p-adic uniformization and an Explicit Jacquet-Langlands isomorphism.

by

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DEDICATION

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ABSTRACT

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In this thesis, we study modular forms on definite and indefinite quaternion algebras. These spaces are a priori very different. On the definite side they are abstract spaces of functions defined on finite sets, whereas on the indefinite side they are sections of an appropriate sheaf on a Shimura curve. We construct an explicit, canonical, and Hecke equivariant isomorphism between these spaces with $\mathbb{Q}_p$-coefficients, where p is a prime dividing the level of the modular forms on the definite quaternion algebra. Our map takes the form:

$$\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \to \mathrm{H}^0(X', \Omega^{\otimes k/2})$$

see Theorem 3.3.2 for details. There are natural $\mathbb{Z}_p$ lattices $\mathcal{M}$ and $\mathcal{N}$ on the left and right respectively. This isomorphism carries $\mathcal{M}$ to $\mathcal{N}$, and for $p > \max(k-2, 3)$ restricts to an isomorphism $\mathcal{M} \cong \mathcal{N}$. The quotient $\mathcal{N}/\mathcal{M}$ is a canonical and finitely generated p-torsion Hecke module. Our isomorphism is an explicit, and $\mathbb{Z}_p$-integral refinement of the Jacquet-Langlands correspondence in our setting.

CHAPTER 1

Introduction

1.1 Modular forms on quaternion algebras

A quaternion algebra H over a field F is a central simple algebra over F of dimension four as an F-vector space. For example, the *split* quaternion algebra over F is the ring of 2×2 matrices $M_2(F)$ with coefficients in F. Let $H/\mathbb{Q}$ be a quaternion algebra. The quaternion algebra H is called *indefinite* if $H \otimes \mathbb{R} \cong M_2(\mathbb{R})$, otherwise it is called *definite*. The discriminant $\operatorname{disc}(H)$ of H is an integer, and determines it up to isomorphism. Let $H_\ell = H \otimes \mathbb{Q}_\ell$ for all finite primes ℓ. If ℓ is a finite prime not dividing $\operatorname{disc}(H)$ then $H_\ell \cong M_2(\mathbb{Q}_\ell)$, and H is said to be *split* at ℓ. If $\ell \mid \operatorname{disc}(H)$, then H is said to be ramified at ℓ.

The finite adeles over $\mathbb{Q}$ are denoted $\mathbb{A}_f$ and $\mathbb{A}_f^\ell$ will denote the projection of $\mathbb{A}_f$ to the primes different from ℓ. The finite adeles of H are denoted $H_f = H \otimes \mathbb{A}_f$ and the corresponding topological group of ideles is denoted H_f^*. Their projections away from ℓ are denoted H_f^ℓ, $H_f^{*,\ell}$ respectively. For ℓ a prime, the absolute value $|\cdot|_\ell$ on $\mathbb{Q}_\ell$ is normalized so that $|\ell|_\ell = 1/\ell$. We omit the subscript ℓ when it is clear from context. The reduced norm on H is denoted $\operatorname{Nr}(h)$. Quaternion algebras are also equipped with an involution, which is called the canonical involution and denoted $b \mapsto \bar{b}$. It is an anti-isomorphism of algebras defined by the property that $b\bar{b}$ is the reduced norm $\operatorname{Nr}(b)$. For $g \in H_f^*$, $\|g\| := \prod_{\ell<\infty} |\operatorname{Nr} g|_\ell$.

1.1.0.1 Eichler orders

Fix a quaternion algebra H over $\mathbb{Q}$. Its discriminant is a square-free integer, which we call N^-. An *order* R in H is a subring of H that is finitely generated as a $\mathbb{Z}$-submodule and such that $\mathbb{Q} \cdot R = H$. It is said to be *maximal* if it is not properly contained in another order, and *Eichler* if it is the intersection of two maximal orders.

We fix isomorphisms $\{ \phi_\ell : H_\ell \to M_2(\mathbb{Q}_\ell) \}_\ell$ for each prime ℓ where H is split, and require that these isomorphisms be chosen such that for some order R' of H, ϕ_ℓ identifies $R' \otimes \mathbb{Z}_\ell$ and $M_2(\mathbb{Z}_\ell)$ for all but finitely many ℓ. If $R \subset H$ is an order, then R is determined by its local behavior - that is, by the collection $\{ \phi_\ell(R \otimes \mathbb{Z}_\ell) \}_{H \text{ split}}$ and $\{ R \otimes \mathbb{Z}_\ell \}_{H \text{ not split}}$. Conversely, if $S_\ell \subset H_\ell$ is an order in H_ℓ for each prime ℓ, and all but finitely many of the S_ℓ are equal to $M_2(\mathbb{Z}_\ell)$, then there is a unique order S in H such that $S \otimes \mathbb{Z}_\ell = S_\ell$ [Vig80, III.5.1]. We use this correspondence to construct various Eichler orders.

In order to determine a maximal order, it suffices to specify one at each finite prime ℓ. If ℓ is a prime, the maximal orders of $M_2(\mathbb{Q}_\ell)$ are exactly the conjugates of $M_2(\mathbb{Z}_\ell)$ in $M_2(\mathbb{Q}_\ell)$. If $\ell \mid N^-$, then H ramifies at ℓ, and H_ℓ has a unique maximal order. Let $\mathcal{R}$ be the order which is $M_2(\mathbb{Z}_\ell)$ at each place where H is split (equivalently, at primes ℓ such that $\ell \nmid N^-$), and the unique maximal order at primes $\ell \mid N^-$. One sees that the above prescription can be used to specify any maximal order of H, and hence all Eichler orders. In particular, all Eichler orders of H are maximal at primes where H ramifies. One can be more explicit.

Fix a prime $\ell \nmid N^-$. Then, every Eichler order of H_ℓ is conjugate (in H_ℓ) to the ring of matrices

$$\left\{ x \in \mathcal{R} \;\middle|\; \phi_\ell(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \ell^n \right\}$$

for some integer n. If N^+ is an arbitrary integer coprime to N^-, then the ring R below is an

Eichler order (of level N^+).

$$R = \left\{ x \in \mathcal{R} \;\middle|\; \phi_\ell(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N^+, \forall \ell \text{ split} \right\}$$

It is clear how to modify this procedure to produce arbitrary Eichler orders of H.

1.1.0.2 Definite quaternion algebras

In this section, we construct the space of modular forms on a definite quaternion algebra, and recall the natural inner product defined on it [DT94, p. 446]. They are the space of functions, valued in a vector space defined below, and satisfying a certain transformation property under an open subgroup of B_f^*.

Let $B/\mathbb{Q}$ be a quaternion algebra. We require that it is definite, namely that $\mathrm{disc}(B)$ is a product of an odd number of primes. Fix a prime p, not dividing N^+ or $\mathrm{disc}(B)$ and let N^+ be an integer coprime to $\mathrm{disc}(B)$ and p. Fix an Eichler order $R \subset B$ of level N^+p and let $U = (R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^* \subset B_f^*$. The subgroup U is open compact in B_f^*. We define the space of modular forms of weight k for U with coefficients in a $\mathbb{Q}_p$-vector space, where $k \geq 2$ is an even integer.

Let $n = k-2$, and $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ be the $\mathbb{Q}_p$ vector space of polynomials in x, y that are homogenous of degree n and coefficients in $\mathbb{Q}_p$. Endow it with a left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the formula

$$\gamma P(x,y) = P((x,y)\cdot\gamma), \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$$

where (x, y) is considered a row vector. Fix an isomorphism $\phi_p : B_p \rightarrow M_2(\mathbb{Q}_p)$ such that R_p is identified with $M_2(\mathbb{Z}_p)$, and endow this space with a B_p^* action via ϕ_p.

The space of weight k modular forms for U is

$$\mathcal{L}_k(U, \mathbb{Q}_p) = \left\{ f : B_f^* \to \mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}) \;\middle|\; \begin{array}{cc} f(bz) = f(z) & \forall z \in B_f^*, b \in B^* \\ f(zu) = u_p^{-1} f(z) & \forall u \in U \end{array} \right\}$$

Here, u_p is the component of the idele u at p. The vector space $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ carries an $\mathrm{SL}_2(\mathbb{Q}_p)$ invariant bilinear form which is defined by the property that for all $0 \leq i, j \leq n$, we have

$$\langle x^i y^{n-i}, x^j y^{n-j} \rangle = (-1)^i i!(n-i)! \delta_{i,n-j}.$$

This induces the following non-degenerate inner product on $\mathcal{L}_k(U, \mathbb{Q}_p)$:

$$\langle f_1, f_2 \rangle = \sum_{[g] \in B^* \backslash B_f^* / U} \langle f_1(g), f_2(g) \rangle (\mathrm{Nr}\, g_p \| \mathrm{Nr}\, g \|)^{k-2}.$$

Remark 1.1.1. If $k = 2$ then $n = 0$ and $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}) = \mathbb{Q}_p$. The modular forms of weight two, $\mathcal{L}_2(U, \mathbb{Q}_p)$, are identified with the $\mathbb{Q}_p$-valued functions on the finite set $B^* \backslash B_f^* / U$.

There is a commutative algebra $\mathbb{T} = \mathbb{Z}[\{ T_m \}_{m \nmid \mathrm{disc}(B) \cdot N^+ p}]$, that acts on this space by linear operators that are diagonalizable over an algebraically closed field. Let $\eta_q \in B_f^*$ be the idele which is one away from q and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ at the q^{th} place. If $U \eta_q U = \coprod_i U x_i$ is a disjoint union decomposition, then the q^{th} Hecke operator of $\mathcal{L}_k(U, \mathbb{Q}_p)$ is the map

$$(T_q f)(z) = \sum_i x_{i,p} \cdot f(z x_i)$$

where $x_{i,p}$ is the component of x_i at p, and the action is via the isomorphism ϕ_p fixed above. There are two natural maps to $\mathcal{L}_k(U, \mathbb{Q}_p)$ from modular forms attached to an Eichler order of level N^+, and the span of their image is called the p-old space. Its orthogonal complement is the p-new space and denoted $\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$. We do not give definitions here, but see Definition 2.5.1 for a precise statement.

1.1.0.3 Indefinite quaternion algebras

In this section, we recall the construction of the space of modular forms attached to an indefinite quaternion algebra.

Let D be an indefinite quaternion algebra over $\mathbb{Q}$, and $\phi_\infty : D \otimes \mathbb{R} \to M_2(\mathbb{R})$ an isomorphism. We assume $D \neq M_2(\mathbb{Q})$. Fix a set of isomorphisms $\{ \phi'_\ell : D_\ell \to M_2(\mathbb{Z}_\ell) \}_{\ell \nmid \mathrm{disc}(D)}$ as in §1.1.0.1, and let R_D be the Eichler order with the property that

$$\phi_\ell(R_D \otimes \mathbb{Z}_\ell) = \left\{ x \in M_2(\mathbb{Z}_\ell) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N^+ \right\}$$

Its subgroup of norm one elements is denoted $R_D^{*,1}$, and $\Gamma = \phi_\infty(R_D^{*,1}) \subset \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. It acts on the complex upper half plane

$$\mathcal{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$$

by Möbius transformations, and the quotient

$$\Gamma \backslash \mathcal{H}$$

is a compact Riemann surface with a canonical model over $\mathbb{Q}$, which we call $X_0^{\mathrm{disc}(D)}(N^+)$, or simply X. It is a *Shimura curve*. The space of modular forms for Γ of weight k, and coefficients in $\mathbb{Q}_p$ is

$$S_k(\Gamma, \mathbb{Q}_p) = \mathrm{H}^0(X \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_p, \Omega^{\otimes k/2}_{X \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{Q}_p})$$

There is a commutative algebra $\mathbb{T} = \mathbb{Z}[\{ T_m \}_{m \nmid \mathrm{disc}(D) \cdot N^+}]$, called the Hecke algebra, that acts on this space by linear operators that are diagonalizable over an algebraically closed field. These endomorphisms are induced by correspondences on the algebraic curve $X/\mathbb{Q}$.

1.1.0.4 The Jacquet-Langlands correspondence

In their book [JL70], Jacquet and Langlands proved a general version of the correspondence that bears their name. Their results are in the language of automorphic representations. Here, we recall the statement of the correspondence in a more classical setting, and define notation that will be used in the remainder of the introduction.

Let $N = N^- N^+ p$ be a squarefree integer, with N^- divisible by an odd number of primes. Let $B/\mathbb{Q}$ be the (definite) quaternion algebra of discriminant N^-, and $D/\mathbb{Q}$ the (indefinite) quaternion algebra of discriminant $N^- \cdot p$. Fix a set of isomorphisms

$$\{ \phi_\ell : B_\ell \to M_2(\mathbb{Z}_\ell) \}_\ell$$

at places where B is split, as in §1.1.0.1 and let R be the Eichler order in B such that

$$\phi_\ell(R \otimes \mathbb{Z}_\ell) = \left\{ x \in M_2(\mathbb{Z}_\ell) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N^+ \cdot p \right\}$$

Fix isomorphisms $\sigma_\ell : D_\ell \to B_\ell$ for each prime ℓ where both B and D split (that is, primes not dividing $N^- \cdot p$), and let R_{indef} be the unique Eichler order of D that agrees with R at all such primes via the fixed isomorphisms. As we noted above, it is necessarily the unique maximal order at all other places. Let $\Gamma_\mathbb{C}$ be the Fuchsian group attached to R_{indef}, U the compact open subgroup of B_f^* attached to R, and $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined above. Let $\mathbb{T}^B$ and $\mathbb{T}^D$ be the Hecke algebras acting on $\mathcal{L}_k(U, \mathbb{Q}_p)$ and $S_k(\Gamma_\mathbb{C}, \mathbb{Q}_p)$ respectively.

Let $f \in \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ be a simultaneous eigenform for the Hecke operators $\mathbb{T}^B$. That is, for each prime $q \nmid N$

$$T_q f = a_q f$$

for some $a_q \in \mathbb{Q}_p$. Then the Jacquet-Langlands correspondence states that there exists $f' \in S_k(\Gamma_{\mathbb{C}}, \mathbb{Q}_p)$ such that the same equation holds, replacing T_q with the q^{th} Hecke operator in $\mathbb{T}^D$, and f by f'.

After tensoring with $\mathbb{C}_p$, one can leverage this to produce non-canonical but Hecke-equivariant isomorphisms between the spaces of modular forms above, by mapping integrally normalized eigenforms to each other. Such an isomorphism is then also an isomorphism of integral lattices spanned by normalized eigenforms. However, the natural integral structures on both sides are not necessarily spanned integrally by eigenforms (on account of congruences between eigenforms).

1.2 Statement of results

Our main result is an explicit isomorphism that exhibits the Jacquet-Langlands correspondence in the above setting, and that respects the natural $\mathbb{Z}_p$-integral structures on both sides. We identify $\mathcal{L}_k(B, \mathbb{Q}_p)^{p-new}$ with sections of a bundle on an explicit twisted form X' of X which is produced in the theory of p-adic uniformization of
Cerednik-Drinfeld [BC91], [Čer76]. We briefly recall the construction of this curve. For a more detailed introduction see §3.1.

1.2.0.1 p-adic Uniformization

Let Ω be the p-adic upper half plane over $\mathbb{Q}_p$, considered as a rigid analytic space. It can be identified with $\mathbb{P}^1 - \mathbb{P}^1(\mathbb{Q}_p)$ and is the generic fiber of a formal scheme $\widehat{\Omega}$, called the formal p-adic upper half plane. The group of matrices $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widehat{\Omega}$ and Ω by
Mobius transformations as usual. Recall that B is a quaternion algebra of discriminant N^-, and that we fixed a set of isomorphisms

$$\{\ \phi_\ell : B_\ell \to M_2(\mathbb{Z}_\ell)\ \}_\ell$$

at places where B is split, as in §1.1.0.1. The map $\phi_p : B_p \to M_2(\mathbb{Q}_p)$ induces an action of B_p^* on Ω via

Mobius transformations. Let R' be the Eichler order such that for each ℓ where B is split,

$$\phi_\ell(R' \otimes \mathbb{Z}_\ell) = \left\{ x \in M_2(\mathbb{Z}_\ell) \;\middle|\; x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N^+ \right\}$$

Define

$$\Gamma' = R'[1/p]^*$$
$$\Gamma = \{z \in \Gamma' | \operatorname{Nr}(z) = 1\}$$
$$\Gamma^{(2)} = \{z \in \Gamma' | \operatorname{Nr}(z) = p^{2k} \text{ for some } k \in \mathbb{Z}\}$$

considered as a subgroups of B_p^*. Let $W = \Gamma'/\Gamma^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ (see Lemma 2.3.1), and let $w_p \in \Gamma'$ be an arbitrary element of norm p (see Lemma 2.3.1). The curve $X \times \operatorname{Spec} \mathbb{Q}_p$ naturally extends to a scheme $\mathcal{X}$ over $\mathbb{Z}_p$ (see §3.1), and its formal completion at p, $\hat{\mathcal{X}}$, admits p-adic uniformization after a field extension (see §3.1.0.3). More precisely, let

$$\hat{\mathcal{X}}' = \Gamma^{(2)} \backslash \widehat{\Omega}$$

be the formal scheme quotient. Let $\mathbb{Z}_p^{(2)} \subset \mathbb{Z}_p^{unr}$ be the ring of integers of the unramified quadratic extension of $\mathbb{Z}_p$, and $\mathbb{Q}_p^{(2)}$ its fraction field. Then,

$$\hat{\mathcal{X}} := W \backslash \left(\hat{\mathcal{X}}' \otimes \mathbb{Z}_p^{(2)} \right)$$

and $\hat{\mathcal{X}}$ is a formal scheme over $\mathbb{Z}_p$ and it is a twisted form of $\hat{\mathcal{X}}'$. The corresponding cocycle in $\mathrm{H}^1(\operatorname{Gal}(\mathbb{Q}_p^{(2)}/\mathbb{Q}_p), \operatorname{Aut}(\hat{\mathcal{X}}'))$ sends the nontrivial element of the Galois group to w_p. Teitelbaum [Tei93, Lemma 31] (see §3.2.2) constructs a sheaf ω on $\widehat{\Omega}$ whose quotient is naturally identified with the formal dualizing sheaf of $\hat{\mathcal{X}}'$. The induced sheaf $\omega \otimes \mathbb{Q}_p$ on the generic fiber X'^{an} is naturally identified with the sheaf of rigid differential forms. Define

$$S_k(X_0^{N^-p}(N^+)', \mathbb{Q}_p) = S_k(X', \mathbb{Q}_p) = \mathrm{H}^0(X'^{an}, \Omega^{\otimes k/2}_{X'^{an}/\mathbb{Q}_p}) = \mathrm{H}^0(\Omega, \omega^{\otimes k/2})^\Gamma$$

This is the space of modular forms that appears in our main theorem, Theorem 3.3.2.

1.2.0.2 Explicit Jacquet-Langlands isomorphism

In this section we state our main results in the context of past work on this problem. The first is an explicit version of the Jacquet-Langlands transfer, Theorem 3.3.2.

