Total Positivity and Network Parametrizations: From Type A to Type C

by

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Dedicated to my mother, Constance Medora Dinges. May her memory be a blessing.

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

ABSTRACT

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Chair: Thomas Lam

The Grassmannian $\mathrm{Gr}(k, n)$ of k-planes in n-space has a stratification by *positroid varieties*, which arises in the study of *total nonnegativity*. The positroid stratification has a rich combinatorial theory, introduced by Postnikov. In the first part of this thesis, we investigate the relationship between two families of coordinate charts, or *parametrizations*, of positroid varieties. One family comes from Postnikov's theory of planar networks, while the other is defined in terms of reduced words in the symmetric group. We show that these two families of parametrizations are essentially the same. In the second part of this thesis, we extend positroid combinatorics to the *Lagrangian Grassmannian* $\Lambda(2n)$, a subvariety of $\mathrm{Gr}(n, 2n)$ whose points correspond to maximal isotropic subspaces with respect to a symplectic form. Applying our results about parametrizations of positroid varieties, we construct network parametrizations for the analogs of positroid varieties in $\Lambda(2n)$ using planar networks which satisfy a symmetry condition.

CHAPTER I

Introduction

1.1 Totally nonnegative matrices

An invertible matrix is *totally positive* if all of its minors are positive real numbers. For example, the matrix

$$\begin{bmatrix} 14 & 7 & 2 \\ 5 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

is totally positive. Similarly, an invertible matrix is *totally nonnegative* if all of its minors are nonnegative real numbers. We denote the *general linear group* of invertible $n \times n$ matrices by $\mathrm{GL}(n)$, the subset of totally positive matrices by $\mathrm{GL}_{>0}(n)$, and the subset of totally nonnegative matrices by $\mathrm{GL}_{\geq 0}(n)$. Note that $\mathrm{GL}_{\geq 0}(n)$ is the closure of $\mathrm{GL}_{>0}(n)$.

Totally positive matrices first appeared in a paper of Schoenberg in 1930 [32]. A few years later, Gantmacher and Krein showed that all eigenvalues of a totally positive matrix are simple, real and positive [9, 10]. Total positivity has proved to be a powerful tool in many areas of mathematics, including analysis, probability, and applied mathematics [12].

Since the 1980's, there has been a great deal of research on the interplay between total positivity and combinatorics [6]. A key result in this area is Lindström's Lemma, which gives a way to construct totally nonnegative matrices from weighted directed graphs [20].

To state Lindström's Lemma, we need some terminology. Let D be a directed acyclic graph with weighted edges. Let u and v be vertices in D, and let π be a directed path from u to v. The *weight* of π, denoted $\mathrm{wt}(\pi)$, is the product of the weights of the edges in π. Let $P_D(u, v)$ be the sum of the weights of all directed paths from u to v in D. For example, in Figure 1.1, we have $P_D(u_1, v_1) = xy + xt$.

Let $\hat{u} = (u_1, \ldots, u_n)$ and $\hat{v} = (v_1, \ldots, v_n)$ be n-tuples of vertices in D. A family of paths $\hat{\pi} = (\pi_1, \ldots, \pi_n)$ from $\hat{u}$ to $\hat{v}$ is a collection of directed paths in D such that π_i starts at vertex u_i and ends at vertex v_i. We say $\hat{\pi}$ is *nonintersecting* if no two paths π_i and π_j share a vertex for $i \neq j$. The *weight* of $\hat{\pi}$ is the product $\mathrm{wt}(\pi_1)\,\mathrm{wt}(\pi_2)\cdots\mathrm{wt}(\pi_n)$. Let $N(\hat{u}, \hat{v})$ denote the sum of the weights of all families of nonintersecting paths from $\hat{u}$ to $\hat{v}$. We say that $\hat{u}$ and $\hat{v}$ are *compatible* if there does not exist a nonintersecting family of paths from $\hat{u}$ to any n-tuple of vertices $\hat{v}'$ obtained by re-ordering the vertices in $\hat{v}$. In Figure 1.1, the pairs (u_1, v_1) and (u_2, v_2) are compatible.

Lemma I.1 (Lindström's Lemma)**.** *Let D be a weighted directed acyclic graph. Let $\hat{u} = (u_1, \ldots, u_n)$ and $\hat{v} = (v_1, \ldots, v_n)$ be compatible n-tuples of vertices in D. Then*

$$N(\hat{u}, \hat{v}) = \det[(P_D(u_i, v_j))_{1 \leq i,j \leq n}]$$

For example, the directed graph in Figure 1.1 corresponds to the matrix

$$A = \begin{bmatrix} xy + xt & x \\ zt & z \end{bmatrix}$$

which has determinant xyz. Notice that xyz is also the weight of the only family of nonintersecting paths from (u_1, u_2) to (v_1, v_2).

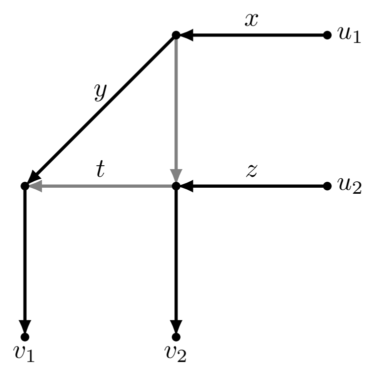


Figure 1.1: A weighted planar directed graph. Unlabeled edges have weight 1. The dark edges show the only nonintersecting family of paths from (u_1, u_2) to (v_1, v_2). Notice that this family of paths has weight xyz.

While Lindström did not state his result in terms of total nonnegativity, the connection is immediate. We say $\hat{u}$ and $\hat{w}$ are *fully compatible* if $(u_{i_1}, \ldots, u_{i_k})$ is compatible with $(v_{j_1}, \ldots, v_{j_k})$ for all $k \neq n$, with

$$1 \leq i_1 \leq \cdots \leq i_k \leq n,$$

$$1 \leq j_1 \leq \cdots \leq j_k \leq n.$$

If $\hat{u}$ and $\hat{v}$ are fully compatible, then every minor of the matrix $U = (P_D(u_i, v_j))_{1 \leq i,j \leq n}$ is a polynomial in the edge weights of D with nonnegative coefficients. If all edges of D have positive real weights, it follows that U is totally nonnegative.

Remarkably, the converse of Lindström's lemma also holds: every totally nonnegative matrix arises from a network. Moreover, we can always take the network to be plana,r as in Figure 1.1. The following theorem is due to Francesco Brenti.

Theorem I.2 ([5])**.** *Let U be an $n \times n$ real matrix. Then U is totally nonnegative if and only if there exists a planar, directed, acyclic graph D with positive real weights, and tuples of vertices $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ in D which are* fully compatible, *such that the entry of U in position (i, j) is given by $P_D(u_i, u_j)$.*

Both Lindström's Lemma and its converse have important application in algebraic and enumerative combinatorics. Lindström's lemma may be used to show that many matrices which arise in combinatorics are totally nonnegative, while the converse yields elegant proofs of certain properties of totally nonnegative matrices [6]. For example, the fact that $\mathrm{GL}_{\geq 0}(n)$ is closed under multiplication follows from the fact that multiplying totally nonnegative matrices is equivalent to concatenating the corresponding networks.

1.2 Total nonnegativity for reductive groups and flag varieties

Anne Whitney, working in the 1950's, studied total nonnegativity from a more algebraic point of view [34]. She showed that $\mathrm{GL}_{\geq}(n)$ is a semigroup, and gave generators for $\mathrm{GL}_{\geq}(n)$. Let $E_{i,j}(t)$ denote the matrix with a nonzero entry t in row i, column j, and 0's everywhere else. Let I_n denote the $n \times n$ identity matrix. We define

$$x_i(t) = I_n + E_{(i,i+1)}(t),$$

$$y_i(t) = I_n + E_{(i+1,i)}(t).$$

Then $\mathrm{GL}_{\geq 0}(n)$ is generated multiplicatively by:

1. Diagonal matrices with positive real entries along the diagonal,
2. All matrices $x_i(t)$ with $1 \leq i \leq n-1$ and t a positive real number,
3. All matrices $y_i(t)$ with $1 \leq i \leq n-1$ and t a positive real number.

Building on Whitney's work, Lusztig extended the notion of total nonnegativity to all split connected reductive groups defined over the real numbers [21]. Let G be such a group. For example, we might take G to be the *special linear group* $\mathrm{SL}(n)$, or the *symplectic group* $\mathrm{Sp}(2n)$. The *totally nonnegative part* $G_{\geq 0}$ of G is a monoid with a set of distinguished generators, which can be defined explicitly using representation theory. In the case of $\mathrm{GL}(n)$, Lusztig's generators coincide with Whitney's.

Elements of $G_{\geq 0}$ satisfy a property which generalizes nonnegativity of minors. The action of $G_{\geq 0}$ on *canonical bases* of certain representations is given by matrices with nonnegative real entries [21]. Fomin and Zelevinsky reformulated this result in more elementary terms, showing that Lusztig's definition is equivalent to positivity of certain *generalized minors* which are defined independently of canonical bases [8].

Lusztig extended the notion of total nonnegativity further to include *flag varieties* [22]. A *flag variety* is a quotient of a reductive group by a *Borel subgroup*. For example, the *complete flag variety* $\mathcal{F}\ell(n)$ is the quotient of the *special linear group* $\mathrm{SL}(n)$ of $n \times n$ matrices of determinant 1 by the subgroup B of upper-triangular matrices. We note that points in $\mathcal{F}\ell(n)$ are in one-to-one correspondence with *flags* of vector spaces

$$V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n\}$$

where V_k has dimension k. The equivalence class of a matrix A in $\mathcal{F}\ell(n)$ corresponds to a flag whose i^{th} subspace is the span of the first i columns of A.

A *partial flag variety* is a quotient of the reductive group by a *parabolic subgroup*. One example is the *Grassmannian* $\mathrm{Gr}(k, n)$, the algebraic variety whose points correspond to k-dimensional subspaces of $\mathbb{C}^n$. We may realize $\mathrm{Gr}(k, n)$ as the quotient of $\mathrm{SL}(n)$ by a subgroup P_k of block-upper-triangular matrices. The equivalence class of a matrix A in the quotient $\mathrm{SL}(n)/P_k$ corresponds to the span of its first k columns. Lusztig defined the totally nonnegative part of a (partial) flag variety to be the *closure* of the image of $G_{\geq 0}$ [22].

1.3 Projected Richardson varieties and total nonnegativity

Every flag variety G/B has a stratification by *Richardson varieties*. Richardson varieties $R_{u,w}$ in G/B are indexed by pairs of elements u, w in the *Weyl group* W of G with $u \leq w$ in the *Bruhat order* on W. For $G = \mathrm{SL}(n)$, the Weyl group is the symmetric group S_n and Richardson varieties are indexed by pairs of permutations. Let B be a Borel subgroup of the reductive group G, and P a parabolic subgroup containing B. There is a natural projection, often denote π_P, from the flag variety G/B to the partial flag variety G/P. For example, we have a projection

$$\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k, n)$$

which takes a flag $V_\bullet$ to the k-dimensional space V_k.

Taking the images of Richardson varieties $R_{u,w}$ under the projection π_P gives a stratification of G/P by *projected Richardson varieties* $\Pi_{u,w}$. For each projected variety Π, there is a family of Richardson varieties $R_{u,w}$ with $\pi_P(R_{u,w}) = \Pi$.

Intersecting projected Richardson varieties with the totally nonnegative part of a partial flag variety gives a decomposition of the totally nonnegative part, which Lusztig conjectured was a *cell decomposition* [22]. Rietsch proved this conjecture using the machinery of canonical bases [29]. Marsh and Rietsch later gave an alternate proof by constructing explicit parametrizations of projected Richardson varieties, thus avoiding canonical bases altogether [24]. We call the totally nonnegative part of a projected Richardson variety a *totally nonnegative cell*.

1.4 The positroid stratification of the Grassmannian

The *Grassmannian* $\mathrm{Gr}(k, n)$ is an algebraic variety whose points are in one-to-one correspondence with k-dimensional subspaces of the vector space $\mathbb{C}^n$. Concretely,

$\mathrm{Gr}(k,n)$ is the space of full-rank $k \times n$ matrices modulo row operations. A matrix corresponds to the span of its rows. Furthermore, $\mathrm{Gr}(k,n)$ is a partial flag variety. We may realize $\mathrm{Gr}(k,n)$ as the quotient $\mathrm{SL}(n)/P_k$, where $\mathrm{SL}(n)$ is the group of $n \times n$ matrices with determinant 1, and P_k is the subgroup of block matrices of the form

$$\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

where X is a $k \times k$ square block, and Z is a $(n-k) \times (n-k)$ square. With these conventions, a matrix representative $A \in \mathrm{SL}(n)$ corresponds to the span of its first k columns. We note that Lusztig's theory of total nonnegativity for partial flag varieties applies to $\mathrm{Gr}(k,n)$. However, the general theory does not immediately yield a concrete description of the totally nonnegative part of $\mathrm{Gr}(k,n)$, or of the stratification of $\mathrm{Gr}(k,n)$ by projected Richardson varieties.

Alexander Postnikov, motivated by combinatorial considerations, gave an alternate definition of total nonnegativity which is specific to $\mathrm{Gr}(k,n)$. Recall that each point in $\mathrm{Gr}(k,n)$ may be represented (non-uniquely) by a full rank $k \times n$ matrix. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ consists of all points in $\mathrm{Gr}(k,n)$ which may be represented by a matrix whose maximal minors are all nonnegative real numbers. For example, the matrix

$$\begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

corresponds to a point in $\mathrm{Gr}_{\geq 0}(2,4)$.

Postnikov introduced a stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ by *positroid cells.* Each positroid cell is defined as the set of points in $\mathrm{Gr}_{\geq 0}(k,n)$ where certain maximal minors vanish. Note that performing row operations on a $k \times n$ matrix does not affect the vanishing of maximal minors, so this is well-defined.

Remarkably, Postnikov's elementary definitions agree with Lusztig's general theory applied to $\mathrm{Gr}(k,n)$ [27]. Postnikov's definition of $\mathrm{Gr}_{\geq 0}(k,n)$ agrees with Lusztig's, and the stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ by positroid cells is precisely the stratification induced by Lusztig's projected Richardson varieties. Allen Knutson, Thomas Lam and David Speyer later introduced a stratification of $\mathrm{Gr}(k,n)$ by *positroid varieties*, defined in terms of rank conditions on matrices, and showed that positroid varieties coincide with Lusztig's projected Richardson varieties [14].

The positroid stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ has a rich combinatorial theory, introduced by Postnikov and developed by many others [27, 28, 14]. There are a number of interesting combinatorial objects which are in bijection with positroid varieties, including *bounded affine pemutations*, *Bruhat intervals*, and Ⅎ-*diagrams* (the symbol Ⅎ is pronounced "le"). Several of these indexing sets have natural poset structures. Whenever this occurs, the partial order on the indexing poset corresponds to closure relations among positroid cells in $\mathrm{Gr}_{\geq 0}(k,n)$. This is analogous to the way in which Young diagrams index Schubert cells in $\mathrm{Gr}(k,n)$, with containment of diagrams encoding closure relations among Schubert cells.

1.5 Parametrizing positroid varieties

A *parametrization* of a positroid variety Π is a birational map $(\mathbb{C}^{\times})^d \to \Pi$ which is a homeomorphism onto its image. There are two remarkable combinatorial constructions which give parametrizations of positroid varieties. The first, due to Postnikov, gives a family of parametrizations for each positroid cell, defined in terms of weighted planar networks. We may view this construction as a combinatorial analog of Theorem I.2, which gives an interpretation of the minors of a totally nonnegative matrix in terms of a planar network. The second method is a special case of Marsh and Rietsch's parametrizations for partial flag varieties [24] Since Marsh and Rietsch drew heavily on work of Deodhar, we call these *Deodhar parametrizations* [7].

1.5.1 The boundary measurement map

Postnikov defined a family of coordinate charts on each positroid cell, using planar bicolored networks called *plabic graphs*. For G a plabic graph, the *boundary measurement map* D_G is a homeomorphism from the space of positive real edge weights of G to some positroid cell $(\Pi_G)_{\geq 0}$. Letting the edge weights range over $\mathbb{C}^\times$ instead of $\mathbb{R}^+$, we obtain a birational map to the positroid variety $\Pi_G \subseteq \mathrm{Gr}(k,n)$ containing $(\Pi_G)_{\geq 0}$ [25]. We have a family of plabic graphs for each positroid cell, which correspond to a family of parametrizations.

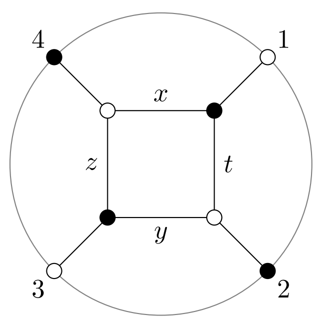


Figure 1.2: A weighted plabic graph. Unlabeled edges have weight 1.

Let ω be a weighting of a plabic graph, and consider the point $D_G(\omega)$ which is the image of ω under the boundary measurement map. Then there is a $k \times n$ matrix representative M for the point $D_G(\omega)$, whose maximal minors are given by summing the weights of certain collections of edges in the graph G. Hence the boundary measurement map gives a Grassmannian analog of Theorem I.2, which states that the minors of a totally nonnegative matrix are given by summing up the weights of certain collections of paths in a planar network. For more on the boundary measurement map, see Section 2.9.

In Chapter III, we focus on a special class of plabic graphs, called *bridge graphs*, which first arose as a computational tool in particle physics [1]. Bridge graphs are constructed by an inductive process, and the corresponding parametrizations are particularly straight-forward. See Figure 1.3 for an example of a bridge graph.

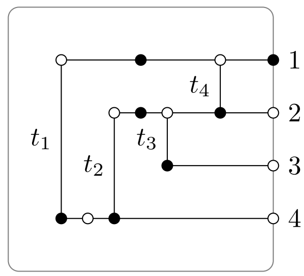


Figure 1.3: A bridge network. All unlabeled edges have weight 1.

Postnikov gave an explicit method for constructing plabic graphs using ⅃-*diagrams* [27]. A *Young diagram* is an finite collection of boxes arranged in left-justified rows, so that the length of the rows is weakly decreasing. A ⅃-*diagram* is a Young diagram in which each box contains either a 0 or a + sign, subject to the following pattern-avoidance condition: no 0 has both a + sign in the same row to its left, and a + in the same column above it. The ⅃-diagrams which fit inside a $k \times (n-k)$ grid are in bijection with positroid cells in $\mathrm{Gr}_{\geq 0}(k,n)$ [27]. Given a ⅃-diagram, Postnikov constructs a plabic graph for the corresponding positroid cell. Figure 1.4 shows an example of Postnikov's construction. The details appear [27], namely in Section 3 and Section 20.

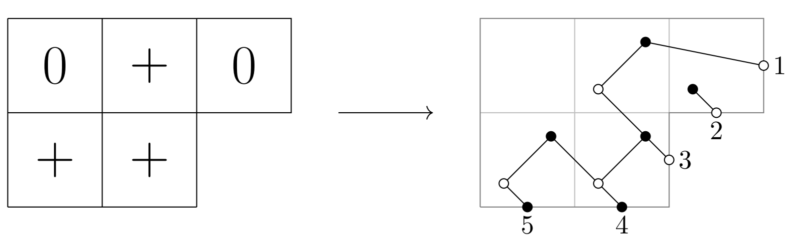


Figure 1.4: Constructing a planar network from a ⅃-diagram for a positroid cell in $\mathrm{Gr}_{\geq 0}(2,5)$

1.5.2 Deodhar parametrizations for positroid varieties

We now discuss *Deodhar parametrizations* in more detail. We note that Deodhar parametrizations may be defined for any partial flag variety. Here, however, we restrict our attention to positroid varieties in $\mathrm{Gr}(k, n)$.

Deodhar's work on partial flag varieties gives a family of decompositions of $\mathrm{Gr}(k, n)$, each of which refines the positroid stratification. Recall that positroid varieties are the images of certain Richardson varieties under the projection $\mathcal{F}\ell(n) \to \mathrm{Gr}(k, n)$, and that Richardson varieties are indexed by pairs of permutations u, w with $u \leq w$ in the Bruhat order on S_n. To give a Deodhar parametrization for $\mathrm{Gr}(k, n)$, we first choose a Richardson variety for each positroid variety. See Section 2.3 for details. Having fixed a Richardson variety $R_{u,w}$ corresponding to the positroid variety Π, we next choose a reduced word $\mathbf{w}$ for the permutation w. The *Deodhar components* of Π are indexed by certain subwords for u in $\mathbf{w}$ called *distinguished subexpressions.* Each positroid variety has a unique top-dimensional Deodhar component, corresponding to the unique *positive distinguished subexpression* for u in $\mathbf{w}$.

Marsh and Rietsch gave explicit parametrizations of Deodhar components of positroid varieties, which we call *Deodhar parametrizations* [24]. Each parametrization is given by a product of matrices corresponding to the factors in a reduced word. See Equation 1.2 for an example, and Section 2.4 for details of the construction. Since we define parametrizations of positroid varieties to be *birational*, we view Marsh and Rietsch's parametrizations of top-dimensional Deodhar components of positroid varieties as parametrizations of the positroid varieties themselves.

1.6 From subexpressions to networks

Chapter III of this thesis explores the relationship between Deodhar parametrizations, which are indexed by positive distinguished subexpressions (PDS's), and plabic graphs. Postnikov's ⅃-diagrams, which are in bijection with positroid cells, provide the first link between these two concepts. We saw above that each ⅃-diagram encodes a plabic graph for the corresponding positroid cell. Moreover, there is a beautiful bijection between ⅃-diagrams and PDS's of *Grassmannian* permutations, permutations with a single descent [27]. We describe this bijection in Section 2.10.

Kelli Talaska and Lauren Williams investigated the link between subexpressions and planar networks further in [33]. They considered Deodhar parametrizations for all components in a fixed Deodhar decomposition of $\mathrm{Gr}(k,n)$. These parametrizations are indexed by *all* distinguished subexpressions of Grassmannian permutations, not just PDS's. Talaska and Williams proved that each of these parametrizations arises from a network, which they constructed explicitly. For components indexed by PDS's, which are precisely the top-dimensional components of positroid varieties, they recovered the planar networks corresponding to Postnikov's ⅃-diagrams; for the remaining components, their networks were not all planar.

In Chapter III, we investigate *all* Deodhar parametrizations for *top-dimensional* Deodhar components of positroid varieties. We show that each of these parametrizations arises from a planar network, which is in fact a plabic graph. The ⅃-diagrams defined by Postnikov, and recovered by Talaska and Williams, give a special case of this result.

1.7 Extending positroid combinatorics

The projected Richardson stratification of an arbitrary partial flag variety may be viewed as a generalization of the positroid stratification of $\mathrm{Gr}(k, n)$. Hence it is natural to ask whether we can generalize positroid combinatorics to other partial flag varieties. Thomas Lam and Lauren Williams explored this question in [19], where they defined analogs of Postnikov's Ⅎ-diagrams for a class of partial flag varieties called *cominiscule Grassmannians*. Since each Ⅎ-diagram corresponds to a plabic graph, generalizing Ⅎ-diagrams may be viewed as a step toward generalizing plabic graphs themselves.

Lam and Williams defined type-B analogs of Postnikov's *decorated permutations* (or equivalently, of bounded affine permutations). These *type-B decorated permutations* index projected Richardson varieties in two different cominiscule Grassmannians: the *odd orthogonal Grassmannian*, a flag variety of type B, and the *Lagrangian Grassmannian*, a flag variety of type C [19]. This follows from the fact that the Weyl groups of types B and C are isomorphic. In Chapter IV of this thesis, we build on the work of Lam and Williams by extending more of the combinatorics of $\mathrm{Gr}_{\geq 0}(k, n)$ to the Lagrangian Grassmannian.

1.8 Summary of results

1.8.1 Bridge graphs and Deodhar parametrizations for positroid varieties

We have seen two different ways to parametrize positroid varieties: via Deodhar parametrizations, which are indexed by certain subexpressions of reduced words in S_n, and via parametrizations from plabic graphs. In Chapter III, we compare Deodhar parametrizations of positroid varieties to parametrizations arising from a special class of plabic graphs called bridge graphs. We show that these two families of parametrizations are essentially the same. Our main result is the following theorem, originally conjectured by Thomas Lam [17].

Theorem I.3. *Let* Π *be a positroid variety in* $\mathrm{Gr}(k,n)$*. For each Deodhar parametrization of* Π*, there is a bridge graph which yields the same parametrization. Conversely, any bridge graph parametrization of* Π *agrees with some Deodhar parametrization.*

Deodhar parametrizations may be defined for any partial flag variety. Postnikov's theory of plabic graphs, however, is specific to the Grassmannian. Theorem I.3 suggests a way to extend the boundary measurement map to other partial flag varieties, by constructing networks that encode Deodhar parametrizations. In Chapter III, we will apply this approach to the Lagrangian Grassmannian.

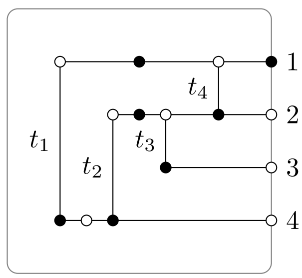


Figure 1.5: A bridge network. All unlabeled edges have weight 1.

To convey the flavor of Theorem I.3, we briefly sketch an example; the details will appear later. Take $k=2$ and $n=4$. Let $u=2134$ and $w=4321$. The Richardson variety $R_{u,w}$ corresponds to the top-dimensional positroid cell in $\mathrm{Gr}(2,4)$. The bridge graph in Figure 1.5 yields a parametrization for this positroid variety, given by

$$(t_1,t_2,t_3,t_4) \mapsto \begin{bmatrix} 1 & t_4 & 0 & -t_1 \\ 0 & 1 & t_3 & t_2 \end{bmatrix} \tag{1.1}$$

We claim that we can obtain the same map from some Deodhar parametrization. Indeed, fix the reduced word $\mathbf{w} = s_1s_2s_3s_2s_1s_2$ for w. The positive distinguished subexpression $\mathbf{u}$ for u in $\mathbf{w}$ comprises the s_3 in position 3 from the left, and the s_1

in position 5, so we have the projected Deodhar parametrization

(1.2)

$$(t_1, t_2, t_3, t_4) \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_2(t_1)\dot{s_1}^{-1}x_2(t_2)\dot{s_3}^{-1}x_2(t_3)x_1(t_4) = \begin{bmatrix} 0 & 1 & t_3 & t_2 \\ -1 & -t_4 & 0 & t_1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 0 & 1 & t_3 & t_4 \\ -1 & -t_4 & 0 & t_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t_4 & 0 & -t_1 \\ 0 & 1 & t_3 & t_2 \end{bmatrix}. \tag{1.3}$$

Hence, the two parametrizations send the point (t_1, t_2, t_3, t_4) to matrices which have the same row space, and hence represent the same point in the $\mathrm{Gr}(2, 4)$

1.8.2 Total positivity for the Lagrangian Grassmannian

In Chapter IV, we extend the combinatorial theory of the positroid stratification of $\mathrm{Gr}(k, n)$ to the *Lagrangian Grassmannian* $\Lambda(2n)$, using results from Chapter III. Note that Lam and Williams had previously defined analogs of Ⅎ-diagrams and bounded affine permutations for $\Lambda(2n)$ [19]. Building on their results, we explicitly describe analogs for $\Lambda(2n)$ of several other combinatorial objects which index positroid varieties. We then use the results of Chapter III to extend the boundary measurement map to $\Lambda(2n)$, using plabic graphs which satisfy a symmetry condition.

Let $\langle \cdot, \cdot \rangle$ be a symplectic form on $\mathbb{C}^{2n}$. A subspace V of $\mathbb{C}^{2n}$ is *isotropic* with respect to $\langle \cdot, \cdot \rangle$ if $\langle v, w \rangle = 0$ for all $v, w \in V$. The *Lagrangian Grassmannian* is the subvariety of $\mathrm{Gr}(n, 2n)$ whose points correspond to maximal isotropic subspaces with respect to $\langle \cdot, \cdot \rangle$. Alternatively, $\Lambda(2n)$ is the quotient of the symplectic group $\mathrm{Sp}(2n)$ by a parabolic subgroup, and is thus a partial flag variety of type C_n. By Lusztig's general theory, $\Lambda(2n)$ has a stratification by projected Richardson varieties, which will be our principal objects of study.

Let $e_1, \ldots, e_{2n}$ denote the standard basis of $\mathbb{C}^{2n}$. In what follows, let $\langle \cdot, \cdot \rangle$ be the symplectic form defined by

$$\langle e_i, e_j \rangle = \begin{cases} (-1)^j & \text{if } j = 2n+1-i \\ 0 & \text{otherwise} \end{cases}.$$

For example, when $n = 2$, the form $\langle \cdot, \cdot \rangle$ is given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The Lagrangian Grassmannian may be realized concretely as the space of full-rank $n \times 2n$ matrices whose rows are orthogonal with respect to $\langle \cdot, \cdot \rangle$, modulo row operations.

The poset $\mathcal{Q}(k, n)$ of *Bruhat intervals* indexes positroid varieties in $\mathrm{Gr}(k, n)$ and has a natural analog for any partial flag variety. We explicitly describe the corresponding poset $\mathcal{Q}^C(2n)$ for $\Lambda(2n)$, and show that it is an induced subposet of $\mathcal{Q}(n, 2n)$. Lam and Williams defined type-B decorated permutations, which index projected Richardson varieties in $\Lambda(2n)$ and serve as an analog of *bounded affine permutations*. We show that these type-B decorated permutations have nice combinatorial properties, similar to those of bounded affine permutations.

