How Mathematical Knowledge for Teaching Intersects with Teaching Practices: The Knowledge and Reasoning Entailed in Selecting Examples and Giving Explanations in Secondary Mathematics

by

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DEDICATION

To all of the teachers who have inspired and supported me along the way, especially my parents and grandparents

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ABSTRACT

Teaching requires both knowing and doing. This dissertation helps bridge between research on teacher knowledge and research on teaching practice by conceptualizing the ways in which teachers draw on their knowledge when enacting specific teaching practices. Recent research on mathematical knowledge for teaching has specified different domains of teachers' knowledge, but has focused less on how teachers use what they know in teaching. Similarly, several teaching practices have been found to matter for student learning, yet researchers have not adequately delved into the knowledge and work entailed in carrying out these practices.

To investigate teachers' knowledge use in practice, this study of eight Algebra II teachers focused on the content of rational expressions and equations and two foundational practices in mathematics teaching – selecting examples and explaining. Data were collected through classroom observations and interviews simulating the two teaching practices. Analyses probed the knowledge used in enacting each practice.

One finding of this study is a description of components of the practice of selecting examples and the knowledge teachers draw on in enacting them. For instance, teachers sequence examples and draw on nuanced understandings of differences across a set of mathematical examples. Second, different categories of explanations were seen and teachers' knowledge use varied by explanation type. Third, across both practices, the knowledge teachers drew on when enacting the practices was associated with differences in how teachers enacted the practice. For example, teachers drew on a broader range of knowledge types when giving *mathematical reasoning explanations* than when giving *procedural explanations*. The research also shows the

complexity of knowledge use when enacting teaching practices. For example, during a single explanation, one teacher drew on common and specialized content knowledge, knowledge of content and students, and pedagogical knowledge.

The findings contribute to theoretical understandings of how teachers use knowledge in teaching by conceptualizing the ways in which teachers draw on their mathematical knowledge for teaching when enacting specific teaching practices. For instance, they better specify types of specialized content knowledge used in practice. This work also has implications for mathematics teacher education and the methods used to study teacher knowledge in practice.

CHAPTER 1: INTRODUCTION

I begin this dissertation with a vignette from a mathematics classroom to highlight the intertwined knowledge and practice demands of teaching. In particular, I focus on the crucial work of selecting and using examples and of explaining mathematical ideas and procedures. The following episode comes from Mr. Baker's Algebra II class. Mr. Baker is in the middle of a unit on rational expressions and equations. In the current lesson, he is introducing ways to simplify rational expressions. During the previous two lessons, Mr. Baker has discussed approaches to graphing rational equations with his students. Later in the unit, the class will cover adding and subtracting rational expressions, solving rational equations, and rational inequalities.

Inside the Everyday Work of Teaching Mathematics

After having students graph a rational equation in groups and working through several problems from the homework, Mr. Baker tells students, "You've done this before". He then displays the fraction $\frac{2+3}{2+5}$ *on a PowerPoint slide. Mr. Baker has chosen to introduce the new material with a simple and familiar example. "Let's consider this good old-fashioned fraction. 2 plus 3 over 2 plus 5. What is this?" he asks. When he hears the correct answer of five sevenths from students, the answer is added on the board* $\frac{2+3}{2+5}=\frac{5}{7}$. *Mr. Baker responds, Five sevenths? Okay, maybe we need to slow down a bit. So let's look at this and you'll notice we have some 2s in common. If we reduce those 2s, what are we going to get?" As he asks this, his PowerPoint shows the fraction a second time and with the click of a button, slashes appear crossing out both of the 2s. At this point, he chooses to emphasize that "when we reduce we are dividing, we are* *not cancelling. I'm not getting rid of the 2s and making them 0s. I'm dividing and making them 1s." Above and below the crossed out 2s, 1s appear on the PowerPoint. When students answer that they get "four sixths," Mr. Baker asks if the two fractions are equivalent and establishes that because they are not equivalent, they are "not allowed" to reduce the 2s in this case.*

Choosing this simple example, which is familiar to students, enables Mr. Baker to explain a complex idea to students in a way that makes the new idea seem less complex. In particular, students might know that they cannot reduce the 2s in the first fraction, but by clearly demonstrating this mistake, they might see in a little more detail why this is an error. Further, students can remember a concrete example of the problems that arise when common terms in a fraction are incorrectly reduced.

To use this example, Mr. Baker might be drawing on knowledge of students' prior experiences with fractions. He might be using his own knowledge of the connections between fractions and rational expressions to connect a common error students make when simplifying rational expressions to a simpler fraction problem, where the error is more obvious to students. He might also be drawing on knowledge of the importance of showing errors to students to prevent them from happening in the future.

Mr. Baker goes through a similar process with the fraction $\frac{2\times 3}{2\times 5}$*, getting* $\frac{6}{10}$ *for the unreduced fraction and* $\frac{3}{5}$ *for the reduced fraction. Because the two fractions are equal, the class decides they are allowed to divide the numerator and denominator by 2. At this point, Mr. Baker introduces the definition of factors. "What are factors? Things that you multiply. And things being a very mathematical term. We love things, right? But I like to just say things because they're things. What are things? They could be numbers. They could be variables. They* *could be polynomials. They could be matrices. They could be things. If you multiply them, they're factors."*

A student jokingly asks, "What about water bottles?"

Mr. Baker's response emphasizes the importance of multiplication in his definition, "if you multiply water bottles, they're factors." Mr. Baker's lesson continues with a definition of terms and a discussion of what it means to simplify. He goes on to use several additional examples in the lesson, starting with $\frac{x^2-2x}{x-2}$ *and* $\frac{2x^2-18}{3-2x-x^2}$. (Mr. Baker 3-27-15)

As with the first problem he used, Mr. Baker has selected a simple example that is familiar to students. In doing so, he is able to demonstrate simplifying a rational expression by reducing common factors. He is also sequencing problems to increase in mathematical complexity, from a simple fraction problem, to a rational expression that requires factoring out a greatest common factor, to a rational expression that requires more complex factoring of quadratics.

In this part of his lesson, Mr. Baker might again be using knowledge of students' prior experiences with fractions. When giving his definition of factors, he might be drawing on knowledge of different types of mathematical objects, such as matrices, functions, and numbers and the properties and structure they share. Mr. Baker might also be drawing on pedagogical knowledge of the importance of giving students a range of mathematical cases for which a definition holds.

What Mr. Baker is doing in this vignette is common across the practice of teaching mathematics. Mathematics teachers are always choosing and using examples; they are also always explaining or supporting students to explain. Examples and explanations can be seen to be the bread and butter of teaching mathematics. As can be seen in this brief glimpse of Mr.

Baker's lesson, there is much more to selecting an example and giving an explanation than might appear on the surface. In particular, being aware of the underlying goal can help teachers be strategic both in their selection of examples and in how they give explanations. The examples teachers select and how they explain mathematical concepts and procedures facilitate student learning. Selecting examples and giving explanations are thus key aspects of the work teachers do to help students develop deeper understanding of the content.

In addition to highlighting the work of teaching mathematics, this brief episode also underscores how much teaching depends on knowing. Mr. Baker has to draw on a range of knowledge, including knowing what factors are, and knowing about students' prior experiences with fractions. In teaching, what teachers do relies on what they know. Teachers draw on knowledge of students when selecting examples that are appropriate for their students based on their prior mathematics learning. When giving an explanation, they rely on their knowledge of important mathematics concepts and common student misconceptions to decide what to emphasize. In short, what teachers know and how they know and use it is integral to the work of teaching.

Background

As the vignette demonstrates, teaching is knowledge-intensive work. What teachers do in the classroom engages them in challenges whose management depends on what they know and how they think about the content (Schoenfeld, 2010). Overall in teaching, teachers are likely drawing on a range of knowledge types, including knowledge of content, knowledge of pedagogy, knowledge of students, and pedagogical content knowledge (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Shulman, 1986). For example, when selecting examples, teachers use knowledge when they consider what they want an example to demonstrate mathematically and how a set of examples will demonstrate the important features of a mathematical concept or procedure. They might also consider how accessible the example is for students based on their prior knowledge, as well as how the example will be implemented during instruction. When giving an explanation, a teacher likely uses knowledge of key mathematical ideas in the lesson and connections between mathematical ideas, including those with which students are already familiar or will learn in the future. For example, when explaining slope, a teacher might consider students' prior experiences with the coordinate plane; connections between slope, the coordinate plane, and variables; and the types of situations in which students will be asked to apply the concept of slope. Further, in adjusting an explanation in the moment based on students' responses and questions, teachers are likely drawing on knowledge of student thinking and nuanced understandings of the mathematics.

However, we do not know much about how teachers draw on different kinds of understanding when enacting particular teaching practices, such as selecting examples and giving explanations, and how they are using different kinds of understanding to carry out the practice. Given that teaching is knowledge-intensive as well as relational and contingent work (Lampert, 2001), part of understanding the work of teaching is to see how teachers draw on knowledge when enacting particular teaching practices. Knowing more about the work of teaching, can position teacher educators to better prepare teachers with the knowledge and skills needed to enact different teaching practices.

There have been strong efforts over the past three decades to better understand teacher knowledge. Beginning with Shulman's (1986) proposal of a broader knowledge base for teaching, which shifted the focus of the field towards subject-specific pedagogical knowledge, researchers have sought to investigate the knowledge needed for teaching particular subject matter areas. This broader knowledge base is important, but what is perhaps even more important is how teachers use their knowledge in teaching. Knowledge is only of value if it is both usable and used. In particular, teachers' knowledge can only impact student learning if teachers actually use this knowledge in teaching. For example, as a mathematics student, I could easily factor complex polynomials. However, that knowledge by itself would not have enabled me to teach high school students to factor quadratics. In my methods course, I learned about the importance of using a range of examples for a particular topic. However, knowing to selecting a range of examples, without selecting them with a particular purpose in mind, would not have supported my students' learning. Rather, in my own teaching, when I taught students to factor quadratics, I drew on a range of knowledge. This included knowledge of students' prior experiences with greatest common factors and the distributive property, as well as a nuanced understanding of how to describe the patterns I looked for when I factored a problem. Only when I drew on this deeper knowledge of what students' knew and of how to decompose my own knowledge could I explain this content to my students. This example underscores the importance of teachers' broad and nuanced knowledge in teaching and how they must draw on it in practice.

Despite increasing clarity about teacher knowledge, little is known about what teachers draw on and how they use what they know in teaching. This dissertation investigates how teachers use knowledge when enacting two teaching practices, selecting examples and giving explanations.

These two practices were chosen to study how teachers use knowledge in teaching because they enable teachers to take their knowledge and convert it into something that can be communicated with students who do not yet understand the mathematics. For instance, when teachers select examples, their mathematical understanding is used to choose problems that can convey particular mathematical ideas. They are also likely drawing on understandings about teaching, learning, and students. Similarly, in giving an explanation, teachers are directly communicating their mathematics knowledge to students. Because these two practices are instances of teaching that require teachers to use their knowledge, they are therefore opportune sites for investigating how teachers draw on their knowledge in teaching. They are also important practices in mathematics teaching.

Explanations and examples are foundational to the teaching of mathematics. They are ubiquitous. As teachers create access to mathematical content and practice, they explain ideas, they show how to carry out procedures, and they model and name. Teachers choose examples to begin a lesson and they come up with others in the course of lessons. Teachers' explanations convey the meaning and importance of mathematical ideas (Leinhardt, 2010). Examples, chosen to make visible key features of a concept or procedure, or unpack an idea, can — when selected and used well — help make content more accessible (Leinhardt, 2001). During instruction, teachers also elicit explanations from students and support them in learning to construct and critique them. In this sense, explanations are also a part of mathematical practice.

Examples are an integral part of mathematics teaching because teachers illustrate concepts or procedures in order to help students see their meaning. Examples are used on a daily basis to teach, practice, and reinforce concepts and procedures. For instance, a teacher might show several linear equations when introducing the concept of slope. Examples might be selected for a range of purposes, including to reinforce general cases of a concept, or to demonstrate exceptional mathematical cases for which students need to think differently about a problem than they would about other related problems. Examples might be sequenced to increase in complexity or to present a range of problem types to students. Given the impact examples can have on the content that is taught, the teaching practice of selecting examples is a critical component of mathematics teaching.

Like examples, explanations are embedded in the daily work of a mathematics classroom. In high school mathematics classrooms, 95% of teachers explain mathematical ideas to their whole class at least once each week (Banilower et al., 2013). Teachers give explanations when introducing a new concept (e.g. what slope is), solving a problem (e.g. how to finding the slope of a line), or remediating a student's misconception (e.g. confusing the x and y coordinates in the slope formula). Explanations must be mathematically accurate, presented at a level that students can understand, and anticipate or respond to student conceptions, misconceptions, and responses. Due to these requirements for explanations, the teaching practice of giving explanations is another critical component of mathematics teaching.

Both teaching practices of selecting examples and giving explanations involve advance planning but both must also be responsive to interactions with students during class. These two practices overlap in teaching in that examples are a necessary part of teachers' explanations of mathematics. Mathematical ideas must be exemplified in some way in order to be shared with students. For example, the concept of slope holds little meaning without concrete examples such as graphs, equations, or mountains. Further, these examples must be explained to make clear the important features of the mathematical object a teacher is exemplifying (Leinhardt, 2001). A key difference between the two practices of selecting examples and giving explanations is that explanations involve a sort of improvisation in that they happen in the moment and must also be responsive and adapt to student responses as they occur. Although a teacher may select an example mid class in response to student understanding, once selected, the example itself is not an improvisation.

Despite their centrality in mathematics teaching, much remains unspecified and tacit about the two practices of selecting examples and giving explanations. Each is used frequently, for a range of purposes, and must be thoughtfully carried out to be effective. A good explanation does not happen by chance; instead, it is deliberate on the part of the teacher (Leinhardt, 2010). This further supports the notion that mathematics teaching is skillful work (Ball & Forzani, 2009). However, we do not know what is involved in carrying out the practices of selecting examples of giving explanations. Given that teaching is skillful deliberate and knowledgeable work, a better understanding of these two practices and the knowledge used in enacting them is important. In particular, if the work involved and the knowledge used in selecting examples and giving explanations are more clearly specified, teachers can be better prepared to give effective explanations and select appropriate examples. This dissertation aims to extend research on these two teaching practices by better specifying what they entail. Further, it aims to extend research on teacher knowledge by describing the knowledge teachers draw on when carrying them out. In doing so, it also contributes to a better understanding of how teachers use knowledge in practice.

There has been a great deal of research on mathematics teachers' knowledge at the elementary level (e.g. Ball et al., 2008; Turner & Rowland, 2008). For example, Ball, Thames and Phelps' (2008) mathematical knowledge for teaching outlines several domains of knowledge used in teaching. However, there have been considerably fewer studies of secondary mathematics teachers' knowledge (e.g. McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). In addition to investigating the knowledge teachers use in enacting particular teaching practices, this dissertation contributes specifically to this understanding at the secondary level. Compared to elementary mathematics teachers, it is often assumed that secondary mathematics teachers possess the mathematics knowledge needed to teach because they have taken several college level mathematics courses. However, research has shown that these teachers can also lack important knowledge for teaching (Knuth, 2002). While secondary mathematics teachers do often have stronger content knowledge because of the many mathematics courses they take, knowledge of mathematics is different than knowledge for teaching mathematics. It is important to more clearly define what this knowledge is and how it is used in teaching.

A study of secondary mathematics teaching could be focused on any number of mathematics topics. In this dissertation, I focus on rational expressions and equations. The topic of rational expressions and equations was selected because it has several features that may make teachers' knowledge and reasoning more visible than they would be with other topics. First, rational expressions and equations involve a range of representations, including algebraic expressions and equations, graphing, and word problems. This range of representations is a rich territory across which teachers' knowledge has more opportunity to vary. Second, rational expressions and equations is a topic that grows conceptually more complex throughout the high school mathematics curriculum and is foundational for later mathematics in calculus. For instance, in Algebra I, students are introduced to the concept of asymptotes. Asymptotes are a simple example of limits, which are an integral component of calculus. The complexity can be mediated both by a teacher's personal understanding of the topic and the decision to teach it in either a more conceptual or more procedural manner. Third, as compared to other topics, rational expressions and equations, and its more conceptual aspects in particular, are an area where teachers may vary in their own understanding. For example, although describing that division by 0 is undefined can be given as a rule, understanding why it is undefined requires deeper conceptual knowledge. In addition to this, the ability to teach it procedurally or in a way that highlights important conceptual understanding provides an interesting space to see how teachers manage the balance between procedural and conceptual teaching. Fourth, due to its challenging conceptual nature, rational expressions and equations is a topic students often have trouble learning, so it provides opportunities to see how teachers reason around student difficulties. Finally, compared to other content areas, such as linear or quadratic equations, there is little research on rational expressions and equations.

Given all of the above, rational expressions and equations is a strategic site for the study of how teachers think and reason about content in the context of teaching. It is also a good match for the practices of giving explanations and selecting examples. In particular, the conceptual complexity requires careful selection of examples and structuring of explanations. There are many nuances that can be highlighted through examples and explanations. For example, a common student misconception when simplifying rational expressions is that a single term in the numerator can reduce with a single term in the denominator. This holds for monomial numerators and denominators, however it will not work for more complex rational expressions. Finally, the range of representations and levels of complexity allow for a broad range of example types and features from which teachers can select examples.

Study Design

In summary, this study seeks to explore the complex relationship between mathematical knowledge for teaching and classroom teaching practices. It does so by looking at how teachers think mathematically in the context of carrying out the teaching practices of selecting examples and giving explanations. I have one main research question and four sub-questions:

- What mathematical knowledge for teaching is entailed by the instructional practices of selecting examples and giving explanations?
    1. What kinds of work do teachers do in carrying out these two teaching practices?

2. What mathematical knowledge for teaching do teachers draw on in carrying out these two teaching practices?
3. How do teachers use this mathematical knowledge for teaching and reasoning in doing this work?
4. Are there differences across the two practices? How are these differences in knowledge and reasoning related to the demands/work of the practices themselves?

To answer these questions, this dissertation considers how Algebra II teachers enact the teaching practices of selecting examples and giving explanations and the knowledge they draw on in doing so. It is a qualitative case study of eight teachers, which used classroom observations and interviews focused on the two teaching practices within the mathematical content of rational expressions and equations.

Contributions of the Study

In further explicating the knowledge and reasoning mathematics teachers use in the practices of selecting examples and giving explanations, this study seeks to contribute to the broader knowledge base around teacher knowledge and teaching practices. A better understanding of teacher knowledge and practices can help in refining and developing frameworks of mathematical knowledge for teaching secondary mathematics, of secondary mathematics teaching practices, and of how mathematical knowledge for teaching is entailed in mathematics teaching practices.

In addition to the empirical contributions, this study contributes to methods of studying teaching with a focus on how knowledge and practice come together in the activities of teaching. It does so by examining teaching across the settings of classroom observations and interviews probing teachers' knowledge and reasoning. This adds to research that has focused primarily on classroom observations.

This study also stands to inform teacher education. Defining the components of selecting examples and giving explanations can facilitate teaching and learning of these practices by pre-service teachers. Further, knowing particular types of knowledge needed for carrying out these teaching practices and the ways knowledge is used in these practices can inform the way teachers are prepared, both in content and methods courses and in classroom placements. Defining the knowledge and practices needed for teaching can also allow certification tests to better measure the knowledge and practices that actually matter in teaching, leading to more qualified and capable teachers entering the field. Similarly, this knowledge might be used to refine professional development.

Organization of the Dissertation

This dissertation is organized into six chapters. Chapter 1 frames the research problem, provides an overview of the study, and outlines this dissertation. In Chapter 2, I review relevant literature on mathematics teacher knowledge, teaching practices, selecting examples, giving explanations, and rational expressions and equations. The methods of the study are described in Chapter 3. Chapters 4 and 5 present the results of my analysis. Chapter 4 presents findings related to the practice of selecting examples. In Chapter 5, I present findings related to the practice of giving explanations, looking at the intersection of practice and knowledge use, across both the interviews and classroom observations. Finally, in Chapter 6, I look across the practices to consider implications of the study and directions for future research.

CHAPTER 2: LITERATURE REVIEW

Introduction

Knowledge and practice are inextricably intertwined. Yet, there are many challenges in studying them simultaneously and in conceptualizing the interplay between them. In this dissertation, I aim to better understand how mathematics teachers use knowledge in practice. To that aim, in this chapter I review what is known about mathematics teacher knowledge and about teaching practice. By “what is known,” I mean the ways in which knowledge and practice have been investigated, as well as what has been learned from this work.

I begin with mathematics teacher knowledge and review how it has been conceptualized and studied, what is “known” about teacher knowledge as a result of this work, and what remains unknown. Next, I take a similar look at teaching practice, reviewing its conceptualization, research findings, and remaining questions. In this section, I also discuss the two focal practices of my study, selecting examples and giving explanations. Finally, I conclude this chapter by reviewing the content area my study focuses on, rational expressions and equations.

Mathematics Teacher Knowledge

Although it is accepted in the field that teacher knowledge matters for student learning, proof of this connection has been elusive. Much of the early work connecting teachers to student learning investigated teachers’ degrees, number of mathematics courses taken, and other proxy measures of knowledge. Overall, there was no consistent relationship between these measures of knowledge and student learning. More recent work puts this research into greater perspective. In particular, there have been important distinctions made about the types of knowledge teachers use in teaching and the mathematical knowledge learned in advanced mathematics courses. This early work was focused on mathematics knowledge and not on the mathematics knowledge used in teaching.

In his AERA presidential address, Shulman proposed a broader knowledge base for teaching, which was specific to the subject being taught. Within this knowledge base for teaching, several domains of knowledge were hypothesized, including three related to content: content knowledge, curricular knowledge, and pedagogical content knowledge (PCK). He coined the term “pedagogical content knowledge” (PCK) to refer to the “special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (Shulman, 1987, p. 8). Pedagogical content knowledge, which resides at the core of expert teaching, is a teacher’s ability to turn content knowledge into pedagogically powerful forms that can be adapted to students’ varying abilities, prior knowledge, and backgrounds. Within pedagogical content knowledge, Shulman included includes “the ways of representing and formulating the subject that make it comprehensible to others” and “the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (1986, p. 9). This proposal of a broader knowledge base for teaching, that differed across content areas and included pedagogical content knowledge, induced a shift in research of mathematics teachers’ knowledge. His ideas acted as an impetus for researchers in education to investigate the knowledge teachers need to teach particular subject matter.

Over the past 20 years, there have been many advances in understanding mathematics teacher knowledge. Several of these advances build on Shulman’s conceptualizations of teacher knowledge. Compared to early research on teacher knowledge, this more recent research is more closely tied to teaching in that it is focused on knowledge for teaching mathematics, as opposed to mathematics knowledge more generally. Even with this more specific focus, the research varies greatly with respect to how closely it is tied to practice and how knowledge has been conceptualized and studied. In this section, I review several programs of research on mathematics teachers' knowledge. In doing so, I focus on the programs' conceptualizations of knowledge, as well as the methods used in their work, which are greatly influenced by their conceptualizations. After reviewing these different programs, I discuss more broadly what has been learned from this more instructional view of teacher knowledge and what remains unknown.

How Has Teacher Knowledge Been Studied and Conceptualized?

Learning Mathematics for Teaching. The Learning Mathematics for Teaching project has investigated mathematical knowledge for teaching (MKT), defined as the knowledge needed to carry out the work of teaching. Their framework represents the efforts of Ball and colleagues to create a practice-based theory of mathematical knowledge for teaching based on Shulman's (1986) conception of pedagogical content knowledge and his initial efforts to define the knowledge needed for teaching. A noticeable distinction of this framework from earlier frameworks is the domain of specialized content knowledge, mathematical knowledge unique to the work of teaching. In investigating the knowledge demands of the tasks of teaching, Ball and colleagues found that several tasks of teaching required teachers to use mathematical knowledge distinct from knowledge of pedagogy or student thinking.

The LMT framework hypothesizes six distinct domains of knowledge, organized within the larger categories of pedagogical content knowledge and subject matter knowledge. Within pedagogical content knowledge are the domains of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum. KCS is knowledge of students' conceptions (both correct and incorrect) of mathematics and is used to anticipate student thinking while planning instruction and to evaluate errors or the mathematical understanding in student work. KCT is the knowledge needed to teach particular content. It includes the ways of teaching particular mathematics topics that best enable student learning, sequencing of tasks and examples, and how content will be presented. Knowledge of content and curriculum (KCC) remains less developed than the other PCK sub-domains, but is thought to include knowledge of standards and benchmarks, and sequencing of topics within and across grades (Ball & Bass, 2009; Ball et al., 2008).

Although the categories contained in pedagogical content knowledge are reflective of categories earlier hypothesized by Shulman, the LMT work extends beyond Shulman's initial work on subject matter knowledge. Subject matter knowledge encompasses the other three subdomains in the LMT framework, Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). CCK is common to other professionals in mathematically intensive fields, where mathematics is used in similar ways to how it is used in teaching. Unlike CCK, SCK is unique to teachers. Other mathematics professionals use mathematics in a compressed and finalized form, but teachers must be able to interpret, understand, and share the uncompressed versions of mathematics knowledge that their students are learning and using. To do so, they must decompose their own mathematical knowledge. Knowing mathematics in these multiple uncompressed forms and mapping between them is a subset of mathematical knowledge unique to teachers and supports the mathematical tasks of teaching, including evaluating student work and conceptions (Ball & Bass, 2009; Ball et al., 2008).

Although CCK is common across others in mathematically intensive fields, using this mathematics knowledge may not require an understanding of the structure of mathematics. However, in addition to teaching students particular mathematics ideas, teachers introduce students to the discipline of mathematics and prepare them for future mathematics learning. To do this, teachers must possess knowledge that exists at the mathematical horizon of students' learning. They must know how the mathematics they teach relates to mathematics students have learned and will learn in the current course and in courses much later in their education. They must also convey disciplinary practices and values to students. This knowledge is all considered to be a part of Horizon Content Knowledge (HCK). Ball and Bass (2009) hypothesize that HCK consists of four types of knowledge: (1) knowledge of the areas of mathematics surrounding current classroom instruction; (2) knowledge of the structure of and major ideas in the discipline of mathematics; (3) knowledge of common mathematics practices; and (4) knowledge of important mathematical values. HCK is unlikely to be taught to students, but it provides a needed perspective for teachers by enabling them to situate the mathematics they are teaching within the larger framework of the discipline and students' future mathematics learning (Ball & Bass, 2009).

The LMT project conducted an analysis of the work of teaching by identifying the places in teaching where teachers would draw on their knowledge and hypothesizing the knowledge teachers might use at those points (Ball & Bass, 2003; Ball et al., 2008). To do so, the group relied on an extensive collection of records of teaching practice and the multidisciplinary expertise of research group. The group also developed items to assess teachers' knowledge. The items reflected a range of content, tasks of teaching, and hypothesized types of knowledge needed to enable student learning. Using quantitative methods, these items were used to better understand the structure of mathematical knowledge for teaching. The results, obtained through the use of factor analysis and item response theory, suggested content knowledge varies by mathematical domain and different types of knowledge exist, including CCK, SCK, and KCS (Hill, Schilling, & Ball, 2004).

Knowledge for Algebra Teaching. The Knowledge for Algebra Teaching (KAT) framework (McCrory et al., 2012) also focuses on how knowledge is used in practice. It is unique in that is contains not only types of mathematical content knowledge, but also mathematical uses of knowledge in teaching. Within this framework, the work of teaching is seen as interactions between types of knowledge and uses of knowledge in teaching around particular mathematical content. The mathematical content knowledge dimension consists of three categories: Knowledge of School Algebra, Knowledge of Advanced Mathematics, and Mathematics-for-Teaching Knowledge. Knowledge of School Algebra contains the content typically taught in high school algebra courses (I and II). Mathematical knowledge extending beyond school algebra, particularly from college level mathematics courses, is considered Knowledge of Advanced Mathematics, which provides teachers with mathematical breadth and depth. Unlike the first two categories, Mathematics-for-Teaching Knowledge is mathematical knowledge used by teachers that is not used in other mathematically proficient domains. Mathematics-for-Teaching Knowledge includes knowing the affordances of each solution method for a particular problem and is also used to interpret the mathematics in student work.

The second dimension of the KAT framework, mathematical uses of knowledge in teaching addresses how teachers use mathematical knowledge in ways different than others. In particular, using mathematical knowledge in teaching involves the three practices of decompressing, trimming, and bridging. Decompressing allows teachers to share sophisticated mathematical ideas with students, who often cannot immediately understand them in their complex mathematical forms, by unpacking implicit pieces of knowledge and exposing underlying mathematical ideas. For example, a teacher might decompress the many interpretations of a word, such as the word *solve* in the context of the directions of a problem. The practice of trimming involves adjusting the level of detail or mathematical rigor to make mathematical ideas accessible to students at their given levels of understanding. When trimming, teachers must include considerations about maintaining mathematical integrity and avoiding simplifications that will lead to future misconceptions. Finally, bridging, involves making connections across different areas of mathematics to see the discipline as an integrated, connected whole. Teachers bridge between different topics and instructional materials and between their advanced mathematical knowledge and the mathematics they teach (McCrory et al., 2012).

In conceptualizing mathematics teacher knowledge, researchers used many artifacts of classroom instruction, including textbooks, teacher interviews, and teaching videos. In each analysis, the goal was to draw inferences about the knowledge teachers would need to teach Algebra based on each of the data sources. The textbook analysis looked for variation in how topics were treated across books. The variety showed the importance of teachers' ability to make connections across different representations of the same mathematics. Teacher interviews focused on student difficulties in learning algebra and helped discern the ways teachers use mathematical knowledge that are unique to teaching. Finally, in the video analysis, researchers considered both what the teacher did and what the teacher could have done. They then made conjectures about the teacher's knowledge based on what they decided to do (McCrory et al., 2012).

The Knowledge Quartet. The knowledge quartet presents a different way of looking at knowledge use in practice. This framework was developed based on observations of teachers and has been used to observe pre-service teachers (Turner & Rowland, 2011). While it was initially developed for elementary teachers, the framework has recently been tried at the secondary level (Rowland, Jared, & Thwaites, 2011). The framework is used to categorize parts of mathematics lessons of specific mathematical content based on teachers' particular mathematics knowledge (Rowland et al., 2011).

The knowledge quartet is made up of four dimensions of teachers' knowledge, *foundation*, *transformation*, *connection*, and *contingency*. *Foundation* includes teachers' mathematical knowledge and understanding, pedagogical theory, and beliefs about mathematics as a discipline. Teachers possess *foundation* knowledge, but may or may not use it in teaching. *Transformation*, the second dimension involves the knowledge used in transforming one's own knowledge in ways that will enable students' learning. This dimension of knowledge is often used in choosing examples. The third dimension of the knowledge quartet, *connection*, involves considering coherence and mathematical connections across lessons. It is used in sequencing examples and instruction. Finally, the fourth dimension of *contingency* is knowledge used in responding to unexpected events during instruction. The quality of a teacher's response depends on the knowledge available to them (Turner & Rowland, 2011).

COACTIV. The COACTIV framework assumes CK is a necessary precursor for PCK. However, these constructs are distinct and researchers have aimed to determine their individual effects on instruction and student achievement. In particular, they hypothesize that the effects of PCK and CK on student achievement are mediated by three attributes of instructional quality: the cognitive level of classroom activities, the alignment of these activities with the curriculum, and teachers' support for individual learning. Within this model, CK is conceptualized as knowing curricular content at a profound, conceptual level. PCK includes three components: (1) "knowledge of mathematical tasks as instructional tools", which enables teachers to select and use tasks in their teaching in ways that are mathematically powerful for students; (2) "knowledge of students' thinking and assessment of understanding", including common misconceptions and strategies; and (3) "knowledge of multiple representations and explanations of mathematical problems", which support students' construction of mathematical understanding (Baumert et al., 2010, p. 142).

Mathematics Teachers' Specialized Knowledge. A group of researchers at the University of Huelva, Spain propose a model of mathematics teachers' knowledge based on challenges they see in Ball et al.'s (2008) MKT framework (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; Montes, Aguilar, Carrillo, & Muñoz-Catalán, 2013). Their model focuses on specialized knowledge unique to mathematics teachers, as opposed to general pedagogical knowledge or general mathematical knowledge. This knowledge is called Mathematics Teachers' Specialized Knowledge (MTSK). The framework is not based on classroom observations, but instead is a proposed theoretical model that can later be tested in practice (Carrillo et al., 2013).

Within the MTSK framework there are six sub-domains, three each in the areas of subject matter knowledge and pedagogical content knowledge. Instead of delineating subject matter knowledge categories based the group of people who might possess them, mathematical knowledge is categorized by attributes of the knowledge itself. Within subject matter knowledge are: (1) *Knowledge of topics*, including concepts, procedures, and underlying mathematical meanings; (2) *Knowledge of the structure of mathematics*, including the larger structure of mathematical knowledge and concepts in the discipline and structure within concepts, connections between ideas related to the same concept and of the same concept across different grade levels; and (3) *Knowledge about mathematics*, which includes the ways of working and generating knowledge within the discipline.

The three topics within pedagogical content knowledge are: (1) *Knowledge of features of learning mathematics*, how students think and react when working on mathematical tasks; (2) *Knowledge of mathematics teaching*, which is used in choosing examples and representations; and (3) *Knowledge of mathematics learning standards*, which include not only knowledge of the curriculum a teacher is using and the standards for student learning at each grade level, but also the larger standards developed from research and used on national achievement tests (Carrillo et al., 2013; Montes et al., 2013).

Knowledge within a framework of learning to teach. Peressini et al. (2004) use a situated perspective to propose a conceptual framework of learning to teach, within which they define three domains of teachers' knowledge, which, they claim, are all intertwined in teaching. Their framework includes one category of mathematics content knowledge in addition to the category of mathematics specific pedagogy, which they consider to be a refinement of Shulman's PCK. Within this category are selecting mathematical tasks and using mathematical discourse in the classroom. The third framework domain is professional identity, which affects how teachers view and respond to problems of practice. This category is reflective of the situated perspective used in their framework. While the domains of mathematics content knowledge and mathematics specific pedagogy are less well-defined than in other, more recent frameworks, the framework's perspective of knowledge as situated provides a useful lens through which to view teacher knowledge.