Theorem 1.2.1. *(Explicit Jacquet-Langlands) Assume $p > 3$. There is an explicit, canonical (up to sign), and Hecke equivariant isomorphism*

$$\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \xrightarrow{\quad \mathrm{JL} \quad} S_k(X', \mathbb{Q}_p).$$

If the weight k is two, then this map has a simple description. The dual graph of $\mathcal{X}' \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$ is naturally a quotient of the Bruhat-Tits tree $\mathcal{T}$. Singular points on $\mathcal{X}' \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$ are necessarily ordinary double points, and their generic fibers (in the sense of Raynaud, that is, points of $X'^{,rig}(\mathbb{Q}_p)$ that specialize to a singular point) are rigid analytic annuli. Elements of $S_k(X', \mathbb{Q}_p)$ are simply $\mathbb{Q}_p$-valued rigid-analytic functions that satisfy a modular transformation property under Γ. If $f \in S_k(X', \mathbb{Q}_p)$, we can consider its residue on each of the annuli defined above. This defines a Γ-invariant function on the edges of the Bruhat-Tits tree of $\mathrm{PGL}(2, \mathbb{Q}_p)$.

By strong approximation, elements of $\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ are naturally identified with functions on the edges of this tree, and this identification is an isomorphism in weight two.

In their article [BD98], M. Bertolini and H. Darmon establish an isomorphism between a space of p-new cusp forms of weight two on a definite quaternion algebra and a space of functions on the edges of the Bruhat-Tits tree of $\mathrm{PGL}(2, \mathbb{Q}_p)$. Their paper, specifically [BD98, Proposition 1.4] inspired our work, and Theorem 2.5.4 should be considered a generalization of their result.

1.2.0.3 $\mathbb{Z}_p$-integral structure

The next result says that this isomorphism is compatible with natural $\mathbb{Z}_p$-integral structures on both sides. Note that elements of the left hand side are identified with p-adic functions on a (possibly finite) set of points, and the integral structure is natural in this setting [DT94]. For example, in weight two, it is defined by scaling p-adic functions on the finite set $B^*\backslash B_f/U$ to take values inside $\mathbb{Z}_p$.

On the right hand side, the integral structure is defined geometrically, namely as global sections of the dualizing sheaf of a certain model over $\mathbb{Z}_p$. See Remark 3.1.2 for a relation between this model and the minimal regular model over $\mathbb{Z}_p$.

Theorem 1.2.2. *If $p > \max(k-2, 3)$ then the isomorphism above takes a natural $\mathbb{Z}_p$-lattice in $\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ to the $\mathbb{Z}_p$-lattice defined by a certain model of (an explicit twisted form X') of the Shimura curve over $\mathbb{Z}_p$.*

For a more precise statement see Theorem 4.2.9. In general, for $p > 3$ and all weight, the explicit Jacquet-Langlands isomorphism induces an inclusion of finite index lattices. Namely, the lattice defined by the dualizing sheaf is always contained in the image of the lattice coming from the definite quaternion algebra. The quotient is a canonically defined torsion module. See Theorem 4.2.9 for a precise statement. There are algorithms, implemented in the software package Sage, that can be used to compute this [FM14].

1.3 Overview of proof

To prove the main theorem, we identify the space of modular forms $\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ and $S_k(X', \mathbb{Q}_p)$ with a space of functions on the edges of the Bruhat-Tits tree of $\mathrm{SL}_2(\mathbb{Q}_p)$. These two maps compose to give Theorem 3.3.2.

The first identification we prove is for $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$, and it is Theorem 2.5.4. This is the main result of Chapter 2. In the first few sections of this chapter, we set up the required notation, and define and prove elementary results about various groups defined in quaternion algebras. The map that appears in Theorem 2.5.4 already appears in Remark 2.3.2, where we also establish it is an isomorphism. Theorem 2.5.4 is then the observation that this map restricts to identify p-new and harmonic functions. To check that, we need the modules defined in §2.3.3, and the inner products defined in §2.4 along with their basic properties. Those results are combined in Lemma 2.5.2, of which Theorem 2.5.4 is essentially a corollary.

In Chapter 3, we identify $S_k(X',\mathbb{Q}_p)$ with a space of harmonic functions. This follows from work of [Sch84], [dS89], [SS91], and [Tei93], and competes the proof of Theorem 3.3.2. The remainder of this article is to check properties of this map. In §4.1 we compute the action of the Hecke algebras $\mathbb{T}^D$ and $\mathbb{T}^B$ on the appropriate spaces of functions, and check that they align. In §4.2 we define an integral structure on $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$ (Definition 4.2.1), which agrees with $\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$ if the involution W_p defined in §2.5.0.2 is trivial on the p-new space. The integral structure coming from a model (of a twisted form) $\mathcal{X}'$ of the Shimura curve X is identified and computed in [Tei93]. Finally, we combine these results to prove a compatibility between the two $\mathbb{Z}_p$ integral structures in Theorem 4.2.9. The desired p-torsion Hecke module arises because Theorem 3.3.2 induces a finite index inclusion of these lattices.

1.4 Further directions

One would like to understand how the isomorphism JL of Theorem 3.3.2 interacts with certain natural inner products on both sides. In particular, we would like to relate the natural inner product $\langle , \rangle_{def}$ on $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$ to a natural bilinear form $\langle \cdot, \cdot \rangle$ defined via p-adic Hodge theory.

The vector space, $S_k(X',\mathbb{Q}_p)$ is identified with a subspace of the (ϕ, N) module $\mathrm{H}^1(X'^{,an}, \mathbb{V}_{k-2})$ associated to a semi-stable Galois representation attached to a certain (self-dual) local system $\mathbb{V}_{k-2}$ on $X^{\prime,an}$, and N is the monodromy operator. For $f', g' \in S_k(X', \mathbb{Q}_p)$, we define $\langle f', g' \rangle = \mathrm{tr}(f' \cup N(g'))$, where cup product followed by the trace map is given by the composite:

$$H^1(X^{an}, \mathbb{V}_{k-2}) \times H^1(X^{an}, \mathbb{V}_{k-2}) \xrightarrow{\cup} H^2(X^{an}, \mathbb{V}_{k-2}^{\otimes 2}) \xrightarrow{\mathrm{tr}} \mathbb{Q}_p$$

Let $\mathrm{JL}_f, \mathrm{JL}_g$ be the images of $f, g \in \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ under JL. Then one may expect a relation of the form:

$$\langle f, g \rangle_{def} = \langle \mathrm{JL}_f, \mathrm{JL}_g \rangle.$$

We hope an equality of this form can be used to establish an analogue of Conj. 4.2 of [Pra08] at the prime p, up to p-adic units.

CHAPTER 2

Modules on the Bruhat-Tits Tree

The main result of §2 is Theorem 2.5.4, and the results outside §2, interact with this section primarily through that theorem, and Lemmas 2.2.5 and 2.5.3.

Theorem 2.5.4 identifies p-new modular forms on a definite quaternion algebra (see Definition 2.5.1) with a space of harmonic functions on the edges of the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$. In §2.1 we setup notation and recall the construction of various rings and modules that play a role throughout this article. In §2.2 we recall basic results about the Bruhat-Tits tree in a form that we use them. The main result of that section is to prove Lemma 2.2.5, and collect supporting lemmas used in this chapter. In §2.3 we construct various modules on the Bruhat-Tits tree. The isomorphism that appears in Theorem 2.5.4 is the restriction of the map that appears in Remark 2.3.1. The results of §2.4 are used to define the space of harmonic forms. Finally, §2.3.3 and §2.4 are combined in §2.5 to prove Lemma 2.5.2 and Lemma 2.5.3 which imply the main result of this section.

2.1 Setup

In this section we construct various oriented and unoriented Eichler orders, and define the space of modular forms on a definite quaternion algebra. The notation introduced here is used throughout the article, except for cases where we indicate otherwise.

2.1.0.1 Quaternion algebras

Let $H/\mathbb{Q}$ a quaternion algebra. For $h \in H^*$ let $h^{(\ell)} \in H_f^*$ denote the idele $(h, \cdots, h, 1, h, \cdots)$ which is h away from the ℓ^{th} place. Define

$$j_\ell^H = j_\ell : H_\ell^* \to H_f^* \qquad j_\ell^H(z) = j_\ell(z) = (1, \cdots, 1, z, 1, \cdots)$$

where $j_\ell(z)$ is the idele which is z at the ℓ^{th} place. When possible, we denote this map simply by j^H or j. The superscript is used to indicate the quaternion algebra, and is surpressed if it is clear from context.

An element $z \in H_p^*$ will be will called *even* (resp. *odd*) if $|\mathrm{Nr}\, z| = p^{2k}$ (resp $|\mathrm{Nr}\, z| = p^{2k+1}$) for some $k \in \mathbb{Z}$. Define sign$(z) = 1$ if z is even and -1 if v is odd.

2.1.0.2 Definite quaternion algebras

Fix a squarefree integer $N = N^-(N^+ \cdot p)$ with p prime, and N^- divisible by an odd number of primes. Throughout this article B will denote the definite quaternion algebra over $\mathbb{Q}$ ramified precisely at the primes dividing the square-free integer N^- and ∞. It is unique up to isomorphism [Vig80, Thm. IV.3.1].

In our context, strong approximation says the following. If $U \subset B_f^*$ is an open subgroup such that $\mathrm{Nr}(U) \supset \prod_{q<\infty} \mathbb{Z}_q^*$ then

$$B^* \cdot B_p^{*,1} \cdot U = B_f^* \tag{2.1}$$

Indeed, LHS contains $B_f^{*,1}$ (ideles that have norm 1 at all places)[Vig80, Theorem III.4.3], and $B^* \cdot U \cdot B_f^{*,1} = B_f^*$.

2.1.0.3 Eichler Orders

Recall that we can construct Eichler orders of B following [Vig80, p.20].

For each integer M coprime to N^-, let $\mathcal{R}(M)$ be the following Eichler order of *level* M [Vig80, §II.2]

$$\mathcal{R}(M) = \left\{ x \in \mathcal{R} \middle| \phi_\ell(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \ell \quad \forall \ell | M \right\}$$

It is maximal at places not dividing M and stable under the canonical involution. Indeed, this can be checked locally [Vig80, III.5.1]. This is clear at split places [Vig80, p.3], and from the discussion above, also at ramified places. Let $R = \mathcal{R}(N^+p)$, and recall that all Eichler orders of level p in B_p are conjugate to

$$R_p = \left\{ x \in \mathcal{R}_p \middle| \phi_\ell(x) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p \right\}$$

2.1.0.4 Orientation

If $\ell \mid N^-$ then an *orientation* [Rob89, Chapter I] on $\mathcal{R}_\ell$ is a homomorphism of rings $\mathfrak{o} : \mathcal{R}_\ell \to \mathbb{F}_{\ell^2}$. If $\ell \nmid N^-$ then it is a homomorphism of rings $\mathfrak{o} : \mathcal{R}_\ell \to \mathbb{Z}/\ell \times \mathbb{Z}/\ell$. A non-maximal oriented Eichler order has exactly two orientations. In the case of R_p they are $\mathfrak{o}(\gamma) = (a, d)$ and $\mathfrak{o}(\gamma) = (d, a)$. That is, the map that takes

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, d) \in \mathbb{Z}/p \times \mathbb{Z}/p$$

(or respectively (d, a)). Choose an orientation $\underline{\mathrm{R}}_\ell$ on R_ℓ for each ℓ, and let $\underline{\mathrm{R}}$ denote the Eichler order R along with these choices of orientations. It is an *oriented Eichler order*.

An automorphism g of the oriented Eichler order $\mathcal{R}_p$ is an automorphism $g : R_p \to R_p$ of the underlying ordinary order such that the following diagram commutes:

$$\begin{tikzcd}
R_p \arrow[d, "g"'] \arrow[dr, "o"] & \\
 & \mathbb{Z}/p \times \mathbb{Z}/p \\
R_p \arrow[ur, "o"'] &
\end{tikzcd}$$

2.1.0.5 Modular Forms

In this section, we define the space of weight k modular forms on B_f^* following [DT94, §2]. Throughout this article, the weight $k \geq 2$ is an even integer, and we let $n = k - 2$.
In our setting, weight k modular forms are functions that take values in a vectorspace of polynomials of degree $n = k - 2$. Recall that the prime p is fixed above.

Definition 2.1.1. Let $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ denote the space of polynomials in x, y that are homogenous of degree n, and let $\mathrm{GL}_2(\mathbb{Q}_p)$ act on this vectorspace on the left by

$$\gamma P(x, y) = P((x, y) \cdot \gamma) \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$$

where (x, y) is considered a row vector. Note that if $\gamma = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ then

$$\gamma P = P \cdot p^n = P \cdot \mathrm{Nr}\, \gamma^{n/2} \tag{2.2}$$

$$P = \gamma P \,\mathrm{Nr}\, \gamma^{-n/2} = \gamma P |\mathrm{Nr}\, \gamma|^{n/2} \tag{2.3}$$

Define the $\mathbb{Q}_p$ bilinear form

$$\langle x^i y^{n-i}, x^j y^{n-j} \rangle = (-1)^i i!(n-i)!\delta_{i,n-j}$$

This inner product is $\mathrm{SL}_2(\mathbb{Q}_p)$ invariant, and satisfies the following transformation property

$\forall F, G \in \mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$:

$$\langle \gamma \cdot F, \gamma \cdot G \rangle = \det \gamma^n \langle F, G \rangle \tag{2.4}$$

Definition 2.1.2. Let $U \subset B_f^*$ be a compact open subgroup. The space of weight k modular forms for U is

$$\mathcal{L}_k(U, \mathbb{Q}_p) = \left\{ f : B_f^* \to \mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}) \;\middle|\; \begin{array}{cc} f(bz) = f(z) & \forall z \in B_f^*, b \in B^* \\ f(zu) = u_p^{-1} f(z) & \forall u \in U \end{array} \right\}$$

Here, u_p is the component of u at p and u_p^{-1} acts on the left on $P_n(\mathbb{Q}_p)$ as the matrix $\phi_p(u_p^{-1})$ where $\phi_p : B_p \xrightarrow{\cong} M_2(\mathbb{Q}_p)$ is the isomorphism fixed above. We embed $B^* \subset B_f^*$ diagonally. Following [DT94, p.446], we endow $\mathcal{L}_k(U, \mathbb{Q}_p)$ with the following non-degenerate bilinear form

$$\langle f_1, f_2 \rangle = \sum_{[g] \in B^* \backslash B_f^* / U} \langle f_1(g), f_2(g) \rangle (\mathrm{Nr}\, g_p \|\mathrm{Nr}\, g\|)^{k-2} \tag{2.5}$$

Remark 2.1.1. If $k = 2$ then $n = 0$ and $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}) = \mathbb{Q}_p$. The modular forms of weight two $\mathcal{L}_2(U, \mathbb{Q}_p)$ are then identified with the $\mathbb{Q}_p$ valued functions on the finite set $B^* \backslash B_f^* / U$.

2.2 The Bruhat-Tits tree

2.2.1 Normalizers

Here we consider the normalizers in B_p^* of various rings. Recall that B is split at p and $\mathcal{R}_p$ corresponds to $M_2(\mathbb{Z}_p)$ under the isomorphism ϕ_p. Let $\beta = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and recall [Vig80, p.40] that the normalizers of $\mathcal{R}_p$ and R_p are

$$N(\mathcal{R}_p) = \mathbb{Q}_p^* \mathcal{R}_p^*$$
$$N(R_p) = \langle \mathbb{Q}_p^* R_p^*, \beta \rangle$$

For convenience, we record that the normalizer of $\underline{\mathrm{R}}_p$ is

$$N(\underline{\mathrm{R}}_p) = \mathbb{Q}_p^* R_p^*$$

and hence $N(R_p) = \langle N(\underline{\mathrm{R}}_p), \beta \rangle$. For a proof, see Lemma 2.2.2 below.

In this section we observe that elements of these groups can be put into a canonical form. First, note that $\beta^2 = p \cdot I_2 = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ and $\beta^{-1} = \beta \cdot p^{-1} = \left(\begin{smallmatrix} 0 & 1/p \\ 1 & 0 \end{smallmatrix}\right)$. Observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \implies \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \beta^{-1} = \beta^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \beta = \begin{pmatrix} d & c/p \\ pb & a \end{pmatrix} \tag{2.6}$$

In particular,

$$R_p = \mathcal{R}_p \cap \beta \, \mathcal{R}_p \, \beta^{-1} = \mathcal{R}_p \cap \beta^{-1} \, \mathcal{R}_p \, \beta$$

We specify elements of B_p as matrices via the isomorphism ϕ_p fixed above.

Lemma 2.2.1. *Every element $b \in N(R_p) = \langle \mathbb{Q}_p^* R_p^*, \beta \rangle$ can be written in exactly one way as $b = \lambda \alpha \beta^m$ and in exactly one way as $b = \lambda \beta^m \alpha'$ with $m \in \{0, 1\}$, $\alpha, \alpha' \in R_p^*$ and $\lambda = \left(\begin{smallmatrix} p^t & 0 \\ 0 & p^t \end{smallmatrix}\right)$ for some $t \in \mathbb{Z}$.*

Proof. Uniqueness is clear because λ and t are determined by $|\mathrm{Nr}\, b|$, which implies that α and α' are also uniquely determined.

To show existence, it suffices to show that if $g \in N(R_p)$ with $|\mathrm{Nr}\, g| = 1$, then, $g \in R_p^*$.

Recall that $\beta^2 = pI_2$. Write g as a finite product $\alpha_1\beta^{m_1}\alpha_2\beta^{m_2}\cdots\alpha_s\beta^{m_s}$ with $\alpha_i \in R_p^*$ and $m_i \in \{-1, 0, 1\}$. Without loss of generality, assume that this expression was taken with minimal s.