Next, we construct network parametrizations of projected Richardson varieties in the Lagrangian Grassmannian. The underlying graphs are plabic graphs which satisfy a symmetry condition, and we impose a corresponding symmetry condition on the edge weights. See Figure 1.6 for an example. Restricting the boundary measurement map to symmetric weightings of symmetric plabic graphs, we obtain parametrizations of projected Richardson varieties in $\Lambda(2n)$. To prove this, we first define *symmetric bridge graphs*, and use results from Chapter III to show that they encode Deodhar parametrizations of projected Richardson varieties in $\Lambda(2n)$. Finally, we use the combinatorics of symmetric plabic graphs to define analogs for the Lagrangian Grassmannian of various posets which index positroid varieties.

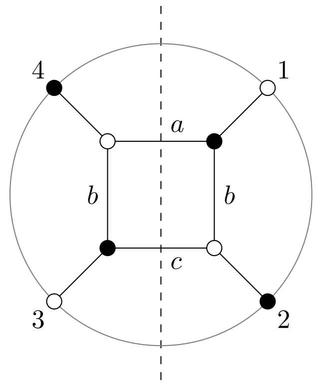


Figure 1.6: A symmetric weighting of a symmetric plabic graph. Unlabeled edges have weight 1.

We now sketch an example showing that the boundary measurement map takes symmetric weightings of symmetric graphs plabic graphs to points in the Lagrangian Grassmannian. Consider the weighted plabic graph in Figure 1.6. Applying the boundary measurement map, this corresponds to the parametrization

$$(a, b, c) \to \begin{bmatrix} b & 1 & c & 0 \\ -a & 0 & b & 1 \end{bmatrix} \tag{1.4}$$

We claim the matrix in Equation 1.4 represents a point in $\Lambda(2n)$ for all values of a, b and c. That is, we claim that the row span of the matrix is isotropic with respect to $\langle \cdot, \cdot \rangle$. For this, it suffices to show that the rows of the matrix are orthogonal with respect to the form $\langle \cdot, \cdot \rangle$. This follows, since

$$\langle (b, 1, c, 0), (-a, 0, b, 1) \rangle = b \cdot 1 - 1 \cdot b + c \cdot 0 - 0 \cdot (-a) = 0.$$

CHAPTER II

Background

2.1 Notation for partitions, root systems and Weyl groups

For $a \in \mathbb{N}$, write $[a]$ for the set $\{1, 2, \ldots, a\} \subseteq \mathbb{N}$, and let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$ for $a \leq b$. For $a > b$ we set $[a, b] = \emptyset$. Let $I, J \subseteq \mathbb{N}$ with

$$I = \{i_1 < i_2 < \ldots < i_m\} \tag{2.1}$$

$$J = \{j_1 < j_2 < \ldots < j_m\}. \tag{2.2}$$

We say $I \leq J$ if $i_r \leq j_r$ for all $1 \leq r \leq m$. We denote the set of all k-element subsets of $[n]$ by $\binom{[n]}{k}$. Let $n \in \mathbb{N}$. For $a \in [2n]$, we write a' to denote $2n + 1 - a$. Let $I \in \binom{[2n]}{n}$. Then we define $R(I) \in \binom{[2n]}{n}$ by setting $R(I) = [2n] \backslash \{a' \mid a \in I\}$.

We number the rows of all matrices from top to bottom, and the columns from left to right. When specifying matrix entries, position (i, j) denotes row i and column j. We denote the elementary matrix with 1's along the diagonal, non-zero entry t at position (i, j), and 0's everywhere else by $E_{(i,j)}(t)$.

Let Φ denote a finite root system with simple roots $\{\alpha_i \mid 1 \leq i \leq m\}$ for some $m \in \mathbb{N}$. Let (W, S) denote the Weyl group of Φ, with simple reflections $S = \{s_i \mid 1 \leq i \leq m\}$ corresponding to the α_i. Let ℓ denote the standard length function on W.

For $u, w \in W$, we write $u \leq w$ to denote a relation in the (strong) Bruhat order. A factorization $u = vw \in W$ is *length additive* if

$$\ell(u) = \ell(v) + \ell(w). \tag{2.3}$$

We use $u \leq_{(r)} w$ to denote a relation in the *right weak order*, so $u \leq_{(r)} w$ if there exists $v \in W$ such that $uv = w$ and the factorization is length additive. Similarly, we write $u \leq_{(l)} w$ to denote *left weak order*, and say $u \leq_{(l)} w$ if there is a length-additive factorization $vu = w$. All functions and permutations act on the left, so $\sigma\rho$ means "first apply ρ, then apply σ to the result."

We fix notation for the root systems of types A_{n-1} and C_n respectively. Let $\{\epsilon_1, \ldots, \epsilon_n\}$ denote the standard basis of $\mathbb{R}^n$. We realize the root system A_{n-1} as a subset of the vector space

$$V := \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \lambda_i = 0 \right\}$$

The roots are given by

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \neq n\}$$

and the simple roots are given by

$$\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\}.$$

The Weyl group of type A_{n-1} is the symmetric group S_n on n letters, acting by permutations on the standard basis. The simple reflection s_i^A corresponding to α_i is the transposition $(i, i+1)$. Let $(S_n)_k$ denote the parabolic subgroup of S_n corresponding to $\{\alpha_i \mid i \neq k\}$. Then $(S_n)_k$ is the Young subgroup $S_k \times S_{n-k}$ consisting of permutations which fix the sets $[k]$ and $[k+1, n]$.

Similarly, we realize the root system of type C_n as a subset of $\mathbb{R}^n$. The roots are given by

$$\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i, j \neq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}$$

and the simple roots by

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & 1 \leq i \leq n-1 \\ 2\epsilon_n & i = n \end{cases}$$

The Weyl group of type C_n consists of permutations σ of $\{\pm i \mid 1 \leq i \leq n\}$ which satisfy $\sigma(-i) = -\sigma(i)$. Permutations act on the standard basis, where we set $\epsilon_{-i} := -\epsilon_i$.

Alternatively, we may realize the Weyl group of type C_n as the subgroup S_n^C of S_{2n} consisting of permutations $\tau \in S_{2n}$ which satisfy

$$\tau(2n+1-a) = 2n+1-\tau(a). \tag{2.4}$$

The simple generators $s_1^C, \ldots, s_n^C$ are given by

$$s_i^C = \begin{cases} s_i^A s_{2n-i}^A & 1 \leq i \leq n-1 \\ s_i^A & i = n \end{cases}$$

and the map $S_n^C \hookrightarrow S_{2n}$ is a Bruhat embedding [4]. With these conventions, the parabolic subgroup $(S_n^C)_n$ corresponding to $\{\alpha_i \mid i \neq n\}$ is the subgroup of permutations in $S_n \times S_n$ which satisfy (2.4). We denote the length functions on S_n and S_n^C by ℓ^A and ℓ^C respectively. For $w \in S_n$ or S_n^C, we let $w([a])$ denote the unordered set $\{w(1), w(2), \ldots, w(a)\}$.

2.2 Flag varieties, Schubert varieties and Richardson varietes

We recall some basic facts about flag varieties, as stated for example in [15]. Let G be a semisimple Lie group over $\mathbb{C}$, and let B_+ and B_- be a pair of opposite Borel subgroups of G. Then G/B_+ is a flag variety, and there is a set-wise inclusion of the Weyl group W of G into G/B_+. The flag variety G/B_+ has a stratification by Schubert cells, indexed by elements of W. For $w \in W$, the *Schubert cell* $\mathring{X}_w$ is B_-wB_+/B_+ while the *Schubert variety* X_w is the closure of $\mathring{X}_w$. The *opposite Schubert cell* $\mathring{Y}_w$ is B_+wB_+/B_+ while the *opposite Schubert variety* Y_w is the closure of $\mathring{Y}_w$. The cells $\mathring{X}_w$ and $\mathring{Y}_w$ are both isomorphic to affine spaces. They have codimension and dimension $\ell(w)$, respectively. Both Schubert and opposite Schubert cells give stratifications of G/B_+.

The *Richardson variety* $\mathring{R}_{u,w}$ is the transverse intersection $\mathring{R}_{u,w} = \mathring{X}_u \cap \mathring{Y}_w$. This is empty unless $u \leq w$, in which case it has dimension $\ell(w) - \ell(u)$. The closure of $\mathring{R}_{u,w}$ is the *Richardson variety* $R_{u,w}$. Open Richardson varieties form a stratification of G/B_+ which refines the Schubert and opposite Schubert stratifications.

Let P be a parabolic subgroup of G containing B_+ and let W_P be the Weyl group of P. Then G/P is a partial flag variety, and there is a natural projection $\pi_P : G/B_+ \to G/P$. We are interested in the images of Richardson varieties under this projection map.

In this paper, we focus on flag varieties of types A and C. In the present chapter and in Chapter III, we use the superscripts A and C to indicate subvarieties of flag varieties of types A and C, respectively. For example, the Schubert cell in the type A flag variety $\mathcal{F}\ell(n)$ corresponding to a permutation w is denoted $\mathring{X}^A_w$. In Chapter II, which deals only with the type A case, we drop the superscripts.

2.2.1 Type A

Let $G = \mathrm{SL}(n)$, a semisimple Lie group of type A. Let B_+ be the subgroup of upper-triangular matrices, and let B_- be the subgroup of lower-triangular matrices. Let $\mathcal{F}\ell(n)$ be the quotient of $\mathrm{SL}(n)$ by the right action of B_+. Then $\mathcal{F}\ell(n)$ is an algebraic variety whose points correspond to flags

$$V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\} \tag{2.5}$$

where V_i is a subspace of $\mathbb{C}^n$ of dimension i. Hence an $n \times n$ matrix $M \in \mathrm{SL}(n)$ represents the flag whose i^{th} subspace is the span of the first i columns of M.

The Weyl group of type A is the symmetric group S_n on n letters. We may represent a permutation w in $\mathcal{F}\ell(n)$ by any matrix $\overline{w}$ with ± 1's in positions $(w(i), i)$, and 0's everywhere else, where the number of -1's is odd or even, depending on the parity of w. These signs are necessarily to ensure that the matrix representative is contained in $\mathrm{SL}(n)$. Then the Schubert cell $\mathring{X}_w^A$ corresponding to w is given by $B_-\overline{w}B_+$, while the opposite Schubert cells $\mathring{Y}_w^A$ is $B_+\bar{w}B_+$.

We have concrete descriptions of the Schubert and opposite Schubert varieties in $\mathcal{F}\ell(n)$, as stated in [14, Section 4]. For a subset J of $[n]$, let $\mathrm{Project}_J : \mathbb{C}^n \to \mathbb{C}^J$ be projection onto the coordinates indexed by J. For a permutation $w \in S_n$, the Schubert cell corresponding to w is given by

$$\mathring{X}_w^A = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[j]}(V_i)) = |w([i]) \cap [j]| \text{ for all } i\} \tag{2.6}$$

The closure of $\mathring{X}_w^A$ is the Schubert variety

$$X_w^A = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[j]}(V_i)) \leq |w([i]) \cap [j]| \text{ for all } i\} \tag{2.7}$$

Similarly, the opposite Schubert cell and opposite Schubert variety respectively are given by

$$\mathring{Y}_w^A = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[n-j+1,n]}(V_i)) = |w([i]) \cap [n-j+1, n]| \text{ for all } i\} \tag{2.8}$$

$$Y_w^A = \{V_\bullet \in \mathcal{F}\ell(n) \mid \dim(\mathrm{Project}_{[n-j+1,n]}(V_i)) \leq |w([i]) \cap [n-j+1, n]| \text{ for all } i\} \tag{2.9}$$

Let P denote the parabolic subgroup of $\mathrm{SL}(n)$ consisting of block-diagonal matrices of the form

$$\begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$$

where the block C is a $k \times k$ square, and E is an $(n-k)\times(n-k)$ square. Then $\mathrm{SL}(n)/P$ is the Grassmannian $\mathrm{Gr}(k, n)$ of k-dimensional linear subspaces of the vector space $\mathbb{C}^n$.

We may realize $\mathrm{Gr}(k, n)$ as the space of full-rank $k \times n$ matrices modulo the left action of $\mathrm{GL}(k)$, the group of invertible $k \times k$ matrices; a matrix M represents the space spanned by its rows. The natural projection $\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k, n)$, carries a flag $V_\bullet$ to the k-plane V_k. If V is a representative matrix for $V_\bullet$ then transposing the first k columns of V gives a representative matrix M for V_k.

The *Plücker embedding*, which we denote p, maps $\mathrm{Gr}(k, n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$ with homogeneous coordinates x_J indexed by the elements of $\binom{[n]}{k}$. For $J \in \binom{[n]}{k}$ let Δ_J denote the minor with columns indexed by J. Let V be an k-dimensional subspace of $\mathbb{C}^n$ with representative matrix M. Then $p(V)$ is the point defined by $x_J = \Delta_J(M)$. This map embeds $\mathrm{Gr}(k, n)$ as a smooth projective variety in $\mathbb{P}^{\binom{n}{k}-1}$. The homogeneous coordinates Δ_J are known as *Plücker coordinates* on $\mathrm{Gr}(k, n)$. The *totally nonnegative Grassmannian*, denoted $\mathrm{Gr}_{\geq 0}(k, n)$, is the subset of $\mathrm{Gr}(k, n)$ whose Plücker coordinates are all nonnegative real numbers, up to multiplication by a common scalar.

2.2.2 Type C

We now outline the same story in type C. Our discussion follows [3, Chapter 3]. However, we use the bilinear form given in [23]. This choice yields a *pinning* of $\mathrm{Sp}(2n)$, defined below, which is compatible with the *standard pinning* for $\mathrm{SL}(n)$. We will use this fact frequently in what follows.

Let V be the complex vector space $\mathbb{C}^{2n}$ with standard basis $e_1, \ldots, e_{2n}$. Let $\langle \cdot, \cdot \rangle$ denote the non-degenerate, skew-symmetric form defined by

$$\langle e_i, e_j \rangle = \begin{cases} (-1)^j & \text{if } j = 2n+1-i \\ 0 & \text{otherwise} \end{cases}$$

Let E be the matrix of this bilinear form. For example, for $n = 2$, we have

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

A subspace $U \subseteq V$ is *isotropic* if $\langle u, v \rangle = 0$ for all $u, v \in V$.

The *symplectic group* $\mathrm{Sp}(2n)$ is the group of matrices $A \in \mathrm{SL}(2n)$ which leave the form $\langle \cdot, \cdot \rangle$ invariant. Alternatively, define a map $\sigma : \mathrm{SL}(2n) \to \mathrm{SL}(2n)$ by

$$\sigma(A) = E(A^t)^{-1}E^{-1}.$$

Then $\mathrm{Sp}(2n)$ is the group of all fixed points of σ. It is a semi-simple Lie group of type C_n.

Let B_+, B_- and P be the subgroups of $\mathrm{SL}(2n)$ given above, where $k = n$. The Borel subgroups B_+ and B_- are both stable under σ, and so is the parabolic P. Let B_+^σ, B_-^σ, and P^σ denote the intersection of B_+, B_- and P respectively with $\mathrm{Sp}(2n)$. Then B_+^σ and B_-^σ are a pair of opposite Borel subgroups of $\mathrm{Sp}(2n)$, while P^σ is a parabolic subgroup of $\mathrm{Sp}(n)$.

The generalized flag variety $\mathrm{Sp}(2n)/B_+^\sigma$ embeds in $\mathcal{F}\ell(2n)$ in the obvious way, and $\mathrm{Sp}(2n)/P^\sigma$ embeds in $\mathrm{Gr}(n, 2n)$. The image of $\mathrm{Sp}(2n)/P^\sigma$ is precisely the subset of $\mathrm{Gr}(n,2n)$ corresponding to maximal isotropic subspaces; that is, the Lagrangian Grassmannian $\Lambda(2n)$. We have a commutative diagram of inclusions and projections, shown in Figure 2.1.

$$\begin{tikzcd}
\mathrm{SL}(2n) \arrow[r, two heads] & \mathcal{F}\ell(2n) \arrow[r, two heads] & \mathrm{Gr}(n,2n) \\
\mathrm{Sp}(2n) \arrow[u, hook] \arrow[r, two heads] & \mathrm{Sp}(2n)/B_+^\sigma \arrow[u, hook] \arrow[r, two heads] & \Lambda(2n) \arrow[u, hook]
\end{tikzcd}$$

Figure 2.1: Realizing $\Lambda(2n)$ as a submanifold of $\mathrm{Gr}(n,2n)$.

For each permutation $w \in S_n^C$, let $\overline{w}$ denote the matrix with $(-1)^{\rho_i}$'s at positions $(w(i),i)$ and 0's everywhere else, where

$$\rho_i = \begin{cases} 1 & \text{if } 1 \le i \le n \text{ and } w(i) \text{ and } i \text{ have opposite parity} \\ 0 & \text{otherwise} \end{cases}.$$

Then the Schubert cell $\mathring{X}_w^C$ corresponding to w is given by $B_-^\sigma \overline{w} B_+^\sigma$, while the opposite Schubert cell $\mathring{Y}_w^C$ is $B_+^\sigma \overline{w} B_+^\sigma$. It is straightforward to check that the following set-theoretic identities hold under the embedding $\mathrm{Sp}(2n)/B_+^\sigma \hookrightarrow \mathcal{F}\ell(2n)$ given above:

$$\mathring{X}_w^C = \mathring{X}_w^A \cap (\mathrm{Sp}(2n)/B_+^\sigma)$$

$$\mathring{Y}_w^C = \mathring{Y}_w^A \cap (\mathrm{Sp}(2n)/B_+^\sigma)$$

$$\mathring{R}_{u,w}^C = \mathring{R}_{u,w}^A \cap (\mathrm{Sp}(2n)/B_+^\sigma).$$

2.3 Projected Richardson varieties and P-order

We now return to the case of general G/P with all notation defined as above. Let $u, w \in W$. The variety $\Pi_{u,w} := \pi_P(R_{u,w})$ is a *projected Richardson variety.* Note that we do not have a one-to-one correspondence between Richardson varieties and projected Richardson varieties. Rather, for each $u \le w$, there is a family of

Richardson varieties $R_{u',w'}$ such that $\Pi_{u',w'} = \Pi_{u,w}$. Projected Richardson varieties have a number of nice geometric properties. In particular, they are normal, Cohen-Macaulay, and have rational singularities [14].

We review the notion of P-Bruhat order, as defined in [15]. We say $u \lessdot_P w$ if $u \lessdot w$ in the usual Bruhat order and $uW_P \neq wW_P$. The *P-Bruhat* order on W is the transitive closure of these relations. As the notation suggests, the relations $\lessdot_P$ are the covering relations in the resulting partial order. We use the symbol $\leq_P$ to denote P-Bruhat order. If $u \leq_P w$, we write $[u,w]_P$ for the P-Bruhat interval $\{v \mid u \leq_P v \leq_P w\}$.

Every (closed) projected Richardson variety may be realized as $\Pi_{u,w}$ with $u \leq_P w$. The projection map π_P is an isomorphism when restricted to $\mathring{R}_{u,w}$ if and only if $u \leq_P w$. In this case, we define the *open projected Richardson variety* $\mathring{\Pi}_{u,w}$ to be $\pi_P(\mathring{R}_{u,w})$. The variety $\mathring{\Pi}_{u,w}$ is irreducible, of dimension $\ell(w)-\ell(u)$ [15]. The projected Richardson variety $\Pi_{u,w}$ is the closure of $\mathring{\Pi}_{u,w}$, and π_P maps $R_{u,w}$ birationally to $\Pi_{u,w}$ [14].

Lemma II.1 ([15])**.** *Suppose we have $u \leq v$ and $x \leq y$ in W, and suppose there is some $z \in W_P$ such that $x = uz$, $y = vz$ with both factorizations length-additive. Then $u \leq_P v$ if and only if $x \leq_P y$.*

Hence there is an equivalence relation on P-Bruhat intervals, which is generated by setting $[u,w]_P \sim [x,y]_P$ if there is some $z \in W_P$ such that $x = uz$ and $y = wz$, with both factorizations length-additive. We write $\langle u,w\rangle_P$ for the equivalence class of $[u,w]_P$ and denote the set of all such classes by $\mathcal{Q}(W,W_P)$.

Define a partial order on $\mathcal{Q}(W,W_P)$ by setting $\langle u,w\rangle_P \leq \langle x,y\rangle_P$ if there exist representatives $[u',w']_P$ of $\langle u,w\rangle_P$ and $[x',y']_P$ of $\langle x,y\rangle_P$ with $[x',y']_P \subseteq [u',w']_P$. The poset $\mathcal{Q}(W,W_P)$ was first studied by Rietsch, in the context of closure relations for totally nonnegative cells in general flag manifolds [30]. Williams proved a number of combinatorial results about this poset directly [35]. The following lemmas are proved in [14].

Lemma II.2. *Suppose $u \leq_P w$, $u = u^P u_P$, and $w = w^P w_P$, where u^P and w^P are minimal-length coset representatives for W_P, and $u_P, w_P \in W_P$. Then $w_P \leq_{(l)} u_P$.*

Lemma II.3. *Suppose $u \leq w$, and w is a minimal-length coset representative for wW_P. Then $u \leq_P w$.*

Lemma II.4. *Each $\langle x, y\rangle_P$ has a unique representative $[x', y']_P$ where y' is of minimal length in its left coset $y'W_P$.*

Hence elements of $\mathcal{Q}(W, W_P)$ are in bijection with pairs (u, w) where $u \leq w$, and w is of minimal length in its coset wW_P. If $u \leq_P w$ and $u' \leq_P w'$, then $\Pi_{u,w} = \Pi_{u',w'}$ if and only if $\langle u, w\rangle_P = \langle u', w'\rangle_P$. Moreover, in this case $\mathring{\Pi}_{u,w}$ and $\mathring{\Pi}_{u',w'}$ are equal. Hence there is a unique open projected Richardson variety $\mathring{\Pi}_{\langle u,w\rangle_P}$ corresponding to $\langle u, w\rangle_P$. The poset $\mathcal{Q}(W, W_P)$ is isomorphic to the poset of projected Richardson varieties, ordered by *reverse* inclusion [30].

When $W = S_n$ and $W_P = S_k \times S_{n-k}$, we recover the k-order introduced by Bergeron and Sottile [2]. We write $u \leq_k w$ to denote a relation in k-order, and write $\mathcal{Q}(k, n)$ for the poset $\mathcal{Q}(W, W_P)$. We write $[u, w]_k$ for the k-Bruhat interval $\{v \mid u \leq_k v \leq_v w\}$, and $\langle u, w\rangle_k$ for the corresponding equivalence class in $\mathcal{Q}(k, n)$. We have the following criterion for comparison in k-Bruhat order.

Theorem II.5. ([2], Theorem A) *Let $u, w \in S_n$. Then $u \leq_k w$ if and only if*

1. *$1 \leq a \leq k < b \leq n$ implies $u(a) \leq w(a)$ and $u(b) \geq w(b)$.*
2. *If $a < b$, $u(a) < u(b)$ and $w(a) > w(b)$, then $a \leq k < b$.*

The minimal-length coset representatives for $S_k \times S_{n-k}$ in S_n are called *Grassmannian permutations* of type (k, n). Concretely, a permutation is Grassmannian of type (k, n) if it is increasing on $[k]$ and on $[k+1, n]$. The following lemma is due to Postnikov [27, Section 20].

Lemma II.6. *Let $u, w \in S_n$, and suppose w is Grassmannian of type (k, n). Then $u \leq w$ if and only if $u(i) \leq w(i)$ for all $i \in [k]$, and $u(i) \geq w(i)$ for all $i \in [k+1, n]$.*

We say a permutation $w \in S_n$ is *anti-Grassmannian* of type (k, n) if it is of *maximal* length in its coset of $S_k \times S_{n-k}$. A permutation is anti-Grassmannian of type (k, n) if it is decreasing on $[k]$ and on $[k+1, n]$.

Proposition II.7. *Each equivalence class $\langle u, w\rangle_k$ in $\mathcal{Q}(k, n)$ has a unique representative with u anti-Grassmannian. If $u \leq w$ with u anti-Grassmannian, then $u \leq_k w$.*

Proof. Let $u' \leq_k w'$ be a representative for $\langle u, w\rangle_k$ with w' Grassmannian. By standard Coxeter-theoretic arguments, there exists a unique $\sigma \in S_k \times S_{n-k}$ such that $u'\sigma$ is anti-Grassmannian and

$$\ell(u'\sigma) = \ell(u') + \ell(\sigma).$$

Setting $u = u'\sigma$ and $w = w'\sigma$ gives the desired representative, and uniqueness follows.

For the second statement, let $u \leq w$ with u anti-Grassmannian. We have a length-additive factorization $w = w^P w_P$, where w^P is Grassmannian and $w_P \in S_k \times S_{n-k}$. Then we have $uw_P^{-1} \leq_k w^P$, and the factorization $u = (uw_P^{-1})(w_P)$ is length-additive. By [15, Lemma 2.3], it follows that $u \leq_k w$ as desired. □

When $W = S_n^C$ and $W_P = (S_n^C)_n$, we denote the P-Bruhat order by $\leq_n^C$ and write $\mathcal{Q}^C(2n)$ for $\mathcal{Q}(W, W_P)$. We denote equivalence classes in $\mathcal{Q}^C(2n)$ by $\langle u, w\rangle_n^C$. In Chapter IV, we relate $\mathcal{Q}^C(2n)$ to the poset $\mathcal{Q}(n, 2n)$.

2.4 Deodhar parametrizations for projected Richardson varieties

2.4.1 Distinguished subexpressions

Deodhar defined a class of cell decompositions for general flag varieties G/B with components indexed by *distinguished subexpressions* of the Weyl group W of G [7]. Let $\mathbf{w} = s_{i_1} \cdots s_{i_m}$ be a reduced word for some $w \in W$. We gather some facts about distinguished subexpressions, borrowing most of our conventions from [24]. A *subexpression* $\mathbf{u}$ of $\mathbf{w}$ is an expression obtained by replacing some of the factors s_{i_j} of $\mathbf{w}$ with the identity permutation, which we denote by 1.

We write $\mathbf{u} \preceq \mathbf{w}$ to indicate that $\mathbf{u}$ is a subexpression of $\mathbf{w}$. We denote the t^{th} factor of $\mathbf{u}$, which may be either 1 or a simple transposition, by u_t, and write $u_{(t)}$ for the product $u_1 u_2 \ldots u_t$. For notational convenience, we set $u_0 = u_{(0)} = 1$. We denote the t^{th} simple transposition in $\mathbf{u}$ by u^t.

Definition II.8. A subexpression $\mathbf{u}$ of $\mathbf{w}$ is called **distinguished** if we have

$$u_{(j)} \leq u_{(j-1)} s_{i_j} \tag{2.10}$$

for all $1 \leq j \leq m$.

Definition II.9. A distinguished subexpression $\mathbf{u}$ of $\mathbf{w}$ is called **positive distinguished** if

$$u_{(j-1)} < u_{(j-1)} s_{i_j} \tag{2.11}$$

for all $1 \leq j \leq m$ We will sometimes abbreviate the phrase "positive distinguished subexpression" to *PDS*.

Given a subexpression $\mathbf{u} \preceq \mathbf{w}$, we say $\mathbf{u}$ is *a subexpression for* $u = u^1 u^2 \cdots u^r$. By abuse of notation, we identify the subexpression $\mathbf{u}$ with the word $u^1 u^2 \cdots u^r$, also denoted $\mathbf{u}$. If the subexpression $\mathbf{u}$ is positive distinguished, then the corresponding word is reduced. The following lemma is an easy consequence of the above definitions.

Lemma II.10. *[24] Let $u \leq w \in W$, and let $\mathbf{w}$ be a reduced word for w. Then the following are equivalent*

1. *$\mathbf{u}$ is a positive distinguished subexpression of $\mathbf{w}$.*
2. *$\mathbf{u}$ is the lexicographically first subexpression for u in $\mathbf{w}$, working from the right.*

In particular, there is a unique PDS for u in $\mathbf{w}$.

Condition 2 means precisely that the factors u^t are chosen greedily, as follows. Suppose $\ell(u) = r$. Working from the right, we set $u^r = s_{i_j}$, where j is the largest index such that $s_{i_j} \leq_{(l)} u$. We then take u^{r-1} to be the next rightmost factor of $\mathbf{w}$ such that $u^{r-1}u^r \leq_{(l)} u$ and so on, until we have $u^1u^2 \cdots u^r = u$.

2.4.2 Deodhar's decomposition

To construct a Deodhar decomposition of G/B, we first fix a reduced word $\mathbf{w}$ for each $w \in W$. There is a Deodhar component $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ for each distinguished subexpression $\mathbf{u} \preceq \mathbf{w}$. For a reduced word $\mathbf{w}$ of w, we have

$$\mathring{R}_{u,w} = \bigsqcup_{\mathbf{u} \preceq \mathbf{w} \text{ distinguished}} \mathcal{R}_{\mathbf{u},\mathbf{w}}. \tag{2.12}$$

Marsh and Rietsch gave explicit parametrizations for the components $\mathcal{R}_{\mathbf{u},\mathbf{w}}$. Rather than using Deodhar's original construction, we define the *Deodhar component* $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ to be the image of the parametrization corresponding to $\mathbf{u} \preceq \mathbf{w}$, as defined in [24]. We give the parametrizations explicitly in Section 2.4.4.

