathcal{V}_X$ *is a homotopy equivalence. Moreover,* $\bar{h} \circ h$ *is homotopic to the identity relative to* $\overline{X \setminus \mathcal{V}_X}$ *and* $h \circ \bar{h}$ *is homotopic to the identity relative to* $\overline{Y \setminus \mathcal{V}_Y}$

In the above lemma, take $X = M$ and $Y = M'(1, n)$, and take $\mathcal{V}_X = \mathcal{V}_Y = W$ and consider $h = i_n \circ f$. By construction, $i_n \circ f$ acts by a lift of h_A on $M' \setminus W$ verifying condition 1. For condition 2, note that $i_n^{-1}(W)$ lies inside of $W \cap \hat{M}'$ and that the preimage $f^{-1}(W \cap \hat{M}')$ lies inside W in M by construction. By the above lemma, we see that $i_n \circ f$ is a homotopy equivalence.

Since $i_n \circ f$ is a homotopy equivalence of M', and ψ_n is a homeomorphism, ρ_n are discrete faithful representations with image $\pi_1(\mathbb{H}^3/\beta_n(\pi_1(\hat{M}')))$, so we see that ρ_n are indeed elements of $int(AH(M))$. Theorem V.4 shows that $M'_n = \hat{M}'(1, n)$ is homeomorphic to M'. However, since f lifts to an embedding of M', the Tameness Theorem and the following result of McCullough-Miller-Swarup gives that N_ρ is homeomorphic to $int(M)$.

Theorem V.7. *(McCullough-Miller-Swarup [34]) Let* N *be an orientable, irreducible 3-manifold with finitely generated fundamental group. If* $i_1 : N_1 \to N$ *and* $i_2 : N_2 \to N$ *are inclusion maps of irreducible compact cores of* N*, then there is a homeomorphism* $h : N_1 \to N_2$ *such that* $h_* = (i_2)_*^{-1} \circ (i_1)_* : \pi_1(N_1) \to \pi_1(N_2)$

We shall see that (N'_ρ, h_ρ) has the same marked homeomorphism type as (M, id) while $(N'_{\rho_n}, h_{\rho_n})$ has the same marked homeomorphism type as (M', h_A) for all n. Observe that, by construction, $f|_{M \setminus W}$ coincides with $h_A|_{M \setminus W}$. Let Y be a regular neighborhood of the boundary component of M containing V_1, we see that $M \setminus Y$ is a deformation retract of M. By composing this deformation retract with $f \circ \psi$, we see that $f \circ \psi$ is a primitive handle shuffle associated to A. We will need another technical lemma, an adaptation of a result from Anderson-Canary-McCullough [4].

Lemma V.8. *(Anderson-Canary-McCullough [4, Lemma 7.1]) Let $h_A^0, h_A^1 : M \to M'$ be primitive handle shuffles associated to a primitive essential annulus A in M. Let W be a regular neighborhood of A as in Chapter IV. If $h_A^0|_{\overline{M \setminus W}} = h_A^1|_{\overline{M \setminus W}}$ then*

1. *there is a homeomorphism $r : M \to M$, which is a composition of Dehn twists about W_l, W_r or V_1 of W, such that $h_A^0 \circ r$ is homotopic to h_A^1 relative to $\overline{M \setminus W}$, and*
2. *there is a homeomorphism $r' : M' \to M'$ which is a composition of Dehn twists about W_l, W_r or V_2 such that $r' \circ h_A^0$ is homotopic to h_A^1 relative to $\overline{M \setminus W}$.*

Since $i_n \circ f$ and h_A are both primitive handle shuffles associated to A, and ψ_n is a homeomorphism, the above lemma shows that

$$[(N_{\rho_n}, \rho_n)] = [(M'_n, i_n \circ f)] = [(M', h_A)]$$

where the first equality comes from the fact that $\rho_n = \psi_n \circ i_n \circ f$ and ψ_n is a homeomorphism (for a more detailed discussion, see the remark at the end of this chapter).

By construction, f lifts to an embedding $f : M \to N_\rho$ into the cover N_ρ of $\hat{N}'$ associated to $\psi_*(f_*(\pi_1(M))) = \rho(\pi_1(M))$. Since $g_* = \rho$ we see that

$$[(N_\rho, h_\rho)] = [(g(M), g)] = [(M, id)]$$

If $[(M, id)]$, and $[(M', h_A)]$ are distinct points in $\mathcal{A}(M)$, then we see that the marked homeomorphism type of (N_{ρ_n}, h_{ρ_n}) changes in the limit (N_ρ, h_ρ).