Suppose m is minimal, and consider the collection S of substrings of this word containing both β and β^{-1}. If β appears in the product then so does β^{-1} (because $\mathrm{Nr}\, g \in \mathbb{Z}_p^*$). Choose the shortest such substring in S. It must contain exactly one β and exactly one β^{-1}, and hence be of the form $\beta\tau\beta^{-1}$ or $\beta^{-1}\tau\beta$ with $\tau \in R_p^*$. However, since $\beta \in N(R_p)$ each of these two expressions are in R_p^*. Thus, if this product contains β, we can modify it to decrease s. That is a contradiction. ■

Lemma 2.2.2. *The normalizer in B_p^* of $\underline{\mathrm{R}}_p$ is*

$$N(\underline{\mathrm{R}}_p) = \mathbb{Q}_p^* R_p^*$$

Proof. By Lemma 2.2.1 it suffices to check that R_p^*, $\mathbb{Q}_p^*$ are in the normalizer, and β is not. It is clear that $\mathbb{Q}_p^*$ is in the normalizer. If $\gamma = \left(\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right)$ and $m = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)$ are the images in $\mathrm{GL}_2(\mathbb{F}_p)$ of elements of R_p^*, then

$$\gamma^{-1} = \frac{1}{a'd'}\begin{pmatrix} d' & -b' \\ & a' \end{pmatrix}$$

and

$$\gamma^{-1} \cdot m \cdot \gamma = \begin{pmatrix} a & * \\ & d \end{pmatrix}$$

That is, $R_p^* \subset N(\underline{\mathrm{R}}_p)$. Finally, from §2.1.0.4, it is clear that $\beta \notin N(\underline{\mathrm{R}}_p)$. Indeed, by (2.6) it interchanges the two orientations. ■

Lemma 2.2.3. *Each $b \in N(\underline{\mathrm{R}}_p) = R_p^*\mathbb{Q}_p^*$ can be uniquely written in the form $\lambda\alpha$ for $\alpha \in R_p^*$ and $\lambda = \begin{pmatrix} p^t & 0 \\ 0 & p^t \end{pmatrix}$ for some $t \in \mathbb{Z}$. Similarly, each $b \in N(\mathcal{R}_p)$ can be written in the form above with $\alpha \in \mathcal{R}_p^*$.*

Proof. Existence is clear since $\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right) \in R_p^* \subset \mathcal{R}_p^*$ for each $u \in \mathbb{Q}_p^*$ with $|u| = 1$. Uniqueness is also clear. If $b = \lambda\alpha$ is an expression as above, $|\mathrm{Nr}\, b| = |\mathrm{Nr}\, \lambda| \cdot |\mathrm{Nr}\, \alpha|$, and $|\mathrm{Nr}\, \alpha| = 1$. This specifies λ uniquely. ■

Corollary 2.2.4. *Let $g \in N(R_p)$ be even (that is, $|\mathrm{Nr}\, g| = p^{2k}$ for some $k \in \mathbb{Z}$). Then, $g \in N(\underline{\mathrm{R}}_p)$.*

Proof. This is clear from Lemma 2.2.1 and Lemma 2.2.3. ■

2.2.2 The Bruhat-Tits Tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$

The Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a tree with vertices $\mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathcal{T})$.

$$\begin{aligned}
\mathcal{V}(\mathcal{T}) &= \{ \text{ maximal orders in } B_p \,\} \\
\mathcal{E}(\mathcal{T}) &= \{ \text{ Eichler orders of level } p \text{ in } B_p \,\} \\
\overrightarrow{\mathcal{E}}(\mathcal{T}) &= \{ \text{ oriented Eichler orders of level } p \text{ in } B_p \,\}
\end{aligned}$$

Each of $\mathcal{V}(\mathcal{T}), \mathcal{E}(\mathcal{T})$ and $\overrightarrow{\mathcal{E}}(\mathcal{T})$ has a natural left B_p^* structure where $z \in B_p^*$ acts by conjugation: $X \mapsto z \cdot X \cdot z^{-1}$. The reader should think of $\overrightarrow{\mathcal{E}}(\mathcal{T})$ as the set of edges of $\mathcal{T}$ with orientation. Recall that we fixed a maximal order $\mathcal{R}_p = \phi_p^{-1}(M_2(\mathbb{Z}_p))$, an Eichler order $R_p \subset \mathcal{R}_p$ of level p, and an orientation on R_p (that is, an oriented Eichler order $\underline{\mathrm{R}}_p$ of level p). Two vertices of $\mathcal{T}$ are connected by an edge iff their intersection is an Eichler order of level p, or equivalently in $\mathcal{E}(\mathcal{T})$. The chosen maximal order $\mathcal{R}_p$ defines a vertex of $\mathcal{T}$. We call a vertex $v \in \mathcal{V}(\mathcal{T})$ *even* (resp. *odd*) if the number of edges in any path connecting $\mathcal{R}_p$ and v is even (resp. odd). Recall that we call an element $z \in B_p^*$ *even* if $|\mathrm{Nr}\, z| = p^{2t}$ for some $t \in \mathbb{Z}$, and *odd* otherwise. An edge $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$ is called *even* if its origin is an even vertex, and *odd* otherwise.

Remark 2.2.1. The tree $\mathcal{T}$ is equivalently the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ [BC91, I.1], [Mus78, §1] or the tree of $\mathrm{SL}_2(\mathbb{Q}_p)$ as described in [Ser80, II]. To make this identification, let $\mathrm{GL}_2(\mathbb{Q}_p)$ act on $\mathbb{Q}_p^2$ by left multiplication. Then, the homothety class of a $\mathbb{Z}_p$ lattice in $\mathbb{Q}_p^2$

defines a maximal compact subgroup of $M_2(\mathbb{Q}_p)$ (its stabilizer), and this correspondence is bijective. Similarly, two vertices are joined by an edge iff their intersection is conjugate to R_p [Ser80, p.77], or equivalently is an Eichler order of level p [Vig80, II.2.4]. In [GvdP80, p.18], the tree $\mathcal{T}$ above is identified with the Tits building of $\mathrm{GL}_2(\mathbb{Q}_p)$

Recall that the goal of this chapter is to prove Theorem 2.5.4, which is a relation between a space of modular forms and certain functions on the edges of the Bruhat-Tits tree. The space of modular forms $\mathcal{L}_k(U, \mathbb{Q}_p)$ can be exhibited as a space of functions on the group B_p^*. We will use the following lemma repeatedly in to identify this space as functions on the edges of the Bruhat-Tits tree. The map d below is a choice of orientation. It embeds $\mathcal{E}(\mathcal{T})$ in $\overrightarrow{\mathcal{E}}(\mathcal{T})$ as the edges that start at even vertices.

Lemma 2.2.5. *There is a commutative diagram*

$$\begin{tikzcd}
B_p^* \arrow[r, "\tilde{D}"] \arrow[d] & B_p^* \arrow[r, bend left, "\tilde{O}"] \arrow[r, bend right, "\tilde{T}"'] \arrow[d] & B_p^* \arrow[d] \\
B_p^*/N(R_p) \arrow[r, "D"] \arrow[d, "a"'] & B_p^*/N(\underline{\mathrm{R}}_p) \arrow[r, bend left, "O"] \arrow[r, bend right, "T"'] \arrow[d, "b"'] & B_p^*/N(\mathcal{R}_p) \arrow[d, "c"] \\
\mathcal{E}(\mathcal{T}) \arrow[r, "d"'] & \overrightarrow{\mathcal{E}}(\mathcal{T}) \arrow[r, bend left, "o"] \arrow[r, bend right, "t"'] & \mathcal{V}(\mathcal{T})
\end{tikzcd}$$

where the maps $a, b, c, d, D, \tilde{D}, o, O, \tilde{O}, t, T$ *and* $\tilde{T}$ *are defined as follows. For all* $z \in B_p^*$, *we define*

$$a(z) = zR_pz^{-1}$$
$$b(z) = z\,\underline{\mathrm{R}}_p\,z^{-1}$$
$$c(z) = z\,\mathcal{R}_p\,z^{-1}$$

$$o(z\,\underline{\mathrm{R}}_p\,z^{-1}) = z\,\mathcal{R}_p\,z^{-1}$$
$$O([z]) = [z]$$
$$\tilde{O}(z) = z$$

$$t(z\,\underline{\mathrm{R}}_p\,z^{-1}) = z\beta\,\mathcal{R}_p(z\beta)^{-1}$$
$$T([z]) = [z\beta]$$
$$\tilde{T}(z) = z\beta$$

For all $z \in B_p^*$, define

$$\tilde{D}(z) = \begin{cases} z & z \textit{ even} \\ z\beta & z \textit{ odd} \end{cases}$$

It is clear that each element of $B_p^*/N(R_p)$ contains a representative $[z]$ with z even, and if such a representative z is chosen, we define $\mathrm{d}(zR_pz^{-1}) = z\,\underline{\mathrm{R}}_p\,z^{-1}$ and $D([z]) = z$.
The maps a, b, c are isomorphisms of left B_p^* sets (the left action on the source is given by left multiplication). For $\alpha \in B_p^*$ even, $d(\alpha z) = \alpha d(z)$ and $D(\alpha z) = \alpha D(z)$. It is clear that O, T, o, t all respect the left B_p^* action.The opposite $\bar{e}$ to an edge $b(z) = e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$ is $b(z\beta)$.
It is clear that $e \mapsto \bar{e}$ commutes with the left B_p^* action.
The map D is an injection, and an edge is in the image of D iff it is even. Since d and D are

identified, the same statement holds for d*.*

Each of the right hand squares are to be interpreted as giving rise to two diagrams with the top (resp. bottom) arrow on the top corresponding to the top (resp. bottom) arrow on the bottom. Unlabeled vertical arrows are canonical.

Proof. Note that commutativity is clear if the maps exist as stated. We show that all the maps exist, and have the stated properties.

a, b, c That these maps are well-defined is clear and injectivity is by definition. Eg, if z, w have the same image under a then $w^{-1}zR_pz^{-1}w = R_p \iff w^{-1}z \in N(R_p) \iff z \in wN(R_p)$. For a, surjectivity is [Vig80, Lemme II.2.4] and for c surjectivity is [Vig80, Lemme II.2.3]. For b, recall that a non-maximal Eichler order has precisely two orientations as given in §2.1.0.2 and conjugation by β above clearly permutes them. Hence, we know that a, b, c are isomorphisms of sets. It is clear that they respect the left B_p^*-module structure.

O, T It is clear that these maps respect the left B_p^* module structure (if they exist). We just need to show existence. Existence for O is clear since $N(\underline{\mathrm{R}}_p) \subset N(\mathcal{R}_p)$. For T we need to show that $\forall \alpha \in N(\underline{\mathrm{R}}_p), z \in B_p^*$,

$$z\alpha\beta N(\mathcal{R}_p) = z\beta N(\mathcal{R}_p)$$

That is, $\beta^{-1}\alpha\beta \in N(\mathcal{R}_p)$. However, if $\alpha = \lambda\alpha'$ is a decomposition as in Lemma 2.2.3 then by definition, $\beta^{-1}\alpha'\beta \in R_p^* \subset \mathcal{R}_p^* \subset N(\mathcal{R}_p)$.

o, t This is the same as for O, T. It is clear that these maps respect the left B_p^* module structure (if they exist). We just need to show existence. For o this is clear. For t we need to show that $\forall \alpha \in N(\underline{\mathrm{R}}_p), z \in B_p^*$

$$z\beta\,\mathcal{R}_p(z\beta)^{-1} = z\alpha\beta\,\mathcal{R}_p(z\alpha\beta)^{-1}$$

That is, $(\beta^{-1}\alpha\beta)\,\mathcal{R}_p(\beta^{-1}\alpha\beta)^{-1} = \mathcal{R}_p$. However, we observed above that $(\beta^{-1}\alpha\beta) \in N(\mathcal{R}_p)$.

d, D We show that d, D are well-defined. Let $z \in B_p^*$. Since $\beta \in N(R_p)$ we can modify z by an element of $N(R_p)$ so that z is even (that is, so $|\mathrm{Nr}\, z| = p^{2k}$ for some $k \in \mathbb{Z}$), hence d, D are defined on their respective domains. If $[z] = [w]$ as elements of $B_p^*/N(R_p)$ (or equivalently $zR_pz^{-1} = wR_pw^{-1}$ because a is an isomorphism) then $z = w\alpha$ for $\alpha \in N(R_p)$. Since z, w are even so is α hence $\alpha \in N(\underline{\mathrm{R}}_p)$ by Corollary 2.2.4. Thus d, D are well-defined.

We show that d, D preserve the left-action by even elements. Let $\omega \in B_p^*$ be even, $z \in B_p^*$ and z' a representative of the coset $zN(R_p)$ with z' even. then, $\omega z'$ is even and a coset representative of ωz. Thus $D(\omega z) = \omega z' = \omega D(z)$. Similarly, $d(\omega z R_p z^{-1}\omega^{-1}) = (\omega z')\,\underline{\mathrm{R}}_p(\omega z')^{-1} = \omega d(zR_pz^{-1})$.

It is clear that D is injective. Indeed, suppose $D([z]) = D([z'])$ for some $z, z' \in B_p^*$. Without loss of generality assume that z and z' are even. Then, $D([z]) = [z]$ and $D([z']) = [z']$. Now,

$$z' \in zN(\underline{\mathrm{R}}_p) \iff z^{-1}z' \in N(\underline{\mathrm{R}}_p) \subset N(R_p) \implies z' \in zN(R_p)$$

and hence the cosets $[z], [z']$ of $B_p^*/N(R_p)$ are the same.

■

Lemma 2.2.6. *For* $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$

$$\mathrm{Stab}_\Gamma\, e = \mathrm{Stab}_\Gamma\, \bar{e}$$

Proof. Since $e \mapsto \bar{e}$ is an involution, we only have to show one containment. Let $e = b(z)$. Then,

$$\mathrm{Stab}_\Gamma\, e = \left\{\, \gamma \in \Gamma \;\middle|\; \gamma z N(\underline{\mathrm{R}}_p) = zN(\underline{\mathrm{R}}_p) \,\right\}$$

The normalizer of $\underline{\mathrm{R}}_p$ is $\mathbb{Q}_p^* R_p^*$ and since $\beta \in N(R_p)$,

$$\beta N(\underline{\mathrm{R}}_p)\beta^{-1} = N(R_p)$$

Observe that $\gamma z N(\underline{\mathrm{R}}_p) = zN(\underline{\mathrm{R}}_p)) \iff \gamma z\beta N(\underline{\mathrm{R}}_p)\beta^{-1} = z\beta N(\underline{\mathrm{R}}_p)\beta^{-1} \iff \gamma z\beta N(\underline{\mathrm{R}}_p) = z\beta N(\underline{\mathrm{R}}_p)$. Since $\overline{b(z)} = b(z\beta)$, this yields the desired conclusion. ■

Lemma 2.2.7. *The map c preserves the notion of parity.*

Proof. This follows from [Ser80, Corollary II.1.2] since our tree $\mathcal{T}$ is the tree of $\mathrm{SL}_2(\mathbb{Q}_p)$, see Remark 2.2.1. ■

2.3 Modules on $\mathcal{T}$

We will use several intermediate modules to establish an isomorphism between the space of p-new forms for the definite quaternion algebra B and a space of harmonic forms. First we define the modules and isomorphisms required to prove Lemma 2.5.2. For the remainder of §2, let $\mathcal{M} = \mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$.

Recall that if $z \in B_p^*$, $j(z) \in B_f^*$ is the idele which is 1 away from the p^{th} place and z at the p^{th} place. If $b \in B^*$, $b^{(p)}$ is the idele which is 1 at p (and b elsewhere). Note that $(p)^{(p)} \in U \subset V$.

2.3.1 Generalities

In this section we recall some results for later use (see [BD98]). Unless otherwise stated,

$$U = \widehat{R}^* = \widehat{\mathcal{R}(N^+p)}^* \tag{2.7}$$

$$V = \widehat{\mathcal{R}(N^+)}^* \tag{2.8}$$

are considered as open subgroups of B_f^*. Note that $\mathrm{Nr}(U)$ and $\mathrm{Nr}(V)$ both contain $\prod_{q<\infty}\mathbb{Z}_q^*$. This is clear at local places q such that $q \nmid N^-$. If $q \mid N^-$ then $R_q \subset \mathcal{O}_K$ where $K/\mathbb{Q}_q$ is isomorphic to the quadratic unramified field extension of $\mathbb{Q}_q$. The conclusion follows since the reduced norm on K agrees with the usual norm of field extensions and $\mathrm{Nr}(\mathcal{O}_K^*) = \mathbb{Z}_q^*$.
Let

$$\begin{aligned}\Gamma' &= \mathcal{R}(N^+)[1/p]^* \\ \Gamma &= \{\, z \in \Gamma' \mid \mathrm{Nr}(z) = 1 \,\} \\ \Gamma^{(2)} &= \left\{\, z \in \Gamma' \;\middle|\; \mathrm{Nr}(z) = p^{2k} \text{ for some } k \in \mathbb{Z} \,\right\}\end{aligned}$$

When identified with a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ via ϕ_p, Γ' is discrete and co-compact. As a subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, $\phi_p(\Gamma')$ is dense and has finite covolume. [Vig80, Thm. IV.1.1] By definition [Vig80, p.109], Γ is a congruence group.
By Lemma 2.3.1 we may and will fix $\tau \in \Gamma'$ with $|\mathrm{Nr}\,\tau| = p$.

Lemma 2.3.1. *There is a short exact sequence*

$$0 \to \Gamma \to \Gamma' \xrightarrow{\gamma \mapsto \mathrm{ord}_p \mathrm{Nr}(\gamma)} \mathbb{Z} \to 0$$

Proof. Injectivity is clear. Observe that Γ' contains $p^k = \left(\begin{smallmatrix} p^k & 0 \\ 0 & p^k \end{smallmatrix}\right), \forall k \in \mathbb{Z}$ because $p = p \cdot 1_{\mathcal{R}(N^+)}$ is a unit in $\mathcal{R}(N^+)[1/p]$ by construction. This group also contains an element of reduced norm an odd power of p [BD98, Lemma 1.5]. Hence we also have surjectivity. If $\gamma \in \Gamma'$ then because B is definite, $\mathrm{Nr}\,\gamma \in \mathbb{Q}$, and $\mathrm{Nr}(\gamma) > 0$ [Vig80, Theorem 4.1]. By [Vig80, Lemma I.4.1], $\mathrm{Nr}(\gamma) \in \mathbb{Z}[1/p]$. ∎

Corollary 2.3.2. *Each element $a \in \Gamma'$ can be written in exactly one way as $a = \lambda_j \gamma \tau^b$ and in exactly one way as $\lambda_k \tau^b \gamma'$ for $\gamma, \gamma' \in \Gamma, b \in \{0, 1\}$, $\lambda_j = \left(\begin{smallmatrix} p^j & 0 \\ 0 & p^j \end{smallmatrix}\right)$, and $j \in \mathbb{Z}$. This element is in $\Gamma^{(2)}$ iff $b = 0$.*

Proof. As $\nu_p(\mathrm{Nr}(\tau^b \lambda_j \gamma)) = 2j + b = \nu_p(\mathrm{Nr}\,a)$, b, j are determined by a. Hence γ, γ' are also uniquely determined. ■

Lemma 2.3.3. *The group* Γ' *is*

$$\Gamma' = \{\, b \in B^* \mid b \in U_\ell \quad \forall \ell < \infty, \ell \neq p \,\}$$

or equivalently,

$$\Gamma' = \left\{\, b \in B^* \;\middle|\; b^{(p)} \in U \,\right\}$$

Proof. Note that $U_\ell = V_\ell$ for $\ell \neq p$. Since $\mathcal{R}(N^+)[1/p] \subset \mathcal{R}(N^+) \otimes \mathbb{Z}_\ell$ (for $\ell \neq p$) the inclusion $\subset$ is clear. If $b \in B_p$ then clearly $p^n b \in R_p$ for $n \gg 0$. Hence, if $b \in RHS$ then $p^n b \in \mathcal{R}(N^+)$ for $n \gg 0$. Thus, $b \in \mathcal{R}(N^+)[1/p]$. Since $b^{-1} \in RHS$, $b \in \mathcal{R}(N^+)[1/p]^*$. ■

Lemma 2.3.4. *The map* $D : B_p^*/N(R_p) \to B_p^*/N(\underline{\mathrm{R}}_p)$ *of Lemma 2.2.5 descends to a bijection*

$$D : \Gamma \backslash B_p^*/N(R_p) \to \Gamma' \backslash B_p^*/N(\underline{\mathrm{R}}_p)$$

Equivalently, $d : \Gamma \backslash \mathcal{E}(\mathcal{T}) \to \Gamma' \backslash \overrightarrow{\mathcal{E}}(\mathcal{T})$ *from Lemma 2.2.5 is a bijection.*

Proof. Since elements of Γ are even, D descends to the map above. Surjectivity is clear, since each coset of $\Gamma' \backslash B_p^*/N(\underline{\mathrm{R}}_p)$ is represented by some even element $z \in B_p^*$. We prove injectivity. Let $z, w \in B_p^*$ be even and suppose $D(z) = D(w)$. Then, $\exists \gamma, \mu \in \Gamma', \alpha, \alpha' \in N(\underline{\mathrm{R}}_p)$ so that

$$\gamma z \alpha = \mu w \alpha'$$

Since z, w, α, α' are even, by Corollary 2.3.2 we can write $\gamma = \tau^b \gamma' \lambda_j, \mu = \tau^b \mu' \lambda_{j'}$ for some $b \in \{0, 1\}$. Cancelling τ^b gives injectivity. ■

Lemma 2.3.5. *There is a bijection*

$$j : \Gamma' \backslash B_p^*/N(\underline{\mathrm{R}}_p) \to B^* \backslash B_f^*/U$$

Here, the map j is induced by

$$j : B_p^* \to B_f^* \qquad j(z) = (1, \cdots, 1, z, 1, \cdots)$$

where $j(z)$ is the idele which is z at p and 1 away from p.