Fix a parabolic subgroup P of G which contains B. If $u, w \in W$ with $u \leq_P w$, then $\mathring{R}_{u,w} \subseteq G/B$ projects isomorphically onto its image in G/P, which is one of Lusztig's projected Richardson strata. To define a Deodhar decomposition of G/P, choose a representative $[u,w]_P$ for each $\langle u',w'\rangle_P \in \mathcal{Q}(W,W_P)$, and a reduced word $\mathbf{w}$ for each chosen w. For each of the selected $u \leq_P w$ and each distinguished subexpression $\mathbf{u} \preceq \mathbf{w}$ where $\mathbf{w}$ is the selected word for w, the *Deodhar component* of G/P corresponding to $\mathbf{u} \preceq \mathbf{w}$ is given by

$$\mathcal{D}_{\mathbf{u},\mathbf{w}} := \pi_P(\mathcal{R}_{\mathbf{u},\mathbf{w}}). \tag{2.13}$$

Since π_P is an isomorphism on $\mathring{R}_{u,w}$, composing Marsh and Rietsch's parametrization of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ with π_P gives a parametrization of $\mathcal{D}_{\mathbf{u},\mathbf{w}}$.

2.4.3 Total nonnegativity.

Let $\mathbf{w}$ be a reduced word of w, let $u \leq_P w$, and let $\mathbf{u} \preceq \mathbf{w}$ be the unique PDS for u in $\mathbf{w}$. Then $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ is the unique top-dimensional Deodhar component of $\mathring{R}_{u,w}$, and $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ is dense $\mathring{R}_{u,w}$ [24]. Hence the corresponding parametrization of $\mathcal{D}_{\mathbf{u},\mathbf{w}}$ gives a birational map to $\mathring{\Pi}_{u,w}$ which is an isomorphism onto its image. We call this a *Deodhar parametrization* of $\mathring{\Pi}_{u,w}$.

The *totally nonnegative part* $(\mathring{R}_{u,w})_{\geq 0}$ of the projected Richardson variety $\mathring{R}_{u,w}$ is the subset of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ where all parameters in the Deodhar parametrization are positive real. Projecting to G/P, we obtain a parametrization of the totally nonnegative part of $\mathring{\Pi}_{u,w}$. Hence $(\mathring{\Pi}_{u,w})_{\geq 0}$ is the part of $\mathcal{D}_{\mathbf{u},\mathbf{w}}$ where all parameters are positive real. Note that the image is independent of our choice of $\mathbf{w}$ [24]. Remarkably, when $G/P = \mathrm{Gr}(k,n)$, this is precisely the locus where all Plücker coordinates are nonnegative real, up to multiplication by a common scalar. We show in Section 4.4 that a similar statement holds for $\Lambda(2n)$.

2.4.4 Parametrizing Deodhar components

We now review Marsh and Rietsch's parametrizations for Deodhar components [24]. Let G/B be a flag variety, and let

$$\Pi = \{\alpha_i \mid i \in I\}$$

be a choice of simple roots for the root system of G. For each $\alpha_i \in \Pi$, there is a group homomorphism $\varphi_i : \mathrm{SL}_2 \to G$, which yields 1-parameter subgroups

$$(2.14) \qquad x_i(m) = \varphi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \qquad y_i(m) = \varphi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \qquad \alpha_i^\vee(t) = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

Such a choice of simple roots and of homomorphisms φ_i is called a *pinning*. For each simple reflection s_i of W, we define a corresponding element of G/B by

$$(2.15) \qquad \dot{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

If $w = s_{i_1} \cdots s_{i_m}$ is a reduced word, define $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}$. This is independent of the choice of reduced expression [24]. For each root α, there is some simple root α_i and $w \in W$ such that $w\alpha_i = \alpha$. The *root subgroup* corresponding to α is given by $\{\dot{w} x_i(t) \dot{w}^{-1} \mid t \in \mathbb{C}\}$.

Marsh and Rietsch give explicit parametrizations for Deodhar components, using products of the elements x_i, y_i, and $\dot{s}_i$ [24]. For $\mathbf{u} \preceq \mathbf{w}$ a subexpression, with $\mathbf{u} = s_{i_1} \cdots s_{i_m}$, they define

$$J_{\mathbf{u}}^{+} = \{k \in \{1, \ldots, m\} \mid u_{(k-1)} < u_{(k)}\}$$

$$J_{\mathbf{u}}^{\circ} = \{k \in \{1, \ldots, m\} \mid u_{(k-1)} = u_{(k)}\}$$

$$J_{\mathbf{u}}^{-} = \{k \in \{1, \ldots, m\} \mid u_{(k-1)} > u_{(k)}\}$$

We assign an element of G to each simple reflection s_{i_k} of $\mathbf{w}$ according to the rule below, where t_k and m_k are parameters, the parameter t_k takes values in $\mathbb{C}^\times$, and m_k takes values on $\mathbb{C}$, by:

$$g_k = \begin{cases} x_{i_k}(m_k)\dot{s}_{i_k}^{-1} & k \in J_{\mathbf{u}}^- \\ y_{i_k}(t_k) & k \in J_{\mathbf{u}}^\circ \\ \dot{s}_{i_k} & k \in J_{\mathbf{u}}^+ \end{cases}$$

Note that if $\mathbf{u}$ is a PDS of $\mathbf{w}$, the third case never occurs. We define a subset of G corresponding to $\mathbf{u} \preceq \mathbf{w}$ by setting

$$G_{\mathbf{u},\mathbf{w}} = \{g_1 \dots g_n \mid t_k \in \mathbb{C}^\times,\ m_k \in \mathbb{C}\} \tag{2.16}$$

Proposition II.11. [24, Proposition 5.2] *The map*

$$(\mathbb{C}^*)^{J_{\mathbf{u}}^\circ} \times \mathbb{C}^{J_{\mathbf{u}}^-} \to G_{\mathbf{u},\mathbf{w}}$$

from (2.16) *is an isomorphism. Projecting to G/B gives an isomorphism*

$$G_{\mathbf{u},\mathbf{w}} \to \mathcal{R}_{\mathbf{u},\mathbf{w}}.$$

2.4.5 A pinning for $\mathrm{SL}(n)$

To write down Deodhar parametrizations of $\mathcal{F}\ell(n)$ explicitly, it is enough to fix a pinning for $\mathrm{SL}(n)$. We use the standard pinning corresponding to our choice of simple roots [33, Section 4]. With this pinning, let $x_i^A(t)$, $y_i^A(t)$, and $\dot{s}_i^A$ denote the matrices in $\mathrm{SL}(n)$ defined as in 2.14 and 2.15 respectively. Then $x_i^A(t) = E_{i,i+1}(t)$ and $y_i^A(t) = E_{i+1,i}(t)$. The matrix $\dot{s}_i^A$ is obtained from the $n \times n$ identity by replacing the 2×2 block whose upper left corner is at position (i,i) with the block matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.17}$$

In Chapter III, where we consider the type A case only, the superscript A's will be dropped.

For $\alpha = \epsilon_i - \epsilon_j$ with $i < j$, the corresponding root subgroup consists of matrices $x^A_\alpha(t) = E_{(i,j)}(t)$, while the root subgroup corresponding to $-\alpha$ consists of matrices $y^A_\alpha(t) = E_{(j,i)}(t)$. For convenience, we write $x^A_{(i,j)}(t)$ for $x^A_\alpha(t)$, and $y^A_{(i,j)}(t)$ for $y^A_\alpha(t)$. Note in particular that $\left(x^A_{(i,j)}(t)\right)^T = y^A_{(i,j)}(t)$.

2.4.6 A pinning for $\mathrm{Sp}(2n)$

We now give a pinning for $\mathrm{Sp}(2n)$, consistent with our previous choice of simple roots. Again, we follow [3, Chapter 3], but with some sign changes to account for our choice of symplectic form. We use the superscript C to denote the symplectic matrices defined using 2.14 and 2.15. Then we have

$$x^C_i(t) = \begin{cases} x^A_i(t) x^A_{2n-i}(t) & 1 \leq i \leq n-1 \\ x^A_n(t) & i = n \end{cases}$$

$$y^C_i(t) = \begin{cases} y^A_i(t) y^A_{2n-i}(t) & 1 \leq i \leq n-1 \\ y^A_n(t) & i = n \end{cases}$$

Similarly, the Weyl group representatives $\dot{s}^C_i$ are given by

$$\dot{s}^C_i = \begin{cases} \dot{s}^A_i \dot{s}^A_{2n-i} & 1 \leq i \leq n-1 \\ \dot{s}^A_n & i = n \end{cases}$$

We now relate Deodhar parametrizations of $\Lambda(2n)$ to Deodhar parametrizations of $\mathrm{Gr}(n, 2n)$. It is not hard to check that the embedding $S^C_n \hookrightarrow S_{2n}$ carries distinguished subexpressions S^C_n to distinguished subexpressions in S_{2n}.

Let $\widetilde{\mathbf{v}} \preceq \widetilde{\mathbf{w}}$ be a distinguished subexpression in S^C_n and let $\mathbf{v} \preceq \mathbf{w}$ be the corresponding distinguished subexpression in S_{2n}. Let $\mathcal{R}^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}}$ be the Deodhar component of $\mathrm{Sp}(2n)/B_+^\sigma$ corresponding to $\widetilde{\mathbf{u}} \preceq \widetilde{\mathbf{w}}$, and let $\mathcal{R}^A_{\mathbf{u},\mathbf{w}}$ be the component of $\mathcal{F}\ell(2n)$ corresponding to $\mathbf{u} \preceq \mathbf{w}$.

Lemma II.12. *We have* $\mathcal{R}^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}} \hookrightarrow \mathcal{R}_{\mathbf{w},\mathbf{u}}$ *under the map* $\mathrm{Sp}(2n)/B_+^\sigma \hookrightarrow \mathcal{F}\ell(n)$*.*

Proof. Consider the parametrization of $\mathcal{R}^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}}$ described above. We expand the formula for each element of $G^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}}$ as a product of matrices $x^A_{i_r}(t_r)$, $y^A_{i_r}(t_r)$, and $\dot{s}^A_{i_r}$. The resulting sequence of matrices in $\mathrm{SL}(n)$ is identical to the sequence of matrices corresponding to $\mathbf{u} \preceq \mathbf{w}$, except that the t_i satisfy some relations of the form $t_i = t_{i+1}$. Hence, $G^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}}$ corresponds to a locally closed subset of $\mathcal{R}^A_{\mathbf{u},\mathbf{w}}$ and the claim follows. $\square$

2.5 Positroid varieties

Let $V \in \mathrm{Gr}_{\geq 0}(k,n)$. The indices of the non-vanishing Plücker coordinates of V give a set $\mathcal{J} \subseteq \binom{[n]}{k}$ called the *matroid* of V. We define the *matroid cell* $\mathcal{M}_{\mathcal{J}}$ as the locus of points $V \in \mathrm{Gr}_{\geq 0}(k,n)$ with matroid $\mathcal{J}$. The nonempty matroid cells in $\mathrm{Gr}_{\geq 0}(k,n)$ are the *positroid cells* defined by Postnikov, and the corresponding matroids are called *positroids*. Positroid cells form a stratification of $\mathrm{Gr}_{\geq 0}(k,n)$, and each cell is homeomorphic to $(\mathbb{R}^+)^d$ for some d [27, Theorem 3.5].

The positroid stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ extends to the complex Grassmannian $\mathrm{Gr}(k,n)$. Taking the Zariski closure of a positroid cell of $\mathrm{Gr}_{\geq 0}(k,n)$ in $\mathrm{Gr}(k,n)$ gives a *positroid variety*. For a positroid variety $\Pi^A \subseteq \mathrm{Gr}(k,n)$, we define the *open positroid variety* $\mathring{\Pi}^A \subset \Pi^A$ by taking the complement in Π^A of all lower-dimensional positroid varieties. The open positroid varieties give a stratification of $\mathrm{Gr}(k,n)$[14].

Positroid varieties in $\mathrm{Gr}(k,n)$ may be defined in numerous other ways. There is a beautiful description of positroid varieties as intersections of cyclically permuted Schubert varieties. In particular, the positroid stratification refines the well-known Schubert stratification of $\mathrm{Gr}(k,n)$ [14].

Remarkably, positroid varieties in $\mathrm{Gr}(k,n)$ coincide with projected Richardson varieties [14, Section 5.4]. Indeed, let $u \leq_k w$. The projection π_k maps $\mathring{R}^A_{u,w}$ homeomorphically onto its image, which is an open positroid variety $\mathring{\Pi}^A_{u,w}$. The closure of $\mathring{\Pi}^A_{u,w}$ is a (closed) positroid variety $\Pi^A_{u,w}$ and we have $\pi_k(R^A_{u,w}) = \Pi^A_{u,w}$. Every positroid variety arises in this way.

Since positroid varieties are projected Richardson varieties, we have an isomorphism between $\mathcal{Q}(k,n)$ and the poset of positroid varieties, ordered by *reverse* inclusion [14, Section 5.4].

2.6 Grassmann necklaces

Grassmann necklaces, first introduced in [27], give another family of combinatorial objects which index positroid varieties.

Definition II.13. A *Grassmann necklace* $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ of type (k,n) is a sequence of k-element subsets of $[n]$ such that the following hold for all $i \in [n]$, with indices taken modulo n:

1. If $i \in I_i$ then $I_{i+1} = (I_i \backslash \{i\}) \cup \{j\}$ for some $j \in [n]$.
2. If $i \notin I_i$, then $I_{i+1} = I_i$.

For $a \in [n]$, let $\leq_a$ denote the cyclic shift of the usual linear order on n given by

$$a < a+1 < \cdots < n < 1 < \cdots < a-1.$$

Note that $\leq_1$ is the usual order $\leq$. We extend this to a partial order on $\binom{[n]}{k}$ by setting $I \leq_a J$ if we have $i_\ell \leq_a j_\ell$ for all $\ell \in [k]$, where

$$I = \{i_1 <_a i_2 <_a \cdots <_a i_k\} \text{ and } J = \{j_1 <_a j_2 <_a \cdots <_a j_k\}.$$

Let $\mathcal{J}$ be a positroid of type (k,n). That is, $\mathcal{J}$ is the matroid of some nonempty positroid cell in $\mathrm{Gr}_{\geq 0}(k,n)$. Then $\mathcal{J}$ is a collection of k-element subsets of $[n]$. For each $1 \leq i \leq n$, let I_i be the minimal element of $\mathcal{J}$ with respect to the shifted linear order $\leq_i$. Then $\mathcal{I} = (I_1, \ldots, I_n)$ is a Grassmann necklace of type (k, n). This procedure gives a bijection between Grassmann necklaces and positroids of type (k, n) [27]. For the inverse bijection, let $\mathcal{I} = (I_1, \cdots, I_n)$ be a Grassmann necklace of type (k, n), and let $\mathcal{J}$ be set of all k-element subsets $J \in \binom{[n]}{k}$ such that $I_a \leq_a J$ for all $a \in [n]$. Then $\mathcal{J}$ is the positroid with Grassmann necklace $\mathcal{I}$ [27, 26].

Positroid varieties are not matroid varieties. Suppose $X \in \mathrm{Gr}(k, n)$ is contained in the open positroid variety $\mathring{\Pi}$ with corresponding positroid $\mathcal{J}$. The matroid of X is contained in $\mathcal{J}$, but may not be equal to $\mathcal{J}$ [14]. However, the two matroids are related. Let $\mathcal{J}_X$ be the matroid of X, and let $(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ be the Grassmann necklace of $\mathcal{J}$. Then I_i is the minimal element of $\mathcal{J}_X$ with respect to the shifted linear order $\leq_i$ for all $i \in [n]$ [15].

We define a partial order on Grassmann necklaces by setting $\mathcal{I} \leq \mathcal{I}'$ if $I_i \leq_i I'_i$ for all i. Hence the poset of Grassmann necklaces is isomorphic to the poset of positroids, ordered by *reverse* containment.

2.7 Bounded affine permutations and Bruhat intervals in type A

Definition II.14. An **affine permutation** of order n is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ which satisfies the condition

$$f(i + n) = f(i) + n \tag{2.18}$$

for all $i \in \mathbb{Z}$. The affine permutations of order n form a group, which we denote $\widetilde{S}_n$.

For a reductive group G, the *extended affine Weyl group* $\widehat{W}$ of G is defined by

$$\widehat{W} = X_* \rtimes W$$

where W is the Weyl group of G and X_* is the cocharacter lattice of G [16]. The group $\widetilde{S}_n$ is the extended affine Weyl group of $\mathrm{GL}(n)$. In particular, we have

$$\widetilde{S}_n \cong \mathbb{Z}^n \rtimes S_n$$

where $\mathbb{Z}^n$ is the cocharacter lattice of $\mathrm{GL}(n)$. Each permutation w acts periodically on $\mathbb{Z}$ with period n, while a *translation element* $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ acts by $i \mapsto i + a_i n$, again extended periodically with period n. As in [11], a cocharacter $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $\mathrm{GL}(n)$ corresponds to the translation element of $\widetilde{S}_n$ is given by $(-\lambda_1, \ldots, -\lambda_n)$. Note that we may realize both $\mathcal{F}\ell(n)$ and $\mathrm{Gr}(k,n)$ as quotients of $\mathrm{GL}(n)$ rather than $\mathrm{SL}(n)$.

Definition II.15. For $k \in \mathbb{Z}$, an affine permutation of order n has type (k, n) if

$$\frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k. \tag{2.19}$$

We denote the set of affine permutations of type (k, n) by $\widetilde{S}_n^k$.

Affine permutations of type $(0, n)$ form an infinite Coxeter group, the *affine symmetric group*. This is the *affine Weyl group* of type A_{n-1}. It has simple generators $\widetilde{s}_1, \ldots, \widetilde{s}_n$, where $\widetilde{s}_i$ is the affine permutation which interchanges $i + rn$ and $i + 1 + rn$ for each $r \in \mathbb{Z}$.

The Bruhat order on $\widetilde{S}_n^0$ extends to a Bruhat order on all of $\widetilde{S}_n$. Each element of $\widetilde{S}_n$ may be written as a product $w\tau$, where τ preserves all simple roots of the affine Weyl group $\widetilde{S}_n^0$, and $w \in \widetilde{S}_n^0$. We say $w'\tau' \leq w\tau$ if $w' \leq w$ in Bruhat order, and $\tau' = \tau$. In our case, the condition $\tau = \tau'$ is equivalent to saying that $w'\tau'$ and $w\tau$ are both of type (k, n) for some $k \in \mathbb{Z}$. For $f = w\tau$, $f \in \widetilde{S}_n^k$ means that τ is the function $x \mapsto x + k$.

Definition II.16. An affine permutation in $\widetilde{S}_n$ is **bounded** if it satisfies the condition

$$i \leq f(i) \leq i+n \text{ for all } i \in \mathbb{Z}. \tag{2.20}$$

For $1 \leq k \leq n$, we write $\mathrm{Bound}(k,n)$ for the set of all bounded affine permutations of type (k,n).

The set $\mathrm{Bound}(k,n)$ inherits the Bruhat order from $\widetilde{S}_n^k$. In fact, $\mathrm{Bound}(k,n)$ is a lower order ideal in $\widetilde{S}_n^k$ [14, Lemma 3.6]. Similarly, the length function ℓ on $\widetilde{S}_n^0$ induces a grading ℓ on $\mathrm{Bound}(k,n)$, defined by $\ell(w\tau) = \ell(w)$. An *inversion* of a bounded affine permutation f is a pair $i < j$ such that $f(i) > f(j)$. Two inversions (i,j) and (i',j') are equivalent if $i' = i + rn$ and $j' = j + rn$ for some $r \in \mathbb{Z}$. We call the resulting equivalence classes *type* $\widetilde{A}$ *inversions*. The length of a bounded affine permutation $f \in \mathrm{Bound}(k,n)$ is the number of type $\widetilde{A}$ inversions of f [14, Theorem 5.9].

For $J \in \binom{[n]}{k}$, we define the *translation element* $t_J \in \widetilde{S}_n$ by setting

$$t_J(i) = \begin{cases} i+n & i \in J \\ i & i \notin J \end{cases} \tag{2.21}$$

for $1 \leq i \leq n$, and extending periodically. Every $f \in \mathrm{Bound}(k,n)$ may be written in the form

$$f = \sigma t_\mu = t_\nu \sigma \tag{2.22}$$

for some $\sigma \in S_n$,, elements μ and ν of $\binom{[n]}{k}$, and t_μ, t_ν translation elements. Both factorizations are unique.

We now give an isomorphism between $\mathcal{Q}(k,n)$ and $\mathrm{Bound}(k,n)$. This isomorphism may be viewed as a special case of Theorem 2.2 from [11]. Let $\langle u, w\rangle_k \in \mathcal{Q}(k,n)$.

The function

$$f_{u,w} = ut_{[k]}w^{-1} \tag{2.23}$$

is a bounded affine permutation of type (k, n). If $[u', w']_k$ is any other representative of $\langle u, w\rangle_k$, then $f_{u',w'} = f_{u,w}$. Hence we have a well-defined map $\mathcal{Q}(k, n) \to \mathrm{Bound}(k, n)$, which is in fact an isomorphism of posets [14, Section 3.4]. Note that for the translation elements $t_{w([k])}$ and $t_{u([k])}$, we have

$$f_{u,w} = uw^{-1}t_{w([k])} = t_{u([k])}uw^{-1} \tag{2.24}$$

This follows easily from (2.23).

The poset of bounded affine permutations is anti-isomorphic to the poset of *decorated permutations*, which Postnikov introduced to index positroid varieties [27]. A decorated permutation of order n is a permutation in S_n with fixed points colored black or white. If f is a bounded affine permutation, and σ is the corresponding decorated permutation, then black fixed points of σ correspond to values $i \in [n]$ such that $f(i) = i$. White fixed points correspond to values i such that $f(i) = i + n$. A decorated permutation σ of order n has type (k, n) if we have

$$k = |\{i \in [n] \mid \sigma^{-1}(i) > i \text{ or } i \text{ is a white fixed point}\}|.$$

Postnikov represented decorated permutations visually using *chord diagrams* [27, Section 16]. Let σ be a decorated permutation. A chord diagram for σ is a circle with vertices labeled $1, 2, \ldots, n$ in clockwise order. We then draw arrows from vertex i to vertex $\sigma(i)$ for all $i \in [n]$. By convention, if i is a white fixed point of σ, we draw a clockwise loop at i; if i is a black fixed point, we draw a counter-clockwise loop.

We say a pair of arrows in a chord diagram represents a *crossing* if they are arranged in one of the configurations shown in Figure 2.2, for $i, j \in [n]$. We say a pair of chords represents an *alignment* if they are arranged in one of the configurations shown in Figure 2.3. Note that rotating any of the diagrams shown in Figure 2.3 gives a valid example of an alignment. The analogous statement holds for crossings.

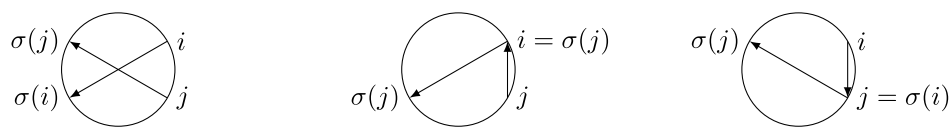


Figure 2.2: Crossings in a chord diagram.

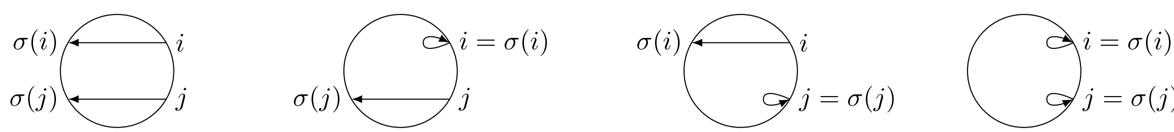


Figure 2.3: Alignments in a chord diagram. Note that loops must be oriented as shown to give a valid alignment.

2.8 Plabic graphs

A *plabic graph* is a planar graph embedded in a disk, with each vertex colored black or white. (Plabic is short for planar bicolored.) The boundary vertices are numbered $1, 2, \ldots, n$ in clockwise order, and all boundary vertices have degree one. We call the edges adjacent to boundary vertices *legs* of the graph. A leaf adjacent to a boundary vertex is called a *lollipop*. A black leaf adjacent to a white boundary vertex is a *black lollipop*, while a *white lollipop* is the opposite. We further assume that every vertex in a plabic graph is connected by some path to a boundary vertex.

Postnikov introduced plabic graphs in [27, Section 11.5], where he used them to construct parametrizations of positroid cells in the totally nonnegative Grassmannian. In this paper, we follow the conventions of [18], which are more restrictive than Postnikov's. In particular, we require our plabic graphs to be bipartite, with the black and white vertices forming the partite sets. An *almost perfect matching* on a plabic graph is a subset of its edges which uses each interior vertex exactly once;

boundary vertices may or may not be used. We consider only plabic graphs which admit an almost perfect matching.

We can write any plabic graph as a union of paths and cycles, as follows. Start with any edge $e = \{u, v\}$. Begin traversing this edge in either direction, say $u \to v$. Turn (maximally) left at every internal white vertex, and (maximally) right at every internal black vertex. The path ends when we either reach a boundary vertex, or find ourselves about to retrace the edge $u \to v$. Repeating this process gives a description of G as a union of directed paths and cycles, called *trips*. Each edge is used twice in this decomposition, once in each direction. Given a plabic graph G with n boundary vertices, we define the *trip permutation* $\sigma_G \in S_n$ of G by setting $\sigma_G(a) = b$ if the trip that starts at boundary vertex a ends at boundary vertex b.

There are a number of *local moves* and *reductions* defined for plabic graphs; again, we use the conventions of [18], which are adapted slightly from those of [27]. The moves are defined as follows. Both moves are reversible, and preserve the trip permutation.

(M1) The "spider," "square," or "urban renewal" move. We may transform the portion of a plabic graph shown at left in Figure 2.4 into the portion shown at right, and vice versa.

(M2) Degree-two vertex removal. If a vertex v has degree 2, we may contract the incident edges (u, v) and (v, u') to a single vertex. Note that if v is adjacent to a boundary vertex b, we cannot contract all the incident edges, since boundary vertices must have degree 1. Hence, we simply remove the vertex v, and reverse the color of b.

Reductions, in contrast, are not reversible, and may change the trip permutation. We have two reductions.

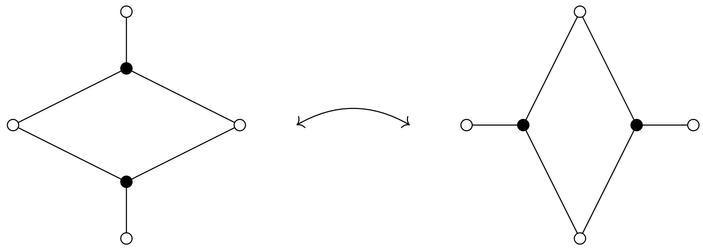


Figure 2.4: A square move

Figure 2.5: A reduction.

(R1) Multiple edges with the same endpoints may be replaced by a single edge. See Figure 2.5.

(R2) Leaf removal. If v is a leaf, and (u, v) the unique edge adjacent to v, we may remove both (u, v) and all edges adjacent to u. However, if u is adjacent to a boundary vertex b, the edge (b, u) is replaced by a boundary edge (b, w), where w has the same color as v, and the color of b flips.

A plabic graph G is *reduced* if it cannot be transformed using the local moves M1-M2 into a plabic graph G' on which we a reduction. If G is a reduced graph, each fixed point of σ_G corresponds to a boundary leaf [27, Section 13]. Suppose G has n boundary vertices, and suppose we have

(2.25)

$$k = |\{a \in [n] \mid \sigma_G(a) < a \text{ or } \sigma_G(a) = a \text{ and } G \text{ has a white boundary leaf at } a\}|$$

Then we can construct a bounded affine permutation $f_G \in \mathrm{Bound}(k, n)$ corresponding to G by setting

$$f_G(a) = \begin{cases} \sigma_G(a) & \sigma_G(a) > a \text{ or } G \text{ has a black boundary leaf at } a \\ \sigma_G(a) + n & \sigma_G(a) < a \text{ or } G \text{ has a white boundary leaf at } a \end{cases} \tag{2.26}$$

Thus we have have a correspondence between plabic graphs and positroid varieties:

to a reduced plabic graph G, we associate the positroid Π_G^A corresponding to f_G. This correspondence is not a bijection. Rather, we have a family of reduced plabic graphs for each positroid variety. Two reduced plabic graphs G and G' have the same bounded affine permutation (and hence, the same associated positroid variety) if and only if we can transform G into G' using a sequence of local moves M1 and M2 [27, Theorem 13.4].

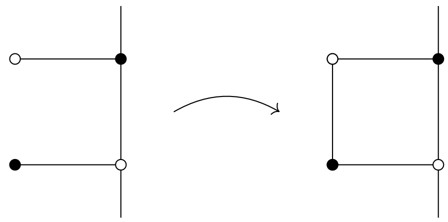
We now describe a way to build plabic graphs inductively by adding new edges, called *bridges*, to existing graphs. The resulting graphs are called *bridge graphs*. This construction appears in [1] and also, in slightly less general form, in [18].

We begin with a plabic graph G. To add a bridge, we choose a pair of boundary vertices $a < b$, such that every $c \in [a+1, b-1]$ is a lollipop. Our new edge will have one vertex on the leg at a, and one on the leg at b. If a (respectively b) is a lollipop, then the leaf at a must be white (respectively black), and we use that boundary leaf as one endpoint of the bridge. If a (respectively b) is not a lollipop, we instead insert a white (black) vertex in the middle of the leg at a (respectively, b). We call the new edge an (a, b)-*bridge*. After adding the new edge, our graph may no longer be bipartite. In this case, we insert additional vertices of degree two or change the color of boundary vertices as needed to obtain a bipartite graph G'. (See Figure 2.6.)

