High-temperature asymptotics of the 4d superconformal index

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Physics)
in the University of Michigan
2016

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To my family.

Acknowledgements

I have been very privileged to have Jim Liu as my thesis advisor. The work described in this dissertation is the result of a long and enjoyable collaboration with him and Phil Szepietowski. I would like to express my deepest gratitude to both of them, for the wonderful and formative time I spent with them.

I am thankful to Finn Larsen and Leo Pando Zayas, with whom I had shorter collaborations, which were nevertheless memorably delightful, and very educational. I am indebted to Finn and Leo also for their continual encouragement and support during my phd.

While carrying out the analysis reported in Chapter 3 of this dissertation, I have benefited from ideas and suggestions of Peter Miller and Eric Rains; I am grateful to both of them for helpful discussions on the subject.

I would like to thank Ratindranath Akhoury, Henriette Elvang, Finn Larsen, Roberto Merlin, Leo Pando Zayas, and Len Sander for the numerous enjoyable scientific conversations we have had, and also for the physics that I have learned from them in their unforgettably nice classes.

I am grateful to Lydia Bieri and Igor Kriz for their inspiring and memorable courses on geometry and topology, through which I learned to appreciate the exciting depth of modern mathematical ideas.

Thanks to all of the graduate students in the physics department at University of Michigan for the fun time we had together. I wish to acknowledge in particular the fruitful scientific interactions with Anthony Charles, Mahdis Ghodrati, Marios Hadjiantonis, Jack Kearney, Gino Knodel, Pedro Lisbao, Alejandro Lopez, Tim Olson, Uttam Paudel, Vimal Rathee, Sam Roland, and Bob Zheng.

Finally, I would especially like to thank all of my friends and family for their unending and unconditional support. I love you all!

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Chapter 1

Introduction

Quantum Field Theory (QFT) is, among other things, the theoretical framework for understanding fundamental particles and their interactions. The particles in a given QFT[1] are divided into *bosons* carrying spin zero or one, and *fermions* carrying spin one-half; matter—in the form common on Earth—typically consists of the fermions, and the bosons mediate interactions between the matter particles.

Despite theoretical successes in the regime where the QFT particles interact weakly and *perturbation theory* accurately describes a wide range of observed phenomena, lack of progress on long-standing strong-interaction problems in QFT (such as the problem of quark confinement) indicates that more powerful *non-perturbative* techniques are needed.

A promising arena wherein to uncover non-perturbative structures in QFT is the realm of *Supersymmetric* (SUSY) QFTs. These are theories enjoying a powerful symmetry that, roughly speaking, exchanges their fermions and bosons. Many of the properties of supersymmetric theories are under better analytic control, sometimes even in the strong-interaction regime, thanks to their large symmetry group.

A subset of SUSY QFTs are yet much more symmetric, and it is natural to start with them in the quest for non-perturbative understanding of QFTs. These are Conformal SUSY QFTs, also known as SuperConformal Field Theories (SCFTs). An SCFT has a Hilbert space that is invariant not only under supersymmetry, but also under the action of the conformal group, which in four space-time dimensions can be described[2] as $\mathrm{SU}(2,2)$; with minimal supersymmetry (i.e. four Poincare and four conformal supercharges) added, the conformal group extends to the $\mathcal{N}=1$ superconformal group $\mathrm{SU}(2,2|1)$.

[1] A specific *quantum system* may be described by a specific *model* in quantum field theory; with an abuse of terminology, we will refer to different "model"s in the QFT framework as different "QFT"s. Also, we will have in mind only conventional QFT models, consisting of fields with spin ≤ 1.

[2] For brevity of exposition, we will not distinguish here between the symmetry group and its universal cover.

Besides serving as toy models of the richer non-conformal SUSY QFTs, SCFTs play a conceptually important role in the renormalization group (RG) approach to SUSY QFT: one can often think of non-conformal SUSY QFTs as describing "flows" between some SCFT in the ultraviolet (UV), and some SCFT in the infrared (IR) regime of energies. Thus, SCFTs can serve as signposts on the landscape of SUSY QFTs.

The main piece of data of an SCFT is the spectrum—of the various quantum numbers—of the states in its Hilbert space. The states of a conformal field theory are in one-to-one correspondence with the local operators in the theory [1]. Therefore an important objective when studying a given SCFT is to classify and count the local operators/states of the theory.

Since the Hilbert space of an SCFT is invariant under the 4d superconformal group, the states/operators in it are labeled by quantum numbers associated to the generators of the maximal compact bosonic subgroup of $\mathrm{SU}(2,2|1)$. A subset of the states/operators (known as BPS states/operators), which have specific relations between their various quantum numbers, sit in short representations of the superconformal algebra. This sector is expected to be protected against "smooth deformations" arising from RG flows or interactions, because the quantum numbers of the short representations do not undergo smooth changes[3]; the states/operators in this sector are hence under better analytic control thanks to this "topological" structure that the superconformal symmetry induces on the Hilbert space. The topological sector consisting of the short representations is sometimes referred to as the *BPS sector* of the SCFT. The *superconformal index* [3, 4] is a particular partition function which efficiently quantifies this controllable sector of an SCFT. As a partition function, it depends on an inverse-temperature parameter[4] β used to weigh various states with Boltzmann-type factors. Investigating the β-dependence of the superconformal index—or *the index*—of various interesting SCFTs is the main goal of this dissertation. More precisely, we would like to understand how the index behaves as the temperature is taken to infinity—or β is taken to zero. (The low-temperature ($\beta \to \infty$) asymptotics of the index is rather trivial; see [5].)

It is worth emphasizing that the "temperature" parameter in the index does not admit a thermal interpretation: in a path-integral picture, the index is computed as the partition function on $S^3 \times S^1$ with *periodic* boundary conditions around the S^1 [6], while the more familiar thermal partition functions are computed with fermions having anti-periodic boundary conditions around the Euclidean time circle. Nevertheless, the Boltzmann-type factors entering the definition of the index [see for instance Eq. 1.7 below] suggest this stretch of

[3]This is true modulo multiplet recombination. The possibilities for recombination are limited though, by the fact that the R-charges of local operators (in a Lagrangian 4d SCFT) are algebraic numbers; c.f. [2].

[4]The index also depends on a squashing parameter b which for simplicity we suppress (i.e. set to unity) in the present chapter.

terminology, and thus we will keep referring to β as 'inverse temperature'.

The reader without prior familiarity with superconformal indices may wonder if investigating the index, which is only a certain measure of a certain sector of an SCFT, is a worthwhile endeavor. The following two remarkable applications of the index respond to this question in the positive.

- Application to supersymmetric duality: the so-called IR dualities in SUSY QFT imply that two differently formulated SCFTs (e.g. two SCFTs with different field contents) are exactly equivalent, even at the *non-perturbative* (!) level; the superconformal index can serve as a probe of this equivalence, since a proposal for duality of two SCFTs may be valid only if the indices of the two theories are equal. The great power of the index in probing supersymmetric dualities was demonstrated in the seminal paper of Dolan and Osborn [8] around 2008.

- Application to holography: according to the AdS/CFT correspondence, the states encoded in (or "counted by") the superconformal index of a holographic SCFT (such as the $\mathcal{N}=4$ SYM) correspond to states of a quantum gravity theory in Anti-de Sitter (AdS) space. In particular, it is expected that the index will help the microscopic counting of high-energy quantum gravity states, such as Giant Gravitons [9].

We will see that understanding the high-temperature asymptotics of the index not only leads to advances in both of the directions itemized above, but also opens up new prospects for understanding the *non-perturbative* (!) low-energy dynamics of 4d supersymmetric gauge theories compactified on a circle. (More precisely, the high-temperature asymptotics of the index of an SCFT formulated as a gauge theory on R^4 in the UV, seems to encode information on the Coulomb branch dynamics of the gauge theory on $R^3 \times S^1$; see subsection 3.3.3.)

In the remaining parts of this chapter, we first introduce the 4d superconformal index, along with its famous precedent, the quantum mechanical Witten index, more elaborately in section 1.1. Then we proceed in section 1.2 to highlight our main result: the high-temperature asymptotics of the superconformal index of finite-rank Lagrangian unitary non-chiral 4d SCFTs. In section 1.3 we discuss the high-temperature asymptotics of the *large-rank limit* of the indices of a class of holographic SCFTs, and explain how our results address (for the class of theories under study) the computation of the Holographic Weyl Anomaly at the subleading order in the $1/N$ expansion.

1.1 From Witten index to the superconformal index

Consider a unitary quantum mechanical system enjoying *supersymmetry*. That is to say, there exists a fermionic operator Q, referred to as the *supercharge* operator, acting on the Hilbert space of the system, and satisfying

$$\{Q,Q^\dagger\}=H \qquad \text{and} \qquad Q^2=0, \tag{1.1}$$

with H the Hamiltonian operator.

Existence of Q implies that states of nonzero energy are paired in the system: it can be easily checked using (1.1) that $Q+Q^\dagger$ provides a one-to-one mapping from the set of bosonic states $|\mathrm{b}\rangle$ with $H|\mathrm{b}\rangle\neq 0$, to the set of fermionic states $|\mathrm{f}\rangle$ with $H|\mathrm{f}\rangle\neq 0$.

But the zero-energy states *are not* necessarily paired:

$$\begin{aligned} 0=\langle \mathrm{b}|H|\mathrm{b}\rangle=\langle \mathrm{b}|\{Q,Q^\dagger\}|\mathrm{b}\rangle=\langle \mathrm{b}|Q\;Q^\dagger|\mathrm{b}\rangle+\langle \mathrm{b}|Q^\dagger\;Q|\mathrm{b}\rangle\\ \Rightarrow (Q+Q^\dagger)|\mathrm{b}\rangle=0,\end{aligned} \tag{1.2}$$

with a similar argument applying to the fermionic zero-energy states. Since the states with zero energy are annihilated by Q and $Q^\dagger$, we say that the zero-energy states are "supersymmetric". Since (1.1) implies that H necessarily has non-negative eigenvalues, we can further say that the zero-energy states are *supersymmetric ground states* of the theory.

Therefore in unitary supersymmetric quantum mechanics, the **Witten index** [7]

$$\begin{aligned}\mathcal{I}^W :=& \sum_i (-1)^F e^{-\hat{\beta}E_i}\\ =& n^b_{\text{z.e.}}-n^f_{\text{z.e.}}\end{aligned} \tag{1.3}$$

(with F the fermion-number operator, evaluating to 1 on fermionic states, and to 0 on bosonic states), receives contributions only from unpaired zero-energy (or supersymmetric ground-) states; the index is thus obviously independent of $\hat{\beta}$.

Less obviously, $\mathcal{I}^W$ is also independent[5] of the *interaction strength* in the system! The reason is that any continuous deformation of the system, such as that induced by RG flows or by variation of the interaction couplings, should cause the supersymmetric ground-states to acquire nonzero energy only in pairs; similarly, any nonzero-energy state that as a result of the continuous deformation becomes a supersymmetric ground-state, should be accompanied by a partner state all along. This independence of the index from continuous deformations gives it a topological character—hence the well-deserved title "index".

[5]Modulo subtleties (see [7]) that are not relevant for our discussion.

Since $\mathcal{I}^W$ is independent of the couplings, it can be computed even in strongly interacting theories. It can thus provide information that would otherwise be inaccessible through (the more conventional) perturbative means. The original work of Witten used this index to probe the ground-state(s) of non-abelian supersymmetric gauge theories. The idea was that a *nonzero index* would mean that either $n^b_{\text{z.e.}} \neq 0$ or $n^f_{\text{z.e.}} \neq 0$, and thus it would imply the existence of supersymmetric ground-states, and hence the *absence of spontaneous supersymmetry breaking*. It is worth emphasizing the remarkable fact that a weak-coupling calculation of $\mathcal{I}^W$ can yield nontrivial information about the (strongly-interacting) ground-state of a non-abelian gauge theory!

The above discussion in the context of supersymmetric quantum mechanics can now help us to extend the concept of an index to unitary 4d SCFTs.

In any 4d SCFT there exists a supercharge operator[6] Q, such that

$$\{Q, Q^\dagger\} = H - 2J_2^z - \frac{3}{2}R \qquad \text{and} \qquad Q^2 = 0, \tag{1.4}$$

with H the Hamiltonian in the radial quantization, J_2^z the third generator of the right-handed[7] Lorentz SU(2), and R the generator of the $\mathrm{U}(1)_R$ inside the $\mathcal{N} = 1$ superconformal group $\mathrm{SU}(2,2|1)$.

Therefore, in analogy with the above quantum mechanical discussion, we can define the following Witten index for unitary 4d SCFTs:

$$\mathcal{I}^W := \mathrm{Tr}(-1)^F e^{-\hat{\beta}(E-2j_2-\frac{3}{2}r)}, \tag{1.5}$$

with E, j_2, r the eigenvalues of H, J_2^z, R, and with the trace taken over the Hilbert space in the radial quantization. Similarly to the quantum mechanical case above, the dependence on $\hat{\beta}$ drops out, since only states with vanishing $E-2j_2-\frac{3}{2}r$ (hence sitting in short representations of $\mathrm{SU}(2,2|1)$) have a chance of surviving bose-fermi cancelations.

It turns out that the combination $H-R/2$ commutes with the supercharge Q used above. We can hence refine $\mathcal{I}^W$ with a fugacity $e^{-\beta}$ for the combination $E-r/2$, without ruining the cancelations underlying its topological character. We refer to this refined Witten index as the **4d superconformal index** [3, 4]:

$$\mathcal{I}(\beta) := \mathrm{Tr}(-1)^F e^{-\beta(E-\frac{r}{2})} e^{-\hat{\beta}(E-2j_2-\frac{3}{2}r)}. \tag{1.6}$$

[6] In fact any 4d SCFT has at least four such operators. The Q that we consider here is one [it doesn't matter which] of the two that transform inside a $(0,1/2)$ representation of the (complexified) Lorentz group.

[7] Had we chosen a Q operator transforming inside a $(1/2,0)$ representation of the Lorentz group, J_2^z would be replaced with J_1^z.

In fact without the refinement with β, the index—as defined in (1.5)—is divergent in interesting SCFTs; thus the Boltzmann-type factor $e^{-\beta(E-\frac{r}{2})}$ is actually necessary as a regulator. Since the superconformal index does not depend on $\hat{\beta}$, we can write

$$\mathcal{I}(\beta) = \sum (-1)^F e^{-\beta(E-\frac{r}{2})}, \tag{1.7}$$

with the sum taken either over the local operators in the SCFT, or equivalently (via the CFT state/operator correspondence) over the states in the radial quantization.

The index (1.7) can be computed in closed form for a wide variety of interesting SCFTs. For instance the index of a free chiral multiplet is given by an *elliptic gamma function* (see appendix A for the definition of the elliptic gamma):

$$\mathcal{I}_\chi(\beta) = \Gamma(e^{-2\beta/3}; e^{-\beta}, e^{-\beta}). \tag{1.8}$$

In theories with several decoupled chiral multiplets, the index would be given by a product of the corresponding elliptic gamma functions. In gauge theories with several chiral multiplets, the index would be given by a product of several chiral-multiplet gamma functions and several vector-multiplet gamma function, integrated (roughly speaking) over the gauge group so as to project the result onto the gauge-singlet sector. We will explain this more carefully in Chapter 2.

1.2 High-temperature asymptotics of the index of finite-N gauge theories

We now summarize the rich structure we find in the high-temperature limit of the superconformal index. Throughout this dissertation we focus on the index of unitary 4d SCFTs that admit a gauge theory description with *non-chiral matter content*; the SU(N) $\mathcal{N}=4$ SYM and the SU(N) SQCD fixed points are the examples that we ask the reader to keep in mind while reading the somewhat abstract discussion below. We emphasize that in the present section we are considering gauge theories with a finite rank; the large-N limit of superconformal indices will be discussed in the next section of the present chapter.

The index of a *Lagrangian SCFT* [by that we mean an SCFT admitting a gauge theory description] is given by an **elliptic hypergeometric integral** (EHI) [8]. This is an expression of the form $\int f(\beta; x_1, \ldots, x_{r_G})\ \mathrm{d}^{r_G}x$, with r_G the rank of the gauge group G of the Lagrangian SCFT, β (> 0) the inverse temperature, and x_i ($\in [-1/2, 1/2]$) the integration variables. The function f is a complicated special function of its arguments, given explicitly as a product of several elliptic gamma functions; moreover, when the SCFT is non-chiral, f is real and positive semi-definite. The integral over $-1/2 < x_i < 1/2$ roughly projects onto the gauge-singlet sector; colloquially speaking, it washes out the contribution of non-gauge-invariant operators to the index[8].

The high-temperature ($\beta \to 0$) limit of the index corresponds to the **hyperbolic limit** of the EHI. This limit has been rigorously analyzed by Eric Rains [10] (around 2006) in certain special EHIs. We put the EHIs studied by Rains in the wider context of the EHIs arising from non-chiral unitary Lagrangian 4d SCFTs. In this generalized framework, the methods of Rains can be extended to uncover a surprisingly rich structure. We find (using, in particular, appropriate uniform estimates (derived in appendix B) for the elliptic gamma function) that in the $\beta \to 0$ limit $\mathcal{I}(\beta)$ simplifies as

$$\mathcal{I}(\beta) = \int f(\beta; \mathbf{x})\, \mathrm{d}^{r_G}x \overset{\beta\to 0}{\longrightarrow} \int e^{-(\mathcal{E}_0^{DK}(\beta)+V^{\mathrm{eff}}(x_1,\ldots,x_{r_G};\beta))}\, \mathrm{d}^{r_G}x, \tag{1.9}$$

with

$$\mathcal{E}_0^{DK}(\beta) = -\frac{16\pi^2}{3\beta}(c-a), \tag{1.10}$$

where c and a are the central charges[9] of the SCFT. We have given a superscript DK to $\mathcal{E}_0$, because a proposal of Di Pietro and Komargodski [11] implies the high-temperature asymptotics $\mathcal{I}(\beta) \approx e^{-\mathcal{E}_0^{DK}(\beta)}$ (see [12] for an earlier hint of this asymptotic formula).

We observe from (1.9) that an effective potential $V^{\mathrm{eff}}(\mathbf{x}; \beta)$ dictates the high-temperature asymptotics of $\mathcal{I}(\beta)$. It turns out that

$$V^{\mathrm{eff}}(\mathbf{x}; \beta) = \frac{4\pi^2}{\beta} L_h(\mathbf{x}), \tag{1.11}$$

with L_h a continuous, real, piecewise linear function of the x_i, which is determined by the matter content of the SCFT (examples can be found in the Figures 3.1, 3.2, and 3.4 below). We will refer to L_h as the *Rains function* of the SCFT. The relations (1.9) and (1.11) imply

[8]This is analogous to how the zeroth (or the "singlet") Fourier component of a periodic real function is obtained by integrating that function.

[9]The central charges ($\in \mathbb{R}^{>0}$) are measures of the number of degrees of freedom in the SCFT. For an SCFT described by an SU(N) gauge theory, c and a are typically of order N^2 at large N; for example, for the SU(N) $\mathcal{N} = 4$ SYM we have $c = a = (N^2 - 1)/4$. See Chapter 3 for a precise expression for $c - a$ in terms of the matter content.

that the index localizes in the $\beta \to 0$ limit to the locus of minima of L_h. We thus find

$$\mathcal{I}(\beta) \approx e^{-(\mathcal{E}_0^{DK}(\beta)+V_{\min}^{\text{eff}}(\beta))}. \tag{1.12}$$

Taking the logarithm of the two sides, we can write this as [the subleading term and the error estimate will be justified in Chapter 3]

$$\ln \mathcal{I}(\beta) = \frac{16\pi^2}{3\beta}(c - a - \frac{3}{4}L_{h\ min}) + \dim\mathfrak{h}_{qu} \ln(\frac{2\pi}{\beta}) + O(\beta^0), \tag{1.13}$$

with $L_{h\ \min}$ (which we will prove to be ≤ 0) the minimum of the Rains function over $-1/2 \leq x_i \leq 1/2$, and $\dim\mathfrak{h}_{qu}$ the dimension of the locus of minima of L_h.

The minimization problem for $L_h(\mathbf{x})$ can often be analytically solved on a case by case basis (as in [10]) using certain generalized triangle inequalities (GTIs); for most SCFTs of interest to us, the required GTI is obtained as a corollary of Rains's GTI, which can be found in appendix C.

Note that the leading piece in (1.13) takes the same form as the Di Pietro-Komargodski formula $\ln \mathcal{I}(\beta) \approx -\mathcal{E}_0^{DK}(\beta) = \frac{16\pi^2}{3\beta}(c-a)$, but with the "shifted $c - a$" defined as

$$(c-a)_{\text{shifted}} := c - a - \frac{3}{4}L_{h\ \min}. \tag{1.14}$$

This last relation appears to be analogous to the equation

$$c_{\text{eff}} = c - 24h_{\min}, \tag{1.15}$$

frequently discussed in the context of non-unitary 2d CFTs (see e.g. [13]).

One application of the result (1.13) is to supersymmetric dualities. Dual SCFTs must have identical partition functions. Comparison of the indices provides one of the strongest tests of any proposed duality between $\mathcal{N} = 1$ SCFTs [8, 14]. The full comparison of the multiple-integrals computing superconformal indices is, however, extremely challenging, except for the few cases (corresponding to various SQCD-type theories [8, 14, 15]) already established in the mathematics literature (e.g. in the celebrated work of Rains [16] (from around 2005) on "transformations" of elliptic hypergeometric integrals). Rather, known dualities are frequently used to conjecture new identities between multi-variable integrals of elliptic hypergeometric type [8, 14, 15, 17].