The following result of Ohshika [42] guarantees that (N_ρ, h_ρ) lies in the closure of the component of $int(AH(M))$ containing points with the same marked homeomorphism type as (N_ρ, h_ρ), provided that (N_ρ, h_ρ) is geometrically finite.

Theorem V.9. *(Ohshika [42, Cor. 6]) Let (N, h) and (N', h') be two points in $AH(M)$ with (N, h) geometrically finite and minimally parabolic and (N', h') geometrically finite. Suppose that there is a homeomorphism $N \to N'$ homotopic to $(h') \circ h^{-1}$. Then (N', h') lies in the closure of the component of $int(AH(M))$ containing (N, h).*

To see that (N_ρ, h_ρ) is geometrically finite, we appeal to the following result, due to Thurson (see Morgan [38, Prop. 7.1]).

Theorem V.10. *(Thurston) Let N be a geometrically finite hyperbolic 3-manifold such that $\partial C(N)$ is nonempty (where $C(N)$ is the convex core of N). Then every covering space N' of N with finitely generated fundamental group is also geometrically finite.*

Therefore the components of $int(AH(M))$ containing (N_ρ, h_ρ) and (N'_n, h_n) bump in $AH(M)$. This completes the proof.

□

We return to the case where M is obtained from $S \times [0, 1]$ by attaching a one-handle. Results of Canary-McCullough [19] (see Theorem III.10) show that $int(AH(M))$ only contains two components in this case. Since we showed that these two components are related by a primitive handle shuffle in Chapter IV, our construction shows that these two components bump in $AH(M)$. In light of the Density Theorem, we see that $AH(M)$ is connected.

Corollary V.11. *Let M be a compact, orientable, hyperbolizable 3-manifold obtained by attaching a one-handle to the boundary of $S \times [0,1]$, then $AH(M)$ is connected.*

□

Remark V.12. In order to not distract from the discussion in the proof of Theorem V.1, we were a little loose with some topological details in establishing that h_A and $i_n \circ f$ produce equivalent markings in $\mathcal{A}(M)$. These details are reproduced here in full.

Recall that $\hat{M}'$ is produced by removing a regular neighborhood V of the core curve c of A from M'. Let Y denote a regular neighborhood of the boundary component containing D_2 in M' that contains V. In the proof of Theorem V.1, we saw that $M' \backslash Y$ is a strong deformation retraction of M' lying entirely within $\hat{M}'$. Alternatively, we may deform M' so that all of the deformation takes place inside of W. Let T denote the introduced torus boundary component of $\hat{M}'$ obtained by drilling V from M' and let $\{m, l\}$ denote a meridian-longitude system of T such that l is parallel in $\hat{M}'$ to the core curve of V_2 and so that m bounds a disk in V. Let A_2 denote an essential annulus in $W \setminus V$ with one boundary component the longitude l on T and the other boundary component lying in V_2. Let M'_c be obtained from $\hat{M}'$ by removing an open regular neighborhood of $A_2 \cup V$ contained in the interior of $W \setminus V$. Then M'_c forms a compact core for M' and there is a strong deformation retraction $\tau : M' \to M'_c$ (see [4, Sec. 10]). Observe that τ is homotopic to an orientation preserving homeomorphism from M' to M'_c.

Since $i_n \circ \tau \circ h_A$ and $i_n \circ f$ agree on $M \setminus W$ and are primitive handle shuffles associated to A, Lemma V.8 shows that there is an orientation preserving homeomorphism $r_n : M \to M$ which is the identity on $M \setminus W$, such that $i_n \circ \tau \circ h_A \circ r_n$ is homotopic to $i_n \circ f$.