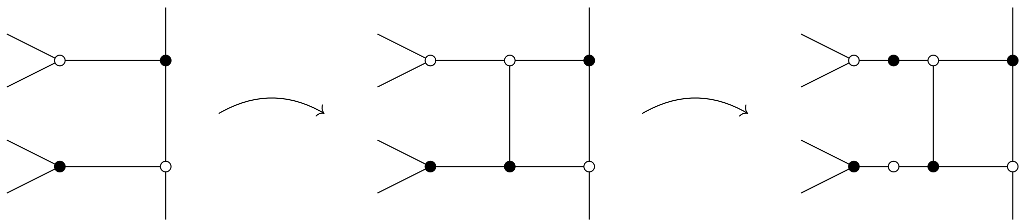
Proposition II.17. *[18] Suppose G is reduced. Choose $1 \leq a < b \leq n$ such that $f_G(a) > f_G(b)$, and each $c \in [a+1, b-1]$ is a lollipop. Let G' be the graph obtained by adding an (a, b)-bridge to G. Then G' is reduced and*

$$f_{G'} = f \circ (a, b) \in \mathrm{Bound}(k, n).$$

Moreover, $f_{G'} \lessdot f_G$ in the Bruhat order on $\mathrm{Bound}(k, n)$, and so Π_G is a codimension-one subvariety of $\Pi_{G'}$.



(a) Adding a bridge between lollipops.



(b) Adding a bridge whose endpoints are not lollipops. Note that after adding the bridge, we add additional vertices of degree 2 to create a bipartite graph.

Figure 2.6: Adding bridges to a plabic graph.

The zero-dimensional positroid varieties correspond to the points in $\mathrm{Gr}(k,n)$ which have a single non-zero Plücker coordinate μ. There is a unique reduced plabic graph for each $\mu \in \binom{[n]}{k}$, which consists of n lollipops. The k lollipops corresponding to elements of μ are white; the rest are black. We call a plabic graph consisting only of lollipops a *lollipop graph.*

A *bridge graph* is a plabic graph which is constructed from a lollipop graph by successively adding bridges

$$(a_1, b_1), \ldots, (a_d, b_d), \tag{2.27}$$

where at each step, (a_i, b_i) satisfies the hypothesis of Proposition II.17 for the graph obtained by adding the first $i-1$ bridges. Hence a bridge graph is always reduced. Let $u \leq_k w$. Then by (2.24) we have

$$f_{u,w} = t_{u([k])} u w^{-1} \tag{2.28}$$

where $t_{u([k])}$ is the translation element corresponding to $u([k])$. To construct a bridge graph for $\Pi_{\langle u,w \rangle_k}$ we begin with the lollipop graph corresponding to $u([k])$, and successively add bridges to obtain a graph with bounded affine permutation $f_{u,w}$. It is perhaps not obvious that every positroid variety has a bridge graph. However, this follows from earlier work on the subject. In particular, Postnikov's Ⅎ-diagrams correspond to a particular choice of bridge graph for each bounded affine permutation.

2.9 Parametrizations from plabic graphs

Let G be a reduced plabic network whose bounded affine permutation has type (k, n). Suppose G has e edges, and assign weights $t_1, \ldots, t_e$ to the edges of G. Postnikov defined a surjective map from the space of positive real edge weights of G to the positroid cell $\left(\mathring{\Pi}_G^A\right)_{\geq 0}$ in $\mathrm{Gr}_{\geq 0}(k, n)$, called the *boundary measurement map* [27, Section 11.5]. Postnikov, Speyer and Williams re-cast this construction in terms of almost perfect matchings [28, Section 4-5], an approach Lam developed further in [18]. Muller and Speyer showed that we can apply the same map to the space of nonzero complex edge weights, and obtain a map to the positroid variety $\mathring{\Pi}_G^A$ in $\mathrm{Gr}(k, n)$ [25]. We use the definition of the boundary measurement map found in [18].

For P an almost perfect matching on a plabic graph G with e edges, let

(2.29)

$$\partial(P) = \{\text{black boundary vertices used in } P\} \cup \{\text{white boundary vertices } \textit{not} \text{ used in } P\}$$

Then $|\partial(P)| = k$, and we define the *boundary measurement map*

$$\partial_G : \mathbb{C}^e \to \mathbb{P}^{\binom{n}{k}-1} \tag{2.30}$$

to be the map which sends $(t_1, \ldots, t_e)$ to the point with homogeneous coordinates

$$\Delta_J = \sum_{\partial(P)=J} t^P \tag{2.31}$$

where the sum is over all matchings P of G, and t^P is the product of the weights of all edges used in P [18].

For positive real edge weights, the boundary measurement map ∂_G is surjective onto the positroid with bounded affine permutation f_G. If instead we let the edge weights range over $\mathbb{C}^\times$, we obtain a well-defined map to the open positive variety $\mathring{\Pi}_G^A$ in $\mathrm{Gr}(k,n)$. The image is an open dense subset of Π_G^A [25].

The boundary measurement map is typically not injective, due to the action of the *gauge group*. Let V be the set of all internal vertices of G. The gauge group $\mathbb{G}^V$ is a copy of $(\mathbb{C}^\times)^{|V|}$ with coordinates indexed by V. Let $\mathbb{G}^E$ denote the space of complex edge weights of G. Then $\mathbb{G}^V$ acts on $\mathbb{G}^E$ by *gauge transformations*. For $\mu \in \mathbb{G}^V$ and $v \in V$, let μ_v be the coordinate of μ corresponding to v. Then the action of μ multiplies the weights of each edge incident to v by μ_v. The weight of an edge (v,w) is thus multiplied by the product $\mu_v\mu_w$. It is easy to see that the action of $\mathbb{G}^V$ preserves the boundary measurement map. Taking the quotient by this action, we obtain a map

$$\mathbb{D}_G : \mathbb{G}^E/\mathbb{G}^V \to \mathring{\Pi}_G^A$$

which carries the image of each weighting $(t_1,\ldots,t_e)$ in $\mathbb{G}^E/\mathbb{G}^V$ to $\partial_G(t_1,\ldots,t_e)$. This map is not only injective, but birational onto its image [25].

Analogous statements hold for positive real edge weights, where the action is by positive real gauge transformations; in this setting, taking the quotient by the gauge group gives an isomorphism onto the positroid cell corresponding to G [27]. We will abuse terminology slightly, and refer to both ∂_G and $\mathbb{D}_G$ as the *boundary measurement map*; it should be clear from context which map is meant.

Taking the quotient by the gauge group is equivalent to specializing an appropriate set of edge weights to 1, and letting the remaining edge weights range over either $\mathbb{R}^+$ or $\mathbb{C}^\times$. Indeed, suppose $F \subset E$ is a set which meets the following conditions.

1. F is a spanning forest of G.

2. Each connected component of F has exactly one vertex on the boundary.

We may construct such a set inductively for any G. It is not hard to show that each point in $\mathbb{G}^E/\mathbb{G}^V$ can be represented uniquely by a weighting of G with all edges in F gauge-fixed to 1. Let $\mathbb{G}^F$ denote the space of all such weightings. Then the natural map $\mathbb{G}^E/\mathbb{G}^V \to \mathbb{G}^F$ is an isomorphism.

If G is a bridge graph, there is a natural specialization of edge weights, and we have a simple procedure for constructing the desired parametrization. Let $\Pi_G^A = \Pi_{\langle u,w \rangle_k}^A$ and let $d = \dim(\Pi_G^A)$. Assign a weight $t_1, \dots, t_d$ to each bridge, in the order the bridges were added, and set all other edge weights to 1. Begin with the $k \times n$ matrix in which the columns indexed by $u([k])$ form a copy of the identity, while the remaining columns contain only 0's. Say the r^{th} bridge is from a_r to b_r with $a_r < b_r$. When we add the r^{th} bridge to the graph, we multiply our matrix on the right by $x_{(a_r,b_r)}^A(\pm t_r)$, the elementary matrix with nonzero entry $\pm t_r$ in row a_r and column b_r. The sign is negative if $|u([k]) \cap [a_r+1, b_r-1]|$ is odd, and positive if $|u([k]) \cap [a_r+1, b_r-1]|$ is even.

Suppose G and G' are related to each other by one of the local moves defined in 2.8. Then the maps ∂_G and $\partial_{G'}$ are related by a birational change of variables. For degree-two vertex removal, we may simply assume that both edges adjacent to the degree-two vertex are fixed to 1, so the change of variables is trivial. For the square move, assume all unlabeled edges in Figure 2.7 are gauge-fixed to 1. Then we have the transformation shown in Figure 2.7, where

$$a' = \frac{a}{ac+bd}, \ b' = \frac{b}{ac+bd}, \ c' = \frac{c}{ac+bd}, \ d' = \frac{d}{ac+bd}.$$

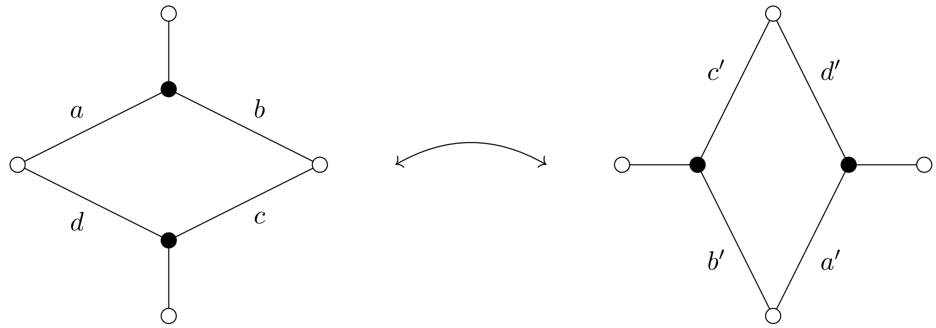


Figure 2.7: A square move with accompanying change of coordinates.

2.10 Ⅎ-diagrams

A Ⅎ-*diagram* is a Young diagram filled with 0's and +'s according to certain rules. (The symbol Ⅎ is pronounced "le," which is "ell" backwards.) Postnikov showed that Ⅎ-diagrams are in bijection with PDS's of Grassmannian permutations, and constructed a plabic graph corresponding to each Ⅎ-diagram [27]. Hence Ⅎ-diagrams provide the first example of the relationship between planar graphs and PDS's. Lam and Williams defined analogs of Ⅎ-diagrams for all *cominiscule Grassmannians* [19]. Since their construction depends only on Weyl group data, the *type B* Ⅎ-*diagrams* of Lam and Williams index projected Richardson varieties in $\Lambda(2n)$. Although Ⅎ-diagrams will not play a major role in our results, they provide important context and motivation for much of this thesis. We review them here for completeness.

To construct a Ⅎ-diagram, begin with a $k \times (n-k)$ rectangle, subdivided into unit boxes. Order the boxes as shown in Figure 2.8. The lower order ideals with respect to this ordering are precisely the *Young diagrams* that fit inside a $k \times (n-k)$ rectangle. (Note that our Young diagrams are drawn using French notation, with minimal elements at the bottom left corner; Postnikov uses the English notation, with minimal elements at the top left.)

We label each box with a simple transpositions S_n as shown in the figure; boxes on the same diagonal are labeled with the same transposition. With these conventions, Young diagrams that fit inside a $k \times (n-k)$ rectangle are in bijection with Grassmannian permutations of type (k, n). The bijection is as follows. The boxes in a Young diagram give a sequence of simple transpositions in S_n. We read off the boxes in increasing order, and list the corresponding transpositions from right to left. This process yields a reduced word for some Grassmannian permutation $w_Y \in S_n$, and the map $Y \to w_Y$ is a bijection. See Figure 2.8 for an example.

$$
\begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}
\quad
\begin{array}{|c|c|c|}\hline s_1 & s_2 & s_3 \\ \hline s_2 & s_3 & s_4 \\ \hline \end{array}
\qquad\qquad
\begin{array}{|c|c|c}\cline{1-2} 0 & 0 & \\ \hline 0 & 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array}
\quad
\begin{array}{|c|c|c}\cline{1-2} 0 & 0 & \\ \hline + & 0 & \multicolumn{1}{c|}{+} \\ \hline \end{array}
$$

Figure 2.8: Constructing ⅃-diagrams for the case $k = 2$, $n = 5$. The diagrams at right correspond to expressions $s_2 s_1 s_4 s_3 s_2$ and $s_2 s_1 1 s_3 1$, respectively

Let Y be a Young diagram, and let $\mathbf{w}_Y$ be the corresponding reduced word for w_Y. We represent subexpressions of $\mathbf{w}_Y$ using $\oplus$-*diagrams* of shape Y, that is, fillings of Y with 0's and +'s. To construct the subexpression corresponding to an $\oplus$-diagram of shape Y, we replace each factor of $\mathbf{w}_Y$ corresponding to a box containing + by the identity permutation 1. An $\oplus$-diagram is a ⅃-*diagram* if it corresponds to a PDS.

Recall that positroid varieties are indexed by pairs $u \leq w$ with w Grassmannian. Further, if $u \leq w$, and $\mathbf{w}$ is a reduced word of w, there is a unique PDS for u in $\mathbf{w}$. It follows that ⅃-diagrams are in bijection with positroid varieties. Moreover, ⅃-diagrams are characterized by a simple pattern-avoidance condition.

Theorem II.18 ([27])**.** *Let Y be a Young diagram. An $\oplus$-diagram of shape Y is a ⅃-diagram if no 0 has both a + in the same row to its left, and a + in the same column below it.*

Note that with Postnikov's conventions, the three boxes in a forbidden pattern forms a backwards L shape, which is the source of the name ⅃-diagram.

The type B construction is analogous. The $k \times (n-k)$ rectangle is replaced with a staircase shape of size n, with boxes ordered and labeled as shown in Figure 2.9. Lower order ideals in the staircase shape give *type B Young diagrams.* They correspond to reduced words for elements of S_n^C which are minimal-length left coset representatives for $S_n^C/(S_n^C)_n$. Let Y be a type B Young diagram. As in the type A case, the $\oplus$-diagrams of shape Y correspond to subexpressions of some reduced word; such a diagram is a *type B ⅃-diagram* if it represents a PDS. Once again, we have a characterization of ⅃-diagrams in terms of pattern avoidance.

4	5	6
2	3	
1		

s_1	s_2	s_3
s_2	s_3	
s_3		

Figure 2.9: We construct type B ⅃-diagrams inside a staircase shape. The figure shows $n = 3$.

Theorem II.19 ([19])**.** *Let Y be a type-B Young diagram. An $\oplus$-diagram of type B is a* type B ⅃-diagram *if the following conditions hold:*

1. *No 0 has both a $+$ in the same row to its left, and a $+$ in the same column below it.*

2. *No 0 which lies in a diagonal box has a $+$ in the same row to its left.*

CHAPTER III

Bridge graphs and Deodhar parametrizations

In this chapter we consider two ways of parametrizing positroid varieties–via bridge graphs, and via projected Deodhar parametrizations–and show they are essentially the same. The bulk of the chapter is devoted to the proof of Theorem I.3, restated below. This result was first conjectured by Thomas Lam [17].

Theorem. Let Π be a positroid variety in $\mathrm{Gr}(k, n)$. For each Deodhar parametrization of Π, there is a bridge graph which yields the same parametrization. Conversely, any bridge graph parametrization of Π agrees with some Deodhar parametrization.

We divide the proof into three parts. In Section 3.2.1 we review some conventions for wiring diagrams and introduce *bridge diagrams*, a class of modified wiring diagram which are essential to the proof. We then give an explicit way to rewrite Deodhar parametrizations so that they more closely resemble parametrizations from bridge graphs. In Section 3.2.2 we prove that every Deodhar parametrization arises from a bridge graph, and in Section 3.2.3 we prove the converse.

The correspondence between Deodhar parametrizations and bridge graphs is not a bijection; rather, we have a family of Deodhar parametrizations for each bridge graph. In Section 3.3 we define an equivalence relation among Deodhar parametrizations such that each equivalence class corresponds to a unique bridge graph.

3.1 Preliminaries

3.1.1 Wiring diagrams

Wiring diagrams allow us to represents words in S_n visually. Fix w in S_n, and let $\mathbf{w}$ be a word of w. The *wiring diagram* for $\mathbf{w}$ has n wires which run from left to right, with some crossings between adjacent wires. We number with the numbers 1 to n the right and left endpoints of the wires respectively, so that the numbers increase from top to bottom. A crossing between two wires in the diagram for $\mathbf{w}$ represents a simple transposition s_i of $\mathbf{w}$, where $i-1$ is the number of wires in the diagram which pass directly above the crossing. However, the crossings in the diagram for $\mathbf{w}$ appear in the opposite order as the simple generators in the word $\mathbf{w}$; the leftmost generator in $\mathbf{w}$ corresponds to the rightmost crossing in the diagram. With these conventions, if $w(s) = t$, the wire with left endpoint s has right endpoint t. A wiring diagram is *reduced* if no two wires cross each other more than once; this occurs if and only if the word $\mathbf{w}$ is reduced. For an example of a reduced wiring diagram, see Figure 3.2a.

3.1.2 Bridge diagrams

We now introduce *bridge diagrams*, which will be an essential tool in our proof of Theorem I.3. Note that bridge diagrams are distinct from bridge graphs. However, the proof of Theorem I.3 shows that they are intimately related. We give an algorithm for constructing a bridge graph corresponding to each bridge diagram in Section 3.2.2.

Let $\mathbf{y}$ be a word in S_n, not necessarily reduced, and let $\mathbf{x}$ be a subword of $\mathbf{y}$. (We reserve the term *subexpression* and the symbol $\preceq$ for subwords of reduced words.) To draw the *bridge diagram* for $\mathbf{x}$, start with a wiring diagram for $\mathbf{y}$, and replace each crossing which is *not* in $\mathbf{x}$ by the "dashed cross" shape shown in Figure 3.1. The result is a wiring diagram for the word $\mathbf{x}$, which we call the *underlying diagram* for $\mathbf{x}$, with some dashed crosses inserted between adjacent wires. We call these dashed crosses *bridges*. For an example, see Figure 3.2b. In the case where $\mathbf{y}$ is a reduced word, and $\mathbf{x}$ a subexpression of $\mathbf{y}$, we will sometimes write $\mathbf{x} \preceq \mathbf{y}$ to denote the bridge diagram corresponding to this subexpression.

Remark III.1. We often construct bridge diagrams by starting with the underlying diagram for $\mathbf{x}$ and inserting dashed crosses, or *bridges*, between the wires. This choice is most natural for various inductive arguments in the proof of Theorem I.3, where we fix the subword $\mathbf{u}$ and induce on the length of $\mathbf{w}$ by successively adding bridges to the diagram.

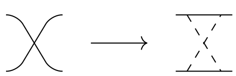
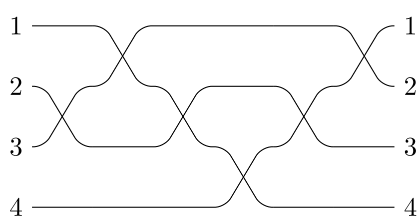
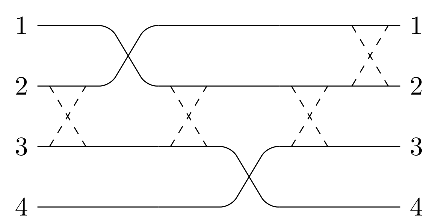


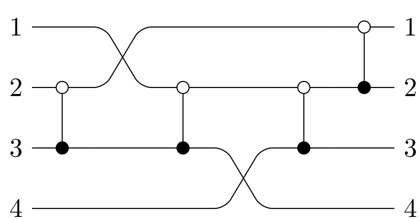
Figure 3.1: Replacing a crossing with a bridge.



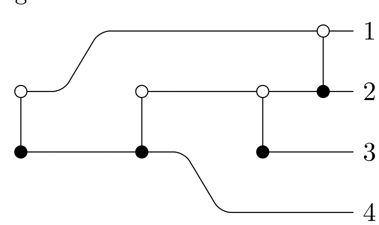
(a) A wiring diagram for the word $\mathbf{w}$.



(b) The solid lines give the wiring diagram for $\mathbf{u}$. The dashed crosses represent bridges.



(c) To construct the bridge graph corresponding to $\mathbf{u} \preceq \mathbf{w}$, we first replace each bridge with a dimer, as shown above.



(d) Next, we delete the tail of each wire up the first dimer on that wire. Adding degree-2 vertices as needed yields the desired plabic graph.

Figure 3.2: Constructing a bridge graph from a bridge diagram. In this example, $\mathbf{w} = s_1s_2s_3s_2s_1s_2$, and $\mathbf{u} = s_3s_1$.

Remark III.2. Let $\mathbf{w}$ be a reduced word, let $\mathbf{u} \preceq \mathbf{w}$, and suppose $\mathbf{u}$. Consider the bridge diagram corresponding to $\mathbf{u} \preceq \mathbf{w}$. The statement that $\mathbf{u}$ is a PDS of $\mathbf{w}$ means precisely that the underlying diagram for $\mathbf{u}$ is reduced, and that each bridge is inserted between two wires in the underlying diagram for $\mathbf{u}$ which never cross again to the right of the bridge.

The following is immediate.

Proposition III.3. *Let* $\mathbf{u} \preceq \mathbf{w}$ *and consider the corresponding bridge diagram. Suppose we insert only the leftmost* r *bridges into the diagram for* $\mathbf{u}$*. Then replacing these bridges with crossings gives a reduced wiring diagram* $\mathbf{v}$ *for some* $v \in S_n$*, and the underlying diagram* $\mathbf{u}$ *represents the PDS for* u *in* $\mathbf{v}$*.*

We introduce some conventions for bridge diagrams. A bridge diagram is *reduced* if it represents a reduced subword $\mathbf{u}$ of a reduced word $\mathbf{w}$. Wires in the underlying diagram for $\mathbf{u}$ are labeled by their right endpoints, so "wire a" means the wire with right endpoint a. Suppose wires a and b cross in the diagram for $\mathbf{u}$, and suppose wire a lies above b to the left of the crossing, and below b to the right. We say wire a crosses wire b "from above," and write $(a \downarrow b)$. Similarly, we say b crosses wire a "from below," and write $(b \uparrow a)$. We say wire a is *isolated* if there are no bridges touching wire a. We refer to a bridge between wires a and b as an (a, b)-bridge. If $a < b$, we call a the *upper wire* of the (a, b)-bridge, and call b the *lower wire*. Note that we may also refer to bridges (a, b) where $a > b$, and a is the lower wire of the bridge.

We use the symbol $\to$ to indicate that one bridge or crossing occurs before (that is, to the left of) another. So for example, $(a, b) \to (c, d)$ means there is a (c, d) bridge to the right of an (a, b) bridge. Similarly, $(a, b) \to (c \downarrow b)$ means that wire c crosses wire b from above after the (a, b)-bridge. For example, we can describe the diagram in Figure 3.2b symbolically by writing

$$(1,4) \to (2 \downarrow 1) \to (2,3) \to (3 \uparrow 4) \to (2,3) \to (1,2)$$

3.2 The Main Result

3.2.1 Rewriting Deodhar parametrizations

The matrices $\dot{s}_i$ and $x_{(a,b)}(\pm t)$ satisfy the following relation, which may be checked directly.

Lemma III.4. *Suppose $s_i(a) < s_i(b)$. Then*

$$\dot{s}_i x_{(a,b)}(t) \dot{s}_i^{-1} = \begin{cases} x_{(s_i(a),s_i(b))}(-t) & i \in \{a-1, b-1\} \\ x_{(s_i(a),s_i(b))}(t) & \textit{otherwise} \end{cases} \tag{3.1}$$

Let $\mathbf{u} \preceq \mathbf{w}$ be a PDS, where $\ell(w) = m$. For $1 \le j \le m$, let

$$\dot{u}_j = \begin{cases} \dot{s_{i_j}} & s_{i_j} \in \mathbf{u} \\ 1 & \text{otherwise} \end{cases} \tag{3.2}$$

Let $d = \ell(w) - \ell(u)$, and let $j_1, \ldots, j_d$ be the indices corresponding to simple transpositions which are not in $\mathbf{u}$. For $1 \le r \le d$, define

$$\bar{r} = d + 1 - r \tag{3.3}$$

and set

$$\beta_{\bar{r}} = (\dot{u}_1 \cdots \dot{u}_{j_r - 1})(x_{i_{j_r}}(t_{j_r}))(\dot{u}_{j_r-1}^{-1} \cdots \dot{u}_1^{-1}). \tag{3.4}$$

Then we can rewrite each $G \in G_{\mathbf{u},\mathbf{w}}$ in the form

$$G = (\dot{u}_m^{-1} \dot{u}_2^{-1} \cdots \dot{u}_1^{-1})(\beta_1 \beta_2 \cdots \beta_d) \tag{3.5}$$

Lemma III.5. *For β_r as above, define:*

$$a = u_{(j_r-1)}(i_{j_r}) \tag{3.6}$$

$$b = u_{(j_r-1)}(i_{j_r}+1) \tag{3.7}$$

$$\theta = |u([k]) \cap [a+1, b-1]| \tag{3.8}$$

Then we have

$$\beta_{\bar{r}} = x_{(a,b)}((-1)^{\theta} t_{j_r}). \tag{3.9}$$

Proof. Since $\mathbf{u}$ is a PDS of $\mathbf{w}$, i_{j_r} is *not* a descent of $u_{(j_r-1)}$. From standard Coxeter arguments, it follows that for each $1 \le q \le m$, we have

$$u_q(u_{q+1} \cdots u_{j_r-1}(i_{j_r})) < u_q(u_{q+1} \cdots u_{j_r-1}(i_{j_r}+1)) \tag{3.10}$$

Hence we can apply (3.1) repeatedly, to obtain $\beta_{\bar{r}} = x_{(a,b)}(\pm t_{j_r})$.

Next, we compute the sign of the parameter $\pm t_{j_r}$. In the language of bridge diagrams, (3.1) implies that we multiply the parameter t_{j_r} by a factor of -1 for each wire c in the diagram for $\mathbf{u}$ such that $(c \downarrow a)$ after the (a,b) bridge; and one for each wire such c' such that $(c' \downarrow b)$ after the (a,b) bridge. After canceling, this yields a factor of -1 for each wire which crosses wire a from above after the (a,b) bridge, and whose right endpoint is between a and b. By Lemma III.10 below, these are precisely the wires c with $c \in \{u([k]) \cap [a+1, b-1]\}$. The claim follows. $\square$

For notational convenience, we renumber our parameters t_{i_j} and define a_r, b_r such that

$$\beta_r = x_{(a_r,b_r)}(\pm t_r) \tag{3.11}$$

where the sign is determined as above. It follows from the proposition that if the simple transposition $s_{i_{j_r}}$ corresponds to an (a,b) bridge in the diagram for $\mathbf{u} \preceq \mathbf{w}$, then $\beta_{\bar{r}} = x_{(a,b)}(\pm t_{\bar{r}})$. Note that the β_r appear in (3.5) in the same order as the corresponding bridges (a_r, b_r) in the diagram for $\mathbf{u} \preceq \mathbf{w}$ (and hence in the opposite order as the factors $s_{i_{j_r}}$ in the word for $\mathbf{w}$).

Example continued

As before, let $n = 4$, $k = 2$. Let $\mathbf{w} = s_1 s_2 s_3 s_2 s_1 s_2 \in S_4$, and let $u = 2143$. The positive distinguished subexpression $\mathbf{u}$ for u in $\mathbf{w}$ comprises the s_3 in position 3 from the left, and the s_1 in position 5, so we have a parametrization of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ by

$$G_{\mathbf{u},\mathbf{w}} = x_2(t_1)\dot{s_1}^{-1}x_2(t_2)\dot{s_3}^{-1}x_2(t_3)x_1(t_4). \tag{3.12}$$

Rewriting this in the form (3.5) and projecting to $\mathrm{Gr}(2,4)$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \dot{s_1}^{-1}\dot{s_3}^{-1}x_{(1,4)}(-t_1)x_{(2,4)}(t_2)x_2(t_3)x_1(t_4) = \begin{bmatrix} 0 & 1 & t_3 & t_2 \\ -1 & -t_4 & 0 & t_1 \end{bmatrix} \tag{3.13}$$

Re-ordering the rows and multiplying the first row by -1, gives

$$\begin{bmatrix} 1 & t_4 & 0 & -t_1 \\ 0 & 1 & t_3 & t_2 \end{bmatrix} \tag{3.14}$$

which is precisely the matrix we obtained from the bridge graph shown in Figure **??**. So these two parametrizations yield the same point in the Grassmannian for each choice of parameters (t_1, t_2, t_3, t_4).

3.2.2 From PDS's to bridge graphs

We next outline a method which produces a bridge graph corresponding to a projected Deodhar parametrization. While the method itself is straightforward, proving that it yields a correct bridge graph requires considerable work. The principle difficulty lies in ensuring that the network we obtain is planar.

We retain all notation from the previous section. In particular, let (a_r, b_r) be defined as in (3.11). Then for $1 \le r \le d$, we have

$$(a_{\bar{r}}, b_{\bar{r}}) = u_{j_r-1}(s_{i_{j_r}})u_{j_r-1}^{-1}. \tag{3.15}$$

It follows that

$$(a_1, b_1)(a_2, b_2)\cdots(a_d, b_d) = uw^{-1} \tag{3.16}$$

and hence

$$t_{u([k])}(a_1, b_1)(a_2, b_2)\cdots(a_d, b_d) = f_{u,w} \tag{3.17}$$

To construct the desired bridge graph, we successively add bridges

$$(a_1, b_1), \cdots, (a_d, b_d)$$

to the lollipop graph with white lollipops indexed by $u([k])$. This is possible, so long as the hypotheses of Proposition II.17 are satisfied at each step. For the moment, let us assume this is the case; that is, the sequence of transpositions in (3.16) corresponds to a bridge graph G. It follows from (3.17) that G has bounded affine permutation $f_{u,w}$ as desired.