Proof. It is clear that j above descends to the given quotient. It is surjective by strong approximation. We show injectivity. Suppose $z, w \in B_p^*$ have the same image. Then, $\exists b \in B^*, u \in U$ such that

$$j(z) = bj(w)u$$

as elements of B_p^*. We have $b \in \Gamma'$ by Lemma 2.3.3 and $u_p \in N(\underline{\mathrm{R}}_p)$ by definition. Hence, injectivity. ∎

Lemma 2.3.6. *There is a bijection*

$$B^* \backslash B_f^* / U \xrightarrow{[z] \mapsto [zj(\beta)]} B^* \backslash B_f^* / U$$

Proof. Suppose $z = bz'u$ are two representatives of a coset. Then,

$$zj(\beta) = bz'uj(\beta) = b(z'j(\beta))j(\beta)^{-1}uj(\beta)$$

and by Lemma 2.2.1, $j(\beta)^{-1}u\beta \in U$. Hence this map is well-defined. It is an involution since $\beta^2 = p$ and $p/j(p) \in U$, and hence bijective. ∎

Lemma 2.3.7. *Let $\gamma \in GL_2(\mathbb{Q}_p)$ and $\tilde{\gamma}$ denote the endomorphism of $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ defined by left multiplication*

$$\tilde{\gamma}F(x, y) = F((x, y)\gamma)$$

Then, $\det \gamma = 1 \implies \det \tilde{\gamma} = 1.$

Proof. Straightforward. ∎

2.3.2 Edges

In this section we identify modular forms for U as spaces of functions on B_p^*. Although the modules defined here play an intermediate role, the maps between them compose to the main isomorphism. The main result here is to identify the space of modular forms $\mathcal{L}_k(U, \mathbb{Q}_p)$ defined above with a space of functions $C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$ that is defined below. Theorem 2.5.4 is then the observation that this isomorphism identifies p-new forms (see §2.5.0.2) and harmonic forms (see §2.5.0.1).

Consider the spaces

$$C^1_{\mathcal{E}}(\mathcal{M}) = \{\, f : \mathcal{E}(\mathcal{T}) \to \mathcal{M} \,\} \tag{2.9}$$

$$C^1_{\overrightarrow{\mathcal{E}}}(\mathcal{M}) = \left\{\, f : \overrightarrow{\mathcal{E}}(\mathcal{T}) \to \mathcal{M} \,\right\} \tag{2.10}$$

of arbitrary $\mathcal{M}$ valued functions on edges (resp. directed edges) of the Bruhat-Tits tree. If f belongs to either of these spaces, let B_p^* act on it as

$$\gamma \cdot f = \gamma f \circ \gamma^{-1} \cdot |\mathrm{Nr}\,\gamma|^{n/2} \qquad \forall \gamma \in B_p^*$$

A function is invariant under $\gamma \in B_p^*$ iff for all w in its domain,

$$(f \circ \gamma)(w) = \gamma f(w) |\mathrm{Nr}\,\gamma|^{n/2}$$

By Lemma 2.2.5, the spaces of Γ and Γ' invariants are

$$C^1_{\overrightarrow{\mathcal{E}}}(\mathcal{M})^{\Gamma'} = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{ll} f(z\alpha) = f(z) & \forall \alpha \in N(\underline{\mathrm{R}}_p), \forall z \in B_p^* \\ f(\gamma z) = \gamma f(z) |\mathrm{Nr}\,\gamma|^{n/2} & \forall \gamma \in \Gamma', \forall z \in B_p^* \end{array} \right\}$$

$$C^1_{\mathcal{E}}(\mathcal{M})^{\Gamma} = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{ll} f(z\alpha) = f(z) & \forall \alpha \in N(R_p), \forall z \in B_p^* \\ f(\gamma z) = \gamma f(z) & \forall \gamma \in \Gamma, \forall z \in B_p^* \end{array} \right\}$$

The following module will play an intermediary role.

$$C_1(\Gamma', k) = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{ll} f(z\alpha) = \alpha^{-1} f(z) |\mathrm{Nr}\,\alpha^{-1}|^{n/2} & \forall \alpha \in N(\underline{\mathrm{R}}_p), \forall z \in B_p^* \\ f(\gamma z) = f(z) & \forall \gamma \in \Gamma', \forall z \in B_p^* \end{array} \right\}$$

Lemma 2.3.8. *If $f \in \mathcal{L}_k(U, \mathbb{Q}_p)$ then $f \circ j \in C_1(\Gamma', k)$, and the association $f \mapsto f \circ j$ is an isomorphism of $\mathbb{Q}_p$ vectorspaces*

$$l_U : \mathcal{L}_k(U, \mathbb{Q}_p) \to C_1(\Gamma', k) \qquad f \mapsto f \circ j$$

Proof. Let $f \in \mathcal{L}_k(U, \mathbb{Q}_p)$. We check that $f \circ j \in C_1(\Gamma', k)$. By Lemma 2.3.3,

$$f \circ j(\gamma z) = f(\gamma^{(p)} j(\gamma z)) = f(\gamma j(z)) = f \circ j(z) \qquad \forall z \in B_p^*, \gamma \in \Gamma'$$

Recall that $|\mathrm{Nr}\,\alpha| = 1$ for $\alpha \in R_p^*$. Hence

$$f \circ j(z\alpha) = \alpha^{-1} f \circ j(z) = \alpha^{-1} f \circ j(z) |\mathrm{Nr}\,\alpha^{-1}|^{n/2}$$

Finally,

$$f \circ j(zp) = f(p^{(p)} j(zp)) = f \circ j(z) = p^{-1} f \circ j(z) |\mathrm{Nr}\,p^{-1}|^{n/2}$$

Hence $f \circ j \in C_1(\Gamma', k)$ and the map l_U is well-defined.

It is clear that l_U is injective by strong approximation. We show surjectivity. For $f \in C_1(\Gamma', k)$, define a $P_{k-2}(\mathbb{Q}_p)$ valued function ψ_f on the subgroup $j(B_p^*) \cdot U \subset B_f^*$ as

$$\psi_f(\alpha) = f(\alpha_p)$$

For $b \in B^*, h \in j(B_p^*)U$ define

$$\psi_f(bh) = f(hp)$$

If $a \in B^*, g \in j(B_p)^*U$ with $ag = bh$ then by Lemma 2.3.3 $b^{-1}a = hg^{-1} \in \Gamma'$ and

$$\psi_f(bh) = f(h_p) = f(b_p^{-1}a_p g_p) = f(g_p)$$

Hence, ψ_f is well-defined on $B^* j(B_p^*)U = B_f^*$. It is clear that $\psi_f \in \mathcal{L}_k(U, \mathbb{Q}_p)$ because it is left B^* invariant, and for $u \in$U, $z \in B_f^*$,

$$\psi_f(zu) = f(z_p u_p) = u_p^{-1} f(z_p) |\mathrm{Nr}\, u_p^{-1}|^{n/2} = u_p^{-1} f(z_p) = u_p^{-1} \psi_f(z)$$

as required. ■

Lemma 2.3.9. *If $f \in C_1(\Gamma', k)$ and $g \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$, define*

$$\tau_f(z) = z f(z) |\mathrm{Nr}\, z|^{n/2} \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$$
$$\sigma_g(z) = z^{-1} g(z) |\mathrm{Nr}\, z|^{-n/2} \in C_1(\Gamma', k)$$

The map

$$C_1(\Gamma', k) \xrightarrow{c_1} C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'} \qquad f \mapsto \tau_f$$

is an isomorphism of $\mathbb{Q}_p$ vector spaces with inverse $\sigma : g \mapsto \sigma_g$

Proof. For $f \in C_1(\Gamma', k)$ and $g \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$, we have $\tau_f \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$ and $\sigma_g \in C_1(\Gamma', k)$.

Indeed, for all $\alpha \in N(\underline{\mathrm{R}}_p)$, $\gamma \in \Gamma'$, and $z \in B_p^*$

$$\tau_f(z\alpha) = z\alpha f(z\alpha)|\mathrm{Nr}\, z\alpha|^{n/2} = \tau_f(z)$$
$$\tau_f(\gamma z) = \gamma z f(\gamma z)|\mathrm{Nr}\, z\gamma|^{n/2} = \gamma\tau_f(z)|\mathrm{Nr}\,\gamma|^{n/2}$$

Hence $\tau_f \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$. Similarly,

$$\sigma_g(z\alpha) = \alpha^{-1}z^{-1}g(z\alpha)|\mathrm{Nr}\, z\alpha|^{-n/2} = \alpha^{-1}\sigma_g(z)|\mathrm{Nr}\,\alpha^{-1}|^{n/2}$$
$$\sigma_g(\gamma z) = z^{-1}\gamma^{-1}g(\gamma z)|\mathrm{Nr}\,\gamma z|^{-n/2} = z^{-1}g(z)|\mathrm{Nr}\, z|^{-n/2} = \sigma_g(z)$$

and $\sigma_g \in C_1(\Gamma', k)$.

It is clear that c_1 and σ compose to the respective identity maps. ∎

Lemma 2.3.10. *Pullback by the map* d *of Lemma 2.2.5 defines a map* $C^1_{\vec{\mathcal{E}}}(\mathcal{M}) \to C^1_{\mathcal{E}}(\mathcal{M})$. *This map descends to an isomorphism of* $\mathbb{Q}_p$ *vector spaces*

$$\sigma : C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'} \to C^1_{\mathcal{E}}(\mathcal{M})^{\Gamma} \qquad \sigma(f)(e) = f(d(e)), f \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$$

As usual, this is expressed as a map on B_p^* *via the map* $\tilde{D}$ *of Lemma 2.2.5. It is*

$$\sigma(f)(z) = f(\tilde{D}(z)) \qquad \forall z \in B_p^*$$

Proof. It is clear that the pullback map descends. Let $F \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$ then for all $\gamma \in \Gamma$ and $e \in \mathcal{E}(\mathcal{T})$,

$$\sigma(F)(\gamma e) = F(d(\gamma e)) = F(\gamma d(e)) = \gamma F(d(e))$$

That is, $\sigma(F) \in C^1_{\mathcal{E}}(\mathcal{M})^{\Gamma}$.

This map is also clearly injective. If $F \in C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'}$ and $\sigma(F) = 0$ then by Lemma 2.2.5,

$F(e) = 0$ for all $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$ even. However, the relation

$$F(\tau e) = \tau F(e) |\mathrm{Nr}\, \tau|^{n/2}$$

implies $F = 0$.

Finally, we show that σ is surjective. If $f \in C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$ then define

$$F(e) = \begin{cases} f(d^{-1}(e)) & \text{if } e \text{ even} \\ \tau^{-1} F(\tau e) |\mathrm{Nr}\, \tau|^{-n/2} & \text{if } e \text{ odd} \end{cases}$$

This is well-defined, because for e even, $d^{-1}(e)$ is well-defined. It is also clear that $F \circ d = f$, and it remains to show that F is Γ' invariant.

Let $\gamma \in \Gamma'$ and by Lemma 2.3.1 write $\gamma = \gamma' \lambda \tau^b$ with $\gamma' \in \Gamma$, $\lambda = p^t$ for some $t \in \mathbb{Z}$, and $b \in \{ 0, 1 \}$. We prove that F is Γ' invariant in cases.

Case: e even, $b = 0$:

$$\begin{aligned} F(\gamma e) = F(\gamma' e) &= f(d^{-1}(\gamma' e)) \\ &= \gamma' f(d^{-1}(e)) = \gamma' F(e) \\ &= \gamma' \lambda F(e) |\mathrm{Nr}\, \gamma|^{n/2} \\ &= \gamma F(e) |\mathrm{Nr}\, \gamma|^{n/2} \end{aligned} \tag{2.2}$$

Case: e even, $b = 1$:

$$\begin{aligned} F(\gamma e) = F(\gamma' \lambda \tau e) &= F(\gamma' \lambda \tau e) \\ &= \tau^{-1} F(\tau \gamma' \lambda \tau e) |\mathrm{Nr}\, \tau|^{-n/2} && \text{since } \gamma' \lambda \tau e \text{ is odd} \end{aligned}$$

Note that $\tau\gamma'\lambda\tau \in \Gamma^{(2)}$, so the above case applies to give this is

$$= \tau^{-1}(\tau\gamma'\lambda\tau)F(e)|\mathrm{Nr}\,\tau|^{-n/2}|\mathrm{Nr}\,\tau\gamma'\lambda\tau|^{n/2}$$
$$= \gamma F(e)|\mathrm{Nr}\,\gamma|^{n/2}$$

Case: e even, $b = 1$: For e odd, we have $F(e) = \tau^{-1}F(\tau e)|\mathrm{Nr}\,\tau|^{-n/2}$. That is, $F(\tau e) = \tau F(e)|\mathrm{Nr}\,\tau|^{n/2}$. The case above applies to give:

$$\begin{aligned} F(\gamma e) &= F(\gamma\tau^{-1}(\tau e)) \\ &= \gamma\tau^{-1}F(\tau e)|\mathrm{Nr}\,\gamma\tau^{-1}|^{n/2} \\ &= \gamma\tau^{-1}\tau F(e)|\mathrm{Nr}\,\tau|^{n/2}|\mathrm{Nr}\,\gamma\tau^{-1}|^{n/2} \\ &= \gamma F(e)|\mathrm{Nr}\,\gamma|^{n/2} \end{aligned}$$

Hence surjectivity. ■

Remark 2.3.1. The maps above compose to an isomorphism of $\mathbb{Q}_p$ vector spaces

$$\mathcal{L}_k(U, \mathbb{Q}_p) \xrightarrow{\ e\ } C^1_{\mathcal{E}}(\mathcal{M})^{\Gamma}$$

Theorem 2.5.4 is the observation that this map restricts to identify p-new forms §2.5.0.2 and harmonic forms §2.5.0.1. In Lemma 2.5.2, we also record explicit formulas for this map.

2.3.3 Vertices

In this section we identify modular forms for the open subgroup V as spaces of functions on B_p^*. The main application of the results here is to prove Lemma 2.5.2, which along with Lemma 2.5.3, implies the main theorem of §2, Theorem 2.5.4. These results here will not be used until §2.5.

Consider the space

$$C^0(\mathcal{M}) = \{\, f : \mathcal{V}(\mathcal{T}) \to \mathcal{M} \,\}$$

of arbitrary $\mathcal{M}$ valued functions on vertices of the Bruhat-Tits tree, and endow it with an action of B_p^* via

$$(\gamma \cdot f)(v) = \gamma f \circ \gamma^{-1}(v) \cdot |\mathrm{Nr}\,\gamma|^{n/2} \qquad \forall \gamma \in B_p^*, v \in \mathcal{V}(\mathcal{T})$$

A function is invariant under $\gamma \in B_p^*$ ifff for all w in its domain,

$$\gamma \cdot f = \gamma f \circ \gamma^{-1} \cdot |\mathrm{Nr}\,\gamma|^{n/2}$$

By Lemma 2.2.5, the spaces of Γ and Γ' invariants are

$$C^0(\mathcal{M})^{\Gamma'} = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{cc} f(z\alpha) = f(z) & \forall \alpha \in N(\mathcal{R}_p), \forall z \in B_p^* \\ f(\gamma z) = \gamma f(z) |\mathrm{Nr}\,\gamma|^{n/2} & \forall \gamma \in \Gamma', \forall z \in B_p^* \end{array} \right\}$$

$$C^0(\mathcal{M})^{\Gamma} = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{ll} f(z\alpha) = f(z) & \forall \alpha \in N(\mathcal{R}_p), \forall z \in B_p^* \\ f(\gamma z) = \gamma f(z) & \forall \gamma \in \Gamma, \forall z \in B_p^* \end{array} \right\}$$

The following module will play an intermediary role:

$$W_k(\Gamma') = \left\{ f : B_p^* \to P_{k-2}(\mathbb{Q}_p) \;\middle|\; \begin{array}{cc} f(z\alpha) = \alpha^{-1} f(z) |\mathrm{Nr}\,\alpha^{-1}|^{n/2} & \forall \alpha \in N(\mathcal{R}_p), \forall z \in B_p^* \\ f(\gamma z) = f(z) & \forall \gamma \in \Gamma', \forall z \in B_p^* \end{array} \right\}$$

Lemma 2.3.11. *If $f \in \mathcal{L}_k(V, \mathbb{Q}_p)$, then $f \circ j \in W_k(\Gamma')$ and the association $f \mapsto f \circ j$ is an isomorphism of $\mathbb{Q}_p$ vectorspaces.*

$$l_V : \mathcal{L}_k(V, \mathbb{Q}_p) \to W_k(\Gamma') \qquad f \mapsto f \circ j$$

Proof. Let $f \in \mathcal{L}_k(V, \mathbb{Q}_p)$. We check that $f \circ j \in W_k(\Gamma')$. By Lemma 2.3.3,

$$f \circ j(\gamma z) = f(\gamma^{(p)} j(\gamma z)) = f(\gamma j(z)) = f \circ j(z) \qquad \forall z \in B_p^*, \gamma \in \Gamma'$$

Recall that $|\mathrm{Nr}\,\alpha| = 1$ for $\alpha \in \mathcal{R}_p^*$. Hence

$$f \circ j(z\alpha) = \alpha^{-1} f \circ j(z) = \alpha^{-1} f \circ j(z) |\mathrm{Nr}\,\alpha^{-1}|^{n/2}$$

Finally,

$$f \circ j(zp) = f(p^{(p)} j(zp)) = f \circ j(z) = p^{-1} f \circ j(z) |\mathrm{Nr}\, p^{-1}|^{n/2}$$

Hence $f \circ j \in W_k(\Gamma')$ and the map l_V is well-defined.

It is clear that l_V is injective by strong approximation. We show surjectivity. For $f \in W_k(\Gamma')$, define a $P_{k-2}(\mathbb{Q}_p)$ valued function ψ_f on the subgroup $j(B_p^*) \cdot V \subset B_f^*$ as

$$\psi_f(\alpha) = f(\alpha_p)$$

For $b \in B^*, h \in j(B_p^*)V$ define

$$\psi_f(bh) = f(hp)$$

If $a \in B^*, g \in j(B_p)^*V$ with $ag = bh$ then by Lemma 2.3.3 $b^{-1}a = hg^{-1} \in \Gamma'$ and

$$\psi_f(bh) = f(h_p) = f(b_p^{-1} a_p g_p) = f(g_p)$$

Hence, ψ_f is well-defined on $B^* j(B_p^*)V = B_f^*$. It is clear that $\psi_f \in \mathcal{L}_k(V, \mathbb{Q}_p)$ because it is left B^* invariant, and for $v \in$V, $z \in B_f^*$,

$$\psi_f(zv) = f(z_p v_p) = v_p^{-1} f(z_p) |\mathrm{Nr}\, v_p^{-1}|^{n/2} = v_p^{-1} f(z_p) = v_p^{-1} \psi_f(z)$$

as required.