Mathematics-for-teaching. Davis and Simmt (2006) focus on mathematics-for-teaching, which signals "the distinct character of teachers' subject matter knowledge" (p. 294). Like Ball and Bass (2003), Davis and Simmt believe the mathematical content knowledge needed for teaching is "a serious and demanding area of mathematical work" that differs from the mathematics their students will learn and further stipulate that much of this knowledge is tacit (2006, p. 295). In this sense, they do not focus on the mathematical demands of teachers' work and the multitude of knowledge embedded in practice, like Ball and colleagues do. They instead recognize teachers' practices as "embodied and enacted understandings" of mathematical content knowledge (2006, p. 316). Teachers' practice can therefore be used as a lens to interpret their tacit mathematical content knowledge.

Davis and Simmt ground their research in complexity science, which "prompts attention toward several dynamic, co-implicated, and integrated levels […] rather than isolated phenomena" (2006, p. 296). In their description of *mathematics-for-teaching*, they therefore do not distinguish between the individual and the collective. Complex systems are represented through nested categories. The layers are not easily distinguishable, but the timescale and number of participants increase moving from the inner to the outer layers. Davis and Simmt's model of *mathematics-for-teaching* contains 4 nested layers, where an individual's *subjective understanding* unfolds within the broader dynamics of *classroom collectivity*, or shared knowledge within a classroom. In a geometry course, these might include standards for reasons in proofs. This *classroom collectivity* is nested in *curriculum structures,* which is encompassed by formal *mathematical objects*. The inner two layers are both categories of knowing, while the outer two layers are categories of knowledge. While knowing can vary across individuals and classrooms, knowledge is shared by larger populations and is therefore more stable than knowing. This model is different from others as it does not define particular types of knowledge, but rather looks at nested levels of mathematical knowledge and how teachers' knowledge is imbedded within larger organizers of mathematical knowledge, like curriculum, and larger mathematical objects defined by the discipline of mathematics. While this model provides a perspective on how student and teacher knowledge in the classroom is nested within disciplinary knowledge of mathematics, it does not define mathematics teacher knowledge, making it a poor fit for research involving items designed to assess teachers' knowledge.

A framework of knowledge development. Silverman and Thompson (2008) focus not on the knowledge embedded in the work of teaching, but on how this knowledge is developed within individual teachers, giving their framework a qualitatively different focus than many of the others. Further, they view mathematical knowledge for teaching not as particular pieces of knowledge, but rather as an underlying conceptual structure formed by a network of mathematical understandings, "that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning" (P. W. Thompson, 2008, p. 46). The belief underscoring this framework is, that for a teacher to be able to develop complex mathematical conceptual structures in her students, she must first posses them herself (P. W. Thompson, Carlson, & Silverman, 2007). Using this frame, mathematical knowledge for teaching is defined as the knowledge enabling teachers to develop these conceptual structures in their students (Silverman & Thompson, 2008).

In summary, while there are numerous frameworks of mathematics teachers' knowledge, some frameworks contain similar domains of knowledge. Many frameworks often include and build on Shulman's content and pedagogical content knowledge. However, there are differences in how different domains of teachers' knowledge are conceptualized. For example, some of the frameworks describe knowledge based on those who possess and use that knowledge (e.g. teachers vs. mathematicians), whereas other frameworks focus on aspects of the knowledge (e.g. knowledge about mathematics vs. mathematical facts). The frameworks also vary in the methods used to conceptualize them and the degree to which these conceptualizations are practice-based. For example, the LMT framework was developed by analyzing occasions in teaching where knowledge might be used. The Silverman and Thompson framework is a more theoretical conceptualization of knowledge.

What Has Been Learned About Teacher Knowledge? The many studies of teacher knowledge have led to important results. First, many components of teacher knowledge have been empirically validated, including CCK, SCK, and KCS (Hill et al., 2004). A later validity study using cognitive interviews[1] confirmed the distinction between content knowledge (CCK and SCK combined) and KCS (Hill, Dean, & Goffney, 2007). Second, mathematical knowledge for teaching has been differentiated from knowledge held by mathematicians (Ball, Lubienski, & Mewborn, 2001; Hill et al., 2007). A study comparing teachers and mathematicians found that while teachers used knowledge of content and students as the main justification on 40.5% of the KCS items, mathematicians did so only on 1.8% of the items, and non-teachers on 15.5%, indicating that this knowledge is distinctive of teachers (Hill et al., 2007).

Finally, while it was considered obvious that teacher knowledge matters for student learning, until recently this relationship remained unproven. In the past 10 years, teacher knowledge has been shown to matter for student learning. Further, this link was made not with proxy measures of knowledge, but rather using a measure of teachers' knowledge (Hill, Rowan,

[1] In a cognitive interview, participants are asked additional questions while taking a survey to ascertain if the survey is being interpreted as intended. Cognitive interviews are recommended as a method of understanding how participants understand and answer surveys (Desimone & Le Floch, 2004).

& Ball, 2005). While mathematical content knowledge, as measured by the MKT items, was significantly associated with a higher increase in student achievement, years of teaching experience and methods and content courses were not. Additionally, there was only a small correlation between math content knowledge and both years of teaching experience and math or methods courses taken, indicating that the MKT items may be a better measure of teachers' knowledge that directly impacts student learning (Hill et al., 2005).

In a study using multilevel structural equation models, the COACTIV group also linked teacher's knowledge to student achievement. To measure teachers' knowledge, the group developed a separate assessment for each of PCK and CK based on findings of earlier factor analysis, which indicated two separate constructs within teachers' knowledge. The PCK assessment was open ended and included questions in three dimensions: (1) identifying multiple solution paths; (2) recognizing student solution strategies and misconceptions; and (3) knowledge of mathematical representations and explanations. The CK assessment was a paper-based test where all items required mathematical reasoning or proof. Student knowledge was assessed using the PISA assessment and test on the standard curriculum. The model also included three measures of instruction: (1) the curricular level of tasks, measured by coding assignments and assessments the teachers used; (2) a measure of instructional quality based on a student rated scale; and (3) classroom management. They found that while teachers' PCK and CK both affect student achievement, PCK has a larger effect. Further, the effects of PCK and CK were mediated by attributes of instructional quality. In particular, PCK affects the cognitive level of classroom activities, the alignment of these activities with the curriculum, and teachers' support for individual learning. However, CK was only found to have an effect on the curricular level of tasks. While it is PCK that has the greater effect on student learning, the model was created with the underlying assumption that CK is a necessary precursor for PCK (Baumert et al., 2010).

Overall, these studies show that teacher knowledge can be measured and is not strongly dependent on coursework or experience. This knowledge is different than knowledge possessed by other mathematically proficient professionals. Further teacher knowledge contains distinct dimensions, both pedagogical and mathematical in nature. Most importantly, teacher knowledge does matter for student learning. Having reviewed what the field has learned about teacher knowledge, I now discuss what remains to be understood about mathematics teachers' knowledge.

What Remains Unknown About Mathematics Teachers Knowledge?

Over the past 25 years, there have been many advances in our understanding of mathematics teachers' knowledge. However, many questions still remain. I focus next on two questions that remain about mathematics teachers' knowledge. First, how do teachers use their knowledge in doing the work of teaching? Second, how do teachers learn to use knowledge in teaching and how is this knowledge developed over time?

How do teachers use their knowledge in teaching? While these efforts do take a more practice-based approach to studying teacher knowledge, there is still a level of hypothesizing distancing them from the knowledge teachers actually use in their classroom teaching. We also do not know how teachers use their knowledge in enacting the work of teaching. This is particularly true at the secondary level, where there has been less research on mathematics teacher knowledge. In this dissertation, I aim not to hypothesize about the knowledge teachers might need for the work they are doing, but to look at the knowledge teachers actually use in doing this work. In doing so, I look both at what this knowledge is and how teachers draw on it in teaching.

Looking at teachers' knowledge from a situated perspective further highlights the importance of looking at how teachers use their knowledge in practice. From a situated perspective, embedded within knowledge are fundamental links to the situations in which it was learned and is used (Brown, Collins, & Duguid, 1989; Greeno, 1997). Further, the embedded contexts within tasks can improve the efficiency with which that task is completed if key features of the context are constant. For example, mathematicians may immediately determine the strategy needed in a proof, yet not able to explain how they knew that particular strategy would work.

Like knowledge and tasks, teaching is situated, in content, students, and classroom settings (Greeno, 1998). Putnam and Borko (2000) argue based on Ball's (1997) claim that students' knowledge is linked to the contexts in which it is learned and used, that teachers' "professional knowledge is developed in context, stored together with characteristic features of the classrooms and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations" (Putnam & Borko, 2000, p. 13). In this view, teacher knowledge is inextricably linked to the contexts, tasks, and practices of teaching. Further, teachers may be better able to use their knowledge for teaching in settings and events that have features in common with teaching than in unrelated settings and events. It seems then, that teachers' knowledge is not only situated in particular contexts, but also varies across teaching and non-teaching situations.

Several studies have documented teachers whose enacted mathematical knowledge in practice differed from the understanding they displayed in research interviews. Interestingly, there is no consistent pattern to these differences. Some teachers have stronger and broader mathematical understanding in classroom contexts than they do in interviews (Hodgen, 2011), while others do not use their documented strong mathematical knowledge in the context of teaching to provide explanations to students, explicitly connect multiple representations, and interpret student work (Borko et al., 1992; Ma, 1999; P. W. Thompson & Thompson, 1994). In other cases, teachers may possess different conceptions of a mathematical concept in and out of a classroom setting (Peressini et al., 2004). Given the differences in mathematics teachers' knowledge across teaching and non-teaching situations, it is necessary to study teacher knowledge as closely to the work of teaching as possible.

Other unknowns about mathematics teacher knowledge. A second area that remains unknown is how teacher knowledge develops and how teachers learn to draw on their knowledge while learning to teach. However, understanding this development is contingent on a strong conceptualization of what this knowledge is and how teachers enact their knowledge in doing the work of teaching.

In order to investigate how teachers use knowledge in practice, I have chosen to focus on specific teaching practices. In the next section, I review research on teaching practices and then focus specifically on my chosen practices, selecting examples and giving explanations.

Teaching Practices

Lampert (2010) gives four distinct meanings of the word practice that are used in education research. First, practice is the opposite of theory. Second, practice is the carrying on of a profession, such as the practice of teaching. Third, practicing as rehearsing in preparation for a future performance. For example, novice teachers often practice components of a lesson and receive feedback before teaching students. Finally, teaching practices are the particular tasks and routines teachers use in carrying out their work. For example, in teacher education there has been work on core or high leverage practices, which are considered to be important for beginning teachers. It is this final meaning that I draw on in this study. In the next section I discuss the ways teaching practices have been conceptualized and studied.

How Are Practices Studied and Conceptualized?

Over the past decade, there has been a renewed interest in practice. The recent focus on teaching practice has taken a few different forms. First, there has been a focus in teacher education on core or high leverage practices, which have the greatest impact on student learning (Ball & Forzani, 2009; Grossman, Hammerness, & McDonald, 2009). The renewed interest in practice has also taken a second form, that of research focused on particular types of teaching. For example, understanding the work involved in equitable mathematics teaching or inquiry based learning, and how these types of teaching require consideration and knowledge of mathematics in relation to the specific pedagogy.

In the first focus on practice, many groups of researchers have curated sets of practices that are most important for beginning teachers as they enter their own classrooms. In curating these collections of practices, teacher educators and researchers considered both the knowledge and practice demands of teaching, particularly for novice teachers. For example, Ball, Sleep, and colleagues (Ball, Sleep, Boerst, & Bass, 2009) discuss the criteria they used for selecting high-leverage practices for mathematics teacher education. They focused on practices that were frequently occurring in teaching, enabled important mathematical work, were aimed to helping all students learn, and could be used within a variety of instructional approaches. They had other considerations due to the context constraints of teacher education, including the accessibility of the practices for novices at their current and future ability levels, in both teacher education courses and field placements, and the ability of the practice to be teachable. From this work, and others, lists of practices have been created. A few of these practices are *leading a group discussion*, *explaining and modeling content, practices, and strategies* and *learning about student understanding* (Ball et al., 2009; Grossman et al., 2009). In addition to curating these sets of practices, much of this work has also considered how novices can be taught these practices (Ball et al., 2009; Grossman et al., 2009). In science, researchers have shown that novices are able to enact these practices as they become beginning teachers (J. Thompson, Windschitl, & Braaten, 2013).

In the second focus on teaching practices, researchers have investigated the impact of particular practices. Research has shown that particular practices do matter for student learning. Some of these practices include cognitively guided instruction (Carpenter & Fennema, 1992), eliciting and interpreting student thinking (Sleep & Boerst, 2012), and particular classroom discourse practices (Hiebert & Wearne, 1993; O'Connor, 1998; Sherin, 2002). Other lines of research have shown that collections of practices matter (e.g. Grossman et al., 2010). While there is research on a few specific teaching practices that have been shown to matter for student learning, overall teaching practices are under researched. In the next section, I discuss some of the areas where the field still lacks understanding around teaching practices.

What Remains Unknown About Teaching Practice?

Given the progress that has been made on teaching practice research, there are several important directions for future work. First, although different lists of core or high-leverage practices have been developed, we do not know which of these practices matter most for student learning. Second, for many of the practices, the components have not been identified empirically and it is unknown if particular components are more or less important for the practice to be successful. Finally, the field lacks understanding of how teachers use knowledge in enacting these practices.

First, there have been several efforts to delineate lists of core or high-leverage practices and to focus in on a few specific practices. These efforts have identified several practices that are important for mathematics instruction and many other practices that are thought to be important. These lists range in length, and while some practices overlap, there are also differences across the lists. Across these lists, we lack an understanding of which practices are most important, both for novices learning to teach and for student learning. More careful study of the individual practices is needed to better understand how they matter in mathematics teaching.

Second, there are many teaching practices that are important for student learning, yet we are only beginning to understand the components of a few of these practices. A better understanding of the components of teaching practices is important not because the pieces matter by themselves, but because understanding the pieces enables greater understanding of the practice as a whole. Further, it will allow for greater emphasis on the most important components in teacher education.

A third area of teaching practice that remains understudied is how teachers draw on knowledge when they enact teaching practices. Given that teachers' knowledge matters for student learning, and instruction also matters, research is needed to understand how teachers use knowledge in enacting particular teaching practices.

This dissertation investigates two specific teaching practices, selecting examples and giving explanations. In the next section I review what is known about each of these two practices.

Selecting Examples

Examples are a critical component of mathematics instruction. In mathematics classrooms they are used on a daily basis to teach, practice, and reinforce concepts and procedures. Examples serve and can be selected for a range of purposes, such as to reinforce general cases of a concept, or to demonstrate exceptional mathematical cases, where students need to think differently about the problem than they would about other related problems. They can be sequenced to increase in complexity or to present a range of problem types to students.

Examples are specific cases of mathematics concepts and procedures, from which students can build a general understanding (Watson & Mason, 2002). In addition to conceptual and procedural examples, examples can also be used for application and reinforcement of underlying ideas (Mohamed & Sulaiman, 2010). "For learning to occur, several examples are needed, not just one; the examples need to encapsulate a range of critical features; and the examples need to be unpacked, with the features that make them an example clearly identified" (Leinhardt, 2001, p. 347). Selecting examples to meet all of these criteria is a complex task, in which teachers draw on many types of knowledge. Because examples are so important to the everyday teaching and learning of mathematics, selecting examples is a foundational practice of the work of teaching mathematics.

Researchers have also looked at teachers' considerations for the examples they chose. Zodik and Zaslavsky (2008) give six types of considerations teachers use when selecting examples. First, teachers might start with an example that is simpler or already familiar to students. Second, teachers select examples that focus on common student errors. Third, teachers select examples that focus on the relevant mathematics being discussed. Fourth, when creating examples during teaching, teachers may try to show an example's generality by selecting values randomly. Fifth, teachers may choose to show unusual cases. Finally, although a less common consideration, teachers also select examples that minimize unnecessary mathematical work.

Despite the importance of examples in classroom instruction, there is much left to understand about how teachers select examples. In studying pre-service elementary teachers' selection and use of examples, Rowland and colleagues (2003) identified three common mistakes: (1) using examples that obscure the role of the variable they are meant to highlight; (2) using randomly generated examples, when purposefully selected examples would be more effective; and (3) using an example for a particular procedure when a different procedure is more appropriate for that example.

Mason and Pimm (1984) discuss that challenges that arise in using specific examples to teach a more general concept. Although a teacher is able to see the general mathematical idea underlying a specific example, students may only be able to see the specific example. Students may therefore learn something about the example, but miss the key mathematical generalization the example was meant to provide. Zaslavsky (2010) describes additional challenges teachers face when using instructional examples. For example, when using random values to make an example generic, the random values chosen can obscure the generality. There may also be a difference between what the teacher is using the example to exemplify and what features of the example students pay attention to.

Despite knowing about the considerations teachers make when selecting examples and the challenges that can arise when using examples, little is known about how teachers carry out the actual practice of selecting examples. Given the important role that examples play in mathematics classroom instruction and learning, and the many issues that can arise when examples are implemented, we need to better understand how teachers select their examples.

Further, teachers' use of examples are thought to reflect their knowledge base (Zaslavsky, Harel, & Manaster, 2006). To fully understand teachers' selection of examples, it is therefore necessary to pay attention to the knowledge teachers use in doing this work.

In terms of their use in classroom instruction, "the generation or selection of examples is a fundamental part of constructing a good explanation" (Leinhardt, 2001, p. 347). I now turn to a discussion of research on explanations.

Giving Explanations

The second practice I investigate is giving explanations. When referring to explanations, I rely on Leinhardt's (2001, 2010) instructional explanations. Explanations are a particular practice that occurs in classrooms and are shared between a teacher and their students. They are ubiquitous in mathematics classrooms. At the high school level, 95% of a nationally representative sample of teachers reported giving an explanation to their class at least once a week (Banilower et al., 2013). Explanations are integral to the work of teaching, and inextricably linked to content. Instructional explanations also convey information about dispositions toward mathematics and how it is done, and norms of mathematical work in a classroom (Schoenfeld, 2010). Unlike selecting examples, explanations must be responsive to students in the moment (Leinhardt, 2001, 2010).

Studies also suggest that good explanations might affect student learning. This is suggested from a case study of one elementary teacher across one unit (Leinhardt, 1987) and through a comparison study of expert and novice teachers (Leinhardt, 1989). In the comparison study, the expert teachers were chosen based on their high student learning gains and there were significant differences across the explanations given by the two groups of teachers.

Leinhardt (2001, 2010) distinguishes between four types of explanations: common explanations, disciplinary explanations, self-explanations, and instructional explanations. Common explanations occur in everyday life in response to naturally arising inquiries. Disciplinary explanations build on common disciplinary knowledge and values. They are used to answer questions of importance in the discipline. Self-explanations are given to oneself and are used in learning at many levels, including memory, meaning, and understanding.

In contrast to common, disciplinary, and self-explanations, instructional explanations have the explicit purpose of teaching content to learners, causing them to differ from other types of explanations. In addition to those given by teachers, instructional explanations can be given by students, textbooks, and other instructional resources. The purpose of an instructional explanation is to teach something to someone who does not yet know it. As such, features of instructional explanations are unique in that, "implicit assumptions need to be made explicit, connections between ideas need to be justified, representations need to be explicitly mapped, and the central query that guides the explanatory discussion must be identified" (Leinhardt 2010, p. 3).

Instructional explanations can be given by a teacher or come about as part of a purposeful classroom discussion involving students. In my analysis, I consider students' contributions to be a part of the explanations. In mathematics, instructional explanations can be prompted by or given about contexts; procedures; representations; properties, such as distributivity; and ways of working in the discipline, such as boundary cases and types of proofs (Leinhardt, 2001).

Instructional explanations usually contain several components. First, they begin with a sense of query around the object (concept or procedure) that will be explained, thereby prompting the explanation. Second, an instructional explanation usually includes an example of the focus of inquiry. Third, the explanation includes a discussion of the focus connecting it to important mathematical ideas or principles. Finally, the explanation includes discussion of the bounds or limits of the concept or procedure, including when it does and does not apply and how it differs from similar concepts or procedures (Leinhardt, 2001).

Leinhardt (2001, 2010) describes instructional explanations as an interrelated system of goals, actions, and knowledge. The goals of an explanation include: establishing the query or problem; connecting the discussion to other relevant knowledge; carefully developing examples; discussing the limitations of the concept or procedure; identifying the central ideas; and discussing potential errors. When giving an explanation, teachers' actions include selecting examples and representations, identifying key features, and making connections. To give an instructional explanation, teachers need the knowledge to accomplish the goals of the explanation, which also includes how to convey the knowledge to students.

Much of the current understanding about explanations is based on comparison studies of expert and novice teachers (e.g. Leinhardt, 1987, 1989) or demonstrates challenges teachers have in giving explanations (e.g. Borko et al., 1992; P. W. Thompson & Thompson, 1994). However, given the prevalence of explanations in mathematics teaching and the impact explanations can have on student learning, more attention must be paid to the practice of giving explanations. Likewise the ways teachers draw on knowledge in giving explanations is understudied. With this dissertation, I aim to better understand the work teachers do and the knowledge they use in giving explanations.

By taking a practice-based view of teacher knowledge, I take the viewpoint that knowledge in inextricably tied to the contexts of practice. In this view, mathematics content can be seen as one component of the context. In this dissertation, I have chosen to focus on the mathematics content of rational expressions and equations at the Algebra II level. In the next section I review literature on this topic.

Rational Expressions and Equations

While there has been a great deal of research on other mathematics topics in the algebra curriculum, such as linear and quadratic equations, rational expressions and equations are understudied. This is in spite of the presences of rational expressions and equations in national standards (National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010).Ruhl and colleagues (2011) documented several different types of errors made by students when simplifying a rational expression by incorrectly reducing terms. Constanta (2012) also found that students made errors resulting from difficulties differentiating terms and factors. Further, a mismatch was found between the errors teachers predict students will make and the errors students do make.

In a study of pre-calculus college students, students had greater difficulty solving algebraic rational expression problems than they did on similar rational expressions containing only numbers (Yantz, 2013). Further, a weak correlation was found between students' ability to solve a numeric item and their ability to solve a similar algebraic item for only one problem set. No correlation was found for the other types of problems, suggesting that the students in the study did not see underlying mathematical relationships between similar numeric and algebraic problems.

The Intersection of Knowledge and Practice

Although the work on teaching practice implicitly considers teachers' knowledge and the work on teacher knowledge is practice-based, there is much left to understand about the relationship between teachers' knowledge and their teaching practice. Many researchers have recognized the deep connections between knowledge and practice, yet it is inherently challenging to investigate their relationship. There has therefore been little research on the relationship between teachers' MKT and their instructional practices. One exception is a recent study by Steele and Rogers (2012) investigating the relationship between mathematical knowledge for teaching and two teachers' classroom teaching practices around proof. Through an exploratory study of one beginning and one experienced teacher, they describe how positioning can mediate teachers' MKT and their instruction. Positioning entails the various roles teachers, students, and outside others take on with respect to the mathematical object being taught, in this case proof and its various roles. This study found that positioning can mediate the relationship between teachers' MKT and their classroom teaching practice. Of note is that this study looked specifically at mathematical knowledge for teaching proof, which was categorized as common and specialized content knowledge (CCK and SCK) in the LMT framework (Ball et al., 2008).

The field has made great progress in recent years in each of the areas of mathematics teacher knowledge and teaching practices. However, much remains unknown about these two areas individually, and about their intersection. In particular, although many teaching practices have been clearly articulated and are viewed as skillful work, the field lacks understanding of the knowledge demands of these practices. Similarly, the field has developed a greater understanding of the types of knowledge teachers use in teaching. Yet, it does not yet know how teachers draw on this knowledge in carrying out the work of teaching, including enacting particular teaching practices. In addition, the field lacks methods for studying the intersection of knowledge and practice.

My Contribution to Research on Teacher Knowledge and Practice

My dissertation contributes to research in mathematics education on teacher knowledge and teacher practice in four ways. First, it is focused specifically on the knowledge teachers use in practice. By using classroom observations with pre- and post-observation interviews to address the knowledge teachers are using in their lessons, this research moves closer to the work of teaching than other studies have been able to.

The second contribution this dissertation makes to the field of mathematics teacher knowledge is methodological. Although knowledge is situated, researching classroom teaching through observations of instruction is more challenging than other types of research due to time and other constraints. By looking at teachers' knowledge in both classroom and cognitive interview settings, this study explores the degree to which mathematics teachers' knowledge and teaching practices, which are situated in practice, can be studied in an interview through situation based items.

Third, it aims to understand components of two ubiquitous practices that have not been the focus of such close study. In this study, I focus on the teaching practices of selecting examples and giving explanations. These two practices are foundational in mathematics teaching as they are used on a daily basis in mathematics classrooms. When presenting content, teachers must exemplify the concept or idea through examples. Similarly, teachers convey the meaning, importance, and use of mathematics content through explanations.

Forth, this dissertation looks at how knowledge is used in enacting specific practices, at both the practice and component level. By looking at both the teaching practices by themselves and in conjunction with teacher knowledge, I aim to delineate the components of enacting each practice and their knowledge demands.

CHAPTER 3: METHODS

Introduction

My study seeks to explore the complex relationship between mathematical knowledge for teaching and classroom teaching practices. It does so by looking at how secondary teachers think mathematically when selecting examples and giving explanations, which are both fundamental to the daily work of mathematics teaching.

I selected these two practices because they each serve a range of purposes and involve multiple domains of knowledge, including specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. Examples are used to present new concepts and procedures, and for practice and reinforcement. They can be chosen to present a range of cases or to increase in mathematical complexity. Like examples, explanations serve a range of purposes for mathematics teachers. They may be given to a whole class while introducing a new topic, to a small group while working on practice problems, or to individual students to remediate a misconception. Some essential features of explanations are that they must be mathematically accurate, at a level that students can understand, and responsive to students in the moment.

In addition to the individual features of each practice, these two practices were selected together because they differ on one key aspect, explanations involve a sort of improvisation in that they happen in the moment and must also be responsive and adapt to student responses as they occur. Although a teacher may select an example mid class in response to student understanding, once selected, the example itself is not an improvisation. These two practices also overlap in teaching in that examples often require explanations and multiple examples can be sequenced as part of a larger explanation.

I investigated these practices within the context of secondary mathematics teaching, specifically, rational expressions and equations. Rational expressions and equations involve a range of mathematical representations and problem types, including algebraic expressions and equations, graphing, and word problems. It is a topic that grows conceptually more complex from its introduction in Algebra I through later use in calculus and requires a careful balance between conceptual and procedural knowledge. The broad range of conceptual and procedural knowledge and representation types are fruitful for this study.

Research Questions

To investigate the relationship between teachers' mathematical knowledge for teaching and how they select examples and give explanations, my main research question is: What mathematical knowledge for teaching is entailed by the instructional practices of selecting examples and giving explanations? To answer this question, I ask four sub-questions:

1. What kinds of work do teachers do in carrying out these two teaching practices?
2. What mathematical knowledge for teaching do teachers draw on in carrying out these two teaching practices?
3. How do teachers use this mathematical knowledge for teaching and reasoning in doing this work?
4. Are there differences across the two practices? How are these differences in knowledge and reasoning related to the demands/work of the practices themselves?

Mathematics Content: Rational Expressions

As previously described, this study is focused on rational expressions and equations at the Algebra II level. Within rational expressions and equations, I selected three subtopics to focus on:

1. Finding discontinuities in and graphing rational equations
2. Simplifying rational expressions, including those involving multiplication and division
3. Solving rational equations and word problems using rational equations

The classroom observations occurred during the teaching of these subtopics and the interview items were based on these subtopics. The three subtopics were selected to address the full scope of mathematics in rational expressions and equations and provide a range of mathematical content, representations, and example types. Keeping the three subtopics constant across the classroom observations allowed for depth across the mathematics that was observed.

Data Collection

To investigate the relationship between teachers' mathematical knowledge for teaching and how they select examples and give explanations, I conducted a qualitative case study of high school Algebra II teachers (Maxwell, 2004). Teachers participated in classroom observations and interviews where they were engaged in the practices of selecting examples and giving explanations. These two methods of data collection were chosen to complement each other. Classroom observations document authentic work of teaching, while interviews can allow for additional insight into participants' in-the-moment knowledge and reasoning.

Participants

The aim of this study was to understand teachers' knowledge use and practice when selecting examples and giving explanations. This is a question of what teachers do and the knowledge they use in everyday high school mathematics teaching. I therefore aimed to recruit everyday high school mathematics teachers. In doing so, I looked for teachers who were currently teaching Algebra II, which included a unit on rational expressions and equations. No other selection criteria were used. Given this selection criterion, what is seen in this data is likely what is regularly occurring in high school mathematics classrooms. However, I do not claim that the data show how these practices should be carried out in mathematics teaching, or that they demonstrate the best ways of doing so. In addition, because this is an analysis of the work of teaching and the knowledge teachers use, it does not evaluate teachers or judge the explanations they gave or their selection of examples.

To recruit participants, I contacted approximately 200 teachers and several district contacts via e-mail. From this recruitment, ten teachers participated in the study. Because two of the teachers had never taught Algebra II, they are not included in the data analysis presented below. The eight Algebra II teachers each taught the topic rational expressions and equations during the year of the study with at least one of their classes. Additional information on teachers' gender, certification, and education can be seen in table 3.1. The teachers had between 3 and 26 years of teaching experience. All of the teachers had taught rational expressions and equations at least twice and half of the teachers had taught rational expressions and equations at least 15 times. Participants' years of teaching experience, approximate years of teaching rational expressions and equations, and school type can be seen in table 3.2.

Table 3.1: Teachers' Certification and Degrees

Participant[a]	Gender	Mathematics Certification	Bachelor's Degree	Master's Degree
Ms. Allen	Female	Secondary Mathematics	Mathematics and Teacher Certification	None
Mr. Baker	Male	Secondary Mathematics	Mathematics Education	Teaching
Mr. Clark	Male	Secondary Mathematics	Statistics and Mathematics	Educational Specialist in Educational Leadership
Mrs. Dayton	Female	Secondary Mathematics	Mathematics and Education	Pure Mathematics
Mr. Johnson	Male	Secondary Mathematics	Mathematics Education	Educational Technology
Mrs. Kelly	Female	Secondary Mathematics	Civil Engineering	Educational Studies with Teacher Certification
Mrs. Stone	Female	Secondary Mathematics	Mathematics Education	Teaching
Mr. Zimmer	Male	Secondary Mathematics	Mathematics and Teacher Certification	None

a. All names are pseudonyms.

Table 3.2: Participants' Teaching Experience and Current School Type

Participant	Years of Teaching Experience	Approximate Years of Teaching Rational Expressions and Equations[a]	School Type
Ms. Allen	5	2	Public High School
Mr. Baker	19	19	Public High School
Mr. Clark	21	21	Public High School
Mrs. Dayton	23	>15	Public High School
Mr. Johnson	22	3	Public High School
Mrs. Kelly	3	2	K-12 Charter School
Mrs. Stone	26	15 - 18	Public High School
Mr. Zimmer	3	3	Public High School

a. The larger numbers are estimates and several participants mentioned teaching rational expression and equations multiple times in some years.

Interviews

I conducted individual interviews with each teacher participant. In the interviews, participants were presented with classroom situations and asked to take on the role of the teacher in the classroom. The interviews allowed for more detailed information about teachers' in-the-moment knowledge and reasoning than would be visible through an observation because the teachers were able to talk about their thinking while engaging in the teaching practices. This method of data collection enabled me to probe more deeply for the ways in which teachers think about and engage in the practices of giving explanations and selecting examples and the knowledge resources they use in doing so.

In this section, I begin by describing my process for writing, piloting, and revising the interview items. I then describe the interviews I conducted with participants using the items. The interviews also included additional questions about participants' teaching and education backgrounds.

Item writing, piloting, and reviewing. I created interview items to address the two practices of selecting examples and giving explanations because there were no existing items. The goal in writing the items was to create items that approximate the work of teaching so that teachers' teaching practice, knowledge, and reasoning could be investigated outside of a classroom context. Each item presented a classroom situation and asked the participant to take on the role of the teacher in that situation. They were either asked to select examples for teaching a particular topic or plan and give a short explanation of a particular concept or procedure. There were three focal content subareas, reflective of the content subareas described above. There were two items focused on each of the first two content subareas (finding discontinuities in and graphing rational equations, and simplifying rational expressions), one addressing selecting examples and the other addressing giving explanations. For the content subarea of solving rational equations and word problems using rational equations, I chose to create items on both solving rational equations and solving word problems using rational equations because it was unclear how often word problems were included in units on rational expressions and equations.

This yielded eight total items. See table 3.3 for more detail on the teaching practice and content subarea addressed by each item. The interview items can be seen in Appendix 2.

Table 3.3: Teaching practices and content subarea focus for each interview item. Bold text indicates the specific area of the item.

Item #	Teaching Practice	Content Subarea
1	Selecting examples	**Finding discontinuities** in and graphing rational equations
2	Selecting examples	Simplifying rational expressions
3	Selecting examples	**Solving rational equations** and word problems using rational equations
4	Giving explanations	Finding discontinuities in and **graphing rational equations (**of the form $y = \frac{a}{x-h} + k$)
5	Giving explanations	Simplifying rational expressions
6	Giving explanations	**Solving rational equations** and word problems using rational equations
7	Selecting examples	Solving rational equations and **word problems using rational equations**
8	Giving explanations	Solving rational equations and **word problems using rational equations**

After drafting the items, they were revised several times based on feedback from other researchers. I then conducted pilot interviews and revised the items further based on the pilot interviews. Pilot interviews were conducted with other graduate students in mathematics education, all of whom were former teachers. These participants were selected because they had experience teaching the content of rational expressions and equations (at the high school or college level) or high school mathematics (but not necessarily rational expressions and equations). There were seven pilot participants and they each piloted different numbers of items. The items were revised during and after the pilot process to make them more realistic to the work of teaching, as well as clearer and less time consuming. More detailed information about how the items were revised can be seen in Appendix 3.