We claim that the parametrization arising from G is precisely the projected Deodhar parametrization corresponding to $\mathbf{u} \preceq \mathbf{w}$. For this, note that we construct both parametrizations by taking a matrix which has a single non-zero Plücker coordinate $\Delta_{u([k])}$ and multiplying on the right by a sequence of factors $\beta_r = x_{(a_r,b_r)}(\pm t_r)$. In each case, the sign of the parameter is negative if $|u([k]) \cap [a_r + 1, b_r - 1]|$ is odd, and positive otherwise, so the parametrizations are the same.

Remark III.6. Our task is to prove that the sequence of transpositions in (3.16) will always correspond to a bridge graph. There are a number of things to check. In particular, suppose that adding the bridges

$$(a_1, b_1), \ldots, (a_{r-1}, b_{r-1}) \tag{3.18}$$

yields a (reduced) bridge graph corresponding to some bounded affine permutation f_{r-1}. To prove that we can add the bridge (a_r, b_r), we must show that the following conditions are met:

1. $f_{r-1}(a_r) > f_{r-1}(b_r)$,
2. If a_r (respectively b_r) is a fixed point of f_{r-1}, then the boundary leaf at a_r is white (respectively, black),
3. Each c with $a_r < c < b_r$ is a lollipop.

Anti-Grassmannian permutations, k-order, and wiring diagrams

We now reduce to the case where u is *anti-Grassmannian.* Let $\mathbf{u} \preceq \mathbf{w}$ be a PDS where $u \leq_k w$. By Proposition II.7, there exists some $z \in S_k \times S_{n-k}$ such that $uz \leq wz$ with uz anti-Grassmannian, and both factorizations are length-additive. Choose a reduced word $\mathbf{w}$ for w and a reduced word $\mathbf{z}$ for z, and let $\mathbf{u}$ be the unique PDS for u in $\mathbf{w}$. Then concatenating $\mathbf{u}$ and $\mathbf{z}$ gives the unique PDS $\mathbf{uz}$ for uz in the reduced word $\mathbf{wz}$. It is clear, however, that the pairs $\mathbf{u} \preceq \mathbf{w}$ and $\mathbf{uz} \preceq \mathbf{wz}$ project to the same Deodhar parametrization of $\Pi_{\langle u,w\rangle_k}$. Hence in what follows, we may always assume that u is an anti-Grassmannian permutation.

Definition III.7. A bridge diagram is *valid* if it corresponds to a pair $\mathbf{u} \preceq \mathbf{w}$ with u anti-Grassmannian, and $\mathbf{u}$ a PDS of $\mathbf{w}$.

Recall that if $u \leq w$ with u anti-Grassmannian, then $u \leq_k w$. Our main result holds only for PDS's $\mathbf{u} \preceq \mathbf{w}$ with $u \leq_k w$. This condition is always satisfied for subexpressions corresponding to valid bridge diagrams.

Planarity

Consider a valid bridge diagram $\mathbf{u} \preceq \mathbf{w}$, as defined in Definition III.7. We describe a way to construct the corresponding bridge graph. For an example, see Figure 3.2. We view the right endpoints of the wires as boundary vertices of a bicolored graph embedded in a disk; the wires themselves as paths with one endpoint on the boundary; and the new crossings as white-black bridges between these paths. Ignore the tail of each wire from the left endpoint up to the first bridge on that wire. Add a white boundary leaf at the right endpoint of each isolated wire with left endpoint $\leq k$, and a black boundary leaf at the right endpoint of each isolated wire with left endpoint $> k$.

We say that a bridge diagram is *planar* if the above process yields a planar embedding of a bridge graph. If this occurs, the graph must necessarily be the bridge graph for $\Pi_{\langle u,w \rangle_k}$ described previously, with its sequence of bridges given by (3.16). Hence, our goal is to prove that a valid bridge diagram will always be planar.

Lemma III.8. *Let a be a non-isolated wire in a valid bridge diagram $\mathbf{u} \preceq \mathbf{w}$, and let t be the left endpoint of wire a. If the first bridge on wire a is a bridge (a, b) with $b > a$, then $t \leq k$. If instead $b < a$, then $t > k$.*

Proof. We argue the first case; the proof of the second case is analogous. Note that wire b must lie below wire a to the right of the (a, b) bridge. Suppose the first bridge on wire a is (a_r, b_r). Let w' be the permutation corresponding to the wiring diagram obtained by adding bridges

$$(a_1, b_1), \ldots, (a_r, b_r)$$

to the diagram for $\mathbf{u}$ and replacing each bridge with a crossing. Then

$$w'(t) = b > a = u(t) \tag{3.19}$$

Since $w' \geq_k u$, Theorem II.5 implies that $t \leq k$. $\square$

Remark III.9. As a corollary, note that the second condition in Remark III.6 will always be satisfied, as long as the sequence of bridges

$$(a_1, b_1), \ldots, (a_d, b_d) \tag{3.20}$$

is planar. This follows from the lemma, since a white lollipop corresponds to an isolated wire whose left endpoint is $\leq k$, and a black lollipop corresponds to an isolated wire whose right endpoint is $> k$.

Note also that our planarity condition is quite strong. We do not assume simply that the sequence of bridges $(a_1, b_1), \ldots, (a_d, b_d)$ gives a planar bridge graph, but that the associated bridge diagram corresponds to a planar embedding of the graph. In particular, for a network corresponding to a planar bridge diagram, the third condition of Remark III.6 is automatic.

Lemma III.10. *Let a, b and c be wires in a valid bridge diagram, with $b < c$. Let t be the left endpoint of wire a. If we have $(b, c) \to (a \downarrow b)$ or $(b, c) \to (a \downarrow c)$, then $t \leq k$. If $(b, c) \to (a \uparrow b)$ or $(b, c) \to (a \uparrow c)$, then $t > k$.*

Proof. We prove the case where $(a \downarrow b)$ or $(a \downarrow c)$. The other case is analogous. If $(b, c) \to (a \downarrow c)$, then we must have

$$(b, c) \to (a \downarrow b) \to (a \downarrow c)$$

so it suffices to consider the case $(a \downarrow b)$. See Figure 3.3.

There are two cases to consider. Suppose wire b satisfies the first condition of Lemma III.8, so that the leftmost bridge on wire b is (b, e) for some $e > b$. Then the left endpoint of b is $\leq k$. Since $(a \downarrow b)$, the same is true of a. Otherwise, we must have $(e, b) \to (b, c)$ for some $e < b$. Hence $(e, b) \to (a \downarrow b)$, and so $(e, b) \to (a \downarrow e)$.

We may now apply the previous argument to e, and so on. Eventually, we must encounter a wire e' such that $(a \downarrow e')$, and e' satisfies the first condition of Lemma III.8. This completes the proof.

□

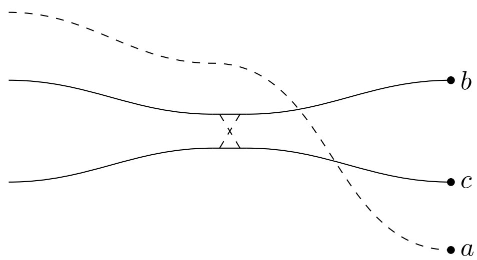


Figure 3.3: If the left endpoint of wire b is $\leq k$, the same must be true for wire a.

Let $\mathbf{u} \preceq \mathbf{w}$ be a valid bridge diagram, and suppose inserting the bridges

$$(a_1, b_1), \ldots, (a_{r-1}, b_{r-1})$$

from the diagram for $\mathbf{u} \preceq \mathbf{w}$ into the diagram for $\mathbf{w}$ gives a planar bridge diagram. We will show that adding the bridge (a_r, b_r) with $a_r < b_r$ preserves planarity. The proof consists of repeatedly apply lemmas III.8 and III.10. There are three cases to consider, depending on whether wires a_r and b_r are isolated.

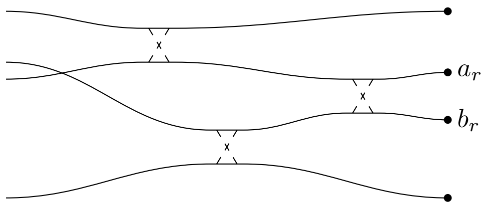
Lemma III.11. *If a_r and b_r are both non-isolated wires, then adding the bridge (a_r, b_r) preserves planarity.*

Proof. Note that we add the (a_r, b_r) bridge to the right of all previous ones, and that wire b_r lies immediately below wire a_r at the location of the (a_r, b_r) bridge. By inductive assumption, the portions of wires a_r and b_r respectively to the right of all the bridges

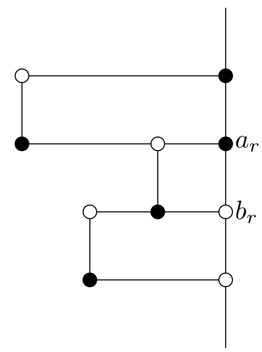
$$(a_1, b_1), \ldots, (a_{r-1}, b_{r-1})$$

correspond to legs of a plabic graph, as described above. Hence, adding the (a_r, b_r)

bridge corresponds to adding a new edge between two adjacent boundary legs of a plabic graph, and the result is planar. See Figure 3.4. □



(a) A portion of a bridge diagram which contains the rightmost bridge (a_r, b_r).



(b) The portion of a plabic network corresponding to the bridge diagram at left.

Figure 3.4: Adding a rightmost bridge (a_r, b_r) between two non-isolated wires corresponds to adding a bridge between two adjacent legs of a planar network.

Lemma III.12. *Suppose exactly one of the wires (a_r, b_r) is isolated. Then adding the (a_r, b_r) bridge gives a planar diagram.*

Proof. We argue the case where b_r is the isolated wire. The other case is analogous. By Lemma III.8, the left endpoint of wire b_r must be $> k$.

Suppose there is a non-isolated wire c with $a_r < c < b_r$, and let (c, c') be a bridge on wire c. Since a_r and c are both non-isolated, the inductive assumption implies that wire c lies below a_r to the right of all the bridges already inserted. In particular, wire c lies below wire a_r at the horizontal position where we insert the (a_r, b_r) bridge, and hence below wire b_r as well. Hence, we must have

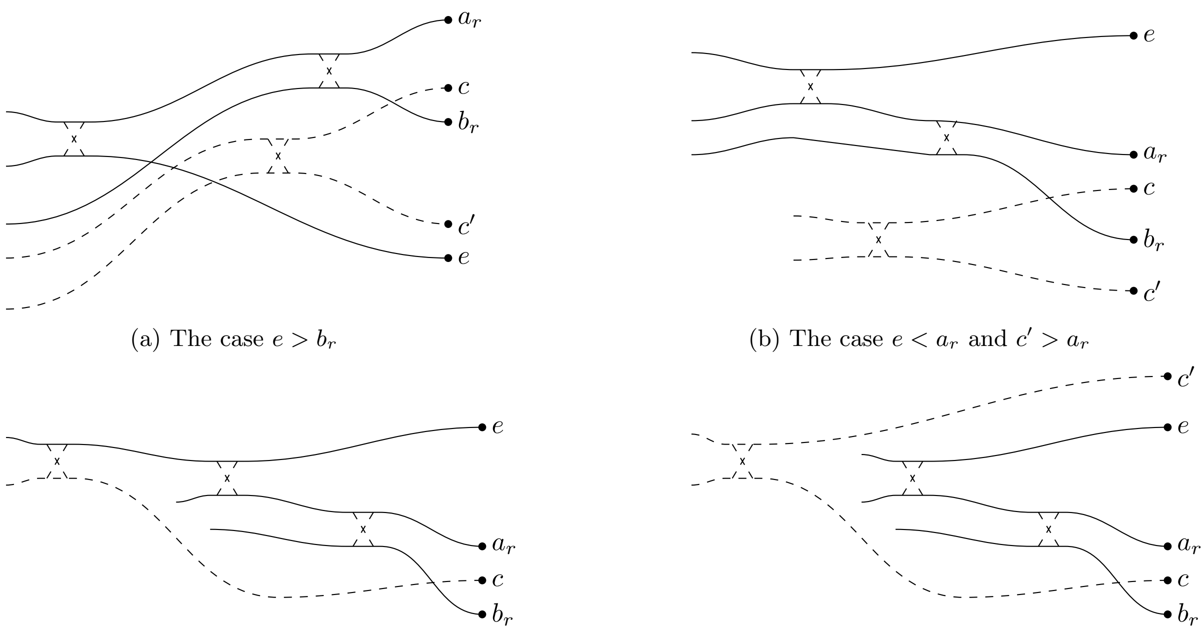
$$(c, c') \to (a_r, b_r) \to (b_r \downarrow c).$$

Since the left endpoint of wire b_r is $> k$, this contradicts Lemma III.10.

We have shown there is no non-isolated wire c with $a_r < c < b_r$. This, together with the fact that the planarity condition holds after adding bridges $(a_1, b_1), \ldots, (a_{r-1}, b_{r-1})$

and the fact that a_r is non-isolated, ensures that adding the (a_r, b_r)-bridge preserves the condition. See Figure 3.5.

□



(a) The case $e > b_r$

(b) The case $e < a_r$ and $c' > a_r$

(c) The case $e < a_r$ and $c' = e$

(d) The case $e < a_r$ and $c' < a_r$

Figure 3.5: Adding a bridge (a_r, b_r), where b_r is an isolated wire, and there is a bridge (a_r, e) for some e. In each case, the existence of a wire c with $a_r < c < b_r$ and a bridge (c, c') forces b_r to cross some non-isolated wire from above, to the right of all bridges on that wire. This yields a contradiction.

Lemma III.13. *If wires a_r and b_r are both isolated, then adding the bridge (a_r, b_r) gives a planar diagram.*

Proof. By Lemma III.8, the left endpoint of a_r is $\leq k$, while the left endpoint of b_r is $> k$. Let c be a non-isolated wire, so that we have a bridge (c, c') to the left of the (a_r, b_r) bridge. Suppose toward a contradiction that $a_r < c < b_r$. Then either we have

$$(c, c') \to (a_r, b_r) \to (a_r \uparrow c)$$

or we have

$$(c, c') \to (a_r, b_r) \to (b_r \downarrow c).$$

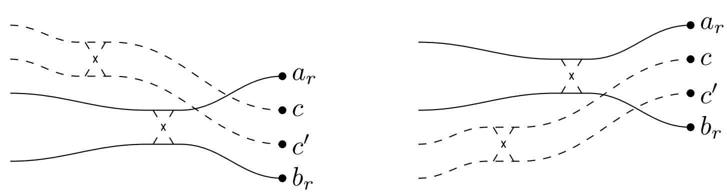
In the first case, Lemma III.10 implies that the left endpoint of a_r is $> k$, and in the second, Lemma III.10 implies that the left endpoint of b_r is $\leq k$. In either case, we have a contradiction, so each wire c with $a_r < c < b_r$ is isolated. See Figure 3.6.

Next, suppose $c > b_r$, where again c is a non-isolated wire. To prove planarity, we must show that wire b_r lies above wire c to the right of the (a_r, b_r) bridge. It is enough to show that the (a_r, b_r) bridge lies above wire c. Suppose the (a_r, b_r) bridge lies below wire c. Then we have

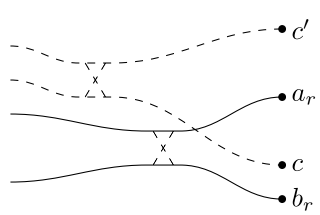
$$(c, c') \to (a_r, b_r) \to (a_r \uparrow c),$$

which gives a contradiction as before.

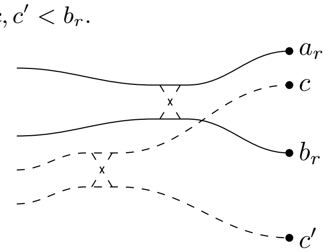
Finally, let c' be a wire which is not isolated, and suppose we have $c' < a_r$. By an analogous argument, the (a_r, b_r) bridge must be inserted below wire c'. Hence, adding a bridge (a_r, b_r) preserves the planarity condition, and the proof is complete. $\square$



(a) The case $a_r < c, c' < b_r$.



(b) The case $c' < a_r < c < b_r$.



(c) The case $a_r < c < b_r < c'$.

Figure 3.6: Adding a bridge (a_r, b_r) between two isolated wires. The existence of a wire c with $a_r < c < b_r$ and a bridge (c, c') forces either $(c, c') \to (a_r \uparrow c)$ or $(c, c') \to (b_r \downarrow c)$ which gives a contradiction.

Combining lemmas $III.13$, $III.11$, and $III.12$, we see that adding the bridge (a_r, b_r) always yields a planar bridge diagram. By induction, we have the following result.

Proposition III.14. *If* $\mathbf{u} \preceq \mathbf{w}$ *is a PDS with* u *anti-Grassmannian, then the bridge diagram for* $\mathbf{u} \preceq \mathbf{w}$ *is planar.*

Note that the third condition of Remark III.6 follows from the proposition. Indeed, consider what happens when we add the bridge (a_r, b_r) to a valid bridge diagram. Any wire c with $a_r < c < b_r$ must cross either wire a_r or wire b_r after the (a_r, b_r) bridge. If c were non-isolated when we added the (a_r, b_r) bridge, this would force a crossing between edges in the corresponding network, violating planarity.

Proving the graph is reduced

We have shown that the construction outlined in Figure 3.2 yields a planar graph. It remains to check that the graph is reduced, or equivalently that the first condition of Remark III.6 is satisfied each time we add a bridge. This follows from the forward direction of the following proposition.

Proposition III.15. *Let* $\mathbf{u} \preceq \mathbf{v}$ *be a valid bridge diagram, and suppose the corresponding plabic graph* G *is reduced. Suppose* $\mathbf{u}' \preceq \mathbf{v}'$ *is obtained from* $\mathbf{u} \preceq \mathbf{v}$ *by adding a new bridge* (a, b) *to the right of all the bridges in* $\mathbf{u} \preceq \mathbf{v}$. *Then the diagram* $\mathbf{u}' \preceq \mathbf{v}'$ *is valid if and only if adding the corresponding bridge to* G *yields a reduced graph.*

Proof. Let f be the bounded affine permutation of G. Let v' be the permutation corresponding to $\mathbf{v}'$. Inserting the bridge (a, b) at the appropriate place in the wiring diagram $\mathbf{v}$ gives a valid wiring digram if and only if $v' \gtrdot v$, in Bruhat order, which holds if and only if $v^{-1}(a) < v^{-1}(b)$. We must show this holds if and only if $f(a) > f(b)$.

By construction, for $c \in [n]$ we have

$$f(c) = \begin{cases} uv^{-1}(c) + n & v^{-1}(c) \leq k \\ uv^{-1}(c) & \text{otherwise} \end{cases} \tag{3.21}$$

Suppose $v^{-1}(a) < v^{-1}(b)$. If $v^{-1}(a) \leq k$ and $v^{-1}(b) > k$, the inequality $f(a) > f(b)$ follows easily from (3.21). Otherwise, we have either

$$v^{-1}(a), v^{-1}(b) \in \{1, \ldots, k\} \text{ or } v^{-1}(a), v^{-1}(b) \in \{k+1, \ldots, n\}. \tag{3.22}$$

The claim then follows since u is anti-Grassmannian, and is hence decreasing on the sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$.

Conversely, if $f(a) > f(b)$, we must have one of the following:

1. $v^{-1}(a) \in [k]$ and $v^{-1}(b) \in [k+1, n]$,
2. $v^{-1}(a), v^{-1}(b) \in [k]$ and $uv^{-1}(a) > uv^{-1}(b)$,
3. $v^{-1}(a), v^{-1}(b) \in [k+1, n]$ and $uv^{-1}(a) > uv^{-1}(b)$.

In the first case, the fact that $v^{-1}(a) < v^{-1}(b)$ is obvious. In the others, it follows from the fact that u is anti-Grassmannian. $\square$

Hence, starting with the lollipop graph for $u([k])$ and adding bridges as in (3.16) gives a reduced bridge graph for $\Pi_{\langle u,w \rangle_k}$. This proves the first direction of Theorem I.3.

Proposition III.16. *Every projected Deodhar parametrization for a positroid variety arises from a bridge graph.*

3.2.3 From bridge graphs to PDS's

We now prove the reverse direction of Theorem I.3.

Proposition III.17. *Every parametrization arising from a bridge graph agrees with some projected Deodhar parametrization.*

Proof. We note that any bridge graph can be constructed iteratively by a sequence of the following operations:

1. Adding bridges between adjacent legs of the graph.
2. Inserting new lollipops along the boundary of the disk.

Let G be a reduced plabic graph, and suppose we have a valid bridge diagram $\mathbf{u} \preceq \mathbf{w}$ corresponding to G. Let G' be a reduced plabic graph obtained from G by either adding a bridge between adjacent legs, or adding a lollipop. It suffices to show that we can modify the diagram $\mathbf{u} \preceq \mathbf{w}$ to give a valid bridge diagram corresponding to G'. The case of adding a bridge follows from Lemma III.15 below, while the case of adding a lollipop follows from Lemma III.18. $\square$

We now introduce some more terminology for discussing bridge diagrams. We call the portion of a bridge diagram to the right of the leftmost bridge, including the bridge itself, the *restricted part* of the diagram; we call the remainder of the diagram the *free part*. An (a, b)-*junction* refers to either an (a, b)-bridge or a crossing between wires a and b.

Lemma III.18. *Let G be a bridge graph, and let G' be a graph obtained from G by adding a lollipop. Suppose we have a valid bridge diagram corresponding to G. Then we can construct a valid bridge diagram for G'.*

Proof. We argue the case of adding a black lollipop; the case of a white lollipop is analogous. By assumption, we have a valid bridge diagram $\mathbf{u} \preceq \mathbf{w}$ corresponding to G, where $\mathbf{u}$ is the PDS for the anti-Grassmannian permutation u in the reduced word $\mathbf{w}$ of w. Adding a black lollipop to G and renumbering boundary vertices gives a reduced bridge graph G' for some $\Pi_{\langle u',w'\rangle_k}$ with u' anti-Grassmannian. Note that u' and w' are uniquely determined by the position of the new lollipop.

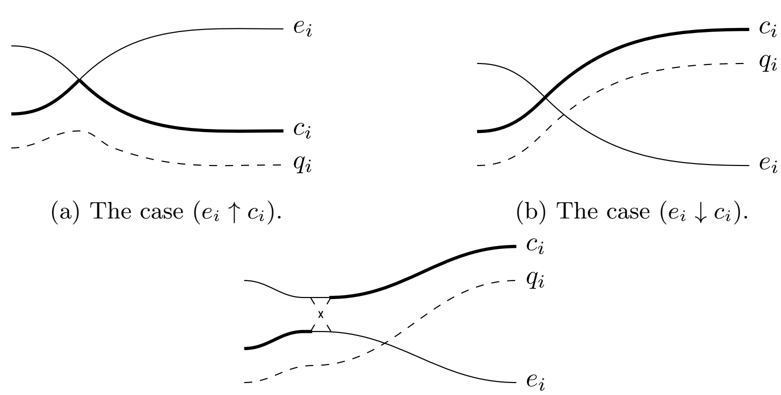
The restricted part of the diagram $\mathbf{u} \preceq \mathbf{w}$ is a reduced bridge diagram B corresponding to some subexpression $\mathbf{x} \preceq \mathbf{y}$. We claim that we can add a new wire to B to produce a bridge diagram B' for some $\mathbf{x}' \preceq \mathbf{y}'$, which we can then extend to a bridge diagram $\mathbf{u}' \preceq \mathbf{w}'$ corresponding to G'.

First, suppose the black lollipop is inserted just counterclockwise of position 1, and the remaining lollipops are re-numbered 2 through n. Then we simply add a new 1-wire which runs straight across the top of the diagram for $\mathbf{x}$, and renumber the endpoints of the existing wires appropriately. Otherwise, the black lollipop is inserted just counterclockwise of position q, for some $q \geq 2$. We add a new right endpoint q directly below $q-1$ in the bridge diagram for $\mathbf{x} \preceq \mathbf{y}$, and renumber the remaining right endpoints $q+1, \ldots, n+1$ accordingly. We then construct the diagram $\mathbf{x}' \preceq \mathbf{y}'$ in sections, working from right to left, as described below.

First, we divide the diagram B into sections as follows. Let $c_0 = q-1$. Find the rightmost point where either c_0 crosses another wire e_0, or there is a bridge (c_0, e_0) with $e_0 > c_0$. Let B_0 denote the portion of B which begins just to the left of the (c_0, e_0)-junction, and extends to the rightmost boundary of B. Let c_1 denote either c_0 or e_0, whichever wire lies below the other immediately to the left of the (c_0, e_0)-junction.

Now, suppose we have already defined sections $B_0, \ldots, B_{i-1}$ and fixed wire c_i. If there is no point to the left of B_{i-1} in B where either c_i crosses another wire e_i or there is a bridge (c_i, e_i) with $e_i > c_i$, then let B_i be the portion of B to the left of B_{i-1}. Otherwise, consider the rightmost junction to the left of B_{i-1} where either c_i crosses another wire e_i or there is a bridge (c_i, e_i) with $e_i > c_i$. Let B_i be the portion of B whose left edge is just to the left of this (c_i, e_i)-junction, and whose right edge is the boundary of B_{i-1}. Let c_{i+1} be either c_i or e_i, whichever is lower to the left of the (c_i, e_i)-junction. See Figure 3.7. Continuing in this fashion, we divide all of B into sections as in Figure 3.8.

Next, we modify each B_i by adding a segment q_i of wire q. The path of q_i depends on the nature of the (c_i, e_i) junction, as shown in Figure 3.7. If $(e_i \uparrow c_i)$ we let q_i lie immediately below e_i to the left of the crossing, and immediately below c_i to the right of the crossing. If $(e_i \downarrow c_i)$ then we let q_i lie below c_i to the left of the crossing; let $(q_i \uparrow e_i)$ immediately to the right of the crossing; then let q_i run below c_i to the boundary of B_i. Finally, if the (e_i, c_i) junction is a bridge with $e_i > c_i$, we let q_i start below wire e_i at left; let $(q_i \uparrow e_i)$ immediately to the right of the bridge; and let q_i run directly below c_i to the boundary of B_i. In each case, we shift the wires below q_i downward to obtain a wiring diagram which satisfies our conventions. See Figure 3.7 and Figure 3.8.



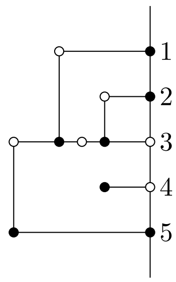
(a) The case $(e_i \uparrow c_i)$.

(b) The case $(e_i \downarrow c_i)$.

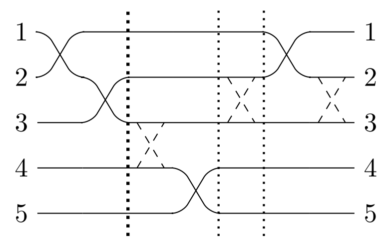
(c) A bridge (e_i, c_i) with $e_i > c_i$.

Figure 3.7: Adding a segment q_i of wire q to the bridge diagram B_i. Here q corresponds to a black lollipop.

Adding the segment q_i to each section B_i of B yields a bridge diagram B' with underlying wiring diagram $\mathbf{x}'$. By construction, the q_i form a unbroken wire q. We claim that B' is reduced. Let $\mathbf{y}'$ be the wiring diagram we obtain from $\mathbf{x}'$ by adding



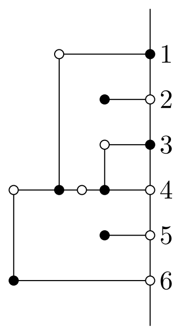
(a) Bridge graph G for $f = [5, 6, 7, 4, 8] \in \text{Bound}(3, 5)$.



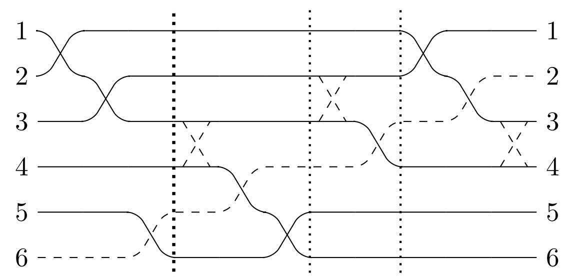
(b) f corresponds to $\langle u, w\rangle_3$ where $u = 32154$, $w = 53214$. We have a bridge diagram for G, corresponding to

$$\mathbf{w} = s_2\boldsymbol{s_1}s_2\boldsymbol{s_4}s_3\boldsymbol{s_2}\boldsymbol{s_1}.$$

Letters in the PDS for u in $\mathbf{w}$ are bolded.



(c) Adding a black lollipop gives a bridge graph G' for $f' = [6, 2, 7, 9, 5, 10] \in \text{Bound}(3, 6)$



(d) f' corresponds to $\langle u', w'\rangle_3$, where $u' = 431652$, $w' = 643152$. We build a bridge diagram for G' by adding a wire (dashed) with right endpoint at position 2. The result corresponds to $\mathbf{w}' = s_3\boldsymbol{s_2 s_1 s_3}s_2\boldsymbol{s_5 s_4}s_3\boldsymbol{s_5 s_2 s_1}$.