We propose comparison of the high-temperature asymptotics of the indices. Since dual SCFTs have equal central charges, the relation (1.13) implies that dual SCFTs must also have equal $L_{h\ \mathrm{min}}$ and $\dim\mathfrak{h}_{qu}$; these new non-trivial tests of supersymmetric dualities were checked in [5] for several specific cases, validating well-known duality conjectures. A few examples of the applications of these tests can be found also in subsection 3.3.1 below. We emphasize that these tests are independent of 't Hooft anomaly matchings (see [5] for a more detailed discussion).

Another application of (1.13) is to holography. For the specific case of the SU(N) $\mathcal{N}=4$ SYM, we have $c-a=L_{h\ \mathrm{min}}=0$ and $\dim\mathfrak{h}_{qu}=N-1$. This means that the asymptotic growth of the index of this SCFT is power-law. Producing this power-law asymptotics from the holographic dual seems to require the state counting of supersymmetric (more precisely, 1/16-BPS) giant gravitons [9]; this appears to be a very interesting objective within reach of current technology. See subsection 3.3.2 for a more detailed discussion.

1.2.1 Relation to previous work

Only two previous works attacked the problem of the high-temperature asymptotics of the 4d superconformal index of rather general Lagrangian SCFTs. (Other papers have considered this problem in special theories; see subsection 1.2 of [5] for references to such papers.)

In the 2013 work of Aharony et. al. [12] an EHI-type expression for the index was considered. Then, assuming that at high temperatures the integrand of the EHI is localized around the unit element of the gauge group, the relation $\mathcal{I}(\beta)\approx e^{-\mathcal{E}_0^{DK}(\beta)}Z_{S^3}$ (or more precisely, an equivariant generalization thereof) was arrived at; Z_{S^3} stands for the three-sphere partition function of the dimensionally reduced daughter of the 4d SCFT (see subsection 3.1.3 below for a matrix-integral expression for Z_{S^3}). The authors of [12] pointed out, however, that the result can not be trusted in general, as Z_{S^3} may be divergent (as the cut-off of the matrix-integral computing it is taken to infinity) due to an unlifted Coulomb branch in the 3d theory; see [18] for an explicit discussion of unlifted Coulomb branches.

In the 2014 work of Di Pietro and Komargodski [11] no explicit form for the index was assumed. But it was assumed that the 4d SCFT is Lagrangian, and that Z_{S^3} is at most power-law divergent with respect to the cut-off ($\propto 1/\beta$) of the high-temperature effective field theory describing the massless sector of the circle-compactified theory living on S^3. It was then intuitively argued that such power-law divergences would modify the asymptotics $\mathcal{I}(\beta)\approx e^{-\mathcal{E}_0^{DK}(\beta)}$ only at the (generically) subleading order in a small-β expansion, such that $\mathcal{I}(\beta)\approx(\frac{1}{\beta})^{n_m}e^{-\mathcal{E}_0^{DK}(\beta)}$, with n_m related to the number of "unlifted moduli".

In the present dissertation (following [5]) we show that Rains's rigorous machinery in [10] can be adapted for a definitive general analysis of the high-temperature asymptotics of the superconformal indices of non-chiral unitary 4d Lagrangian SCFTs. We derive results that clarify the following points:

- [explicit study of various examples leads to the conjecture that] in theories where Z_{S^3} is power-law divergent, the (generically) subleading power-law asymptotics of $\mathcal{I}(\beta)$ can be most nicely associated with a "Coulomb branch" picture *in the crossed channel* (see subsection 3.3.3);

- in some of the most interesting SCFTs (more specifically, in certain interacting $\mathcal{N}=1$ SCFTs with $c<a$), Z_{S^3} is *exponentially divergent*, and as a result even the leading asymptotics $\mathcal{I}(\beta)\approx e^{-\mathcal{E}_0^{DK}(\beta)}$ receives a modification, with the correct asymptotics reading $\mathcal{I}(\beta)\approx e^{-(\mathcal{E}_0^{DK}(\beta)+V_{\min}^{\text{eff}})}$ (see section 3.1, and subsections 3.2.5 and 3.2.6).

1.3 Taking the large-N limit of the index first

In holography, or more specifically in the AdS/CFT correspondence, the large-N limit of gauge theories plays an important role. We will focus on a certain class of holographic SCFTs when discussing the large-N limit; these are SCFTs arising from toric quiver gauges theories. One of their important features is that they are dual to IIB string theory on $\mathrm{AdS}_5\times\mathrm{SE}_5$, with SE_5 a *toric* Sasaki-Einstein 5-manifold.

Taking the large-N limit of the index of these theories one obtains the *multi-trace index* of the SCFT [19]; this is the index of the multi-trace operators of the SCFT in the planar limit. This index is holographically dual to the multi-particle index of the gravity side; the multi-particle index receives contributions from multi-particle Kaluza-Klein (KK) states in the bulk. The multi-particle index can be related through simple combinatorial procedures (namely via *plethystic* exponentials/logarithms [20]) to the single-particle index of the gravity theory, which receives contributions only from the bulk single-particle KK states.

In a series of papers written by Jim Liu, Phil Szepietowski, and the author, it was discovered that the high-temperature asymptotics of the single-particle index encodes the bulk KK fields' contribution to the *subleading holographic Weyl anomaly* [21, 22, 23]. The problem of holographic Weyl anomaly is to reproduce the central charges a and c of a holographic SCFT from its gravitational dual[10]. In a large-N expansion, the leading ($O(N^2)$) piece of the central charges can be holographically obtained using Einstein gravity on the AdS side; this was done around 1998 [24]. Obtaining the subleading ($O(1)$) piece of the central charges from the gravity side was more challenging, until the relation with the superconformal index was understood [21, 22, 23].

The holographic connection between the subleading central charges and the single-particle index is derived roughly as follows. First of all, long multiplets of $\mathrm{SU}(2,2|1)$ in the bulk KK spectrum do not contribute to either the single-particle index or the holographic central charges. Next, for short multiplets, *irrespective of the type* (which could be chiral, anti-chiral, conserved, semi-long I, or semi-long II), the holographic contribution to the central charges takes a simple form, determined by the high-temperature asymptotics of the contribution of the multiplet to the single-particle index. Summing up the contributions of all the KK particles in the bulk, one concludes that the high-temperature asymptotics of the single-particle index is related to the subleading holographic Weyl anomaly. This relation will be discussed further in Chapter 4; there we will explain how the relation leads to a solution to the problem of Holographic Weyl Anomaly in toric quiver SCFTs.

1.4 Overview of the publications this dissertation is based on

- A. A. Ardehali, J. T. Liu, and P. Szepietowski, *$c - a$ from the $\mathcal{N} = 1$ superconformal index,* JHEP **1412**, 145 (2014) [arXiv:1407.6024 [hep-th]]. (Listed as reference [21].)

  This work established a holographic relation between the difference of the central charges (i.e. $c - a$) and the single-particle index, in the context of 4d SCFTs dual to IIB theory on $\mathrm{AdS}_5\times\mathrm{SE}_5$ (with SE_5 a Sasaki-Einstein 5-manifold). The relation was then checked explicitly for toric quiver SCFTs (with $\mathrm{SU}(N)$ nodes) without adjoint matter and with a smooth dual SE_5; this successful check can be considered a test of AdS/CFT at the subleading order (in $1/N$) for an infinite class of holographic SCFTs.

  The paper also conjectured the holographically derived relation between $c - a$ and the index to hold for all (not necessarily holographic) 4d SCFTs; this conjecture was ruled out later in [5].

[10] The expression “Weyl anomaly” is used because the central charges determine, among other things, the anomalous behavior of the SCFT partition function under Weyl re-scalings of the spacetime metric; see e.g. [24].

- A. A. Ardehali, J. T. Liu, and P. Szepietowski, *Central charges from the $\mathcal{N} = 1$ superconformal index,* Phys. Rev. Lett. **114**, 091603 (2015) [arXiv:1411.5028 [hep-th]]. (Listed as reference [22].)

  This work extended the holographic result of the previous paper to expressions for the $O(N^0)$ pieces of a and c separately. (Note that for SCFTs dual to $\mathrm{AdS}_5\times\mathrm{SE}_5$ we always have $c - a = O(N^0)$.) The relations were then explicitly checked for toric quiver SCFTs (with $\mathrm{SU}(N)$ nodes) without adjoint matter and dual to smooth SE_5; this check constitutes a very strong and general test of AdS/CFT at the subleading order in the $1/N$ expansion.

  The paper also presented general conjectures for extracting the central charges of any (finite rank, not necessarily holographic) 4d SCFT from its index, but those conjectures were later ruled out in [5].

- A. A. Ardehali, J. T. Liu, and P. Szepietowski, *High-temperature expansion of supersymmetric partition functions,* JHEP **1507**, 113 (2015) [arXiv:1502.07737 [hep-th]]. (Listed as reference [23].)

  This work generalized the above-mentioned AdS/CFT matching of the subleading central charges to all toric quivers with $\mathrm{SU}(N)$ nodes (even to quivers with adjoint matter fields and/or with singular dual SE_5; there were two extra assumptions made though, as explained in Chapter 4 below).

  This paper contains also the first correct calculation of the SUSY Casimir energy in the literature; it thereby clarified the connection between the 4d superconformal index, and its corresponding SUSY partition function computed by path-integration over $S_b^3 \times S_\beta^1$ (with S_b^3 the unit three-sphere with squashing parameter b, and β the circumference of the circle). This result appeared shortly afterwards also in the independent work of Assel et. al. [25].

  The paper [23] also proposed a conjecture for the high-temperature asymptotics of the indices of general (finite-rank) 4d SCFTs; that conjecture was ruled out later in [5].

- A. A. Ardehali, *High-temperature asymptotics of supersymetric partition functions,* [arXiv:1512.03376 [hep-th]]. (Listed as reference [5].)

  This paper extended Rains's analysis [10] to study the high- (and low-) temperature asymptotics of the index of Lagrangian SCFTs with a semi-simple gauge group (under some extra simplifying assumptions spelled out at the beginning of the Discussion section in [5]).

1.5 Novel results

There are three previously unpublished results in the present dissertation.

The first is an improved derivation (compared to the original one in [5]) of the asymptotics of the indices of *non-chiral* SCFTs. This derivation is given in Chapter 3, and leads to Eq. (3.24), which is our main result. The original derivation (reported in [5]) of Eq. (3.24) was based on the physically expected—but mathematically unjustified—assumption that certain cancelations do not occur in the high-temperature limit of the EHIs arising from SUSY gauge theories (see the comments below Eq. (3.15) of [5]).

The second previously unpublished result is the asymptotics, shown in (3.66), of the index of the puncture-less SU(2) class-$\mathcal{S}$ theories of genus $g \geq 2$; the result is interesting: these $\mathcal{N}=2$ SCFTs satisfy the Di Pietro-Komargodski formula, even though they famously have the unusual balance $c < a$ between their central charges. These theories are thus to be contrasted with the $\mathcal{N}=1$ SCFTs with $c < a$ discussed in subsections 3.2.5 and 3.2.6, which do not satisfy the Di Pietro-Komargodski formula.

The third novel result is the relation between the high-temperature asymptotics of the single-trace and multi-trace indices, shown in (D.9), and its corollary in Eq. (D.14). Part of the relation (D.14) was given as an ansatz in [23]; we not only prove that ansatz in appendix D, but also derive a piece of it that was left undetermined in [23].

Chapter 2

The 4d superconformal index

The goal of this chapter is to write down—and to explain—the explicit expression for the elliptic hypergeometric integral (EHI) whose high-temperature asymptotics we will analyze (under certain simplifying conditions) in the next chapter. This expression can be found in Eq. (2.12) below.

In the physical context, the EHI in Eq. (2.12) may arise as the superconformal index of a 4d Lagrangian SCFT. Elaborating on the physical context is the purpose of the following two sections. The reader not interested in—or already familiar with—this physical context can skip directly to the third section below (i.e. section 2.3) where the EHI of our interest is spelled out.

2.1 Background: The building blocks of a unitary 4d Lagrangian SCFT

The Hilbert space of a 4d SCFT is invariant under the action of the 4d $\mathcal{N}=1$ superconformal group $\mathrm{SU}(2,2|1)$. The generators of this group constitute the 4d superconformal algebra.

The bosonic part of the 4d superconformal algebra consists of the 4d conformal algebra and a U(1) automorphism referred to as the $\mathrm{U}(1)_R$. We denote the charge of a state under H (the generator of dilations, which in the radial quantization becomes the Hamiltonian) by E, the charge under R (the generator of the $\mathrm{U}(1)_R$) by r, and the charges under J_1^z and J_2^z (the Cartan generators of the left and right SU(2) spins of the Lorentz group) by j_1 and j_2. All these charges are real numbers, j_1 and j_2 are half-integers, and unitarity implies $E \geq 0$.

The fermionic part the 4d superconformal algebra consists of the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$, and their conformal partners S^α, $\bar{S}^{\dot{\alpha}}$. Importantly, we have $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}$, and

$\{S^\alpha, \bar{S}^{\dot\alpha}\} = 2K^{\alpha\dot\alpha}$, with P and K respectively the generators of translations and special conformal transformations.

A computationally efficient description of a Lagrangian SCFT is provided, however, not through the Hilbert space perspective, but by the field content of the gauge theory that flows to it. The field content of a supersymmetric gauge theory is organized inside supermultiplets. Focusing on interacting unitary 4d Lagrangian SCFTs with fields of spin ≤ 1, we are left with two possible supermultiplets: chiral multiplets and vector multiplets.

A chiral multiplet consists of a complex scalar and a Weyl fermion, whereas a vector multiplet consists of a vector boson and a Weyl fermion. (Note that since we are interested in SCFTs, we are restricting our attention to QFTs with massless field content.)

The scalar inside a chiral multiplet χ has an R-charge that we denote by r_χ; the R-charge of the supersymmetric partner (the Weyl fermion in the same multiplet) is $r_\chi - 1$. On the other hand, a vector boson has zero R-charge, and its superpartner (the Weyl fermion in the same multiplet, referred to as the gaugino) has R-charge 1.

The interaction of massless vector bosons is described by a gauge theory. This means, among other things, that the vector boson field transforms in the adjoint representation of a gauge group G (which we take to be a compact matrix Lie group with a semi-simple algebra). A chiral multiplet χ in the theory may transform in a representation $\mathcal{R}_\chi$ of the gauge group G.

With the above background in mind, and for our purposes below, **we take the following as the defining data of a unitary 4d Lagrangian SCFT**: i) a *gauge group* G, which we take to be a compact semi-simple matrix Lie group of rank r_G, denote its typical root vector by $\alpha := (\alpha_1, \ldots, \alpha_{r_G})$, and denote the set of all the roots by Δ_G; ii) a finite number of *chiral multiplets* $\chi_j := \{\mathcal{R}_j, r_j\}$, with $j = 1, \ldots, n_\chi$, where $\mathcal{R}_j$ is a finite-dimensional irreducible representation of G, whose typical weight vector we denote by $\rho^j := (\rho^j_1, \ldots, \rho^j_{r_G})$, and the set of all the weights of $\mathcal{R}_j$ we denote by Δ_j, while r_j ($\in]0, 2[$) is the *R-charge* of the chiral multiplet χ_j.

We further demand that the following *anomaly cancelation* conditions be satisfied by the $\mathcal{R}_j$ and the r_j:

$$\sum_j \sum_{\rho^j \in \Delta_j} \rho^j_l \rho^j_m \rho^j_n = 0, \quad \text{(for all } l, m, n) \tag{2.1}$$

$$\sum_j \sum_{\rho^j \in \Delta_j} \rho^j_l = 0, \quad \text{(for all } l) \tag{2.2}$$

$$\sum_j (r_j - 1) \sum_{\rho^j \in \Delta_j} \rho^j_l \rho^j_m + \sum_{\alpha \in \Delta_G} \alpha_l \alpha_m = 0, \quad \text{(for all } l, m) \tag{2.3}$$

$$\sum_j (r_j-1)^2 \sum_{\rho^j\in\Delta_j} \rho_l^j = 0 \quad \text{(for all } l\text{)}. \tag{2.4}$$

These relations correspond respectively to cancelation of the following anomalies: *i*) the gauge^3 anomaly; *ii*) the gauge-gravitational-gravitational anomaly; *iii*) the $\mathrm{U}(1)_R$-gauge-gauge anomaly; and *iv*) the gauge-$\mathrm{U}(1)_R$-$\mathrm{U}(1)_R$ anomaly.

Note that it would be more appropriate to say that the above data defines 'a SUSY gauge theory with a U(1) R-symmetry', and not necessarily an SCFT. In particular, the above conditions on the data do not guarantee that the r_χ are the *superconformal* R-charges of the chiral multiplets in the IR fixed point of the SUSY gauge theory defined by the above data; for instance, the $\mathrm{SU}(N_c)$ SQCD with R-charge assignment $r_\chi = 1 - N_c/N_f$, for $N_f > N_c$ but outside the conformal window, *does* satisfy the above conditions, even though its IR fixed point is free, with emergent accidental symmetries mixing with its $\mathrm{U}(1)_R$ in the infrared. Therefore we keep in mind that only a subset of the SUSY gauge theories defined by the above data lead to SCFTs with the chiral multiplets in the IR having superconformal R-charges r_χ. On the other hand, any SUSY gauge theory with U(1) R-symmetry—as defined by the above data—can be assigned an EHI via Eq. (2.12) below; for non-conformal theories the resulting EHI can be thought of as arising from path-integration (c.f. [6]), rather than from a "superconformal" index calculation.

2.2 Definition of the index

The superconformal index is defined as

$$\mathcal{I}(b,\beta) = \mathrm{Tr}\left[(-1)^F e^{-\hat{\beta}(E-2j_2-\frac{3}{2}r)} p^{j_1+j_2+\frac{1}{2}r} q^{-j_1+j_2+\frac{1}{2}r}\right], \tag{2.5}$$

with $p = e^{-b\beta}$ and $q = e^{-b^{-1}\beta}$; we take $b, \beta > 0$, and refer to b as the *squashing parameter*, and β as the *inverse temperature* (the reason for these names will become clear shortly); the special case with $b = 1$ corresponds to the index introduced in Chapter 1. The trace in the above relation is over the Hilbert space of the theory on $S^3 \times \mathbb{R}$, with S^3 the round unit three-sphere, and $\mathbb{R}$ the time direction. The index is independent of $\hat{\beta}$ because it only receives uncanceled contributions from states with $E - 2j_2 - \frac{3}{2}r = 0$. In a superconformal theory, these states correspond to operators that sit in short representations of the superconformal algebra. The index of an SCFT thus encodes exact (non-perturbative) information about the operator spectrum of the underlying theory.

The exponents of p and q correspond to operators that commute with the supercharge used in the definition of the index: the expression $E - 2j_2 - \frac{3}{2}r$ is Q-exact for a particular supercharge Q, and the combinations $J_1^z + J_2^z + R/2$ and $-J_1^z + J_1^z + R/2$ both commute with that Q. Therefore p and q refine the Witten index $\mathcal{I}^W = \mathrm{Tr}[(-1)^F e^{-\hat{\beta}(E-2j_2-\frac{3}{2}r)}]$ without ruining the cancelations underlying its topological character. In fact without refinement with p and q, the index $\mathcal{I}^W$ is often divergent, and thus p and q are necessary as regulators.

2.3 Evaluation of the index

There are two ways to compute the index of a Lagrangian SCFT. The Hamiltonian route goes through the so-called Romelsberger prescription [26]. The Lagrangian route uses the supersymmetric localization of the path-integral on $S_b^3 \times S_\beta^1$, where S_b^3 is the unit three-sphere with squashing parameter $b > 0$, and $\beta > 0$ is the circumference of the circle [6].

Originally, the indices of Lagrangian SCFTs were computed using the Romelsberger prescription; see for instance the work of Dolan and Osborn from 2008 [8]. Later on, supersymmetric localization caught up, and not only reproduced the correct expression for the index, but also gave an extra Casimir-type factor which is of physical significance; see [27] for the localization computation for the $\mathcal{N} = 4$ theory, and the 2014 paper of Assel et. al. [6] for the result for the case with more general matter content (the correct evaluation of the Casimir-type factor was done later in [23] and [25]).

In the present section we evaluate the index of a general unitary Lagrangian 4d SCFT (defined as in section 2.1) using the Romelsberger prescription (see [26, 8]). According to the prescription, one starts with adding up the *single-letter* indices of various multiplets, and then plethystically exponentiates the result. To project onto the gauge-singlet sector though, one should *i*) make the single-letter indices character-valued, and *ii*) integrate the result of the plethystic exponentiation against the Haar measure of the gauge group.

A chiral multiplet $\chi = (\phi_r, \psi_{r-1})$, along with its CP-conjugate multiplet $\bar{\chi} = (\bar{\phi}_{-r}, \bar{\psi}_{-r+1})$, contributes

$$i_\chi(z; p, q) = \sum_{\rho^\chi \in \Delta_\chi} \frac{(pq)^{r_\chi/2} z^{\rho^\chi} - (pq)^{1-r_\chi/2} z^{-\rho^\chi}}{(1-p)(1-q)}, \tag{2.6}$$

to the total single-letter index. Recall that the set Δ_χ consists of as many weights ρ^χ as the dimension of the representation $\mathcal{R}_\chi$. Also, our symbolic notation z^{ρ^χ} should be understood as $z_1^{\rho_1^\chi} \times \cdots \times z_{r_G}^{\rho_{r_G}^\chi}$, where $\rho^\chi \equiv (\rho_1^\chi, \ldots, \rho_{r_G}^\chi)$, with r_G the rank of the gauge group.