Teacher interviews. The interviews ranged from approximately one hour and twenty minutes to two hours. Each interview was video and audio recorded and all artifacts (copies of the items with participants' written work) were collected. The full protocol can be seen in Appendix 1. Most participants were presented with all of the items, but due to time constraints, three participants did not complete the final 1-2 items.

In the interviews, I asked each participant about their background and teaching experiences. Participants were also asked about their experiences with the two practices during their teacher preparation and any professional development they participated in. A few of the participants recalled discussing selecting examples during their teacher preparation and one participant recalled discussing explanations. None of the participants had participated in professional development around either selecting examples or giving explanations. Participants' experiences with the topics of selecting examples and giving explanations during their teacher education and professional development can be seen in table 3.4. Participants reported using a wide range of sources and resources for the examples and explanations they use in their teaching, including textbooks, colleagues, and the internet. The resources used by each participant can be seen in table 3.5. Because the interviews were specifically focused on rational expressions and equations, I asked participants about the topics included in their regularly taught unit. All participants reported that their unit included graphing rational equations; simplifying rational expressions; and solving rational equations. Most teachers also included finding discontinuities in rational equations in their unit. While a few teachers reported that their unit included using rational equations to solve word problems, other teachers said they included few or no word problems. The topics included in each participant's unit can be seen in table 3.6.

Table 3.4: Participants' encounters with selecting examples and giving explanations during their teacher education and professional development. Participants were asked if they discussed either practice during their teacher preparation and if they had participated in professional development focused on either practice

	Teacher Education		Professional Development	
Participant	Selecting Examples	Giving Explanations	Selecting Examples	Giving Explanations
Ms. Allen	Yes	No	No	No
Mr. Baker	No	No	No	No
Mr. Clark	No	No	No	No
Mrs. Dayton	No	No	No	No
Mr. Johnson	No	No	No	No
Mrs. Kelly	Yes	Yes	No	No
Mrs. Stone	Maybe	No	No	No
Mr. Zimmer	No	No	No	No

Table 3.5: Sources and resources used by participants when selecting examples of planning explanations

Participant	Current textbook and related resources	Other textbooks	Colleagues	Make up own problems	Internet resources	Problem generating software	Teaching experience	Own experience as a student	Resources from research projects	Graphing software
Ms. Allen					X	X			X	X
Mr. Baker	X	X	X	X			X			
Mr. Clark	X			X			X	X		
Mrs. Dayton	X	X	X			X	X			
Mr. Johnson			X	X		X	X	X		
Mrs. Kelly	X		X	X	X					
Mrs. Stone	X	X	X				X			
Mr. Zimmer	X		X		X			X	X	

Table 3.6: Topics included in each participant’s unit on rational expressions and equations

Participant	Finding discontinuities in rational equations	Graphing rational equations	Simplifying rational expressions	Adding and subtracting rational expressions	Multiplying and dividing rational expressions	Solving rational equations	Using rational equations to solve word problems	Rational Inequalities	Inverse Variation	Joint variation
Ms. Allen	X	X	X	X	X	X	X			
Mr. Baker	X	X	X	X	X	X	X	X		
Mr. Clark		X	X	X	X	X			X	
Mrs. Dayton	X	X	X	X	X	X	Some			
Mr. Johnson	X	X	X	X	X	X				
Mrs. Kelly	X	X	X	X	X	X	Some			
Mrs. Stone		X	X	X	X	X	Not much		X	X
Mr. Zimmer	X	X	X	X	X	X	Not much			

Classroom Observations

The classroom observations allowed me to observe the teaching practices as they were carried out in teaching. The pre- and post-observations interviews provided additional insight into the knowledge and reasoning teachers used in carrying out these practices during teaching, including the changes they made during the lesson.

Five of the teachers were observed one to three times while teaching lessons on rational expressions. This number was chosen to provide a range of data from the observations while keeping data collection and analysis manageable. All observed lessons were classified within the three subtopics described above: (1) Finding discontinuities in and graphing rational equations; (2) Simplifying rational expressions, including those involving multiplication and division; and (3) Solving rational equations or solving word problems using rational equations. See table 3.7 for the lesson topics for each observed lesson by teacher. Two of the observations, both of the same teacher were not included in the analysis because the observations did not contain any instances of whole class explanations.

Table 3.7: Lesson Topics for Each Observed Lesson by Teacher

Participant	Observation 1	Observation 2	Observation 3
Mr. Baker	Graphing rational equations	Simplifying rational expressions	Solving rational equations
Mr. Clark	Graphing rational equations	Simplifying rational expressions	Solving rational equations
Mr. Johnson	Graphing rational equations	Graphing rational equations	
Mrs. Stone	Solving rational equations		

In addition to the classroom observation, teachers participated in pre- and post-observation interviews for each lesson. All observed lessons were video recorded and the interviews were audio recorded. Artifacts from each lesson, such as lesson plans, PowerPoint slides, and student worksheets were collected. A few of the teachers also provided copies of student homework or assessments, which will be used in a future study.

I conducted pre-observation interviews either the day before or the day of the observation, once teachers had planned their lesson. The pre-observation interview focused on the teacher's decisions about the lesson in three areas: the teacher's overall plans for the lesson, the examples the teacher planned to use, and the explanations they planned to give. Questions on the overall lesson plan included lesson goals, parts of the lesson the teacher anticipated might be more challenging for students, and how the lesson fit into the current unit. In addition to being asked about the specific examples they planned to use, teachers were asked where the examples came from, why they were selected or created, and how the examples were related to the overall lesson goal. Finally, with respect to their explanations, teachers were asked the extent to which the explanations were planned out, the key components they planned to include, where their explanations came from, and what they thought about while planning their explanation. See Appendix 4 for the full interview protocol.

After the lesson, I selected four segments of each lesson that were part of an explanation or showed an example being used. I created a short video clip of each segment, and these clips were used in the post-observation interview. I conducted the post-observation interviews as soon after the observation as they could be scheduled. For some observations, it was possible to do the post observation interview later in the day. Most of the nine post-observation interviews were scheduled within a week, with two taking place over a week later. Each post-observation interview began by asking the teacher to share their thoughts on the lesson. I then asked each teacher about their examples and explanations, how they were similar to or different from what they had planned, and what they might do differently if they were to teach this lesson again in the future. During the rest of the interview, teachers were shown the video clips of their teaching and asked about their thoughts and decisions at that point in the lesson. Depending on the content of the video clip, the questions focused on the explanation being given, the example that was used, or both. See Appendix 5 for the full interview protocol.

Data Analysis

The data analysis looked at the components of each teaching practice and the knowledge teachers used in enacting each practice. Portions of the interviews, where participants gave explanations and selected examples were professionally transcribed, as were the pre- and post-observation interviews. I used a grounded theory approach to data analysis (Corbin & Strauss, 2008). Coding was an iterative process (Miles & Huberman, 1994) and took different paths for each of the practices. As mentioned above, not all of the data collected were included in the analyses. Table 3.8 shows the subset of data used for each analysis. Two additional teachers participated in the interviews, but their data were not included in the analysis because they did not teach Algebra II at the time of the study. In addition, the two word problem interview items, one of which was focused on selecting examples and one on giving explanations, were not included in either of the analyses because several of the teachers were not shown these two items due to interview time constraints. In addition, several of the teachers did not include word problems in their unit on rational expressions and equations and were therefore less familiar and comfortable discussing their teaching of this content. In the two sections that follow which detail the different analyses, I describe in more detail how I decided which data to use.

Table 3.8: Data Used in Analyses

Teaching Practice	Data Source	Data Collected	Total Collected	Group for Analysis
Selecting Examples	Interviews	Video and artifacts from interview	3-4 items per teacher for 10 teachers	3 items per teacher for 8 teachers[a, b]
Giving Explanations	Classroom Observations	Video and artifacts from instruction Pre- and post-observation interviews	11 observations across 5 teachers	Video and artifacts from 9 observations across 4 teachers[c]
Giving Explanations	Interviews	Video and artifacts from interview	3-4 items per teacher for 10 teachers	3 explanations per teacher for 4 teachers[b, d]

a. Two teachers were not included in the analysis because they did not rational expressions and equations at the time of the study.
b. The word problem item was not included because not all teachers had time for the item in their interviews.
c. Two observations were not included because they did not contain any instances of whole class explanations.
d. Interview data were included only for teachers who were also observed.

Selecting Examples

Although the intention was to look both at how teachers select examples in their actual teaching practice and in interview settings, the classroom observation data, as well as the pre- and post-observation interviews, were not used in the final analysis. Many of the teachers had taught the same content before. In the observations, because they were familiar with teaching the content and had materials prepared from previous years, the teachers were not selecting new examples for their lessons in the unit. It is challenging to observe the practice of selecting examples when teachers are not selecting new examples for their lessons, but are instead using examples from a previous year or course. The analysis focused instead on the interview items, where participants were asked to select new examples for particular purposes within the unit of rational expressions and equations. These interview items provided the opportunity to observe participants engage in the practice of selecting examples for content they were familiar with.

Unlike the classroom observations, participants engaged in the full work of selecting examples instead of starting with a set of examples they had used the previous year and modifying the set as needed.

I began the selecting examples analysis by open coding components of the practice of selecting examples and of the knowledge teachers drew on in carrying out the practice. When identifying the knowledge teachers drew on, I used the lens of mathematical knowledge for teaching (Ball et al., 2008). I developed two sets of codes from the open coding, one for components of the practice and another for knowledge and then applied them iteratively to sections of the transcript. Codes were clarified and modified, and additional codes were added as needed.

For the final coding, segments of transcript were coded based on the component(s) of the practice the teacher was enacting. Although in some cases more than one practice component applied to a particular transcript segment, the segments were split so that as few components as possible applied to each segment. Often, this was only one component. Segments ranged from part of a sentence to several paragraphs in length. On a second pass through the transcripts, each segment was also labeled with all of the knowledge types the teacher was drawing on in that segment.

After all of the transcripts were coded for both components of the practice and knowledge, I looked for patterns across teachers in the components of the practice they enacted and the knowledge they drew on. To do this, I first analyzed the components of the practice. In particular, I looked at frequencies of different components of the practice in three ways: overall, by teacher, and by item. I also created a table to see the frequencies of co-occurrences of components of the practice, which were few. Second, I focused on the knowledge codes, looking at the frequencies of different types of knowledge overall, by teacher, and by item. I also looked at co-occurrences of knowledge codes. Third, I looked at the types of knowledge teachers drew on when enacting different components of the practice using a table that showed the frequencies of each type of knowledge for each component of the practice. Finally, I looked at tables showing the frequencies of each type of knowledge for each component of the practice for each individual teacher. Many of the numbers in these tables were small. Although these data are likely too fine-grained and specific to the individual teacher to be analyzed closely, I used these tables not to draw new conclusions, but to look for patterns confirming those seen in the larger data set.

The aim in this analysis was to look closely at the practice of selecting examples to begin to understand the work involved in carrying it out. Similarly, I aimed to understand the kinds of knowledge teachers used in doing this work. My findings suggest that there are different patterns in how teachers carry out this practice and the knowledge they draw on in doing so. However, the data is not large enough to support statistical testing of these patterns. Future work with a larger data set may be able to determine if these patterns and differences are statistically significant. I did not include this in my study design because my focus was on better understanding the work of selecting examples and not on differences across teachers.

Giving Explanations

My initial attempts at analyzing the practice of giving explanations were focused on creating a coding scheme of components of the practice and then a second coding scheme focused on the knowledge teachers used. While an initial coding scheme was created, it was challenging to apply to the transcripts of the explanations because multiple components of the practice could be applied to most parts of the transcript. The same was true of the knowledge teachers used in their explanations. Having so many components of the practice apply to each part of the transcript made an analysis of how teachers were enacting these components and their knowledge use unfeasible. While looking and trying to code the transcripts, I noticed different patterns in what parts of the mathematics teachers explained and the mathematical depth of these explanations. The final analysis focused on the types of explanations teachers gave and the levels of mathematical reasoning they included.

The classroom observation data provided a large set of explanations to analyze. In contrast, while there were 110 explanations across the nine classroom observations, each teacher gave only three explanations in the interview. Given the large difference in quantity for teachers who were and were not observed, I chose to only look at the interview explanations for the four teachers who were also observed during their classroom instruction. I wanted to give depth to the analysis and be able to see larger patterns in how teachers give explanations. For teachers who were not observed, three explanations was too small of a sample size compared to the other teachers, who had anywhere from 15 to 45 classroom explanations to analyze. In addition, including the interview data for these four teachers allowed me to compare how the teachers' explanations in the interview setting compared to the explanations they gave during classroom instruction.

Analysis of the explanations began by marking explanations in the transcript and labeling their purpose. A segment of the transcript was considered one explanation if there was one instructional purpose for the entire segment. Purposes included how to find the domain of a function, how to simplify a rational expression, and how to graph a rational equation given known values. Explanations ranged in length from a sentence to several pages. For each of the interview items, each participant's entire response to the prompt, not including any follow up questions, was considered one explanation.

After all of the explanations were marked, explanations were categorized based on the level of mathematical depth they contained. The explanation categories are described in greater detail in chapter 5. I looked for patterns in the types of explanations given by each teacher in both the interview and their observations. Two of the explanation types, procedural explanations and mathematical reasoning explanations, were looked at more closely to better understand the knowledge teachers drew on in giving the explanations. Of the four explanation categories, procedural explanations have the lowest level of reasoning while still being mathematically correct. In contrast, mathematical reasoning explanations contained the highest level of reasoning. These two types of explanations were both looked at to see if expected differences occurred in the knowledge teachers drew on in giving each type of explanation. In particular, it was expected that teachers would draw on a wider range of knowledge when giving mathematical reasoning explanations and that they might not draw on specialized content knowledge when giving procedural explanations. The types of explanations, frequencies of explanation type by teacher, and patterns seen in teachers' knowledge use across these two types of explanations are described in Chapter 5.

CHAPTER 4: THE PRACTICE OF SELECTING EXAMPLES

I begin this chapter by presenting two cases of teachers selecting examples. I then discuss components of the practice of selecting examples. In this section, I describe the components I found and look at patterns in how teachers enact different components of the practice. In the third section, I focus on teachers' knowledge use while selecting examples. I both describe the knowledge on which teachers seemed to draw and look at patterns in knowledge use. In the fourth section, I look at the relationship between components of the practice of selecting examples and the knowledge on which teachers draw in enacting the practice. In the fifth and final section of this chapter, I look at differences across how teachers select examples and the knowledge they draw on in doing so.

Cases of Teachers Selecting Examples

In this section, I present two cases from my study of teachers selecting examples for a specific purpose. These two cases are included to show some of the different things teachers did when selecting examples. In the first case, Mrs. Kelly selected examples to demonstrate different types of discontinuities (Example Item 1). In the second case, Mr. Zimmer selected examples to teach solving rational equations (Example Item 3). The full prompts for each interview item can be seen in Appendix 2.

Mrs. Kelly and Finding Discontinuities: A Case of Demonstrating Specific Mathematical Ideas

After reading the prompt asking her to select examples to demonstrate different types of discontinuities, Mrs. Kelly immediately mentioned three different mathematical cases she would show students and the order in which she would use them, “They've seen what the graph for a rational function looks like, so I would probably start with asymptotes first and then holes, and then combine the two.” For her first example, showing an asymptote, Mrs. Kelly created the problem $y = \frac{3}{x-4}$, which is an example of a vertical asymptote. She also wanted her example to show the new content without distracting students with other mathematics and created this problem with a constant numerator and a linear denominator “because then we're just looking at discontinuities instead of having to involve factoring also.”

For her second example, Mrs. Kelly selected the problem $y = \frac{x^2 - 8x + 12}{x - 2}$, which has a hole, or removable discontinuity. It also requires students to factor the numerator before they can simplify the fraction. She discussed how she would use the problem with her students to bring up the mathematical concept of holes.

> So they have to factor and then realize that once they've canceled out factors that it's linear but there's this problem with the denominator so then we can talk about what would happen in that space, and we can talk about how there's a hole there.

Finally, Mrs. Kelly selected the problem $y = \frac{x^2 + 5x + 6}{x^2 - x - 12}$ as an example of an equation with both types of discontinuities, holes, which divide out, and asymptotes, which do not. She wanted students to “see what happens when you had to cross something out like you had discontinuity, there's like a hole and then you had an asymptote.” When asked what problem she might use if she were to pick a fourth example, Mrs. Kelly said she “would probably do one with two asymptotes” and decided on the problem $y = \frac{2}{x^2 - 4}$.

She summarized her overall selection as, “I would do one with asymptotes, one with holes, one with both and then probably start talking about two asymptotes.” Each of Mrs. Kelly’s examples was selected to match a particular type of problem she wanted to show students. She also sequenced these problems in a specific order, first using an example that has an asymptote, then one that has a hole, and finally showing an example with both an asymptote and a hole.

Mr. Zimmer and Solving Rational Equations: A Case of Common Student Misconceptions

After reading the prompt asking him to select examples to teach solving rational equations, Mr. Zimmer spent about a minute and a half looking over the set of problems and said he would use problems (a) [$\frac{-8x+15}{x^2-3x}=\frac{x}{3-x}$], (b) [$\frac{2}{x+3}-\frac{1}{x}=\frac{1}{4x}$], and (c) 26 [$\frac{3}{x^2-7x+10}+2=\frac{x-4}{x-5}$]. Mr. Zimmer said he chose each of the problems "based on what problems my students run into" and more generally described that he selects problems by anticipating what students will have difficulty with or have struggled with in the past. He said he would start with problem (c), then use problem (b), and end with problem (a) and talked about the problems in that order when he described why he selected each of them.

When talking about problem (c), Mr. Zimmer began by describing the two methods he has shown students for solving rational equations:

> Sometimes we get a common denominator and then kinda go through and cross multiply. Sometimes we find a common denominator, but use it to reduce or cancel out all of my denominators. That's what I've gone with recently, is just find something we could multiply straight across by everything.

Although he wasn't entirely sure that problem (c) would lead to a quadratic equation, he selected it because,

> When they end up with a quadratic left, they don't know what to do with it. […] I would use that so that when it ends being a quadratic, they still see ok it's a quadratic we can still solve it, just like we've done before and not get the sense that every problem is going to be a linear equation that we end up with.

Mr. Zimmer selected problem (b) because he has seen students have difficulty differentiating the factors of x and $x+3$, “Sometimes I’ll have students want to multiply this [the term $\frac{1}{x}$] by 3.” He went on to clarify that students would think they do not need a factor of x in the least common denominator because x is part of the factor $x+3$. Students would therefore use the least common denominator of $4(x+3)$ and when multiplying by the term $\frac{1}{x}$, they would incorrectly reduce the xs in the terms x and $4(x+3)$.

Finally, Mr. Zimmer selected problem (a) because the denominator of the second fraction, $3-x$ has a negative x, which is something students typically struggle with, “They typically having a lot of trouble dealing with that. They don't know to switch it and then factor out the negative.” After deciding to use problem (a) as the last problem, he also mentioned that he would not start with a problem where students can solve using cross multiplication because,

> Sometimes I get nervous about cross multiplying and that they're just going to resort to that every time. Showing them that this method exists and they kind of transfer over and use it here [on problems with three different terms].

In selecting examples, Mr. Zimmer picked problems that would bring common student misconceptions and errors up before students could make the errors themselves. Like Mrs. Kelly, he also sequenced the problems, but his purpose in sequencing was focused on preventing student misconceptions. In particular, he saved the proportion problem for last because he did not want his students to misapply cross multiplication to other problems.

These two cases demonstrate some of the things teachers did when selecting examples, including sequencing problems, thinking about common student misconceptions, and thinking about pieces of mathematics that are part of the larger purpose or goal of the lesson. In the next section, I describe the different things teachers did when selecting examples.

Components of the Practice of Selecting Examples and Teachers' Enactments of the Components

In this section, I discuss the components of the practice of selecting examples. I then look at patterns in how teachers enact these components. This analysis is based on teachers' responses to the selecting examples items in the interview.

Components of the Practice of Selecting Examples

The components of the practice of selecting examples can be seen in table 4.1. As discussed in the Chapter 3, these components were developed based on what the teachers did when they selected examples in the interviews. Below I describe each component with a few examples from the data.

Table 4.9: Components of the Practice of Selecting Examples

Components of Selecting Examples
Thinking about the end goal
Thinking about pieces of the end goal (components, sub goals, or scaffolds)
Thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem
Noticing a subset of problems with a particular characteristic and deciding they fit a new sub goal
Noticing a subset of problems with a particular characteristic and deciding not to use that type of problem
Finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic
Looking at a set of problems for interesting features in the set
Evaluating the features of a problem, including comparing to one or more other problems
Creating or modifying a problem to match a desired (sub) goal
Sequencing problems
Describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.)
Solving a problem

Thinking about the end goal. Each of the interview item prompts discussed a lesson goal before asking teachers to select examples for that purpose. The first interview item asked teachers to select examples to demonstrate different types of discontinuities. The goal of the second problem was for students to be able to simplify rational expressions, including those involving multiplication and division. The third interview item goal was for students to be able to solve rational equations. The full prompts for each interview item can be seen in Appendix 2.

Several of the teachers discussed an overall goal that they wanted students to learn from a particular lesson. For example, when responding to the prompt asking her to select examples to teach simplifying rational expressions, including those involving multiplication, Ms. Allen said, “The goal of learning to multiply and divide polynomials is really understanding the structure of reducing. Not so much the action of the multiplying and dividing because multiplying and dividing is a cover phrase for factor and cancel” (Ms. Allen Example Item 2). Similarly, when responding to the prompt asking the teacher to select examples to introduce students to different types of discontinuities, Mr. Clark stated that, “The main purpose for me is to make sure they understand the bottom needs to be equal to 0” (Mr. Clark Example Item 1). For the same interview item, Mrs. Stone stated, “Our goal is just simplifying” (Mrs. Stone Example Item 1). References to the end goal were similar across the three interview items. Two teachers more frequently discussed the end goal, while others did so infrequently or not at all.

Thinking about pieces of the end goal (components, sub goals, or scaffolds). All of the teachers also talked about particular pieces of the larger goal that they saw as important or planned to address. This sometimes took the form of components or sub goals. After reading the first interview item, which asked teachers to select example to teach different types of discontinuities, but before looking at any of the problems, Mr. Zimmer listed several sub goal, “I'm thinking about horizontal asymptotes. I'm also thinking about holes and having something that would reduce to be one. I'm also thinking about multiple vertical asymptotes” (Mr. Zimmer

Example Item 1). Holes and asymptotes are two common types of discontinuities and addressing them would likely be sub goals for this topic. In some cases, teachers discussed sub goals as scaffolds to be used in teaching the particular topic. For example, when prompted to select problems to teach solving rational equations, Mrs. Stone mentioned that she wanted a problem that was a proportion and another where one of the three terms was a constant, with a denominator of 1.

Thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem. As they were selecting examples, the teachers all thought about common student errors and places where they might struggle, as well as places where the students were likely to have success. This occurred both within specific problems. For example, when evaluating the problem $\frac{x^2-25}{x^2-10x+25}$, Mrs. Stone was focused on a common student misconception:

> You see a *25* in the numerator and denominator […] This is a common mistake that kids have. They don't factor. They see things that are similar and they go "Oh, I can cancel." So they want to eliminate. In [this problem] they would just cross out the *25*s. "Those are the same. If I cancel out the *x* squared I have *1* over negative *10x*" and they're done with the question. (Mrs. Stone Example Item 2)

In the misconception Mrs. Stone describes, students often see and reduce common terms, which is not mathematically valid, instead of first factoring the problem and then reducing common factors.

Teachers also made broader comments about students' errors and places where they would have difficulty or success. For example, when deciding what types of factoring she wanted to include in examples for simplifying rational expressions, Mrs. Kelly commented that, "Most of my students can factor smaller things really quickly" (Mrs. Kelly Example Item 2). She chose to include problems where the factoring was less complex to put the focus on the new content of simplifying.

Noticing a subset of problems with a particular characteristic and deciding they fit a new sub goal. When looking at the resource of problems given with a particular interview prompt, some of the teachers noticed a particular type of problem. They then decided that this problem types was something that they wanted to show their students. For example, when prompted to select examples to teach solving equations, both Mr. Baker and Mrs. Dayton noticed several problems where two rational expressions were set equal and decided that they wanted to include an example where students can cross multiply. These problems are different than the rest of the problems in the set, which cannot be solved using cross multiplication. When prompted to select examples for simplifying rational expressions, Mr. Zimmer noticed a set of problems where the factoring only involved greatest common factors, with either a binomial in the numerator and a monomial in the denominator, or the reverse. He then decided to include one of each type.

Noticing a subset of problems with a particular characteristic and deciding not to use that type of problem. Similar to the previous component, teachers noticed a particular type of problem. However, instead of deciding to show the problem to their students, they decided they did not what to show students a problem with that particular characteristic. For example, when prompted to select examples to teach solving rational equations. Mr. Clark noticed the set of equations that could be solved using cross multiplication and decided not to include one of them. He did so because, “My kids are going to misuse it [cross multiplication] and so I almost treat those as two different problems, the way I've always taught that” (Mr. Clark Example Item

3). He did not want his students to mis-apply the cross multiplication procedure to problems where it would not work and instead chose to teach cross multiplication as a separate topic.

Finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic. In this component of the practice, after naming a particular sub goal, or type of problem, teachers looked for one or more problems that matched the sub goal. Alternately, teachers also picked one problem out of a subset with a particular characteristic. In some cases, they had particular reasons for focusing in on a specific problem. For instance, after describing her criteria for a first example for simplifying rational expressions, Mrs. Kelly decided on a particular problem. Her criteria included an example where the factoring was easier, so that students would be able to focus on the simplifying.

> It might be something like [problem] 21 where it's not too difficult factoring but they'll factor, they'll be able to see there's common factors. So *x* squared plus *7x* plus *12*. Or *x* squared minus *6x* minus *27*. Because then they could factor. (Mrs. Kelly Example Item 2)

In other cases, teachers just chose a problem that met the overall goal without much consideration of the particular features of the problem. For example, after Ms. Allen decided that she wanted a rational equation where one of the terms was a constant, the interviewer asked which problem she would pick. Ms. Allen responded that she would pick $\frac{5}{x^2+x-2}=\frac{1}{x+2}+1$, "for no reason. I have no good reason to choose that over the others. Just picking one" (Ms. Allen Example Item 3). This problem did have a constant term, which met the criterion she discussed, however other problems also met this criterion but differed from the problem she picked in other ways.

Looking at a set of problems for interesting features in the set. A few of the teachers looked more broadly across a set of problems for interesting features instead of selecting a particular type of problem or sub goal ahead of time. These features included numbers or expressions likely to bring out common student misconceptions or require a specific type of factoring, or even which answers "are good outcomes" (Mr. Johnson Example Item 2). These teachers looked across the set after first reading the prompt to find problem features that stood out and the also looked across the set of problems after selecting a few problems, "looking for something that provides a new twist" or "looking for a skill that I might not have caught" (Mr. Clark Example Item 2).

Evaluating the features of a problem, including comparing to one or more other problems. When evaluating the features of a problem, teachers were focused on the particular features of one problem, although they did often compare that problem to other problems that were similar in some way. For example, when evaluating the problem $\frac{2-x}{x^2-4}$, Mrs. Stone commented that she liked the problem because the factors "have the reverse order, the opposites, the *2* minus *x* and *x* minus *2*" (Mrs. Stone Example Item 2). When prompted to select examples to teach solving rational equations, Mrs. Kelly compared problem 9 [$x+2=\frac{48}{x}$] with problem 27 [$\frac{12}{x^2+x-20}=\frac{2x+6}{x+5}-3$], which both contain terms with a denominator of one, before deciding on problem 27.

> I'd probably do [problem] 9, because I liked that it was, the solving, you have to solve a quadratic. […] I thought there was probably a better one. Let me see. I might do something like [problem] 27 actually. I think [problem] 9, I wouldn't want to do [problem] 9 because it's really, they're already had to multiply by a binomial and then this would feel like I'm going backwards because they're multiplying, the common denominator is really easy to find. (Mrs. Kelly Example Item 3)

After multiplying to clear the fractions, both problems result in a quadratic equation. Mrs. Kelly decided on problem 27 instead of problem 9 because the work of clearing the fractions was a bit more complex in problem 27. In problem 9, to clear the fractions, students would only need to multiply $x+2$ by x, which involves multiplying a single term by a binomial. Because they would have already done more complex multiplication in a previous example Mrs. Kelly picked, she decided that this would be moving the difficulty level in the wrong direction.

Creating or modifying a problem to match a desired (sub) goal. When selecting examples, a few of the teachers made up their own problems instead of selecting problems from the problem set provided. For example, Mr. Baker created his own problems to teach discontinuities, starting with the problem $y = \frac{2}{x+1}$ and later using the problem $y = \frac{x^2 - 4}{x^2 + 3x + 2}$, which were not on the list of problems. "So the first thing I would probably do is look at a function where we're just looking at maybe a basic *2* over *x* plus *1* where we just have one vertical asymptote" (Mr. Baker Example Item 1). He later described another problem he would use, which he created:

> I might give them a function where it's maybe say *x* squared minus *4* over *x* squared plus *3x* plus *2*, where now suddenly there's a factor that reduces, so not only are we looking at discontinuities with asymptotes, we're looking at a hole in the graph, a removable discontinuity. (Mr. Baker Example Item 1)

Teachers also modified problems from the set, picking a problem they liked and then adjusting particular values. For example, Mrs. Dayton decided to modify the problem $\frac{2}{x+3} + \frac{1}{x} = \frac{1}{4x}$.

"Maybe have, rather than *x* plus *3*, an *x*, and a *4x*, I might have an *x* plus *3*, an *x*, and an *x* plus *3* for the denominator" (Mrs. Dayton Example Item 3). In the modified version of the problem, the least common denominator would only have two factors, $x+3$ and x. I was therefore less complex than the given problem, which would have a least common denominator with three factors.

Sequencing problems. When picking their examples, teachers explicitly talked about how they would use the examples to develop larger mathematical ideas, this included discussing the order in which they would use their examples. For example, Mr. Zimmer said he would use the problem $\frac{3x^2+21x-90}{3x^2+31x+10}$ as his first example for simplifying rational expressions "because I like the idea of them seeing breaking, writing it down into factors, and then using that same idea to break these down" (Mr. Zimmer Example Item 2). Mr. Baker discussed how he would sequence problems to teach solving rational equations.

> Here I would probably start with maybe something similar to number 14 [$\frac{1}{3}+\frac{x}{6}=\frac{20}{x}$], but I would maybe include integers in all the denominators to begin with just because it essentially becomes a linear equation with fractions. We did that back in the beginning of class where we talk about we have fractions, they don't like fractions, we get rid of the fractions. We multiply, use the property of equality to multiply and get rid of the fraction, so I might start out with a problem like that where it's something they're familiar with, and then all of the sudden I would come to something like number 17 [$\frac{2}{x}+\frac{2}{2x}=3$] where now here are some variables. Let's do the same thing with the variables. (Mr. Baker Example Item 3)

For these problems, Mr. Baker chose to start with something that would be familiar to students, an example with no variables in the denominators, which simplifies to a linear equation. He then would move to a similar example that is more complex because the fractions did have variables in the denominators.

Describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.). As they selected examples to use, some of the teachers talked in great detail about how they would use the problem with their students. This included things they might point out to students and patterns they would like students to see. For example, when prompted to selecting examples to teach solving rational equations, Mrs. Dayton described how she would work through the problem $\frac{x}{3}+\frac{x^2}{2}=\frac{1}{6}$ with students. "The first step that I have them do is, 'What's the common denominator?' And set it up like that. Say, 'Okay, the common denominator is *6*. Now, what did you multiply by to go from here to here?'" (Mrs. Dayton Example Item 3).

Solving a problem. While deciding which problems to use with students, a few of the teachers solved the problems, or worked through part of the problem to be able to see some of the problem' features. For instance, Ms. Allen factored all of the problems in the set for the interview item on finding discontinuities.

How Teachers Enact Components of the Practice of Selecting Examples

The percent of segments where teachers were carrying out each practice component can be seen in table 4.3. The aggregated values by interview item can be seen in table 4.2. The four most frequent components of the practice of selecting examples were (1) evaluating the features of a problem, including comparing to one or more other problems, (2) sequencing problems, (3) thinking about pieces of the end goal (components, sub goals, or scaffolds), and (4) thinking about the end goal.