Figure 3.8: Adding a lollipop to the bridge graph G gives a bridge graph G'. We construct a bridge diagram for G' by adding a new wire (dashed) to a bridge diagram for G. The portion of each bridge diagram to the right of the thick vertical line is the restricted part. The vertical lines divide the restricted part of each diagram into segments, as in the proof of Lemma III.18.

the bridges inherited from the diagram B, and replacing each bridge with a crossing. Since B is reduced, it suffices to show that wire q does not cross any wire more than once, in either $\mathbf{x}'$ or $\mathbf{y}'$. This follows, since every crossing involving wire q has the form $(q \uparrow c)$ for some c.

Hence, we have a reduced bridge diagram B' corresponding to a subexpression $\mathbf{x}' \preceq \mathbf{y}'$, which we obtained from B by adding an isolated wire q. By construction, the sequence of bridges

$$(a_1, b_1), \ldots, (a_d, b_d)$$

in B' is precisely the sequence of bridges in the graph G'. It suffices to show that we can add additional crossings on the left side of B' to create a valid bridge diagram $\mathbf{u}' \preceq \mathbf{w}'$ whose restricted part is B'.

The free part of the diagram $\mathbf{u} \preceq \mathbf{w}$ is a reduced wiring diagram $\mathbf{v}$ for some $v \in S_n$. Let x' and y' be the permutations corresponding to $\mathbf{x}'$ and $\mathbf{y}'$ respectively. By construction, we have

$$x^{-1}u = y^{-1}w = v. \tag{3.23}$$

Now, u' and w' are uniquely determined by u and w; the fact that u' is anti-Grassmannian; and the fact that

$$u'^{-1}(q) = w'^{-1}(q) > k. \tag{3.24}$$

It follows that

$$x'^{-1}u' = y'^{-1}w' = v' \tag{3.25}$$

for some $v' \in S_{n+1}$. Consider the concatenation of a reduced diagram $\mathbf{v}'$ for v' and the diagram B'. We claim that this is the desired bridge diagram $\mathbf{u}' \preceq \mathbf{w}'$. It suffices to check that the resulting diagram is reduced and represents a PDS; the other needed properties follow from the previous discussion.

First, we show that the wiring diagram $\mathbf{u}'$ obtained by concatenating $\mathbf{v}'$ and $\mathbf{x}'$ is reduced. For this, it is enough to show that the factorization $u' = x'v'$ is length-additive, or equivalently, that $x' \leq_{(r)} u'$.

By an *inversion* of a permutation σ, we mean a pair of values $a < b$ with $\sigma^{-1}(a) > \sigma^{-1}(b)$. By the usual criterion for comparison in the right weak order, we must show that every inversion of x' is an inversion of u'. This follows by construction for any inversion which does not involve the value q. The remaining inversions correspond to wires b which cross wire q. Since $(q \uparrow b)$ for each such b, we have only pairs $b > q$

with $x'^{-1}(b) < x'^{-1}(q)$. We claim $u'^{-1}(b) < u'^{-1}(q)$. If $b \in u'([k])$, this is obvious, since $u'^{-1}(q) > k$; otherwise, $u'^{-1}(b)$, $u'^{-1}(q) \in [k+1, n]$, and the result follows from the fact that u' is anti-Grassmannian.

Next, let $\mathbf{u}' \preceq \mathbf{w}'$ be the diagram obtained by adding the bridges inherited from the diagram $\mathbf{u} \preceq \mathbf{w}$ to the diagram $\mathbf{u}'$. We must show that $\mathbf{w}'$ is reduced. Suppose adding crossings to the diagram $\mathbf{u}'$ corresponding to bridges

$$(a_1, b_1), \ldots, (a_{r-1}, b_{r-1})$$

gives a reduced wiring diagram $\mathbf{w}^*$, where the endpoints of the bridges have been renumbered to reflect the addition of wire q. Consider what happens when we add the crossing corresponding to the bridge (a_r, b_r). It follows from Remark III.2 and the fact that $\mathbf{u}$ is the PDS for u in $\mathbf{w}$ that wires a_r and b_r do not cross in $\mathbf{u}'$ to the right of the (a_r, b_r) bridge, so the same holds in $\mathbf{w}^*$. Hence, it suffices to show that wires a_r and b_r do not cross in the diagram $\mathbf{w}^*$ to the left of the (a_r, b_r) bridge. Since $a_r, b_r \neq q$, it follows from our construction that such a crossing occurs if and only if the corresponding wires cross in the corresponding part of the diagram for $\mathbf{w}$. Since $\mathbf{w}$ is reduced, this cannot happen, and $\mathbf{w}'$ is reduced by induction.

We have thus constructed a reduced bridge diagram $\mathbf{u}' \preceq \mathbf{w}'$. From the previous paragraph and Remark III.2 we see that $\mathbf{u}'$ is a PDS of $\mathbf{w}'$, so the diagram is valid. This completes the proof. $\square$

3.3 Local moves for bridge diagrams

Let $u \leq w \in S_n$, let $\mathbf{w}$ be a reduced word for w, and let $\mathbf{u} \preceq \mathbf{w}$ be the PDS for u in $\mathbf{w}$. Performing a Coxeter move on $\mathbf{w}$ yields a new word $\mathbf{w}'$ for w. Let $\mathbf{u}'$ be the unique PDS for u in $\mathbf{w}'$. Then $\mathbf{u}' \preceq \mathbf{w}'$ can be obtained from $\mathbf{u} \preceq \mathbf{w}$ by a local transformation, which we call a *PDS move* [31]. To perform a PDS move on the diagram $\mathbf{u} \preceq \mathbf{w}$, we first perform the desired Coxeter move on the diagram for $\mathbf{w}$. We then choose some of the affected crossings to be bridges in our new diagram, as described below.

Rietsch exhibits complete sets of PDS moves for all finite Weyl groups in [31], without using the language of bridge diagrams. We recall her result for S_n. For a commutation move $s_i s_j = s_j s_i$ with $|i-j| > 1$, the factor s_i on the left is contained in the PDS $\mathbf{u} \preceq \mathbf{w}$ if and only if it is contained in the PDS $\mathbf{u'} \preceq \mathbf{w'}$, and similarly for s_j. For braid moves, the situation is summarized in table 3.1 below. (The terms "legal" and "illegal" will be explained in the next section.) In each case, the generators contained in the PDS are bolded.

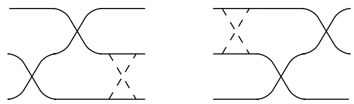
Table 3.1: Braid moves for PDS's. Factors in the PDS are bolded.

Legal Moves	Illegal Moves
$s_{i+1}\boldsymbol{s_i}\boldsymbol{s_{i+1}} \leftrightarrow \boldsymbol{s_i}\boldsymbol{s_{i+1}}s_i$	$s_{i+1}s_i\boldsymbol{s_{i+1}} \leftrightarrow s_i\boldsymbol{s_{i+1}}s_i$
$\boldsymbol{s_{i+1}}\boldsymbol{s_i}s_{i+1} \leftrightarrow s_i\boldsymbol{s_{i+1}}\boldsymbol{s_i}$	$s_{i+1}\boldsymbol{s_i}s_{i+1} \leftrightarrow s_i s_{i+1}\boldsymbol{s_i}$
$\boldsymbol{s_{i+1}}\boldsymbol{s_i}\boldsymbol{s_{i+1}} \leftrightarrow \boldsymbol{s_i}\boldsymbol{s_{i+1}}\boldsymbol{s_i}$	$s_{i+1}s_i s_{i+1} \leftrightarrow s_i s_{i+1} s_i$

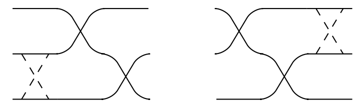
Rietsch's PDS moves correspond to local transformations of bridge diagrams, shown in Figure 3.9. The following is immediate.

Proposition III.19. *Let* $\mathbf{u} \preceq \mathbf{w}$, *where* $\mathbf{u}$ *is a PDS for* $\mathbf{w}$. *Let* $\mathbf{w'}$ *be obtained from* $\mathbf{w}$ *by performing a braid or commutation move, and let* $\mathbf{u'} \preceq \mathbf{w'}$ *be the PDS for* u *in* $\mathbf{w'}$. *Then performing the corresponding local move from Table 3.1 on the bridge diagram for* $\mathbf{u} \preceq \mathbf{w}$ *yields the bridge diagram for* $\mathbf{u'} \preceq \mathbf{w'}$.

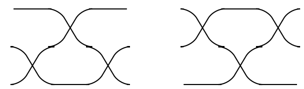
Since any reduced word for a permutation w can be transformed into any other using Coxeter moves, any bridge diagram for $\Pi_{u,w}$ can be transformed into any other using PDS moves.



(a) $s_{i+1}\boldsymbol{s_i s_{i+1}} \leftrightarrow \boldsymbol{s_i s_{i+1}} s_i$

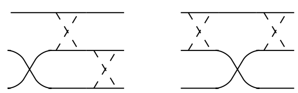


(b) $\boldsymbol{s_{i+1} s_i} s_{i+1} \leftrightarrow s_i \boldsymbol{s_{i+1} s_i}$

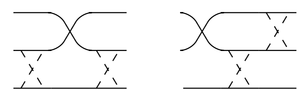


(c) $\boldsymbol{s_{i+1} s_i s_{i+1}} \leftrightarrow \boldsymbol{s_i s_{i+1} s_i}$

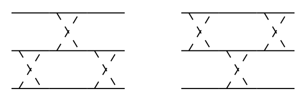
(1) Legal braid moves for bridge diagrams. These moves preserve isotopy class.



(d) $s_{i+1} s_i \boldsymbol{s_{i+1}} \leftrightarrow s_i \boldsymbol{s_{i+1}} s_i$



(e) $s_{i+1} \boldsymbol{s_i} s_{i+1} \leftrightarrow s_i s_{i+1} \boldsymbol{s_i}$



(f) $s_{i+1} s_i s_{i+1} \leftrightarrow s_i s_{i+1} s_i$

(2) Illegal braid moves for bridge diagrams. These moves change isotopy class.

Figure 3.9: Braid moves for PDS's

3.3.1 Isotopy classes of bridge diagrams

Let $\mathbf{w}$ and $\mathbf{w}'$ be reduced words for w, and let $\mathbf{u} \preceq \mathbf{w}$ and $\mathbf{u}' \preceq \mathbf{w}'$ be the PDS's for u in $\mathbf{w}$ and $\mathbf{w}'$ respectively. We say the bridge diagrams $\mathbf{u} \preceq \mathbf{w}$ and $\mathbf{u}' \preceq \mathbf{w}'$ are *isotopic* if they have the same sequence of bridges

$$(a_1, b_1), \ldots, (a_d, b_d).$$

If u is anti-Grassmannian, the valid bridge diagrams $\mathbf{u} \preceq \mathbf{w}$ and $\mathbf{u}' \preceq \mathbf{w}'$ correspond to isotopic bridge graphs.

We call a PDS move *legal* if it preserves isotopy class and illegal otherwise. Any commutation move is legal. From Figure 3.9, we see that a braid move is legal if and only if it involves at most one bridge; or equivalently, if and only if it involves as least two factors of the PDS. We say two bridge diagrams for PDS's are *move-equivalent* if we can transform one into the other by a sequence of legal moves. We will show that any two valid isotopic bridge diagrams are move equivalent. Thus the legal moves define equivalence classes of valid bridge diagrams, and these equivalence classes are in bijection with bridge graphs.

For $x, y \in S_n$, and $\mathbf{y}$ a reduced word for y, let $\mathbf{x} \preceq \mathbf{y}$ be a bridge diagram representing the PDS for x in $\mathbf{y}$. Let (a, b) be a bridge in B, and let $a < c < b$. Then we have either $(a, b) \to (c \downarrow a)$ or $(a, b) \to (c \uparrow b)$. We say B is *k-divided* if, for all such a, b and c, we have

$$(c \downarrow a) \text{ if } x^{-1}(c) \leq k$$

$$(c \uparrow b) \text{ if } x^{-1}(c) > k$$

Remark III.20. By Lemma III.10, any valid bridge diagram is k-divided.

Proposition III.21. *Let $u \leq w$. Let $B_1 = \mathbf{u} \preceq \mathbf{w}$ and $B_2 = \mathbf{u}' \preceq \mathbf{w}'$ denote isotopic, k-divided bridge diagrams corresponding to PDS's for u in reduced words of w. Then B_1 and B_2 are move-equivalent.*

Proof. We induce on the number of bridges in the diagrams. For the base case, note that any move involving at most one bridge is legal. Hence, any two isotopic bridge diagrams with at most one bridge are move-equivalent. Assume the claim holds for diagrams with at most $d-1$ bridges, and suppose B_1 and B_2 each have d bridges.

Suppose the rightmost bridge of each B_i is a bridge (a, b) where $a < b$. For $i = 1, 2$, let B_i^0 be the portion of B_i which extends from the rightmost bridge to the right edge of the diagram. Then we have bridge diagrams

$$B_1^0 = \bar{\mathbf{u}} \preceq \bar{\mathbf{w}} \tag{3.26}$$

$$B_2^0 = \bar{\mathbf{u}}' \preceq \bar{\mathbf{w}}' \tag{3.27}$$

for some $\bar{u}, \bar{u}', \bar{w}, \bar{w}' \in S_n$, and each B_i^0 corresponds to a PDS. Next, we construct a reduced bridge diagram B^0 with a single bridge (a, b) adjacent to its left edge, which satisfies all of the following for each $c \in [n]$:

1. If $c < a$ or $c > b$, then wire c runs straight from the left endpoint labeled c to the right, and does not cross any other wires.
2. If $a < c < b$, we have

$$(a, b) \to (c \downarrow a) \text{ if } u^{-1}(c) \leq k$$

$$(a, b) \to (c \uparrow b) \text{ if } u^{-1}(c) > k$$

3. If $a < c, c' < b$ and wires c and c' cross, then $u^{-1}(c') \leq k$ and $u^{-1}(c) > k$.

The fact that such a diagram exists follows easily from the proof of Lemma III.18, and we have

$$B^0 = \mathbf{u}_* \preceq \mathbf{w}_* \tag{3.28}$$

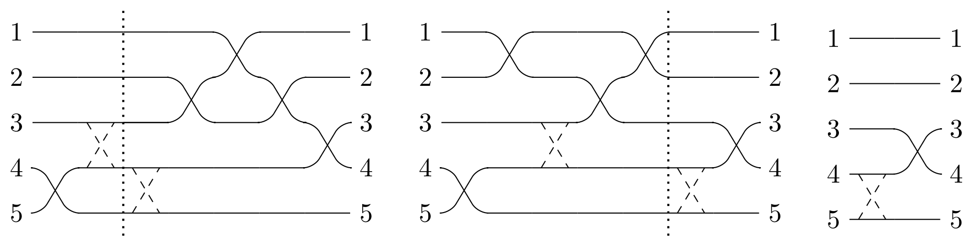
for some $u_*, w_* \in S_n$. See Figure 3.10.

Note that every inversion of u_* is also an inversion of $\bar{u}$ and $\bar{u}'$, since B_1 and B_2 are k-divided. By the usual criterion, it follows that $u_* \leq_{(r)} \bar{u}, \bar{u}'$. Hence we can add additional crossings to the diagram $\mathbf{u}_*$ on the left to build a reduced diagram $\bar{\mathbf{u}}_*$ for $\bar{u}$, or alternatively to build a reduced diagram $\bar{\mathbf{u}}'_*$ for $\bar{u}'$.

Since the reduced diagrams B_1^0 and B_2^0 each have only one bridge (a, b), it follows that wires a and b do not cross in the wiring diagrams $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}'$, and thus do not cross in the diagrams $\bar{\mathbf{u}}_*$ and $\bar{\mathbf{u}}'_*$. Hence we can add an (a, b) bridge to each of $\bar{\mathbf{u}}_*$ and $\bar{\mathbf{u}}'_*$, with the bridge just to the left of the copy of $\mathbf{u}_*$ in each diagram. The result is a pair of reduced diagrams C_1^0 and C_2^0 for PDS's, which are isotopic to B_1^0 and B_2^0 respectively, and whose rightmost entries form a copy of B^0.

By the base case, C_i^0 is move-equivalent to B_i^0 for $i = 1, 2$. So we can transform B_1, B_2 into bridge diagrams B_1', B_2' whose rightmost entries form a copy B^0. Let B_i^* be the part of B_i' to the left of the copy of B^0, for $i = 1, 2$. Then B_1^* and B_2^* are isotopic, k-divided bridge diagrams with $d-1$ bridges, and are hence move-equivalent by induction. Thus B_1' and B_2' are move-equivalent, and hence so are B_1 and B_2. $\square$

Corollary III.22. *Any two isotopic valid bridge diagrams are move equivalent.*



(a) $B_1 = \boldsymbol{s_3 s_2 s_1 s_2} s_4 s_3 \boldsymbol{s_4}$ (b) $B_2 = \boldsymbol{s_3} s_4 \boldsymbol{s_1 s_2} s_3 \boldsymbol{s_1 s_4}$ (c) $B_0 = \boldsymbol{s_3} s_4$

Figure 3.10: B_1 and B_2 are isotopic valid bridge diagrams with $u = 42153$ and $w = 42531$. For $i = 1, 2$, the portion of B_i to the right of the dashed line is B_i^0.

CHAPTER IV

Total positivity for the Lagrangian Grassmannian

In this chapter, we extend the combinatorial theory of $\mathrm{Gr}_{\geq 0}(k,n)$ to $\Lambda_{\geq 0}(2n)$. We begin with a discussion of the poset $\mathcal{Q}^C(2n)$ of type-C Bruhat intervals. In Section 4.1, we realize $\mathcal{Q}^C(2n)$ as a subposet of $\mathcal{Q}(n,2n)$, and define bounded affine permutations for the Lagrangian Grassmannian. We then use these results to relate the geometry of the projected Richardson stratification of $\Lambda(2n)$ to the geometry of the positroid stratification of $\mathrm{Gr}(n,2n)$.

In Section 4.2 we define the Lagrangian Grassmannian analogs of bridge graphs, and show that they encode Deodhar parametrizations for projected Richardson varieties in $\Lambda(2n)$. In Section 4.3 we extend this construction to give a Lagrangian Grassmannian analogue of the boundary measurement map, defined in terms of *symmetric plabic graphs*. In Section 4.4, we relate our network parametrizations to total positivity in $\Lambda(2n)$. Finally, in Section 4.5, we describe several more combinatorial indexing sets for projected Richardson varieties in $\Lambda(2n)$. Namely, we give type C analogs of *Grassmann necklaces*, *dual Grassmann necklaces*, and a class of matroids called *positroids*.

4.1 Bounded affine permutations and Bruhat intervals in type C

By [35, Corollary 6.4], the poset $\mathcal{Q}^C(2n)$ is graded, with rank function

$$\rho(\langle u, w\rangle_n^C) = \frac{n(n+1)}{2} - (\ell^C(w) - \ell({}^C u)).$$

Recall that $\Pi_{u,w}^C \subseteq \Lambda(2n)$ has dimension $\ell^C(w) - \ell^C(u)$ [15]. Since the dimension of $\Lambda(2n)$ is $\frac{n(n+1)}{2}$ it follows that $\Pi_{u,w}^C$ has codimension $\frac{n(n+1)}{2} - (\ell^C(w) - \ell^C(u))$. Hence the rank of an element of $\mathcal{Q}^C(2n)$ is the *codimension* of the corresponding projected Richardson variety in $\Lambda(2n)$.

Our goal is to show that $\mathcal{Q}^C(2n)$ is isomorphic to a subposet of $\mathcal{Q}(n, 2n)$. The image of $(S_n^C)_n$ in S_{2n} is the group of permutations in $S_n \times S_n$ which satisfy Equation 2.4. We note that $w \in S_n^C$ is a left coset representative of $(S_n^C)_n$ of minimal (respectively, maximal) length if and only if w is a left coset representative of $S_n \times S_n$ in S_{2n} of minimal (respectively, maximal) length. This follows easily from the discussion in [4, Chapter 8].

Proposition IV.1. *Let $u, w \in S_n^C$, where we view S_n^C as a subgroup of S_{2n}. Then $u \leq_n^C w$ in S_n^C if and only if $u \leq_n w$ in S_{2n}.*

Proof. For the forward direction, it is enough to show this when $u \lessdot_n^C w$ in S_n^C. There are two cases to consider. Recall that $a' = 2n+1-a$ for $a \in [2n]$. Either $w = us_{(a,a')}^A$ for some $a \in [n]$, or $w = us_{(a,b)}^A s_{(b',a')}^A$ for some $a \in [n]$, $b \in [n+1, 2n]$. In the first case, the claim is immediate; in the second, it is easy to see that

$$u \lessdot_n us_{(a,b)}^A \lessdot_n us_{(a,b)}^A s_{(b',a')}^A$$

so $u \leq_n w$ as desired.

For the reverse direction, let $u, w \in S_n^C$, and suppose $u \leq_n w \in S_{2n}$. Then w factors uniquely as $w^n w_n$ where $w_n \in S_n \times S_n$ and w^n is a minimal-length coset representative of $S_n \times S_n$ in S_{2n}. Moreover, we must have $w^n, w_n \in S_n^C$. Let $u' = uw_n^{-1}$. Then by Lemma II.2 we have $u' \leq_n w^n$, and $\langle u', w^n \rangle_n = \langle u, w \rangle_n$. Now w^n is a minimal-length coset representative for $S_n \times S_n$ in S_{2n}, and hence a minimal-length coset representative for $(S_n^C)^n$ in S_n^C. Thus by Lemma II.3 we have $u' \leq_n^C w^n$. Moreover, since the inclusion of S_n^C into S_{2n} is a Bruhat embedding, the factorizations $u = u'w_n$ and $w = w^n w_n$ are both length-additive in S_n^C. Hence $u \leq_n^C w$ as desired.

□

Again, let $u, w, x, y \in S_n^C$. It is clear that $\langle u, w \rangle_n = \langle x, y \rangle_n$ if and only if $\langle u, w \rangle_n^C = \langle x, y \rangle_n^C$. Hence, $\mathcal{Q}^C(2n)$ is a subset of $\mathcal{Q}(n, 2n)$. Moreover, it is easy to see that if $\langle u, w \rangle_n^C \leq \langle x, y \rangle_n^C$, then $\langle u, w \rangle_n \leq \langle x, y \rangle_n$. We claim that the converse holds, so that $\mathcal{Q}^C(2n)$ embeds as a sub-poset of $\mathcal{Q}(n, 2n)$.

Proposition IV.2. *Suppose $\langle u, w \rangle_n \leq \langle x, y \rangle_n$ with $u, w, x, y \in S_n^C$. Then $\langle u, w \rangle_n^C \leq \langle x, y \rangle_n^C$.*

Proof. We may assume for simplicity that w is Grassmannian. Since $\langle u, w \rangle_n \leq \langle x, y \rangle_n$ we have some representative $[x', y']_n$ of $\langle x, y \rangle_n$ such that

$$u \leq x' \leq_n y' \leq w.$$

It suffices to show that we can choose $x', y' \in S_n^C$. Write $y = y^n y_n$ where y^n is Grassmannian and $y_n \in S_n \times S_n$. Note that $y_n = ts$ where t fixes $[n]$ and s fixes $[n+1, 2n]$. Let s' be a permutation in $S_n \times S_n$ which fixes $[n]$, such that $ss' \in S_n^C$. Since w is Grassmannian, for $v, w \in S_{2n}$, we have $v \leq w$ if and only if $v(i) \leq w(i)$ for $i \in [n]$, while $v(i) \geq w(i)$ for $i \in [n+1, 2n]$. By symmetry, since $w \in S_n^C$, we have $y^* = y^n ss' \leq w \in S_{2n}$. Moreover, since $x \leq_n y$ with $x \in S_n^C$, we have $x = x^n rr'ss'$ where r fixes $[n+1, 2n]$, r' fixes $[n]$, $rr' \in S_n^C$, and rs is a length-additive factorization.

Let

$$x^* = x^n rr'ss'$$

Then $x^* \leq_n y^*$, and we have

$$\langle x^*, y^* \rangle_n = \langle x, y \rangle_n.$$

It remains to show that $u \leq x^*$. Let $\mathbf{x}$ be a reduced word for x in S_{2n} whose leftmost portion corresponds to a reduced word $\mathbf{q}$ of $x_n rr'$ under the embedding $S_n^C \hookrightarrow S_{2n}$. Consider the lexicographically leftmost subexpression $\mathbf{u}$ for u in $\mathbf{x}$. As in the construction of PDS's, we choose the factors of $\mathbf{u}$ greedily, working from the left. Since $\mathbf{q}$ corresponds to a reduced word in S_n^C, it follows that the portion of $\mathbf{u}$ which is contained in $\mathbf{q}$ does also. The remaining factors in $\mathbf{u}$ multiply to give some element of $z \in (S_n^C)_n$ such that st contains a reduced word for z in S_{2n}. Let $\mathbf{s}$ be reduced word for s, and $\mathbf{s}'$ a reduced word for s'. Then there is a reduced subexpression for z in the reduced word $\mathbf{ss}'$. Hence we can find a reduced word for u in the reduced word $\mathbf{qss}'$ of x^*. This completes the proof. □

Corollary IV.3. *The poset $\mathcal{Q}^C(2n)$ embeds as a sub-poset of $\mathcal{Q}(n, 2n)$.*

These results give a compact description of projected Richardson varieties in $\Lambda(2n)$.

Proposition IV.4. *Let $\mathring{\Pi}^C_{u,w}$ be an open projected Richardson variety in $\Lambda(2n)$, where $u, w \in S_n^C$ with $u \leq_n^C w$. Then $u \leq_n w \in S_{2n}$ and set-theoretically we have*

$$\mathring{\Pi}^C_{u,w} = \mathring{\Pi}^A_{u,w} \cap \Lambda(2n)$$

where $\mathring{\Pi}^A_{u,w}$ is the open positroid variety corresponding to $\langle u, w \rangle_n$ in $\mathrm{Gr}(n, 2n)$.

The set-theoretic intersection of an open positroid variety $\mathring{\Pi}^A_{u,w}$ *with* $\Lambda(2n)$ *is empty unless* $\langle u, w\rangle_n$ *has a representative* $\langle u', w'\rangle_n$ *where* $u', w' \in S^C_n$*, in which case the intersection is* $\mathring{\Pi}^C_{u',w'}$*.*

Finally, the closure partial order on projected Richardson varieties in $\Lambda(2n)$ *is induced by the closure partial order on positroid varieties in the obvious way. In particular, for* $u \leq^C_n w$*, we have*

$$\Pi^C_{u,w} = \Pi^A_{u,w} \cap \Lambda(2n).$$

Proof. Let $\langle u, w\rangle^C_n \in \mathcal{Q}^C(2n)$. Recall that we have

$$\mathring{R}^C_{u,w} = \mathring{R}^A_{u,w} \cap \mathrm{Sp}(2n)/B^\sigma_+$$

since $u \leq w \in S^C_n$. Since $u \leq^C_n w$, we have $u \leq_n w$, and the projection $\pi_n : \mathcal{F}\ell(2n) \to \mathrm{Gr}(n, 2n)$ carries $\mathring{R}_{u,w}$ isomorphically to $\mathring{\Pi}^A_{u,w}$. It follows that $\mathring{\Pi}^C_{u,w} \subseteq \mathring{\Pi}^A_{u,w} \cap \Lambda(2n)$.

Since open projected Richardson varieties stratify $\Lambda(2n)$, while open positroid varieties stratify $\mathrm{Gr}(n, 2n)$, we have

$$\Lambda(2n) = \bigsqcup_{\langle u,w\rangle^C_n \in \mathcal{Q}^C(2n)} \mathring{\Pi}^A_{u,w} \cap \Lambda(2n)$$

which in turn implies

$$\mathring{\Pi}^C_{u,w} = \mathring{\Pi}^A_{u,w} \cap \Lambda(2n)$$

for all $\langle u, w\rangle^C_n$.

Since the open positroid varieties of the form $\mathring{\Pi}^A_{u,w}$ for $\langle u, w\rangle^C_n$ cover $\Lambda(2n)$, it follows that $\mathring{\Pi}^A_{x,y} \cap \Lambda(2n)$ is empty if $\langle x, y\rangle_n$ is not contained in the image of the embedding $\mathcal{Q}^C(2n) \hookrightarrow \mathcal{Q}(n, 2n)$.

The final statement follows from Lemma IV.2, since the partial orders on $\mathcal{Q}^C(2n)$ and $\mathcal{Q}(n, 2n)$ give the reverse of the closure partial orders on projected Richardson varieties in $\Lambda(2n)$ and $\mathrm{Gr}(n, 2n)$, respectively. $\square$

We now define a type C analog of the poset $\mathrm{Bound}(k, n)$. Recall that we have a well-defined isomorphism $\mathcal{Q}(k, n) \to \mathrm{Bound}(k, n)$ given by $\langle u, w\rangle_k \mapsto f_{\langle u,w\rangle_k}$ where

$$f_{\langle u,w\rangle_k} = u t_{[k]} w^{-1}.$$

Definition IV.5. The set $\mathrm{Bound}^C(2n)$ of *type C bounded affine permutations* is the image of $\mathcal{Q}^C(2n)$ under the map $\mathcal{Q}(n, 2n) \to \mathrm{Bound}(n, 2n)$.

Recall that bounded affine permutations are elements of the extended affine Weyl group of $\mathrm{GL}(n)$. We show that a similar statement holds for $\mathrm{Bound}^C(2n)$. A *symplectic similitude* $A \in \mathrm{GL}(2n)$ is a linear transformation such that for all $v, w \in \mathbb{C}^{2n}$ we have

$$\langle Av, Aw\rangle = \mu\langle v, w\rangle$$

for a fixed nonzero scalar μ. Let $\mathrm{GSp}(2n)$ denote the group of symplectic similitudes, which is a reductive group of type C_n.