■

Lemma 2.3.12. *If $f \in W_k(\Gamma')$ and $g \in C^0(\mathcal{M})^{\Gamma'}$ define*

$$\tau_f(z) = zf(z)|\mathrm{Nr}\, z|^{n/2} \in C^0(\mathcal{M})^{\Gamma'}$$

$$\sigma_g(z) = z^{-1}g(z)|\mathrm{Nr}\, z|^{-n/2} \in W_k(\Gamma')$$

The map

$$W_k(\Gamma') \xrightarrow{w} C^0(\mathcal{M})^{\Gamma'} \qquad f \mapsto \tau_f$$

is an isomorphism of $\mathbb{Q}_p$ vector spaces with inverse $\sigma : g \mapsto \sigma_g$

Proof. For $f \in W_k(\Gamma')$ and $g \in C^0(\mathcal{M})^{\Gamma'}$, we have $\tau_f \in C^0(\mathcal{M})^{\Gamma'}$ and $\sigma_g \in W_k(\Gamma')$. Indeed, for all $\alpha \in N(\mathcal{R}_p)$, $\gamma \in \Gamma'$, and $z \in B_p^*$

$$\tau_f(z\alpha) = z\alpha f(z\alpha)|\mathrm{Nr}\, z\alpha|^{n/2} = \tau_f(z)$$

$$\tau_f(\gamma z) = \gamma z f(\gamma z)|\mathrm{Nr}\, z\gamma|^{n/2} = \gamma\tau_f(z)|\mathrm{Nr}\,\gamma|^{n/2}$$

Hence $\tau_f \in C^0(\mathcal{M})^{\Gamma'}$. Similarly,

$$\sigma_g(z\alpha) = \alpha^{-1}z^{-1}g(z\alpha)|\mathrm{Nr}\, z\alpha|^{-n/2} = \alpha^{-1}\sigma_g(z)|\mathrm{Nr}\,\alpha^{-1}|^{n/2}$$

$$\sigma_g(\gamma z) = z^{-1}\gamma^{-1}g(\gamma z)|\mathrm{Nr}\,\gamma z|^{-n/2} = z^{-1}g(z)|\mathrm{Nr}\, z|^{-n/2} = \sigma_g(z)$$

and $\sigma_g \in W_k(\Gamma')$.

It is clear that w and σ compose to the respective identity maps.

■

Lemma 2.3.13. *We have a linear map of $\mathbb{Q}_p$-vector spaces:*

$$\psi : C^0(\mathcal{M})^{\Gamma'} \oplus C^0(\mathcal{M})^{\Gamma'} \to C^0(\mathcal{M})^{\Gamma}$$

defined by

$$\psi(f,g)(z) = \begin{cases} f(z) & \textit{if } z \textit{ even} \\ g(z) & \textit{if } z \textit{ odd} \end{cases}$$

As a map on vertices,

$$\psi(f,g)(v) = \begin{cases} f(v) & \text{if } v \text{ even} \\ g(v) & \text{if } v \text{ odd} \end{cases}$$

Proof. $\psi(f,g)$ is a well-defined function on B_p^*. We need to show that it satisfies the two properties required by $C^0(\mathcal{M})^\Gamma$. Let $z \in B_p^*$ and $\alpha \in N(\mathcal{R}_p)$. Since α is even, the parity of $z\alpha$ is that of α, hence $\psi(f,g)(z) = f(z\alpha)$ (resp $\psi(f,g)(z) = g(z\alpha)$) if z is even (resp. odd). However, $f(z\alpha) = f(z)$ and $g(z\alpha) = g(z)$. Hence $\psi(f,g)(z\alpha) = \psi(f,g)(z)$. Similarly, left-multiplication by $\gamma \in \Gamma$ does not change parity. Hence $\psi(f,g) \in C^0(\mathcal{M})^\Gamma$. ∎

2.4 Adjointness

The goal of this section is to define the space of harmonic forms (2.17) in a general setting. One of our main results, Theorem 3.3.2, identifies the space of p-new modular forms on a Shimura curve with with the harmonic forms constructed here.

Although we will eventually take the space of coefficients M that appears below to be $P_n(\mathbb{Q}_p)$, the results hold in more generality, in particular if M is not a vector space but only a module over certain rings. It is also possible to deduce these results, not just for the Bruhat-Tits tree $\mathcal{T}$, but for connected trees $T \subset \mathcal{T}$ that are not necessarily subtrees (that is, trees T that do not contain all vertices intermediate between vertices of T). If Γ is a Schottky group these correspond to different models of the corresponding Shimura curve, and one can deduce if the corresponding model is the minimal regular model from properties of T. See [Mum72, p.164] for a beautiful description. For Γ not Schottky, see [Kur79, §3]. We allow more general coefficients, and leave it to the reader to consider more general trees.

The results here are essentially contained in [dS89, §5]. We include proofs because in our setting Γ may contain torsion while de Shalit assumes Γ is torsion free.

2.4.0.1 Setup

Let R be a commutative domain with 1, $\Gamma \subset B_p^* \to \mathrm{GL}_2(\mathbb{Q}_p)$ the subgroup defined above, and M an arbitrary left $R[\Gamma]$ module. Recall that $\mathrm{Nr}\,\gamma = 1$ for all $\gamma \in \Gamma$, and that Γ acts without inversions. That is, for any $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$, the parity of $o(\gamma e)$ is the same as that of $o(e)$. The graph $\Gamma \backslash \mathcal{T}$ is finite. That is, it has finitely many vertices and finitely many edges [Kur79, §3].

Recall that we denote the vertices of $\mathcal{T}$ by $\mathcal{V}(\mathcal{T})$, and the directed edges of $\mathcal{T}$ by $\overrightarrow{\mathcal{E}}(T)$. We will identify the set of un-directecd edges $\mathcal{E}(\mathcal{T})$ with pairs $(e, \bar{e})$. If $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$, then its inverse is denoted $\bar{e}$. If $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$ then $o(e)$ and $t(e)$ denote its origin and target respectively. By definition, $t(\gamma e) = \gamma t(e), o(\gamma e) = \gamma o(e)$ for all $\gamma \in \Gamma$.

Define modules

$$C^0(M) = \{\, f : \mathcal{V}(\mathcal{T}) \to M \,\} \tag{2.11}$$

$$C^1(M) = \left\{\, f : \overrightarrow{\mathcal{E}}(\mathcal{T}) \to M \;\middle|\; f(\bar{e}) = -f(e) \quad \forall e \in \overrightarrow{\mathcal{E}}(\mathcal{T}) \,\right\} \tag{2.12}$$

and maps

$$\partial^* : C^0(M) \to C^1(M) \qquad \partial_* : C^1(M) \to C^0(M) \tag{2.13}$$

$$\partial^*(f)(e) = f(t(e)) - f(o(e)) \qquad \partial_*(g)(v) = \sum_{t(e)=v} g(e) \tag{2.14}$$

$$\partial^*(f)(z) = f(\tilde{T}(z)) - f(\tilde{O}(z)) \qquad \partial_*(g)(z) = \sum_{\substack{w \in B_p^*/N(\underline{\mathrm{R}}_p) \\ T(w)=z}} g(w) \tag{2.15}$$

defined for all $z \in B_p^*$, $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$, and $v \in \mathcal{V}(\mathcal{T})$. Functions on edges or vertices are identified with functions on B_p^* via Lemma 2.2.5 as usual. Let Γ act on $C^0(M)$ and $C^1(M)$

on the left via

$$(\gamma f)(z) = \gamma \circ f \circ \gamma^{-1}(z) \cdot |\mathrm{Nr}\,\gamma|^{n/2} \tag{2.16}$$

Remark 2.4.1. Note that since $\Gamma \xhookrightarrow{\phi_p} \mathrm{SL}_2(\mathbb{Q}_p)$, $|\mathrm{Nr}\,\gamma|^{n/2} = 1$ for all $\gamma \in \Gamma$, and the definition of the action above can be simplified. We state the action in this more general form because later M will be a $\mathbb{Q}_p$ module, and have an action of not just Γ but all of B_p^*. In this case, the maps ∂^* and ∂_* above preserve not just the Γ action, but also B_p^* action:

$$\begin{aligned}
\gamma(\partial^*(f))(e) &= \gamma\,\partial^*(f)(\gamma^{-1}e)|\mathrm{Nr}\,\gamma|^{n/2} && \forall f \in C^0(M)\\
&\gamma f(t(\gamma^{-1}e))|\mathrm{Nr}\,\gamma|^{n/2} - \gamma f(o(\gamma^{-1}e))|\mathrm{Nr}\,\gamma|^{n/2}\\
&\gamma f(\gamma^{-1}t(e))|\mathrm{Nr}\,\gamma|^{n/2} - \gamma f(\gamma^{-1}o(e))|\mathrm{Nr}\,\gamma|^{n/2}\\
&\partial^*(\gamma f)(e)\\
\gamma\,\partial_*(g)(v) &= \gamma\left(\sum_{t(e)=v} g(e)\right)(v) && \forall g \in C^1(M)\\
&= \gamma \sum_{t(e)=\gamma^{-1}v} g(e)|\mathrm{Nr}\,\gamma|^{n/2}\\
&= \gamma \sum_{t(\gamma e)=v} g(e)|\mathrm{Nr}\,\gamma|^{n/2}\\
&= \gamma \sum_{t(e)=v} g(\gamma^{-1}e)|\mathrm{Nr}\,\gamma|^{n/2}\\
&= \partial_*(\gamma g)(v)
\end{aligned}$$

2.4.0.2 Adjointness

The goal of this section is to prove Lemma 2.4.6. This is the main ingredient input in identifying Γ-invariants of the space of harmonic functions (which will be defined below).

Lemma 2.4.1. *Let G be a free group on g generators, M an arbitrary left $R[G]$ module, and*

X a set on which G acts freely by left multiplication. Let

$$C^0(X) = \{f : X \to M\}$$

the collection of arbitrary M-valued functions and G act on $C^0(X)$ via $\gamma(f)(x) = \gamma(f(\gamma^{-1} \cdot x))$. Then,

$$H^1(G, C^0(X)) = 0$$

Proof. Straightforward. ■

For the remainder of this section we assume that $\mathbb{Q} \subset R$.

Corollary 2.4.2. *The first cohomology group vanishes:*

$$H^1(\Gamma, C^0(X)) = 0$$

Proof. The group Γ contains a Schottky subgroup of finite index [Kur79, §3], and the desired result follows from the lemma above, and the inflation-restriction sequence

$$0 \to \mathrm{H}^1(\Gamma/G, M^G) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(\Gamma, M) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(G, M)$$

Since M is an R-module, and $\mathbb{Q} \subset R$, $\mathrm{H}^1(\Gamma/G, M^G) = 0$ [Ser79, Corollary VIII.1] ■

Remark 2.4.2. We assume $\mathbb{Q} \subset R$ so that $\mathrm{H}^1(\Gamma, C^0(X)) = 0$ below. With further knowledge one can hope to take $R = \mathbb{Z}_\ell$ for various primes ℓ. For example, one could hope to have bounds on the index of a Schottky group of Γ, or alternatively the subgroup of torsion elements $\Gamma_{tors} \subset \Gamma$. The latter subgroup is also normal, and the quotient Γ/Γ_{tors} is free [vdP92, §3].

Lemma 2.4.3. *The sequence*

$$0 \to M \to C^0(M) \xrightarrow{\partial^*} C^1(M) \to 0$$

is exact.

Proof. Elements of M define the constant maps in $C^0(M)$. Since $\mathcal{T}$ is connected, the kernel of ∂^* is precisely M. It is clear that the image of ∂^* is contained in $C^1(M)$. Surjectivity is also clear. Let $f \in C^1(M)$ and choose $v \in \mathcal{T}$ and $m \in M$ arbitrary. Define $g(w) = \sum_e f(e)$ where the sum ranges over the unique non-repeating collection of edges of $\mathcal{T}$ going from v to w. Then, $\partial^*(g) = f$. ■

Applying cohomology we have the long exact sequence

$$0 \to M^\Gamma \to C^0(M)^\Gamma \xrightarrow{\partial^*} C^1(M)^\Gamma \to H^1(\Gamma, M) \to 0$$

Let $\langle , \rangle : M \times M \to R$ be a symmetric Γ-invariant (that is, $\langle \gamma x, \gamma y \rangle = \langle x, y \rangle$) R-bilinear form, and this pairing $\langle , \rangle$ is non-degenerate, in the sense that if $x \in M$, the function $\langle x, - \rangle$ on M is zero iff $x = 0$.

Lemma 2.4.4. *Under these hypotheses we have well-defined non-degenerate R-bilinear pairings*

$$\langle f, g \rangle = \sum_{[v] \in \Gamma \backslash \mathcal{V}(\mathcal{T})} \langle f(v), g(v) \rangle \qquad f, g \in C^0(M)^\Gamma$$
$$\langle f, g \rangle = \sum_{[e],[\bar{e}] \in \Gamma \backslash \mathcal{E}(\mathcal{T})} \langle f(e), g(e) \rangle \qquad f, g \in C^1(M)^\Gamma$$

That is, the function $\langle f, \cdot \rangle$ on $C^0(M)^\Gamma$ (resp. $C^1(M)^\Gamma$) is the zero function if and only if $f \in C^0(M)^\Gamma$ (resp. in $C^1(M)^\Gamma$) is zero.

Proof. Well-definedness:

Suppose $v' = \gamma v$ for some $\gamma \in \Gamma$. Then, $\langle f(v'), g(v') \rangle = \langle f(\gamma v), g(\gamma v) \rangle = \langle \gamma f(v), \gamma g(v) \rangle = \langle f(v), g(v) \rangle$. Note that in the second sum we take the value of a function on one of e, $\bar{e}$. The result does not depend on this choice by the defining properties of $C^1(M)$. Similarly, the

R-bilinear form on $C^1(M)^\Gamma$ is well-defined. These sums are finite, because $\Gamma\backslash\mathcal{T}$ is a finite graph.

Non-degeneracy: Suppose $f \in C^0(M)^\Gamma$ is nonzero. Choose a vertex v such that $f(v) \neq 0$ and let $x \in M$ such that $\langle f(v), x\rangle \neq 0$. Define $g : \mathcal{V}(\mathcal{T}) \to M$ by

$$g(w) = \begin{cases} \sum_{\mu \in \operatorname{Stab}_\Gamma v} \gamma\mu x & \text{if } w = \gamma v \text{ for } \gamma \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

Observe that g is well-defined: if $w = \gamma v = \gamma' v$ then $\gamma^{-1}\gamma'$ is in the stabalizer of v, hence $\gamma' = \gamma$. It is also invariant under Γ: $(\gamma g)(w) = \gamma g(\gamma^{-1} w) = g(w), \forall w \in \mathcal{V}(\mathcal{T})$. Finally,

$$\langle f, g\rangle = \langle f(v), \sum_{\mu \in \operatorname{Stab}_\Gamma v} \mu x\rangle = \sum_{\mu} \langle f(v), \mu x\rangle = |\operatorname{Stab}_\Gamma v| \langle f(v), x\rangle$$

Similarly, let $f \in C^1(M)^\Gamma$ and by non-degeneracy on M, fix $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$ and $x \in M$ such that $\langle f(e), x\rangle \neq 0$. Define $g : \overrightarrow{\mathcal{E}}(\mathcal{T}) \to M$ by

$$g(w) = \begin{cases} \gamma \sum_{\mu \in \operatorname{Stab}_\Gamma e} \mu x & \text{if } w = \gamma e \\ -\gamma \sum_{\mu \in \operatorname{Stab}_\Gamma e} \mu x & \text{if } w = \gamma \bar{e} \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined: if $w = \gamma' e = \gamma e$ then $\gamma^{-1}\gamma' \in \operatorname{Stab}_\Gamma e$. Similarly, if $w = \gamma' \bar{e} = \gamma \bar{e}$ then $\gamma^{-1}\gamma' \in \operatorname{Stab}_\Gamma \bar{e} = \operatorname{Stab}_\Gamma e$ by Lemma 2.2.6. It is also clear that $g \in C^1(M)^\Gamma$. Non-degeneracy follows, because

$$\langle f, g\rangle = \sum_{\mu \in \operatorname{Stab}_\Gamma e} \langle f(e), \mu x\rangle = |\operatorname{Stab}_\Gamma e| \langle f(e), x\rangle \neq 0$$

■

Lemma 2.4.5. *Let $f \in C^1(M)$. Then, via Lemma 2.2.5, f corresponds to a function on B_p^**

and is determined as an element of $C^1(M)$ *by its values on* $B_p^{*,1}$.

Proof. Let $e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$. Then, since $f(\bar{e}) = -f(e)$ and exactly one of $e, \bar{e}$ is in the image of $d : \mathcal{E}(\mathcal{T}) \to \overrightarrow{\mathcal{E}}(\mathcal{T})$, f is determined by its values on $\mathcal{E}(\mathcal{T})$. Under the isomorphism

$$a : B_p^*/N(R_p) \to \mathcal{E}(\mathcal{T})$$

from Lemma 2.2.5, each $e \in \mathcal{E}(\mathcal{T})$ is represented by some $z \in B_p^{*,1}$ (since, $B_p^{*,1} \cdot N(R_p) = B_p^*$ by Lemma 2.2.1). ■

Lemma 2.4.6. *Let* $f \in C^0(M)^\Gamma, g \in C^1(M)^\Gamma$. *Then,*

$$\langle f, \partial_* g \rangle = \langle \partial^* f, g \rangle$$

Proof. The proof is by induction on the number of edges. To simplify language, we simplify notation and sometimes refer to the pair $e, \bar{e}$ (for e and edge) simply as an edge.

(*) A subset $X \subset \overrightarrow{\mathcal{T}}$ is said to satisfy condition (*) if it is a collection of edges and vertices stable under inversion, Γ, taking endpoints, and such that $\deg_X v \geq 1$ for all $v \in V(X)$.

Here, $\deg_X v$ is the number of edges of X terminating at $v \in \mathcal{V}(X)$. This depends only on the Γ-equivalence class of v. Suppose X satisfies (*) and $\Gamma \backslash X$ contains exactly one edge. Let $e \in X$ an edge, $f \in C^0(X)^\Gamma, g \in C^1(X)^\Gamma$ (the latter groups are given the same definitions as above, replacing $\mathcal{T}$ with X). $\deg_X(o(e)) = \deg_X(t(e)) = 1$. Hence,

$$\langle f, \partial_* g \rangle = \langle f(t(e)), g(e) \rangle - \langle f(o(e)), g(e) \rangle = \langle \partial^* f, g \rangle$$

Inductively, suppose that the result holds for subsets $X \subset \overrightarrow{\mathcal{T}}$ which satisfy (*) and such that $\Gamma \backslash X$ contains at most n edges. Choose $e \in X$, and let $Y = X - (\Gamma e \cup \Gamma \bar{e})$. Remove $\Gamma o(e)$ (resp. $\Gamma t(e)$) if $\deg_X o(e) = 1$ (resp. if $\deg_X t(e) = 1$). It is clear that Y is stable under Γ

and inversion, since the subset we removed is. The vertices in $\Gamma o(e), \Gamma t(e)$ are the only ones whose degree in Y (possibly) differs from that in X. Hence, it is clear that $\deg_Y v \geq 1$ for all $v \in V(Y)$. Finally, it is clear that Y is closed under taking terminal vertices, because we removed only vertices that met only removed edges.