Table 4.10: Frequencies of Each Component of the Practice of Selecting Examples by Item and Overall

	Item							
	1		2		3		All	
Component of the Practice	n	%	n	%	n	%	n	%
Thinking about the end goal	7	4.8	6	3.4	7	4.5	20	4.2
Thinking about pieces of the end goal (components, sub goals, or scaffolds)	30	20.5	27	15.3	33	21.0	90	19.0
Thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem	14	9.6	30	16.9	25	15.9	69	14.6
Noticing a subset of problems with a particular characteristic and deciding they fit a new sub goal	0	0.0	4	2.3	2	1.3	6	1.3
Noticing a subset of problems with a particular characteristic and deciding not to use that type of problem	0	0.0	0	0.0	5	3.2	5	1.1
Finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic	11	7.5	14	7.9	19	12.1	44	9.3
Looking at a set of problems for interesting features in the set	3	2.1	3	1.7	1	0.6	7	1.5
Evaluating the features of a problem, including comparing to one or more other problems	46	31.5	38	21.5	23	14.6	107	22.6
Creating or modifying a problem to match a desired (sub) goal	10	6.8	15	8.5	11	7.0	36	7.6
Sequencing problems	33	22.6	37	20.9	23	14.6	93	19.7
Describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.)	14	9.6	17	9.6	23	14.6	54	11.4
Solving a problem	2	1.4	0	0.0	6	3.8	8	1.7
Total number of segments	146		177		157		480	

Table 4.11: Frequencies of Each Component of the Practice of Selecting Examples by Teacher

	Participant															
	Ms. Allen		Mr. Baker		Mr. Clark		Mrs. Dayton		Mr. Johnson		Mrs. Kelly		Mrs. Stone		Mr. Zimmer	
Component of the Practice	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Thinking about the end goal	7	9.3	0	0.0	7	9.6	1	1.4	1	1.7	0	0.0	3	4.8	1	3.2
Thinking about pieces of the end goal (components, sub goals, or scaffolds)	8	10.7	9	23.7	7	9.6	22	31.4	15	25.9	17	23.6	9	14.3	3	9.7
Thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem	15	20.0	3	7.9	11	15.1	5	7.1	7	12.1	7	9.7	12	19.0	9	29.0
Noticing a subset of problems with a particular characteristic and deciding they fit a new sub goal	1	1.3	2	5.3	0	0.0	1	1.4	0	0.0	0	0.0	1	1.6	1	3.2
Noticing a subset of problems with a particular characteristic and deciding not to use that type of problem	0	0.0	0	0.0	3	4.1	1	1.4	1	1.7	0	0.0	0	0.0	0	0.0
Finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic	6	8.0	2	5.3	5	6.8	12	17.1	6	10.3	4	5.6	7	11.1	2	6.5
Looking at a set of problems for interesting features in the set	0	0.0	0	0.0	5	6.8	0	0.0	1	1.7	0	0.0	0	0.0	1	3.2
Evaluating the features of a problem, including comparing to one or more other problems	18	24.0	6	15.8	22	30.1	7	10.0	5	8.6	14	19.4	23	36.5	12	38.7
Creating or modifying a problem to match a desired (sub) goal	5	6.7	7	18.4	0	0.0	3	4.3	12	20.7	7	9.7	2	3.2	0	0.0
Sequencing problems	5	6.7	11	28.9	12	16.4	12	17.1	15	25.9	23	31.9	12	19.0	3	9.7
Describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.)	10	13.3	8	21.1	5	6.8	14	20.0	5	8.6	8	11.1	1	1.6	3	9.7
Solving a problem	3	4.0	0	0.0	0	0.0	0	0.0	0	0.0	5	6.9	0	0.0	0	0.0
Total number of segments	75		38		73		70		58		72		63		31	

When aggregated by interview item or by teacher, the teachers thought about pieces of the end goal more often than they thought about an overall end goal, with the exception of Mr. Clark, who thought about both equally. Looking at teachers' selection of examples on the individual interview items, there were 6 instances on particular interview items where a teacher thought about the end goal and pieces of the end goal with the same frequency, and on interview item 2, which prompted teachers to select examples for simplifying rational expressions, Mr. Clark thought about the end goal more often than he did pieces of the end goal.

When aggregated by question, the frequencies of teachers' enactments of different components of selecting examples were overall similar. There was a difference in how often teachers evaluated the features of a problem, including comparing to one or more other problems. They did so in 31.5% of transcript segments for the first interview item, 21.5% of transcript segments for the second interview item, and 14.6% of transcript segments for the third interview item. There were differences in how often different components of selecting examples were enacted across teachers. These differences will be discussed later in the chapter.

Teachers' Knowledge Use When Selecting Examples

Similar to the last section on components of the practice of selecting examples, I begin this section by discussing the different types of knowledge teachers use in selecting examples. I then present overall findings about the types of knowledge teachers used.

Knowledge Codes

In creating the knowledge codes, I went through several of the participants' responses to each interview item. I looked at each segment that was labeled with one or more practice codes and labeled the domain of knowledge being used and then described what that knowledge was. From this initial coding, I created a set of knowledge codes. Each code is contained within an

MKT domain, however some domains have multiple codes. The codes and their corresponding domain of MKT, are listed in table 4.4. In the sections that follow, which are organized by MKT domain, I describe each code and provide examples from the data.

Table 4.12: Knowledge Codes Used to Analyze the Practice of Selecting Examples

MKT Domain	**Specific Code**
CCK	Knowledge of how to carry out a procedure
	Knowledge of mathematical ideas
SCK	Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set
	Knowledge of simple problems that show a more complex idea
	Knowledge of multiple procedures (ways) to solve a problem
	Recognizing artificial patterns in problems
KCT	Knowledge of how to strategically sequence examples for a purpose
	Knowledge of how a simple example can be used to help students learn a more complex idea
	Knowledge of how examples and the values used in them should not demonstrate an artificial pattern
	Knowledge of how errors can be used for particular purposes
KCS	Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget
	Knowledge of students' strengths and abilities
KCC	Knowledge of the curriculum, including past and future learning
Incorrect Knowledge	Knowledge that is mathematically incorrect

Common Content Knowledge (CCK). There are two knowledge codes within the domain of CCK, *knowledge of how to carry out a procedure* and *knowledge of mathematical ideas*. These two categories represent the distinction between knowing pieces of mathematics knowledge and knowing how to apply knowledge to solve a problem or carry out a procedure. Participants demonstrated *knowledge of how to carry out a procedure* when they solved the mathematics problems or described how they would talk about a procedure with students. For instance, when discussing how she would use the problem $\frac{1}{2x}+\frac{3}{2x^2}=\frac{1}{6x}$ with students, Mrs.

Kelly drew on knowledge of the process of multiplying a rational equation by the least common denominator.

> On this one, when I tell them we're going to multiply both sides, or the whole equation by *6x* squared we can talk about, 'Okay, if I'm doing *6x* squared times *3x*, and dividing it by *6x* squared, those *6x* squareds are going to disappear because we're taking out that factor, we're dividing 6x squared by itself and that's *1*.' (Mrs. Kelly Example Item 3)

In this quote, Mrs. Kelly demonstrates knowledge of the procedure for solving a rational equation.

Teachers frequently used *knowledge of mathematical ideas* while selecting examples. They demonstrated this knowledge by talking about specific mathematics concepts, such as end behavior, the zero-product property, or proportions. For example, on the first interview item, which asked teachers to select examples to demonstrate different types of discontinuity, several teachers specifically mentioned that they would want to show both vertical asymptotes and holes to students. Both vertical asymptotes and holes are types of discontinuities and would therefore fit the prompt.

Specialized Content Knowledge (SCK). There are four different categories within the domain of SCK. The first code is *recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set*. Teachers demonstrated this knowledge when they talked about breaking a topic down into different types of problems, with different features, that range mathematical difficulty. For example, when responding to the prompt to select examples for solving rational equations, Mr. Clark saw equations where two rational fractions were set equal as distinct from equations where the sum of two rational expressions is equal to a third rational expression, because the first set of equations can be solved using cross multiplication. He therefore taught them separately, "I almost treat those as two different problems, the way I've always taught that" (Mr. Clark Example Item 3). Others might see all of these problems as rational equations.

The second code within the domain of SCK is *knowledge of simple problems that show a more complex idea*. Several teachers were able to name simple problems that showed a more complex idea. For example, when responding to the prompt to select examples for simplifying rational expressions, Mr. Baker talked about the problems $\frac{2+3}{2+5}$ and $\frac{2 \bullet 3}{2 \bullet 5}$ as simple rational expressions that can be simplified. Both of these fractions are rational expressions. However, because they do not have variables in their denominators, they are also problems that are simpler. This knowledge code is included in the domain of SCK because it is mathematics knowledge unlikely to be used by anyone besides teachers.

The third code within the domain of SCK is *knowledge of multiple procedures (ways) to solve a problem*. In their interviews teachers occasionally mentioned multiple methods of solving equations. These distinctions between methods of solving rational equations might be less visible to others than they are to teachers working with students. For example, when responding to the prompt to select examples for solving rational equations, Mr. Zimmer described two different methods he has taught for solving rational equations. "Sometimes we get a common denominator and they go through and cross multiply. Sometimes we find a common denominator, but use it to reduce or cancel out all the denominators" (Mr. Zimmer Example Item 3). In both cases, all of the fractions are multiplied by the common denominator to reduce the fractions. However, in once case the fractions are first changed to have common denominators. Ms. Allen also discussed the method of getting a common denominator before multiplying by the common denominator over one to clear the fractions. In addition, she mentioned an additional method she has elected not to use, where two of the fractions are combined and the problem can then be solved using cross multiplication.

The fourth code within the domain of SCK is *recognizing artificial patterns in problems*. Two teachers mentioned artificial patterns that they wanted to avoid in their examples. Artificial patterns are derived sets of cases that show a pattern that is not mathematically valid and does not hold in other instances. For example, when factoring x^2+4x+4 into $(x+2)(x+2)$, a student might assume that all equations of the form $x^2+a^2x+a^2$ factor as $(x+a)(x+a)$, however $x^2+a^2x+a^2=(x+a)(x+a)$ only when a=*2*. Mr. Johnson discussed that he is "always careful to change the signs, change the numbers" so that there are no artificial patterns in his examples and that he tries to avoid numbers like "*2*s and *4*s and *1*s and things that are always going to be the same" (Mr. Johnson Examples Item 1). This knowledge code is included in the domain of SCK because it is about mathematics ideas and values in each problem. However, consideration of these artificial patterns is likely something unique to the work of teaching.

Knowledge of Content and Teaching (KCT). Within the domain of KCT, there are four knowledge codes. The first code is *knowledge of how to strategically sequence examples for a purpose*. Teachers frequently discussed how they would sequence the examples for several purposes. These purposes included introducing only one type of discontinuity at a time when showing students different types of discontinuities, starting with the most complex problem, and having problems increase in mathematical complexity. For example, when responding to the prompt to select examples to demonstrate different types of discontinuities, Mrs. Kelly selected a first example that focused students in on the idea of discontinuities, instead of a problem that would require complex factoring and take the focus off of the discontinuity. "I might start with something like *3* over *x* minus *4* because then we're just looking at discontinuities instead of having to involve factoring also” (Mrs. Kelly Example Item 1). When selecting a first example for solving rational equations, Mr. Zimmer decided on $\frac{3}{x^2-7x+10}+2=\frac{x-4}{x-5}$ to avoid starting with cross multiplication, which students would be likely to misapply to other problems. “Sometimes I get nervous about cross multiplying and that they're just going to resort to that every time. Showing them that this method exists and they try to transfer over and use it here [on a problem where the method does not work]. I would be hesitant to start with this one for that reason” (Mr. Zimmer Example Item 3). He chose not to start with an example where students could use cross multiplication because if it was the first thing they learned, he thought they would try to use cross multiplication to solve problems that cannot be solved using that method.

The second code in this domain is *knowledge of how a simple example can be used to help students learn a more complex idea*. Several teachers mentioned that they would start a new topic using an example that was already familiar to students. Given the content of rational expressions, many of these examples involved fractions with integer denominators. These fractions are examples of rational expressions, but because they do not have variables in their denominators, they are also just fractions, which students have seen before and are familiar with. For instance, when prompted to select examples to teach simplifying rational expressions, Mr. Baker discussed how he would start with the simpler, familiar problems $\frac{2+3}{2+5}$ and $\frac{2\bullet 3}{2\bullet 5}$ to demonstrate the difference between factors and terms. Although this code is related to the SCK code of *knowledge of simple problems that show a more complex idea*, it is included in the domain of KCT because it is knowledge of the pedagogical purpose and value of using such an example, and not knowledge of a simpler mathematics problem.

The third KCT code is *knowledge of how examples and the values used in them should not demonstrate an artificial pattern*. Two teachers explicitly mentioned that they thought about incorrect patterns that students might inadvertently draw from a series of problems when selecting examples for a particular purpose. According to Mr. Johnson, “I don't want students to ever see artificial patterns so I'm always careful to change the signs, change the numbers so it doesn't seem like oh yeah, every time those are all positive, oh that's a coincidence so I don't want that to happen” (Mr. Johnson Example Item 3). When selecting examples, Mr. Johnson was careful to avoid choosing problems where students might make see patterns that do not actually exist. Although this code is related to the SCK code of *recognizing artificial patterns in problems*, it is included in the domain of KCT because it is knowledge of the pedagogical importance of avoiding artificial patterns across a set of examples.

The fourth code within the domain of KCT is *knowledge of how errors can be used for particular purposes*. A few teachers mentioned explicitly showing students errors so that students are aware of the error. When discussing how she would use a problem with students, Mrs. Kelly mentioned “Then I'd even do something like make a mistake on this problem […] so the kids can stop and say like, ‘Oh, no that's wrong.’ So they can figure out where I went wrong also” (Mrs. Kelly Example Item 3). In this quote, Mrs. Kelly shows that she knows to use errors in her teaching for the purpose of having students identify and understand the error, potentially preventing students from making it in the future.

Knowledge of Content and Students (KCS). Within the domain of KCS, there are two codes, both of which are about knowledge of what students know and often do. The first is *knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget*. Frequently during the interviews, teachers described common student errors and misconceptions, as well as places where they thought students might struggle with the mathematics. Several teachers mentioned the common error of reducing terms in a rational expression or equation, instead of factors. Teachers also mentioned that students often misapply the method of cross multiplication to problems where it is not applicable, or have difficulty recognizing how to factor particular types of quadratics, such as those with a leading coefficient or the difference of squares. All of these are common student difficulties that arise in the content area.

The teachers also demonstrated *knowledge of students' strengths and abilities*, the second code in the domain of KCS. In particular, teachers demonstrated knowledge of particular skills that their students were able to do. When discussing how students would solve the problem $y = \frac{x^2 + 5x + 6}{x^2 - x - 12}$, Mr. Clark commented that he liked that the denominator was factorable. "That's something that my guys can handle and they understand the Zero Product Property" (Mr. Clark Example Item 1). Mrs. Kelly said that her students "would have already done addition and subtraction, so they would know how to find a common denominator" (Mrs. Kelly Example Item 3). These are both examples of teachers showing knowledge of the abilities their students would have in this topic.

Knowledge of Content and Curriculum. (KCC) There was only one code for the domain of KCC. Occasionally teachers mentioned knowledge of what students had learned in other years, what they learned earlier that school year, or what they would learn in a future course. For example, when discussing the examples she would pick in response to the prompt on solving rational equations, Mrs. Stone mentioned that students learned the concept of equivalent fractions in elementary school. This shows knowledge of what students had already learned. Mr.

Clark also discussed that removable discontinuities are a calculus concept. This shows knowledge of what students would continue to learn in the future.

Incorrect Knowledge. The final knowledge code was that of incorrect knowledge. On a few occasions, participants demonstrated mathematical errors or incorrect knowledge. For example, Ms. Allen discussed that extraneous solutions to rational equations always come from problems where one of the rational expressions is not fully reduced before solving. Such a problem will result in an extraneous solution, often in addition to the original solutions. However, extraneous solutions also exist in problems where the rational expressions have no common factors, such as $\frac{10}{x^2-2x}+\frac{4}{x}=\frac{5}{x-2}$. Thus, Ms. Allen demonstrated incorrect mathematics knowledge.

Teachers' Knowledge Use

Having described each of the knowledge codes, I now look at how their frequencies in the data. In this section, I discuss the types of knowledge teachers used in solving the interview items. I also look at patterns in knowledge use across the three interview items. Finally, I look at types of knowledge that were frequently used together. In addition to differences across the interview items, there were differences in the frequency with which they used particular types of knowledge. These differences will be discussed in greater detail in the final section of this chapter.

Types of Knowledge Used. Table 4.5 shows the number of segments that were coded as each type of knowledge as well as the percent of total segments for each interview item and all of the interview items. Table 4.6 shows the same information aggregated by teacher. All of the teachers used both types of common content knowledge (knowledge of how to carry out a procedure and knowledge of mathematical ideas) on each interview item. Overall, the teachers demonstrated knowledge in the domains of KCS, KCT, and SCK across the majority of the interview items. With the exception of Ms. Allen on the second interview item, all of the teachers recognized nuanced differences/ subsets within a larger set of problems that others would group as one set on each of the interview items. Similarly, with the exception of Mr. Baker on the first and third interview items, all of the teachers drew on knowledge of common student errors or misconceptions, or places students generally struggle. All of the teachers drew on knowledge of sequencing for a purpose on each interview item, with the exception of Mr. Johnson on item 1.

Table 4.13: Frequencies of Each Knowledge Code by Item and Overall

MKT Domain	Specific Code	Item 1 n	Item 1 %	Item 2 n	Item 2 %	Item 3 n	Item 3 %	All n	All %
CCK	Knowledge of how to carry out a procedure	64	43.8	82	46.3	52	33.1	198	41.3
	Knowledge of mathematical ideas	88	60.3	69	39.0	37	23.6	194	40.4
SCK	Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	73	50.0	75	42.4	71	45.2	219	45.6
	Knowledge of simple problems that show a more complex idea	0	0.0	11	6.2	5	3.2	16	3.3
	Knowledge of multiple procedures (ways) to solve a problem	1	0.7	1	0.6	7	4.5	9	1.9
	Recognizing artificial patterns in problems	4	2.7	0	0.0	0	0.0	4	0.8
KCT	Knowledge of how to strategically sequence examples for a purpose	37	25.3	57	32.2	54	34.4	148	30.8
	Knowledge of how a simple example can be used to help students learn a more complex idea	0	0.0	11	6.2	5	3.2	16	3.3
	Knowledge of how examples and the values used in them should not demonstrate an artificial pattern	4	2.7	0	0.0	0	0.0	4	0.8
	Knowledge of how errors can be used for particular purposes	0	0.0	0	0.0	2	1.3	2	0.4
KCS	Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	15	10.3	43	24.3	25	15.9	83	17.3
	Knowledge of students' strengths and abilities	11	7.5	7	4.0	11	7.0	29	6.0
KCC	Knowledge of the curriculum, including past and future learning	1	0.7	3	1.7	3	1.9	7	1.5
Incorrect Knowledge	Knowledge that is mathematically incorrect	2	1.4	0	0.0	3	1.9	5	1.0
	Total number of segments	146		177		157		480	

Table 4.14: Frequencies of Each Knowledge Code by Teacher

MKT Domain	Specific Code	Participant															
		Ms. Allen		Mr. Baker		Mr. Clark		Mrs. Dayton		Mr. Johnson		Mrs. Kelly		Mrs. Stone		Mr. Zimmer	
		n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
CCK	Knowledge of how to carry out a procedure	30	40.0	27	71.1	33	45.2	22	31.4	28	48.3	24	33.3	15	23.8	19	61.3
	Knowledge of mathematical ideas	38	50.7	22	57.9	30	41.1	25	35.7	27	46.6	21	29.2	15	23.8	16	51.6
SCK	Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	15	20.0	22	57.9	37	50.7	39	55.7	28	48.3	39	54.2	29	46.0	10	32.3
	Knowledge of simple problems that show a more complex idea	0	0.0	5	13.2	0	0.0	1	1.4	2	3.4	7	9.7	1	1.6	0	0.0
	Knowledge of multiple procedures (ways) to solve a problem	1	1.3	0	0.0	3	4.1	1	1.4	1	1.7	2	2.8	0	0.0	1	3.2
	Recognizing artificial patterns in problems	0	0.0	0	0.0	0	0.0	2	2.9	2	3.4	0	0.0	0	0.0	0	0.0
KCT	Knowledge of how to strategically sequence examples for a purpose	6	8.0	12	31.6	21	28.8	36	51.4	19	32.8	33	45.8	17	27.0	4	12.9
	Knowledge of how a simple example can be used to help students learn a more complex idea	0	0.0	5	13.2	0	0.0	1	1.4	2	3.4	7	9.7	1	1.6	0	0.0
	Knowledge of how examples and the values used in them should not demonstrate an artificial pattern	0	0.0	0	0.0	0	0.0	2	2.9	2	3.4	0	0.0	0	0.0	0	0.0
	Knowledge of how errors can be used for particular purposes	1	1.3	0	0.0	0	0.0	0	0.0	0	0.0	1	1.4	0	0.0	0	0.0
KCS	Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	22	29.3	3	7.9	11	15.1	8	11.4	9	15.5	6	8.3	11	17.5	13	41.9
	Knowledge of students' strengths and abilities	1	1.3	2	5.3	6	8.2	4	5.7	2	3.4	7	9.7	7	11.1	0	0.0
KCC	Knowledge of the curriculum, including past and future learning	1	1.3	1	2.6	1	1.4	0	0.0	0	0.0	1	1.4	3	4.8	0	0.0
Incorrect Knowledge	Knowledge that is mathematically incorrect	3	4.0	0	0.0	1	1.4	0	0.0	1	1.7	0	0.0	0	0.0	0	0.0

Total number of segments	75	38	73	70	58	72	63	31

Knowledge Use by Item. When comparing teachers' knowledge use across the three interview items, teachers drew on similar knowledge. However, there were two noticeable differences across the interview items. First, teachers more frequently drew on knowledge of student errors and struggles on interview item 2. Teachers demonstrated this knowledge on 24.3% of the transcript segments for interview item 2, compared to 10.3% of the transcript segments for interview item 1 and 15.9% of the transcript segments for interview item 3. Second, the teachers demonstrated both knowledge of a simple problem that shows a complex idea and knowledge of how a simple example can be used to help students learn a more complex idea on interview items 2 and 3. However, none of the teachers demonstrated this knowledge on the first interview item.

Co-occurrences of Knowledge. The teachers drew on multiple types of knowledge for each interview item. Each teacher demonstrated knowledge in at least four categories for each interview item, with some teachers demonstrating up to ten categories on one item. Some types of knowledge co-occurred frequently. Table 4.7 shows how frequently different pairs of knowledge co-occurred in the same segment. Both of the categories within CCK, knowledge of how to carry out a procedure and knowledge of mathematical ideas, co-occurred with all of the other types of knowledge, except for the KCT code of showing errors for a purpose, which was only used twice. In addition, in 81.8% of instances where teachers drew on the KCT knowledge of how to strategically sequence examples for a purpose, they also recognized nuanced differences/ subsets within a larger set of problems that others would group as one set.

There were two pairs of knowledge codes that overlapped completely: (1) knowledge of simple problems that show a more complex idea and knowledge of how a simple example can be used to help students learn a more complex idea, and (2) recognizing artificial patterns in problems and knowledge of how examples and the values used in them should not demonstrate an artificial pattern. These two pairs of knowledge completely overlapped because when teachers described using a simple example or avoiding artificial patterns, they both demonstrated the specialized mathematics knowledge of the simple example or possible artificial pattern, and the pedagogical knowledge recognizing the impact on student learning that such examples or artificial patterns can have.

Table 4.15: Co-occurrences of Knowledge Codes. All numbers are percentages, with the exception of the totals.

	CCK		SCK				KCT				KCS		KCC
	Knowledge of how to carry out a procedure	Knowledge of mathematical ideas	Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	Knowledge of simple problems that show a more complex idea	Knowledge of multiple procedures (ways) to solve a problem	Recognizing artificial patterns in problems	Knowledge of how to strategically sequence examples for a purpose	Knowledge of how a simple example can be used to help students learn a more complex idea	Knowledge of how examples and the values used in them should not demonstrate an artificial pattern	Knowledge of how errors can be used for particular purposes	Knowledge of common student errors or struggles	Knowledge of students' strengths and abilities	Knowledge of the curriculum, including past and future learning
Knowledge of how to carry out a procedure	--	65.5	44.3	43.8	66.7	25.0	31.1	7.0	25.0	0.0	43.4	51.7	42.9
Knowledge of mathematical ideas	64.1	--	42.5	31.3	22.2	25.0	30.4	5.0	25.0	0.0	33.7	31.0	57.1
Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	49.0	47.9	--	12.5	0.0	50.0	81.8	2.0	50.0	0.0	22.9	31.0	28.6
Knowledge of simple problems that show a more complex idea	3.5	2.6	0.9	--	0.0	0.0	2.7	16.0	0.0	0.0	2.4	3.4	28.6
Knowledge of multiple procedures (ways) to solve a problem	3.0	1.0	0.0	0.0	--	0.0	0.0	0.0	0.0	0.0	3.6	6.9	0.0
Recognizing artificial patterns in problems	0.5	0.5	0.9	0.0	0.0	--	1.4	0.0	100.0	0.0	2.4	0.0	0.0
Knowledge of how to strategically sequence examples for a purpose	23.2	23.2	55.3	25.0	0.0	50.0	--	4.0	50.0	0.0	15.7	13.8	14.3
Knowledge of how a simple example can be used to help students learn a more complex idea	3.5	2.6	0.9	100.0	0.0	0.0	2.7	--	0.0	0.0	2.4	3.4	28.6
Knowledge of how examples and the values used in them should not demonstrate an artificial pattern	0.5	0.5	0.9	0.0	0.0	100.0	1.4	0.0	--	0.0	2.4	0.0	0.0
Knowledge of how errors can be used for particular purposes	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	--	1.2	0.0	0.0
Knowledge of common student errors struggles	18.2	14.4	8.7	12.5	33.3	50.0	8.8	2.0	50.0	50.0	--	24.1	28.6
Knowledge of students' strengths and abilities	7.6	4.6	4.1	6.3	22.2	0.0	2.7	1.0	0.0	0.0	8.4	--	0.0
Knowledge of the curriculum, including past and future learning	1.5	2.1	0.9	12.5	0.0	0.0	0.7	2.0	0.0	0.0	2.4	0.0	--

Total number of segments	198	194	219	16	9	4	148	16	4	2	83	29	7

The Intersection of Knowledge Use and Practice: What Knowledge Do Teachers Use in Enacting Different Components of the Practice of Selecting Examples?

In this section, I look at the intersection of knowledge and practice to examine the knowledge teachers draw on as they enact different components of the practice of selecting examples. This analysis looks across all of the teachers to see what knowledge is used within each component of the practice. I focus on four findings. First, teachers drew on multiple types of knowledge in enacting each component of the practice of selecting examples. The second finding considers the use of common and specialized content knowledge when enacting different components of selecting examples. The third finding considers teachers' use of knowledge of how to strategically sequence examples for a purpose when enacting difference components of the practice. The fourth finding focuses on the knowledge teachers use when enacting the component of thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem.

Multiple Types of Knowledge

Overall, teachers drew on a range of knowledge when enacting each component of the practice of selecting examples. The frequency with which teachers drew on knowledge when enacting each component can be seen in table 4.8. For each component, the teachers collectively drew on at least 5 different types of knowledge. For example, across all of the instances of evaluating the features of a problem, including comparing to one or more other problems, teachers demonstrated eight different types of knowledge, including knowledge of how to carry out a procedure, knowledge of mathematical ideas, recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set, knowledge of common student errors or struggles, and knowledge of students' strengths and abilities. In the rest of this section, I focus on patterns that emerged in the types of knowledge teachers drew on when enacting specific components of the practice of selecting examples.

Table 4.16: Occurrences of Knowledge Codes Within Each Component of the Practice of Selecting Examples. All numbers are percentages, with the exception of the totals.

	Thinking about the end goal	Thinking about pieces of the end goal	Thinking about common student errors or places where they will have difficulty or success	Noticing a subset of problems with a particular characteristic and deciding they fit a new sub goal	Noticing a subset of problems with a particular characteristic and deciding not to use that type of problem	Finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic	Looking at a set of problems for interesting features in the set	Evaluating the features of a problem, including comparing to one or more other problems	Creating or modifying a problem to match a desired (sub) goal	Sequencing problems	Describing how the problem will be used with students	Solving a problem
Knowledge of how to carry out a procedure	35.0	38.9	40.6	50.0	40.0	20.5	14.3	57.0	55.6	24.7	75.9	100.0
Knowledge of mathematical ideas	40.0	45.6	31.9	16.7	20.0	18.2	71.4	57.0	61.1	32.3	68.5	25.0
Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	15.0	70.0	23.2	66.7	80.0	70.5	57.1	62.6	66.7	66.7	9.3	12.5
Knowledge of simple problems that show a more complex idea	0.0	6.7	1.4	0.0	0.0	0.0	0.0	0.9	16.7	4.3	9.3	0.0
Knowledge of multiple procedures (ways) to solve a problem	0.0	0.0	4.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	13.0	0.0
Recognizing artificial patterns in problems	0.0	2.2	1.4	0.0	0.0	0.0	0.0	0.0	2.8	0.0	0.0	0.0
Knowledge of how to strategically sequence examples for a purpose	10.0	47.8	13.0	16.7	20.0	38.6	0.0	29.0	41.7	79.6	13.0	12.5
Knowledge of how a simple example can be used to help students learn a more complex idea	0.0	6.7	1.4	0.0	0.0	0.0	0.0	0.9	16.7	4.3	9.3	0.0
Knowledge of how examples and the values used in them should not demonstrate an artificial pattern	0.0	2.2	1.4	0.0	0.0	0.0	0.0	0.0	2.8	0.0	0.0	0.0
Knowledge of how errors can be used for particular purposes	0.0	0.0	1.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.9	0.0
Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	5.0	5.6	82.6	16.7	20.0	2.3	14.3	18.7	0.0	4.3	18.5	0.0

Knowledge of students' strengths and abilities	0.0	3.3	23.2	16.7	0.0	2.3	0.0	4.7	2.8	1.1	11.1	12.5
Knowledge of the curriculum, including past and future learning	0.0	2.2	1.4	0.0	0.0	0.0	14.3	0.0	2.8	0.0	5.6	0.0
Total number of segments	20	90	69	6	5	44	7	107	36	93	54	8

Common and Specialized Content Knowledge

Teachers drew on both components of CCK, knowledge of how to carry out a procedure and knowledge of mathematical ideas, as well as the specialized content knowledge of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set when enacting each of the components of the practice of selecting examples. However, there are differences in which types of knowledge were used more frequently across different components of the practice. Knowledge of how to carry out a procedure was most frequently used when teachers enacted the practice components of solving a problem (100%), describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.) (75.9%), evaluating the features of a problem, including comparing to one or more other problems (57.0%), and creating or modifying a problem to match a desired (sub) goal (55.6%). When enacting the following practices, teachers drew most frequently on knowledge of mathematical ideas: looking at a set of problems for interesting features in the set (71.4%), describing how the problem will be used with students (e.g. what they would point out, how to get students to notice features, etc.) (68.5%), creating or modifying a problem to match a desired (sub) goal (61.1%), and evaluating the features of a problem, including comparing to one or more other problems (57.0%).

Teachers recognized nuanced differences/ subsets within a larger set of problems that others would group as one set more than 50% of the time when enacting each component of the practice except describing how the problem will be used with students (9.3%), solving a problem (12.5%), thinking about the end goal (15.0%), and thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem (23.2%), They most frequently recognized these nuances when enacting the components of noticing a subset of problems with a particular characteristic and deciding not to use that type of problem (80.0%), finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic (70.5%), and thinking about pieces of the end goal (components, sub goals, or scaffolds) (70.0%).

In addition to the differences in how often each knowledge type was used across different components, there were differences in which type of knowledge teachers drew on most heavily within each component. For several components of the practice, teachers drew more frequently on both areas of CCK than they did on the specialized content knowledge of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set. These practice components include thinking about the end goal, thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem, and describing how the problem will be used with students. For example, when describing how the problem will be used with students, teachers drew on knowledge of how to carry out a procedure 75.9% of the time and knowledge of mathematical ideas 68.5% of the time, yet they only recognized nuanced differences/ subsets within a larger set of problems that others would group as one set 9.3% of the time.

For several practice components, including thinking about pieces of the end goal (components, sub goals, or scaffolds), finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic, and sequencing problems, the opposite was true. Teachers more frequently recognized nuanced differences/ subsets within a larger set of problems that others would group as one set, compared to drawing on knowledge of how to carry out a procedure or of mathematical ideas. For example, when carrying out the practice component of finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic, teachers recognized nuanced differences/ subsets within a larger set of problems that others would group as one set 70.5% of the time, yet drew on knowledge of how to carry out a procedure 20.5% of the time and knowledge of mathematical ideas only 18.2% of the time.

For a few of the practice components, there were larger differences between the frequencies with which teachers drew on each of the types of common content knowledge. For the two practice components involving noticing a subset of problems with a given feature and deciding to include or exclude them, teachers most frequently used the specialized content knowledge of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set, followed by knowledge of how to carry out a procedure, and drew even less frequently on knowledge of mathematical ideas. There were only a few instances overall of each of these practice components. For example, when teachers noticed a subset of problems with a particular characteristic and decided not to use that type of problem, they recognized nuanced differences/ subsets within a larger set of problems that others would group as one set 80% of the time, drew on knowledge of how to carry out a procedure 40% of the time, and drew on knowledge of mathematical ideas 20% of the time. When looking at a set of problems for interesting features in the set, teachers drew on knowledge of mathematics ideas 71.4% of the time, recognized nuanced differences/ subsets within a larger set of problems that others would group as one set 57.1% of the time, and drew on knowledge of how to carry out a procedure only 14.3% of the time. Finally, as might be expected, when solving problems, teachers drew on knowledge of how to carry out a procedure 100% of the time, but infrequently drew on knowledge of mathematical ideas and the specialized content knowledge of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set.

Knowledge of How to Sequence Examples Strategically

Teachers drew on knowledge of how to sequence examples in ways that were strategic for a particular purpose when enacting each of the practice components except for looking at a set of problems for interesting features in the set. As might be expected, teachers drew on this knowledge most often when they were sequencing problems. In addition to this component of the practice, teachers also drew on knowledge of how to strategically sequence examples frequently when enacting other practice components. They drew on this knowledge 47.8% of the time when they were thinking about pieces of the end goal (components, sub goals, or scaffolds), 41.7% of the time when they were creating or modifying a problem to match a desired (sub) goal, and 38.6% of the time when they were finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic.