Recall that the sequence $\beta_n : \pi_1(\hat{M}') \to PSL_2(\mathbb{C})$ uniformizes hyperbolic 3-manifolds N'_n homeomorphic to the interior of $\hat{M}'(1,n)$ and recall that $i_n : \hat{M}' \to \hat{M}'(1,n)$ denotes the inclusion map. For each n, let i'_n be an embedding of $\hat{M}'$ into the interior of $\hat{M}'(1,n)$ which is isotopic to i_n. Then $\psi_n(i'_n(M'_c))$ is a compact core for N'_n. Furthermore, $s_n = \psi_n \circ i'_n \circ \tau \circ h_A \circ r_n$ is a homotopy equivalence from M to N'_n with image $\psi_n(i'_n(M'_c))$ such that $(s_n)_*$ is conjugate to ρ_n. This implies that

$$[(\psi_n(i'_n(M'_c)), s_n)] = [(M'_c, \tau \circ h_A \circ r_n)] = [(M', h_A \circ r_n)]$$

with the last equality following from the fact that τ is homotopic to an orientation-preserving homeomorphism from M' to M'_c. Since $h_A \circ r_n$ and h_A are both primitive handle shuffles with respect to A that agree on $M \setminus W$, Lemma V.8 shows that there exists an orientation-preserving homeomorphism $r'_n : M' \to M'$ such that $r'_n \circ f$ is homotopic to $h_A \circ r_n$. We see that

$$[(M', h_A \circ r_n)] = [(M', r'_n \circ h_A)] = [(M', h_A)]$$

BIBLIOGRAPHY

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[1] Ian Agol. Tameness of hyperbolic 3-manifolds. *arxiv preprint: arXiv:math.GT/0405568*, 2004.

[2] Lars Ahlfors and Lipman Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960.

[3] James Anderson and Richard Canary. Algebraic limits of Kleinian groups which rearrange the pages of a book. *Inventiones Mathematicae*, 126(2):205–214, 1996.

[4] James W Anderson, Richard Canary, and Darryl McCullough. The topology of deformation spaces of Kleinian groups. *Annals of Mathematics*, 152(3):693–741, 2000.

[5] Lipman Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc.*, 66(2):94–97, 1960.

[6] Lipman Bers. Spaces of Kleinian groups. In *Several Complex Variables, I (Proc. Conf., Univ. of Maryland, College Park, Md., 1970)*, pages 9–34. Springer, Berlin, 1970.

[7] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, 124(1):71–158, 1986.

[8] Francis Bonahon and Jean-Pierre Otal. Variétés hyperboliques à géodésiques arbitrairement courtes. *Bull. London Math. Soc.*, 20(3):255–261, 1988.

[9] B. H. Bowditch. Geometrical finiteness for hyperbolic groups. *J. Funct. Anal.*, 113(2):245–317, 1993.

[10] Jeffrey Brock and Kenneth Bromberg. On the density of geometrically finite Kleinian groups. *Acta Mathematica*, 192(1):33–93.

[11] Jeffrey Brock, Richard Canary, and Yair Minsky. The classification of Kleinian surface groups, II: The ending lamination conjecture. *Ann. of Math. (2)*, 176(1):1–149, 2012.

[12] Jeffrey F. Brock, Kenneth W. Bromberg, Richard D. Canary, and Yair N. Minsky. Local topology in deformation spaces of hyperbolic 3-manifolds. *Geom. Topol.*, 15(2):1169–1224, 2011.

[13] Kenneth Bromberg. Hyperbolic Dehn surgery on geometrically infinite 3-manifolds. *arxiv preprint: arxiv:math.GT/0009150*, 2000.

[14] Kenneth Bromberg. The space of Kleinian punctured torus groups is not locally connected. *Duke Math. J.*, 156(3):387–427, 2011.

[15] Kenneth Bromberg and John Holt. Self-bumping of deformation spaces of hyperbolic 3-manifolds. *Journal of Differential Geometry*, 57(1):47–65, 2001.

[16] Danny Calegari and David Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 19(2):385–446, 2006.

[17] Richard Canary. A covering theorem for hyperbolic 3-manifolds and its applications. *Topology*, 35(3):751–778, 1996.

[18] Richard Canary. Introductory bumponomics: the topology of deformation spaces of hyperbolic 3-manifolds. In *Teichmüller theory and moduli problem*, volume 10 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 131–150. Ramanujan Math. Soc., Mysore, 2010.

[19] Richard Canary and Darryl McCullough. *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups*, volume 812. American Mathematical Soc., 2004.

[20] Richard D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6(1):1–35, 1993.

[21] Timothy Comar. *Hyperbolic Dehn surgery and convergence of Kleinian groups*. ProQuest LLC, Ann Arbor, MI, 1996. Thesis (Ph.D.)–University of Michigan.