The extended affine Weyl group $\widetilde{S}_n^C$ of $\mathrm{GSp}(2n)$ may be realized as a subgroup of $\widetilde{S}_{2n}$, and the inclusion is a Bruhat embedding. Concretely, the extended affine Weyl group of $\mathrm{GSp}(2n)$ consists of all affine permutations wt, with $w \in S_n^C$ and $t = (a_1, \ldots, a_{2n})$ a translation element satisfying

$$a_i + a_{i'} = a_j + a_{j'}$$

for all $1 \leq i, j \leq n$. For details, see [16]. In particular, the Bruhat order on $\widetilde{S}_n^C$ induces a partial order on $\mathrm{Bound}^C(2n)$ which agrees with the partial order inherited from $\mathrm{Bound}(n, 2n)$.

Each element of $f \in \mathrm{Bound}^C(2n)$ satisfies

$$f(2n + a - 1) = 4n + 1 - f(a)$$

for all $a \in [2n]$. We claim that every $f \in \mathrm{Bound}(n, 2n)$ satisfying this condition must in fact be contained in $\mathrm{Bound}^C(2n)$. For this, it suffices to show that each such bounded affine permutation has the form $f = ut_{[n]}w^{-1}$ where w is Grassmannian, and $u, w \in S_n^C$. Certainly, we know that $f = ut_{[n]}w^{-1}$ for some $u, w \in S_{2n}$, with w Grassmannian. Since

$$f(2n + a - 1) = 4n + 1 - f(a)$$

for each a, exactly one of each pair $\{a, a'\}$ must be in $w^{-1}[n]$. From this, it follows that $w \in S_n^C$. But this, in turn, forces $u \in S_n^C$, since $uw^{-1}(a') = uw^{-1}(a)'$ for all $a \in [n]$. We now have the following.

Proposition IV.6. *The poset* $\mathrm{Bound}^C(2n)$ *consists of all bounded affine permutations* $f \in \mathrm{Bound}(n, 2n)$ *which satisfy*

$$f(2n + a - 1) = 4n + 1 - f(a)$$

for all $a \in [2n]$*.*

Let $\ell^{\widetilde{C}}$ denote the Bruhat order on $\widetilde{S}_n^C$. For $f \in \widetilde{S}_{2n}$, we define an equivalence relation on inversions of f by setting two inversions (a, b) and (c, d) equivalent if either

$$(c, d) = (a + 2rn, b + 2rn)$$

for some $r \in \mathbb{Z}$ or

$$(c, d) = (2n + 1 - a, 2n + 1 - b).$$

We call the resulting equivalence classes *type* $\widetilde{C}$ *inversions.* The following is an immediate consequence of the discussion in [4, Chapter 8].

Proposition IV.7. *Let* f *be a bounded affine permutation in* $\mathrm{Bound}^C(2n)$*. Then* $\ell^{\widetilde{C}}(f)$ *is the number of type* $\widetilde{C}$ *inversions of* f*. Alternatively* $\ell^{\widetilde{C}}(f)$ *is the number of* *type $\widetilde{A}$ inversions of f which have a representative of one of the following forms, for $n+1 \leq i \leq j \leq 2n$, and r a positive integer: (i,j), (i',j), $(i,j+2rn)$, $(i,j'+2rn)$, or $(j',i+2rn)$.*

We note the type C analog of Theorem 3.16 from [15]. The proof is entirely analogous to the type A version.

Proposition IV.8. *If $f = ut_{[n]}w^{-1} \in \mathrm{Bound}^C(2n)$, then f has length*

$$\frac{n(n+1)}{2} - (\ell^C(w) - \ell^C(u))$$

so the bijection from $\mathcal{Q}^C(2n)$ to $\mathrm{Bound}^C(2n)$ is graded. The codimension of Π_f^C in $\Lambda(2n)$ is equal to $\ell^{\widetilde{C}}(f)$.

Remark IV.9. He and Lam gave a construction which yields analogs of $\mathrm{Bound}(k,n)$ for many partial flag varieties [11]. We focus here on a special case, which suffices for our purposes. Let G be a quasi-simple reductive group with Weyl group (W,S) and extended affine Weyl group $\widehat{W}$. We may assume that G is *adjoint*, so that $\widehat{W}$ is as large as possible. In particular, for each simple root α_i of G, the cocharacter lattice of G contains a *fundamental coweight* λ_i which is dual to α_i.

Let α_i be a simple root of W. Let $J = S\backslash\{\alpha_i\}$, let W^J denote the set of minimal-length representatives of W/W_J, and let λ_i be the fundamental coweight which is dual to α_i. Let P_J be the parabolic subgroup of G corresponding to J. Then the desired indexing set for projected Richardson varieties in G/P_J is given by

$$\mathcal{Q} = \{ut^{-\lambda_i}w^{-1} \mid u \in W,\ w \in W^J \text{ and } u \leq w\} \subseteq \widehat{W}$$

where $t^{-\lambda_i}$ is the *translation element* corresponding to $-\lambda_i$.

Since the pairs (u,w) listed above are a complete set of representatives for the set of W_J-Bruhat intervals $\langle u,w\rangle_J$ in W, the set $\mathcal{Q}$ has a poset structure inherited from the poset of W_J-Bruhat intervals. Moreover, this coincides with the partial order induced on $\mathcal{Q}$ by the Bruhat order on $\widehat{W}$. The poset $\mathcal{Q}$ is graded by the length function on $\widehat{W}$, and the length of an element $ut^{-\lambda_i}w^{-1}$ of $\mathcal{Q}$ is equal to the codimension of the projected Richardson variety corresponding to $\langle u, w\rangle_J$ [11].

Note that we realize $\mathrm{Bound}(k,n)$ as a subset of the extended affine Weyl group of $\mathrm{GL}(n)$, and $\mathrm{Bound}^C(2n)$ as a subset of the extended affine Weyl group of $\mathrm{GSp}(2n)$. Since $\mathrm{GL}(n)$ and $\mathrm{GSp}(2n)$ are not quasi-simple, we cannot apply the results of [11] in this setting. Rather, we must look at the extended affine Weyl groups of the adjoint quasi-simple Lie groups of types A_{n-1} and C_n, respectively.

The adjoint Lie group of type A_{n-1} is $\mathrm{PSL}(n)$, the quotient of $\mathrm{SL}(n)$ by the subgroup of scalar matrices. Similarly, the adjoint Lie group $\mathrm{PSp}(2n)$ of type C_n is the quotient of $\mathrm{Sp}(2n)$ by the subgroup of symplectic scalar matrices. Taking the quotient of $\mathrm{PSL}(n)$ by the image of the subgroup of upper-triangular matrices gives $\mathcal{F}\ell(n)$, and similarly for $\mathrm{PSp}(2n)$. Let $\widehat{W}_n^A$ denote the extended affine Weyl group of $\mathrm{PSL}(n)$, and $\widehat{W}_n^C$ denote the extended affine Weyl group of $\mathrm{PSp}(2n)$.

It is not hard to show that He and Lam's result applied to $G = \mathrm{PSL}(n)$ and the simple root α_k gives the desired isomorphism of graded posets between $\mathcal{Q}(k,n)$ and $\mathrm{Bound}(k,n)$. The translation element $t_{[k]}$ in $\widetilde{S}_n^A$ plays an analogous role to the translation element $t^{-\lambda_k}$ in $\widehat{W}_n^A$, where λ_k is the fundamental coweight which is dual to α_k. The situation is similar in type C. Here $t_{[n]}$ in $\widetilde{S}_n^C$ plays an analogous role to the translation element $t^{-\lambda_n}$ in $\widehat{W}_n^C$, where λ_n is the fundamental coweight dual to α_n.

Remark IV.10. In [19], the authors introduced the poset of *type B decorated permutations* of order $2n$, denoted $\mathcal{D}_n^B$, and showed that it indexes projected Richardson varieties in both the odd orthogonal Grassmannian $\mathrm{OG}(n, 2n+1)$ (a flag variety of type B_n) and the Lagrangian Grassmannian $\Lambda(2n)$. The correspondence between decorated permutations and bounded affine permutations maps $\mathcal{D}_n^B$ isomorphically to $\mathrm{Bound}^C(2n)$. Lam and Williams also give an isomorphism from $\mathcal{Q}^C(2n)$ to $\mathcal{D}_n^B$ which is equivalent to our isomorphism from $\mathcal{Q}^C(2n)$ to $\mathrm{Bound}^C(2n)$. What is new in the present paper is the realization of $\mathcal{Q}^C(2n)$ as an induced subposet $\mathcal{Q}(n,2n)$, and the description of $\mathrm{Bound}^C(2n)$ in terms of $\widetilde{S}_n^C$.

4.2 Bridge graphs and Deodhar parametrizations for $\Lambda(2n)$

We now define bridge graphs for projected Richardson varieties in $\Lambda(2n)$, and show that they encode Deodhar parametrizations. Recall that $a' = 2n+1-a$ for all $a \in [2n]$. Let $f \in \mathrm{Bound}^C(2n)$. Suppose for some $a \in [n]$, we have

1. $f(a') > f(a)$.
2. Every $c \in [a+1, a'-1]$ is a fixed point of f.

Then f has a *symmetric bridge* at (a, a'). Alternatively, for $a, b \in [n]$ with $a < b$, we have

1. $f(a) > f(b)$ and $f(b') > f(a')$.
2. Every $c \in [a+1, b-1] \cup [b'+1, a']$ is a fixed point of f.

Then we say f has a *symmetric pair of bridges* at (a, b) and (b', a').

Now suppose that for $f, g \in \mathrm{Bound}^C(2n)$, we have $g = f s_{(a,a')}$ where f has a symmetric bridge at (a, a'). Then $g < f$ in the Bruhat order on $\mathrm{Bound}(n, 2n)$, and hence in the Bruhat order order on $\mathrm{Bound}^C(2n)$. It follows from [4, Proposition 8.4.1] that in fact $g \lessdot f$. Similarly, if $g = f s_{(a,b)} s_{(b',a')}$ where f has a symmetric pair of bridges at (a, b) and (b', a'), then $g \lessdot f$ in the Bruhat order on $\mathrm{Bound}^C(2n)$.

Let $f = u t_{[n]} w^{-1}$, where $u, w \in S_n^C$. Then a *symmetric bridge graph* for $f \in \mathrm{Bound}^C(n, 2n)$ is a graph obtained by starting with the symmetric lollipop graph corresponding to $t_{u[n]}$ and repeatedly adding either symmetric bridges (a, a') or symmetric pairs of bridges $(a, b)(b', a')$ until we obtain a graph with bounded affine permutation f. See Figure 4.1 for an example. It is perhaps not obvious that each $f \in \mathrm{Bound}^C(n)$ corresponds to a symmetric bridge decomposition. However, this will follow from our results.

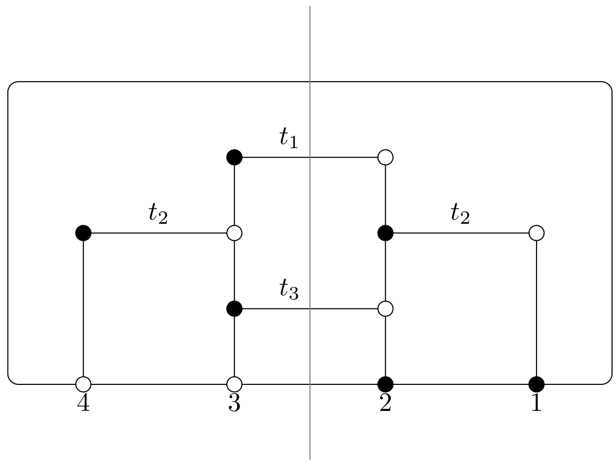


Figure 4.1: A symmetric bridge graph with symmetric weights.

Let B be a symmetric bridge graph for f. Weight each symmetric bridge (a_i, b_i) with an indeterminate t_i. For each symmetric pair of bridges, weight both bridges (a_j, b_j) and (b'_j, a'_j) with the same indeterminate t_j. Applying the boundary measurement map, we get a parametrization of a locally closed subset of $\mathring{\Pi}^f_A$. We claim that the image lies in $\Lambda(2n)$. In fact, we prove something stronger: a Lagrangian analog of Theorem I.3.

Proposition IV.11. *Let $f = ut_{[n]}w^{-1} \in \mathrm{Bound}^C(2n)$, where $u \leq^C_n w$. Then every Deodhar parametrization for $\mathring{\Pi}^C_f$ corresponds to a parametrization arising from a symmetric bridge graph. Conversely, every parametrization of $\mathring{\Pi}^C_f$ from a symmetric bridge graph arises from some Deodhar parametrization.*

Proof. The proof is nearly identical to the type A version. We sketch the argument here, and refer the reader to Chapter III for details. Let $\widetilde{\mathbf{w}}$ be a reduced word for w in S_n^C, and let $\widetilde{\mathbf{u}} \preceq \widetilde{\mathbf{w}}$ be the PDS for u in $\widetilde{\mathbf{w}}$. Let $\mathbf{u} \preceq \mathbf{w}$ be the corresponding PDS in S_{2n}, which is unique up to commutation moves. Note that $u \leq_n w$ as elements of S_{2n}. Hence $\mathbf{u} \preceq \mathbf{w}$ corresponds to a unique bridge graph; labeling the bridges with parameters gives the Deodhar parametrization of $\mathcal{D}^A_{\mathbf{u},\mathbf{w}}$ corresponding to $\mathbf{u} \preceq \mathbf{w}$. As in the proof of Lemma II.12, setting the weights on the two bridges in each symmetric pair equal to each other gives a parametrization of $\mathcal{D}^C_{\widetilde{\mathbf{u}},\widetilde{\mathbf{w}}}$, which we view as a parametrization of $\mathring{\Pi}^C_f$.

As an aside, we note that the planarity of the resulting graph puts restrictions on the sequence of factors x^C_α which may appear in the parametrization corresponding to a PDS $\mathbf{u} \preceq \mathbf{w}$ when the parametrization is written as in Equation 3.5. In particular, the only roots α which appear are of the form

$$\epsilon_j - \epsilon_i = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_{j-1}$$

for $1 \leq i < j \leq n$, and those of the form

$$-2\epsilon_i = -2\alpha_i - 2\alpha_{i+1} - \cdots - \alpha_n$$

for $1 \leq i \leq n$. Indeed, a factor x_α for $\alpha = \epsilon_i + \epsilon_j$ would correspond to a pair of bridges (i, j') and (j, i'), contradicting the planarity of the bridge graph.

For the reverse direction, it is enough to show that every symmetric bridge graph corresponds to a so-called *bridge diagram*, defined in Chapter III, which is symmetric up to isotopy with respect to reflection through the horizontal axis. The proof is entirely analogous to the type A case. As in type A, we build the desired bridge diagrams iteratively, by either adding bridges or lollipops; in the type C case, to maintain the symmetry of our diagrams, we always add lollipops in symmetric black-white pairs. $\square$

We say a symmetric bridge graph has *symmetric weights* if whenever two bridges form a symmetric pair, their weights are equal. So the above result says that symmetric bridge graphs with symmetric weights give parametrizations of projected Richardson varieties in $\Lambda(2n)$. Let B be a symmetric bridge graph with symmetric weights, corresponding to $\mathring{\Pi}^C_B$. By the properties of Deodhar parametrizations, restricting the weights of the bridges in B to $\mathbb{R}^+$ gives a parametrization of the totally nonnegative part of $\mathring{\Pi}^C_B$.

4.3 The Lagrangian boundary measurement map

4.3.1 Symmetric plabic graphs.

We now define symmetric plabic graphs, first introduced in [13]. Just as ordinary plabic graphs yield parametrizations of positroid varieties, symmetric plabic graphs give parametrizations of projected Richardson varieties in $\Lambda(2n)$.

Remark IV.12*.* Throughout this paper, we require plabic graphs to be bipartite. Postnikov's original definition allows plabic graphs which are not bipartite; however, these graphs can be made bipartite by either contracting unicolor edges, or adding degree-2 vertices. Our construction of the boundary measurement map only applies to bipartite graphs; the general case requires a different construction, given in [27]. We believe that our results extend naturally to the non-bipartite case, but have not checked the details.

Definition IV.13. A symmetric plabic graph G is a plabic graph with $2n$ boundary vertices, which has a distinguished diameter d such that the following hold:

1. The diameter d has one endpoint between boundary vertices $2n$ and 1, and the other between n and $n+1$.

2. No vertex of G lies on d, although some edges may cross d.

3. Reflecting the graph G through the diameter d gives a graph G' which is identical to G, but with the colors of vertices reversed.

See Figure 4.2 for an example. The following is an immediate consequence of Theorem 3.1 from [13].

Lemma IV.14. *Let G be a symmetric plabic graph. Then $f_G \in \mathrm{Bound}^C(2n)$. Conversely, for every $g \in \mathrm{Bound}^C(2n)$, there is a symmetric plabic graph G such that $f_G = g$.*

Let G be a symmetric plabic graph, with vertex set V. We define a map $r : V \to V$ which maps each vertex $v \in V$ to its image under reflection through d.

Definition IV.15. A weighting μ of a symmetric plabic graph G is *symmetric* if μ assigns the same weight to (u, v) and $(r(u), r(v))$ for each edge (u, v) of G.

We will show that for a symmetric plabic graph G, the boundary measurement map takes a symmetric weighting of G to a point in $\Lambda(2n)$. We first characterize points of $\Lambda(2n)$ in terms of Plücker coordinates. Recall that for $i \in [2n]$, we have $i' = 2n + 1 - i$.

Lemma IV.16. *Let $V \in \mathrm{Gr}(n, 2n)$, and let $I = \{i_1, \ldots, i_n\}$ be the lex-first non-zero Plücker coordinate of V. Then $V \in \Lambda(2n)$ if and only if the following hold:*

1. *For each $1 \leq i \leq n$, we have $i \in I$ if and only if $i' \notin I$.*
2. *For each $j < k \in [n]$ with $i_j < i'_k$, we have*

$$\Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}.$$

Proof. Let V be an n-dimensional subspace of $\mathbb{C}^{2n}$. Represent V by a $k \times n$ matrix in reduced row-echelon form, so the columns indexed by I form a copy of the identity matrix. For $j < k$, let v_j and v_k represent rows j and k of M, and say

$$v_j = (a_1, \ldots, a_{2n})$$

$$v_k = (b_1, \ldots, b_{2n})$$

Then we have

$$\langle v_j, v_k \rangle = \sum_{r=1}^{n} (a_{(2r-1)} b_{(2r-1)'} - a_{(2r)} b_{(2r)'}) \tag{4.1}$$

Now, $a_\ell = 0$ for $\ell < i_j$ and $b_{\ell'} = 0$ for $\ell > i'_k$, and $a_{i_j} = b_{i_k} = 1$. Suppose $i'_k = i_j$, so that V violates condition (1) above. Then $\langle v_j, v_k \rangle = \pm 1$, and V is not Lagrangian.

Assume now that V satisfies condition (1). We will show that V is Lagrangian if and only if V satisfies condition (2). By Equation (4.1), we have $\langle v_j, v_k \rangle = 0$ unless $i_j \leq i'_k$.

Suppose $i_j \leq i_{k'}$. Since $i'_k \neq i_j$ by hypothesis, we have $i_j < i_{k'}$. For each of $\ell \notin \{i_j, i_k, i'_j, i'_k\}$, either $a_\ell = 0$ or $b_{\ell'} = 0$, since either ℓ or ℓ' is a pivot column of M. Hence, we have

$$\langle v_j, v_k \rangle = (-1)^{i_j+1} b_{i'_j} + (-1)^{i'_k+1} a_{i'_k}.$$

Thus V is Lagrangian if and only if, for all $j < k$ with $i_j < i'_k$, we have

$$b_{i'_j} = \begin{cases} a_{i'_k} & i_j \text{ and } i_k \text{ have the same parity} \\ -a_{i'_k} & i_j \text{ and } i_k \text{ have opposite parity} \end{cases}$$

or equivalently, we have

$$b_{i'_j} = (-1)^{(i_k - i_j)} a'_{i_j}.$$

In the language of Plücker coordinates, this is equivalent to a collection of relations of the form

$$\Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = \pm \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}$$

for all i, j with $i_j < i'_k$. We calculate the relative sign in each case. Since $i_j < i'_k$ we have

$$a_{i'_k} = (-1)^{|I \cap [i_j+1, i'_k-1]|} \Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}}$$
$$b_{i'_j} = (-1)^{|I \cap [i_k+1, i'_j-1]|} \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}$$

Consider first the case where $i_j < i_k \leq n$. Then

$$(-1)^{|I \cap [i_j+1, i'_k-1]|} \Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = (-1)^{(i_k - i_j)} (-1)^{|I \cap [i_k+1, i'_j-1]|} \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}$$

The pivot columns that lie strictly between i_k and i'_k contribute at factor of -1 to each side of this equation. Canceling these factors, we are left with

$$(-1)^{|I \cap [i_j+1, i_k]|} \Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = (-1)^{(i_k - i_j)} (-1)^{|I \cap [i'_k, i'_j-1]|} \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}$$

Note that $\ell \in I \cap [i_j + 1, i_k]$ is contained in I and only if $\ell' \in [i'_k, i'_j - 1]$ is *not* contained in I. Hence

$$|I \cap [i_j + 1, i_k]| + |I \cap [i'_k, i'_j - 1]| = |[i_j + 1, i_k]| = i_k - i_j$$

and we have

$$\Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}} = \Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}}$$

Now, take the case where $n \leq i_k \leq i'_j$. Again, we have

$$(-1)^{|I \cap [i_j+1, i'_k-1]|} \Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = (-1)^{(i_k - i_j)} (-1)^{|I \cap [i_k+1, i'_j-1]|+1} \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}$$

By a similar argument to the above, we have

$$|I \cap [i_j + 1, i'_k - 1]| + |I \cap [i_k + 1, i'_j - 1] = |[i_j + 1, i'_k - 1]| = i'_k - i_j - 1$$

Since i'_k and i_k have opposite parity, it follows that $(-1)^{(i'_k - i_j - 1)} = (-1)^{(i_k - i_j)}$, and so

$$\Delta_{(I \setminus \{i_j\}) \cup \{i'_k\}} = \Delta_{(I \setminus \{i_k\}) \cup \{i'_j\}}.$$

This completes the proof. $\square$

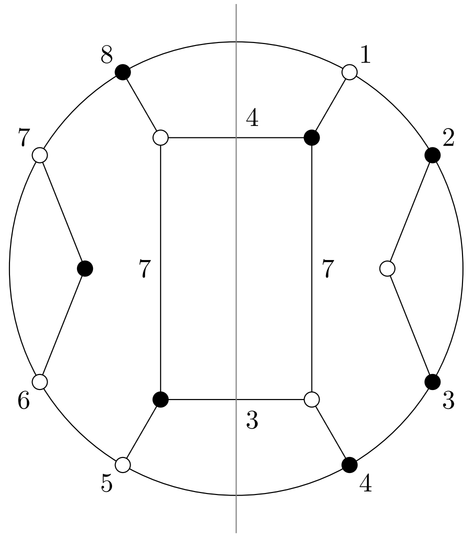


Figure 4.2: A symmetric weighting of a symmetric plabic graph. All unlabeled edges have weight 1.

Proposition IV.17. *Let G be a symmetric plabic graph with a symmetric weighting μ, and suppose G is reduced as an ordinary plabic graph. Then the point $P = \partial_G(\mu)$ is contained in $\Lambda(2n)$.*

Proof. Let I be the lex-first element of the matroid of P. Then by results of [27, Section 16], we have

$$I = \{i \in [n] \mid \sigma_G^{-1}(i) > i \text{ or } \sigma_G(i) \text{ is a white fixed point}\}.$$

Since G is a symmetric plabic graph, the chord diagram of σ_G is symmetric about the distinguished diameter d. It follows that $i \in I$ if and only if $i' \notin I$. Hence exactly one member of each pair (i, i') is contained in I.

Recall that for $J \in \binom{[2n]}{n}$, we have $R(J) = [2n] \backslash \{j' \mid j \in J\}$. It follows from the discussion in [13, Section 3] that $\Delta_J(P) = \Delta_{R(J)}(P)$ for all $J \in \binom{[2n]}{n}$. While the statement in that paper is only for positive real edge weights, the same argument holds for nonzero complex weights.

Let $1 \le j < k \le n$ such that $i'_k > i_j$. Let $J = I \backslash \{i_j\} \cup \{i'_k\}$ and let $J' =$

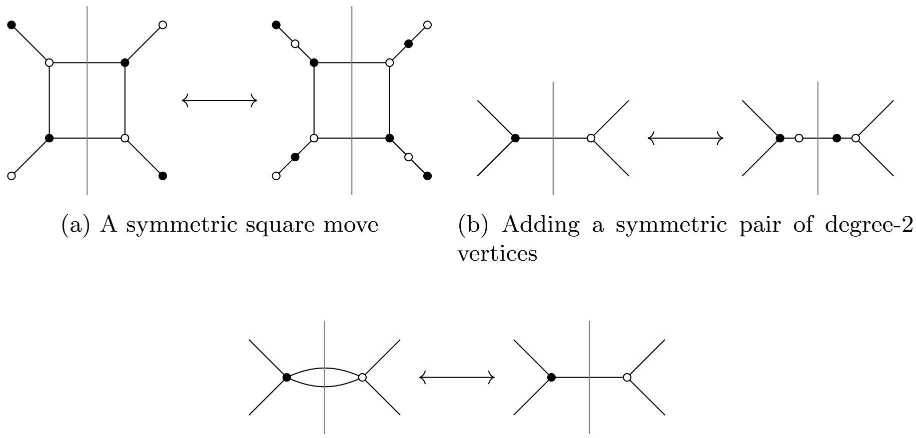
$I \backslash \{i_k\} \cup \{i'_j\}$. Then $J' = R(J)$, so $\Delta_J(P) = \Delta_{J'}(P)$. By the lemma above, it follows that the symmetric weighting of G indeed corresponds to a point in $\Lambda(2n)$, and the proof is complete. □

4.3.2 Local moves for symmetric plabic graphs

Our goal is to show that symmetric plabic graphs with symmetric weights give parametrizations of projected Richardson varieties in $\Lambda(2n)$, just as ordinary plabic graphs give parametrizations of positroid varieties in $\mathrm{Gr}(k, n)$. In this section, we define local moves and reductions for symmetric plabic graphs.

For each move or reduction from [27], we have a corresponding symmetric move or reduction, defined as follows. Let G be a symmetric plabic graph. Suppose we can perform an ordinary move or reduction on G, such that the affected portion of the graph lies entirely on one side of the distinguished diameter R. Then simultaneously performing the corresponding move or reduction on the opposite side of R yields a new symmetric plabic graph, equivalent to the first. This gives two moves and two reductions, corresponding to those of Postnikov.

We have two additional moves and one additional reduction, shown in Figure 4.3. For the first additional move, if we have a square face which is bisected by the diameter d, performing a square move at that face and contracting or uncontracting edges as in Figure 4.3a yields a symmetric plabic graph. For the second, suppose we have an edge which crosses the midline, both of whose vertices have valence two. Then removing both of these vertices again yields a symmetric plabic graph; conversely, we may add a pair of two-valent vertices of opposite colors to an edge which crosses the midline. For the additional reduction, if we have a pair of parallel edges which span d, performing a parallel edge reduction again yields the desired graph. Finally, we may perform symmetric gauge transformations, by applying the



(a) A symmetric square move

(b) Adding a symmetric pair of degree-2 vertices

(c) A symmetric reduction.

Figure 4.3: A symmetric reduction.

same gauge transformation to a pair of vertices v_1 and v_2 which are symmetric with respect to the midline.

Definition IV.18. A symmetric plabic graph is *reduced* if it cannot be transformed by symmetric moves into a graph on which one can perform a symmetric reduction. A symmetric plabic graph is *strongly reduced* if it is reduced as an ordinary plabic graph.

Each of our moves or reductions are compositions of Postnikov's moves for ordinary plabic graphs. Suppose G is a strongly reduced. Then performing a symmetric move, and transforming the edge weights according to Postnikov's rules, carries symmetric weightings of G to symmetric weightings.

The goal of this section is to show that a symmetric plabic graph is reduced if and only if it is strongly reduced, and that reduced symmetric plabic graphs with the same bounded affine permutation are equivalent via symmetric local moves. We make extensive use of the following lemma, which is Lemma 13.5 from [27]. Note that earlier, we required a bridge from i to $i+1$ to have a white vertex on the leg at i, and a black one at $i+1$. In this section, we also allow bridges which have a black vertex on the leg at i, and a white vertex on the leg at $i+1$.

Lemma IV.19. *Let G be a reduced plabic graph with trip permutation π_G, where π_G has no fixed points. Let $i < j$ be indices such that $\pi_G(i) = j$ or $\pi_G(j) = i$. Suppose there is no pair $a, b \in [i+1, j-1]$ such that $\pi_G(a) = b$. Then G is move-equivalent to a graph with a bridge from i to $i+1$. If $\pi_G(i) = j$ and $\pi_G(j) = i$, then i and j are connected by a path whose non-boundary vertices all have degree 2.*

The lemma below follows from results of Postnikov. A proof, in the language of bounded affine permutations, may be found in [18].

Lemma IV.20. *Let G be a reduced plabic graph with decorated permutation π_G and bounded affine permutation f_G, and assume π_G has no fixed points. Suppose the chords $i \to \pi_G(i)$ and $i+1 \to \pi_G(i+1)$ represent a crossing in G. Then we may transform G into a graph with a bridge that is white at i, and black at $i+1$.*

We recall Postnikov's criterion for reducedness, which is Theorem 13.2 of [27].