The first term in the numerator of (2.6) is the contribution $(pq)^{r_\chi/2}$ that ϕ_r makes to the index, multiplied by the character $\sum_{\rho^\chi\in\Delta_\chi} z^{\rho^\chi}$ of the representation $\mathcal{R}_\chi$ of G under which χ transforms. The second term in the numerator of (2.6) is the contribution $(pq)^{1-r_\chi/2}$ of $\bar{\psi}_{-r+1}$ to the index, multiplied by the character of the representation $\bar{\mathcal{R}}_\chi$ of G under which $\bar{\chi}$ transforms. The denominator of (2.6) comes from summing up the geometric series arising from adding the contributions of the conformal descendants of ϕ_r and $\bar{\psi}_{-r+1}$ (see section 2 of [8] for the details).

The plethystic exponential of $i_\chi(z;p,q)$ is given by a product of several elliptic gamma functions:

$$\mathcal{I}_\chi(z;p,q) := \exp(\sum_{n=1}^{\infty} \frac{i_\chi(z^n;p^n,q^n)}{n}) = \prod_{\rho^\chi\in\Delta_\chi} \Gamma((pq)^{r_\chi/2} z^{\rho^\chi}). \tag{2.7}$$

The elliptic gamma function $\Gamma(*)$ is a special function explained in appendix A.

The vector multiplets in the theory contribute to the total single-letter index as

$$\begin{aligned} i_v(z;p,q) &= \left(-\frac{p}{(1-p)(1-q)} - \frac{q}{1-q} + \frac{pq}{(1-p)(1-q)}\right)[r_G + \sum_{\alpha_+}(z^{\alpha_+} + z^{-\alpha_+})] \\ &= \frac{2pq-p-q}{(1-p)(1-q)}[r_G + \sum_{\alpha_+}(z^{\alpha_+} + z^{-\alpha_+})]. \end{aligned} \tag{2.8}$$

The α_+ are the positive roots of G. By z^{α_+} we mean $z_1^{\alpha_1} \times \cdots \times z_{r_G}^{\alpha_{r_G}}$, where $\alpha_+ \equiv (\alpha_1, \ldots, \alpha_{r_G})$.

Inside the brackets on the RHS of the first line of (2.8) we have the character of the adjoint representation of G. Inside the parentheses on the RHS of the first line of (2.8) we have respectively the contribution of the first gaugino, the second gaugino, and the gauge field, along with their conformal descendants; the p-descendants of the second gaugino are not taken into account because the equation of motion relates them to the q-descendants of the first gaugino (see section 2 of [8] for the details).

The plethystic exponential of $i_v(z;p,q)$ yields a product of Pochhammer symbols and elliptic gamma functions:

$$\mathcal{I}_v(z;p,q) := \exp(\sum_{n=1}^{\infty} \frac{i_v(z^n;p^n,q^n)}{n}) = \frac{(p;p)^{r_G}(q;q)^{r_G}}{\prod_{\alpha_+}(1-z^{+\alpha_+})(1-z^{-\alpha_+})\Gamma(z^{\pm\alpha_+})}. \tag{2.9}$$

The *Pochhammer symbol* $(*;*)$ is a special function explained in appendix A.

Multiplying the contribution of the various chiral multiplets $\prod_\chi \mathcal{I}_\chi(z;p,q)$ by the contribution of the vector multiplet(s) $\mathcal{I}_v(z;p,q)$ we obtain [alternatively we could have summed up the character-valued single-letter indices of various multiplets, and then plethystically exponentiated the result]

$$\mathcal{I}(z;p,q) = (p;p)^{r_G}(q;q)^{r_G}\frac{\prod_\chi \prod_{\rho^\chi\in\Delta_\chi}\Gamma((pq)^{r_\chi/2}z^{\rho^\chi})}{\prod_{\alpha_+}(1-z^{+\alpha_+})(1-z^{-\alpha_+})\Gamma(z^{\pm\alpha_+})}. \tag{2.10}$$

The above index receives contributions from non-gauge-invariant operators. By integrating it against the Haar measure of the gauge group

$$\mathrm{d}\mu = \frac{1}{|W|}\mathrm{d}^{r_G}x\prod_{\alpha_+}(1-z^{+\alpha_+})(1-z^{-\alpha_+}), \tag{2.11}$$

we arrive at the contribution of only the gauge-singlet sector. On the RHS of the above relation, $|W|$ is the order of the Weyl group of G, and $z_j = e^{2\pi i x_j}$.

The end result is the following elliptic hypergeometric integral [for comparison with [10] note that $\omega_{1\ \mathrm{there}} = ib_{\mathrm{here}}$, $\omega_{2\ \mathrm{there}} = ib_{\mathrm{here}}^{-1}$, and $v_{\mathrm{there}} = \frac{\beta_{\mathrm{here}}}{2\pi}$]:

$$\boxed{\mathcal{I}(b,\beta) = \frac{(p;p)^{r_G}(q;q)^{r_G}}{|W|}\int \mathrm{d}^{r_G}x\frac{\prod_\chi \prod_{\rho^\chi\in\Delta_\chi}\Gamma((pq)^{r_\chi/2}z^{\rho^\chi})}{\prod_{\alpha_+}\Gamma(z^{\pm\alpha_+})}.} \tag{2.12}$$

The integral is over the unit hypercube $x_j \in [-1/2, 1/2]$ in the Cartan subalgebra (or alternatively, over the maximal torus of G in the space of z_j).

Since the expression in Eq. (2.12) might seem a bit complicated, let us specialize it to a very simple case: the SU(2) SQCD with three flavors. The gauge group SU(2) has rank $r_G = 1$. The Weyl group of SU(N) is the permutation group of N elements, so it has order $N!$, which for SU(2) becomes 2. We have three chiral quark multiplets with $\rho_1^{\chi_1}, \rho_1^{\chi_2}, \rho_1^{\chi_3} = \pm 1$, and three chiral anti-quark multiplets with $\rho_1^{\chi_4}, \rho_1^{\chi_5}, \rho_1^{\chi_6} = \mp 1$ (each of the chiral multiplets has two weights (± 1), because they sit in two-dimensional representations of the gauge group). All the chiral multiplets have R-charge $r_\chi = 1/3$. Finally, the group SU(2) has two roots, corresponding to the raising and lowering operators of the 3d angular momentum, and the positive root (the raising operator) has $\alpha_+ = 2$. All in all, we get for this simple example

$$\mathcal{I}_{N_c=2,N_f=3}(b,\beta) = \frac{(p;p)(q;q)}{2}\int_{-1/2}^{1/2}\mathrm{d}x\frac{\Gamma^6((pq)^{1/6}z^{\pm 1})}{\Gamma(z^{\pm 2})}. \tag{2.13}$$

Many explicit expressions for the index $\mathcal{I}(b,\beta)$ of specific 4d SCFTs can be found in [14, 15, 5]. A few specific examples will be spelled out in the next chapter as well.

Miscellaneous remarks

To further clarify the notation we are using for the roots and weights, we add that with our notation the three-dimensional representation of SU(3) has weights $(\rho_1, \rho_2) = (1,0), (0,1), (-1,-1)$, and the positive roots of SU(3) are $\alpha_+ = (1,-1), (2,1), (1,2)$.

If a Lagrangian 4d SCFT has emergent accidental symmetries mixing with its ultraviolet $\mathrm{U}(1)_R$ to give the superconformal $\mathrm{U}(1)_R$ in the infrared, the Romelsberger prescription can not be applied to it. For such SCFTs, the EHI in (2.12) can be interpreted as arising from path-integration of the UV gauge theory using the ultraviolet $\mathrm{U}(1)_R$, but the EHI would not coincide with the superconformal index of the IR SCFT. In this dissertation we do not discuss the superconformal index of such SCFTs.

The EHIs studied by Rains in [10] correspond to the $\mathrm{Sp}(2N)$ and $\mathrm{SU}(N)$ supersymmetric quantum chromodynamics theories [8]. (Note that for simplicity we are focusing on the special case where all the u_r and v_r in [10] are equal.)

For a mathematically oriented introduction to the EHIs studied in [10] see [29].

Chapter 3

High-temperature asymptotics of the index of non-chiral theories

We now focus on *non-chiral* SCFTs: those in which nonzero ρ^χ come in pairs with opposite signs. With this restriction, the hyperbolic limit of the EHI shown in (2.12) can be analyzed completely reliably, as described below.

3.1 General analysis

3.1.1 Step 1: simplifying the EHI to an ordinary integral

The high-temperature asymptotics of the index (2.12) of a non-chiral SCFT is found as follows. Using (B.4), the Pochhammer symbols in the prefactor of (2.12) can be immediately replaced with their asymptotic expressions. We have

$$(p;p)^{r_G}(q;q)^{r_G} \simeq e^{-\pi^2(b+b^{-1})r_G/6\beta} \times \left(\frac{2\pi}{\beta}\right)^{r_G} \times e^{\beta(b+b^{-1})r_G/24}, \qquad (\text{as } \beta \to 0) \tag{3.1}$$

with the symbol $\simeq$ as defined in appendix B.

The asymptotics of the integrand of (2.12) can be obtained from the estimates in (B.8). With the aid of (3.1) and (B.8) we find the $\beta \to 0$ asymptotics of $\mathcal{I}$ as[1]

$$\mathcal{I}(b,\beta) \simeq \frac{1}{|W|}\left(\frac{2\pi}{\beta}\right)^{r_G} e^{-\mathcal{E}_0^{DK}(b,\beta)} W_0(b) e^{\beta E_{\text{susy}}(b)} \int_{\mathfrak{h}_{cl}} \mathrm{d}^{r_G}x \; e^{-V^{\text{eff}}(\mathbf{x};b,\beta)} W(\mathbf{x};b,\beta), \tag{3.2}$$

[1]Compared to the expression in (3.9) of [5], the RHS of (3.2) lacks a phase $i\Theta$ in the exponent because (as explained in [5]) in non-chiral theories $\Theta = 0$. Also, the RHS of (3.2) has the extra factors $1/|W|$, $W_0(b)$, $e^{\beta E_{\text{susy}}(b)}$, and $W(\mathbf{x};b,\beta)$ which were absent in [5]; these arise here because in the analysis below we are using estimates that are stronger than the estimates used in [5].

with $\mathfrak{h}_{cl}$—which in the path-integral picture can be interpreted [5] as the "classical" moduli-space of the holonomies around S^1_β—denoting the unit hypercube $x_i \in [-1/2, 1/2]$, and with

$$\mathcal{E}_0^{DK}(b,\beta) = \frac{\pi^2}{3\beta}(\frac{b+b^{-1}}{2})\mathrm{Tr}R, \tag{3.3}$$

$$V^{\mathrm{eff}}(\mathbf{x}; b, \beta) = \frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_h(\mathbf{x}), \tag{3.4}$$

$$E_{\mathrm{susy}}(b) = \frac{1}{6}(\frac{b+b^{-1}}{2})^3\mathrm{Tr}R^3 - (\frac{b+b^{-1}}{2})(\frac{b^2+b^{-2}}{24})\mathrm{Tr}R. \tag{3.5}$$

The 't Hooft anomalies in the above relations are given by

$$\begin{aligned} \mathrm{Tr}R &:= \mathrm{dim}G + \sum_\chi (r_\chi - 1)\mathrm{dim}\mathcal{R}_\chi = -16(c-a), \\ \mathrm{Tr}R^3 &:= \mathrm{dim}G + \sum_\chi (r_\chi - 1)^3\mathrm{dim}\mathcal{R}_\chi = \frac{16}{9}(5a-3c). \end{aligned} \tag{3.6}$$

We have also defined $W_0(b)$, and the *real* functions $L_h(\mathbf{x})$ and $W(\mathbf{x}; b, \beta)$ via

$$L_h(\mathbf{x}) := \frac{1}{2}\sum_\chi (1-r_\chi)\sum_{\rho^\chi\in\Delta_\chi}\vartheta(\langle\rho^\chi\cdot\mathbf{x}\rangle) - \sum_{\alpha_+}\vartheta(\langle\alpha_+\cdot\mathbf{x}\rangle), \tag{3.7}$$

$$W_0(b) = \prod_\chi \prod_{\rho^\chi=0}\Gamma_h(r_\chi\omega), \tag{3.8}$$

$$W(\mathbf{x}; b, \beta) = \prod_\chi\prod_{\rho^\chi_+}\frac{\psi_b(-\frac{2\pi i}{\beta}\{\langle\rho^\chi_+\cdot\mathbf{x}\rangle\} + (r_\chi - 1)\frac{b+b^{-1}}{2})}{\psi_b(-\frac{2\pi i}{\beta}\{\langle\rho^\chi_+\cdot\mathbf{x}\rangle\} - (r_\chi - 1)\frac{b+b^{-1}}{2})}\prod_{\alpha_+}\frac{\psi_b(-\frac{2\pi i}{\beta}\{\langle\alpha_+\cdot\mathbf{x}\rangle\} + \frac{b+b^{-1}}{2})}{\psi_b(-\frac{2\pi i}{\beta}\{\langle\alpha_+\cdot\mathbf{x}\rangle\} - \frac{b+b^{-1}}{2})}. \tag{3.9}$$

The function $\vartheta(x)$ in (3.7) is defined as $\vartheta(x) := \{x\}(1-\{x\})$ (with the fractional part function defined as $\{x\} := x - \lfloor x\rfloor$). In (3.8), the second product is over the zero weights of $\mathcal{R}_\chi$ (the adjoint representation, for instance, has r_G such weights), and ω is defined as $\omega := i(b+b^{-1})/2$. The ρ^χ_+ in (3.9) denote the positive weights of $\mathcal{R}_\chi$. The *non-compact quantum dilogarithm* $\psi_b(*)$ and the *hyperbolic gamma* $\Gamma_h(*)$ are special functions explained in appendix A.

That $L_h(\mathbf{x})$ is real should be obvious from the definition of $\vartheta(x)$; that $W(\mathbf{x}; b, \beta)$ is real follows from (A.6) and (A.7).

Note that in (3.2) we are claiming that *the matrix-integral is approximated well with the* *integral of its approximate integrand.* This is true because the estimates we have used inside the integrand are *i*) uniform, and *ii*) accurate up to exponentially small corrections of the type $e^{-1/\beta}$; these two strong conditions—on the integrand estimates—were not satisfied in the treatment of [5].

Now, from (A.6) it follows that $W_0(b)$ is a real number; it is moreover nonzero and finite, as we are assuming $r_\chi \in]0,2[$ (the zeros and poles of the hyperbolic gamma function are described in appendix A). We would thus make an $O(\beta^0)$ error in the asymptotics of $\ln \mathcal{I}(b,\beta)$ by setting $W_0(b)$, along with $|W|$ and $e^{\beta E_{\text{susy}}(b)}$, to unity. In other words,

$$\mathcal{I}(b,\beta) \approx \left(\frac{2\pi}{\beta}\right)^{r_G} e^{-\mathcal{E}_0^{DK}(b,\beta)} \int_{\mathfrak{h}_{cl}} \mathrm{d}^{r_G}x\; e^{-V^{\text{eff}}(\mathbf{x};b,\beta)} W(\mathbf{x};b,\beta), \tag{3.10}$$

with an $O(\beta^0)$ error upon taking the logarithm of the two sides.

We are hence left with the asymptotic analysis of the integral $\int_{\mathfrak{h}_{cl}} e^{-V} W$. From here, standard methods of asymptotic analysis can be employed.

3.1.2 Step 2: asymptotic analysis of the simplified integral

Before continuing our asymptotic analysis further, we note that the star of our show, the real function L_h which determines the effective potential[2] $V^{\text{eff}}(\mathbf{x};b,\beta)$, is *piecewise linear*; the quadratic terms in it cancel because of the ABJ $\mathrm{U}(1)_R$-gauge-gauge anomaly cancelation:

$$\frac{\partial^2 L_h(\mathbf{x})}{\partial x_i \partial x_j} = \sum_\chi (r_\chi - 1) \sum_{\rho^\chi \in \Delta_\chi} \rho_i^\chi \rho_j^\chi + \sum_\alpha \alpha_i \alpha_j = 0. \tag{3.11}$$

Also, L_h is continuous, is even under $\mathbf{x} \to -\mathbf{x}$, and vanishes at $\mathbf{x} = 0$; these properties follow from the properties of the function $\vartheta(x)$ defined above. We refer to $L_h(\mathbf{x})$ as the *Rains function* of the SCFT. This function has been analyzed by Rains [10] in the special cases of the elliptic hypergeometric integrals associated to $\mathrm{SU}(N)$ and $\mathrm{Sp}(N)$ SQCD theories.

Writing V^{eff} in terms of the Rains function L_h, (3.10) simplifies to

$$\mathcal{I}(b,\beta) \approx \left(\frac{2\pi}{\beta}\right)^{r_G} e^{-\mathcal{E}_0^{DK}(b,\beta)} \int_{\mathfrak{h}_{cl}} \mathrm{d}^{r_G}x\; e^{-\frac{4\pi^2}{\beta}\left(\frac{b+b^{-1}}{2}\right) L_h(\mathbf{x})} W(\mathbf{x};b,\beta). \tag{3.12}$$

[2]Somewhat surprisingly, L_h also appears in the $n \to 1$ limit of the zero-point energy associated to nonzero spatial holonomies on $S^1 \times S^3/\mathbb{Z}_n$; c.f. Eq. (29) of the arXiv preprint of [30] (with ν, a in there set to zero). It might be possible to clarify this coincidence by analytically continuing the results of [30] (see also [31, 32]) to non-integer n, and then using modular properties of the generalized elliptic gamma functions employed in that work.

It will be useful for us to know that $W(\mathbf{x};b,\beta)$ is a positive semi-definite function of $\mathbf{x}$; this follows from (A.6) and (A.7).

To analyze the integral in (3.12), first note that the integrand is not smooth over $\mathfrak{h}_{cl}$. We hence break $\mathfrak{h}_{cl}$ into sets on which L_h is linear. These sets can be obtained as follows. Define

$$\begin{gathered}\mathcal{S}_g := \bigcup_{\alpha_+}\{\mathbf{x}\in\mathfrak{h}_{cl}|\langle\alpha_+\cdot\mathbf{x}\rangle\in\mathbb{Z}\},\qquad \mathcal{S}_\chi := \bigcup_{\rho_+^\chi}\{\mathbf{x}\in\mathfrak{h}_{cl}|\langle\rho_+^\chi\cdot\mathbf{x}\rangle\in\mathbb{Z}\},\\ \mathcal{S} := \bigcup_\chi \mathcal{S}_\chi\cup\mathcal{S}_g.\end{gathered}\tag{3.13}$$

It should be clear that everywhere in $\mathfrak{h}_{cl}$, except on $\mathcal{S}$, the function L_h is guaranteed to be linear—and therefore smooth.

The set $\mathcal{S}$ consists of a union of codimension one affine hyperplanes inside the space of the x_i. These hyperplanes chop $\mathfrak{h}_{cl}$ into (finitely many, convex) polytopes $\mathcal{P}_n$. The integral in (3.12) then decomposes to

$$\mathcal{I}(b,\beta)\approx e^{-\mathcal{E}_0^{DK}(b,\beta)}\sum_n\left(\frac{2\pi}{\beta}\right)^{r_G}\int_{\mathcal{P}_n}\mathrm{d}^{r_G}x\; e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_h(\mathbf{x})}W(\mathbf{x};b,\beta).\tag{3.14}$$

Let $\mathcal{S}_g^{(\beta)}$ denote the set of all points in $\mathfrak{h}_{cl}$ that are at a distance less than $N_0\beta$ from $\mathcal{S}_g$, with some fixed $N_0>0$. We divide $\mathcal{P}_n$ into *i*) $\mathcal{P}_n\cap\mathcal{S}_g^{(\beta)}$, and *ii*) the rest of $\mathcal{P}_n$, which we denote by $\mathcal{P}'_n$. Now, by taking N_0 to be large enough, we can push $\mathcal{P}'_n$ away from the zeros of ψ_b, and thus make $w_i<W(\mathbf{x};b,\beta)<w_s$ over $\mathcal{P}'_n$ (with some $0<w_i$ and some $w_s<\infty$). Therefore the contribution that the nth summand in (3.14) receives from $\mathcal{P}'_n$ is well approximated (with an $O(\beta^0)$ error upon taking the logs) by

$$J_n := \left(\frac{2\pi}{\beta}\right)^{r_G}\int_{\mathcal{P}'_n}\mathrm{d}^{r_G}x\; e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_h(\mathbf{x})}.\tag{3.15}$$

Let's further replace $\mathcal{P}'_n$ in (3.15) with $\mathcal{P}_n$; we will shortly see that this replacement introduces a negligible error. We would hence like to estimate

$$I_n := \left(\frac{2\pi}{\beta}\right)^{r_G}\int_{\mathcal{P}_n}\mathrm{d}^{r_G}x\; e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_h(\mathbf{x})}.\tag{3.16}$$

Since L_h is linear on each $\mathcal{P}_n$, its minimum over $\mathcal{P}_n$ is guaranteed to be realized on $\partial\mathcal{P}_n$. Let us assume that this minimum occurs on the kth j-face of $\mathcal{P}_n$, which we denote by j_n-$\mathcal{F}_n^k$. We denote the value of L_h on this j-face by $L_{h\ \mathrm{min}}^n$. Equipped with this notation, we can write (3.16) as

$$I_n = \left(\frac{2\pi}{\beta}\right)^{r_G} e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_{h\ \min}^n} \int_{\mathcal{P}_n} \mathrm{d}^{r_G}x\ e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})\Delta L_h^n(\mathbf{x})}, \tag{3.17}$$

where $\Delta L_h^n(\mathbf{x}) := L_h(\mathbf{x}) - L_{h\ \min}^n$ is a linear function on $\mathcal{P}_n$. Note that $\Delta L_h^n(\mathbf{x})$ vanishes on j_n-$\mathcal{F}_n^k$, and it increases as we go away from j_n-$\mathcal{F}_n^k$ and into the interior of $\mathcal{P}_n$. [The last sentence, as well as the rest of the discussion leading to (3.24), would receive a trivial modification if $j_n = r_G$ (corresponding to constant L_h over $\mathcal{P}_n$).] Therefore as $\beta \to 0$, the integral in (3.17) localizes around j_n-$\mathcal{F}_n^k$.