Now, $\langle f, \partial_* g \rangle$ and $\langle \partial^* f, g \rangle$ differ from $\langle f|_Y, \partial_*(g|_Y) \rangle$ and $\langle \partial^*(f|_Y), g|_Y \rangle$ respectively by $\langle f(t(e)), g(e) \rangle - \langle f(o(e)), g(e) \rangle$. Hence, the result holds for X. ■

2.4.0.3 Harmonic functions

Finally, define the space of harmonic functions

$$C^1_{\text{har}}(M) = \ker \partial_* = \left\{ f : \overrightarrow{\mathcal{E}}(\mathcal{T}) \to M \;\middle|\; f(\bar{e}) = -f(e), \sum_{t(e)=v} f(e) = 0 \,\forall v \in \mathcal{V}(\mathcal{T}), e \in \overrightarrow{\mathcal{E}}(\mathcal{T}) \right\} \tag{2.17}$$

By Lemma 2.4.6, the Γ-invariants of this space have the following description

$$C^1_{\text{har}}(M)^\Gamma = \partial^*(C^0(M)^\Gamma)^\perp = \left\{ f \in C^1(M) \;\middle|\; \langle f, \partial^* g \rangle = 0 \forall g \in C^0(M)^\Gamma \right\} \tag{2.18}$$

where $\partial^*(C^0(M)^\Gamma)^\perp$ represents the orthogonal complement is taken in $C^1(M)^\Gamma$. This is clear

$$\begin{aligned} f \in \ker(\partial_* |_{C^1(M)^\Gamma}) &\iff \langle \partial_* f, g \rangle = 0 && \forall g \in C^0(M)^\Gamma \\ &\iff \langle f, \partial^* g \rangle = 0 \\ &\iff f \in \partial^*(C^0(M)^\Gamma)^\perp \end{aligned}$$

2.5 Harmonic forms

The goal of this section is to prove Theorem 2.5.4, which identifies a space of harmonic functions with the space of p-new modular forms on a definite quaternion algebra. This is used in our explicit Jacquet-Langlands isomorphism, Theorem 3.3.2, which identifies modular forms on a Shimura curve with the same space.

2.5.0.1 Harmonic forms

In this section, we specialize the results above to define the space of harmonic forms $C^1_{\mathcal{E},\text{har}}(\mathcal{M})^\Gamma$, with coefficients in $\mathcal{M}$, which appears in the main result of this chapter, Theorem 2.5.4.

Lemma 2.5.1. *Let $C^1_{\mathcal{E}}(\mathcal{M})$ and $C^1(\mathcal{M})$ be the $\mathbb{Q}_p[B_p^*]$ modules defined in* (2.9) *and* (2.11) *respectively. Let d be as in Lemma 2.2.5, and consider the subgroup $B_p^{*,2} = \left\{ z \in B_p^* \;\middle|\; z \text{ even } \right\}$ of even elements of B_p^*. The map given by restriction along d is an isomorphism of left $R[B_p^{*,2}]$ modules*

$$C^1(M) \xrightarrow{d^*} C^1_{\mathcal{E}}(M)$$

satisfying the following explicit formulas for $f \in C^1(M)$ and $g \in C^1_{\mathcal{E}}(M)$

$$d^*(f)(z) = f \circ \tilde{D}(z) = \begin{cases} f(z) & z \text{ even} \\ f(z\beta) & z \text{ odd} \end{cases}$$

$$d^{*-1}(g)(z) = \begin{cases} g(z) & z \text{ even} \\ -g(z\beta) & z \text{ odd} \end{cases}$$

Proof. It is clear that this map is injective and surjective because elements of $C^1(M)$ are determined by their values on the image of $d : \mathcal{E}(\mathcal{T}) \to \overrightarrow{\mathcal{E}}(\mathcal{T})$, and arbitrary maps specified on the image of d can be lifted to $C^1(M)$. It remains to check that this is a map of left $R[B_p^{*,2}]$

modules, which follows from the following computation.

$$\gamma d^*(f)](z) = \gamma d^*(f)(\gamma^{-1}z)|\mathrm{Nr}\,\gamma|^{n/2} = \gamma f(d(\gamma^{-1}z))|\mathrm{Nr}\,\gamma|^{n/2} = \gamma f(\gamma^{-1}d(z))|\mathrm{Nr}\,\gamma|^{n/2}$$
$$d^*(\gamma f)(z) = d^*(\gamma f \circ \gamma^{-1} \cdot |\mathrm{Nr}\,\gamma|^{n/2})(z) = \gamma f(\gamma^{-1}d(z))|\mathrm{Nr}\,\gamma|^{n/2}$$

■

Define $C^1_{\mathcal{E},\mathrm{har}}(\mathcal{M})$ as the image of $C^1_{har}(\mathcal{M})$ under the isomorphism of Lemma 2.5.1. It's space of Γ-invariants is identified with $\mathrm{C}^1_{\mathrm{har}}(\mathcal{M})^\Gamma$.

2.5.0.2 p-new forms

The source of the isomorphism of Theorem 2.5.4 is the space of p-new forms $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$. We recall its construction here.

Consider the linear endomorphism W_p of $\mathcal{L}_k(U,\mathbb{Q}_p)$

$$W_p(f)(z) = \beta f(zj(\beta))p^{-n/2} = \beta f(zj(\beta))|\mathrm{Nr}\,\beta|^{n/2} \qquad z \in B_f^*$$

Here β is as in (2.6). It is clear that $W_p(f)$ is left B^* invariant and transforms correctly under U:

$$W_p(f)(zu) = \beta f(zu\beta)p^{-n/2} = \beta f(z\beta\beta^{-1}u\beta)p^{-n/2} = u_p^{-1}\beta f(z\beta)p^{-n/2} = u_p^{-1}W_p(f)(z)$$

hence an endomorphism of $\mathcal{L}_k(U,\mathbb{Q}_p)$ as claimed. It is also an involution

$$(W_p \circ W_p f)(z) = \beta^2 f(z\beta^2)p^{-n} = f(z)$$

and an isometry. To see this, let $f, g \in \mathcal{L}_k(U,\mathbb{Q}_p)$ and $z_1\beta, \cdots, z_r\beta$ a system of representatives of $B^*\backslash B_f^*/U$. Then by the usual argument,

$$\begin{aligned}
\langle W_p f, W_p g\rangle &= \sum_{[z]\in B^*\backslash B_f^*/U} \langle W_p(f)(z), W_p(g)(z)\rangle \left(\mathrm{Nr}\, z_p \|\mathrm{Nr}\, z\|\right)^n \\
&= \sum_{i=1}^{r} p^{-n}(\mathrm{Nr}\,\beta)^n \langle f(z_i), g(z_i)\rangle \left(\mathrm{Nr}(z_{i,p})\|\mathrm{Nr}\, z_i\|\right)^n \\
&= \sum_{i=1}^{r} \langle f(z_i), g(z_i)\rangle \left(\mathrm{Nr}\, z_{i,p}\|\mathrm{Nr}\, z_i\|\right)^n = \langle f, g\rangle
\end{aligned}$$

The last equality holds because by Lemma 2.3.6 $\{z_i\}_i$ is a system of representatives of $B^*\backslash B_f^*/U$.

Definition 2.5.1. Define

$$\mathcal{L}_k(V,\mathbb{Q}_p)\oplus\mathcal{L}_k(V,\mathbb{Q}_p) \xrightarrow{\psi} \mathcal{L}_k(U,\mathbb{Q}_p) \qquad \psi(f,g)(z) = W_p(g)(z) - f(z)$$

We call its image the p-old subspace $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-old}$ and its orthogonal complement the p-new subspace

$$\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new} = (\mathcal{L}_k(U,\mathbb{Q}_p)^{p-old})^{\perp}$$

Note that W_p stabalizes $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-old}$, and hence is an involution of $\mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$.

2.5.0.3 Identification

Recall that the map appearing in Theorem 2.5.4 was constructed above (see Remark 2.3.1). Here, we identify the p-old forms (Definition 2.5.1) and the image of a module under ∂^* (2.13).

Lemma 2.5.2. *The diagram commutes*

$$\begin{tikzcd}
\mathcal{L}_k(V,\mathbb{Q}_p)\oplus\mathcal{L}_k(V,\mathbb{Q}_p) \arrow[r, "l_V\oplus l_V", "\cong"'] \arrow[d, "\psi"'] & W_k(\Gamma')\oplus W_k(\Gamma') \arrow[r, "w\oplus w", "\cong"'] & C^0(\mathcal{M})^{\Gamma'}\oplus C^0(\mathcal{M})^{\Gamma'} \arrow[r] & C^0(\mathcal{M})^{\Gamma} \arrow[d, "d^*\circ\partial^*"'] \\
\mathcal{L}_k(U,\mathbb{Q}_p) \arrow[r, "\cong", "l_U"'] & C_1(\Gamma',k) \arrow[r, "\cong", "c_1"'] & C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma'} \arrow[r, "\cong", "\sigma"'] & C^1_{\vec{\mathcal{E}}}(\mathcal{M})^{\Gamma}
\end{tikzcd}$$

The map which is the composition of the arrows appearing in the bottom row will be called e. For later reference, we record the following formulas, which hold for all $z \in B_p^*$. If $f \in \mathcal{L}_k(U, \mathbb{Q}_p)$ the image of f under e is denoted $e_f(z)$, and

$$e_f(z) = \begin{cases} z \cdot f \circ j(z) |\mathrm{Nr}\, z|^{n/2} & z \text{ even} \\ z \cdot W_p(f) \circ j(z) |\mathrm{Nr}\, z|^{n/2} = z\beta \cdot f \circ j(z\beta) |\mathrm{Nr}\, z\beta|^{n/2} & z \text{ odd} \end{cases} \tag{2.19}$$

If $f \in C^0(\mathcal{M})^\Gamma$, then its image under the right vertical arrow is

$$d^* \circ \partial^*(f)(z) = \begin{cases} f(z\beta) - f(z) & z \text{ even} \\ f(z) - f(z\beta) & z \text{ odd} \end{cases}$$

If $(f, g) \in \mathcal{L}_k(V, \mathbb{Q}_p) \oplus \mathcal{L}_k(V, \mathbb{Q}_p)$ then its image, in $C^0(\mathcal{M})^\Gamma$, under the top row is

$$\alpha_{f,g}(z) = \begin{cases} z \cdot f \circ j(z) |\mathrm{Nr}\, z|^{n/2} & z \text{ even} \\ z \cdot g \circ j(z) |\mathrm{Nr}\, z|^{n/2} & z \text{ odd} \end{cases}$$

If $f, g \in \mathcal{L}_k(V, \mathbb{Q}_p) \oplus \mathcal{L}_k(V, \mathbb{Q}_p)$ then the left vertical arrow is $\psi(f, g)(z) = W_p(g)(z) - f(z)$ (Definition 2.5.1).

Proof. The proof is straightforward, we first verify the formulas above, and then check the composition.

If $(f, g) \in \mathcal{L}_k(V, \mathbb{Q}_p) \oplus \mathcal{L}_k(V, \mathbb{Q}_p)$, then under the top row, this pair maps to the following function of $z \in B_p^*$

$$\begin{aligned}
&\psi \circ w \oplus w \circ l_V \oplus l_V \circ (f,g)(z) \\
&= \psi \circ w \oplus w \circ (f \circ j, g \circ j)(z) \\
&= \psi \circ (z \cdot j \circ j(z)|\mathrm{Nr}\, z|^{n/2}, z \cdot g \circ j(z)|\mathrm{Nr}\, z|^{n/2} \\
&= \alpha_{f,g}(z)
\end{aligned}$$

If $f \in \mathcal{L}_k(U, \mathbb{Q}_p)$, then the image of$f$ under the bottom row is the following function of $z \in B_p^*$

$$\begin{aligned}
&\sigma \circ c_1 \circ l_U \circ f \circ (z) \\
&= \sigma \circ c_1 \circ f \circ j(z) \\
&= \sigma \circ (z \cdot f \circ j(z)|\mathrm{Nr}\, z|^{n/2}) \\
&= \tilde{D}(z) \cdot f \circ j \circ \tilde{D}(z)|\mathrm{Nr}\, \tilde{D}(z)|^{n/2} \\
&= e_f(z)
\end{aligned}$$

If $f \in C^0(\mathcal{M})^\Gamma$, then its image under the right vertical arrow is the following function of $z \in B_p^*$

$$\begin{aligned}
d^* \circ \partial^*(f)(z) &= d^* \circ (f \circ \tilde{T} - f \circ \tilde{O})(z) \\
&= f \circ \tilde{T} \circ \tilde{D}(z) - f \circ \tilde{O} \circ \tilde{D}(z)
\end{aligned}$$

which is the stated function.

Finally, we verify that the diagram commutes. Let $(f,g) \in \mathcal{L}_k(V, \mathbb{Q}_p) \oplus \mathcal{L}_k(V, \mathbb{Q}_p)$. Then, the image in $C^0(\mathcal{M})^\Gamma$ taken via the top arrow is

$$d^* \circ \partial^* \circ\alpha_{f,g}(z) = \begin{cases} z\beta \cdot g \circ j \circ (z\beta)|\mathrm{Nr}\, z\beta|^{n/2} - z \cdot f \circ j(z)|\mathrm{Nr}\, z|^{n/2} & z \text{ even} \\ z \cdot g \circ j(z)|\mathrm{Nr}\, z|^{n/2} - z\beta \cdot f \circ j(z\beta)|\mathrm{Nr}\, z\beta|^{n/2} & z \text{ odd} \end{cases}$$

Similarly, the image in $C^0(\mathcal{M})^\Gamma$ taken via the bottom arrow is

$$e_{W_p(g)-f}(z) = \begin{cases} z \cdot W_p(g) \circ j(z)|\mathrm{Nr}\, z|^{n/2} - z \cdot f \circ j(z)|\mathrm{Nr}\, z|^{n/2} & z \text{ even} \\ z\beta \cdot W_p(g) \circ j(z\beta)|\mathrm{Nr}\, z\beta|^{n/2} - z\beta \cdot f \circ j(z\beta)|\mathrm{Nr}\, z\beta|^{n/2} & z \text{ odd} \end{cases}$$

The compositions agree, and hence the diagram commutes. ■

2.5.0.4 Inner product

The space of modular forms $\mathcal{L}_k(U, \mathbb{Q}_p)$ is equipped with a natural non-degenerate inner product (2.5), as is $C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$ (Lemma 2.5, Lemma 2.5.1). We show that these inner products agree under the isomorphism

$$\mathcal{L}_k(U, \mathbb{Q}_p) \xrightarrow{\ e\ } C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$$

of Lemma 2.5.2, or equivalently Remark 2.3.1. The main theorem of this part, Theorem 2.5.4, then follows easily.

The inner product (2.4) is non-degenerate, and hence by Lemma 2.4.4 and Lemma 2.5.1, induce non-degenerate $\mathbb{Q}_p$ bilinear forms on $C^0(\mathcal{M})^\Gamma$ and $C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$. They satisfy the following formulas:

$$\langle f, g\rangle = \sum_{[v]\in\Gamma\backslash\,\mathcal{V}(\mathcal{T})} \langle f(v), g(v)\rangle = \sum_{[z]\in\Gamma\backslash B_p^*/N(\mathcal{R}_p)} \langle f(z), g(z)\rangle \tag{2.20}$$

$$\langle f, g\rangle = \sum_{[e]\in\Gamma\backslash\,\mathcal{E}(\mathcal{T})} \langle f(e), g(e)\rangle = \sum_{[z]\in\Gamma\backslash B_p^*/N(R_p)} \langle f(z), g(z)\rangle \tag{2.21}$$

Here, f, g are taken in $C^0(\mathcal{M})^\Gamma$ for the first formula, and in $C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$ for the second formula. As usual, function on edges or vertices of the Bruhat-Tits tree are identified with functions on B_p^* via Lemma 2.2.5.

Lemma 2.5.3. *The isomorphism obtained in Lemma 2.5.2*

$$\mathcal{L}_k(U, \mathbb{Q}_p) \xrightarrow{\ e\ } C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$$

preserves inner products. That is, $\langle f, g \rangle = \langle e_f, e_g \rangle$.

Proof. Recall the bijections from Lemma 2.3.4 and Lemma 2.3.5 compose to

$$\Gamma \backslash B_p^* / N(R_p) \xrightarrow{D} \Gamma' \backslash B_p^* / N(\underline{\mathrm{R}}_p) \xrightarrow{j} B^* \backslash B_f^* / U$$

In particular, if $z_1, \cdots, z_r \in B_p^*$ is a collection of even representatives of $\Gamma \backslash B_p^* / N(R_p)$ then $j(z_1), \cdots, j(z_r)$ is a set of representatives of $B^* \backslash B_f^* / U$. Let $f, g \in \mathcal{L}_k(U, \mathbb{Q}_p)$. Then,

$$\begin{aligned}
\langle f, g \rangle &= \sum_{i=1}^{r} \langle f \circ j(z_i), g \circ j(z_i) \rangle (\mathrm{Nr}\, z_i |\mathrm{Nr}\, z_i|)^n \\
\langle e_f, e_g \rangle &= \sum_{i=1}^{r} \langle z_i f \circ j(z_i) |\mathrm{Nr}\, z_i|^{n/2}, z_i g \circ j(z_i) |\mathrm{Nr}\, z_i|^{n/2} \rangle \\
&= \sum_{i=1}^{r} (\mathrm{Nr}\, z_i)^n |\mathrm{Nr}\, z_i|^n \langle f \circ j(z_i), g \circ j(z_i) \rangle
\end{aligned}$$

The last equality is by (2.4). ■

Finally, the main result of this chapter is

Theorem 2.5.4. *The map e of Lemma 2.5.2 restricts to an isomorphism of $\mathbb{Q}_p$ vector spaces*

$$\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \xrightarrow{\ e\ } C^1_{\mathcal{E},har}(\mathcal{M})^\Gamma$$

given by $f \mapsto e_f$, *where*

$$e_f(z) = \tilde{D}(z) f \circ j(\tilde{D}(z)) |\mathrm{Nr}\, \tilde{D}(z)|^{n/2}$$

See (2.19) *and Lemma 2.5.2 for explicit formulas.*

Proof. From Lemma 2.5.2 we already know that e is an isomorphism $\mathcal{L}_k(U, \mathbb{Q}_p) \to C^1_{\mathcal{E}}(\mathcal{M})^\Gamma$. The natural inner products on each side agree by Lemma 2.5.3. Finally, the p-new space, and the harmonic forms are the orthogonal complements of the p-old forms, and its image under e respectively. ∎

CHAPTER 3

Modular forms on Shimura Curves

The goal of this chapter is to prove one of our main results, Theorem 3.3.2.

In the previous chapter we studied weight k modular forms of level U (that are p-new) on a definite quaternion algebra B and in Theorem 2.5.4 wrote them as a space of harmonic forms $C^1_{\mathcal{E},\text{har}}(\mathcal{M})^\Gamma$. Here, we introduce the indefinite quaternion algebra and Shimura curve that are associated to the triple (B, U, p). The explicit Jacquet-Langlands isomorphism, Theorem 3.3.2, will identify the modular forms studied above with a space of modular forms on this Shimura curve. That one can express differential forms in terms of their residues is work of [Sch84], [dS89], [Tei90], [Tei93], [SS91]. See also [DS01] and [AdS03].

In §3.1, we give an introduction to p-adic uniformization in the form we will use it, following [BC91]. In §3.2 we recall the construction of the relative dualizing sheaf following [Tei93]. Finally, in §3.3 we collect these results to prove Theorem 3.3.2. Compatibilities of this map are checked in Chapter 4.

3.1 p-adic uniformization

For more details on the results of this section see [BC91, III.5]. [1]

[1] For a detailed history, see the introduction to [BC91].

3.1.0.1 Setup

Let $D/\mathbb{Q}$ be the unique (up to isomorphism) indefinite quaternion algebra of discriminant $N^{-}p$. Throughout this section, $\mathcal{U} \subset D_f^*$ will be a compact open subgroup of the form $\mathcal{U} = \mathcal{U}_p \cdot \mathcal{U}^p$ where $\mathcal{U}^p \subset B_f^{*,p}$ is a compact open subgroup and $\mathcal{U}_p \subset D_p^*$ is the open subgroup defined by the unique maximal order of D_p. Fix an isomorphism $\phi_\infty : D \otimes_{\mathbb{Q}} \mathbb{R} \to M_2(\mathbb{R})$. The action of $d \in D^*$ on $\mathcal{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ is via the Möbius transformation associated to $\phi_\infty(d)$.