Theorem IV.21. *Let G be a plabic graph which has no leaves, except perhaps some leaves attached to boundary vertices. Then G is reduced if and only if the following conditions hold.*

1. *G has no round-trips.*
2. *G has no trips which use the same edge more than once, except perhaps for trips corresponding to boundary leaves.*
3. *G has no pair of trips which cross twice at edges e_1 and e_2, where e_1 and e_2 appear in the same order in both trips.*
4. *If $\pi_G(i) = i$ then G has a lollipop at i.*

Note in particular that while a trip may have a self-intersection at a vertex, no trip in a reduced plabic graph may cross itself. The following is an easy topological consequence of the above result. See Figure 2.3 for the concept of alignment.

Lemma IV.22. *Let G be a reduced plabic graph, and suppose the chords $i \to \pi_G(i)$ and $j \to \pi_G(j)$ are aligned in the sense of [27, Section 16]. Then the trips from i to $\pi_G(i)$ and j to $\pi_G(j)$ do not cross in G.*

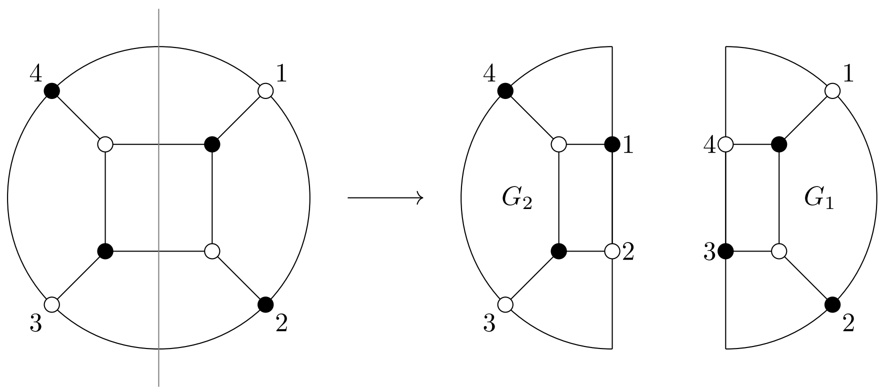


Figure 4.4: The distinguished diameter d divides a symmetric plabic graph into two regions, which correspond to a pair of plabic graphs G_1 and G_2.

Let G be a symmetric plabic graph. The diameter d of the disc divides G into two regions. Consider the region to the right of d. Drawing a new boundary segment along d, and adding boundary vertices wherever an edge of G intersects d, we obtain a new plabic graph G_1. Similarly, we may define a plabic graph G_2 corresponding to the region of G to the left of d. See Figure 4.4.

Notice that any move or reduction we may perform in G_1 or G_2 corresponds to a valid move or reduction in G. By inserting pairs of degree-two vertices along edges that cross d, we may assume that a move or reduction in G_1 does not affect G_2, and vice versa. Hence performing the corresponding move on each side of d gives a symmetric move or reduction in G. Thus if G is a reduced symmetric plabic graph, G_1 and G_2 are reduced as ordinary plabic graphs. Since strongly reduced implies reduced, this is also true for strongly reduced graphs.

Proposition IV.23. *Let G be a strongly reduced symmetric plabic graph with more than one face. Then G may be transformed by symmetric moves into a symmetric plabic graph which either has a symmetric pair of bridges not crossing the midline d, or a single bridge which does cross d. In the latter case, we may assume the two endpoints of the bridge are connected by a path containing only two-valent vertices.*

Proof. Since G is strongly reduced, every fixed point of G corresponds to a boundary leaf, so we may assume without loss of generality that G has no fixed points. Let G_1 and G_2 be as above. Let B_1 be the segment of the boundary of G_1 which lies on the boundary of G, and define B_2 similarly. Suppose some pair of boundary vertices $1 \leq i < j \leq n$ on B_1 satisfies the hypotheses of Lemma IV.19. Then we can transform G_1 into a graph with a bridge (i, k) for some $i < k < j$. Performing the corresponding sequence of symmetric moves in G yields a symmetric plabic graph with two commuting bridges (i, k) and (k', i').

Next, suppose no such pair (i, j) exists. Then π_{G_1} does not map any vertex on B_1 to another vertex in B_1. Note that since G is strongly reduced, no trip in G may cross the midline more than once. Indeed, suppose T is such a trip, and suppose T crosses the midline at edges e_1 and e_2. Let T' be the trip which is the mirror image of T' (such a trip exists, since G is symmetric). Then T' also crosses the midline at edges e_1 and e_2, in that order; this contradicts the reducedness criterion. Hence the trip permutation cannot map any boundary vertex of G_1 which lies on d to another vertex on d. By assumption, since no pair of vertices on B_1 satisfies the conditions of Lemma IV.19, the permutation π_{G_1} cannot map any vertex on B_1 to another vertex on B_1. Hence π_{G_1} maps each vertex on B_1 to a vertex on d, and vice versa.

Number the vertices of G_1 clockwise so that each vertex on B_1 comes before each vertex on d. Let r_1 be the first vertex on d, and let $t_1, \ldots, t_n$ be the vertices on

B_1. If $t_a = \pi_{G_1}(r_1) \neq t_n$ then the pair (t_a, r_1) satisfies the condition of Lemma IV.19, so we can transform G_1 into a graph with a bridge adjacent to t_a. Performing the corresponding sequence of symmetric moves transforms G to a graph with two symmetric bridges that do not cross the midline, and we are done in this case. Similarly, if $t_b = \pi_{G_1}^{-1}(r_1) \neq t_n$, then (t_b, r_1) satisfies the condition, and we are done as above.

There remains the case where $\pi_{G_1}(t_n) = r_1$ and $\pi_{G_1}(r_1) = t_n$. In this case, we may transform G_1 to a graph where t_n and r_1 are connected by a two-valent path. Performing the corresponding moves on G_2, see that G is move equivalent to a symmetric graph where the vertices corresponding to t_n and its reflection over the midline d are connected by a path with all vertices two-valent, and we are done. $\square$

Proposition IV.24. *A symmetric plabic graph G is reduced if and only if it is strongly reduced.*

Proof. A symmetric plabic graph which is strongly reduced is certainly reduced, since all symmetric reductions are composition of Postnikov's reductions. We must prove the converse. That is, suppose G is a symmetric plabic graph with no non-boundary leaves which does not satisfy the conditions of Theorem IV.21. We claim that we can transform G into a graph G' upon which we may perform a symmetric reduction. We induce on the number of faces of G. If G has a single face, then G is a lollipop graph, and the result is trivial. Suppose G has m faces, and suppose the result holds for graphs with fewer than m faces.

Let G, G_1 and G_2 be as above. If G_1 is non-reduced as a plabic graph, we may transform G_1 into a graph on which we can perform some reduction. Hence we may transform G by symmetric moves into a graph where we may perform some symmetric reduction, and G is non-reduced as a symmetric graph. We may thus assume G_1 and G_2 are reduced.

Suppose there is a trip T_1 in G which crosses the midline d more than once, with consecutive crossings e_1 and e_2. Let T_2 be the trip which is the mirror image of T_1 in G. (Such a trip exists, since G is symmetric.) Then T_1 and T_2 cross at edges e_1 and e_2, which appear in the same order in both trips. Note that since G_1 and G_2 are reduced, any round trip, trip which uses the same edge twice, or trip which starts and ends at the same point in G must cross d at least twice, and hence induce some instance of this configuration.

Let T_1 and T_2 be as above. Let S_1 by the segment of T_1 between e_1 and e_2 inclusive, and define S_2 similarly. Uncontracting edges, we may eliminate any self-intersections of S_1 or S_2. Hence we may assume the area bounded by S_1 and S_2 is homeomorphic to a disk [27, Section 13]. Let H be the subgraph of G bounded by S_1 and S_2. We may further uncontract edges to ensure that each vertex on S_1 or S_2 is adjacent to at most one vertex inside H. Hence the subgraph H of G bounded by S_1 and S_2 is a reduced symmetric plabic graph, which is strongly reduced by induction. Note that since G has no leaves, the trip permutation of H has no fixed points. We claim that we can reduce to the case where H has only one face.

Suppose without loss of generality that S_1 is a clockwise trip. The trip T_1 alternates between black and white vertices. All vertices on S_1 incident to edges in H are white, while those adjacent to edges outside of H are black. Moreover, by contracting and un-contracting edges, we may ensure that each vertex on S_1 or S_2 has degree 3.

Suppose H has more than once face. Then we may transform H into a graph with either a pair of commuting bridges on either side of the midline, or a valent-two path between two neighboring vertices on either side of the midline, by Lemma IV.23.

If, after these transformations, the graphs G_1 and G_2 are no longer reduced, we are done, so suppose G_1 and G_2 remained reduced.

Suppose the first case holds, so that H has a pair of commuting bridges. After contracting some edges, each bridge gives a square face of G, two of whose sides are on the boundary of H. (Note that we must obtain such a square face when we contract edges; the other option, a pair of parallel edges, would contradict the fact that G_1 is reduced after the transformation.) Performing a square move at this face, and the symmetric move on the other side, we reduce the number of faces of H by two.

Next, suppose the second case holds. Then we have a two-valent path between the boundary vertex of H on S_1 and its reflection through the midline. Symmetrically removing two-valent vertices and contracting edges, we obtain a symmetric square face in G, three of whose edges are part of the boundary of H. Performing a symmetric square move at this face reduces the number of faces of H by 1.

Hence H has a single face. But this, in turn, means that e_1 and e_2 are a pair of parallel edges. Hence we may perform a parallel edge reduction, and this case is complete.

Next suppose that no two trips in G cross the midline more than once. Since G is not reduced, G must have two trips T_1 and T_2 which cross twice, with the two crossings occurring in the same order in both trips. Moreover, T_1 and T_2 must cross once at an edge to the right of d, and once at and edge the left of d. By symmetry, we may assume that T_1 and T_2 originate to the left of d, cross once to the left of d, and then cross again to the right of d. Hence, two trips in G_1 which originate on d must cross.

Let $r_1, \ldots, r_m$ be the boundary vertices of G_1 which lie on d, from bottom to top.

Let B_1 be the segment of the boundary of G_1 which coincides with the boundary of G. Each trip in G_1 which originates on d must end on B_1. Hence each pair of trips originating on d represents either an alignment or a crossing. Moreover, the trips originating at $r_1, \ldots, r_{i+1}$ cannot all represent alignments: by Lemma IV.22 and the previous paragraph, some of these trips must cross.

Consider the trips in G_1 which originate at $r_1, \ldots, r_n$. Let i be the first index such that π_{G_1} such that the trips originating at r_i and r_{i+1} form a crossing. Then the trips originating at $r_1, \ldots, r_i$ are pairwise aligned. Assume that i is maximal among all symmetric graphs which are equivalent to G by symmetric moves, and which satisfy the conditions that no trip crosses d more than once and the subgraph on each side of d is reduced. We may transform G_1 into a graph with a bridge (r_i, r_{i+1}) which is white at r_i and black at r_{i+1}, and perform the corresponding transformation on G_2. Performing a symmetric square move in G then yields a new graph G^*. Note that no trip in G^* crosses the midline more than once.

Now G_1^* is identical to G_1, except that the bridge at (r_i, r_{i+1}) is now black at r_i and white at r_{i+1}. If G_1^* is non-reduced, then G is non-reduced as a symmetric plabic graph, and we are done. Otherwise, suppose the trips starting at r_i and r_{i-1} are now aligned. Then the trips originating at $r_1, \ldots, r_{i+1}$ are pairwise aligned, contradicting the maximality of i. Hence the trips which originate at r_{i-1} and r_i must form a crossing. We may thus transform G_1^* into a graph with a bridge at $(i-1, i)$, and perform a symmetric square move in G^*. Continuing in this fashion, we may successively uncross trips starting at r_ℓ and $r_{\ell-1}$, for $\ell = i, i-1, \ldots, 2$. This process must eventually yield a graph $\overline{G}$ where $\overline{G}_1$ is non-reduced. Otherwise, we could may transform G by symmetric moves into a graph $\overline{G}$ with $\overline{G}_1$ reduced, no trip crossing the midline more than once, and trips originating at $r_1, \ldots, r_{i+1}$ pairwise aligned, a contradiction. This completes the inductive step, and with it the proof. □

Proposition IV.25. *Let G and G' be two reduced symmetric plabic graphs. Then G and G' have the same bounded affine permutation if and only if we can transform G into G' by a series of symmetric moves.*

Proof. Each symmetric move is a composition of one or more of Postnikov's moves for plabic graphs. Hence by Lemma 13.1 of [27], the symmetric moves do not change the decorated permutation of G, and the forward direction is clear.

For the reverse direction, note that since G is reduced, G is strongly reduced. In particular, no trip in G crosses the midline more than once. Let G_1 and G_2 be defined as above, and let $r_1, \ldots, r_m$ be the vertices of G_1 which lie along d, numbered from bottom to top. By the argument in the previous proof, the trips in G_1 originating on d represent pairwise alignments or crossings. Moreover, we claim that we may reduce to the case where these trips are pairwise aligned.

Suppose the trips starting at r_i and r_{i+1} are the first pair which represent a crossing, so that the trips starting at $r_1, \ldots, r_i$ are pairwise aligned. Assume that we have chosen i maximal in the move-equivalence class of G. Then we can transform G_1 into a graph with a white-black bridge at (r_i, r_{i+1}) and perform a square move. As above, this uncrosses the trips originating at r_i and r_{i+1}, while leaving the other trips which originate on d unchanged. Since G is reduced, the result must be a graph G^* with G_1^* and G_2^* reduced. If r_i and r_{i-1} now represent a crossing, we iterate this process, and uncross r_i and r_{i-1}. Continuing in this fashion, we may transform G^* into a new graph $\overline{G}$ where the trips originating at $r_1, \ldots, r_{i+1}$ are pairwise aligned. This is a contradiction, since i was assumed to be maximal.

Hence we may transform G by symmetric moves into a configuration where the trips originating at $r_1, \ldots, r_m$ are aligned, and the symmetric statement is true for

G_2. Note that with this condition, the trip permutations π_{G_1} and π_{G_2} are uniquely determined by π_G.

Repeating the argument, we may transform G' into the same form. But now G_1 and G_1' are reduced plabic graphs with the same trip permutation, and similarly for G_2 and G_2'. Hence we can transform G_1 into G_1' by a series of moves. Performing the corresponding symmetric moves on G yields G', and the proof is complete. $\square$

We have shown that two symmetric plabic graphs have the same bounded affine permutation, and hence are associated to the same cell in the Lagrangian Grassmannian, if and only if they are equivalent by symmetric local moves. Hence the combinatorics of symmetric plabic graphs neatly parallels the combinatorics of ordinary plabic graphs, as desired.

4.3.3 Network parametrizations for projected Richardson varieties in $\Lambda(2n)$.

Let G be a reduced symmetric plabic graph with edge set E, and let Π_G^A denote the corresponding positroid variety in $\mathrm{Gr}(n, 2n)$. Let $\mathbb{G}^E$ denote the space of edge weightings of G, and let $\mathbb{G}_C^E$ be the space of symmetric weightings of G. Let $\mathbb{G}^V$ denote the group of gauge transformations of G, and let $\mathbb{G}_C^V$ denote the group of symmetric gauge transformations; that is, gauge transformations which act by the same value on v and $r(v)$ for each internal vertex v of G.

Lemma IV.26. *The image of* $\mathbb{G}_C^E$ *is closed in* $\mathbb{G}^E/\mathbb{G}^V$.

Proof. First, we choose a subset F of the edges of G which meets all of the following conditions:

1. F is the disjoint union of a collection of trees in G, each of which contains exactly one boundary leg of G.
2. F covers each vertex of G exactly once.

3. F is symmetric about the midline d of G; that is, an edge e of G is contained in F if and only if the same is true for $r(e)$.

It is not hard to show that such a subset exists, since G is symmetric and reduced.

Working inward from the boundary, we may then successively gauge-fix each remaining edge in F to 1. We call the resulting weighting of G an F-weighting. Each point in $\mathbb{G}^E/\mathbb{G}^V$ may be represented by a unique F-weighting, and the weights of edges in $E - F$ give coordinates on a dense subset of Π_G^A.

Consider a weighting w of G which lies in $\mathbb{G}_C^E$. Since the forest F is symmetric about the midline, the weighting w may be transformed into an F-weighting by a series of symmetric gauge transformations, which preserve $\mathbb{G}_C^E$. Every gauge-equivalence class contains a unique F-weighting, so an equivalence class $\overline{w} \in \mathbb{G}^E/\mathbb{G}^V$ is in the image of $\mathbb{G}_C^E$ if and only if the corresponding F-weighting is symmetric. Moreover, the map

$$\omega_F : \mathbb{G}^E \to \mathbb{G}^{E \setminus F}$$

which carries each weighting w to the corresponding F-weighting is continuous.

Now, let w be a weighting of G which is *not* contained in the image of the space $\mathbb{G}_C^E$ of symmetric weightings, so that the F-weighting w' corresponding to $\overline{w}$ is not symmetric. Then there is a neighborhood of the point $\omega_F(w')$ in $\mathbb{G}^{E\setminus F}$ which corresponds to F-weightings which likewise are not symmetric; taking the preimage in $\mathbb{G}^E$, we have an open subset of G^E containing the preimage of $\overline{w}$, which does not intersect the image of $\mathbb{G}_C^E$. This completes the proof. $\square$

We have shown that the image of $\mathbb{G}_C^E$ is closed in $\mathbb{G}^E/\mathbb{G}^V$. Moreover, two elements of $\mathbb{G}_C^E$ map to the same point in $\mathbb{G}^E/\mathbb{G}^V$ if and only if they are related by a symmetric gauge transformation. Let $\mathbb{G}_C^V$ be the group of symmetric gauge transformations.

Then we obtain a Lagrangian boundary measurement map

$$\mathbb{D}_G^C : \mathbb{G}_C^E/\mathbb{G}_C^V \to \Pi_G^C$$

simply by composing the boundary measurement map with the obvious embedding of $\mathbb{G}_C^E/\mathbb{G}_C^V \hookrightarrow \mathbb{G}^E/\mathbb{G}^V$.

Theorem IV.27. *The Lagrangian boundary measurement map $\mathbb{D}^C$ takes $\mathbb{G}_C^E/\mathbb{G}_C^V$ birationally to a dense subset of Π_G^C.*

Proof. By the main theorem of [25], the boundary measurement map

$$\mathbb{D}_G : \mathbb{G}^E/\mathbb{G}^V \to \Pi_G^A$$

is a birational map, which is regular on its domain of definition. Moreover, $\mathbb{D}_G^{-1}$ is defined precisely on the image of $\mathbb{D}_G$, which is open in Π_G^A. It follows that $\mathbb{D}_G(\mathbb{G}^E/\mathbb{G}^V) \cap \Pi_G^C$ is an open, nonempty subset of the irreducible algebraic set Π_G^C and is therefore dense in Π_G^C. Now, the torus $\mathbb{G}_C^E/\mathbb{G}_C^V$ embeds as a closed subset of $\mathbb{G}^E/\mathbb{G}^V$. Hence, the image $\mathbb{D}_G^C(\mathbb{G}_C^E/\mathbb{G}_C^V)$ is a locally closed subset of $\mathbb{D}_G(\mathbb{G}^E/\mathbb{G}^V) \cap \Pi_G^C$.

We show that $\mathbb{D}_G^C(\mathbb{G}_C^E/\mathbb{G}_C^V) = \mathbb{D}_G(\mathbb{G}^E/\mathbb{G}^V) \cap \Pi_G^C$ and hence $\mathbb{D}_G^C(\mathbb{G}_C^E/\mathbb{G}_C^V)$ is open and dense in Π_G^C. For this, in turn, it suffices to show

$$\dim\left(\mathbb{G}_C^E/\mathbb{G}_C^V\right) = \dim \Pi_G^C \tag{4.2}$$

since a full-dimensional closed subset of the irreducible algebraic set $\mathbb{D}_G(\mathbb{G}^E/\mathbb{G}^V) \cap \Pi_G^C$ must be the entire set.

To check (4.2), it is enough to consider the case of a bridge graph. The image of each symmetric bridge graph is indeed full-dimensional in Π_G^C, since bridge graphs encode Deodhar parameterizations. This completes the proof. □

Hence symmetric plabic graphs parametrize projected Richardson varieties in $\Lambda(2n)$, just as ordinary plabic graphs parametrize positroid varieties in $\mathrm{Gr}(n, 2n)$. From this result, we obtain a collection of relations which cut out $\Lambda(2n)$ in $\mathrm{Gr}(n, 2n)$.

Theorem IV.28. *Let $P \in \mathrm{Gr}(n, 2n)$. Then P lies in $\Lambda(2n)$ if and only if, for each $I \in \binom{[2n]}{n}$, we have*

$$\Delta_I(P) = \Delta_{R(I)}(P).$$

Proof. We first show that the desired relations hold on all of $\Lambda(2n)$. It is enough to show that the desired relations hold on each projected Richardson variety Π^C of $\Lambda(2n)$, since projected Richardson varieties form a stratification of $\Lambda(2n)$. Clearly, these relations are satisfied by any point corresponding to a symmetric plabic graph with a symmetric weighting. Let G be a symmetric plabic graph corresponding to Π^C. Then the Lagrangian boundary measurement map $\mathbb{D}^C$ takes the space of Lagrangian weightings of G to a dense subset of Π^C. Since Π^C is an irreducible algebraic set, the relations hold on all of Π^C, and we are done.

Conversely, let $P \in \mathrm{Gr}(n, 2n)$ be a point which satisfies

$$\Delta_I(P) = \Delta_{R(I)}(P)$$

for all $I \in \binom{[2n]}{n}$. Let J be the lex-minimal basis of P. We claim that J contains exactly one element of each pair $\{a, a'\}$ with $a \in [2n]$. Suppose the claim holds. Then P satisfies all of the relations in the statement of Lemma IV.16, and so $P \in \Lambda(2n)$ and we are done. It remains to check the claim.

Suppose the claim fails. Let a be the smallest element of $[2n]$ such that either $\{a, a'\} \subseteq J$ or $\{a, a'\} \subseteq [2n] \backslash J$. Now $\Delta_{R(J)}(P) = \Delta_J(P)$, so $J \leq R(J)$ in the lex order on $\binom{[2n]}{n}$. Note that if $a > 1$, then we have

$$J \cap ([1, a-1] \cup [a'+1, 2n]) = R(J) \cap ([1, a-1] \cup [a'+1, 2n]).$$

If $a, a' \notin J$, then $a, a' \in R(J)$, so $R(J) < J$ in lex order, a contradiction. Thus, we have $a, a' \in J$. But this, in turn, forces $a, a' \notin R(J)$.

Let M be a matrix representative for P. Then the minor of M indexed by $R(J)$ includes precisely the pivot columns of M indexed by $[1, a-1] \cup [a'+1, 2n]$, together with some subset of the columns $[a+1, a'-1]$. Hence, the span of these columns is contained in the span of $J \backslash \{a'\}$, and the corresponding minor vanishes, a contradiction. It follows that J contains exactly one of each pair $\{a, a'\}$, and the proof is complete. $\square$

4.4 Total nonnegativity for $\Lambda(2n)$

In the Grassmannian case, positivity of Plücker coordinates agrees with Lusztig's notion of total nonnegativity for partial flag manifolds. Moreover, plabic graphs with positive real edge weights parametrize totally nonnegative cells in $\mathrm{Gr}(k, n)$. We now prove analogous statements for $\Lambda(2n)$.

Proposition IV.29. *Let $\mathring{\Pi}^C$ be a projected Richardson variety in $\Lambda(2n)$. Then set theoretically,*

$$\mathring{\Pi}^C_{\geq 0} = \mathring{\Pi}^C \cap \mathrm{Gr}_{\geq 0}(k, n).$$

Proof. Let $\langle u, w \rangle_n^C \in \mathcal{Q}^C(2n)$ be the equivalence class corresponding to $\mathring{\Pi}^C$, and consider the corresponding class $\langle u, w \rangle_n \in \mathcal{Q}(n, 2n)$. Let $\widetilde{\mathbf{w}}$ be a reduced word for w in S_n^C, and let $\widetilde{\mathbf{u}} \preceq \widetilde{\mathbf{w}}$ denote the unique PDS for u in w. Let $\mathbf{u}$ and $\mathbf{w}$ denote the images of $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{w}}$, respectively, under the embedding $S_n^C \hookrightarrow S_{2n}$; this is uniquely determined up to commutation moves. The totally nonnegative part of $\mathring{\Pi}^C$ is the image in $\Lambda(2n)$ of the subset of $\mathcal{R}^C_{\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}}}$ where all parameters take nonnegative real values.

Embed $\mathrm{Sp}(2n)/B_+^\sigma$ in $\mathcal{F}\ell(n)$ as before. It follows from the proof of Lemma II.12

that each Deodhar component of $\mathring{R}^C_{u,w}$ is the subset of the corresponding Deodhar component of $\mathring{R}^A_{u,w}$ where the parameters satisfy a number of conditions of the form $t_i = t_{i+1}$. In particular, the totally nonnegative part of $\mathring{R}^C_{u,w}$ is the locally closed subset of $\mathcal{R}_{\mathbf{u},\mathbf{w}}$ cut out by these equalities, and is hence the intersection of $\mathring{R}^C_{u,w}$ with the totally nonnegative part of $\mathring{R}^A_{u,w}$. Projecting to $\Lambda(2n) \subseteq \mathrm{Gr}(n, 2n)$ gives the desired result. □

Since open projected Richardson varieties form a stratification of $\Lambda(2n)$, it follows that set-theoretically we have

$$\Lambda_{\geq 0}(2n) = \Lambda(2n) \cap \mathrm{Gr}_{\geq 0}(n, 2n)$$

with our given choice of embedding and of symplectic form. Hence, the totally nonnegative part of $\Lambda(2n)$ is precisely the *symmetric part* of $\mathrm{Gr}_{\geq 0}(k, n)$ studied in [13].

Let G be a symmetric plabic graph, with corresponding projected Richardson variety Π^C_G. Consider the space $(\mathbb{G}^E_C)_{\geq 0}$ of Lagrangian weightings of G such that all edges of G have positive real weights. Let $(\mathbb{G}^V_C)_{\geq 0}$ denote the group of symmetric, positive real gauge transformations of G. Then we have

$$(\mathbb{G}^E_C)_{\geq 0}/(\mathbb{G}^V_C)_{\geq 0} \hookrightarrow \mathbb{G}^E_C/\mathbb{G}^V_C.$$

Theorem IV.30. *For G a symmetric plabic graph, restricting the map $\mathbb{D}^C$ to $(\mathbb{G}^E_C)_{\geq 0}/(\mathbb{G}^V_C)_{\geq 0}$ gives an isomorphism of real semi-algebraic sets*

$$(\mathbb{G}^E_C)_{\geq 0}/(\mathbb{G}^V_C)_{\geq 0} \cong (\Pi^C_G)_{\geq 0}.$$

Proof. As long as we restrict internal edge weights to positive real numbers, all symmetric local moves induce isomorphisms–not simply birational maps–between the spaces of symmetric edge weightings of symmetric plabic graphs. Hence, it is enough to prove the claim for a single choice of G. For this, we simply choose a symmetric bridge graph. Symmetric bridge graphs with Lagrangian weightings encode Deodhar parametrizations, so the claim follows easily in this case, and the proof is complete. $\square$

Corollary IV.31. *The totally nonnegative cells of* $\Lambda(2n)$ *are precisely the nonempty matroid cells of* $\Lambda_{\geq 0}(2n)$.

4.5 Indexing projected Richardson varieties in $\Lambda(2n)$

We now state a theorem which gives the type C versions of the major combinatorial indexing sets for positroid varieties.

Theorem IV.32. *Each of the following are in bijection with projected Richardson varieties in* $\Lambda(2n)$. *In cases where the indexing set has a natural poset structure, the partial order corresponds to the* reverse *of the closure order on projected Richardson varieties in* $\Lambda(2n)$.

1. *Type* B Ⅎ*-diagrams which fit inside a staircase shape of size* n.
2. *The poset* $\mathcal{Q}^C(2n)$.
3. *The poset* $\mathrm{Bound}^C(2n)$.
4. *Equivalence classes of reduced symmetric plabic graphs, where the equivalence relation is given by symmetric moves.*
5. *The poset of positroids* $\mathcal{J}$ *which satisfy*

$$I \in \mathcal{J} \Leftrightarrow R(I) \in \mathcal{J}$$

ordered by reverse *containment.*

6. *The poset of Grassmann necklaces* $\mathcal{I} = (I_1, \ldots, I_{2n})$ *which satisfy* $I_i = R(I_{i'+1})$, *ordered by setting* $\mathcal{I} \leq \mathcal{K}$ *if* $I_i \leq_i \mathcal{K}_i$ *for all* i.

Proof. Part (1) was proved in [19]. We showed (2) and (3) in Section 4.1, while (4) follows from the discussion in Section 4.3. For (5) note that the matroids of the totally nonnegative cells in $\Lambda_{\geq 0}(2n)$ are precisely the positroids corresponding to symmetric plabic graphs. The characterization of these positroids, and their corresponding Grassmann necklaces, follows from Theorem 3.1 of [13]

For the statement about partial orders, note that each of the given posets embeds in the corresponding type A poset which indexes positroid varieties. In each case, the type C poset indexes the open positroid varieties which intersect $\Lambda(2n)$. There are canonical isomorphisms among the type-A posets, which preserve the correspondences with positroid varieties. Hence it is enough to show the claim for any *one* of the type C posets. Lemma IV.4 says precisely this for the poset $\mathcal{Q}^C(2n)$. $\square$

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