To further simplify (3.17), we now adopt a set of new coordinates—affinely related to x_i and with unit Jacobian—that are convenient on $\mathcal{P}_n$. We pick a point on j_n-$\mathcal{F}_n^k$ as the new origin, and parameterize j_n-$\mathcal{F}_n^k$ with $\bar{x}_1, ..., \bar{x}_{j_n}$. We take x_{in} to parameterize a direction perpendicular to all the $\bar{x}$s, and to increase as we go away from j_n-$\mathcal{F}_n^k$ and into the interior of $\mathcal{P}_n$. Finally, we pick $\tilde{x}_1, ..., \tilde{x}_{r_G-j_n-1}$ to parameterize the perpendicular directions to x_{in} and the $\bar{x}$s. Note that, because ΔL_h^n is linear on $\mathcal{P}_n$, it does not depend on the $\bar{x}$s; they parameterize its flat directions. By re-scaling $\bar{x}, x_{\text{in}}, \tilde{x} \mapsto \frac{\beta}{2\pi}\bar{x}, \frac{\beta}{2\pi}x_{\text{in}}, \frac{\beta}{2\pi}\tilde{x}$, we can absorb the $(\frac{2\pi}{\beta})^{r_G}$ factor in (3.17) into the integral, and write the result as

$$I_n = \int_{\frac{2\pi}{\beta}\mathcal{P}_n} \mathrm{d}^{j_n}\bar{x}\ \mathrm{d}x_{\text{in}}\ \mathrm{d}^{r_G-j_n-1}\tilde{x}\ e^{-2\pi(\frac{b+b^{-1}}{2})\Delta L_h^n(x_{\text{in}},\tilde{\mathbf{x}})}. \tag{3.18}$$

To eliminate β from the exponent, we have used the fact that ΔL_h^n depends homogenously on the new coordinates. We are also denoting the re-scaled polytope schematically by $\frac{2\pi}{\beta}\mathcal{P}_n$.

Instead of integrating over all of $\frac{2\pi}{\beta}\mathcal{P}_n$ though, we can restrict to $x_{\text{in}} < \epsilon/\beta$ with some small $\epsilon > 0$. The reason is that the integrand of (3.18) is exponentially suppressed (as $\beta \to 0$) for $x_{\text{in}} > \epsilon/\beta$. We take $\epsilon > 0$ to be small enough such that a hyperplane at $x_{\text{in}} = \epsilon/\beta$, and parallel to j_n-$\mathcal{F}_n^k$, cuts off a prismatoid $P_{\epsilon/\beta}^n$ from $\frac{2\pi}{\beta}\mathcal{P}_n$. After restricting the integral in (3.18) to $P_{\epsilon/\beta}^n$, the integration over the $\bar{x}$s is easy to perform. The only potential difficulty is that the range of the $\bar{x}$ coordinates may depend on x_{in} and the $\tilde{x}s$. But since we are dealing with a prismatoid, the dependence is linear, and by the time the range is modified significantly (compared to its $O(1/\beta)$ size on the re-scaled j-face $\frac{2\pi}{\beta}(j_n$-$\mathcal{F}_n^k)$), the integrand is exponentially suppressed. Therefore we can neglect the dependence of the range of the $\bar{x}$s on the other coordinates in (3.18). The integral then simplifies to

$$I_n \approx \left(\frac{2\pi}{\beta}\right)^{j_n} \mathrm{vol}(j_n\text{-}\mathcal{F}_n^k) \int_{\hat{P}_{\epsilon/\beta}^n} \mathrm{d}x_{\text{in}}\ \mathrm{d}^{r_G-j_n-1}\tilde{x}\ e^{-2\pi(\frac{b+b^{-1}}{2})\Delta L_h^n(x_{\text{in}},\tilde{\mathbf{x}})}, \tag{3.19}$$

where $\hat{P}^n_{\epsilon/\beta}$ is the pyramid obtained by restricting $P^n_{\epsilon/\beta}$ to $\bar{x}_1 = ... = \bar{x}_{j_n} = 0$. The logarithms of the two sides of (3.19) differ by $O(\beta)$, with the error mainly arising from our neglect of the possible dependence of the range of the $\bar{x}$ coordinates in (3.18) on x_{in} and the $\tilde{x}s$. (Recall that the other error, arising from restricting the integral in (3.18) to $P^n_{\epsilon/\beta}$, is exponentially small.)

We now take $\epsilon \to \infty$ in (3.19). This introduces an exponentially small error, as the integrand is exponentially suppressed (as $\beta \to 0$) for $x_{\text{in}} > \epsilon/\beta$. The resulting integral is strictly positive, because it is the integral of a strictly positive function. We denote by A_n the result of the integral multiplied by vol(j_n-$\mathcal{F}^k_n$). Then I_n can be approximated as

$$I_n \approx e^{-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L^n_{h\ \min}} \left(\frac{2\pi}{\beta}\right)^{j_n} A_n. \tag{3.20}$$

We are now in a position to argue $J_n \approx I_n$. If we had integrated over $\mathcal{P}'_n$, then we would end up with an expression similar to (3.20), in which $L^n_{h\ \min}$ would be replaced with the minimum of L_h over $\mathcal{P}'_n$; but since L_h is piecewise linear, the difference between the new minimum and $L^n_{h\ \min}$ would be $O(\beta)$, which translates to an $O(\beta^0)$ multiplicative difference between J_n and I_n. Other sources of difference between J_n and I_n similarly introduce negligible error; more precisely, we have $\ln I_n = \ln J_n + O(\beta^0)$.

The dominant contribution to $\mathcal{I}(b, \beta)$ comes, of course, from the terms/polytopes whose $L^n_{h\ \min}$ is smallest. If these terms are labeled by $n = n^1_*, n^2_*, ...$, we can introduce $\mathfrak{h}_{qu}$ and dim$\mathfrak{h}_{qu}$ via

$$\mathfrak{h}_{qu} := \bigcup_{n_*} j_{n_*}\text{-}\mathcal{F}^k_{n_*}, \quad \text{dim}\mathfrak{h}_{qu} := \max(j_{n_*}). \tag{3.21}$$

Put colloquially, if $\mathfrak{h}_{qu}$ has multiple connected components, by dim$\mathfrak{h}_{qu}$ we mean the dimension of the component(s) with greatest dimension, while if a connected component consists of several intersecting flat elements inside $\mathfrak{h}_{cl}$, by its dimension we mean the dimension of the flat element(s) of maximal dimension.

Our final estimate for the contribution to $\mathcal{I}(b, \beta)$ from $\cup_n \mathcal{P}'_n$ is thus

$$Be^{-\mathcal{E}^{DK}_0(b,\beta)-\frac{4\pi^2}{\beta}(\frac{b+b^{-1}}{2})L_{h\ \min}} \left(\frac{2\pi}{\beta}\right)^{\text{dim}\mathfrak{h}_{qu}}, \tag{3.22}$$

where $L_{h\ \min} := L^{n_*}_{h\ \min}$, and B is some positive real number.

We are left with determining the contribution to $\mathcal{I}(b, \beta)$ coming from $\mathcal{S}^{(\beta)}_g$. Over $\mathcal{P}_n \cap \mathcal{S}^{(\beta)}_g$, the simple estimate $W(\mathbf{x}; b, \beta) = O(1)$ (which follows from the fact that $W(\mathbf{x}; b, \beta)$ is uniformly bounded on $\mathcal{S}^{(\beta)}_g$) suffices for our purposes; we thus learn that the contribution that the integral (3.14) receives from $\mathcal{P}_n \cap \mathcal{S}_g^{(\beta)}$ is not only positive, but also

$$O\left(\int_{\frac{2\pi}{\beta}(\mathcal{P}_n\cap\mathcal{S}_g^{(\beta)})} \mathrm{d}^{j_n}\bar{x}\ \mathrm{d}x_{\mathrm{in}}\ \mathrm{d}^{r_G-j_n-1}\tilde{x}\ e^{-2\pi(\frac{b+b^{-1}}{2})\Delta L_h^n(x_{\mathrm{in}},\tilde{\mathbf{x}})}\right). \tag{3.23}$$

Now, the argument of the O above is nothing but the difference between I_n and J_n, which we already argued to be negligible. Thus the contribution to $\mathcal{I}(b,\beta)$ coming from $\mathcal{S}_g^{(\beta)}$ is negligible.

Using the explicit expression (3.3) for $\mathcal{E}_0^{DK}(b,\beta)$, and noting that (3.22) is an accurate estimate for $\mathcal{I}(b,\beta)$ up to a multiplicative factor of order β^0, we arrive at our main result:

$$\boxed{\ln\mathcal{I}(b,\beta) = -\frac{\pi^2}{3\beta}(\frac{b+b^{-1}}{2})(\mathrm{Tr}R + 12L_{h\ \mathrm{min}}) + \dim\mathfrak{h}_{qu}\ln(\frac{2\pi}{\beta}) + O(\beta^0).} \tag{3.24}$$

3.1.3 Connection with the S^3 partition function

In this subsection we comment on the connection between the asymptotics of the index of a 4d SCFT, and the divergence of the S^3 partition function Z_{S^3} of the dimensionally reduced daughter of the 4d theory.

We will show below that the degree of divergence of Z_{S^3} (as the cut-off of the matrix-integral computing it is taken to infinity) is determined by the behavior of the Rains function L_h near the origin of $\mathfrak{h}_{cl}$. In particular

- if the origin is an isolated local minimum of L_h, then Z_{S^3} is finite;
- if the origin is part of an extended locus where L_h is locally minimized, then Z_{S^3} is power-law divergent;
- if the origin is not a local minimum of L_h, then Z_{S^3} is exponentially divergent.

Note that it is the *local* behavior of L_h near the origin that determines the degree of divergence of Z_{S^3}. On the other hand, according to (3.24), the asymptotics of the 4d index is determined by the *global* properties of L_h. Therefore, at least until theorems relating the local and global properties of L_h are established, the asymptotics of the 4d index is not as tightly connected to the divergence of Z_{S^3} as one may have wished.

For instance, *we can not say* (in absence of theorems of the kind discussed in the previous paragraph) that 'the Di Pietro-Komargodski asymptotics applies to the index if Z_{S^3} is finite'; it may happen that in a (non-chiral unitary Lagrangian) 4d SCFT (with $r_\chi \in]0,2[$) *the origin is an isolated local, but not global, minimum* of L_h; that the origin is an isolated local minimum would imply that Z_{S^3} is finite; that L_h is minimized somewhere else would imply—according to (3.24)—that the Di Pietro-Komargodski formula receives a modification.

However, *we can say with certainty* that (in a non-chiral unitary 4d Lagrangian SCFT with $r_\chi \in]0,2[$) 'if Z_{S^3} is exponentially divergent, then the Di Pietro-Komargodski formula receives a modification'; this is simply because if L_h is not locally minimized at the origin, it is certainly not globally minimized there either.

We now demonstrate the three propositions itemized above.

The starting point is the observation that the function $\vartheta(x)$ featuring in L_h simplifies if its argument is "small enough":

$$\vartheta(x) = |x| - x^2 \qquad \text{for } x \in [-1,1]. \tag{3.25}$$

Using the above simplification in the expression (3.7) for L_h, we learn that for small enough $|\mathbf{x}|$ the Rains function simplifies to the following *homogenous* function[3]:

$$\tilde{L}_{S^3}(\mathbf{x}) = \frac{1}{2}\sum_{\chi}(1-r_\chi)\sum_{\rho^\chi\in\Delta_\chi}|\langle\rho^\chi\cdot\mathbf{x}\rangle| - \sum_{\alpha_+}|\langle\alpha_+\cdot\mathbf{x}\rangle|. \tag{3.26}$$

Note that there is no quadratic term in $\tilde{L}_{S^3}$, thanks to the cancelation of the $\mathrm{U}(1)_R$-gauge-gauge anomaly.

Next, we consider (recall $\omega := i(b+b^{-1})/2$)

$$Z_{S^3}(b;\Lambda) := \frac{1}{|W|}\int_\Lambda \mathrm{d}^{r_G}x\ \frac{\prod_\chi\prod_{\rho^\chi\in\Delta_\chi}\Gamma_h(r_\chi\omega+\langle\rho^\chi\cdot\mathbf{x}\rangle)}{\prod_{\alpha_+}\Gamma_h(\pm\langle\alpha_+\cdot\mathbf{x}\rangle)}, \tag{3.27}$$

which is the matrix-integral computing the squashed-three-sphere partition function of the dimensionally reduced daughter (c.f. Eq. (5.23) of [12]), assuming the same R-charge assignments as those directly descending from the parent 4d theory. We are keeping the cut-off Λ explicit, emphasizing that the integration is over the hypercube $|x_i| < \Lambda$.

To study the convergence/divergence of $Z_{S^3}(b;\Lambda)$ as Λ is taken to infinity, we use the estimate (B.5) for the hyperbolic gamma functions in the integrand of (3.27). We find that

[3] Interestingly, on a discrete subset of its domain (corresponding to the cocharacter lattice of the gauge group G), the function $\tilde{L}_{S^3}$ coincides (up to normalization) with the $S^2\times S^1$ Casimir energy ϵ_0 [33] associated to monopole sectors of the 3d $\mathcal{N}=2$ theory obtained from dimensional reduction of the 4d $\mathcal{N}=1$ gauge theory. In the context of 3d $\mathcal{N}=4$ theories, a different connection between $\tilde{L}_{S^3}$ and 3d monopoles was discussed in [34].

the integrand of $Z_{S^3}(b;\Lambda)$ can be estimated, as $|\mathbf{x}|\to\infty$, by

$$\frac{\prod_\chi \prod_{\rho^\chi\in\Delta_\chi}\Gamma_h(r_\chi\omega+\langle\rho^\chi\cdot\mathbf{x}\rangle)}{\prod_{\alpha_+}\Gamma_h(\pm\langle\alpha_+\cdot\mathbf{x}\rangle)}\approx e^{-2\pi(\frac{b+b^{-1}}{2})\tilde{L}_{S^3}(\mathbf{x})}, \tag{3.28}$$

with $\tilde{L}_{S^3}$ the homogeneous function defined above.

Note that whether the integrand of Z_{S^3} decays or grows at large $|\mathbf{x}|$, is determined by the behavior $\tilde{L}_{S^3}(\mathbf{x})$, and does not depend on b (recall that we take $b>0$).

Here comes the crucial point: since $\tilde{L}_{S^3}(\mathbf{x})$ is homogenous, its sign at large $|\mathbf{x}|$ is the same as its sign at small $|\mathbf{x}|$. Since at small enough $|\mathbf{x}|$, the two functions $\tilde{L}_{S^3}(\mathbf{x})$ and $L_h(\mathbf{x})$ coincide, the large-$|\mathbf{x}|$ behavior of the integrand of Z_{S^3} is connected to the behavior of the Rains function near the origin of $\mathfrak{h}_{cl}$. Therefore,

- if the origin is an isolated local minimum of L_h, then L_h, and hence $\tilde{L}_{S^3}$, is positive near the origin, and since $\tilde{L}_{S^3}(\mathbf{x})$ is homogeneous, it is positive also for large $|\mathbf{x}|$, leading in combination with (3.28) to the conclusion that the integrand of Z_{S^3} decays exponentially at large $|\mathbf{x}|$, and implying that Z_{S^3} is finite as $\Lambda\to\infty$;

- if the origin is part of an extended locus where L_h is locally minimized, then $\tilde{L}_{S^3}(\mathbf{x})$ has flat directions near the origin, and hence at large $|\mathbf{x}|$, and therefore the integrand of Z_{S^3} does not decay in certain directions, leading to the conclusion that Z_{S^3} is power-law divergent in Λ as $\Lambda\to\infty$;

- if the origin is not a local minimum of L_h, then L_h, and hence $\tilde{L}_{S^3}$, is negative somewhere near the origin, and since $\tilde{L}_{S^3}(\mathbf{x})$ is homogeneous, it is negative also for large $|\mathbf{x}|$ in certain directions, leading in combination with (3.28) to the conclusion that the integrand of Z_{S^3} grows exponentially at large $|\mathbf{x}|$ in certain directions, and implying that Z_{S^3} is exponentially divergent in Λ as $\Lambda\to\infty$.

3.2 Illustrative examples

3.2.1 A_k SQCD theories with $N_f>\frac{2N}{k+1}$

Take now the example of A_k SQCD with SU(N) gauge group. This theory has a chiral multiplet with R-charge $r_a=\frac{2}{k+1}$ in the adjoint, N_f flavors in the fundamental with R-charge $r_f=1-\frac{2}{k+1}\frac{N}{N_f}$, and N_f flavors in the anti-fundamental with R-charge $r_{\bar{f}}=r_f$. For r_f to be positive we must have $N_f>2N/(k+1)$.

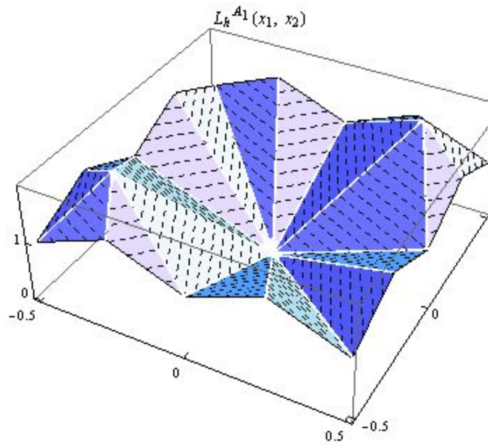

Figure 3.1: The Rains function of the A_1 SU(3) theory—also known as SU(3) SQCD.

We also assume that we are in the right range of parameters, so we are inside the conformal window of this theory.

The superconformal index of this theory is (c.f. [8])

$$\mathcal{I}_{A_k}(b,\beta) = \frac{(p;p)^{N-1}(q;q)^{N-1}}{N!}\Gamma^{N-1}((pq)^{r_a/2}) \int \mathrm{d}^{N-1}x \left(\prod_{1\leq i<j\leq N} \frac{\Gamma((pq)^{r_a/2}(z_i/z_j)^{\pm 1})}{\Gamma((z_i/z_j)^{\pm 1})}\right) \prod_{i=1}^{N} \Gamma^{N_f}((pq)^{r_f/2} z_i^{\pm 1}), \tag{3.29}$$

with $\prod_{i=1}^{N} z_i = 1$.

The Rains function of the theory is

$$\begin{aligned} L_h^{A_k}(x_1,\ldots,x_{N-1}) &= N_f(1-r_f)\sum_{i=1}^{N}\vartheta(x_i) + (1-r_a)\sum_{1\leq i<j\leq N}\vartheta(x_i-x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i-x_j) \\ &= \frac{2}{k+1}(N\sum_i \vartheta(x_i) - \sum_{1\leq i<j\leq N}\vartheta(x_i-x_j)). \end{aligned} \tag{3.30}$$

The x_N in the above expression is constrained by $\sum_{i=1}^{N} x_i \in \mathbb{Z}$, although since $\vartheta(x)$ is periodic with period one we can simply replace $x_N \to -x_1 - \cdots - x_{N-1}$. For $k=1$ and $N=3$, the resulting function is illustrated in Figure 3.1.

We recommend that the reader convince herself that the Rains function in (3.30) can be easily written down by examining the integrand of (3.29). Whenever the index of a theory is available in the literature, a similar examination of the integrand quickly yields the theory's

L_h function.

Using Rains's generalized triangle inequality (C.1), in the special case where $d_i = 0$, we find that the above function is minimized when all x_i are zero. This establishes that the integrand of (3.29) is localized around $x_i = 0$, and is exponentially suppressed everywhere else, as $\beta \to 0$. Therefore $L_{h\ \min}^{A_k} = 0$ and $\dim \mathfrak{h}_{qu}^{A_k} = 0$. We thus arrive at

$$\ln \mathcal{I}_{A_k}(b,\beta) = -\frac{\pi^2}{3\beta}(\frac{b+b^{-1}}{2})(\mathrm{Tr}R) + O(\beta^0). \tag{3.31}$$

Much more precise asymptotics

A more careful study shows [5] (see appendix B for the definition of the symbol $\sim$)

$$\ln \mathcal{I}_{A_k}(b,\beta) \sim -\frac{\pi^2}{3\beta}(\frac{b+b^{-1}}{2})(\mathrm{Tr}R) + \ln Z_{S^3}^{A_k}(b) + \beta E_{\mathrm{susy}}(b), \tag{3.32}$$

where

$$Z_{S^3}^{A_k}(b) = \frac{\Gamma_h^{N-1}(r_a\omega)}{N!}\int \mathrm{d}^{N-1}x \left(\prod_{1\leq i<j\leq N}\frac{\Gamma_h(r_a\omega \pm (x_i - x_j))}{\Gamma_h(\pm(x_i - x_j))}\right)\prod_{i=1}^{N}\Gamma_h^{N_f}(r_f\omega \pm x_i), \tag{3.33}$$

with the integral over $-\infty < x_i < \infty$.