The associated Shimura curve, $S_{\mathcal{U}}$ over $\mathbb{Q}$ has complex points

$$S_{\mathcal{U}}(\mathbb{C}) = D^* \backslash \left[\mathcal{H}^{\pm} \times D_f^*/\mathcal{U}\right].$$

Since D and B are obtained by interchanging invariants at p and ∞, algebras, we fix an isomorphism $\sigma : D_f^p \to B_f^p$. If $q \neq p$ is a prime, and $\sigma_q : D_q \to B_q$ is the induced isomorphism, we require that the collection $\{\ \phi_q \circ \sigma_q\ \}_{q \text{ split}, q \neq p}$ satisfies the usual compatibility. To simplify notation, we may treat σ as an equality. Let σ' be the anti-isomorphism $\sigma' : D_f^p \xrightarrow{\sigma} B_f^p \xrightarrow{z \mapsto \bar{z}} B_f^p$ (as usual, $\bar{z} = \mathrm{Nr}(z) z^{-1}$ denotes the canonical involution applied to z). Let

$$\mu : D_f^{*,p} \xrightarrow{\sigma'} B_f^{*,p} \xrightarrow{g \mapsto g^{-1}} B_f^{*,p} \qquad x \mapsto \sigma(x)\,\mathrm{Nr}(\sigma(x))^{-1}$$

be the isomorphism on units induced by composing with $g \mapsto g^{-1}$. Recall that we fixed an isomorphism $\phi_p : B_p \to M_2(\mathbb{Q}_p)$, and note that the Eichler orders $R \subset \mathcal{R} \subset B$ define orders $R_{indef} \subset \mathcal{R}_{indef} \subset D$. The isomorphisms $\{\sigma_q\}_{q \neq p}$ specify the order at finite places away from p, and we set $R_{indef,p} = \mathcal{R}_{indef,p}$ as the unique maximal order at p. Since R and $\mathcal{R}$ are stable under the canonical involution 2.1.0.3 it is equivalent to use $\{\sigma'_q\}_{q \neq p}$ instead of $\{\sigma_q\}_{q \neq p}$. The isomorphism μ identifies $\mathcal{U}^p$ with an open subgroup of $B_f^{*,p}$.

3.1.0.2 Small enough $\mathcal{U}^p$

Let $\mathcal{U}(M) \subset \widehat{\mathcal{R}}_{indef}$ be the subgroup of units congruent to 1 modulo M. If $\mathcal{U} \subset \mathcal{U}(M)$ for some integer $M \geq 3$, or more generally $\mathcal{U}$ is *small enough* (see [BC91, III.1.3] for a definition), $S_{\mathcal{U}}$ represents a moduli problem over $\mathbb{Q}$ [BC91, III.1.1.4] and this functor extends to $\mathbb{Z}_p$ [BC91, III.3]. The scheme representing the moduli problem over $\mathbb{Z}_p$ is projective and flat [BC91, III.3.4]. Since we will primarily work over $\mathbb{Z}_p$, $S_{\mathcal{U}}$ will henceforth refer to this scheme.

Let $\widehat{\Omega}$ the formal p-adic upper half plane, $\mathrm{Fr} : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ the p-th power map $x \mapsto x^p$, and $\widetilde{\mathrm{Fr}} : \mathbb{Z}_p^{nr} \to \mathbb{Z}_p^{nr}$ the associated lift. The action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}$ is [BC91, Thm. II.9.3] obtained from the natural action of g on $\widehat{\Omega}$ and the action of $\widetilde{\mathrm{Fr}}^{-ord_p(\mathrm{Nr}\, g)}$ on $\mathbb{Z}_p^{nr}$. If $V \subset B_f^*$ is a compact open subgroup, let

$$Z_{V^p} = Z_V = V^p \backslash B_f^* / B^*$$

and

$$Z_{\mathcal{U}} = Z_{\mu(\mathcal{U}^p)}$$

The group $B_p^* \xrightarrow{\phi_p} \mathrm{GL}_2(\mathbb{Q}_p)$ acts on $Z_{\mathcal{U}}$ by left multiplication.

Theorem 3.1.1 (Čerednik-Drinfeld)**.** *Let $\mathcal{U} \subset D_f^*$ an open subgroup as above, and suppose that $\mathcal{U}^p$ is small enough [BC91, III.1.3]. Then, there is an isomorphism of formal $\mathbb{Z}_p$-schemes*

$$\hat{S}_{\mathcal{U}} \cong \mathrm{GL}_2(\mathbb{Q}_p) \backslash [\widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{\mathcal{U}}]$$

Here, $\hat{S}_{\mathcal{U}}$ is the formal completion of $S_{\mathcal{U}}$ along its special fiber. The isomorphism is compatible, as $\mathcal{U}^p$ varies, with the projection maps, and is compatible with the actions of the idelic groups $D_f^{*,p}$, $B_f^{*,p}$ on each side via the isomorphism $D_f^{*,p} \xrightarrow{\mu} B_f^{*,p}$.

The natural action of $D_f^{*,p}$ on the projective system $\{S_{\mathcal{U}}\}$ is a right action, while the natural action of $B_f^{*,p}$ on the projective system $\{Z_{\mathcal{U}}\}$ is a left-action. In order to compare them, we change the action on $\{Z_{\mathcal{U}}\}$ to a right action by definining

$$z \cdot g = g^{-1} \cdot z \qquad \forall g \in B_f^{*,p},\, z \in \{Z_{\mathcal{U}}\}$$

Then, the theorem above states that the isomorphism ϕ of projective systems above is compatible with the group actions in the sense that $\forall g \in D_f^{*,p}$,

$$\phi(x \cdot g) = \phi(x) \cdot \mu(g) = \mu(g)^{-1}\phi(x) = \mathrm{Nr}(\sigma(g))\sigma(g)^{-1}\phi(x)$$

3.1.0.3 General $\mathcal{U}^p$

Even if $\mathcal{U}$ is not small enough, there exists a distinguished subgroup of finite index $\mathcal{U}_1^p \subset \mathcal{U}^p$ that is small enough. Let $\mathcal{U}_1 = \mathcal{U}_1^p \mathcal{U}_p$ and define $S_{\mathcal{U}}$ over $\mathbb{Z}_p$ as $S_{\mathcal{U}_1}/(\mathcal{U}/\mathcal{U}_1)$. Apply the theorem above to $\mathcal{U}_1$, and pass to the quotient by the finite group $\mathcal{U}^p/\mathcal{U}_1^p$. We get an isomorphism [BC91, III.5.3.2]

$$\hat{S}_{\mathcal{U}} \cong \mathrm{GL}_2(\mathbb{Q}_p)\backslash[\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{\mathcal{U}}]$$

Remark 3.1.1. If $\mathcal{U}_p$ is not maximal then Drinfeld's approach still gives p-adic uniformization of the corresponding Shimura curves. See [BC91, Theorem III.5.5].

Recall that we identify open subgroups of $D_f^{*,p}$ and $B_f^{*,p}$ via μ. Via this identification, set $\mathcal{U} = U^p \cdot \mathcal{U}_p$, $S = S_{\mathcal{U}}$, and $\hat{S} = \hat{S}_{\mathcal{U}}$. The following conditions are satisfied [BC91, III.5.3.3]:

1. The image of $\mathcal{U}^p$ under the reduced norm is maximal, that is, equal to $\prod_{\ell \neq p} \mathbb{Z}_\ell^*$

2. The p-adic valuation of $\mathbb{Q}$ maps the intersection of $\mathcal{U}^p$ and the center $\mathbb{Q}^*$ of D surjectively to $\mathbb{Z}$.

Indeed, (1) is clear by the same argument in 2.3.1, and (2) can be checked for the intersection of U^p and the center of B which is done in Lemma 2.3.1.

Recall that $\Gamma' = U^p \cap B^*$, and $\Gamma^{(2)} \subset \Gamma'$ is the subgroup of elements whose reduced norm has even p-adic valuation. These subgroups are identified with subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ via the isomorphism $\phi_p : B_p \to M_2(\mathbb{Q}_p)$ above. Then,

$$\hat{S} = \Gamma' \backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}$$

Let $W = \Gamma'/\Gamma^{(2)} \cong \mathbb{Z}/2\mathbb{Z}$, and fix $w_p \in \Gamma' - \Gamma^{(2)}$ with $\operatorname{ord} \operatorname{Nr} w_p = 1$. Let

$$\hat{S}' = \Gamma^{(2)} \backslash \widehat{\Omega}$$

Let $\mathbb{Z}_p^{(2)} \subset \mathbb{Z}_p^{unr}$ be the ring of integers of the unramified quadratic extension of $\mathbb{Z}_p$, and $\mathbb{Q}_p^{(2)}$ its fraction field. Then,

$$\hat{S} = W \backslash \left(\hat{S}' \otimes \mathbb{Z}_p^{(2)} \right)$$

is a quotient of a Mumford curve by a finite group, and $\hat{S}$ is a twisted form of $\hat{S}'$. The corresponding cocycle in $\mathrm{H}^1(\mathrm{Gal}(\mathbb{Q}_p^{(2)}/\mathbb{Q}_p), \mathrm{Aut}(\hat{S}'))$ sends the nontrivial element of the Galois group to w_p.

Remark 3.1.2. The curve S' is normal, proper and flat over $\mathbb{Z}_p$. Its geometric special fiber is reduced, connected, and has at most ordinary double points as singularities. The normalizations of the components of the special fiber are $\mathbb{F}_p$-rational curves, and the singular points of the fiber are $\mathbb{F}_p$-rational. See [Kur79, Proposition 3.2], which also explains how to produce the minimal regular model from properties of the graph $\Gamma \backslash \mathcal{T}$, and gives a formula for the genus in terms of a graph associated to the special fiber.

3.2 Relative dualizing sheaf

In this section, we state an explicit isomorphism between modular forms on a Shimura curve and modular forms on an indefinite quaternion algebra. Later, in Theorem 4.1.9, we will show that this isomorphism is compatible with the Hecke action on both sides. Theorem 4.2.9 will explain how it relates the natural integral structures.

3.2.1 Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Fix an even integer $k \geq 2$, let $n = k-2$ and $\mathbf{P}^{\mathrm{r}}_{\mathrm{k}}(\mathbb{Q}_{\mathrm{p}})$ be the left $\mathrm{GL}_2(\mathbb{Q}_p)$ module with underlying vector space the set of homogenous polynomials in x, y of degree k with coefficients in $\mathbb{Q}_p$. An element $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ acts as

$$\gamma P(x,y) = P\left(\gamma^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \mathrm{Nr}\,\gamma^r$$

Explicitly, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then $\gamma^{-1} = \frac{1}{\det\gamma}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and $\gamma P(x,y) = P(dx - by, -cx + ay)\det\gamma^{r-k}$. Similarly, let $\mathbf{P}^{\mathrm{r}}_{\mathrm{k}}(\mathbb{Z}_{\mathrm{p}}) \subset \mathbf{P}^{\mathrm{r}}_{\mathrm{k}}(\mathbb{Q}_{\mathrm{p}})$ the collection of polynomials with coefficients in $\mathbb{Z}_p$ and left $\mathrm{GL}_2(\mathbb{Z}_p)$ action determined by the formula above. If $\phi : \mathbf{P}^{\mathrm{r}}_{\mathrm{k}}(\mathbb{Q}_{\mathrm{p}}) \to P_k(\mathbb{Q}_p) = \mathrm{P}^0_k(\mathbb{Q}_p)$ is the identity map, then clearly $\phi(\gamma P) = \gamma\phi(P)\det\gamma^{-r}$.

Lemma 3.2.1. *Recall that $\mathrm{GL}_2(\mathbb{Q}_p)$ modules $P^r_{k-2}(\mathbb{Q}_p)$ and $P_{k-2}(\mathbb{Q}_p)$ defined at the beginning of §3.2.1 and Definition 2.1.1 respectively. The map*

$$\phi : P_{k-2}(\mathbb{Q}_p) \to P^r_{k-2}(\mathbb{Q}_p) \qquad \phi(P) = P(y, -x)$$

is an isomorphism of $\mathbb{Q}_p$ vector spaces with

$$\phi(\gamma P) = \gamma\phi(P)\det\gamma^s \qquad \gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$$

for $s = n - r = k - 2 - r$. In particular, if $r = k - 2$ then ϕ is an isomorphism of left $\mathrm{GL}_2(\mathbb{Q}_p)$ modules and if $r = n/2$ then $s = n/2$.

If $\gamma = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ and $F \in P^r_n$ then $\gamma F = F \cdot p^{2r-n}$. In particular, if $r = n$ this is multiplication by p^n and γ is the identity map if $r = n/2$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\lambda = \det\gamma$, and $\gamma^{-1} = \lambda^{-1}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Then,

$$\phi(P) = P(y, -x)$$
$$\gamma P(x, y) = P((x, y)\gamma) = P(ax + cy, bx + dy)$$
$$\phi(\gamma P) = P(ay - cx, by - dx)$$
$$\gamma\phi(P) = P(y, -x)(\gamma^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)) = P(y, -x)(dx - by, -cx + ay)\lambda^{r-k+2} = P(-cx + ay, by - dx)\lambda^{r-k+2}.$$

■

3.2.2 Teitelbaum's sheaf

Let $\widehat{\Omega}$ the formal p-adic upper half plane and $\Omega = \mathbb{P}^1 - \mathbb{P}^1(\mathbb{Q}_p)$ its generic fiber. It is the p-adic upper half plane considered as a rigid analytic space. Let z be a coordinate function on $\mathbb{P}^1$. Then, $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on Ω by Mobius transformations in z. The harmonic functions, $\mathrm{C}^1_{\mathrm{har}}(\mathbf{P}^{\mathrm{n}/2}_{\mathrm{n}}(\mathbb{Q}_\mathrm{p}))$, with coefficients in $\mathbf{P}^{\mathrm{n}/2}_{\mathrm{n}}(\mathbb{Q}_\mathrm{p})$ play a special role, and will be denoted

$$\mathrm{C}_{\mathrm{har}}(k) = \left\{ f : \overrightarrow{\mathcal{E}}(\mathcal{T}) \to \mathbf{P}^{\mathrm{n}/2}_{\mathrm{n}}(\mathbb{Q}_\mathrm{p}) \;\middle|\; f(\bar{e}) = -f(e), \quad \sum_{t(e)=v} f(e) = 0 \quad \forall v \in \mathcal{V}(\mathcal{T}) \right\}$$

We recall properties of the invertible sheaf $\omega = \omega_{\mathbb{Z}_p}$ that is defined [Tei93, Def. 10] on the formal scheme $\widehat{\Omega}$. On the generic fiber[2] Ω of $\widehat{\Omega}$, $\mathbb{Q}_p \otimes \omega$ is naturally isomorphic to the sheaf of rigid differential forms [Tei93, Lem. 13], and on the special fiber of $\widehat{\Omega}$, $\omega/p\omega$ is the sheaf of regular differentials [Tei93, Lem. 12]. Pullback of differentials under the $\mathrm{GL}_2(\mathbb{Q}_p)$ action gives $\Gamma(\Omega, \omega^{\otimes k/2})$ the structure of a right $\mathrm{GL}_2(\mathbb{Q}_p)$ module.

We can identify $\mathrm{H}^0(\widehat{\Omega}, \omega^{k/2})^\Gamma$ with sections of the dualizing sheaf of $\hat{S}'$ over $\mathbb{Z}_p$ via the following lemma [Tei93, Lemma 31]

Lemma 3.2.2 (Teitelbaum)**.** *Assume that $p > 3$ and recall that $\hat{S}' = \Gamma^{(2)}\backslash\widehat{\Omega} = \Gamma\backslash\widehat{\Omega}$. Let ω_Γ denote the relative dualizing sheaf of $\hat{S}' \to \mathrm{Spf}\,\mathbb{Z}_p$. Then, the natural map $\pi : \widehat{\Omega} \to \Gamma\backslash\widehat{\Omega}$*

[2]The generic fiber is taken in the sense of Raynaud

induces an isomorphism of finitely generated and torsion free $\mathbb{Z}_p$-modules

$$H^0(\hat{S}', \omega_\Gamma^{k/2}) \xrightarrow{\pi^*} H^0(\widehat{\Omega}, \omega^{k/2})^\Gamma$$

Remark 3.2.1. The completion of the relative dualizing sheaf of $S' \to \operatorname{Spec} \mathbb{Z}_p$ is the relative dualizing sheaf of $\hat{S}' \to \operatorname{Spf} \mathbb{Z}_p$. See [Yek98, Example 5.12] and [ATJLL03, Example 2.5.2(2)] for precise definitions.

Now we recall basic properties of the residue map. Its compatibility with the $\mathbb{Z}_p$-integral structure is discussed in §4.2. Let $\mathcal{O}(k)$ the space of global rigid functions on Ω endowed with a right $\mathrm{GL}_2(\mathbb{Q}_p)$ action. An element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$, acts by

$$f|_\gamma(z) = \frac{\det(\gamma)^{k/2}}{j(\gamma, z)^k} f\left(\frac{az+b}{cz+d}\right) \qquad \text{where } j(\gamma, z) = cz + d$$

and the following map is an isomorphism of right $\mathrm{GL}_2(\mathbb{Q}_p)$ modules

$$\mathrm{H}^0(\Omega, \omega^{k/2}) \to \mathcal{O}(k) \qquad f(z)dz^{k/2} \to f(z)$$

Consider $\mathcal{O}(k)$ and $\mathrm{H}^0(\Omega, \omega^{k/2})$ as left $\mathrm{GL}_2(\mathbb{Q}_p)$ modules via the anti-isomorphism $\gamma \mapsto \gamma^{-1}$ and give $\mathrm{C}_{\mathrm{har}}(\mathrm{k})$ the structure of a left $\mathrm{GL}_2(\mathbb{Q}_p)$ module via $\gamma f = \gamma \cdot f \circ \gamma^{-1}$.

Remark 3.2.2. Note that this is different from the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on harmonic functions considered in the last chapter (2.16). The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ here is determined by the pullback action on differential forms (see Lemma 3.2.3 below). However, since Γ consists of norm one elements, the restrictions of the actions to Γ are the same.

Recall that Schneider's residue map [dS89, §3] [Tei93, p569-70] [SS91, p.97] is

$$I : \mathcal{O}(k) \to \mathrm{C}_{\mathrm{har}}(\mathrm{k}) \qquad I(f)(e) = \operatorname{Res}_e \beta_\omega(z)$$

where $\beta_\omega(z) = (x - yz)^n f(z)dz$, and

$$\mathrm{Res}_e\, \beta_\omega(z) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i y^{n-i}\, \mathrm{Res}_e\, z^{n-i} f(z)dz$$

The map I is surjective, but not injective. See [Tei93, Thm. 15], [SS91] for a description of the kernel. It is injective when restricted to $\mathrm{H}^0(\widehat{\Omega}, \omega^{\otimes k/2})$ [Tei93, Thm. 19], and Teitelbaum identifies the image $\mathrm{Z_{har}(k)}$ of this subspace (see §4.2).