[22] Richard Evans and John Holt. Non-wrapping of hyperbolic interval bundles. *Geom. Funct. Anal.*, 18(1):98–119, 2008.

[23] J. Holt. Bumping and self-bumping of deformation spaces. In *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pages 269–284. Amer. Math. Soc., Providence, RI, 2004.

[24] John Holt. Multiple bumping of components of deformation spaces of hyperbolic 3-manifolds. *American Journal of Mathematics*, 125(4):691–736, 2003.

[25] Kentaro Ito. Convergence and divergence of Kleinian punctured torus groups. *Amer. J. Math.*, 134(4):861–889, 2012.

[26] William H. Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.*, 21(220):viii+192, 1979.

[27] Klaus Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[28] M. Kapovich. *Hyperbolic Manifolds and Discrete Groups*. Modern Birkhäuser Classics. Birkhäuser Boston, 2009.

[29] Irwin Kra. On spaces of Kleinian groups. *Comment. Math. Helv.*, 47:53–69, 1972.

[30] Aaron D. Magid. Deformation spaces of Kleinian surface groups are not locally connected. *Geom. Topol.*, 16(3):1247–1320, 2012.

[31] Albert Marden. The geometry of finitely generated Kleinian groups. *Ann. of Math. (2)*, 99:383–462, 1974.

[32] Bernard Maskit. On boundaries of Teichmüller spaces and on Kleinian groups. II. *Ann. of Math. (2)*, 91:607–639, 1970.

[33] Bernard Maskit. Self-maps on Kleinian groups. *Amer. J. Math.*, 93:840–856, 1971.

[34] D McCullough, A Miller, and GA Swarup. Uniqueness of cores of non-compact 3-manifolds. *Journal of the London Mathematical Society*, 2(3):548–556, 1985.

[35] Curtis McMullen. Complex earthquakes and Teichmüller theory. *Journal of the American Mathematical Society*, 11(2):283–320, 1998.

[36] Yair N. Minsky. Harmonic maps into hyperbolic 3-manifolds. *Trans. Amer. Math. Soc.*, 332(2):607–632, 1992.

[37] Yair N. Minsky. End invariants and the classification of hyperbolic 3-manifolds. In *Current developments in mathematics, 2002*, pages 181–217. Int. Press, Somerville, MA, 2003.

[38] John W. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 37–125. Academic Press, Orlando, FL, 1984.

[39] G.D. Mostow. Quasi-conformal mappings in n-space and the rigidity of the hyperbolic space forms. *Publ. Math. IHES*, 1968.

[40] Hossein Namazi and Juan Souto. Non-realizability and ending laminations: Proof of the density conjecture. *Acta Mathematica*, 209(2):323–395, 2012.

[41] Ken-Ichi Ohshika. Divergence, exotic convergence and self-bumping in quasifuchsian spaces. *preprint*.

[42] Ken-Ichi Ohshika. Ending laminations and boundaries for deformation spaces of Kleinian groups. *Journal of the London Mathematical Society*, s2-42(1):111–121, 1990.

[43] Ken-Ichi Ohshika. Realising end invariants by limits of minimally parabolic, geometrically finite groups. *Geometry & Topology*, 15(2):827–890, 2011.

[44] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *arxiv preprint: arxiv:math.GT/0307245*, 2003.

[45] G. Perelman. Ricci flow with surgery on three-manifolds. *arxiv preprint: arxiv:math.GT/0303109*, 2003.

[46] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *arxiv preprint: arxiv:math.GT/0211159*, 2002.

[47] Gopal Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.*, 21:255–286, 1973.

[48] Peter Scott. Compact submanifolds of 3-manifolds. *J. London Math. Soc.*, 7:246–250, 1973.

[49] Dennis Sullivan. Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.*, 155(3-4):243–260, 1985.

[50] William P. Thurston. Hyperbolic structures on 3-manifolds, ii: Surface groups and 3-manifolds which fiber over the circle. *arxiv preprint: arXiv:math.GT/9801045*.

[51] William P. Thurston. *The Geometry and topology of 3-manifolds.* Princeton University Lecture Notes, 1982.

[52] William P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.

[53] William P. Thurston and S. Levy. *Three-dimensional Geometry and Topology.* Princeton University Press, 1997.