3.2.2 SO($2N+1$) SQCD with $N_f > 2N-1$

Consider the SO(n) SQCD theories with N_f chiral matter multiplets of R-charge $r = 1 - \frac{n-2}{N_f}$ in the vector representation. For the R-charges to be greater than zero, and the gauge group to be semi-simple, we must have $0 < n-2 < N_f$.

We also assume that we are in the right range of parameters, so we are inside the conformal window of this theory.

We perform the analysis for odd n; the analysis for even n is completely analogous, and the result is similar. The index of SO($2N+1$) SQCD is given by (c.f. [8])

$$\begin{aligned}\mathcal{I}_{SO(2N+1)}(b,\beta) =& \frac{(p;p)^N(q;q)^N}{2^N N!}\Gamma^{N_f}((pq)^{r/2}) \\ &\times \int \mathrm{d}^N x \frac{\prod_{j=1}^{N}\Gamma^{N_f}((pq)^{r/2}z_j^{\pm 1})}{\prod_{j=1}^{N}\Gamma(z_j^{\pm 1})\prod_{i<j}(\Gamma((z_i z_j)^{\pm 1})\Gamma((z_i/z_j)^{\pm 1}))}.\end{aligned} \tag{3.34}$$

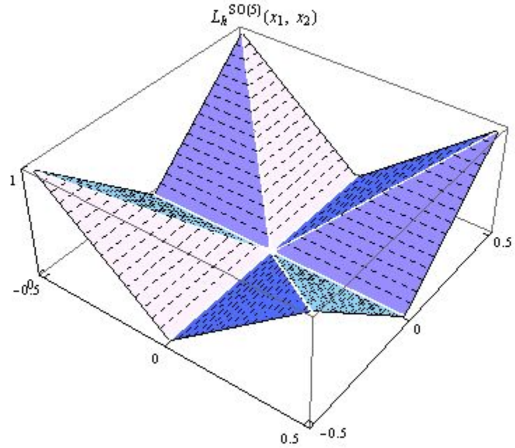

Figure 3.2: The Rains function of the SO(5) SQCD.

The Rains function of the theory is

$$L_h^{SO(2N+1)}(\mathbf{x}) = (2N-2)\sum_{j=1}^{N}\vartheta(x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i+x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i-x_j). \qquad (3.35)$$

For the case $N=2$, corresponding to the SO(5) theory, this function is illustrated in Figure 3.2.

To find the minima of the above function, we need the following result. For $-1/2 \leq x_i \leq 1/2$

$$\begin{aligned}(2N-2)\sum_{1\leq j\leq N}\vartheta(x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i+x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i-x_j) &= 2\sum_{1\leq i<j\leq N}\min(|x_i|,|x_j|)\\ &= 2(N-1)\min(|x_i|) + 2(N-2)\min{}_2(|x_i|) + \cdots + 2\min{}_{N-1}(|x_i|),\end{aligned} \qquad (3.36)$$

where $\min(|x_i|)$ stands for the smallest of $|x_1|,\ldots,|x_N|$, while $\min_2(|x_i|)$ stands for the next to smallest element, and so on. To prove (3.36), one can first verify it for $N=2$, and then use induction for $N>2$.

Applying (3.36) we find that the Rains function in (3.35) is minimized to zero when one (and only one) of the x_j is nonzero, and the rest are zero. This follows from the fact that $\max(|x_i|)$ does not show up on the RHS of (3.36). Therefore, unlike for the theories of the previous subsection, here the matrix-integral is not localized around the origin of the x_i

space, but *localized around the axes*. Equation (3.24) thus simplifies to

$$\ln \mathcal{I}_{SO(2N+1)}(b,\beta) = -\mathcal{E}_0^{DK}(b,\beta) + \ln\left(\frac{2\pi}{\beta}\right) + O(1) \qquad (\text{as } \beta \to 0). \tag{3.37}$$

The discussion in subsection 3.1.3 implies that the three-sphere partition function Z_{S^3} of the dimensionally reduced daughter of this theory diverges as $Z_{S^3} \approx \Lambda$ (as the cut-off Λ of the corresponding matrix-integral is taken to infinity); this power-law divergence is closely related to the (generically) subleading logarithmic term on the RHS of (3.37). See subsection 3.2 of [5] for a more detailed discussion of the relation between the power-law divergence of Z_{S^3} and the subleading asymptotics of the index.

Much more precise asymptotics for the SO(3) theory with $N_f = 2$ when $b = 1$

Luckily, for the special case of $N = 1, N_f = 2, b = 1$, the asymptotic expansion in (3.37) can be completed to all orders, with the result reading [5] (see appendix B for the definition of the symbol $\sim$ used below)

$$\ln \mathcal{I}_{SO(3)}(\beta) \sim \ln\left(\frac{\pi}{2\beta} - \frac{1}{2\pi}\right) + \frac{3}{8}\beta \quad (\text{as } \beta \to 0). \tag{3.38}$$

3.2.3 SU(N) $\mathcal{N} = 4$ SYM

The SU(N) $\mathcal{N} = 4$ theory has the following index [35]:

$$\begin{aligned}\mathcal{I}_{\mathcal{N}=4}(b,\beta) =& \frac{(p;p)^{N-1}(q;q)^{N-1}}{N!}\Gamma^{3(N-1)}((pq)^{1/3})\\ &\times \int \mathrm{d}^{N-1}x \prod_{1\le i<j\le N} \frac{\Gamma^3((pq)^{1/3}(z_i/z_j)^{\pm1})}{\Gamma((z_i/z_j)^{\pm1})},\end{aligned} \tag{3.39}$$

with $\prod_{i=1}^N z_i = 1$.

Recall that for the A_k SQCD theories the integrand of the matrix-integral was everywhere exponentially smaller than in the origin of the x_i space; in other words, the integral localized at a point. We will shortly find that for the $\mathcal{N} = 4$ theory the matrix-integral does not localize at all.

The Rains function of the theory is

$$L_h^{\mathcal{N}=4} = 3(1-\frac{2}{3}) \sum_{1\le i<j\le N} \vartheta(x_i - x_j) - \sum_{1\le i<j\le N} \vartheta(x_i - x_j) = 0. \tag{3.40}$$

In other words, there is no effective potential, and the matrix-integral does not localize: $\mathfrak{h}_{qu} = \mathfrak{h}_{cl}$. Eq. (3.24) thus dictates

$$\ln \mathcal{I}_{\mathcal{N}=4}(b,\beta) = (N-1)\ln(\frac{2\pi}{\beta}) + O(\beta^0). \tag{3.41}$$

There is no $O(1/\beta)$ term on the RHS, because $\mathrm{Tr}R = 0$ for the $\mathcal{N} = 4$ theory (and also $L_{h\ \min}^{\mathcal{N}=4} = 0$).

The discussion in subsection 3.1.3 implies that the three-sphere partition function Z_{S^3} of the dimensionally reduced daughter of this theory diverges as $Z_{S^3} \approx \Lambda^{N-1}$ (as the cut-off Λ of the corresponding matrix-integral is taken to infinity); this power-law divergence is closely related to the logarithmic term on the RHS of (3.41). See subsection 3.2 of [5] for more details.

More precise asymptotics

A more careful treatment shows that [5]

$$\ln \mathcal{I}_{\mathcal{N}=4}(b,\beta) = (N-1)\ln(\frac{2\pi}{\beta}) + 3(N-1)\ln\Gamma_h(\frac{2}{3}\omega) - \ln N! + o(1) \quad (\text{as } \beta \to 0). \tag{3.42}$$

3.2.4 The $\mathbb{Z}_2$ orbifold theory

We now study a quiver gauge theory, to illustrate how easily Rains's method generalizes to theories with more than one simple factor in their gauge group.

Consider the $\mathbb{Z}_2$ orbifold of the $\mathcal{N} = 4$ SYM with SU(N) gauge group. The theory consists of two SU(N) gauge groups, with one chiral multiplet in the adjoint of each, and one doublet of bifundamental chiral multiplets from each gauge group to the other. All the chiral multiplets have R-charge $r = 2/3$.

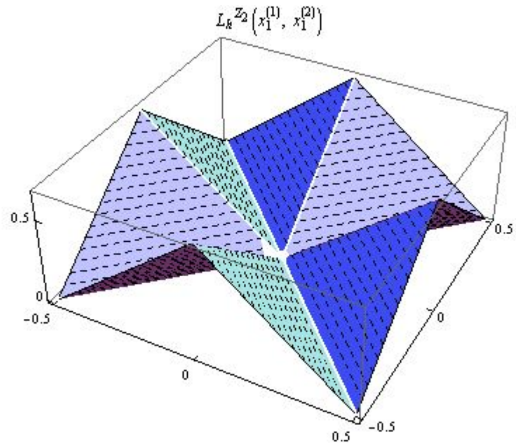

Figure 3.3: The Rains function of the SU(2)×SU(2) orbifold theory.

The superconformal index is given by (c.f. [19])

$$\mathcal{I}_{\mathbb{Z}_2}(b,\beta) = \Big(\prod_{k=1,2}\Big[\frac{(p;p)^{N-1}(q;q)^{N-1}}{N!}\Gamma^{N-1}((pq)^{1/3})\int \mathrm{d}^{N-1}x^{(k)} \\ \left(\prod_{1\le i<j\le N}\frac{\Gamma((pq)^{1/3}(z_i^{(k)}/z_j^{(k)})^{\pm 1})}{\Gamma((z_i^{(k)}/z_j^{(k)})^{\pm 1})}\right)\Big]\Big) \times \prod_{i,j=1}^{N}\left(\Gamma((pq)^{1/3}(z_i^{(1)}/z_j^{(2)})^{\pm 1})\right), \tag{3.43}$$

with $\prod_{i=1}^N z_i^{(1)} = \prod_{i=1}^N z_i^{(2)} = 1$.

The Rains function of the theory is

$$L_h^{\mathbb{Z}_2}(\mathbf{x}^{(1)},\mathbf{x}^{(2)}) = -\frac{2}{3}\sum_{1\le i<j\le N}\vartheta(x_i^{(1)}-x_j^{(1)}) - \frac{2}{3}\sum_{1\le i<j\le N}\vartheta(x_i^{(2)}-x_j^{(2)}) + \frac{2}{3}\sum_{i,j=1}^{N}\vartheta(x_i^{(1)}-x_j^{(2)}). \tag{3.44}$$

For the case $N=2$, corresponding to the SU(2)×SU(2) theory, this function is illustrated in Figure 3.3.

The generalized triangle inequality (C.1) applies with $c = x^{(1)}, d = x^{(2)}$, and implies that $L_h^{\mathbb{Z}_2}$ is positive semi-definite. It moreover shows that $L_h^{\mathbb{Z}_2}$ vanishes if the $x_i^{(1)}, x_j^{(2)}$ can be permuted such that either of (C.2) or (C.3) holds. For simplicity we consider all $x_i^{(1)}$ to be positive and very small, except for $x_N^{(1)} = -x_1^{(1)} - \cdots - x_{N-1}^{(1)}$ being negative and very small, and similarly for $x_j^{(2)}$. Assuming either (C.2) or (C.3), we conclude that $x_i^{(1)} = x_i^{(2)}$. Based on this result, and also the $N=2$ case whose Rains function is displayed in Figure 3.3, we conjecture that for the $\mathbb{Z}_2$ orbifold theory $\dim\mathfrak{h}_{qu} = N-1$, and thereby

$$\ln \mathcal{I}_{\mathbb{Z}_2}(b,\beta) = -\mathcal{E}_0^{DK}(b,\beta) + (N-1)\ln\left(\frac{2\pi}{\beta}\right) + O(1) \qquad (\text{as } \beta \to 0). \tag{3.45}$$

The discussion in subsection 3.1.3 implies that the three-sphere partition function Z_{S^3} of the dimensionally reduced daughter of this theory diverges as $Z_{S^3} \approx \Lambda^{N-1}$ (as the cut-off Λ of the corresponding matrix-integral is taken to infinity); this power-law divergence is related to the subleading logarithmic term on the RHS of (3.45). See subsection 3.2 of [5] for more details.

3.2.5 The SU(2) ISS model

There are two famous interacting Lagrangian SCFTs with $c < a$. The first is the Intriligator-Seiberg-Shenker (ISS) model of dynamical SUSY breaking [36]. The theory is formulated in the UV as an SU(2) vector multiplet with a single chiral multiplet in the four-dimensional representation of the gauge group. Although originally suspected to confine (and to break supersymmetry upon addition of a tree-level superpotential) [36], the theory is currently believed to flow to an interacting SCFT in the IR [37, 38], where the chiral multiplet has R-charge 3/5. The IR SCFT has $c - a = -7/80$.

The index of this theory is (c.f. [39])

$$\mathcal{I}_{ISS}(b,\beta) = \frac{(p;p)(q;q)}{2}\int \mathrm{d}x \frac{\Gamma((pq)^{3/10}z^{\pm 1})\Gamma((pq)^{3/10}z^{\pm 3})}{\Gamma(z^{\pm 2})}. \tag{3.46}$$

The Rains function of the theory is

$$L_h^{ISS}(x) = \frac{2}{5}\vartheta(x) + \frac{2}{5}\vartheta(3x) - \vartheta(2x). \tag{3.47}$$

This function is plotted in Figure 3.4.

A direct examination reveals that $L_h^{ISS}(x)$ is minimized at $x = \pm 1/3$, and $L_h^{ISS}(\pm 1/3) = -2/15$. The asymptotics of $\mathcal{I}_{ISS}$ is hence given according to (3.24) by

$$\ln \mathcal{I}_{ISS}(b,\beta) = \frac{\pi^2}{15\beta}(\frac{b+b^{-1}}{2}) + O(\beta^0). \tag{3.48}$$

In other words we have $(c-a)_{\text{shifted}} = c - a + 1/10 = 1/80$.

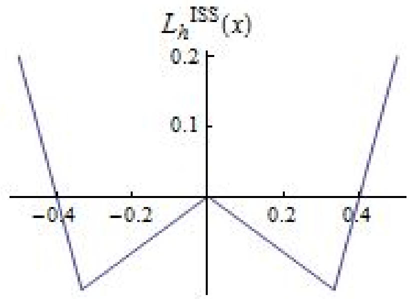

Figure 3.4: The Rains function of the SU(2) ISS theory.

The discussion in subsection 3.1.3 implies that the three-sphere partition function of the dimensionally reduced daughter of this theory is exponentially divergent; this severe divergence is related to the modification that the Di Pietro-Komargodski formula receives in this case.

Much more precise asymptotics

A more careful study shows [5] (see appendix B for the definition of the symbol $\sim$)

$$\ln \mathcal{I}_{ISS}(b,\beta) \sim \frac{16\pi^2}{3\beta}(c-a)_{\text{shifted}}\left(\frac{b+b^{-1}}{2}\right) + \ln Y_{S^3}^{ISS}(b) + \beta E_{\text{susy}}(b), \quad (\text{as } \beta \to 0) \tag{3.49}$$

with

$$Y_{S^3}^{ISS}(b) = \int_{-\infty}^{\infty} \mathrm{d}x' e^{-\frac{4\pi}{5}(b+b^{-1})x'} \times \Gamma_h(3x' + (3/5)\omega)\Gamma_h(-3x' + (3/5)\omega), \tag{3.50}$$

and $(c-a)_{\text{shifted}} = (c-a) + 1/10 = 1/80$. A numerical evaluation using

$$\ln \Gamma_h(ix; i, i) = (x-1)\ln(1-e^{-2\pi i x}) - \frac{1}{2\pi i} Li_2(e^{-2\pi i x}) + \frac{i\pi}{2}(x-1)^2 - \frac{i\pi}{12}, \tag{3.51}$$

yields $Y_{S^3}^{ISS}(b=1) \approx .423$.

3.2.6 The SO($2N+1$) BCI model with $1 < N < 5$

The second famous example of interacting SCFTs with $c < a$ is provided by the "misleading" SO(n) theory of Brodie, Cho, and Intriligator [40]. This is an $\mathcal{N}=1$ SO(n) gauge theory with a single chiral multiplet in the two-index symmetric traceless tensor representation of the gauge group. The theory is asymptotically free if $n \geq 5$. For $5 \leq n < 11$ the corresponding

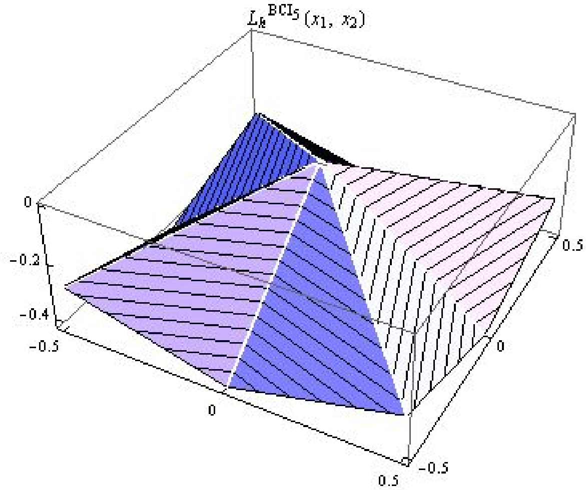

Figure 3.5: The Rains function of the SO(5) BCI theory.

interacting IR SCFT has $c - a = -(n-1)/16$ (for greater values of n the R-symmetry of the IR SCFT is believed to mix with an emergent accidental symmetry, and thus more care is called for; c.f. [41]).

For the SO$(2N+1)$ theory (with $1 < N < 5$) we have (c.f. [39])

$$\begin{aligned}\mathcal{I}_{BCI}(b,\beta) =& \frac{(p;p)^N(q;q)^N}{2^N N!}\Gamma^N((pq)^{2/(2N+3)})\int \mathrm{d}^N x \\ & \prod_{i<j}\frac{\Gamma((pq)^{2/(2N+3)}z_i^{\pm 1}z_j^{\pm 1})}{\Gamma(z_i^{\pm 1}z_j^{\pm 1})}\prod_{j=1}^{N}\frac{\Gamma((pq)^{2/(2N+3)}z_j^{\pm 1},(pq)^{2/(2N+3)}z_j^{\pm 2})}{\Gamma(z_j^{\pm 1})}.\end{aligned} \tag{3.52}$$

The Rains function of the theory is

$$L_h^{BCI}(x) = \frac{4}{2N+3}\left((\frac{2N-1}{4})\sum_j \vartheta(2x_j) - \sum_j \vartheta(x_j) - \sum_{i<j}\vartheta(x_i+x_j) - \sum_{i<j}\vartheta(x_i - x_j)\right). \tag{3.53}$$

For $N=2$, corresponding to the SO(5) theory, this function is plotted in Figure 3.5.

To find the minima of the above function, we need the following result, valid for $-1/2 \leq x_i \leq 1/2$:

$$\begin{aligned}&(\frac{2N-1}{4})\sum_{1\leq j\leq N}\vartheta(2x_j) - \sum_{1\leq j\leq N}\vartheta(x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i+x_j) - \sum_{1\leq i<j\leq N}\vartheta(x_i-x_j) = \\ & -\frac{3}{2}\sum_{i<j}\max(|x_i|,|x_j|) + \frac{1}{2}\sum_{i<j}\min(|x_i|,|x_j|) = \sum_j(-\frac{3N}{2}+2j-\frac{1}{2})\mathrm{min}_{N-j+1}(|x_i|),\end{aligned} \tag{3.54}$$

with $\min_N(|x_i|) := \max(|x_i|)$. The proof of (3.54) is similar to that of (3.36).

Note that the coefficient of the jth term on the RHS of (3.54) is negative if $j < \frac{3N+1}{4}$, and positive otherwise. This implies that the Rains function (3.53) is minimized when $\lfloor \frac{3N+1}{4} \rfloor$ of the $|x_i|$ are maximized (i.e. $x_i = \pm 1/2$), and the rest of the $|x_i|$ are minimized (i.e. $x_i = 0$). Consequently, the minimum of the Rains function is

$$L_{h\ \min}^{BCI} = -\frac{1}{2N+3} \sum_{1 \leq j \leq \lfloor \frac{3N+1}{4} \rfloor} (3N+1-4j). \tag{3.55}$$

This is less than zero for any $N > 1$. Therefore the Di Pietro-Komargodski formula needs to be modified in the SO$(2N+1)$ BCI model with $1 < N < 5$.

The discussion in subsection 3.1.3 implies that the three-sphere partition function of the dimensionally reduced daughter of this model (with $1 < N < 5$) is exponentially divergent; this severe divergence is related to the modification that the Di Pietro-Komargodski formula receives in this case. See subsection 3.3 of [5] for more details.

Consider now the concrete case of the SO(5) theory corresponding to $N = 2$. This theory has $c - a = -1/4$. From Eq. (3.55) we have in this case $L_{h\ min}^{BCI}(x) = -3/7$. The asymptotics of $\mathcal{I}$ is therefore given according to (3.24) by

$$\ln \mathcal{I}_{BCI_5}(b,\beta) = \frac{8\pi^2}{21\beta}(\frac{b+b^{-1}}{2}) + O(\beta^0). \tag{3.56}$$

In other words $(c-a)_{\text{shifted}} = c - a + 9/28 = 1/14$.