Knowledge Used When Enacting the Component of Thinking About Common Student Errors or Places Where They Will Have Difficulty or Success

When enacting the practice of thinking about common student errors or places where they will have difficulty or success, both broadly and within a specific problem, teachers drew heavily on knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget. They used this knowledge 82.6% of the time. Although they also drew on knowledge of students' strengths and abilities, they did so far less often, only 23.2% of the time, which is also how often they recognized nuanced differences/ subsets within a larger set of problems that others would group as one set. When enacting this practice, teachers drew on the common content knowledge areas of how to carry out a procedure 40.6% of the time and mathematical ideas 31.9% of the time.

Differences Between Teachers in Their Knowledge Use When Enacting Different Components of the Practice of Selecting Examples

As mentioned earlier, there were variations across teachers both in terms of the components of the practice of selecting examples they enacted and the knowledge they drew on in doing so. Looking at both sets of data together, three different patterns of selecting examples and drawing on knowledge emerged. In particular, Mr. Baker and Mrs. Kelly were focused on key mathematical ideas in their lesson and were least focused on student misconceptions. Ms. Allen and Mr. Zimmer were most focused on selecting examples that would address and prevent common student misconceptions. Mr. Clark and Mrs. Stone fell somewhere in the middle. In the subsections that follow, I describe these patterns in more detail and discuss the frequencies with which different teachers enacted different practice components and drew on different knowledge in doing so. I draw both on tables discussed earlier in the chapter, which each look at two of the three dimensions of knowledge, practice, and teachers. I also look at all three variables together by looking at differences in individual teacher's knowledge when enacting a particular practice component. The number and frequency of segments coded with each component of the practice of selecting examples are shown in table 4.3 in the first section of this chapter. The number and frequency of segments that received each knowledge code can be seen in table 4.6 in the second section of this chapter. Several additional tables in this section show frequencies of knowledge used by each teacher and of co-occurrences of practice components for each teacher when enacting different components of the practice.

Mr. Baker and Mrs. Kelly: Focusing on Key Mathematical Ideas

The first pattern is seen in the practice components enacted and the knowledge drawn on by Mr. Baker and Mrs. Kelly. Overall, Mr. Baker and Mrs. Kelly were focused on the key mathematical ideas and sub goals they planned to discuss and were least focused on student misconceptions. In terms of the components of the practice that they enacted, these two teachers did not think about an overall end goal, but they did each think about pieces of the end goal (components, sub goals, or scaffolds). Mr. Baker did so in 23.7% of the marked segments in his transcripts and Mrs. Kelly did so in 23.6% of the marked segments in her transcripts (see table 4.3). They each thought about pieces of the end goal more frequently than four of the other participants, but less frequently than Mrs. Dayton and Mr. Johnson. Both Mr. Baker and Mrs. Kelly had the highest frequencies for the practice of sequencing problems, which they did in 28.9% and 31.9% of their marked transcript segments, respectively (see table 4.3). As might be expected, Mr. Baker and Mrs. Kelly drew on knowledge of how to strategically sequence examples for a purpose more frequently than some of the teachers (31.6% and 45.8%, respectively, see table 4.6). However, Mrs. Dayton drew on this knowledge more frequently than both Mr. Baker and Mrs. Kelly, and Mr. Johnson did so more frequently that Mr. Baker. Mr. Baker and Mrs. Kelly both recognized nuanced differences/ subsets within a larger set of problems that others would group as one set more frequently than several of the other teachers. Mr. Baker drew on this knowledge 57.9% of the time and Mrs. Kelly drew on this knowledge 54.2% of the time, which was less than only one other participant, Mrs. Dayton at 55.7% (see table 4.6).

Table 4.9 shows each individual teacher's knowledge use for the practice component of thinking about pieces of the end goal. It also shows co-occurrences of the practice component of sequencing examples. When enacting the practice component of thinking about pieces of the end goal, Mr. Baker and Mrs. Kelly had the highest co-occurrence frequency of sequencing problems. Mr. Baker sequenced problems in 66.7% of the transcript segments in which he was thinking about pieces of the end goal and Mrs. Kelly did so in 58.8% of the segments where she was thinking about pieces of the end goal (see table 4.9). Both Mr. Baker and Mrs. Kelly also more frequently drew on knowledge of how to strategically sequence examples for a purpose when they were thinking about pieces of the end goal compared to the other teachers. They did so during 44.4% and 58.8% of the segments in which they were thinking about pieces of the end goal, respectively (see table 4.9). Only Mrs. Dayton had a higher frequency; she drew on knowledge of how to strategically sequence examples for a purpose during 86.4% of the segments where she was thinking about pieces of the end goal (see table 4.9). In addition, when enacting the practice component of thinking about pieces of the end goal, Mr. Baker recognized nuanced differences within a larger set of problems that others would group as one set 88.9% of the time, more than any of the other teachers (see table 4.9).

Table 4.17: Frequencies of Each Knowledge Code or Co-occurring Practice Component Used When Carrying Out the Practice Component of Thinking About Pieces of the End Goal by Teacher

	Participant															
	Focusing on Key Mathematics Ideas												Focusing on Common Student Misconceptions			
	Mr. Baker		Mrs. Kelly		Mrs. Dayton		Mr. Johnson		Mrs. Stone		Mr. Clark		Ms. Allen		Mr. Zimmer	
Component of the Practice or Knowledge Code	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Sequencing problems	6	66.7	10	58.8	7	31.8	4	26.7	2	22.2	3	42.9	1	12.5	0	0.0
Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	8	88.9	11	64.7	17	77.3	11	73.3	6	66.7	6	85.7	4	50.0	0	0.0
Knowledge of how to strategically sequence examples for a purpose	4	44.4	10	58.8	19	86.4	4	26.7	2	22.2	3	42.9	1	12.5	0	0.0
Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	0	0.0	0	0.0	1	4.5	1	6.7	0	0.0	0	0.0	2	25.0	1	33.3
Total number of segments	9		17		22		15		9		7		8		3	

Mr. Baker and Mrs. Kelly show similar patterns for the practice component of sequencing problems. Table 4.10 presents teachers' knowledge use for the practice component of sequencing problems. It also shows co-occurrences of the practice component of thinking about pieces of the end goal. When sequencing problems, Mr. Baker and Mrs. Kelly also enacted the practice component of thinking about pieces of the end goal more frequently than all of the other teachers (54.5% and 43.5% respectively), with the exception of Mrs. Dayton, who did so in 58.3% of the segments in which she sequenced problems (see table 4.10). In addition, when sequencing problems, Mr. Baker recognized nuanced differences within a larger set of problems that others would group as one set 81.8% of the time, more than all of the other teachers except for Mr. Clark, who did so 83.3% of the time when he was sequencing problems (see table 4.10).

Table 4.18: Frequencies of Each Knowledge Code or Co-occurring Practice Component Used When Carrying Out the Practice Component of Sequencing Problems by Teacher

	Participant															
	Focusing on Key Mathematics Ideas												Focusing on Common Student Misconceptions			
	Mr. Baker		Mrs. Kelly		Mrs. Dayton		Mr. Johnson		Mrs. Stone		Mr. Clark		Ms. Allen		Mr. Zimmer	
Component of the Practice or Knowledge Code	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Thinking about pieces of the end goal (components, sub goals, or scaffolds)	6	54.5	10	43.5	7	58.3	4	26.7	2	16.7	3	25.0	1	20.0	0	0.0
Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	9	81.8	15	65.2	8	66.7	10	66.7	8	66.7	10	83.3	2	40.0	0	0.0
Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	0	0.0	1	4.3	1	8.3	0	0.0	0	0.0	0	0.0	1	20.0	1	33.3
Total number of segments	11		23		12		15		12		12		5		3	

One interesting feature of Mr. Baker and Mrs. Kelly's practice is that they less frequently engaged in the practice component of thinking about common student errors or places where they will have difficulty or success, which occurred 7.9% of the time for Mr. Baker and 9.7% of the time for Mrs. Kelly. Only Mrs. Dayton engaged in this component less, 7.1% of the time (see table 4.3). Both Mr. Baker and Mrs. Kelly drew on knowledge of common student errors or struggles less frequently than all of the other teachers (7.9% and 8.3% respectively, see table 4.6). What is most interesting is that while they did not have the highest frequencies for drawing on knowledge of students' strengths and abilities, they were the most balanced of the teachers in drawing on knowledge of both errors/struggles and strengths. In fact, Mrs. Kelly was the only teacher to more frequently draw on knowledge of students' strengths and abilities (9.7%%) than on knowledge of student errors or struggles (8.3%, see table 4.6).

These results are echoed when looking specifically at the knowledge teachers used and practice components that co-occurred with the practice component of thinking about common student errors or places where they will have difficulty or success. Table 4.11 shows frequencies of knowledge used by each teacher and of co-occurrences of practice components for each teacher when enacting the practice component of thinking about common student errors or places where they will have difficulty or success. When enacting this practice, Mrs. Kelly had the highest frequency of drawing on knowledge of students' strengths (71.4%) and the lowest frequency of drawing on student errors or places where they may struggle (57.1%, see table 4.11). She was also the only teacher to think about students' strengths more frequently than common student errors or difficulties. When enacting the practice component of thinking about common student errors or places where they will have difficulty or success, Mr. Baker had the third highest frequency of drawing on knowledge of students' strengths and abilities, 33.3% (see table 4.11).

Table 4.19: Frequencies of Each Knowledge Code or Co-occurring Practice Component Used When Carrying Out the Practice Component of Thinking About Common Student Errors or Places Where They Will Have Difficulty or Success by Teacher

	Participant															
	Focusing on Key Mathematics Ideas												Focusing on Common Student Misconceptions			
	Mr. Baker		Mrs. Kelly		Mrs. Dayton		Mr. Johnson		Mrs. Stone		Mr. Clark		Ms. Allen		Mr. Zimmer	
Component of the Practice or Knowledge Code	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	3	100.0	4	57.1	4	80.0	6	85.7	8	66.7	8	72.7	15	100.0	9	100.0
Knowledge of students' strengths and abilities	1	33.3	5	71.4	1	20.0	1	14.3	5	41.7	3	27.3	0	0.0	0	0.0
Total number of segments	3		7		5		7		12		11		15		7	

Despite more frequently engaging in the practice components of thinking about pieces of the end goal and sequencing problems, and recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set, both Mr. Baker and Mrs. Kelly engaged in the practice component of finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic less frequently than all of the other teachers. Mr. Baker engaged in this practice component 5.3% of the time and Mrs. Kelly 5.6% of the time (see table 4.3). Similarly, Mr. Baker and Mrs. Kelly less frequently evaluated the features of a problem, including comparing to one or more other problems, than some, but not all of the other teachers. They did so 15.8% and 19.4% of the time, respectively (see table 4.3).

Although they evaluated the features of a problem less frequently than the other teachers, when enacting this practice component, Mr. Baker and Mrs. Kelly also enacted the practice component of sequencing problems more frequently than any of the other teachers. When evaluating the features of a problem, Mr. Baker also sequenced problems 50.0% of the time and Mrs. Kelly did so 14.3% of the time. These values can be seen in table 4.12, which presents teachers' knowledge use for the practice component of sequencing problems. It also shows co-occurrences of the practice component of thinking about pieces of the end goal. In addition, when evaluating the features of a problem, Mr. Baker and Mrs. Kelly recognized nuanced differences within a larger set of problems that others would group as one set more frequently than any of the other teachers (100.0% and 85.7% of the time, respectively, see table 4.12), and drew on knowledge of how to strategically sequence examples more frequently than most of the other teachers. Mr. Baker did so most often (83.3%), Mr. Johnson did so 80% of the time, and Mrs. Kelly did so 50.0% of the time (see table 4.12). Finally, when evaluating the features of a problem, Mr. Baker was the only teacher who did not draw on knowledge of student errors and Mrs. Kelly drew on this knowledge less than several of the other teachers (14.3%, see table 4.12).

Table 4.20: Frequencies of Each Knowledge Code or Co-occurring Practice Component Used When Carrying Out the Practice Component of Evaluating the Features of a Problem by Teacher

	Participant															
	Focusing on Key Mathematics Ideas												Focusing on Common Student Misconceptions			
	Mr. Baker		Mrs. Kelly		Mrs. Dayton		Mr. Johnson		Mrs. Stone		Mr. Clark		Ms. Allen		Mr. Zimmer	
Component of the Practice or Knowledge Code	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Thinking about common student errors or places where they will have difficulty or success	0	0.0	1	7.1	1	14.3	0	0.0	3	13.0	0	0.0	3	16.7	6	50.0
Sequencing problems	3	50.0	2	14.3	0	0.0	0	0.0	2	8.7	2	9.1	0	0.0	0	0.0
Recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set	6	100.0	12	85.7	3	42.9	3	60.0	16	69.6	12	54.5	7	38.9	8	66.7
Knowledge of how to strategically sequence examples for a purpose	5	83.3	7	50.0	1	14.3	4	80.0	6	26.1	7	31.8	0	0.0	1	8.3
Knowledge of common student errors or misconceptions, areas were students may struggle with the mathematics, or information students commonly forget	0	0.0	2	14.3	1	14.3	1	20.0	4	17.4	1	4.5	4	22.2	7	58.3
Total number of segments	6		14		7		5		23		22		18		12	

Finally, in terms of other knowledge they each drew on when selecting examples, both Mr. Baker and Mrs. Kelly drew on knowledge of content and curriculum, which not all of the teachers used. They also more frequently drew both on knowledge of simple problems that show a more complex idea and knowledge of how a simple example can be used to help students learn a more complex idea. Mr. Baker drew on each of these types of knowledge 5 times, while Mrs. Kelly did so 7 times. Only three other teachers also drew on these two types of knowledge and only did so once or twice (see table 4.6).

Two other participants, Mrs. Dayton and Mr. Johnson were similar in some ways to Mr. Baker and Mrs. Kelly. Mrs. Dayton engaged in the practice component of thinking about common student errors or places where they will have difficulty or success and drew on knowledge of students' strengths and abilities with the same frequently as Mr. Baker and Mrs. Kelly. Mr. Johnson engaged in this practice component slightly more frequently that Mr. Baker and Mrs. Kelly, and drew on knowledge of students' strengths and abilities slightly less frequently (see tables 4.3 and 4.6). Both Mrs. Dayton and Mr. Johnson drew on knowledge of common student errors or struggles more frequently than Mr. Baker and Mrs. Kelly. Mrs. Dayton and Mr. Johnson both drew on knowledge of how to strategically sequence examples for a purpose as much as, or more frequently than Mr. Baker and Mrs. Kelly. Finally, Mrs. Dayton recognized nuanced differences/ subsets within a larger set of problems that others would group as one set as frequently as Mr. Baker and Mrs. Kelly, while Mr. Johnson did so less frequently (see table 4.6).

Despite the similarities, Mrs. Dayton and Mr. Johnson's enactments of components of the practice of selecting examples and knowledge used were different enough in some of the components that they were not included in the same group. Mrs. Dayton and Mr. Johnson each engaged in the practice component of thinking about the end goal once. Compared to Mr. Baker and Mrs. Kelly, they also more frequently thought about pieces of the end goal. They more frequently enacted the component of finding problem(s) that match a particular feature or (sub) goal, or picking one of a subset of problems with a particular characteristic. In addition, they less frequently enacted the components of evaluating the features of a problem and sequencing problems (see table 4.3). Neither Mrs. Dayton nor Mr. Johnson drew on knowledge of content and curriculum. Finally, although both Mrs. Dayton and Mr. Johnson drew on knowledge of simple problems that show a more complex idea and knowledge of how a simple example can be used to help students learn a more complex idea, they did so less frequently than Mr. Baker and Mrs. Kelly (see table 4.6).

Compared to Mr. Baker and Mrs. Kelly, when enacting the component of thinking about pieces of the end goal, Mrs. Dayton had a lower co-occurrence frequency of sequencing problems, yet when sequencing problems, she had a higher co-occurrence frequency of thinking about pieces of the end goal (see tables 4.9 and 4.10). In addition, when thinking about pieces of the end goal, Mrs. Dayton also drew on knowledge of how to strategically sequence examples for a purpose (see table 4.9). This suggests differences in Mrs. Dayton's knowledge and practice compared to Mr. Baker and Mrs. Kelly. In particular, that when she was thinking about pieces of the end goal, she was sequencing mathematical ideas that would be used in problems and not sequencing specific problems.

Mr. Johnson enacted the practice component of evaluating the features of a problem less frequently than Mr. Baker and Mrs. Kelly. However, when enacting this component, he drew on knowledge of how to strategically sequence examples for a purpose more frequently than all of the other teachers except Mr. Baker. Mr. Johnson drew on this knowledge during 80% of the transcript segments in which he was evaluating the features of a problem and Mr. Baker did so 83.3% of the time when enacting the practice component (see table 4.12).

Overall, Mr. Baker and Mrs. Kelly were focused on the key mathematical ideas they wanted to get across in their lesson. In doing so, they also were least focused on student misconceptions and thought about students' strengths more proportionately than any of the other teachers.

Ms. Allen and Mr. Zimmer: Focusing on Common Student Misconceptions

The second pattern that emerged can be seen in the practice components enacted and the knowledge drawn on by Ms. Allen and Mr. Zimmer. Ms. Allen and Mr. Zimmer were most focused on selecting problems that would surface common student misconceptions and were less focused on the key mathematical ideas they might discuss for a particular topic. Both Ms. Allen and Mr. Zimmer enacted the practice component of thinking about common student errors or places where they will have difficulty or success more frequently than any of the other teachers. Ms. Allen did so 20.0% of the time and Mr. Zimmer did so 29.0% of the time (see table 4.3). They drew on knowledge of common student errors or misconceptions more frequently than any of the other teachers (Ms. Allen did so 29.3% of the time and Mr. Zimmer did so 41.9% of the time), and also drew on knowledge of students' strengths and abilities less frequently than any of the other teachers. Ms. Allen drew on this knowledge once and Mr. Zimmer did not demonstrate it at all (see table 4.6). In addition, both Mr. Zimmer and Ms. Allen enacted the practice component of evaluating the features of a problem more frequently than several of the other teachers. Ms. Allen did so 24.0% of the time and Mr. Zimmer did so 38.7% of the time, more than any of the other teachers (see table 4.3).

As might be expected, both Ms. Allen and Mr. Zimmer drew on knowledge of common student errors or places where they may struggle every time they enacted the practice component of thinking about common student errors or places where they will have difficulty or success. Neither of them drew on knowledge of students' strengths and abilities when enacting this practice component (see table 4.11). In addition, when enacting the practice components of thinking about pieces of the end goal, sequencing problems, and evaluating the features of a problem, Ms. Allen and Mr. Zimmer drew on knowledge of common student errors or struggles more frequently than all of the other teachers. For each of the three practices, Ms. Allen drew on this knowledge at least 20% of the time and Mr. Zimmer did so at least 33.3% of the time (see tables 4.9, 4.10, and 4.12). Similarly, when enacting the component of evaluating the features of a problem, Ms. Allen and Mr. Zimmer had the highest frequencies of co-occurrence of the practice component of thinking about common student errors or places where they will have difficulty or success (16.7% for Ms. Allen and 50.0% for Mr. Zimmer, see table 4.12).

Compared to the other teachers, Ms. Allen and Mr. Zimmer less frequently engaged in practice components that involved breaking down the content the examples were intended to address. Both Ms. Allen and Mr. Zimmer enacted the practice component of thinking about pieces of the end goal (components, sub goals, or scaffolds) less frequently than the other teachers, with the exception of Mr. Clark. Ms. Allen thought about pieces of the end goal 10.7% of the time, Mr. Zimmer did so 9.7% of the time, and Mr. Clark did so 9.6% of the time (see table 4.3). Ms. Allen and Mr. Zimmer also engaged in the practice of sequencing less frequently than all of the other teachers (6.7% and 9.7%, respectively, see table 4.3). In terms of the knowledge related to breaking down the content, Ms. Allen and Mr. Zimmer drew on knowledge of how to strategically sequence examples for a purpose and recognized nuanced differences/

subsets within a larger set of problems that others would group as one set less frequently than any of the other teachers. Mr. Zimmer drew on knowledge of how to strategically sequence examples for a purpose 12.9% of the time and Ms. Allen did so 8.0% of the time (see table 4.6). Mr. Zimmer recognized nuanced differences/ subsets within a larger set of problems that others would group as one set 32.3% of the time and Ms. Allen did so 20.0% of the time (see table 4.6).

When enacting the practice component of thinking about pieces of the end goal, Ms. Allen and Mr. Zimmer had the lowest frequencies of drawing on knowledge of how to strategically sequence examples for a purpose and recognizing nuanced differences within a larger set of problems that others would group as one set. Mr. Zimmer did not draw on either type of knowledge when enacting this practice. Ms. Allen drew on knowledge of how to strategically sequence examples for a purpose in 12.5% of the instances in which she thought about pieces of the end goal. Other teachers did so between 22.2% and 86.4% of the time. Similarly, Ms. Allen recognized nuanced differences within a larger set of problems that others would group as one set in 50% of the instances in which she thought about pieces of the end goal, while other teachers did so between 64.7% and 88.9% of the time (see table 4.9). Similarly, Ms. Allen had the lowest frequencies of drawing on these two types of knowledge when enacting the practice component of evaluating features of a problem (33.3% for recognizing nuanced differences within a larger set of problems that others would group as one set and 38.9% for knowledge of how to strategically sequence examples for a purpose, see table 4.12). Mr. Zimmer had the second lowest frequency (8.3%) of drawing on knowledge of how to strategically sequence examples for a purpose when enacting the practice component of evaluating the features of a problem (see table 4.12). Ms. Allen and Mr. Zimmer also had the lowest frequencies of recognizing nuanced differences within a larger set of problems that others would group as one set when enacting the practice component of sequencing problems (see table 4.10).

Overall, Ms. Allen and Mr. Zimmer were most focused on student errors and preventing those errors. They most frequently thought about common student misconceptions or places where they would struggle or succeed. In doing so, they frequently drew on knowledge of student errors and misconceptions, but infrequently drew on knowledge of student strengths. Compared to the other teachers, they less frequently thought about pieces of the end goal and sequenced examples.

Mr. Clark and Mrs. Stone

The third pattern that emerged can be seen in the practice components enacted and the knowledge drawn on by Mr. Clark and Mrs. Stone. These two teachers were generally between Mr. Baker/Mrs. Kelly and Ms. Allen/Mr. Zimmer in terms of their frequencies enacting different components of the practice of selecting examples and drawing on particular types of knowledge. Both Mr. Clark and Mrs. Stone thought about the end goal frequently. Mr. Clark did so during 9.6% of the time, more than any other participant, and Mrs. Stone did so 4.8% of the time when selecting examples (see table 4.3). Mr. Clark also thought about pieces of the end goal (components, sub goals, or scaffolds) 9.6% of the time, less than any other participant. Mrs. Stone thought about pieces of the end goal 14.3% of the time, which was more frequent than both Ms. Allen and Mr. Zimmer, but much less frequent than the other teachers (see table 4.3). Both Mr. Clark and Mrs. Stone were in the middle in terms of their frequency of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set. However, their frequencies of 50.7% and 46.0%, respectively, were closer to those of Mr. Baker and Mrs. Kelly than to those of Ms. Allen and Mr. Zimmer (see table 4.6).

Despite his overall lower frequency of thinking about pieces of the end goal, when Mr. Clark enacted this practice, he had the second highest frequency of recognizing nuanced differences within a larger set of problems that others would see as one set (85.7%). Only Mr. Baker had a higher frequency (88.9%, see table 4.9). Mrs. Stone's frequency of drawing on this knowledge when enacting the practice component of thinking about pieces of the end goal was in the middle of the other teachers.

Mr. Clark and Mrs. Stone enacted the practice component of thinking about common student errors or places where they will have difficulty or success more frequently than Mr. Baker and Mrs. Kelly, but less frequently than Ms. Allen and Mr. Zimmer (see table 4.3). They also drew on knowledge of common student errors or struggles more frequently that Mr. Baker and Mrs. Kelly, but less frequently than Ms. Allen and Mr. Zimmer. Mrs. Stone drew on knowledge of students' strengths and abilities more frequently than any of the other teachers (11.1% of the time) and, with the exception of Mrs. Kelly (9.7%), Mr. Clark drew on this knowledge more frequently than the rest of the teachers (8.2%, see table 4.6). With the exception of Mr. Zimmer (38.7%), both Mr. Clark and Mrs. Stone enacted the practice component of evaluating the features of a problem more than all other teachers. They did so 30.1% and 36.5% of the time, respectively (see table 4.3).

When enacting the practice component of thinking about common student errors or places where they will have difficulty or success, Mrs. Stone and Mr. Clark were more balanced than most of the other teachers in their use of knowledge of students' strengths and abilities compared to their use of knowledge of common student errors or places where they may struggle (see table 4.11). In addition, Mrs. Stone had the second highest frequency of drawing on students' strengths and abilities (41.7%) when enacting the practice component of thinking about common student errors or places where they will have difficulty or success (see table 4.11).

Mr. Clark enacted the practice component of sequencing problems 16.4% of the time and Mrs. Stone did so 19% of the time (see table 4.3). They engaged in this practice more frequently than Ms. Allen and Mr. Zimmer, and less frequently than Mr. Baker and Mrs. Kelly. Similarly, Mr. Clark and Mrs. Stone drew on knowledge of how to strategically sequence examples for a purpose more frequently than Ms. Allen and Mr. Zimmer, and less frequently than all of the other teachers. Both Mr. Clark and Mrs. Stone also drew on knowledge of the curriculum (see table 4.6).

When enacting the practice components of thinking about pieces of the end goal and evaluating the features of a problem, Mr. Clark and Mrs. Stone's frequencies of recognizing nuanced differences within a larger set of problems that others would group as one set were in the middle of the other participants (see tables 4.9 and 4.12). However, when enacting the component of sequencing problems, Mr. Clark more frequently recognized nuanced differences within a larger set of problems that others would group as one set than any of the other teachers (he did so in 83.3% of the instances in which he enacted the practice). Mrs. Stone's frequency for using this knowledge when sequencing problems was in the middle of the other teachers (see table 4.10).

Overall, Mr. Clark and Mrs. Stone were similar to Ms. Allen and Mr. Zimmer in some ways and similar to Mr. Baker and Mrs. Kelly in others. They frequently thought about the end goal and drew on knowledge of students' strengths and abilities. They also more frequently evaluated the features of different problems than any of the other teachers.

Summary

In this chapter, I sought to better understand the complex practice of selecting examples. In doing so, I described the components of the practice of selecting examples and the knowledge teachers drew on in enacting those components. I then looked at the knowledge teachers drew on when enacting particular components. Finally, I discussed patterns across teachers as they selected examples and the knowledge they drew on in doing so.

The four most frequently enacted components of the practice of selecting examples are evaluating the features of a problem, sequencing problems, thinking about pieces of the end goal, and thinking about common student errors or places where they will have difficulty or success. When selecting examples, teachers most frequently drew on common content knowledge, both knowledge of how to carry out a procedure and of mathematical ideas, and the specialized content knowledge of recognizing nuanced differences/ subsets within a larger set of problems that others would group as one set. Teachers also frequently drew on knowledge of how to strategically sequence examples for a purpose. When selecting a set of examples for a particular purpose, teachers drew on multiple domains of knowledge. Across the set of explanations, at least four different types of knowledge occurred when enacting each component.

Patterns emerged across the teachers in the components of the practice the enacted and the knowledge they drew on in selecting examples. Compared with the other teachers, Ms. Allen and Mr. Zimmer more frequently thought about common student errors or places where students would have difficulty or success. They more frequently drew on knowledge of common student errors and infrequently drew on knowledge of students' strengths. They also sequenced problems less frequently than all of the other teachers and thought about pieces of the end goal less frequently than all but one of the other teachers. This suggests that they were selecting examples to surface common misconceptions, or places where they thought students might have difficulty, instead of focusing their examples on key mathematical ideas, or sub goals in the lesson, which was more frequently seen among the other teachers.

In contrast to Ms. Allen and Mr. Zimmer, two other teachers did the opposite. Mr. Baker and Mrs. Kelly sequenced problems more frequently than all of the other teachers and thought about pieces of the end goal more frequently than several of the other teachers. They were the only two teachers who did not think about a more general end goal and thought about common student errors or places where students would have difficulty or success less frequently than all but one other teacher. This suggests that they were selecting examples based on key mathematical ideas, or sub goals in the lesson, instead of focusing solely on demonstrating common student misconceptions. It further suggests that they had a good understanding of the key mathematical ideas of the lesson, not just the overall end goal, as well as how to sequence them for students.

In short, both Ms. Allen and Mr. Zimmer appeared to be selecting their examples with the purpose of avoiding common misconceptions. In contrast, Mr. Baker and Mrs. Kelly appear to be focused on the mathematical goals of the lesson. In particular, they are focused not on an overarching goal, such as solving rational equations, but on smaller pieces of the goal that can be approached with individual problems. For example, instead of focusing on an overarching goal of selecting problems to teaching solving rational, they saw the goal as a sequence of different features of increasing mathematical complexity, and selected examples to match those individual features.

CHAPTER 5: THE PRACTICE OF GIVING EXPLANATIONS

In this chapter, I look at the practice of giving explanations. I begin with two cases of teachers giving explanations and then discuss what I considered an explanation in the data. Next, I describe the four categories of explanations, with examples of each type. I then focus on two categories of explanations and look at the knowledge teachers demonstrated when giving explanations of each type. Finally, I look at patterns in the types of explanations given and knowledge used across teachers.

What Does it Mean to Give an Explanation?

In this section, I present two cases from my study of teachers giving explanations. In the first case, Mr. Clark explains to students why cross multiplication works using the problem $\frac{4}{x} = \frac{9}{7}$. In the second case, Mr. Baker explains how to find the x-intercepts of a function and why the function $f(x) = \frac{1}{x-1}$ has no x-intercepts. I then discuss what I considered to be an explanation in my analysis.

Case 1:

> Mr. Clark: Since you've seen it before in Algebra I, the bottoms come to the tops, on the other side. Cross multiplying ends up doing that. Why does it work? In algebra, aren't we supposed to do something to both sides of the equation? Well, what are we doing? We're not doing anything to both sides of the equation. We're just taking stuff and moving it around. It's magic. Yes, that's not even math right now, it seems like it's just cheating.
>
> What you're actually doing is not magic. But, let's assume to start this problem, I multiply both sides by *x*, what would be the final result? On the left, what would happen? The *x*s cancel, don't they? Didn't I just do something to both sides? That's one of our algebra rules, right? Multiplying both sides by *x*, those cancel. Didn't this *x* kind of magically

move up there? What's the next thing I'm going to multiply both sides by? *4*, no. 7. Very good. When I multiply both sides by *7*, these cancel, and so the *7* came up here, and the *x* came up there.

That's really what you're doing. Cross multiplying is just multiplying both sides by two different things at the same time, and being too lazy to write down what they were.

Student: The first way is easier, though.

Mr. Clark: The first way's easier, right? Totally agree. That's why we do it that way. But, it's helpful to understand why it works. In fact, here's what's even more clever, ready for blow my mind time? I would like you to take note of this, *7x*, do you see that little expression that's on both sides right now, even though that's in the wrong order, do you see that *7x*? Look at those two fractions that you see at the beginning of the problem. How does *7x* have to do with those two fractions?

Student: Common denominator.

Mr. Clark: Oh. If I told you to give me the LCD [least common denominator] of those two fractions that you see, what would it be?

Student: *7x*.

Mr. Clark: *7x*. Put it on both sides, that is going to be the rule for this section, as we get more complex. Find the lowest common denominator, multiply both sides by it. Cross multiplying is just a fast version of doing exactly that. You don't have to write all of it down, you can just cross multiply because you're all smart, and whatever, but the point of the matter is that you should have an idea of what you're doing. (Mr. Clark 4-30-15 Lines 40 – 79)

Mr. Baker and x-intercepts: An explanation of no x-intercepts

Mr. Baker: Next, intercepts, intercepts, also known as?

Student: Zeros.

Mr. Baker: Zeros, there we go. x-intercepts for the zeros. To review, how do we find the zeros of a function?

Student: Set it equal to 0.

Mr. Baker: Set it.

Student: The function.

Mr. Baker: The function. Set the function equal to zero. Don't use your its's and the's. We want to set f of x equal to zero.

Student: Just multiply it by *x* minus *1*.

Mr. Baker: Here is the thing we don't have to do that. When is a fraction going to equal *0*?

Student: When the top equals *0*.

Mr. Baker: When the numerator is *0*. A fraction is going to equal *0* when our numerator is equal to *0*. *0* divided by a number, *0* divided by anything.

Student: *0*.

Mr. Baker: Anything divided by *0*?

Student: [inaudible 00:03:40].

Mr. Baker: Bad, the world explodes right. Bad. In this case we want the numerator equal to zero. Let's set the numerator equal to *0*. *1* is never going to equal. *1* will never equal *0*, so what does that tell us? This has no x-intercepts, no x-intercepts. What's that?

Student: Ah, no, no, no.

Mr. Baker: Because *1* is a constant.

Student: Yeah.

Mr. Baker: There is no variable. I can't turn a *1* into a *0*. (Mr. Baker 3-25-15 Lines 43 – 87)

What is Considered an Explanation?

As demonstrated by the vignette at the beginning of this dissertation, explanations are an integral part of mathematics teaching. Drawing on Leinhardt's (2001) instructional explanations, I consider explanations to be instances in which the teacher is communicating mathematics content to students. This content can include mathematical concepts or procedures, as well as information about the discipline of mathematics and how mathematics is conducted within the discipline. These explanations may be completely teacher led, or may include student participation through questions. Although the teachers are giving explanations about mathematics, the explanations in my data are not looked at as purely mathematical explanations. It would be expected that these explanations are mathematically accurate and contain mathematical reasoning. However, the purpose of these explanations is not to make a mathematical claim, but instead to communicate mathematics with students. A purely mathematical explanation would not consider this implied pedagogical purpose.