Much more precise asymptotics for the SO(5) BCI theory

A more careful treatment shows [5]

$$\ln \mathcal{I}_{BCI_5}(b,\beta) \sim \frac{16\pi^2}{3\beta}(c-a)_{\text{shifted}}(\frac{b+b^{-1}}{2}) + \ln Y_{S^3}^{BCI_5}(b) + \beta E_{\text{susy}}(b), \quad (\text{as } \beta \to 0) \tag{3.57}$$

with

$$\begin{aligned} Y_{S^3}^{BCI_5}(b) =& \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x_1'\ \Gamma_h((4/7)\omega \pm 2x_1') \times \\ & \frac{\Gamma_h^2((4/7)\omega)}{2} \int_{-\infty}^{\infty} \mathrm{d}x_2\ \frac{\Gamma_h((4/7)\omega \pm x_2)\Gamma_h((4/7)\omega \pm 2x_2)}{\Gamma_h(\pm x_2)}, \end{aligned} \tag{3.58}$$

and $(c-a)_{\text{shifted}} = (c-a) + 9/28 = 1/14$. A numerical evaluation using (3.51) yields $Y_{S^3}^{BCI_5}(b=1) \approx .026$.

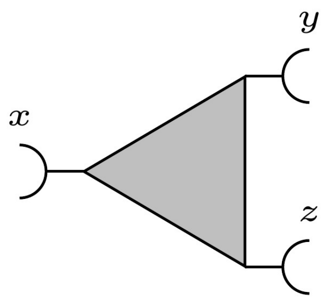

Figure 3.6: The building block of the puncture-less Lagrangian class-$\mathcal{S}$ theories.

3.2.7 Puncture-less SU(2) class-$\mathcal{S}$ theories

An interesting class of Lagrangian $\mathcal{N}=2$ SCFTs arise from quiver gauge theories associated to Riemann surfaces of genus $g \geq 2$, without punctures (see e.g. [42] for a discussion of the indices of these theories). These quivers can be constructed from fundamental blocks of the kind shown in Figure 3.6. The triangle in Figure 3.6 represents eight chiral multiplets of R-charge 2/3 transforming in the tri-fundamental representation of the three SU(2) gauge groups[4] represented by the (semi-circular) nodes; more precisely, when two semi-circular nodes are connected together to form a circle, they represent an $\mathcal{N}=2$ SU(2) vector multiplet. A class-$\mathcal{S}$ theory of genus g arises when $2g-2$ of these blocks are glued back-to-back (and forth-to-forth) along a straight line, with the leftmost and the rightmost blocks having two of their half-circular nodes glued together; see Figure 3.7 for an example.

An $\mathcal{N}=2$ SU(2) vector multiplet contributes to the Rains function of the SCFT as

$$L_h^{\mathcal{N}=2\ v}(x) = -\frac{2}{3}\vartheta(2x). \tag{3.59}$$

A semi-circular node contributes half as much, and thus the three semi-circular nodes in Figure 3.6 contribute together as

$$L_h^{\mathrm{semi-nodes}}(x,y,z) = -\frac{1}{3}\left(\vartheta(2x)+\vartheta(2y)+\vartheta(2z)\right). \tag{3.60}$$

The eight chiral multiplets represented by the triangle in Figure 3.6 contribute to the

[4] We focus on class-$\mathcal{S}$ theories constructed from T_2, and leave the study of higher-rank theories constructed from $T_{N>2}$ to future work.

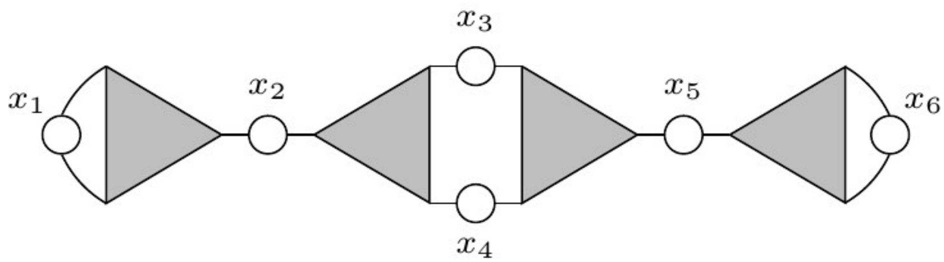

Figure 3.7: The quiver diagram of the $g = 3$ class-$\mathcal{S}$ theory.

Rains function of the theory as

$$L_h^{T_2}(x, y, z) = \frac{1}{3}\left(\vartheta(x+y+z) + \vartheta(x+y-z) + \vartheta(x-y+z) + \vartheta(-x+y+z)\right). \tag{3.61}$$

Adding up (3.60) and (3.62) we obtain the contribution of a single block to the Rains function:

$$\begin{aligned} L_h^{\text{block}}(x, y, z) =& \frac{1}{3}[\vartheta(x+y+z) + \vartheta(x+y-z) + \vartheta(x-y+z) + \vartheta(-x+y+z) \\ &- \vartheta(2x) - \vartheta(2y) - \vartheta(2z)]. \end{aligned} \tag{3.62}$$

With the Rains function of the block at hand, we can now write down the Rains function of genus g class-$\mathcal{S}$ theories. For example, the Rains function of the $g = 2$ theory is given by

$$L_h^{\mathcal{S}_{g=2}}(x_1, x_2, x_3) = L_h^{\text{block}}(x_1, x_1, x_2) + L_h^{\text{block}}(x_2, x_3, x_3), \tag{3.63}$$

and the Rains function of the $g = 3$ theory (illustrated in Figure 3.7) is obtained as

$$\begin{aligned} L_h^{\mathcal{S}_{g=3}}(x_1, x_2, x_3, x_4, x_5, x_6) =& L_h^{\text{block}}(x_1, x_1, x_2) + L_h^{\text{block}}(x_2, x_3, x_4) \\ &+ L_h^{\text{block}}(x_3, x_4, x_5) + L_h^{\text{block}}(x_5, x_6, x_6). \end{aligned} \tag{3.64}$$

Importantly, Rains's GTI (C.1), with $c_1 = x + y,\ c_2 = x - y,\ d_1 = z,\ d_2 = -z$, implies that

$$L_h^{\text{block}}(x, y, z) \geq 0. \tag{3.65}$$

It is not difficult to show that the equality holds in a finite-volume subspace of the x, y, z space; take for instance $x, y, z \approx .1$ within $.01$ of each other, and use the fact that for small argument L_h reduces to $\tilde{L}_{S^3}$ to show that L_h vanishes in the domain just described.

Since the Rains function of a $g \geq 2$ class-$\mathcal{S}$ theory is the sum of several block Rains functions, the positive semi-definiteness of L_h^{block} guarantees the positive semi-definiteness of $L_h^{\mathcal{S}_{g\geq 2}}(x_i)$; moreover, taking all x_i to be around .1 and within .01 of each other we can easily conclude (as in the previous paragraph) that for the genus g theory $\dim\mathfrak{h}_{qu} = 3(g-1)$. The relation (3.24) thus yields

$$\ln \mathcal{I}_{\mathcal{S}_{g\geq 2}}(b,\beta) = \frac{16\pi^2}{3\beta}(c-a)(\frac{b+b^{-1}}{2}) + 3(g-1)\ln(\frac{2\pi}{\beta}) + O(\beta^0), \tag{3.66}$$

with $c - a = -(g-1)/24$.

3.3 Applications

3.3.1 Supersymmetric dualities

Dual QFTs must have equal partition functions. As a trivial corollary, the high-temperature asymptotics of the index of dual 4d SCFTs must match.

Assume now that both sides of the duality are non-chiral 4d Lagrangian SCFTs with a semi-simple gauge group. The relation (3.24) then yields two quantities to be matched between the theories: $L_{h\ \min}$ and $\dim\mathfrak{h}_{qu}$. Comparison of $L_{h\ \min}$ can rule out for instance the confinement scenario for the SU(2) ISS model: on the gauge theory (UV) side, as discussed above, we have $L_{h\ \min} = -2/15$, while on the mesonic (IR) side[5] we have no gauge group and thus $L_h = 0$.

As another example, consider the recent E_7 SQCD duality of [17, 44]. In that case a direct examination reveals that $L_{h\ \min} = \dim\mathfrak{h}_{qu} = 0$, both on the electric and the magnetic side. Their proposal hence passes both our tests.

The case of the interacting $\mathcal{N} = 1$ SCFTs with $c < a$ (namely the IR fixed points of the ISS model, and the BCI_{2N+1} model with $1 < N < 5$) is particularly interesting. A dual description for these theories is currently lacking. Our results for $L_{h\ \min}$ and $\dim\mathfrak{h}_{qu}$ on the electric side might help to test future proposals for magnetic duals of these theories.

[5]Following [39], we are assuming that an index can be consistently assigned to the proposed IR theory, even though the IR chiral multiplet would have R-charge $12/5 \notin]0,2[$. This assignment requires an analytic continuation of the kind discussed in [5]; the small-β asymptotics of the resulting function can be obtained as in [23]. See [43] for an alternative take on this problem.

3.3.2 Holography

In the present chapter we analyzed the high-temperature asymptotics of the indices of various gauge theories at *finite* N. The finite-N indices of holographic SCFTs are expected to encode information about micro-states of the supersymmetric Giant Gravitons of the dual string theories [9]. Take for instance the SU(N) $\mathcal{N}=4$ SYM. One of the novel results of [5] is the following high-temperature asymptotics for the superconformal index of this theory (see Eqs. (3.41) and (3.42) above):

$$\mathcal{I}(b=1,\beta)=\sum_{\text{operators}}(-1)^F e^{-\beta(\Delta-\frac{1}{2}r)}\approx(\frac{1}{\beta})^{N-1}. \tag{3.67}$$

The above *canonical* relation can be transformed to the *micro-canonical* ensemble to yield the asymptotic (fermion-number weighted) degeneracy of the protected high-energy operators in the $\mathcal{N}=4$ theory:

$$N(E)\approx E^{N-2}, \tag{3.68}$$

with $E=\Delta-r/2$. This result should presumably be reproduced by geometric quantization of the 1/16 BPS Giant Gravitons of IIB theory on $\mathrm{AdS}_5\times S^5$, along the lines of [45]. It would be interesting to see if this expectation pans out.

3.3.3 Quantum Coulomb branch dynamics on $R^3\times S^1$

Take a non-chiral 4d SCFT with a semi-simple gauge group, and with $r_\chi\in]0,2[$. Its superconformal index $\mathcal{I}(b,\beta)$ can be computed by a path-integral on $S^3_b\times S^1_\beta$, with S^3_b the unit-radius squashed three-sphere. We now replace the S^3_b with the round three-sphere $S^3_{r_3}$ of arbitrary radius $r_3>0$. The path-integral on the new space gives $\mathcal{I}(\beta;r_3)=\mathcal{I}(b=1,\beta/r_3)$; i.e. the resulting partition function only depends on the ratio β/r_3, as the theory is conformal. Thus, as far as $\mathcal{I}(\beta;r_3)$ is concerned, shrinking the S^1 (i.e. the high-temperature limit) is equivalent to decompactifying the S^3. We hence fix β, and send r_3 to infinity. In this limit we expect the unlifted zero-modes on $S^3_{r_3}\times S^1_\beta$ to roughly correspond to the quantum zero-modes on $R^3\times S^1$. Therefore at high temperatures the unlifted holonomies of the theory on $S^3_{r_3}\times S^1_\beta$ should be in correspondence with (a real section of) the quantum Coulomb branch of the 3d $\mathcal{N}=2$ theory obtained from compactifying the 4d theory on the circle of $R^3\times S^1$. In particular, we expect $\dim\mathfrak{h}_{qu}$ to be equal to the (complex-) dimension of the quantum Coulomb branch of the 3d theory. (Recall that the Coulomb branch of the circle-compactified theory living on R^3 consists not just of the holonomies around the S^1, but also of the dual

3d photons; hence our references above to “a real section” and “complex-dimension”.)

We do not expect to recover the $R^3 \times S^1$ Higgs branch from the zero-modes on $S^3_{r_3} \times S^1_\beta$: for any (arbitrarily small) curvature on the S^3, curvature couplings presumably lift the Higgs-type zero-modes on $S^3_{r_3} \times S^1_\beta$.

From the point of view of $R^3 \times S^1$, picking one of the R^3 directions as time[6], we can relate $\mathcal{E}_0^{DK}$ to the Casimir energy associated to the spatial manifold $R^2 \times S^1$: we reintroduce r_3 in $\mathcal{E}_0^{DK}$ (by replacing its β with β/r_3), set in it $b=1$, interpret $\tilde{\beta} := 2\pi r_3$ as the circumference of the crossed channel thermal circle, and write

$$\mathcal{E}_0^{DK}(\beta; r_3) = \tilde{\beta} E_0^{R^2\times S^1}(\beta), \quad \text{with} \quad E_0^{R^2\times S^1}(\beta) = \frac{\pi}{6\beta}\mathrm{Tr}R. \tag{3.69}$$

Now $E_0^{R^2\times S^1}(\beta)$ admits an interpretation as the (regularized) Casimir energy associated to the spatial $R^2 \times S^1_\beta$. Similarly, resurrecting the r_3 in V^{eff}, and setting in it $b=1$, we obtain what can be loosely regarded as $\tilde{\beta}$ times the quantum effective potential on (a real section of) the crossed channel Coulomb branch. From this perspective, the two tests we advocated in subsection 3.3.1 would not really be new, but would correspond to the comparison of low-energy properties on $R^3 \times S^1$.

The discussion in the previous three paragraphs is rather intuitive, and should be considered suggestive at best. It is desirable to have it made more precise. Nevertheless, in the examples of the SU(N), Sp($2N$), and SO($2N+1$) SQCD theories, and the SU(N) $\mathcal{N}=4$ SYM, it turns out [5] that (upon quotienting by the Weyl group) $\mathfrak{h}_{qu}$ *does indeed resemble* (a real section of) the $R^3 \times S^1$ quantum Coulomb branch; see [12, 18] and [47]. We therefore conjecture that the relation between $\mathfrak{h}_{qu}$ and the unlifted Coulomb branch on $R^3 \times S^1$ continues to remain valid, at least for all the theories with a positive semi-definite Rains function. In particular, we predict that, when placed on $R^3 \times S^1$, the SU(N) A_k SQCD theories (in the appropriate range of their parameters such that all their r_χ are in $]0,2[$) have no quantum Coulomb branch, and the $\mathbb{Z}_2$ orbifold of the SU(N) $\mathcal{N}=4$ theory has an $(N-1)$-dimensional unlifted Coulomb branch.

For theories whose Rains function is not positive semi-definite, on the other hand, it seems like this connection with $R^3 \times S^1$ fails. The Rains function of the SU(2) ISS model does not have a flat direction, and appears to suggest a Higgs vacuum for the theory on

[6] The following discussion is in the spirit of the arguments in [46], though our treatment is not as precise. We are approaching R^3 from S^3, rather than from T^3 (as in [46]). While on T^3 each of the circles can be picked as the time direction, picking a time direction along the S^3 makes the spatial sections time-dependent, rendering our arguments in the paragraph of this footnote somewhat hand-wavy. I thank E. Shaghoulian for several helpful conversations related to the subject of the present subsection.

$R^3 \times S^1$. However, the study of [38] indicates that this theory possesses an unlifted Coulomb branch on $R^3 \times S^1$, and in particular does not necessarily break the gauge group at low energies. It would be nice to understand if this conflict is only a manifestation of the sloppiness of our intuitive arguments above, or it has a more interesting origin.

Chapter 4

Taking the large-N limit first: the holographic Weyl anomaly from the index

4.1 The large-N limit of the 4d superconformal index: the multi-trace index

It is often the case that asymptotically at large N a hierarchy appears in the spectrum of local operators of an SCFT. This hierarchy is expected to be reflected in the superconformal index. Take for instance the $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM, which has the following *Schur index* [9] (see [48] for the definition of the Schur index; in the present chapter we discuss the Schur index only as a toy model of the superconformal index):

$$\mathcal{I}_{Schur}(\beta)=\frac{(q;q)}{(q^{1/2};q^{1/2})^2}\sum_{n=0}^{\infty}(-1)^n\left[\binom{N+n}{N}+\binom{N+n-1}{N}\right]q^{(nN+n^2)/2}, \tag{4.1}$$

with $q=e^{-\beta}$. In the above expression, n clearly has an interpretation as a soliton-counting number. Of course, these solitons are naturally interpreted in the gravity dual to the $\mathrm{U}(N)$ $\mathcal{N}=4$ SYM. They presumably correspond to Giant Gravitons of the IIB theory on $\mathrm{AdS}_5\times S^5$ [9].

The large-N limit suppresses (energetically) all the $n\neq 0$ terms in (4.1), and yields

$$\mathcal{I}^{N\to\infty}_{Schur}(\beta)=\frac{(q;q)}{(q^{1/2};q^{1/2})^2}. \tag{4.2}$$

This is the "multi-trace" Schur index of the U(N) $\mathcal{N}=4$ SYM; it can be obtained by summing over multi-trace operators of the gauge theory in the planar limit.

Another general point that our toy model can help illustrate is that the high-temperature asymptotics of the multi-trace index may be (and generically is) very different from the asymptotics of the finite-N index (which we focused on in the previous chapter). Our toy model index (4.1) has the high-temperature asymptotics (see [5] for the similar asymptotic analysis of the Schur index of the SU(N) $\mathcal{N}=4$ SYM)

$$\ln \mathcal{I}_{Schur}(\beta) = N\ln(\frac{2\pi}{\beta}) + O(\beta). \tag{4.3}$$

Taking the $N\to\infty$ limit before the $\beta\to 0$ limit, changes the high-temperature asymptotics drastically. The high-temperature asymptotics of the large-N Schur index (in (4.2)) is found as (see appendix B for the definition of the symbol $\sim$ used below)

$$\ln \mathcal{I}^{N\to\infty}_{Schur}(\beta) = \ln(q;q) - 2\ln(q^{1/2};q^{1/2}) \sim \frac{\pi^2}{2\beta} + \frac{1}{2}\ln(\frac{\beta}{8\pi}), \qquad (\text{as } \beta\to 0) \tag{4.4}$$

differing significantly from the asymptotics of the finite-N index in (4.3).

The purpose of the above discussion was to help orient the reader for the following analysis of toric quiver gauge theories with SU(N) nodes. Toric quiver theories are a much-studied subset of supersymmetric gauge theories whose field content can be efficiently summarized using quiver diagrams. The latter are directed graphs with nodes representing vector multiplets and edges representing chiral multiplets. The nodes at the ends of an edge represent vector multiplets under which the chiral multiplet (represented by the edge) is charged. The direction of the edge encodes further information about the representation of the gauge group according to which the chiral multiplet transforms. The toric condition puts further constraints on the theory, thereby guaranteeing some nice properties such as existence of a non-trivial IR fixed point with a holographic dual describable by "toric geometry" (see for instance [49]). A canonical example is the $\mathcal{N}=4$ SYM with SU(N) gauge group, which can be represented by one node (standing for the SU(N) vector multiplet), and three directed edges (standing for the three chiral multiplets in the adjoint) that both emanate from and end on that one node.

Similarly to the case of the Schur index discussed above, *in the large-N limit the index of toric quivers simplifies to the multi-trace index.* The multi-trace index is the one obtained by summing over multi-trace operators in the SCFT. These operators correspond to the multi-particle KK states in the gravity dual.

The superconformal index of these SCFTs is studied in [19, 50, 51, 23], and its large-N limit is found to be

$$\mathcal{I}^{N\to\infty}_{quiver}(b,\beta) = \frac{1}{\prod_{i=1}^{n_z}((pq)^{r_i/2};(pq)^{r_i/2})\prod_{adj}\Gamma((pq)^{R_{adj}/2};p,q)}, \tag{4.5}$$

with the first product in the denominator being over the n_z extremal BPS mesons (with R-charge r_i), and the second product over the chiral multiplets (with R-charge R_{adj}) in the adjoints of various nodes. For example, the SU(N) $\mathcal{N}=4$ SYM has $n_z=3$, and $r_{1,2,3}=R_{adj}=2/3$, with three adjoint chiral multiplets in total.

Just as in the case of the Schur index discussed above, the high-temperature asymptotics of $\mathcal{I}^{N\to\infty}_{quiver}(b,\beta)$ is quite different from the asymptotics of the same quiver at finite N; the latter asymptotics can be obtained (for non-chiral quivers) from the results of the previous section. Finding the asymptotics of $\mathcal{I}^{N\to\infty}_{quiver}(b,\beta)$, on the other hand, requires separate calculations.

4.2 From the multi-trace index to the single-trace index

The single-trace index is defined as the plethystic log [20] of the multi-trace index

$$I_{s.t.}(b,\beta) \equiv \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\mathcal{I}^{N\to\infty}(b,n\beta), \tag{4.6}$$

where $\mu(n)$ is the Möbius function. The adjective "single-trace" is particularly appropriate for theories that admit a planar limit in which single-trace operators are weakly interacting. For such cases if in the definition of the index in (2.5) one restricts the trace to the "single-trace states" in the Hilbert space, one obtains the single-trace index as defined above. In AdS/CFT, the weakly interacting mesons of the large-N SCFT at large 't Hooft coupling map to the KK supergravity modes in the bulk. Therefore the boundary single-trace index is (according to AdS/CFT) equal to the bulk single-particle index, with the latter receiving contributions only from the bulk single-particle KK states.

The single-trace index of SU(N) toric quiver SCFTs can be easily computed by taking the plethystic logarithm of the two sides of (4.5); the result is

$$I_{s.t.\ quiver}(b,\beta) = \sum_{i=1}^{n_z} \frac{(pq)^{r_i/2}}{1-(pq)^{r_i/2}} - \sum_{adj} \frac{(pq)^{R_{adj}/2} - (pq)^{1-R_{adj}/2}}{(1-p)(1-q)}. \tag{4.7}$$

In the next section we will see that in its high-temperature asymptotics, the above index encodes the subleading Weyl anomaly of the underlying SU(N) toric quiver SCFT.

An interesting problem, which is not relevant to the main discussion of the present chapter, is the connection between the small-β asymptotics of $I_{s.t.}(b,\beta)$ and $\mathcal{I}^{N\to\infty}(b,\beta)$; this problem is addressed in appendix D.

4.3 Asymptotics of the single-trace index and the holographic Weyl anomaly

In this section we present holographic results implying that the subleading central charges of a holographic SCFT are encoded in the high-temperature asymptotics of its large-N index.

We focus on SCFTs whose dual geometry is of the form $\mathrm{AdS}_5\times\mathrm{SE}_5$, with SE_5 a Sasaki-Einstein 5-manifold. The KK spectrum of the IIB theory on $\mathrm{AdS}_5\times\mathrm{SE}_5$ organizes itself into representations of the 4d $\mathcal{N}=1$ superconformal group $SU(2,2|1)$.

The shortened multiplets of $SU(2,2|1)$ are listed in Table 4.1, along with their contributions to the single-trace index. For convenience, we have introduced $t \equiv 1/\sqrt{pq}$, and $y \equiv \sqrt{p/q}$. The chiral and SLII multiplets (on the 2nd and the last row, respectively) contribute to the right-handed index[1], while the CP-conjugate multiplets, namely the anti-chiral and SLI multiplets (on the 3nd and the 4th row, respectively), contribute to the left-handed index. Conserved multiplets (on the 1st row), which are CP self-conjugate, contribute to both.

We begin with relating $c-a$ to the index. First consider the chiral and SLII multiplets. The contribution to $c-a$ from a generic chiral multiplet $\mathcal{D}(E_0,j_1,0;r)$ in the bulk KK spectrum is given by the following holographically derived expression [52, 21]

$$(c-a)\big|_{\text{chiral}} = -\frac{1}{192}(-1)^{2j_1}(2E_0-3)(2j_1+1)\left(1-8j_1(j_1+1)\right). \tag{4.8}$$

[1] The index defined in (2.5) is the right-handed index. One can also define the left-handed index in which one replaces r with $-r$ and swaps j_1 and j_2 in the definition of the index in (2.5). The index in Table 4.1 is defined as $I^+_{s.t.} \equiv \frac{1}{2}(I^R_{s.t.} + I^L_{s.t.})$, in terms of the left and right single-trace indices. For toric quivers $I^+_{s.t.} = I^R_{s.t.} = I^L_{s.t.}$.

Shortening condition	Representation	$(1-t^{-1}y)(1-t^{-1}y^{-1})I^{+}_{s.t.}$
$E_0 = 2+j_1+j_2,\ \frac{3}{2}r = j_1 - j_2$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_2+2)/3}\chi_{j_1}(y) + (j_1 \leftrightarrow j_2)$
$E_0 = \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, 0, r)$	$\frac{1}{2}(-1)^{2j_1}t^{-2E_0/3}\chi_{j_1}(y)$
$E_0 = -\frac{3}{2}r$	$\mathcal{D}(E_0, 0, j_2, r)$	$\frac{1}{2}(-1)^{2j_2}t^{-2E_0/3}\chi_{j_2}(y)$
$E_0 = 2+2j_1 - \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_1+2)/3}\chi_{j_2}(y)$
$E_0 = 2+2j_2 + \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_2+2)/3}\chi_{j_1}(y)$

Table 4.1: Contributions to the superconformal index from the various shortened multiplets.

Similarly, a generic SLII multiplet $\mathcal{D}(E_0, j_1, j_2; r)$ in the bulk spectrum contributes [52, 21]

$$(c-a)\big|_{\mathrm{SLII}} = \frac{1}{192}(-1)^{2j_1+2j_2}(2E_0+2j_2-1)(2j_1+1)\left(1-8j_1(j_1+1)\right). \tag{4.9}$$

It is now possible to see how these expressions may be obtained from the contributions to the right-handed index given in Table 4.1. Since the SU(2) character $\chi_j(y)$ is given by

$$\chi_j(y) = \frac{y^{2j+1} - y^{-(2j+1)}}{y - y^{-1}}, \tag{4.10}$$

the differential operator $(6(y\partial_y)^2 - 1)$ acting on the contributions to the index gives $(2j+1)[8j(j+1)-1]$ when y is set to one. The operator $(t\partial_t + 1)$ then produces the E_0-dependent factors in (4.8) and (4.9).

The CP conjugate multiplets (anti-chiral and SLI) contribute similarly to (4.8) and (4.9) with the appropriate replacement of quantum numbers, and are accounted for in the left-handed index. Finally, since conserved multiplets contribute as the sum of one SLI and one SLII multiplet, they are implicitly included in both the left- and right-handed indices. Our key observation is that the contribution to $c-a$ has a uniform expression for every single bulk multiplet. Hence a single differential operator acting on the index can yield the appropriate contribution to $c-a$ regardless of the shortening condition. Summing over all the bulk KK multiplets, one finally arrives at

$$\begin{aligned} c-a &= \lim_{t\to 1} -\frac{1}{32}\left(t\partial_t + 1\right)\left(6(y\partial_y)^2 - 1\right) \\ &\quad \times \left[(1-t^{-1}y)(1-t^{-1}y^{-1})I^{+}_{s.t.}(t,y)\right]\Big|_{y=1}^{\text{finite}}, \end{aligned} \tag{4.11}$$

where the fugacities are set to one after acting with the differential operator on the index. Note that the factor $(1-t^{-1}y)(1-t^{-1}y^{-1})$ multiplying the single-trace index removes the contribution from descendant states. The result obtained is often divergent, as we are working in the large-N limit, so the prescription is that the finite term in an expansion about

$t = 1$ yields the value of $c - a$.

A few remarks are now in order.

- The $t \to 1$ limit corresponds to the high-temperature limit $\beta \to 0$. Therefore the prescription (4.11) extracts $c - a$ from the high-temperature asymptotics of $I^+_{s.t.}$.
- The index in (4.11) is the single-particle supergravity index, which is—according to the AdS/CFT conjecture—equal to the single-trace index of the SCFT.
- In the prescription (4.11), the index provides a natural regulator for the Kaluza-Klein sums encountered in the holographic $c - a$ calculations of [55, 54, 53].

Following a similar approach, but using holographic expressions for the individual central charges, one arrives at [52, 22]

$$\begin{aligned} \delta a &= \frac{1}{32}(t\partial_t + 1)(-\tfrac{9}{2}t\partial_t(t\partial_t + 2) + \tfrac{9}{2}(y\partial_y)^2 - 3)\hat{I}(t, y), \\ \delta c &= \frac{1}{32}(t\partial_t + 1)(-\tfrac{9}{2}t\partial_t(t\partial_t + 2) - \tfrac{3}{2}(y\partial_y)^2 - 2)\hat{I}(t, y), \end{aligned} \tag{4.12}$$

where $\hat{I} = (1 - yt^{-1})(1 - y^{-1}t^{-1})I^+_{s.t.}$ is the single-trace index with descendants removed, and δ indicates that we are referring to the $O(N^0)$ part of the central charges (and not their leading $O(N^2)$ piece). The fugacities are set to one after acting with the differential operators on $\hat{I}$; we are thus again dealing with the high-temperature limit of the index.

In principle, a successful application of Eq. (4.12) to a holographic SCFT can be viewed as a one-loop test of AdS/CFT. This can be easily done for arbitrary SU(N) toric quiver SCFTs without adjoint matter that are dual to smooth Sasaki-Einstein 5-manifolds. The single-trace index of such a toric theory is [50]

$$I_{s.t.} = \sum_i \frac{1}{t^{r_i/3} - 1}, \tag{4.13}$$

where r_i are the R-charges of extremal BPS mesons. Applying (4.12) to (4.13) gives

$$\delta a = -\frac{27}{32(t-1)^2}\sum_{i=1}^{n_v}\frac{1}{r_i} - \frac{1}{32}\sum_{i=1}^{n_v} r_i + \cdots \tag{4.14}$$

in an expansion about $t = 1$. Noting that $\sum r_i = 6$(# nodes in the quiver), and keeping only the finite part, we obtain

$$\delta a = -\frac{3}{16}(\#\text{ nodes in the quiver}). \tag{4.15}$$

This matches the expected result for the $O(1)$ part of a based on the decoupling of a U(1) at each node in the quiver; since there are no adjoint matter fields in the quiver, there are no additional $O(1)$ contributions to a in the field theoretical computation through $a = \frac{1}{32}(9\mathrm{Tr}R^3 - 3\mathrm{Tr}R)$.

The successful matching for the $O(1)$ part of c can be deduced from a similar application of the second relation in Eq. (4.12) to (4.13).

The prescriptions in Eq. (4.12) can also be successfully applied to the single-trace index of an arbitrary SU(N) toric quiver, as given in (4.7). However, the result would count as a successful test of AdS/CFT only up to the following two assumptions: *i*) a combinatorial conjecture [51] that has gone into the derivation of (4.7), which although strongly supported in [51], is not yet proven; *ii*) the assumption that the index (4.7) (which is derived as the single-trace index of the SCFT) equals the single-particle index of the gravity dual (which is what goes on the RHS of the prescriptions in Eq. (4.12)). The equality in the assumption *ii* is not yet proven [50] when the toric quiver has adjoint matter or is dual to singular toric SE_5.

Finally, the study of [23] shows that the validity of the prescriptions in Eq. (4.12) is guaranteed if the high-temperature asymptotics of the single-trace index of SCFTs dual to $\mathrm{AdS}_5\times\mathrm{SE}_5$ has the following form (see appendix B for the definition of the symbol $\sim$ used below):

$$\begin{aligned} I_{s.t.} \sim & \frac{2H}{\beta\,(b+b^{-1})} + \frac{G\,(b+b^{-1})}{2\beta} + C \\ & - \beta\left(\frac{4}{27}(b+b^{-1})^3\,(3\delta c - 2\delta a) + \frac{4}{3}(b+b^{-1})\,(\delta a - \delta c)\right), \end{aligned} \tag{4.16}$$

with G,H,C constants that are insignificant for the formulas in Eq. (4.12), except that H determines the pole terms that according to the prescription of [22] one should drop. The above asymptotics was explicitly verified in [23] for the single-trace index of the SU(N) toric quivers, shown in (4.7).

Chapter 5

Concluding remarks

5.1 Summary of the high-temperature content of the index

5.1.1 Finite-N non-chiral theories

We have shown that the high-temperature expansion of the superconformal index of finite-rank non-chiral SCFTs (having all their r_χ inside $]0,2[$) looks like

$$\ln \mathcal{I}(b,\beta) = \frac{A(b)}{\beta} + B\ln(\frac{2\pi}{\beta}) + C(b) + o(\beta^0), \qquad (\text{as } \beta \to 0) \tag{5.1}$$

with

$$A(b) = \frac{16\pi^2}{3}(\frac{b+b^{-1}}{2})(c-a-\frac{3}{4}L_{h\ min}), \tag{5.2}$$

$$B = \dim\mathfrak{h}_{qu}, \tag{5.3}$$

and $C(b)$ some real function of b that we have not found a general expression for.

Based on various examples that we have looked at, it seems like whenever $L_{h\ min} = 0$, the (complex-) dimension of the quantum Coulomb branch of the theory on $R^3 \times S^1$ coincides with $\dim\mathfrak{h}_{qu}$. Thus we can say the following.

For theories with positive semi-definite Rains function, the high-temperature expansion of $\ln\mathcal{I}(b,\beta)$ *encodes i) in its order-*$1/\beta$ *term the difference of the central charges* $c-a$*, and ii) in its order-*$\ln(1/\beta)$ *term the (complex-) dimension of the quantum Coulomb branch of the theory on* $R^3 \times S^1$*.*

Note that while we have proven item i above, item ii is only a conjecture based on various examples studied in [5].

Moving on to the subleading terms, the following statement was demonstrated in [5] for $C(b)$. (We define $Z_{S^3}(b) := Z_{S^3}(b;\infty)$; see subsection 3.1.3 for the definition of $Z_{S^3}(b;\Lambda)$.)

For theories whose Rains function is minimized only at the origin of $\mathfrak{h}_{cl}$ (hence have $L_{h\ min} = \dim\mathfrak{h}_{qu} = 0$), the high-temperature expansion of $\ln\mathcal{I}(b,\beta)$ encodes in its order-β^0 term the logarithm of the squashed three-sphere partition function $Z_{S^3}(b)$ of the dimensionally reduced theory; in other words, for the said theories $C(b) = \ln Z_{S^3}(b)$.

[The above statement was claimed in [5] to hold even for chiral theories; however, while for non-chiral theories it is straightforward to show $Z_{S^3}(b) \neq 0$, for chiral theories we have not been able to show that $Z_{S^3}(b)$ is non-zero; we thus emphasize that the above statement is demonstrated in [5] for chiral theories assuming $Z_{S^3}(b) \neq 0$.]

Although we have not been able to make general statements about the $o(\beta^0)$ terms on the RHS of (5.1), based on the examples studied in [5] it seems that

$$\ln\mathcal{I}(b,\beta) = \frac{A(b)}{\beta} + B\ln(\frac{2\pi}{\beta}) + C(b) + D(b)\beta + O(\beta^2) \qquad (\text{as } \beta \to 0). \tag{5.4}$$

For theories whose Rains function is minimized on a set of isolated points, the above asymptotics can actually be demonstrated (with $B = 0$, of course); it can moreover be shown that the error term is not just $O(\beta^2)$, but beyond all orders (and of the type $e^{-1/\beta}$) [5]. Furthermore, in those theories $D(b)$ coincides with the SUSY Casimir energy[1] (encountered also in (3.5) above)

$$E_{\text{susy}}(b) = \frac{2}{27}(b+b^{-1})^3(3c-2a) + \frac{2}{3}(b+b^{-1})(a-c). \tag{5.5}$$

Therefore we can say the following [5].

For theories whose Rains function is minimized on a set of isolated points in $\mathfrak{h}_{cl}$, the high-temperature expansion of $\ln\mathcal{I}(b,\beta)$ takes the form shown in (5.4), with $B = 0$, and with the error being not only $O(\beta^2)$ but also exponentially small. Moreover, the order-β term encodes the SUSY Casimir energy; in other words, for the said theories $D(b) = E_{\text{susy}}(b)$.

[1] The SUSY Casimir energy relates the superconformal index $\mathcal{I}(b,\beta)$ to its corresponding partition function $Z^{\text{SUSY}}(b,\beta)$ computed via path-integration on $S^3_b \times S^1_\beta$ [23, 25]: $Z^{\text{SUSY}}(b,\beta) = e^{-\beta E_{\text{susy}}(b)}\mathcal{I}(b,\beta)$.

The above statement implies that (whenever the Rains function is minimized on a set of isolated points) the central charges a and c—and hence the 't Hooft anomalies $\mathrm{Tr}R$ and $\mathrm{Tr}R^3$—are both encoded in the order-β term in the high-temperature expansion of $\ln \mathcal{I}(b,\beta)$. It can actually be shown that introducing flavor fugacities $u_a = e^{i\beta m_a}$ in the superconformal index, the relation $D(b) = E_{\text{susy}}(b)$ generalizes to $D(b;m_a) = E_{\text{susy}}(b;m_a)$, with $E_{\text{susy}}(b;m_a)$ the equivariant SUSY Casimir energy (which encodes all the 't Hooft anomalies in the theory [56]); thus (whenever the Rains function is minimized on a set of isolated points) all the 't Hooft anomalies are encoded in the order-β term in the high-temperature expansion of $\ln \mathcal{I}(b,\beta;m_a)$. This statement is related (but not equivalent) to some of the claims in [28], which were made there in the context of SU(N) SQCD.

5.1.2 Large-N toric quivers

It was shown in [23] that for SU(N) toric quiver SCFTs (see appendix B for the definition of the symbol $\sim$ used below)

$$\begin{aligned}\ln \mathcal{I}^{N\to\infty}_{quiver}(b,\beta) \sim{} & \frac{\pi^2}{6\beta(\frac{b+b^{-1}}{2})}\sum_{i=1}^{n_z}\frac{1}{r_i} + \frac{16\pi^2(\frac{b+b^{-1}}{2})}{3\beta}\sum_{adj}(\delta c_{adj} - \delta a_{adj}) + \frac{n_z}{2}\ln(\beta/2\pi) + \ln Y_b \\ & + \beta\left(\frac{2}{27}(b+b^{-1})^3(3\delta c - 2\delta a) + \frac{2}{3}(b+b^{-1})(\delta a - \delta c)\right),\end{aligned} \tag{5.6}$$

where the notation is similar to that in (4.5), except for $\ln Y_b = \frac{1}{2}\sum_{i=1}^{n_z}\ln(r_i(\frac{b+b^{-1}}{2})) - \sum_{adj}\ln\Gamma_h(iR_{adj}(\frac{b+b^{-1}}{2}))$, with $\Gamma_h(*)$ a special function explained in appendix A.

Based on various specific examples, it was conjectured in [21, 23] that

$$\sum_{i=1}^{n_z}\frac{1}{r_i} = \frac{3}{16\pi^3}\left(19\mathrm{vol}(SE) + \frac{1}{8}\mathrm{Riem}^2(SE)\right), \tag{5.7}$$

where SE denotes the Sasaki-Einstein 5-manifold dual to the quiver gauge theory. The above conjecture was motivated by the finding in [57] that one can "hear the shape of the dual geometry" in the asymptotics of the Hilbert series of mesonic operators in the SCFT. We note that the leading high-temperature behavior of the index of the quivers is contained in the first two terms of (5.6). The first term, according to (5.7), is dictated by the geometry of the dual internal manifold, while the second is given by the $O(1)$ part of the contribution of adjoint matter to $c-a$. The latter is hence the only part of the finite-N Di Pietro-Komargodski formula that escapes metamorphosis into "geometry" in the planar limit.

Interestingly, the order-β term on the RHS of (5.6) is β times $\delta E_{\text{susy}}(b)$, where by the latter we mean $E_{\text{susy}}(b)$ as in (5.5) but with the central charges in it replaced with their $O(N^0)$ pieces. Therefore the order-β term in the high-temperature expansion of $\ln \mathcal{I}^{N\to\infty}_{quiver}(b,\beta)$ is somewhat similar to the corresponding term for the finite-N non-chiral theories whose Rains function is minimized on a set of isolated points (see the previous subsection).

Note that the discussion below (4.16) (combined with relation (D.9)) implies that the holographic computation of the subleading central charges can be thought of as extracting δc and δa from the order-β term of the high-temperature expansion of $\ln \mathcal{I}^{N\to\infty}(b,\beta)$. In the case of the toric quivers, the holographic prescriptions in (4.12) thus extract δc and δa from $\delta E_{\text{susy}}(b)$.

5.2 Future directions

The main result of this dissertation is the high-temperature asymptotics of the EHIs arising as the superconformal index of unitary non-chiral 4d Lagrangian SCFTs. The most important extension of our work would be to chiral SCFTs; the preliminary investigation of [5] seems to indicate that the extension would not be straightforward.

A particularly interesting outcome of our work has been the connection between the high-temperature asymptotics of the index, and the Coulomb branch dynamics on $R^3 \times S^1$; see subsection 3.3.3. This is undoubtedly worth pursuing more carefully. Even before aiming at establishing the connection in a general context, it would be nice to validate it by examining the Coulomb branch dynamics of the A_k SQCD (with SU(N) gauge group) and the Z_2 orbifold theory (with SU(N)$\times$SU(N) gauge group), to see if the (complex-) dimension of their quantum Coulomb branch on $R^3 \times S^1$ coincides with their $\dim\mathfrak{h}_{qu}$, which we have found to be respectively zero and $N-1$.

Appendix A

Useful special functions

The **Pochhammer symbol** ($|q| \in]0,1[$)

$$(a;q) := \prod_{k=0}^{\infty}(1-aq^k), \tag{A.1}$$

is related to the more familiar Dedekind eta function via

$$\eta(\tau) = q^{1/24}(q;q), \tag{A.2}$$

with $q = e^{2\pi i\tau}$.

The eta function has an $\mathrm{SL}(2,\mathbb{Z})$ modular property that will be useful for us: $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$.

The Pochhammer symbol $(q;q)$ equals the inverse of the generating function of integer partitions. It also appears in the index of 4d SUSY gauge theories that contain vector multiplets.

The **elliptic gamma function** is defined as $(\mathrm{Im}(\tau), \mathrm{Im}(\sigma) > 0)$

$$\Gamma(x;\sigma,\tau) := \prod_{j,k\geq 0} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \tag{A.3}$$

with $z := e^{2\pi i x}$, $p := e^{2\pi i\sigma} = e^{-\beta b}$, and $q := e^{2\pi i\tau} = e^{-\beta b^{-1}}$. The above expression gives a meromorphic function of $x \in \mathbb{C}$. For generic choice of τ and σ, the elliptic gamma has simple poles at $x = l - m\sigma - n\tau$, with $m, n \in \mathbb{Z}^{\geq 0}$, $l \in \mathbb{Z}$.