Although explanations can be more broadly thought of as answering the question of why within a specific domain (Leinhardt, 2001), in my data, not all of the explanations contain discussion of an underlying why. In some instances, the teachers shared information, but did not discuss the underlying mathematical reasoning of why or how. For example, in some instances, teachers talked through the steps of a procedure, but did not discuss why these steps were valid or why they accomplished a particular goal. I consider this to be an explanation because the teacher is communicating knowledge of the procedure and how to carry it out to students.

When distinguishing explanations within the classroom observation data, I identified instances of explanations based on the purpose of the explanation being given. By purpose I mean the main piece of mathematics the teacher was sharing with students. For example, a teacher might explain how to solve a particular rational equation, the definition of the word domain, or why reducing terms in a rational expression is not mathematically valid. Some explanations were short in length. For example, at the beginning of a lesson on removable discontinuities, Mr. Johnson explained what discontinuity means. “Discontinuity means something discontinues. It goes along, goes along, goes along and then it stops, right? Discontinued” (Mr. Johnson 4-2-15 Lines 24 – 25). He went on to mention that discontinuities are a fundamental idea in Calculus, but the entire explanation was only a few sentences in length.

Other explanations were significantly longer, taking fifteen or more minutes. For example, Mr. Baker’s explanation of how to graph a rational equation based on some known pieces of information was one of the longer explanations. In it, he described how to use the intercepts and asymptotes of a rational function, which the class had already found, to begin to graph the rational function. He then described how to pick different values for *x* that could be substituted into the equation to find additional coordinate points that are part of the graph. Using these additional points, Mr. Baker showed students how to reason using the known information to sketch in the rest of the graph. Each of these pieces could potentially be considered an explanation by itself. However, Mr. Baker's overall purpose was to show students how to graph the rational equation. Each of these components can therefore be seen as contributing to the larger purpose and therefore form one explanation.

After splitting the observation transcripts into explanations, there were 110 explanations across the 9 observations. Each of the interview prompts gave a single purpose. Teachers' responses to each prompt were therefore considered a single explanation. Four teachers responses to the three prompts yielded 12 explanations from the interviews. The distribution of explanations by teacher for the observations and interviews can be seen in table 5.1. In the next section, I describe how I categorized these explanations.

Table 5.21: Number of Explanations Given by Teacher for Classroom Observations and Interviews

	Observation Explanations				Interview Explanations	Total Explanations
Teacher	1	2	3	Total		
Mr. Baker	15	11	9	35	3	38
Mr. Clark	19	14	12	45	3	48
Mr. Johnson	9	6	--	15	3	18
Mrs. Stone	15	--	--	15	3	18
Total	--	--	--	110	12	122

Categorizing Explanations

In this section, I begin by describing each of the four types of explanations. I then look at patterns in the types of explanations each teacher gave in their observed lessons and during the interviews.

Categories

As described above, the instances of explanations in the data correspond to the teacher communicating mathematics to students. However, these explanations contained varying amounts of mathematical reasoning. In analyzing these explanations, I began by looking at the reasoning in each explanation. Explanations were categorized into one of four groups, which are ordered from least to most mathematical reasoning; *problematic*, *procedural*, *superficial reasoning*, and *mathematical reasoning*. Brief descriptions of each type of explanation are given in table 5.2 below. Given the lengths of many of the explanations, one explanation could potentially fit into multiple categories. Each explanation was placed into only one category. This was determined based on the purpose of the explanation and the way that component of the explanation was explained.

Table 5.22: Explanation Types

Explanation Type	Description
Problematic	Explanations which are imprecise, confusing, or mathematically incorrect
Procedural	Explanations which go through procedural steps and do not provide any reasoning about why the procedure is carried out the way it is
Superficial Reasoning	Explanations which contain some superficial reasoning by (1) describing conditions when something happens instead of why it happens; or (2) using short hand descriptions without discussion of deeper ideas (these short hand descriptions may reference deeper ideas)
Mathematical Reasoning	Explanations that contain deep mathematical reasoning explaining the why or how underlying a mathematical concept or procedure The deep explanation is about the purpose of the explanation, other parts of the explanation may provide more superficial reasoning or be procedural Explanations are also responsive to students

Problematic explanations. Explanations were categorized as problematic for three reasons. First, a problematic explanation might be imprecise. For example, when explaining what a rational function is, one teacher described rational functions as related to fractions. He then stated, "a rational function means we've got a top and a bottom" (Mr. Clark 4-15-15 Line 556). Although it is true that rational functions are fractional equations with polynomials in the numerator and denominator, top and bottom are not mathematically precise terminology. Instead, for the explanation to be considered precise, the teacher could have used the terms numerator and denominator.

Second, explanations were labeled as problematic when they would be confusing from the perspective of a student. For example, when discussing the domain and range of the function $y = \frac{-4}{x}$, one teacher described the domain and range as the same. While the domain and range in this case are both all real numbers except zero, domain and range are two distinct concepts. Domain is the set of values that the independent variable, x, can be and range is the set of values that the dependent variable, *y*, can be for the given function. Because the domain and range correspond to different variables, they are often different sets of values. By describing them as the same, students may believe the domain of any rational function will be the same as the range of the function, which would be a misconception.

Finally, explanations were labeled as problematic if they were mathematically incorrect. For example, when explaining how to graph the equation $y = \frac{2}{x-3} - 4$, a student asked how the *2* affected the graph. The teacher responded that they did not need to do anything with the *2*, but that it would matter if the value were negative. This is not correct. For an equation of the form $y = \frac{a}{x-h} + k$, when the value of *a* is negative, the graph of the equation is reflected over the x-axis. However, other values of *a* affect the graph by stretching or shrinking it.

Seven of the 110 explanations given during the classroom observations were categorized as problematic. In some cases, when a longer explanation contained an instance of problematic terminology or a small mathematical error, which was not directly related to the purpose of the explanation, it was labeled as one of the other categories based on the part of the explanation focused on the overall purpose. None of the interview explanations were categorized as problematic.

Procedural explanations. Explanations were tagged as "procedural" if the steps of a procedure were shown or worked through, but without reasoning about why the procedure works or is carried out the way it is. For example, near the end of her lesson on solving rational equations, Mrs. Stone solved the problem $\frac{2}{x-3} = \frac{1}{x^2 - 2x - 3}$ with students after they had an opportunity to try the problem on their own:

Mrs. Stone: Your least common multiple being the *x* minus *3* and the *x* plus *1*. On the left, we end up with a *2x* plus *2*. On the right, we end up with a *1*. Does negative one half work?

Student: Yeah, no.

Mrs. Stone: No, yes?

Student: Yeah.

Mrs. Stone: Yes. What can we not have for an answer? *3*? Or a negative *1* for this guy. [Points to the denominator of the second fraction.] (Mrs. Stone 3-26-15 Lines 466 – 480)

In this explanation, Mrs. Stone briefly talked through some of the steps of the procedure and showed the steps on the board, but did not explain why any of the steps were carried out. Although, Mrs. Stone does ask students if the answer works, there is no explanation of why *3* and *-1* being restricted values means that the answer of $-\frac{1}{2}$ works. She might have explained that the only restricted values are *3* and *-1*, which cause one of the denominators to be 0. Since the final answer is neither of those values, $-\frac{1}{2}$ is a valid solution. She might also have suggested students check the answer to make sure it is correct. Because she talked through steps of the procedure but did not discuss why the steps were used, this explanation was categorized as procedural. Out of 110 explanations teachers gave during the classroom observations, 38 were categorized as procedural. Only one of the interview explanations was labeled as procedural.

Superficial reasoning explanations. Superficial reasoning explanations contained some amount of reasoning, but this reasoning did not address the mathematical why or how underlying the content being explained. Explanations were categorized as superficial reasoning for two reasons. First, the teacher explained the conditions in which a mathematical phenomenon happens instead of why it happens. For example, when explaining how to graph

$f(x) = \frac{1}{2(x+3)} - 2$, Mr. Johnson uses the equation of the general form of a rational equation,

$y = \frac{a}{x-h} + k$, to explain how the values of *a*, *h*, and *k* affect the graph. Before this explanation, Mr. Johnson had discussed with students how the values of *a*, *h*, and *k* in the general forms of different functions shift and stretch the graph of the parent function.

> Mr. Johnson: So, the question here is *a*. What does *a* stand for in this expression? [Student], what have you got?
>
> Student: The stretch or shrink.
>
> Mr. Johnson: So let's write that down, stretch or shrink. Thanks boss. What about this denominator *x* minus *h*? What does it tell us [Student]?
>
> Student: Horizontal shift
>
> Mr. Johnson: Boss, this is a horizontal shift, yeah! This is what moves this function horizontally, left to right. Nice boss. Horizontal shift. The reason that I thought that that was a big deal that I said that like that was because a lot of times we get confused because really, when we actually make this line from this, what's the line going to look like? When we draw the line on the graph. I see somebody's doing that thing with their hand to show it. What are they doing? What's the line going to look like. Hey, [Student].
>
> Student: Vertical
>
> Mr. Johnson: Yeah. It's going to be a vertical line. That was what [student] was doing, this thing going up or down with his hand. That means [Student] has a thing, it's a horizontal shift, and not get that confused with a vertical line, so that's why I was impressed with that. This one over here, the last one, *k*? What's that, [Student]?
>
> Student: Vertical shift.
>
> Mr. Johnson: And that's the vertical shift. So vertical shift. (Mr. Johnson 4-2-15 Lines 149 - 175)

In this quote explains that the *h* value causes a horizontal shift in the graph, and that the related line on the graph is vertical. In doing so, Mr. Johnson is describing the conditions that cause the graph to shift. However, he does not give a mathematical explanation as to why the *h* value causes the graph to shift or why the related line is vertical. The explanation was therefore categorized as superficial reasoning.

Second, explanations were labeled as superficial reasoning when teachers used short hand descriptions that could potentially reference deeper ideas, but did not discuss the deeper ideas. For example, Mr. Baker frequently referred to division by zero being undefined as, “We cannot divide by *0*. Cannot divide by *0*, the world ends” (Mr. Baker 3-25-15 Line 37). By itself, this statement might be considered problematic. However, in the context of this lesson, Mr. Baker repeatedly used this phrase to signify that division by *0* is undefined, making it a short hand description that references the mathematical idea that division by *0* is undefined, but does not explain why. This explanation was therefore categorized as superficial reasoning. Out of the 110 explanations teachers gave during the classroom observations, 52 were categorized as superficial reasoning. Eight of the twelve interview explanations were categorized as superficial reasoning.

Mathematical reasoning explanations. Deep mathematical reasoning occurred in 13 of the 110 explanations given during the classroom observations and three of the twelve interview explanations. These explanations provide mathematical reasoning addressing the underlying why or how of a mathematical concept or procedure. After solving the rational equation $\frac{3}{u+2} = \frac{1}{u-2}$ with his students, Mr. Baker explained why students must check for extraneous solutions when solving rational equations. His explanation was categorized as a mathematical reasoning explanation. He explained:

> Anytime we change the type of equation, we have to check for extraneous solutions because there may be situations where we solve this equation and that solution is not a solution to the original equation. We said, using the multiplication property of equality, the equations are going to be equivalent. The solutions to this [$\frac{3}{u+2} = \frac{1}{u-2}$] will be the solutions to this equation [$3u-6 = u+2$]. That is true, but sometimes we get more

> solutions. That's why we have to check, because we're changing the type of equation (Mr. Baker 3-31-15 Lines 156 – 161).

This explanation was categorized as mathematical reasoning because the teacher provided a mathematical explanation about why extraneous solutions occur when solving rational equations. In particular, extraneous solutions can occur when the form of the equation is changed. Although this could have been explained in more detail, because Mr. Baker gave a mathematical reason about why extraneous solutions occur when solving rational equations, this explanation was categorized as mathematical reasoning. In particular, he drew on the multiplication property of equality, which students were familiar with. If Mr. Baker had instead said that extraneous solutions occur when solving rational equations, the explanation would have been categorized as superficial reasoning.

For an explanation to be coded as mathematical reasoning, the focus of the explanation needed to be explained in mathematical detail. However, other parts of the explanation might contain superficial reasoning or be procedural in nature. These explanations are labeled mathematical reasoning because the underlying why or how of the mathematics is addressed. However, this label does not imply that the explanation of the underlying why or how is mathematically complete. Mathematical reasoning explanations are also attuned and responsive to students. In particular, the mathematics is discussed at a level that is appropriate for students in the class. In the explanation above, Mr. Baker drew on the multiplication property of equality, making his explanation mathematically accurate and, because his students had knowledge of this property, accessible for students.

Differences Across Teachers

There were differences across teachers in terms of the types of explanations they gave during their classroom teaching and during the interviews. The number and frequency of each type of explanation by teacher for both the classroom observation and interview explanations can be seen in tables 5.3 and 5.5. Table 5.4 provides the category given to each explanation given during the interview.

Table 5.23: Frequency of Explanation Types During Classroom Observations by Teacher

			Explanations of Each Type							
			Problematic		Procedural		Superficial Reasoning		Mathematical Reasoning	
Teacher	Observations	Total Explanations	n	%	n	%	n	%	n	%
Mr. Baker	3	35	0	0.0	5	14.3	21	60.0	9	25.7
Mr. Clark	3	45	7	15.6	27	60.0	11	24.4	0	0.0
Mr. Johnson	2	15	0	0.0	1	6.7	10	66.7	4	26.7
Mrs. Stone	1	15	0	0.0	5	33.3	10	66.7	0	0.0
Total	9	110	7	6.4	38	34.5	52	47.3	13	11.8

Table 5.24: Explanation Types for Interview Explanations

	Explanation		
Teacher	Graphing a Rational Equation	Simplifying a Rational Expression	Solving a Rational Equation
Mr. Baker	Superficial Reasoning	Mathematical Reasoning	Mathematical Reasoning
Mr. Clark	Superficial Reasoning	Superficial Reasoning	Superficial Reasoning
Mr. Johnson	Superficial Reasoning	Superficial Reasoning	Mathematical Reasoning
Mrs. Stone	Procedural	Superficial Reasoning	Superficial Reasoning

Table 5.25: Frequency of Interview Explanation Types by Teacher

	Interview Explanations of Each Type							
	Problematic		Procedural		Superficial Reasoning		Mathematical Reasoning	
Teacher	n	%	n	%	n	%	n	%
Mr. Baker	0	0.0	0	0.0	1	33.3	2	66.7
Mr. Clark	0	0.0	0	0.0	3	100.0	0	0.0
Mr. Johnson	0	0.0	0	0.0	2	66.7	1	33.3
Mrs. Stone	0	0.0	1	33.3	2	66.7	0	0.0
Total	0	0.0	1	11.1	8	88.9	3	33.3

All of the teachers gave at least one explanation that was labeled as procedural and several explanations that included superficial reasoning. In both the interview and observation settings, only Mr. Baker and Mr. Johnson gave explanations that were categorized as mathematical reasoning. Mr. Clark was the only teacher who gave problematic explanations.

There were also differences in the amount of explaining done by each teacher during one observation. This was not always a function of how many discrete explanations were given. Several of Mr. Johnson's explanations were long, and he therefore gave fewer explanations per observed lesson. Mr. Clark and Mrs. Stone gave, on average, more explanations in one observed lesson than Mr. Baker and Mr. Johnson.

Explanations given in the interviews were more frequently categorized as superficial reasoning or mathematical reasoning than were the explanations given during observations of classroom instruction. These explanations potentially include stronger reasoning because they were the only explanation the teacher gave about a specific concept or procedure. In contrast, during classroom instruction, teachers' explanations across one lesson frequently covered similar content, where a teacher might not provide significant depth on the same concept or procedure multiple times. During several of the classroom observations, when a teacher gave several explanations about the same concept or procedure, the first explanation contained deeper reasoning than the subsequent explanations. This is perhaps because the teacher focused on why or how a procedure worked in their initial explanation to expose students to the underlying mathematical reasoning, but in later explanations focused instead on students' ability to execute the procedure.

There were no problematic explanations given during the interviews and only one procedural explanation. Although each teacher only gave three explanations during the interviews, the types of explanations they gave in the interviews were all types of explanations seen in the observations. For example, Mr. Johnson's interview explanations were categorized as superficial reasoning and mathematical reasoning. He gave both of these types of explanations during his observed lessons. I now look specifically at the patterns in each teacher's explanations.

Mr. Clark. The majority of Mr. Clark's classroom explanations (60%) were categorized as procedural. Several of his classroom explanations (24.4%) were categorized as superficial reasoning. In contrast, all three of Mr. Clark's interview explanations were labeled as superficial reasoning, perhaps because they were all explanations in which a new mathematical idea or procedure is being introduced and he therefore included more explanation of the underlying mathematics he was discussing.

When teaching, Mr. Clark tended to introduce a topic using superficial reasoning and then give several shorter explanations where he worked through similar examples with students. For example, in his lesson on graphing rational equations, Mr. Clark's first explanation of the lesson provided some reasoning about what happens to the graph of $y = \frac{3}{x}$ as x approaches infinity:

> Mr. Clark: Is there someone that can remember why does it make sense that as we go this way, the graph gets closer and closer and closer to the x-axis, but it doesn't appear to hit the x-axis. Think about the equation.
>
> Student: Because x is always getting bigger, so the denominator is getting bigger.
>
> Mr. Clark: He said as x is getting bigger, think about what's happening with this fraction. If x is *1*, what's y? What's *3* divided by *1*?
>
> Student: *3*.
>
> Mr. Clark: *3*, normal, but as x gets huge, what happens? *3* over a thousand, *3* over a million, right? It's going to get really tight. It won't hit *0*. (Mr. Clark 4-15-15 Lines 42 – 55)

This explanation was categorized as superficial reasoning because Mr. Clark explained that for this equation, dividing *3* by larger and larger values of *x* will cause the graph to get close to the x-axis, but will never be *0* based on the pattern of numbers he discussed. In doing so, he gave a reason for what happened, but he did not go beyond the pattern and address the mathematics underlying why this happened. In particular, that as *x* approaches infinity, *3* is being divided by larger and larger numbers. $\frac{3}{x}$ will therefore get smaller and smaller and approach *0*, but will never reach it, causing the asymptote at y=0. Later in the lesson, several of his explanations consisted of finding the asymptotes from the equation, drawing them, and then sketching in the shape of the graph. However, how the graph approaches the asymptotes and why it does not touch them were not discussed during these explanations. These explanations were therefore categorized as procedural.

Mr. Clark was the only teacher to give explanations that were categorized as problematic. In several of these explanations, this was due to imprecise language. For example, when introducing rational functions, Mr. Clark mentioned that the beginning of the word rational is ratio and should get students thinking about fractions and division. He then stated, “a rational function means we've got a top and a bottom. That's all that's saying. There should be a top and a bottom to it” (Mr. Clark 4-15-15 Lines 556 – 557). A more precise definition might have used the terms numerator and denominator.

Some of Mr. Clark’s other explanations, which were categorized as procedural or superficial reasoning, also contained imprecise language. For example, he used the term “de-foil” when talking about factoring a quadratic. “Foil” is an acronym used to help students remember how to distribute when multiplying two binomials, but is not a verb. “De-foil” is not a mathematical phrase. However, in these cases, the imprecise language was not directly related to the main purpose of the explanation. Instead, the explanation was categorized based on other features of the explanation more closely related to the explanation's purpose, such as greater mathematical detail or superficial reasoning.

Mrs. Stone. The majority of Mrs. Stone's explanations (66.7%) contained superficial reasoning. For example, when explaining why students need to check for extraneous solutions when solving rational equations, Mrs. Stone explains that extraneous solutions arise when you get a solution that *x* cannot equal and that they are caused by violating rules. Two rules she might be referring to are that the denominator of a fraction cannot be *0*, or that you cannot take the logarithm of a negative number.

> Remember an extraneous solution when we go and do the mathematics part of this, sometimes what we're doing with of different things like multiplying both sides or dividing both sides […] that mathematically it seems to make sense, but we're violating rules when dealing with functions. When you go back to plug in the original question, it doesn't really make any sense. For instance when we did logarithms, we had questions where maybe it was something like the log of I don't know, *2x*. We came out with an answer that *x* was equal to a negative *4*. We plugged it back in and we said that doesn't work because you can't take a logarithm with negative numbers. That was extraneous, like we talked about. We're doing the same thing when we deal with rational expressions. Anytime we have to deal with things like *x*s in denominators, think about what we did with asymptotes, restricted domains, when there are certain values that *x* cannot be equal to. We just have to go back and double check to make sure we're not having a problem with that. (Mrs. Stone 3-26-15 Lines 109 - 128)

One third of Mrs. Stone's explanations were categorized as procedural and she was the only teacher of the four to give a procedural explanation during the interview. The types of explanations Mrs. Stone gave during the interview were consistent with the types of explanations she gave during her observed lessons. The majority of Mrs. Stone's procedural explanations during her observed lesson were given when solving a problem that students had already tried on their own. Like Mr. Clark, these explanations frequently came after similar explanations on the same content, which often contained more reasoning.

Mr. Johnson. The majority of Mr. Johnson's explanations (66.7% of both the observed classroom explanations and the interview explanations) contained superficial reasoning. He gave one procedural explanation during an observed lesson, and the rest of his explanations were categorized as mathematical reasoning. Mr. Johnson's explanations were often long, using a single problem in great detail to focus in on a concept.

In his lessons, Mr. Johnson placed an emphasis on reasoning. The majority of his explanations were structured as a string of questions for students, which frequently included why. For example, "Why did we draw a line at that point? [...] What is it about this value, negative *3*, that forces us to put a vertical line over there?" (Mr. Johnson 4-2-15 Lines 283 – 285), "Why are we setting it equal to *0*" (Mr. Johnson 4-2-15 Line 537), and "Why is negative *4* a discontinuity?" (Mr. Johnson 4-14-15 Line 126).

Some of his explanations included deeper mathematical reasoning, including an explanation of why the function $f(x)=\dfrac{x^2-2x-8}{2x^2-8x}$ has a hole, or removable discontinuity, at *-4* and not a vertical asymptote. Before this explanation, Mr. Johnson had only discussed vertical asymptotes with students and had not introduced holes.

> Mr. Johnson: So what's going on with the *4*, what happened to the *4*? Chief?
>
> Student: In the calculator it canceled out the *x* minus *4*
>
> Mr. Johnson: It did, that's right. The calculator said no no no, don't do that. The calculator said just get rid of those because that can be divided. But wait a minute, hold on a second, isn't it true still though, that if we put in *4* we'll still have an undefined function? Isn't that right, yeah. [...] I blew up the graph and this is what the graph looks like. Look what's over there at *4*. What's that?
>
> Student: Oh it's a hole.
>
> Mr. Johnson: That's a hole. That's what we actually call a hole. A hole is the second type of discontinuity. So let me write over here, second type of discontinuity is what's known as a hole. What happens with a hole is that because the function divides out that factor there's no need for a vertical asymptote. However, as the function travels along and it hits

> the point 4, all the sudden what happens to the function. Here's the function, right there. What happens when the *4* gets in here. (Makes alarm noise) Right, it's an error. As soon as it passes through the hole to the other side, everything's cool. So it jumps over that hole. It's an inconsistency in the smooth function. (Mr. Johnson 4-2-15 Lines 759 – 782)

This explanation was categorized as mathematical reasoning because Mr. Johnson discusses the mathematics underlying why a hole occurs in the graph at $x=4$ because there is a common factor of $x-4$ in the numerator and the denominator. In particular, although the common factor can be reduced for most values of x, when x is 4, both of the factors are 0, making the function undefined. There is therefore a discontinuity, but only at that point, causing a hole in the graph.

Other explanations included reasoning that was more superficial. For example, when asking students what will make a function undefined, Mr. Johnson explained that the denominator of the function $f(x)=\dfrac{x^2-2x-8}{2x^2-8x}$ cannot be 0, but did not explain why a denominator of 0 causes the function to be undefined. This explanation came out of a discussion of how to find the vertical asymptotes by looking for the values that x cannot be.

> Mr. Johnson: So when a function is undefined, like if I have a function n divided by? What would make it undefined?
>
> Student: *0*
>
> Mr. Johnson: Yeah, that's all I need to know. Right, it was what would make the denominator *0*? (Mr. Johnson 4-2-15 Lines 417 – 422)

By explaining that a function is undefined when the denominator is 0, Mr. Johnson is giving a condition for when functions are undefined instead of addressing the underlying reasoning. This explanation was therefore labeled as superficial reasoning and not mathematical reasoning.

Mr. Baker. The majority (60.0%) of Mr. Baker's explanations during his classroom observations were categorized as superficial reasoning, and 25.7% were categorized as mathematical reasoning. Two of the three explanations Mr. Baker gave during the interview contained mathematical reasoning and the third contained superficial reasoning. Only a few

(14.3%) of his classroom explanations were procedural. Like the other teachers, Mr. Baker's procedural explanations often occurred when working through a problem similar to a previous example that was explained with more reasoning.

Mr. Baker's explanation of how to solve a rational equation using the problem $\frac{z^2}{3} - \frac{z}{6} = 1$ was categorized as mathematical reasoning.

> Mr. Baker: Why? I'm just going to multiply everything by *6*, just because. That's our favorite explanation. Do we have a property?
>
> Student: Yes, the identity property.
>
> Mr. Baker: Not the identity, the identity property is what we used yesterday when we multiplied by one. This is an equation, so it's a property of ...
>
> Student: Equality.
>
> Mr. Baker: Equality. What property of equality allows us to multiply both sides of an equation?
>
> Student: Multiplication property of equality.
>
> Mr. Baker: Wow […], great job. Great memory. We're going to use the multiplication property of equality alright and we're going to multiply both sides by *6*. Multiplication property of equality. *If a* is equal to *b*, then *a times c* is equal to *b* times *c*. Are these fractions ... Is the value of this equation, are these fractions going to be the same as those fractions?
>
> Students: Yes.
>
> Mr. Baker: Are we keeping it, are we keeping them equal? Is *6 times z over z* squared over *3*, equal *to z* squared over *3*?
>
> Students: No.
>
> Mr. Baker: No, they're not. This is not equivalent to that. What we're doing is we're making the equation equivalent because, what is our goal here?
>
> Student: To get rid of the fractions.
>
> Mr. Baker: To solve for *z*. Yesterday, we wanted to have one equivalent fraction. We wanted to make sure the fraction was always the same, had the same value. This is completely different. We're solving. I'm able to multiply by *z* or multiply by *6* because I'm not concerned about keeping each fraction equal. I want to keep the equation equal

> and this is still equivalent. Still going to have an equivalent fraction, so when we do this, we're going to distribute and everything is going to get multiplied by *6*. (Mr. Baker 3-31-15 Lines 27 – 67)

This explanation was labeled as mathematical reasoning because Mr. Baker discussed the underlying mathematics properties that allowed students to multiply to eliminate the denominators. He also differentiated between simplifying expressions, where the value of the expression must be maintained, and solving an equation, where equality must be preserved.

Knowledge Used by Teachers When Giving Explanations

In this section, I describe the different types of knowledge used by teachers as they gave explanations categorized as procedural and explanations categorized as mathematical reasoning. I begin with the procedural explanations.

Knowledge Used in Procedural Explanations

When giving procedural explanations, teachers drew primarily on common content knowledge. However, knowledge of content and students and specialized content knowledge were seen in a few of the explanations. Other types of knowledge were not seen. See table 5.6 for frequencies of knowledge use in the procedural explanations. Below I describe the ways in which teachers used common content knowledge, knowledge of content and students, and specialized content knowledge when giving procedural explanations.

Table 5.26: Types of Knowledge Used in Procedural Explanations by Teacher

	Knowledge Type		
Teacher	Common Content Knowledge	Knowledge of Content and Students	Specialized Content Knowledge
Mr. Baker	5	0	0
Mr. Clark	27	7	3
Mr. Johnson	1	0	0
Mrs. Stone	6	0	0
Total	39	7	3

Common content knowledge. Teachers drew on common content knowledge in all of the procedural explanations. As might be expected, they most often drew on knowledge of how to carry out a procedure. Teachers drew on knowledge of a wide range of procedures, including finding the domain of a function, finding the intercepts of a function, shifting the asymptotes of the graph of a basic rational function, factoring, simplifying a rational expression, and solving a rational equation.

On occasion, teachers demonstrated knowledge of mathematical concepts and ideas, including asymptotes, domain, and the distributive property. For example, early in his lesson on graphing rational equations, Mr. Baker defined domain as "all *x*s such that our denominator does not include *0*" (Mr. Baker 3-25-15 Lines 153-154). Later in his lesson, Mr. Baker explains how to find the domain of the rational function $g(x)=\frac{x^2-9}{x^2-x-2}$. In this procedural explanation, Mr. Baker draws on knowledge of the concept of domain and references the earlier definition, "Do exactly what our definition states. If our denominator does not equal *0*, set your denominator to not equal *0*. Simple as that. Factor and what can *x* not equal?" (Mr. Baker 3-25-15 Lines 346-348).

Knowledge of content and students (KCS). In seven of the procedural explanations, which were all given by the same teacher, Mr. Clark, the teacher demonstrated KCS by mentioning a common student error or misconception. All of these explanations were given by the same teacher, Mr. Clark. Although in these cases, Mr. Clark said that the error or misconception was incorrect, he did not explain why it was incorrect. For example, after a student gave the correct answer of $\frac{2}{x}$ when simplifying the fraction $\frac{8}{4x}$, Mr. Clark brought a common misconception to his students' attention. When simplifying a fraction that ends up with a simple numerator and a more complex denominator, students sometimes move the expression in the denominator into the numerator. For example, an answer of $\frac{1}{x+5}$ might become x+5. "What I worry about is people come up with the answer of *2x*. Somehow magically the answer floats to the top. I don't know why this occurs" (Mr. Clark 4-17-15 Lines 39-40).

Specialized content knowledge (SCK). There was some evidence of SCK in three of the procedural explanations, all given by Mr. Clark. However, these instances show only small pieces of SCK, compared to the wider range of SCK demonstrated in the mathematical reasoning explanations. Two particular types of SCK knowledge were used in the procedural explanations.

First, in two different explanations, Mr. Clark differentiated between different types of problems involving simplifying rational expressions, a distinction that is likely unimportant outside of teaching, where the problems might all be seen as similar. When discussing the procedure for simplifying rational expressions, Mr. Clark gave his students three steps: (1) Factor, (2) Multiply, and (3) Simplify. Partway through the lesson, while explaining how to simplify $\frac{4x^2}{3x} \bullet \frac{9x}{8x^5}$, Mr. Clark mentioned to students that there was nothing to factor, so they could move immediately to step two. He then differentiated between the type of problem they had worked on at the beginning of class, the type of problem they were currently working on, and the type of problem they would work on: "The first set of problems we did, we did step one and then three. Now we're going to do step two and three. Eventually, all three" (Mr. Clark 4-17-15 Lines 430-431). The first set of problems involved simplifying a single fraction that had common factors and some of the problems required students to factor a greatest common factor out of the numerator or denominator before simplifying. The second set of problems involved simplifying the product of two rational fractions, which each had monomial terms in their numerators or denominators. The final set of problems involved simplifying the product of two rational fractions, which had more complex polynomials in their numerators and denominators. In most cases, the polynomials were linear or quadratic and required factoring a quadratic or factoring a greatest common factor before simplifying.

The second way SCK knowledge was seen in the procedural explanations was when a teacher drew on knowledge of two different, nuanced understandings of the relationship between a product (e.g. *2x*) and its two factors (e.g. *2* and *x*). This type of knowledge appeared in two of the procedural explanations. While simplifying the expression $\frac{2x}{3x-6} \bullet \frac{2x-4}{x^2}$, Mr. Clark discussed with students how to factor $2x-4$. After establishing that *2* is a common factor of both terms, Mr. Clark factored the binomial by thinking of each of the terms as the product of the common factor, *2*, and some other unknown value.

> Mr. Clark: *2* times what is *2x*? *2* times something is *2x*.
>
> Student: *2* times *x*.
>
> Mr. Clark: Yes. *2* times something is negative *4*. Negative *2*. (Mr. Clark 4-17-15 Lines 689-694)

He then provided an alternative way for students to think about the factoring: "Or you can think of it like division and you can say *2x* divide by *2* is *x* and negative *4* divide by *2* is negative *2*" (Mr. Clark 4-17-15 Lines 694-696). In this example, Mr. Clark is demonstrated two different understandings of the relationship between the product *2x* and its factors *2* and *x*, one viewing the relationship as multiplicative (i.e. *2x* = *2* times *x*) and one viewing a division relationship (i.e. *2x* divided by *2* is *x*).

Knowledge Used in Mathematical Reasoning Explanations

In the explanations that were categorized as mathematical reasoning, the teachers drew on a wide variety of knowledge. They also showed evidence of deep mathematical and pedagogical understanding. In the subsections below, I provide examples of teachers' common content knowledge, specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. I then provide two examples, which show how the teachers drew on multiple domains of knowledge simultaneously during the mathematical reasoning explanations.

Common content knowledge. All of the teachers drew on common content knowledge for each of the mathematical reasoning explanations they gave. Compared to the procedural explanations, teachers drew on a more extensive range of common mathematical knowledge. In every explanation, the teachers drew on mathematical ideas and concepts, such as the multiplicative identity, the multiplication property of equality, intercepts, holes, asymptotes, and the end behavior of a function. Teachers also drew on knowledge of many different procedures in all of the explanations, including factoring, solving a rational equation, graphing an equation given known information, and polynomial long division.

In addition, unlike the knowledge used during procedural explanations, when giving mathematical reasoning explanations, teachers also drew on knowledge of connections between mathematical ideas, such as how the values in the rational equation $y = \frac{a}{x-h} + k$ affect the graph in the same way they affect other parent functions, or the difference between the multiplicative identity and the multiplication property of equality. These connections were not present in the procedural explanations. For example, in his interview, Mr. Johnson began his explanation of how to graph $y = \frac{1}{x-2} + 1$ by discussing parent functions and transformation more broadly. In an explanation of how to solve a rational equation, Mr. Baker differentiated between the multiplication property of equality, which was being used to solve the equation and maintains equality of an equation, and the multiplicative identity, which maintains the value of an expression and is used to simplify expressions.

Specialized content knowledge. The teachers drew on specialized content knowledge in 13 of the 16 mathematical reasoning explanations. This knowledge showed up in several different forms. First, specialized content knowledge was most frequently seen in the form of nuanced mathematical understandings. In particular, the teachers decomposed pieces of mathematics to a level that was mathematically appropriate for students and they did so in ways that mathematicians might not be able to verbalize. For example, when explaining how to simplify the expression $\frac{x^2 - 2x}{x-2}$, Mr. Baker showed a nuanced understanding of terms and factors, which he used to help students see what they can and cannot do to simplify the problem. This example was the first problem Mr. Baker used that had variables in the denominator. Prior to this example, he had used the fractions $\frac{2+3}{2+5}$ and $\frac{2\times 3}{2\times 5}$ to show that factors can be reduced, but terms cannot. He also defined factors, terms, and rational expressions and discussed what it meant to simplify a rational expression.

> Mr. Baker: The fact that those [the *2* and the *x* in *2x*] are multiplied together ... they're being multiplied, right? Are they factors? Absolutely. But what is this whole *2x* doing? don't have any factors.
>
> Student: It’s being subtracted
>
> Mr. Baker: It's being subtracted from *x* squared so that makes the *2x* a term. If you're adding and subtracting to other things it is a term. Now within that term you can have factors, but I cannot reduce any part of this term. Even though those are multiplied

> together, I can't reduce part of it because that entire *2x* is a term being subtracted from *x* squared. So keep that in mind. Same thing with this *2* on the bottom. Same thing with the *x* and the *x* squared, they're all terms because they're all being added or subtracted. (Mr. Baker 3-27-15 Lines 135 – 147)

Mr. Baker's nuanced understanding of factors and terms can be seen here when he discusses that while the *2* and *x* in the *2x* in the numerator are factors, the *2x* as a whole is a term.

The second way teachers drew on specialized content knowledge was in using a simple example to show a more complex idea. In particular, by using a simple example to show a complex idea, teachers were demonstrating knowledge of the simpler, related problem. For example, Mr. Baker used the problems $\frac{2+3}{2+5}$ and $\frac{2\times 3}{2\times 5}$ in his introductory explanation to simplifying rational expressions to show students that terms cannot be reduced, but factors can. He used the two problems together during both his classroom instruction and his interview. Both problems are familiar to students, yet directly address key mathematical ideas involved in simplifying rational expressions. Mr. Baker also used the equation $\frac{z^2}{3}-\frac{z}{6}=1$ in his first explanation on solving rational equations. The equation has no variables in the denominator and is a type of problem students have solved before in previous chapters.

Finally, teachers drew on specialized content knowledge when they saw nuanced differences across different problems that others might see as the same type of problem. For example, in his interview, Mr. Baker began his explanation of how to solve a rational equation by differentiating between the problem that had just been solved, $\frac{2}{x}=\frac{5}{x-3}$ and the new problem, $\frac{15}{x^2-3x}+\frac{2}{x}=\frac{3}{x-3}$. He saw the first problem as part of a subset of proportion problems that could be solved using a different method, cross multiplication, which students were already familiar with:

> We just finished solving the equation, the rational equation, *2* divided by *x* is equal to *5* over *x* minus *3*. We said the equation when two ratios are set equal to each other, this is a proportion and we know that solving a proportion we can cross multiply; solve our equation. If we take this one step further and I take that same equation: *2* divided by *x* is equal to *5* over *x* minus *3*, however now I add an extra fraction. Now we have, *15* divided by *x* squared minus *3x*. Is this a proportion? Can we solve this using the same methods we used over here? No, so let's talk about how we would go about solving this equation. (Mr. Baker Explanation Item 3)

This new problem cannot be solved using cross multiplication because it is not a proportion where one fraction is equal to a second fraction. Students would therefore need a new method to solve the problem.

In addition to the different types of SCK already described, teachers might be using SCK to pick strategic examples. However, the teachers' explanations did not reveal how or why they selected their examples. It was therefore difficult to determine in many cases if teachers were drawing on SCK to pick examples for particular purposes.

Knowledge of content and students (KCS). Teachers drew on knowledge of content and students in just over half (nine out of 16) of the mathematical reasoning explanations. In these explanations, KCS was most commonly seen in the form of expected misconceptions or difficulties. These misconceptions and difficulties included reducing terms instead of factors, confusing the procedure used in simplifying and solving, not using parenthesis when typing equations into a calculator, factoring quadratics with a negative leading coefficient, and not recognizing factors that differ by a factor of negative 1. For example, after factoring the expression $\frac{2x^2-18}{3-2x-x^2}$ to get $\frac{2(x-3)(x+3)}{(1+x)(3-x)}$, Mr. Baker asks students, "Is there anything in common here that we can reduce? Any common factors?" (Mr. Baker 3-27-15 Lines 245 – 246).

He receives mixed answers of no and "*3* minus *x* and *x* minus *3*" and goes on to explain how the factors *3-x* and *x-3* are related and can be reduced by factoring a negative 1 out of one of the factors. Students often have difficulty noticing that a term with a negative *x* can be rewritten by factoring out the negative because writing it as *–x* is as acceptable as writing *-1x*. They may therefore overlook common factors in an expression because the factors do not look the same.

Knowledge of content and teaching (KCT). Teachers drew on knowledge of content and teaching in multiple ways. I focus here on two specific types of KCT of knowledge that were visible in teachers' explanations. First, teachers drew on the KCT knowledge of how a simple example can be used to show a more complex idea. For example, as described in the SCK section above, Mr. Baker used the equation $\frac{z^2}{3} - \frac{z}{6} = 1$ to begin his lesson on solving rational equations. In doing so, he was not only drawing on the SCK knowledge that allowed him to create such an example or know that it was a simpler example. He was also drawing on the KCT knowledge of how a simple example can be used for the pedagogical purpose of showing a more complex idea.

The second type of KCT visible in teachers' explanations was knowledge of how to strategically use student errors or misconceptions. For example, Mr. Baker began his explanation of simplifying rational expressions by deliberately making the error of reducing terms in the fraction $\frac{2+3}{2+5}$ to show that the error was wrong:

> Mr. Baker: All right, good. Let's consider this good old fashioned fraction: *2* plus *3* over *2* plus *5*. What is this?
>
> Students: Five sevenths
>
> Mr. Baker: Five sevenths? Okay maybe we need to slow down a bit. So let's look at this and you'll notice we have some *2*s in common. If we reduce those *2*s, what are we going to get?

When we reduce we are dividing, we are not cancelling. I'm not getting rid of the *2*s and making them 0s. I'm dividing and making them *1*s. So if we divide and get a *1*, what is this going to give us?

Students: Four sixths.

Mr. Baker: Four sixths. Are those equivalent?

Students: No.

Mr. Baker: No. Four sixths is not equal to Five sevenths. So are we allowed to do this?

Students: No. (Mr. Baker 3-27-15 Lines 38 – 62)

The error Mr. Baker describes is that of reducing the *2*s in the numerator and denominator. Because they are terms, they cannot be reduced. This is obvious to students when the fraction has two different, non-equivalent values. However, when simplifying a rational expression, such as $\frac{x+3}{x+5}$, students are very likely to make the error of reducing the *x*s without seeing why their answer is incorrect. Mr. Baker is strategically using the simpler example of $\frac{2+3}{2+5}$ to make the error more recognizable by students, who know four sixths and five sevenths are not equivalent.

In one of his explanations, Mr. Johnson's strategic use of misconceptions took a slightly different form. When introducing removable discontinuities to his students, Mr. Johnson had students find the discontinuities and x-intercepts of the equation $f(x)=\frac{x^2-2x-8}{2x^2-8x}$. Before using their calculators to graph the equation, Mr. Johnson asked students where the vertical asymptotes and x-intercepts would be. By doing so, he set his students up to expect their graphs to look a particular way. When the graph looked different than expected, students were curious about why the graph did not match their expectations:

Mr. Johnson: Where are we going to see vertical asymptotes? Thank you [Student]. Go ahead boss, where's one?

Student: *0*

Mr. Johnson: There's going to be a vertical asymptote and *0*, that's one of our excluded values. Where's the other one [Student]?

Student: *4*

Mr. Johnson: *4*, right. So we're going to see vertical asymptotes there because that's what would make the function undefined. What about our x intercepts? Where are we going to see the function cross the x-axis? Chief?

[…]

Student: *4* and negative *2*

Mr. Johnson: *4* and negative *2*. Sure that's our solutions. That was what [Student] explained, right? That's where the function crosses the x-axis. [Student], what have you got, boss?

Student: You said that the vertical asymptotes were *4* and *0* but how come you have two vertical asymptotes?

Mr. Johnson: Because there were two values here that would make this equal to *0*. One of them was 4 and the other was *0*, so those would both make this *0*, which would make the function undefined. So we have two. So let's go ahead and graph them and let's look for those and see if everything looks right, and we'll know we did it correctly and there it is, and here comes the graph. There goes the graph, so what's going on here? What was supposed to happen? What do you- do you see an asymptote? Yes, were do you see an asymptote?

Student: At *0*

Mr. Johnson: At *0*, right. You can see it at *0*. Just like our parent graph. Wasn't there supposed to be another asymptote? Where.

Student: *4*

Mr. Johnson: At *4*? There's no asymptote at *4*, is there. Hey let's go on the left side of this graph. Do you see where the graph kind of crosses the x-axis in the second and third quadrants? I don't know if you guys have ever done this with your calculator but we talked about how you use your calculator, like check your work and make sure you're doing stuff right and everything. A way to understand how things shift. If you change different numbers it'll move a little bit and then it kind of gives you an idea of how it behaves.

I'm going to push the trace button, see the trace button right here? I'm going to push the arrow to the left and the reason I'm going to do that is because I want to put my trace button right over here, I want to put it right there. Because it looks to me like that's where

> the function crosses the x-axis. If it really is where the function crosses the x-axis, does anybody know what that value of *x* should be if it's right? Maybe I should do this. Does anybody know what the value of *x* should be if it's right?
>
> Student: Negative *2*.
>
> Mr. Johnson: Negative *2*, that was one of our solutions. We had another solution of *4*. Does it appear to be crossing the x-axis at *4*? So we have no x-intercept at *4*, and we have no vertical asymptote at *4* even though we had both of those answers over here, so there's something wrong. (Mr. Johnson 4-2-15 Lines 591 - 654)

This was coded as KCT because Mr. Johnson is strategically setting students up to expect there to be two asymptotes. However, he knows that there will only be one. In doing so, he is engaging students to ask why the graph looks different than they would expect.

Multiple types of knowledge at once. The teachers drew on many domains of knowledge, and different pieces of knowledge within those domains, when giving mathematical reasoning explanations. Perhaps even more important is that in giving these explanations teachers were drawing on many of these different types of knowledge simultaneously. In this section, I look closely at two explanations that were categorized as mathematical reasoning to describe teachers' use of multiple different types of knowledge.

The first example is part of Mr. Baker's explanation of how to simplify the rational expression during his interview. The prompt asked him to explain how to simplify a rational expression using the example $\frac{x^2-16}{x^2-x-20}$.

> Let's take a look at our first example that we're going to go over today with our rational expressions. We want to reduce *x* squared minus *16* over *x* squared minus *x* minus *20*. As I know this, as I look at this, the first thing I'm going to see is all the terms.
>
> In our numerator we have an *x* squared. That's a term, a negative *16* that's a term, our denominator *x* squared, *x*, and *20*, all terms. As much as you want to, reduce the *x* squared and *x* squared, as we saw from our first problem, because they are being added to a negative *16* and add to a negative *x*, they are terms and we cannot reduce them. By definition, if we want to reduce factors, then what should we do first here? If we want to reduce factors, then it would make sense for us to factor each of these polynomials and see what product we get.

> Our numerator of course is the difference of the two squares. We're going to factor that into *x* minus *4* and *x* plus *4*. Our denominator is going to factor as an *x* minus *5* and *x* plus *4*. Now that we have things multiplied, and it's something you should know this year, within each factor that *x* is a term and the *4* is a term, so we're still adding terms, but together the *x* minus *4* is being multiplied times the *x* plus *4*, which in turn makes these two factors. They're being multiplied. *x* minus *5* and the *x* plus *4* are factors, and when we reduce we're not going to reduce the *x*s, and the *4*s, and the *5*s.
>
> We're going to reduce the entire factor. As we look from numerator to denominator, we want to see any factors that are in common that we can reduce. I'm not a big fan. I don't like to cancel. Cancel means we're getting rid of. What we're doing is we are going to take these common factors, *x* plus *4*, and I'm going to divide them. *x* plus *4* over *x* plus *4* is *1*. I put a *1* here just to remind me that they're not gone, they're not *0*s, they're *1*s. When we divide the common factor out of the problem, we're going to be left with an *x* minus *4* over an *x* minus *5*. As much as you want to, as much as you want to reduce those *x*s, once again, that *x* is being added to a negative *4*, *x* is being added to a negative *5*. These are terms. We can't reduce terms. We simplified this fraction to its lowest terms by reducing the common factors. (Mr. Baker Explanation Item 1)

In this explanation, Mr. Baker draws on common content knowledge of how to simplify a rational expression. When explaining that although the *x* and *4* are terms, *x-4* is a factor, he is using the SCK of nuanced understandings of the mathematics. In differentiating between terms and factors, Mr. Baker is also drawing on KCS of the common student error of reducing terms, which cannot be reduced. Although not seen in this part of his explanation, Mr. Baker also drew on SCK of a simpler problem that shows a more complex idea and KCT of how such an example can be used to show the complex idea. He did so by beginning his explanation by simplifying the fractions $\frac{2+3}{2+5}$ and $\frac{2\times 3}{2\times 5}$. With these examples, Mr. Baker also drew on KCS of the common misconception of reducing terms and KCT of how an error can be used strategically.

The second example comes from Mr. Johnson's explanation of why a hole occurs when a rational equation has a common factor in the numerator and denominator. Part of this explanation is shown in the above section on KCT. In addition to the KCT knowledge described in the previous section, Mr. Johnson drew on the common content knowledge of different types of discontinuities and how to graph a rational function. Later in the same explanation, he discussed how the calculator approximates values, "the way the calculator is doing this, very interesting, is it's using a principle from calculus" (Mr. Johnson 4-2-15 Lines 700 – 701). In doing so, he demonstrated knowledge of mathematical connections. Mr. Johnson also drew on KCS when he set students up to expect two vertical asymptotes. This required knowing what students know about discontinuities and what they would overlook, specifically that it is impossible to have a vertical asymptote and an x-intercept with the same value.

Summary

In line with the overall goal of this dissertation, in this chapter, I sought to describe the work teachers do when giving an explanation and the knowledge they use in doing so. I did this by delineating four categories of explanations, *problematic*, *procedural*, *superficial reasoning*, and *mathematical reasoning*, and then looking more closely at, and comparing, the knowledge teachers drew on when giving procedural explanations and mathematical reasoning explanations.

Explanations containing superficial reasoning were most common, followed by procedural explanations. Mathematical reasoning explanations were less frequent. All of the mathematical reasoning explanations were given by two of the teachers, and they occurred about 25% of the time for each of those two teachers. Finally, problematic explanations were the least frequent and were all given by the same teacher.

When giving procedural explanations, teachers drew on common content knowledge, most often knowledge of how to carry out a procedure. They also drew on the KCS knowledge of common student errors. Finally, teachers drew on two specific types of specialized content knowledge, recognizing different types of problems that others would consider the same and nuanced understandings of a mathematical idea (e.g. seeing the relationship between factors and their product as a multiplication problem and as a division problem).

When giving explanations that were categorized as mathematical reasoning, teachers drew on common content knowledge, specialized content knowledge, and knowledge of content and students. They also drew on the pedagogical knowledge of knowledge of content and teaching, which was not seen in the procedural explanations. Teachers also drew on a wider and deeper range of knowledge within each domain. For example, in all of the mathematical reasoning explanations, teachers drew on common content knowledge of both concepts and procedures. When giving mathematical reasoning explanations teachers used multiple domains of knowledge simultaneously.

CHAPTER 6: DISCUSSION AND CONCLUSIONS

Teacher knowledge matters for student learning. Yet, understanding how teacher knowledge and student learning are linked requires several intermediate steps. One important step is understanding how teachers' knowledge is enacted in teaching. Through a study of eight Algebra II teachers focused on the content of rational expressions and equations, this dissertation describes how teachers' knowledge is enacted while carrying out the teaching practices of selecting examples and explaining. Findings from this study broadly inform research on teacher knowledge and teaching practices, as well as their intersection. These findings also contribute to the range of methods used to study teacher knowledge and teaching practices, and provide support for looking more deeply at how knowledge is enacted in teaching.

I begin this chapter with a summary of the findings. In this summary, I discuss how the teachers enacted each practice of selecting examples and giving explanations, the different types of knowledge they drew on in doing so, and patterns in teachers' knowledge and practice. In the second section, I look across the practices to more broadly discuss the knowledge teachers use in practice. I describe the implications of this work in the third section of this chapter, including theoretical and methodological implications, as well as implications for teacher education. Finally, I conclude this chapter with a discussion of the limitations of this study and areas for future research.

Summary of Findings

In this section, I begin by reviewing the aims of this dissertation. I then discuss findings related to the practice of selecting examples, including components of the practice and knowledge teachers drew on. Next I focus on findings related to the practice of giving explanations, specifically the different categories of explanations and the knowledge teachers drew on when giving different types of explanations. Finally, I look across the two practices to compare how teachers draw on mathematical knowledge for teaching in enacting each practice.

Study Aims

This dissertation sought to better understand the work of mathematics teaching by looking closely at two teaching practices that are integral to and embedded within the daily work of classroom teaching. Although teachers select examples and give explanations throughout mathematics teaching, there are many unknowns as to what is entailed in carrying out each of these practices and how teachers draw on mathematical knowledge for teaching in carrying out each practice. I aimed to investigate these practices through the following research questions:

- What mathematical knowledge for teaching is entailed by the instructional practices of selecting examples and giving explanations?
    1. What kinds of work do teachers do in carrying out these two teaching practices?
    2. What mathematical knowledge for teaching do teachers draw on in carrying out these two teaching practices?
    3. How do teachers use this mathematical knowledge for teaching and reasoning in doing this work?
    4. Are there differences across the two practices? How are these differences in knowledge and reasoning related to the demands/work of the practices themselves?

Despite similar aims for understanding each of the two practices, the analyses of the practice of selecting examples and the practice of giving explanations were different. Differences in what is known in the field about these practices, the occurrences of each practice in this study, and the practices themselves, including the in-the-moment demands and the simultaneity of different components of the practice, led to differences in the analyses of the work involved in each practice. In both cases, the analysis sought to better understand what is involved in the work of carrying out each practice. For the practice of selecting examples, the analysis focused on understanding the components of the practice. For the practice of giving explanations, the analysis focused on the different types of explanations mathematics teachers give during whole class instruction. Although there were differences in how the practices were analyzed, the knowledge used in carrying out each practice was looked at through the same lens of mathematical knowledge for teaching. Though the practices were analyzed differently, this does not imply that comparisons cannot me made across the practices or that future research must analyze each teaching practice separately from other practices. In this study, given what was known about each practice already and the specific instances of each practice that comprised the data in this study, the analyses of the two practices varied. In the sections that follow, I discuss the findings of this study, beginning with the practice of selecting examples.

Selecting Examples

The practice of selecting examples is more complex than it may appear. When selecting examples, teachers are doing a range of things, from thinking about their goals, sub goals, and students, to solving a problem. Most frequently, teachers enacted the following components of the practice: evaluating the features of a problem, sequencing problems, thinking about pieces of the end goal, and thinking about common student errors or places where they will have difficulty or success.

When selecting examples, teachers most frequently drew on common content knowledge, both of how to carry out a procedure and of mathematical ideas. They also drew on the specialized content knowledge of recognizing nuanced differences within a larger set of problems that others would group as one set and on knowledge of how to sequence examples strategically for a purpose. When responding to each individual interview item, each teacher drew on at least four different types of knowledge, with some teachers drawing on several additional types of knowledge.

Different types of knowledge co-occurred frequently. Given the mathematics involved in all of the examples and the frequency with which teachers drew on common content knowledge, it is logical that CCK co-occurred with almost all of the other types of knowledge at least once. Further, there were two pairs of knowledge codes that overlapped completely: (1) knowledge of simple problems that show a more complex idea and knowledge of how a simple example can be used to help students learn a more complex idea, and (2) recognizing artificial patterns in problems and knowledge of how examples and the values used in them should not demonstrate an artificial pattern. As discussed in Chapter 4, the two knowledge codes in each pair are linked. For example, when teachers described using a simple example to show a more complex idea they demonstrated mathematics knowledge of the simple related example and pedagogical knowledge of the purpose of using an example and how to do so. These two overlapping pairs of knowledge suggest an intrinsic link between pieces of specialized content knowledge and knowledge of content and teaching. Specifically, one such link is that knowing specific types of problems that are pedagogically powerful is linked to the knowledge that such problems can be used for a specific pedagogical purpose.

Looking at specific components of the practice of selecting examples, the teachers collectively drew on five or more different types of knowledge across all instances of each component. Further, there were several types of knowledge that were draw on when enacting each component by at least one teacher. These include the common content knowledge of how to carry out a procedure and of mathematical ideas, as well as the specialized content knowledge of recognizing nuanced differences within a larger set of problems that others would group as one set. In addition, knowledge of how to sequence examples strategically was used for every component of the practice with the exception of looking at a set of problems for interesting features in the set. Taken with the range of knowledge each teacher drew on, and the co-occurrences of different knowledge codes, this suggests that selecting examples is an important site for mathematical work in teaching because it requires teachers to draw on a large range of knowledge types simultaneously, and use this knowledge across the different components of the practice. I return later to the complexity of teachers' knowledge use when enacting this practice.

Giving Explanations

Explanations were categorized into four groups, ordered from least to most mathematical reasoning: *problematic*, *procedural*, *superficial reasoning*, and *mathematical reasoning*. Explanations containing superficial reasoning were most common. Almost half of the explanations that occurred during classroom observations were categorized as superficial reasoning, as were eight of the 12 interview explanations. Procedural explanations were less frequent than superficial reasoning explanations, showing up in just over one third of the classroom observation explanations. All of the mathematical reasoning explanations were given by two of the teachers, and they occurred about 25% of the time for each of those two teachers. Finally, problematic explanations were the least frequent and were all given by the same teacher.

Explanations using more reasoning occurred when introducing content that was new to students and subsequent examples of the same content often had a lower level of reasoning. When introducing an idea for the first time, teachers may want to provide students with an understanding of the key features of a mathematical idea, how an idea is related to students' prior mathematics knowledge and experience, how a concept is different than a related concept, or why a procedure works. In later explanations of the same content, they may be less likely to use the same amount of reasoning because students have already been exposed to the reasons underlying the concept or procedure. They may instead be focusing on helping students gain fluency with the more procedural elements of the content. Within the content area of rational expressions and equations, this often took the form of the teacher explaining a new procedure with a detailed explanation of why certain steps are used and what cannot be done. Later in the lesson, when going through subsequent examples of the same procedure, the teacher gave less reasoning and focused instead on executing the steps of the procedure correctly.

As might be expected, across both procedural and mathematical reasoning explanations, teachers drew on common content knowledge for each explanation. As might also be expected, teachers drew on a much wider range of knowledge and showed greater depth of knowledge within particular domains when they gave explanations that were categorized as mathematical reasoning, compared to when they gave procedural explanations. When giving procedural explanations, teachers drew on the domains of common content knowledge, knowledge of content and students, and specialized content knowledge. In contrast, when giving explanations that were categorized as mathematical reasoning, teachers also drew on knowledge of content and teaching. Further, within each of the MKT domains, teachers drew on a wider range of knowledge.

During the procedural explanations, the common content knowledge drawn on was most frequently knowledge of how to carry out a procedure. However, teachers did occasionally draw on knowledge of mathematical concepts and ideas. In contrast, when giving mathematical reasoning explanations, teachers drew on knowledge of mathematical ideas and of how to carry out a procedure during each explanation. This difference supports the distinction between the two types of explanations, and might therefore be expected. Procedural explanations are focused on working through a procedure, with no explanation about why the procedure is being carried out or why the steps of the procedure work. In contrast, mathematical reasoning explanations provide deep mathematical reasoning related to the underlying why or how of a mathematical concept or procedure. As such, teachers would be expected to draw on knowledge of both mathematical ideas and how to carry out procedures.

Comparing procedural and mathematical reasoning explanations, it might be expected that teachers would not draw on specialized content knowledge when giving procedural explanations because they are merely going through the steps of a procedure and carrying out a mathematical procedure does not require specialized mathematical knowledge. If the teacher were to unpack what they were doing to make sense of the procedure for students and help them learn to carry it out, the teacher would likely be discussing some reasoning underlying the procedure, which would make the explanation a superficial reasoning or mathematical reasoning explanation. However, teachers drew on two different types of specialized content knowledge when giving procedural explanations, a nuanced understanding of the relationship between a product and its factors, and knowledge of nuanced differences within a larger set of problems that others would see as the same. The fact that teachers drew on specialized content knowledge when giving a procedural explanation suggests that even when an explanation is procedural, there may be something specialized about the work teachers are doing to give that explanation.

It might be expected that teachers would draw on specialized content knowledge when giving mathematical reasoning explanations, as these explanations are attuned to students' mathematical understanding. Giving an explanation that uses mathematical reasoning to students requires a teacher to decompose their own knowledge to a level that students can make sense of, which is one component of specialized content knowledge. Teachers' nuanced understandings of mathematical concepts were seen throughout the mathematical reasoning explanations.

In addition to these nuanced understandings, when giving mathematical reasoning explanations teachers drew on knowledge of content and teaching, which was not seen in the procedural explanations. In particular, when giving mathematical reasoning explanations, teachers strategically used errors to show students what not to do. They also used knowledge of how to use a simple familiar problem to demonstrate a more complex concept.

In comparison, teachers did not draw on knowledge of content and teaching during the procedural explanations. It is likely that if they had, the explanations would have moved to a different category, because when drawing on knowledge of content and teaching during an explanation, teachers often went beyond the steps of a procedure and began to reason about the mathematics involved, either through a simple related example or using errors.

Looking at teachers' use of knowledge of content and students across the two explanation types, it is not unexpected that knowledge of content and students was only seen during the procedural explanations in the form of mentioning a common error or misconception. Had a teacher used the errors in some way, the explanation would likely have moved to a category involving reasoning because for an error to be used strategically, the teacher would need to explain in some way why it is an error and would therefore be using reasoning in the explanation.

During the mathematical reasoning explanations, when teachers drew on knowledge of content and students, they not only mentioned a common misconception or error, but went into some detail about how to avoid the error, or how students might recognize something they might otherwise overlook. Further, one teacher even used his knowledge of content and students to engage students in a problem by setting them up to expect something that wasn't there and question why it was missing. Specifically, after teaching students about vertical asymptotes, Mr. Johnson introduced removable discontinuities by giving his students a problem with both a vertical asymptote and a removable discontinuity. Because they had only learned about vertical asymptotes, students expected both of the discontinuities to be asymptotes, but when they looked at the graph of the equation, they saw that there was only one vertical asymptote. By using his knowledge of students and setting the problem up in this way, Mr. Johnson caused his students to question what was happening in the equation to cause the graph to differ from their expectations.

Overall, comparing teachers' knowledge use across the two types of explanations, as might also be expected, the teachers drew on a wider range of knowledge when giving mathematical reasoning explanations than they did when giving procedural explanations. Further, when giving mathematical reasoning explanations, as described above, within several of the domains, the knowledge teachers drew on was more detailed and teachers drew on multiple domains of knowledge at once. Given that individual teachers' explanations varied in type and the knowledge they drew on varied across explanations, even of the same content, this study suggests that teachers are deciding, either explicitly or implicitly, when to draw on more extensive knowledge in an explanation and go into greater mathematical detail, and when to use a straightforward procedural explanation.

Interpreting the Findings

In this section, I look beyond the findings related to each practice to more broadly discuss the knowledge teachers use in practice. First, I look at how different types of knowledge are associated with differences in how teachers enact practices. Second, I look at the complexity of teachers' knowledge use. Third, I focus specifically at the places where specialized content knowledge does and does not show up in practice. Finally, I discuss the impact different methods can have on studying teacher knowledge and practice.

Teachers Use Different Types of Knowledge When Enacting the Practices, Which is Associated with Differences in How They Enact the Practices

Across both practices, the knowledge teachers drew on when enacting the practices led to differences in how teachers enacted the practice. For example, when selecting examples, teachers who drew most heavily on knowledge of student misconceptions were more focused on student misconceptions than they were on pieces of the end mathematics goal. In contrast, teachers who more frequently drew on knowledge of how to strategically sequence examples for a purpose and recognized nuanced differences within a larger set of problems that other would see as one set were more focused on pieces of the end mathematics goal than they were on student misconceptions.

The knowledge teachers drew on also influenced the explanations they gave. When giving explanations, teachers drew on a broader range of knowledge types when giving mathematical reasoning explanations than when giving procedural explanations. As described earlier, drawing on this broader range of knowledge might directly impact the type of explanation a teacher gives. For example, if a teacher draws on knowledge of mathematical ideas, they are likely discussing those mathematical ideas and their explanation would likely be categorized as superficial or mathematical reasoning. Similarly, if a teacher draws on knowledge of how a simple problem can be used to show a complex idea, they will likely discuss how the simple problem demonstrates the more complex idea. Their explanation would therefore likely be categorized as superficial or mathematical reasoning.

Complexity of Knowledge Use

In addition to the many types of knowledge teachers used when enacting each practice, teachers drew on several types of knowledge simultaneously. This suggests that there is a complex relationship between different domains of knowledge and that teachers' knowledge use in teaching is complex and multifaceted.

Where Does Specialized Content Knowledge Show Up in Practice?

This dissertation contributes to the field's understanding of specialized content knowledge by identifying places where teachers draw on it in practice. Teachers drew on different types of specialized content knowledge when enacting each of the components of the practice of selecting examples and when giving mathematical reasoning explanations. Unexpectedly, teachers drew on two different types of specialized content knowledge when giving procedural explanations. In identifying places where teachers draw on this knowledge, I also distinguish several specific types of specialized content knowledge, including recognizing nuanced differences within a larger set of problems that others would see as one set, knowledge of simple problems that show a more complex idea, and knowledge of multiple procedures to solve a problem.

Although I was able to identify several types of specialized content knowledge, the most frequent type of SCK teachers drew on was nuancing differences in problems that others would see as the same. Other types of SCK were infrequent and it is possible that there are other types of SCK knowledge that this study did not capture. There might be something about the content of rational expressions and equations that makes it more challenging to see specialized content knowledge. In particular, most subtopics within rational expressions and equations are taught in a specific way. For example, although there are multiple procedures for solving a rational equation, they all involve using multiplication to eliminate the fractions. Because there are typical ways this content is frequently taught, there may be less opportunities to see how teachers draw on knowledge in making decisions about which examples to use and what to emphasize in their explanations. It is also possible that the more complex nature of the topic means that teachers may have a less robust understanding of the nuances of the content and are therefore less likely to have specialized content knowledge related to this topic.

The Impact of Different Methods of Studying Teacher Knowledge and Practice

This dissertation used two different methods for collecting data, as well as different methods of analysis for each practice. The interview setting used to study the relationship between teacher knowledge and teaching practices provided a different view of these two teaching practices than that of a classroom observation. For the practice of selecting examples in particular, the interviews made the work teachers do more visible than it often is during classroom instruction. By using both methods of interviews and classroom observations, I was able to learn about the work of enacting these two practices. In this section, I describe the strengths and weaknesses of these two research methods and of the different analysis methods used in this study.

For both of the practices, the interviews allowed for a more standardized analysis of the practice because all of the teachers were selecting examples for the same purpose or giving an explanation to meet the same learning objective. This similarity allowed for a closer comparison across the teachers. Because the task was held constant, claims about differences in teachers' knowledge use and practice are more robust. In the classroom observations, despite holding mathematics content constant, the learning goals varied across teachers and their different classes, creating greater variation in the explanations given by different teachers.

Both sets of interview items asked the participants to take on the role of the teacher in the classroom situation described in the item. In the case of selecting examples, this required the teacher to talk through their thought process when picking examples. The items focused on giving explanations required a bit more of the teachers. In particular, not only were teachers asked to talk about their thought process, but they were also asked to then model or perform the explanation as if there were students in the room. One challenge that occurred with the explanation items is that a few of the teachers tended found giving an explanation with no students present to be awkward. On occasion this came across in their explanation and a few teachers resorted to telling what they would do and needed to be prompted to explain it like they would to students. In contrast, other teachers' explanations closely mirrored their classroom explanations and would have been difficulty for an observer to notice that there were no students present. In a few of these cases, teachers even maintained a dialogue in their explanation by adding in expected student contributions. Future work might consider how interview items can be revised to encourage teachers to give an explanation as though they were talking to a class of students.

Looking specifically at the methods used for the practice of selecting examples, the items set up the practice of selecting examples slightly differently than it might be set up in teaching because each item explicitly stated a goal in the prompt. In doing so, the items enabled a view of teachers' selection of examples once they have a specific goal in mind. However, determining the goal for a set of examples is likely an important first step when teachers select examples in their own teaching.

The analysis of the practice of selecting examples included describing the different types of work teachers do when carrying out this practice. The identified components allowed for a comparison of different patterns in how teachers select examples, and represent an important contribution to the field's understanding of how teachers select examples. Although previous work has looked at considerations teachers make in selecting examples (e.g., Zodik & Zaslavsky, 2008), these components describe the work involved in carrying out the practice. These components can provide a foundation for decomposing the practice when discussing selecting examples in teacher education. They can also be used to cultivate an understanding of how teachers develop the practice of selecting examples over time, including which components novices carry out more frequently and which more experienced teachers use most often. Further, this analysis identified more specific types of knowledge that teachers use in carrying out this practice. These specific pieces of knowledge are another step in understanding the knowledge teachers use in practice and the specific ways knowledge is drawn on within particular components of the work. A similar analysis of other practices might allow for comparisons across different practices to determine if there are shared components. Further, this analysis might identify patterns in how teachers use knowledge across practices.