We sometimes write $\Gamma(x;\sigma,\tau)$ as $\Gamma(z;p,q)$, or simply as $\Gamma(z)$. Also, the arguments of elliptic gamma functions are frequently written with "ambiguous" signs (as in $\Gamma(\pm x;\sigma,\tau)$);

by that one means a multiplication of several gamma functions each with a "possible" sign of the argument (as in $\Gamma(+x;\sigma,\tau)\times\Gamma(-x;\sigma,\tau)$). Similarly $\Gamma(z^{\pm 1}) := \Gamma(z;p,q)\times\Gamma(z^{-1};p,q)$.

The elliptic gamma function appears in the exact solution of some important 2d integrable lattice models. It also features in the index of 4d Lagrangian SUSY QFTs that contain chiral multiplets.

Following Rains [10], we define the **hyperbolic gamma function** by

$$\Gamma_h(x;\omega_1,\omega_2) := \exp\left(\mathrm{PV}\int_{\mathbb{R}} \frac{e^{2\pi i x w}}{(e^{2\pi i\omega_1 w}-1)(e^{2\pi i\omega_2 w}-1)}\frac{\mathrm{d}w}{w}\right). \tag{A.4}$$

The above expression makes sense only for $0 < \mathrm{Im}(x) < 2\mathrm{Im}(\omega)$, with $\omega := (\omega_1+\omega_2)/2$. In that domain, the function defined by (A.4) satisfies

$$\Gamma_h(x+\omega_2;\omega_1,\omega_2) = 2\sin(\frac{\pi x}{\omega_1})\Gamma_h(x;\omega_1,\omega_2). \tag{A.5}$$

This relation can then be used for an inductive meromorphic continuation of the hyperbolic gamma function to all $x\in\mathbb{C}$. For generic ω_1,ω_2 in the upper half plane, the resulting meromorphic function $\Gamma_h(x;\omega_1,\omega_2)$ has simple zeros at $x=\omega_1\mathbb{Z}^{\geq 1}+\omega_2\mathbb{Z}^{\geq 1}$ and simple poles at $x=\omega_1\mathbb{Z}^{\leq 0}+\omega_2\mathbb{Z}^{\leq 0}$.

For convenience, we will frequently write $\Gamma_h(x)$ instead of $\Gamma_h(x;\omega_1,\omega_2)$, and $\Gamma_h(x\pm y)$ instead of $\Gamma_h(x+y)\Gamma_h(x-y)$.

The hyperbolic gamma function has an important property that can be easily derived from the definition (A.4):

$$\Gamma_h(-\mathrm{Re}(x)+i\mathrm{Im}(x);\omega_1,\omega_2) = (\Gamma_h(\mathrm{Re}(x)+i\mathrm{Im}(x);\omega_1,\omega_2))^*, \tag{A.6}$$

with $*$ denoting complex conjugation.

We also define the non-compact quantum dilogarithm ψ_b (c.f. the function $e_b(x)$ in [58]; $\psi_b(x)=e_b(-ix)$) via

$$\psi_b(x) := e^{-i\pi x^2/2+i\pi(b^2+b^{-2})/24}\Gamma_h(ix+\omega;\omega_1,\omega_2), \tag{A.7}$$

where

$$\omega_1 := ib, \quad \omega_2 := ib^{-1}, \quad \text{and} \quad \omega := (\omega_1+\omega_2)/2. \tag{A.8}$$

For generic choice of b, the zeros of $\psi_b(x)^{\pm 1}$ are of first order, and lie at $\pm((b+b^{-1})/2 + b\mathbb{Z}^{\geq 0}+b^{-1}\mathbb{Z}^{\geq 0})$. Upon setting $b=1$ we get the function $\psi(x)$ of [59]; i.e. $\psi_{b=1}(x)=\psi(x)$.

An identity due to Narukawa [60] implies the following important relation between $\psi_b(x)$ and the elliptic gamma function (see also Appendix A of [23])

$$
\begin{aligned}
\Gamma(z;\sigma,\tau) &= \frac{e^{2i\pi Q_-(x;\sigma,\tau)}}{\psi_b(\frac{2\pi i x}{\beta}+\frac{b+b^{-1}}{2})}\prod_{n=1}^{\infty}\frac{\psi_b(-\frac{2\pi i n}{\beta}-\frac{2\pi i x}{\beta}-\frac{b+b^{-1}}{2})}{\psi_b(-\frac{2\pi i n}{\beta}+\frac{2\pi i x}{\beta}+\frac{b+b^{-1}}{2})} \\
&= e^{2i\pi Q_+(x;\sigma,\tau)}\psi_b(-\frac{2\pi i x}{\beta}-\frac{b+b^{-1}}{2})\prod_{n=1}^{\infty}\frac{\psi_b(-\frac{2\pi i n}{\beta}-\frac{2\pi i x}{\beta}-\frac{b+b^{-1}}{2})}{\psi_b(-\frac{2\pi i n}{\beta}+\frac{2\pi i x}{\beta}+\frac{b+b^{-1}}{2})},
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
Q_-(x;\sigma,\tau) =& -\frac{x^3}{6\tau\sigma}+\frac{\tau+\sigma-1}{4\tau\sigma}x^2-\frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{12\tau\sigma}x \\
&-\frac{1}{24}(\tau+\sigma-1)(\tau^{-1}+\sigma^{-1}-1), \\
Q_+(x;\sigma,\tau) =& Q_-(x;\sigma,\tau)+(x-\frac{\tau+\sigma}{2})^2/2\tau\sigma-(\tau^2+\sigma^2)/24\tau\sigma,
\end{aligned}
\tag{A.10}
$$

Appendix B

Some asymptotic analysis

We say $f(\beta) = O(g(\beta))$ as $\beta \to 0$, if there exist positive real numbers C, β_0 such that for all $\beta < \beta_0$ we have $|f(\beta)| < C|g(\beta)|$. We say $f(x,\beta) = O(g(x,\beta))$ *uniformly* over S as $\beta \to 0$, if there exist positive real numbers C, β_0 such that for all $\beta < \beta_0$ and all $x \in S$ we have $|f(x,\beta)| < C|g(x,\beta)|$.

We will write $f(\beta) = o(g(\beta))$, if $f(\beta)/g(\beta) \to 0$ as $\beta \to 0$.

We use the symbol $\sim$ when writing the all-orders asymptotics of a function. For example, we have

$$\ln(\beta + e^{-1/\beta}) \sim \ln\beta, \qquad (\text{as } \beta \to 0) \tag{B.1}$$

because we can write the LHS as the sum of $\ln\beta$ and $\ln(1+e^{-1/\beta}/\beta)$, and the latter is beyond all-orders in β.

More precisely, we say $f(\beta) \sim g(\beta)$ as $\beta \to 0$, if we have $f(\beta) - g(\beta) = O(\beta^n)$ for any (arbitrarily large) natural n.

We will write $f(\beta) \simeq g(\beta)$ if $\ln f(\beta) \sim \ln g(\beta)$ (with an appropriate choice of branch for the logarithms). By writing $f(x,\beta) \simeq g(x,\beta)$ we mean that $\ln f(x,\beta) \sim \ln g(x,\beta)$ for all x on which $f(x,\beta), g(x,\beta) \neq 0$, and that $f(x,\beta) = g(x,\beta) = 0$ for all x on which either $f(x,\beta) = 0$ or $g(x,\beta) = 0$.

With the above notations at hand, we can asymptotically analyze the Pochhammer symbol as follows. The low-temperature ($T \to 0$, with $q = e^{-1/T}$) behavior is trivial:

$$(q;q) \simeq 1 \qquad (\text{as } 1/\beta \to 0). \tag{B.2}$$

The high-temperature ($\beta \to 0$, with $q = e^{-\beta}$) asymptotics is nontrivial. It can be obtained using the $\mathrm{SL}(2,\mathbb{Z})$ modular property of the eta function, which yields

$$\ln \eta(\tau = \frac{i\beta}{2\pi}) \sim -\frac{\pi^2}{6\beta} + \frac{1}{2}\ln(\frac{2\pi}{\beta}) \qquad (\text{as } \beta \to 0). \tag{B.3}$$

The above relation, when combined with (A.2), implies

$$\ln(q;q) \sim -\frac{\pi^2}{6\beta} + \frac{1}{2}\ln(\frac{2\pi}{\beta}) + \frac{\beta}{24} \qquad (\text{as } \beta \to 0). \tag{B.4}$$

For the hyperbolic gamma function, Corollary 2.3 of [10] implies that when $x \in \mathbb{R}$

$$\ln \Gamma_h(x + r\omega; \omega_1, \omega_2) = -\frac{i\pi}{2} x|x| - i\pi(r-1)\omega|x| + O(1), \qquad (\text{as } |x| \to \infty) \tag{B.5}$$

for any fixed real r, and fixed $b > 0$.

From the asymptotics of the hyperbolic gamma function, it follows that for fixed $\mathrm{Re}(x)$ and fixed $b > 0$

$$\ln \psi_b(x) \sim 0, \qquad (\text{as } \beta \to 0, \text{ for } \mathrm{Im}(x) = -1/\beta) \tag{B.6}$$

with a transcendentally small error, of the type $e^{-1/\beta}$.

The above estimate can be combined with (A.9) to yield the small-β estimates

$$\begin{aligned} \Gamma(x;\sigma,\tau) &\simeq \frac{e^{2i\pi Q_-(x;\sigma,\tau)}}{\psi_b(\frac{2\pi i x}{\beta} + \frac{b+b^{-1}}{2})}, \qquad (\text{for } -1 < \mathrm{Re}(x) \le 0) \\ &\simeq e^{2i\pi Q_+(x;\sigma,\tau)} \psi_b(-\frac{2\pi i x}{\beta} - \frac{b+b^{-1}}{2}), \qquad (\text{for } 0 \le \mathrm{Re}(x) < 1) \end{aligned} \tag{B.7}$$

with the range of $\mathrm{Re}(x)$ explaining our subscript notations for Q_+ and Q_-. As a result of (B.7) we have for $x \in \mathbb{R}$, as $\beta \to 0$:

$$\begin{aligned} \Gamma(-x + (\frac{\tau+\sigma}{2})r;\sigma,\tau) &\simeq \frac{e^{2i\pi Q_-(-\{x\} + (\frac{\tau+\sigma}{2})r;\sigma,\tau)}}{\psi_b(-\frac{2\pi i \{x\}}{\beta} - (r-1)\frac{b+b^{-1}}{2})}, \\ \Gamma(x + (\frac{\tau+\sigma}{2})r;\sigma,\tau) &\simeq e^{2i\pi Q_+(\{x\} + (\frac{\tau+\sigma}{2})r;\sigma,\tau)} \psi_b(-\frac{2\pi i\{x\}}{\beta} + (r-1)\frac{b+b^{-1}}{2}), \end{aligned} \tag{B.8}$$

with $\{x\} := x - \lfloor x \rfloor$. The above estimates are first obtained in the range $0 \le x < 1$, and then extended to $x \in \mathbb{R}$ using the periodicity of the LHS under $x \to x+1$.

Appendix C

Generalized triangle inequalities

Define $\vartheta(x) := \{x\}(1-\{x\})$. The Lemma 3.2 of [10] says that for any sequence of real numbers $c_1,\ldots,c_n$, $d_1,\ldots,d_n$, the following inequality holds:

$$\sum_{1\le i,j\le n}\vartheta(c_i-d_j)-\sum_{1\le i<j\le n}\vartheta(c_i-c_j)-\sum_{1\le i<j\le n}\vartheta(d_i-d_j)\ge\vartheta(\sum_{1\le i\le n}(c_i-d_i)), \qquad \text{(C.1)}$$

with equality iff the sequence can be permuted so that either

$$\{c_1\}\le\{d_1\}\le\{c_2\}\le\cdots\le\{d_{n-1}\}\le\{c_n\}\le\{d_n\}, \qquad \text{(C.2)}$$

or

$$\{d_1\}\le\{c_1\}\le\{d_2\}\le\cdots\le\{c_{n-1}\}\le\{d_n\}\le\{c_n\}. \qquad \text{(C.3)}$$

The proof can be found in [10].

Re-scaling with $c_i,d_i\mapsto vc_i,vd_i$, taking $v\to 0^+$, and using the relation $\vartheta(vx)=v|x|-v^2x^2$ (which holds for small enough v), Rains obtains the following corollary of (C.1):

$$\sum_{1\le i,j\le n}|c_i-d_j|-\sum_{1\le i<j\le n}|c_i-c_j|-\sum_{1\le i<j\le n}|d_i-d_j|\ge|\sum_{1\le i\le n}(c_i-d_i)|, \qquad \text{(C.4)}$$

with equality iff the sequence can be permuted so that either

$$c_1\le d_1\le c_2\le\cdots\le d_{n-1}\le c_n\le d_n, \qquad \text{(C.5)}$$

or

$$d_1\le c_1\le d_2\le\cdots\le c_{n-1}\le d_n\le c_n. \qquad \text{(C.6)}$$

The fact that the inequality (C.4) arise as a corollary of (C.1) justifies the name "generalized triangle inequality" for the latter.

Various generalized triangle inequalities (GTIs) allow us to analytically address the minimization problems for the piecewise linear functions L_h arising in Chapter 3. In several physically interesting cases, the required GTI is a corollary of Rains's GTI shown in (C.1) above.

Appendix D

Proof of an ansatz in [23]

The universal property of large-N SCFTs that allows a systematic study of their high-temperature asymptotics is the *large-N factorization*. The factorization implies that the index of large-N theories is conveniently expressed in terms of the single-trace index as

$$\ln \mathcal{I}^{N\to\infty}(\beta, b) = \sum_{n=1}^{\infty} \frac{1}{n} I_{s.t.}(n\beta, b). \tag{D.1}$$

Let us now review a useful technique in asymptotic analysis, which we will find useful when studying large-N indices expressible as in (D.1).

Say we are interested in the small-β asymptotics of a real function $F(\beta)$ that can be written in the form

$$F(\beta) = \sum_{m=1}^{\infty} f(m\beta), \tag{D.2}$$

with $f(\beta)$ a real function having the $\beta \to 0$ asymptotic development

$$f(\beta) \sim \sum_{\lambda \geq -1}^{\infty} b_\lambda \beta^\lambda. \tag{D.3}$$

Assume moreover that $f(\beta)$ and all its derivatives decay faster than $1/\beta^{1+\varepsilon}$ as $\beta \to \infty$, for some $\varepsilon > 0$. Then, according to Zagier [61], the $\beta \to 0$ asymptotics of $F(\beta)$ is given by

$$F(\beta) \sim \frac{1}{\beta}\left(b_{-1} \ln(\frac{1}{\beta}) + I_f^*\right) + \sum_{\lambda > -1} b_\lambda \zeta(-\lambda)\ \beta^\lambda, \tag{D.4}$$

with $I_f^* := \int_0^\infty (f(x) - b_{-1} e^{-x}/x) \mathrm{d}x$.

Equation (D.1) has a remarkable resemblance to the sums to which Zagier's method applies. In fact, dividing both sides of (D.1) by β, we arrive at

$$\frac{\ln \mathcal{I}^{N\to\infty}(\beta)}{\beta} = \sum_{n=1}^{\infty} f(n\beta), \tag{D.5}$$

with $f(\beta) = I_{s.t.}(\beta)/\beta$.

In all the examples we are aware of, $I_{s.t.}(\beta)$ has a leading asymptotics of the form I_{-1}/β. Therefore $f(\beta)$ defined above has a leading asymptotics of the form I_{-1}/β^2, and thus Zagier's formula (D.4) does not immediately apply to it. However, defining $\tilde{f}(\beta) := f(\beta) - I_{-1}/\beta^2$, we obtain

$$\frac{\ln \mathcal{I}^{N\to\infty}(\beta)}{\beta} = \frac{\pi^2}{6\beta^2} I_{-1} + \sum_{n=1}^{\infty} \tilde{f}(n\beta), \tag{D.6}$$

and now Zagier's method can be applied to find the asymptotics of the sum on the RHS of the above relation. The result is

$$\frac{\ln \mathcal{I}^{N\to\infty}(\beta)}{\beta} \sim \frac{\pi^2}{6\beta^2} I_{-1} + \frac{1}{\beta}(I_0 \ln(\frac{1}{\beta}) + I^*_{\tilde{f}}) + \sum_{m=0}^{\infty} \tilde{f}_m \zeta(-m)\beta^m, \tag{D.7}$$

where $I^*_{\tilde{f}} := \int_0^\infty (\frac{I_{s.t.}(x) - I_{-1}/x}{x} - I_0 \frac{e^{-x}}{x})\mathrm{d}x$, and $\tilde{f}_m$ is the coefficient of β^m in the asymptotics of $\tilde{f}(\beta)$. Also I_0 is the β-independent term in the asymptotic expansion of $I_{s.t.}$.

Let I_n be the coefficient of β^n in the asymptotics of the single-trace index $I_{s.t.}$. From

$$\tilde{f}(\beta) = I_{s.t.}(\beta)/\beta - I_{-1}/\beta^2, \tag{D.8}$$

we obtain $\tilde{f}_n = I_{n+1}$ for $n = 0, 1, \dots$. Therefore we can write (D.7) as

$$\ln \mathcal{I}^{N\to\infty}(\beta) \sim \frac{\pi^2}{6\beta} I_{-1} + I_0 \ln(\frac{1}{\beta}) + I^*_{\tilde{f}} + \sum_{m=1}^{\infty} I_m \zeta(-m+1)\beta^m. \tag{D.9}$$

This relation is the main result of this appendix. It expresses the all-orders small-β asymptotics of $\ln \mathcal{I}^{N\to\infty}$ in terms of data that can be found from the single-trace index.

As an application of the result (D.9), we derive the ansatz given in [23] for the asymptotics of the index of the A_k SQCD in the Veneziano limit.

The single-trace index of the A_k SQCD in the Veneziano limit is given by [21]

$$I^{A_k}_{s.t.} = \frac{\tau^{-\frac{2}{k+1}}}{1-\tau^{-\frac{2}{k+1}}} + \frac{\tau^{-\frac{4k}{k+1}}}{1-\tau^{-\frac{4k}{k+1}}} - \frac{\tau^{-\frac{2k}{k+1}}}{1-\tau^{-\frac{2k}{k+1}}} - \frac{\left(\tau^{-\frac{2}{k+1}} - \tau^{-\frac{2k}{k+1}}\right) - N_f^2 \frac{\left(\tau^{\frac{2N_c}{(k+1)N_f}} - \tau^{-\frac{2N_c}{(k+1)N_f}}\right)^2}{\tau^2\left(1-\tau^{-\frac{2}{k+1}}\right)\left(1+\tau^{-\frac{2k}{k+1}}\right)}}{(1-p)(1-q)}, \tag{D.10}$$

where $\tau := (pq)^{-1/2}$.

Expanding $I^{A_k}_{s.t.}$ at high temperatures we find a series of the form

$$I^{A_k}_{s.t.}(\beta) = \frac{I_{-1}}{\beta} + I_0 + \sum_{m\ \ odd>0} I_m \beta^m. \tag{D.11}$$

Note that no positive even powers of β show up in the expansion. This is because $I^{A_k}_{s.t.}(\beta)$ is "almost" an odd function of β: one can directly check from (D.10) that $I^{A_k}_{s.t.}(\beta)+I^{A_k}_{s.t.}(-\beta) = -1$.

Plugging (D.11) in (D.9) we find that

$$\ln \mathcal{I}^{N\to\infty}_{A_k}(\beta) \sim \frac{\pi^2}{\beta} I_{-1} + I_0 \ln\left(\frac{1}{\beta}\right) + I^*_{\tilde{f}} + I_1 \beta. \tag{D.12}$$

Using the actual values

$$\begin{aligned}
I_{-1} &= \frac{2k^3+3k^2-1}{4k(1+k)}\left(\frac{1}{(\frac{b+b^{-1}}{2})}\right) + \frac{16kN_c^2 - 8k^2 + 8k}{4k(1+k)}\left(\frac{b+b^{-1}}{2}\right)\\
I_0 &= -\frac{1}{2}\\
I_1 &= -\left(\frac{4}{27}(b+b^{-1})^3(3c-2a) + \frac{4}{3}(b+b^{-1})(a-c)\right).
\end{aligned} \tag{D.13}$$

we obtain

$$\begin{aligned}
\ln \mathcal{I}^{N_c\to\infty}_{A_k}(\beta, b) \sim & \frac{2k^3+3k^2-1}{4k(1+k)}\left(\frac{\pi^2}{6\beta(\frac{b+b^{-1}}{2})}\right) + \frac{16kN_c^2-8k^2+8k}{4k(1+k)}\left(\frac{\pi^2(\frac{b+b^{-1}}{2})}{6\beta}\right)\\
& - \frac{1}{2}\ln\left(\frac{1}{\beta}\right) + I^*_{\tilde{f}}(b) + \beta\left(\frac{2}{27}(b+b^{-1})^3(3c-2a) + \frac{2}{3}(b+b^{-1})(a-c)\right).
\end{aligned} \tag{D.14}$$

This is the ansatz of [23], now rigorously derived, and supplemented with the β-independent term $I^*_{\tilde{f}}(b)$ which was left undetermined in that work.

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