The analysis of the practice of giving explanations began by splitting the classroom observations into explanations based on the overall purpose of the explanation. One contribution of this analysis is the way in which the explanations were split. The explanations were then categorized based on the type of reasoning involved.

One take away from this study is that there were few explanations that were deeply mathematical. This raises the question of whether this is due to differences in explanations in high school mathematics classrooms and explanations within the discipline of mathematics. In mathematics teaching, a good explanation requires more than common content knowledge because learners are unlikely to understand a mathematical idea in its fully compressed final form. Instead, teachers are decomposing mathematical ideas to share the nuances with students, requiring them to draw on specialized content knowledge. Despite the use of specialized content knowledge in many of the mathematical reasoning explanations, there were often important mathematical details that were not addressed. Many things are also "taken-as-shared" (Yackel & Cobb, 1996) at the high school level and these taken-as-shared ideas may be covering for more complex mathematics. For example, Mr. Baker frequently referred to division by zero as causing the world to end instead of describing it as undefined. In doing so, he is glossing over the deeper mathematics underlying why division by zero is problematic, and is teaching this aspect of division as a rule without reason.

In contrast to a high school mathematics explanation, a mathematical explanation might require only common mathematics content knowledge, as the audience is not learners, but others who are competent in the field. Thus, the nuanced details and reasoning required in a high school mathematics explanation are not necessary. Often it is these nuanced details that are taken-as-shared and therefore not discussed. A future study might look at what it means for something to be explained mathematically and how a mathematical explanation is different than an explanation that might be given in a high school mathematics classroom.

Implications

This study contributes in three distinct areas. First, from an empirical standpoint, the major contribution of this study is a better conceptualization of the teaching practices of selecting examples and giving explanations and the mathematical knowledge and reasoning used in carrying them out. This conceptualization includes particular components of the work teachers do in selecting examples, different types of explanations teacher give, and the types of mathematical knowledge they draw on in doing so. Further, conceptualizing teachers' use of mathematical knowledge and reasoning in practice enables the field to ask new questions about the development and trajectory of this knowledge, reasoning, and the ability to enact these practices. Future research can also build on this knowledge by investigating other teaching practices across grade levels and mathematics content. Doing so will provide greater understanding of how teachers use mathematical knowledge and reason in doing the work of teaching and how their knowledge use and reasoning are or are not dependent on content area and grade level.

In addition, this study shows the influence teachers knowledge can have on their practice and how important it is for teachers to have the knowledge needed to enact components of the practices. The two teachers who selected examples based most heavily on common student misconceptions they wanted to avoid appeared to have a weaker understanding of the key mathematical ideas that made up the larger goal in the prompt. This is supported by their less frequent recognition of nuanced differences in the set of problems. Because they lacked knowledge of the important sub goals underlying the overall mathematical goal of each item, they instead focused their example selection on showing common errors to prevent students from making them. However, showing students several errors and discussing why they are problematic may leave students with an incomplete knowledge of the important mathematics ideas in a lesson. This suggests that to be able to enact these components of the practice, teachers need a strong, and nuanced, understanding of the mathematics content.

The second contribution of this study is to the field of teacher education. Knowing the particular types of mathematical knowledge needed for carrying out particular teaching practices and the ways knowledge is used in these practices can inform teacher training and education, both in teacher education programs and in professional development. This study stands to inform content and methods courses individually, by contributing to knowledge that might be taught in a content course and teaching practices that might be taught in a methods course. However, and perhaps more importantly, this study highlights the interdependent nature of knowledge and practice in teaching. It therefore highlights the potential of integrating knowledge and practice in teacher education instead of keeping them separate. A greater understanding of the interrelated nature of knowledge and practice also affords the opportunity for certification tests to better measure the knowledge and practices that actually matter in teaching, leading to more qualified and capable teachers entering the field. All of these have the potential to help teachers to provide better instruction, thereby improving students' learning.

Finally, this study has methodological implications. In particular, it contributes to the variety of methods used to study teachers' mathematical knowledge and reasoning. In a classroom setting, teachers must be responsive to the many in-the-moment demands of teaching. Classroom observations allow researchers to observe teachers' practice and how these demands affect it. However, because classroom teaching occurs in a "live" setting, with other participants, it is not possible to ask teachers about or record their thinking while teaching. Post observation interviews can delve into teachers' mathematical knowledge and reasoning, but only retrospectively. In contrast to these settings, the interviews used in this study offer a setting where teachers' mathematical knowledge and reasoning can be probed as they are used, providing access to knowledge otherwise obscured by the demands of classroom teaching. However, these interviews are only useful for assessing knowledge use in teaching if we understand how teachers' mathematical knowledge and reasoning during such interviews are related to their mathematical knowledge and reasoning while enacting the work of teaching. This dissertation contributes to this needed understanding.

Limitations

There are three main limitations of this study. First, the sample of teachers is too small to make claims about all secondary mathematics teachers. However, there is enough data to better understand how the teachers in this study drew on knowledge when selecting examples and giving explanations for the content of rational expressions and equations. Further, these findings suggest patterns in how the larger population of secondary mathematics teachers might enact knowledge within these particular teaching practices.

The second limitation is the focus on a specific mathematical domain. This limitation has two components. First, teachers may draw on different types of knowledge when teaching different content areas. This might further vary based on the complexity of the topic, or its prevalence in the curriculum. Second, there may be limitations specific to the content of rational expressions and equations. Compared to a topic such as linear equations, the content area of rational expressions and equations may have more prototypical example types, whereas other topics may have more variation in the types of examples used to teach them. For instance, when teaching simplifying rational expressions, all of the teachers who started with a simple, familiar problem used a basic fraction simplification example. A content area like linear equations has a wider variety of simple familiar problems, such as slope in everyday life or a simple unit rate problem. In addition, there is a lack of variation in the procedures used for simplifying rational expressions and solving rational equations. There are a few different procedures for solving rational equations, but they all involve multiplication to remove the fractions in the problem. Other content areas, such as linear equations, have a wider variety of procedures and concepts that might be explained, allowing for a wider range of possible explanations. Despite these limitations, the data suggest particular patterns in how teachers select examples and give explanations for the content of rational expressions and equations, which provides a basis for investigating teachers' knowledge use when enacting these practices in other content areas.

Finally, the third limitation of this study is the small number of mathematical reasoning explanations. Given the wider range of knowledge teachers drew on when giving these explanations, a larger number of mathematical reasoning explanations would have enabled a more detailed analysis of how teachers enact knowledge when giving explanations. The content of rational expressions and equations may have also limited the number of mathematical reasoning explanations in some way. However, the data did include some mathematical reasoning explanations, which do suggest patterns in how teachers enact knowledge when giving these explanations. These patterns can be further investigated in future research. Given the small number of mathematical reasoning explanations, it is also possible that there is a category of explanations that contain a deeper level of mathematical reasoning than those in the highest category of my study. However, my data suggest that such explanations are more rare in secondary mathematics teaching, which may be why an example of such an explanation was not captured.

In addition to the listed limitations, some might say that there is an additional limitation, the lack of measurement of participants' knowledge. The purpose of this study was not to measure teachers' knowledge but to look generally at how teachers draw on knowledge when enacting particular teaching practices. The aim was therefore not to investigate the knowledge individual teachers hold, but rather the knowledge used in practice. The lack of measurement of participants' knowledge is therefore not a limitation of this study, but rather a purposeful design feature of the study.

Future Research

There are two lines of research that can build on this dissertation. The first line is focused on developing a more robust conceptualization of teachers' knowledge use in practice. Building on the work of this study, future research might look at the knowledge teachers use when they select examples and give explanations within other mathematics content areas and grade levels. This would allow for a generalization of teachers' knowledge use within these two teaching practices. In addition, for the practice of giving explanations, additional content areas might provide the opportunity to see explanations with deeper mathematical reasoning, perhaps even an additional category of reasoning that was not observed in this data. Future research might also look beyond these two teaching practices to study how teachers use knowledge when enacting other teaching practices, allowing for generalization of knowledge use in practice. In addition to broadening the scope of investigation around teachers' knowledge use when enacting particular teaching practices, future research might look at the differences between the knowledge teachers hold and the knowledge they use in teaching. Understanding this distinction might begin to answer the question, how can we make teachers' robust, but tacit knowledge more readily available in their teaching practice?

The second research line that stems from this dissertation looks at the development of teachers' knowledge and ability to carry out teaching practices over time. In particular, future research might investigate whether there are particular experiences that can be leveraged to develop teachers' knowledge and ability to carry out teaching practices. It might also investigate if there are certain types of knowledge that are most useful and useable for novice teachers. Looking at teaching practices, this research might investigate whether there are particular teaching practices novices can master earlier on in their training, or if the ability to carry out a particular practice enables novices to carry out other practices more easily.

Conclusion

Teacher knowledge matters for student learning and teachers draw on their knowledge in teaching when enacting teaching practices. This dissertation contributes to the literature by describing two foundational practices in mathematics teaching, selecting examples and giving explanations, and analyzing the knowledge used in enacting each practice. The findings add to theoretical understandings of mathematical knowledge for teaching and mathematics teaching practices. This research helps bridge between research on teacher knowledge and research on teaching practice by conceptualizing the ways in which teachers draw on their knowledge when enacting specific teaching practices. This work also has implications for mathematics teacher education and the methods used to study teacher knowledge in practice.

APPENDIX 1: INTERVIEW PROTOCOL

Introduction

I first want to thank you again for participating in this study. As you know, the purpose of the study is to learn more about the work of teaching and its mathematical demands. We're not really studying you. We are trying to learn about how teachers select examples and give explanations. So I am hoping to find out how you do these two practices. I'm really curious about how you do these two practices in your teaching.

The purpose of this interview is to gain a better understanding of how secondary mathematics teachers think. I will present you with questions and ask you to talk through your reasoning in solving them. The responses you provide will only be used by our research group and will not be shared with anyone else. If any question makes you feel uncomfortable, feel free to tell me to skip it. Also if at any point you decide you don't want to participate, let me know and we'll stop.

Before beginning the interview, do you have any questions about the study or what you will be doing? Is there anything else before we get started?

Teaching Experience

1. What courses/ grade levels are you currently teaching?

2. How many years have you been teaching? (If this is not the teacher's first year, ask about other grade levels.)

3. How many years/ times have you taught rational expressions and equations?

4. What topics are included in your unit on rational expressions and equations?

5. What type of credential (or certificate) do you have? When did you earn it?

6. What is your BA/ MA degree in?

7. Do you remember discussing selecting examples or giving explanations in your teacher preparation?

8. Have you participated in any professional development about selecting examples or giving explanations?

Examples and Explanations

9. Where do you typically get your examples? Do you generally use examples from a curriculum or other resource, or create your own examples?

10. Where do your explanations generally come from? What resources do you use in planning an explanation?

Interview items

The goal of this research is to better understand how teachers select examples and give explanations. I will be presenting you with several items, which describe a classroom situation and ask you to take on the role of the teacher in that classroom. As you work through each item, I ask that you talk aloud through your thinking. At the end of each problem, I may ask follow up questions to better understand your thinking.

For the example questions:

Possible Probes

- What order would you use the examples in? Why?

For the explanation questions:

The goal of this question is to better understand how teachers plan for and give explanations and the knowledge they draw on in doing so. For the purposes of this interview, explaining is not an interactive process: While students in a real classroom might participate by asking questions and responding to prompts from the teacher, in this interview the explanation is given by the teacher without direct student involvement.

While I will be observing your explanation, there will be no students. The goal of this task is to understand how teachers plan and give explanations, not how you interact with students. Therefore, it is not necessary for you to engage with me during your explanation. While your explanation may be different than your actual classroom practice because there are no students present, you should still give an explanation that would be appropriate for Algebra II students.

You will have about 8 minutes to plan your explanation. Once you have planned your explanation, you will then have 12 minutes to give your explanation. You may use less than the 12 minutes and it is okay if you do not finish your explanation. My goal is to understand the work that teachers do preparing for and giving explanations. I will then ask you a few follow up questions about your explanation and your planning process.

Possible Probes

- What was your goal?
- Why did you decide to do _______ first?
- Why did you decide to highlight ______?
- Why did you decide to show that common error?

APPENDIX 2: INTERVIEW ITEMS

In a unit on rational expressions and equations, you have taught one lesson on graphing rational equations of the form $y = \frac{c}{x-h} + k$. In doing so, you have discussed with students what a discontinuous graph looks like. For the second lesson in the unit, you want to introduce students to different types of discontinuities. You have defined rational expressions as the quotient of two polynomials and have defined discontinuities as values where the denominator is zero. You plan to work through a few different examples demonstrating different types of discontinuities. What are three examples you might use?

1. $y=\dfrac{2}{x^2-4}$

2. $y=\dfrac{4x}{x^3+2x}$

3. $y=\dfrac{x^2+5x+6}{x^2-x-12}$

4. $y=\dfrac{-3x+3}{x^2+9}$

5. $y=\dfrac{x^2-7x+12}{6}$

6. $y=\dfrac{3x^2+2x}{x}$

7. $y=\dfrac{x+6}{x^2-5x}$

8. $y=\dfrac{x-7}{x-7}$

9. $y=\dfrac{x^2-8x+12}{x-2}$

In the unit on rational expressions and equations you are introducing simplifying rational expressions. Your goal is for students to be able to simplify rational expressions, including those involving multiplication and division. What are three examples you would use in your lesson?

Simplify the following rational expressions.

1. $\dfrac{2x^2+(4x)^2}{(3x)^3}$ 2. $\dfrac{42x^2+48x}{36x^2}$ 3. $\dfrac{10x^5}{2x^3+2x^2}$ 4. $\dfrac{8x^2}{4x^2-2x}$ 5. $\dfrac{2-x}{x^2-4}$

6. $\dfrac{2x-4}{x-2}$ 7. $\dfrac{x-4}{3x^2-12x}$ 8. $\dfrac{28}{8x-16}$ 9. $\dfrac{x^2-16}{x^2-x-20}$ 10. $\dfrac{x^2+2x}{x^2-4}$

11. $\dfrac{x+6}{x^2+5x-6}$ 12. $\dfrac{3x-6}{x^2-5x+6}$ 13. $\dfrac{x^2+4x}{x^2+2x-8}$ 14. $\dfrac{x^2-25}{x^2-10x+25}$

15. $\dfrac{6x^2+18x}{2x^3+6x^2-36x}$ 16. $\dfrac{8x^3-16x^2}{2x^3-2x^2-4x}$ 17. $\dfrac{5x^3-20x}{x^4+5x^3+6x^2}$ 18. $\dfrac{9x^2+81x}{x^3+8x^2-9x}$

19. $\dfrac{3x^2+3x-6}{x^2+3x-4}$ 20. $\dfrac{x^2-9x+14}{x^2+2x-8}$ 21. $\dfrac{x^2+7x+12}{x^2-6x-27}$ 22. $\dfrac{x^2-5x-14}{x^2+4x+4}$

23. $\dfrac{x^2+2x-8}{3x^2-2x-8}$ 24. $\dfrac{x^2+9x+18}{x+6}$ 25. $\dfrac{12x^2-32x-12}{3x^2+10x+3}$ 26. $\dfrac{2x^2-3x-2}{x^2-5x+6}$

27. $\dfrac{3x^2+5x-2}{7x^2+12x-4}$ 28. $\dfrac{x^3+4x^2+4x}{-x^2-5x-6}$ 29. $\dfrac{3x^4-x^3+2x^2}{6x^2-2x+4}$ 30. $\dfrac{x^2+3x-4}{7x^3+7x^2-14x}$

31. $\dfrac{-x^2-2x+8}{3x^3+9x^2-30x}$ 32. $\dfrac{x^3-x^2-42x}{2x^2-20x+42}$ 33. $\dfrac{3x^2-39x+90}{x^2-3x-70}$ 34. $\dfrac{x^2+2x-80}{2x^3-24x^2+64x}$

35. $\dfrac{3x^2+26x-9}{5x^2+40x-45}$ 36. $\dfrac{3x^2+21x-90}{3x^2+31x+10}$ 37. $\dfrac{x^3+11x^2+18x}{x^2+x-2}$ 38. $\dfrac{5x^2-57x+70}{2x^2-16x-40}$

Answers: 1. $\dfrac{2}{3x}$ 2. $\dfrac{7x+8}{6x}$ 3. $\dfrac{5x^3}{x+1}$ 4. $\dfrac{4x}{2x-1}$ 5. $\dfrac{-1}{x+2}$ 6. 2 7. $\dfrac{1}{3x}$ 8. $\dfrac{7}{2(x-2)}$ 9. $\dfrac{x-4}{x-5}$ 10. $\dfrac{x}{x-2}$ 11. $\dfrac{1}{x-1}$ 12. $\dfrac{3}{x+3}$ 13. $\dfrac{x}{x-2}$ 14. $\dfrac{x+5}{x-5}$ 15. $\dfrac{3}{x-3}$ 16. $\dfrac{4x}{x+1}$ 17. $\dfrac{5(x-2)}{x(x+3)}$ 18. $\dfrac{9}{x-1}$ 19. $\dfrac{3(x+2)}{x+4}$ 20. $\dfrac{x-7}{x+4}$ 21. $\dfrac{x+4}{x-9}$ 22. $\dfrac{x-7}{x+2}$ 23. $\dfrac{x+4}{3x+4}$ 24. $x+3$ 25. $\dfrac{4(x-3)}{x+3}$ 26. $\dfrac{2x+1}{x-3}$ 27. $\dfrac{3x-1}{7x-2}$ 28. $\dfrac{x(x+2)}{-1(x+3)}$ 29. $\dfrac{x^2}{2}$ 30. $\dfrac{x+4}{7x(x+2)}$ 31. $\dfrac{-1(x+4)}{3x(x+5)}$ 32. $\dfrac{x(x+6)}{2(x-3)}$ 33. $\dfrac{3(x-3)}{x+7}$ 34. $\dfrac{x+10}{2x(x-4)}$ 35. $\dfrac{3x-1}{5(x-1)}$ 36. $\dfrac{3(x-3)}{3x+1}$ 37. $\dfrac{x(x+9)}{x-1}$ 38. $\dfrac{2(5x-7)}{x+2}$

In the unit on rational expressions and equations you are introducing solving rational equations. Students are familiar with simplifying rational expressions, including those involving multiplication and division. They are also comfortable adding and subtracting rational expressions. Your goal is for students to be able to solve rational equations, building to the case where the sum of two rational fractions is equal to a third rational fraction. What are three examples you would use in your lesson?

Solve the following rational equations.

1. $\frac{1}{x}=\frac{5}{x-4}$ 2. $\frac{x}{-4}=\frac{x+1}{x}$ 3. $\frac{4}{x}=\frac{7}{x-3}$

4. $\frac{x}{-3}=\frac{2x+3}{x}$ 5. $\frac{14}{x+5}=\frac{21}{3(x+1)}$ 6. $\frac{10}{x+4}=\frac{15}{4(x+1)}$

7. $\frac{-8x+15}{x^2-3x}=\frac{x}{3-x}$ 8. $x+3=\frac{54}{x}$ 9. $x+2=\frac{48}{x}$

10. $x+1=\frac{56}{x}$ 11. $\frac{1}{3}+\frac{x}{6}=\frac{20}{x}$ 12. $\frac{1}{2}+\frac{x}{6}=\frac{18}{x}$

13. $\frac{3}{x}+\frac{3}{3x}=4$ 14. $\frac{2}{x}+\frac{2}{2x}=3$ 15. $\frac{x}{x+1}=2-\frac{x}{x-2}$

16. $\frac{2x-3}{x-3}-2=\frac{12}{x+3}$ 17. $\frac{2}{x+3}-\frac{1}{x}=\frac{1}{4x}$ 18. $\frac{3}{x+2}-\frac{1}{x}=\frac{1}{5x}$

19. $\frac{3}{x+2}-\frac{2}{x}=\frac{-6}{x(x+2)}$ 20. $\frac{15}{x^2-3x}+\frac{2}{x}=\frac{3}{x-3}$

21. $\frac{10}{x^2-2x}+\frac{4}{x}=\frac{5}{x-2}$ 22. $\frac{5}{x^2+2x}+\frac{2}{x}=\frac{-1}{x+2}$

23. $\frac{7}{x^2-5x}+\frac{2}{x}=\frac{3}{2x-10}$ 24. $\frac{10}{x^2-4}=1-\frac{1}{x+2}$

25. $\frac{5}{x^2+x-2}=\frac{1}{x+2}+1$ 26. $\frac{3}{x^2-7x+10}+2=\frac{x-4}{x-5}$

27. $\frac{12}{x^2+x-20}=\frac{2x+6}{x+5}-3$ 28. $\frac{x}{x-2}+\frac{1}{x-4}=\frac{2}{x^2-6x+8}$

29. $\frac{2}{x-5}+\frac{x}{x-4}=\frac{-8}{x^2-9x+20}$ 30. $\frac{3}{x-5}+\frac{x}{x-3}=\frac{26}{x^2-8x+15}$

Answers:

1. $x=-1$ 2. $x=-2$ 3. $x=-4$ 4. $x=-3$ 5. $x=3$ 6. $x=4/5$ 7. $x=5$ 8. $x=-9,6$
9. $x=-8,6$ 10. $x=-8,7$ 11. $x=-12,10$ 12. $x=-12,9$ 13. $x=1$ 14. $x=1$ 15. $x=-4$
16. $x=5$ 17. $x=5$ 18. $x=4/3$ 19. No soln. 20. $x=9$ 21. No soln. 22. $x=-3$ 23. $x=6$
24. $x=-3,4$ 25. $x=-4,2$ 26. $x=3$ 27. $x=-8,3$ 28. $x=-1$ 29. $x=0,3$ 30. $x=-5,7$

In the unit on rational expressions and equations you are introducing word problems involving rational equations. Students are familiar with solving rational equations. Your goal is for students to be able to set up and solve word problems using rational equations. What are three examples you would use in your lesson?

1. Three pipes are filling a pool. Working by itself, pipe A will take 5 days, pipe B will take 6 days, and pipe C will take 2 days. How long would it take to fill the pool if all three pipes are on?

2. Ashley and Callee are planting flowers. Ashley plants flowers four times faster than Callee. If they can plant 100 flowers in 30 minutes working together, how long will it take each of them to plant 100 flowers working alone?

3. A canoe trip took 8 hours. The trip was 12 miles each way (upstream and downstream). If the canoe's speed in still water is four miles per hour, how fast is the current flowing?

4. A roundtrip flight from Cleveland to Boston took 5 hours. The plane flies at a speed of 300 miles/hour with no wind. There was no wind on the way to Boston. If the distance between the two cities is 720 miles, what was the windspeed on the flight to Cleveland? Which direction was it blowing?

5. Suppose one painter can paint the entire house in twelve hours, and the second painter takes eight hours. How long would it take the two painters together to paint the house?

6. One pipe can fill a pool 1.25 times faster than a second pipe. When both pipes are opened, they fill the pool in five hours. How long would it take to fill the pool if only the slower pipe is used?

7. Huckleberry Finn and Tom Sawyer have a raft race down the Mississippi River. They start from a dock, race to a buoy that is 300 feet away and return. Tom makes the round trip in 5 minutes with a river current of 25 feet/minute downstream. What was his speed in still water?

The goal of this question is to better understand how teachers plan for and give explanations and the knowledge they draw on in doing so. For the purposes of this interview, explaining is not an interactive process: While students in a real classroom might participate by asking questions and responding to prompts from the teacher, in this interview the explanation is given by the teacher without direct student involvement.

While I will be observing your explanation, there will be no students. The goal of this task is to understand how teachers plan and give explanations, not how you interact with students. Therefore, it is not necessary for you to engage with me during your explanation. While your explanation may be different than your actual classroom practice because there are no students present, you should still give an explanation that would be appropriate for Algebra II students.

You will have about 8 minutes to plan your explanation. Once you have planned your explanation, you will then have 12 minutes to give your explanation. You may use less than the 12 minutes and it is okay if you do not finish your explanation. My goal is to understand the work that teachers do preparing for and giving explanations. I will then ask you a few follow up questions about your explanation and your planning process.

In your Algebra II class, you began your introductory lesson in a unit on rational expressions and equations by graphing $y = \frac{1}{x}$. In the next lesson, you plan to teach continuity, identifying different types of discontinuities from equations and sketching more complex rational functions using important points in the graph, such as asymptotes and zeros. In your current lesson, you now plan to graph $y = \frac{1}{x-2} + 1$ with your students and want them to be able to graph similar equations in the future. Please take 8 minutes to plan your explanation. When you are done, I will ask you to give your explanation.

You began a unit on rational expressions and equations in your Algebra II class by graphing rational equations and using algebraic methods to find and classify discontinuities in rational equations. Your goal for today is for students to be able to simplify rational expressions. You plan to explain simplifying rational expressions to your students using the following example: $\frac{x^2-16}{x^2-x-20}$. Please take 8 minutes to plan your explanation. When you are done, I will ask you to give your explanation.

In a unit on rational expressions and equations, you have recently covered adding and subtracting rational expressions. Your goal for today is for students to be able to solve rational equations. You have just completed solving the problem $\frac{2}{x} = \frac{5}{x-3}$. You now plan to use the following example to explain to your class how to solve a rational equation using the problem $\frac{15}{x^2 - 3x} + \frac{2}{x} = \frac{3}{x-3}$. Please take 8 minutes to plan your explanation. When you are done, I will ask you to give your explanation.

At the end of a unit on rational expressions and equations, you are beginning a lesson solving word problems using rational equations. Your goal is for students to be able to set up and solve word problems using a rational equation. You plan to use one of the examples below. Please take 8 minutes to plan your explanation. When you are done, I will ask you to give your explanation.

- *A canoe trip took 8 hours. The trip was 12 miles each way (upstream and downstream). If the canoe's speed in still water is four miles per hour, how fast is the current flowing?*

- *Suppose one painter can paint the entire house in twelve hours, and the second painter takes eight hours. How long would it take the two painters together to paint the house?*

APPENDIX 3: INTERVIEW ITEM REVISIONS

Example Problems

In the pilot interviews, several participants spent extensive time solving all of the problems in items 2 and 3. Not only was this very time consuming (in one case a participant spent nearly 45 minutes on one of the items), it was also not a realistic representation of the work teachers do in selecting examples. A classroom teacher rarely solves all possible examples before selecting the ones they plan to use. Rather, classroom teachers frequently select from a larger resource of problems, such as a textbook. Many of these resources also include an answer key. Items 2 and 3 were further modified to better reflect the resources a teacher would have in their daily practice of selecting an example, by including several more examples and an answer key. By modifying the item in this way, participants did not solve all of the items. Although some participants did solve or partially solve a few of the problems they were deciding between, overall they instead focused on the features of the items.

Item 1 was not revised to include more problems or an answer key because the problems do not require extensive time to solve. The problems in item 1 were also designed to provide a specific range of choices related to the mathematics content, such as a problem where the only discontinuity is removable, and providing answers would have provided the participants with information about the potential purpose of each problem, making it more difficult to assess what participants based their selection on.

Problem 3 was revised again after the first few interviews to better reflect the types of problems teachers used in the classroom observations. The revision involved replacing a few problems that were similar in form to other problems with problems where one of the terms in the equation was an integer, while the other two terms were fractions with a variable in the denominator, such as $\frac{10}{x^2-4}=1-\frac{1}{x+2}$.

Explanation Problems

The pilot testing was used to determine the amount of time needed for participants to plan and give the specified explanations. In the interviews, participants were given 8 minutes to plan their explanations and 12 minutes to give them. Participants were told that it was fine if they did not finish their explanation in the 12 minutes and if the explanations ran long, they were often given time to finish them. Participants all had ample time to give their explanations. The 12 minute time limit was included in the item prompt to constrain the scope of participants' explanations, compared to the scope of a classroom explanation, which might be 30 – 45 minutes in length.

APPENDIX 4: PRE-OBSERVATION INTERVIEW PROTOCOL

Introduction

Thank you again for participating in this study and for letting me observe and video record this lesson. The purpose of this interview is to find out more about your plans for your next math lesson, what you hope to accomplish with your students, and what kinds of things you thought about in your planning. I am particularly interested in how you selected the examples and planned the explanations you will give in the lesson.

Do you have any questions before we begin?

1. What is today's lesson about? What are the goals of today's lesson?
   - Have you taught this lesson or a similar lesson before? If so, has your previous experience influenced your plans for this lesson?
   - What should students be able to do at the end of class?

2. How does this lesson fit into the current unit?
   - Is there anything about the unit that might explain the examples and explanations you're using today?

3. Where did the lesson come from?
   - Can you give me a sense of how much was there (level of detail)?
   - I'm interested in how much materials do or do not help in your planning, would it be possible to get a copy of them?

4. Are there parts of the lesson or goal(s) that you expect will be hard (more challenging) for students?
   - What did you think about in planning these parts of the lesson?
   - Did this affect the examples you plan to use or explanations you plan to give?

5. I was hoping you could walk me through today's lesson. What will students do first?

6. How specifically have you determined your examples for today's lesson?
   - What examples will you be using in today's lesson?
     - Where did these examples come from?
     - How did you decide on these examples?
       - What were you considering when you decided to use these examples?
       - Why these particular numbers?
       - Is there a particular mathematical idea you are trying to highlight with this example?

        - How are these examples related to your goal or the parts of the lesson you expect will be hard for students?
    - Were there other examples you considered using?
        - Why did you decide against using them?

7. What concepts or procedures will you be explaining in today's lesson or having your students explain?
    - To what extent have you planned how you're going to explain this?/ Is there a particular way you're thinking of explaining this?
    - What are the key components you plan to include in this explanation?
        - How did you decide to focus on these components?
    - Where did your explanations for these concepts/ procedures come from?
    - What did you think about in creating (deciding on the key components of/ considering what would count as an) your explanations?
    - Were there other things you considered including?
    - Why did you decide against including them?

8. Is there anything else about this lesson that you're particularly thinking about or that I should know?

APPENDIX 5: POST-OBSERVATION INTERVIEW PROTOCOL

Introduction

I thought we'd start by having you share your thoughts about the lesson, and I'll ask some follow-up questions as you talk. After that, we'll watch a few short clips from your lesson together and talk about your thinking and decisions at that point in the lesson. Does that sound okay?

While some of my questions are about your lesson as a whole, I'm focused in particular on your examples and explanations.

General reactions to lesson

1. Why don't you start by sharing your thoughts about the lesson.
   (Probes if needed)
   - Did you accomplish what you were hoping to accomplish in today's lesson?
   - How closely did the lesson match what you were expecting? Were there any surprises for you?
   - Did any of the students do or say anything you didn't expect or that didn't seem to make sense? Can you describe what happened?
   - Did anything take more or less time than you thought it would?

   - Was there anything that felt particularly hard or easy about teaching this lesson? Why do you think that was particularly easy?
   - Is there anything you would change about the lesson if you could teach it again?

2. What do you think students learned in today's lesson? How do you know? Do you think some students learned more than others?
   - Was there anything that was difficult for some of your students in terms of the math? *(Probe for specific details about the mathematics.)* Why do you think that was difficult for them in terms of the math?
     - Did it shape the examples you used or your explanations?

   - And what about the opposite? Were there aspects of the math that you think were particularly easy for students? Why do you think that was easy for them?
     - Do you think the examples you picked or the explanations you gave contributed to this?

3. What do you think of the examples you used?
   - Did the examples you planned to use change during the lesson?
   - Did your examples play out as you expected or did they change during the lesson?
   - Did the examples fit your expectations for the lesson?

   - How did you decide to include/ exclude ___ example?
   - Is there a particular mathematical idea you were trying to highlight with ____ example?
   - Why did you choose to use an example that showed _____ (non-example, contradiction, etc.)?

   - Did you consider other examples/ numbers? Why did you decide on ____?
   - If you were to teach this lesson again, would you use the same examples or would you change them?
     - o How? What would you add? Which would you not include?

4. What do you think of your explanation of _____?
   - What do you think students took away from the explanation?

5. Did your planned explanations play out as you expected or did they change during the lesson?
   - How did these explanations change?
   - How did you decide to make these changes? What information did you use to make this change?

   - If you were to teach this lesson again, would you use the same explanations or would you change them?
     - o How? What would you add? Which would you not include?

5b. If given by student, What did you think of the student's explanation of ___ ? How good/accurate/appropriate was it?

Video clips

First clip: As we've been talking about, my goal is to understand how teachers select examples and give explanations in their teaching. One important part of this work is the thinking that happens while the lesson is happening. While this is a really important part of teaching, it is difficult to see by observation alone. We'll watch a clip, I will have you describe what was happening at this point in the lesson, and then I'll ask you some follow up questions.

Later clips: Here is another clip that shows a different part of your lesson.

[Watch clip.]

1. Can you describe what was happening at this point in the lesson?

2. Do you remember what you were thinking about at this point in the lesson?

3. Why did you decide to use this example/ give this explanation?

   If not mentioned, ask: Did you consider other options?

4. Do you think it accomplished what you had hoped? How do you know?

5. How did this differ from what you had initially planned? How did you decide to make these changes?

6. Would you do anything differently if you were teaching the lesson again?

Possible probes:

Explanations of main mathematical concepts and procedures:

- What do you think of your explanation for ____?
- Can you explain why ____ works?
- What would you say if a student asked why ____?
- What did you think about the student's explanation for _____?
- Why did you decide to mention ____?
- Was there something you were particularly trying to highlight in your explanation?

Examples/numbers:

- How did you decide to use those examples/numbers in the problem?
- Is there a particular mathematical idea you are trying highlight with the example?
- Did you consider using other examples/numbers? If so, why?
- Why do you think that for this particular problem it was important for students to see both right and wrong answers?

Concluding question

6. Were there any other specific parts of the lesson that you wanted to talk about, or any parts of the video you'd like to see?

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