Lemma 3.2.3. *The $\mathbb{Q}_p$-linear map I is equivariant for the $\mathrm{GL}_2(\mathbb{Q}_p)$ action.*

Proof. Let $\mu = f(z)dz^{k/2}$, and define

$$\alpha_\mu(z) = f(z)dz \qquad \beta_\mu(z) = (x - yz)^n f(z)dz = (x - yz)^n \alpha_\mu(z)$$

For $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$,

$$\beta_\mu(\gamma z) = \frac{\gamma(u - vz)^n \det \gamma^{k-2}}{j(\gamma, z)^n \det \gamma^{k/2-1}} \frac{f(\gamma z) \det \gamma dz}{j(\gamma, z)^2} = \gamma(x - yz)^n \alpha_{\gamma^* \mu}(z)$$

Hence,

$$\beta_\mu(\gamma^{-1} z) = \gamma^{-1}(x - yz)^n \alpha_{\gamma^{-1*}\mu}(z)$$
$$(\gamma I(\beta_\mu))(e) = \gamma\, \mathrm{Res}_{\gamma^{-1}e}\, \beta_\mu(z) = \gamma\, \mathrm{Res}_e\, \beta_\mu(\gamma^{-1} z) = \mathrm{Res}_e (x - yz)^n \alpha_{\gamma^{-1*}\mu}(z)$$
$$I(\beta_{\gamma\mu}(z))(e) = \mathrm{Res}_e\, \beta_{\gamma^{-1*}\mu}(z) = \mathrm{Res}_e (x - yz)^n \alpha_{\gamma^{-1*}\mu}(z)$$

This is the desired equality. ∎

3.3 Explicit Jacquet-Langlands isomorphism

Finally, we state the Explicit Jacquet-Langlands isomorphism.

Theorem 3.3.1 (Explicit Jacquet-Langlands)**.** *There is an isomorphism of $\mathbb{Q}_p$ vector spaces*

$$\mathrm{JL} : \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \xrightarrow{f \mapsto \mathrm{JL}_f} \mathrm{C}_{har}(\mathrm{k})^{\Gamma}$$

where

$$\mathrm{JL}_f(z) = \phi \circ d^{*-1} \circ e_f(z) = (-1)^{sign\, z} \tilde{D}(z) \cdot \phi \circ f \circ j \circ \tilde{D}(z) \cdot |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2}$$

.

Explicitly,

$$\mathrm{JL}_f(z) = \begin{cases} z \cdot \phi \circ f \circ j(z) |\mathrm{Nr}\, z|^{n/2} \det z^{n/2} & z \text{ even} \\ -z\beta \cdot \phi \circ f \circ j(z\beta) |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2} & z \text{ odd} \end{cases}$$

Recall that sign$(z) = 1$ or -1 if z is even or odd respectively, $\beta = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, and $\tilde{D}$ is defined in Lemma 2.2.5. The map j naturally identifies B_p^* with ideles that are 1 away from p.

Proof. This follows from Theorem 2.5.4, Lemma 2.5.1 and Lemma 3.2.1. ■

Assume that $p > 3$. This assumption is only used by Lemma 3.2.2. Let $\omega_{S'}$ be the relative dualizing sheaf of $S' \to \operatorname{Spec} \mathbb{Z}_p$. Then, there are canonical identifications

$$\mathrm{H}^0(S', \omega_{S'/\mathbb{Z}_p}^{\otimes k/2}) = \mathrm{H}^0(\hat{S}', \widehat{\omega_{S'/\mathbb{Z}_p}}^{\otimes k/2}) = \mathrm{H}^0(\widehat{\Omega}, \omega^{\otimes k/2})^{\Gamma}$$

In light of Remark 3.2.1, the first equality follows from [Ill05, Thm. 2.2] and the second equality follows from Lemma 3.2.2. Let

$$S_k(S', \mathbb{Q}_p) = \mathrm{H}^0(S', \omega_{S'/\mathbb{Z}_p}^{\otimes k/2}) \otimes \mathbb{Q}_p$$

Under the identifications above, the residue map restricts to an isomorphism and we have the following

$$S_k(S', \mathbb{Q}_p) \to \mathrm{C}_{\text{har}}(\mathrm{k})^{\Gamma}$$

Theorem 3.3.2. *The space of harmonic forms* $\mathrm{C}_{har}(\mathrm{k})^{\Gamma}$ *is identified with a space of differential forms via Lemma 3.2.2, and the isomorphism* JL *of Theorem 3.3.1 takes the form*

$$\mathrm{JL} : \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \xrightarrow{f \mapsto \mathrm{JL}_f} \mathrm{C}_{har}(\mathrm{k})^{\Gamma} \cong S_k(S', \mathbb{Q}_p)$$

Proof. This follows from Theorem 3.3.1. ■

CHAPTER 4

Compatibility

In this chapter we prove that the isomorphism in Theorem 3.3.2 is compatible with the action of the Hecke operators, and natural $\mathbb{Z}_p$-integral structures.

4.1 Hecke action

In this section, we observe that the isomorphism in Theorem 3.3.2 is compatible with the natural Hecke actions on both sides.

Throughout this section, fix a prime $q \nmid N$, and let T_q be the q^{th} Hecke operator acting as an endomorphism of $\mathrm{H}^0(S_{\mathcal{U}}, \Omega^{\otimes k/2})$. We will compute T_q in terms of pullback by elements of $\mathrm{GL}_2(\mathbb{R})$ acting on the complex upper half plane, and by elements of $\mathrm{GL}_2(\mathbb{Q}_p)$ acting on the p-adic upper half plane. First, we prove some required lemmas. In this section, as is usual, $\coprod$ represets the union of sets that are pairwise disjoint.

Let $\mathrm{GL}_2(\mathbb{R})^+ = \{ \gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0 \}$. As usual, this group acts on the upper half plane $\mathcal{H}$ by Möbius transformations. Namely, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$ acts as $\gamma z = \frac{az+b}{cz+d}$. Set $j(\gamma, z) = cz + d$, then $d(\gamma z) = \det(\gamma) \cdot j(\gamma, z)^{-2}$. For $f : \mathcal{H} \to \mathbb{C}$ holomorphic, define

$$f|_\gamma(z) = \det \gamma^{k/2} j(\gamma, z)^{-k} f(\gamma z)$$

Let $\mathcal{O}_{\mathcal{H}}$ be the space of holomorphic functions on $\mathcal{H}$ and $\Omega_{\mathcal{H}} = d\mathcal{O}_{\mathcal{H}} = \mathcal{O}_{\mathcal{H}} dz$ the $\mathcal{O}_{\mathcal{H}}$ module of differentials. For each Fuchsian group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ of the first kind, let $S_k(\Gamma)$

be the space of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ such that $f|_\gamma = f \; \forall \gamma \in \Gamma$. The quotient map $\pi : \mathcal{H} \to \Gamma \backslash \mathcal{H}$ identifies $\Omega^{\otimes k/2}_{\Gamma\backslash \mathcal{H}}$ with $(\Omega^{k/2}_{\mathcal{H}})^\Gamma$ [BN81]. The map $f(z) \mapsto f(z) dz^{\otimes k/2}$ is an isomorphism [BN81, Thm 1.4]

$$S_k(\Gamma) \to \mathrm{H}^0(\Gamma \backslash \mathcal{H}, \Omega^{\otimes k/2})$$

As a Riemann surface, the Shimura curve $S_{\mathcal{U}}$ above is the quotient of $\mathcal{H}$ by a Fuchsian group $\Gamma_{\mathbb{C}}$ of the first kind [Miy89, Thm 5.2.13]

$$\Gamma_{\mathbb{C}} = \{ \, z \in R_{indef} \mid \mathrm{Nr}(z) = 1 \, \} = \mathcal{U} \cap D^{>0} \subset \mathrm{SL}_2(\mathbb{R})$$

where $D^{>0} = \{ \, z \in D^* \mid \mathrm{Nr}(z) > 0 \, \}$, and $\Gamma_{\mathbb{C}}$ is identified with a subgroup of $\mathrm{SL}_2(\mathbb{R})$ via the isomorphism $\phi_\infty : D \otimes_{\mathbb{Q}} \mathbb{R} \to M_2(\mathbb{R})$ fixed above. If $x \in D^*_f$, let $\Gamma_{x,\mathbb{C}}$ be $x\mathcal{U}x^{-1} \cap D^{>0}$ considered as a subgroup of $\mathrm{SL}_2(\mathbb{R})$ via ϕ_∞.

4.1.0.1 Coset decompositions

Let $\eta_q \in B^*_f$ the idele which is 1 away from the q^{th} place, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ at the q^{th} place. Let $\Delta = \{ \, r \in R_{indef} \mid \mathrm{Nr}(r) > 0 \, \}$

Lemma 4.1.1. *Let $\alpha \in R_{indef}$ with $\mathrm{Nr}(\alpha) = q$, and*

$$\Gamma_{\mathbb{C}} \alpha \Gamma_{\mathbb{C}} = \coprod_i \Gamma_{\mathbb{C}} \alpha_i$$

Then,

1. *Identify η_q with an element of D^*_f via σ_q (see §3.1.0.1). Then,*

$$\mathcal{U} \eta_q \mathcal{U} = \coprod_i \mathcal{U} \alpha_i$$

2.

$$\mathrm{GL}_2(\mathbb{Z}_q)\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)\mathrm{GL}_2(\mathbb{Z}_q)=\coprod_i \mathrm{GL}_2(\mathbb{Z}_q)\alpha_i$$

3.

$$U\eta_q U=\coprod_i U j_q^B(\alpha_i)$$

Proof. This proof follows from the properties of [Miy89, Thm. 2.7.6] which are checked in [Miy89, 5.3.5]. Since $\Gamma_{\mathbb{C}}\subset\mathcal{U}$, and $\mathcal{U}\alpha\Gamma_{\mathbb{C}}=\mathcal{U}\alpha\mathcal{U}$,

$$\mathcal{U}\alpha\mathcal{U}=\mathcal{U}\alpha\Gamma_{\mathbb{C}}=\cup_i\mathcal{U}\alpha_i$$

This is a disjoint decomposition. Indeed, fix indices j,j'. It is clear that the cosets $\mathcal{U}\alpha_j,\mathcal{U}\alpha_{j'}$ meet iff they are equal. By [Miy89, 2.7.6iii]

$$\mathcal{U}\alpha_j=\mathcal{U}\alpha_{j'}\implies\alpha_{j'}\in\mathcal{U}\alpha_j\cap\Delta=\Gamma_{\mathbb{C}}\alpha_j\implies\alpha_{j'}\in\Gamma_{\mathbb{C}}\alpha_j\implies\Gamma_{\mathbb{C}}\alpha_j=\Gamma_{\mathbb{C}}\alpha_{j'}\implies j=j'$$

That is, the elements of the union above are disjoint.

1. Note that $\alpha^{-1}=\bar{\alpha}q^{-1}\in R^*_{indef,\ell}$ for $\ell\neq q$, since each local order is stable under the canonical involution. Let $\xi=j_q(\alpha)\in D_f^*$. Then, $\mathcal{U}\xi\mathcal{U}=\mathcal{U}\alpha\mathcal{U}$. Since $\mathrm{Nr}(\alpha)=q$, the Smith Normal Form of $\alpha\in M_2(\mathbb{Z}_q)$ is $\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)$ and $\mathcal{U}\alpha\mathcal{U}=\mathcal{U}\eta_q\mathcal{U}$. This proves the first part.

2. The above decomposition can be written

$$\mathcal{U}\xi\mathcal{U}=\coprod_i\mathcal{U}j_q(\alpha_i)$$

   and projecting on the q^{th} place gives the second part.

3. The third equality follows from interpreting $j_q(\alpha_i)$ as an element of B_f^*.

∎

We need a decomposition of the last double coset $U\eta_q U$ in terms of elements of $R[1/p]$. This is provided by the following lemma.

Lemma 4.1.2. *Let $\mathcal{O} = R[1/p]$. There exists a collection $\{ \xi_i \}_i$ of elements in $\mathcal{O}$ of norm q such that*

$$U\eta_q U = \coprod_i U j_q(\xi_i)$$

is a disjoint union. It is clear that $\xi_i \in R_\ell^*$ for $\ell \neq p, q$.

Proof. Let $U' = \hat{\mathcal{O}}^* \subset B_f^*$, and

$$U'\eta_q U' = \coprod_i U'\zeta_i$$

where each ζ_i is an idele that is 1 away from the q^{th} place. The associated left $\mathcal{O}$-ideals $I_i \subset B$ [Vig80, III.5B] are principal. Indeed, the ideal classes are naturally in bijection with

$$U' \times B_\infty^* \backslash B_{\mathbb{A}}^* / B^*$$

which is a singleton [Vig80, Cor III.5.7 (1)], and so any left $\mathcal{O}$-ideal I can be written in the form $I = \mathcal{O}b$ with $b \in B^*$. Let $I_i = \mathcal{O}\xi_i$. Since $U'\zeta_i = U'\xi_i$ by definition, $\mathrm{Nr}(\xi_i)$ is an ℓ-adic unit for $\ell \neq p, q$. Modify ξ_i (via left-multiplication by an element of $\mathcal{O}^*$) so that $\mathrm{Nr}(\xi_i) \in \mathbb{Q}^*$ is supported away from p. We can do this by Lemma 2.3.1. Hence, $\mathrm{Nr}(\xi_i) = \pm q$. Since B is definite, the norm is positive [Vig80, Thm. III.4.1]:

$$\mathrm{Nr}(\xi_i) = q$$

The claim follows. Indeed, projecting $U'\eta_q U' = \coprod_i U'\zeta_i = \coprod_i U'\xi_i$ to the q^{th} place gives

$$U'_q \eta_{q,q} U'_q = \coprod_i U'_q \zeta_{i,q} = \coprod_i U'_q \xi_i$$

Since $U'_q = U_q$,

$$U\eta_q U = \coprod_i Uj_q(\xi_i)$$

■

The two disjoint union decompositions

$$U\eta_q U = \coprod_i Uj_q^B(\alpha_i) = \coprod_i Uj_q^B(\xi_i)$$

are indexed over the same set (because left cosets are equal iff they meet). Assume that $Uj_q(\alpha_i) = Uj_q(\xi_i)$ for each index i.

Lemma 4.1.3. *For each index i, there exists $y_i \in Uj_q(\xi_i)$ such that $y_{i,\ell} = 1$ for $\ell \neq q$, $y_i U = j_q(\bar{\xi}_i)U$, $Uy_i = Uj_q(\xi_i)$, and*

$$U\eta_q U = \coprod_i Uy_i = \coprod_i y_i U = \coprod_i U\bar{y}_i$$

Proof. It suffices to show that $Uj_q(\xi_i) \cap j_q(\bar{\xi}_i)U \neq \emptyset$. If this is the case, choose any y_i in the intersection such that $y_{i,\ell} = 1$ for $\ell \neq q$. Then, $Uy_i = Uj_q(\xi_i)$ and $y_i U = j_q(\bar{\xi}_i)U$. Since $U = \bar{U}$,

$$\overline{U\eta_q U} = U\eta_q U = \coprod_i U\xi_i = \coprod_i \bar{\xi}_i U$$

The result follows.

It remains only to show that $U_q\xi_i \cap \bar{\xi}_i U_q \neq \emptyset$, or equivalently,

$$\bar{\xi}_i^{-1} U_q \xi_i \cap U_q \neq \emptyset$$

There exists $u, v \in U_q$ such that $\xi_i = u \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right) v$ and $\bar{\xi}_i^{-1} = \xi/q = u \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/q \end{smallmatrix}\right) v$. It is clear that

$$u \begin{pmatrix} 1 & 0 \\ 0 & 1/q \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_q) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} v \cap \mathrm{GL}_2(\mathbb{Z}_q) \neq \emptyset$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/q \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_q) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} qa & b \\ c & d/q \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q) \right\}$$

■

4.1.0.2 Hecke operators on Shimura Curves

The goal of this section is to compute the action of T_q on $\mathrm{H}^0(S_{\mathcal{U}}, \Omega^{\otimes k/2})$ in terms of an action on the p-adic upper half plane. Classically, Hecke operators are given by pulling back (tensor powers of) differential forms. The relevant maps are expressed idelically in Lemma 4.1.4, and the Theorem 3.1.1 of Čerednik-Drinfeld is used to transfer the action to the p-adic upper half plane. Recall the isomorphisms σ and μ defined above in §3.1.0.1.

Let $\alpha \in \Delta$ an element of norm q. Since, by Lemma 4.1.1, the double coset $\Gamma_{\mathbb{C}} \alpha \Gamma_{\mathbb{C}}$ does not depend on the choice of α, the q^{th} Hecke operator is

$$T_q = \Gamma_{\mathbb{C}} \alpha \Gamma_{\mathbb{C}}$$

Fix a finite disjoint union decomposition $\Gamma_{\mathbb{C}} \alpha \Gamma_{\mathbb{C}} = \coprod_i \Gamma_{\mathbb{C}} \alpha_i$. This double coset acts on $S_k(\Gamma_{\mathbb{C}})$ [Miy89, §2.8] by

$$f|_{[\Gamma_{\mathbb{C}} \alpha \Gamma_{\mathbb{C}}]} = \det(\alpha)^{k/2-1} \sum_i f|_{\alpha_i}(z) = q^{k/2-1} \sum_i f|_{\alpha_i}(z) = q^{k-1} \sum_i j(\alpha_i, z)^{-k} f(\alpha_i z)$$

Note that for each index i, and prime $\ell \neq q$, both α and α_i are in $R^*_{indef,\ell}$ (by definition these elements are in $R_{indef,\ell}$, and the local order order is stable under the canonical involution). The corresponding action on the complex upper half plane is computed via the following lemma.

Lemma 4.1.4. *Let $x \in D_f^{*,p}$, and fix $\gamma \in D^{>0} = \{\, z \in D^* \mid \mathrm{Nr}(z) > 0 \,\}$ such that $\gamma x \in$*

$\mathcal{U}$. Consider the map on Shimura cures below, given on complex valued points by right multiplication by x. The diagram commutes.

$$\begin{tikzcd}
S_{x\mathcal{U}x^{-1}}(\mathbb{C}) \arrow[r, "x"] & S_{\mathcal{U}}(\mathbb{C}) \\
\Gamma_{x,\mathbb{C}}\backslash\mathcal{H} \arrow[u, "\cong"] \arrow[r, "\gamma"] & \Gamma_{\mathbb{C}}\backslash\mathcal{H} \arrow[u, "\cong"]
\end{tikzcd}$$

Here, the vertical arrows are induced by the natural identifications of the complex upper half plane with $\mathcal{H}\times[1]$, see §3.1.0.1.

Proof. Recall that $D^*\backslash D_f^*/\mathcal{U}$ is identified as the set of left-ideal classes of R_{indef}, and this set contans exactly one element [Vig80, Thm. III.4.1, Cor 5.7]. The order R_{indef} is stable under the canonical involution, and hence contains an element λ with $\mathrm{Nr}(\lambda) = -1$ [Vig80, Cor III.5.9] If $z \in D_f^*$, choose $d \in D^*, u \in \mathcal{U}$ such that $dzu = 1$. One can conjugate by λ to insure that $d \in D^{>0}$. Hence, $D_f^* = D^{>0}\mathcal{U}$.
The diagram above is

$$\begin{tikzcd}
D^*\backslash[\mathcal{H}^{\pm}\times D_f^*/x\mathcal{U}x^{-1}] \arrow[r, "x"] & D^*\backslash[\mathcal{H}^{\pm}\times D_f^*/\mathcal{U}] \\
D^{>0}\cap x\mathcal{U}x^{-1}\backslash\mathcal{H} \arrow[u, "\cong"] \arrow[r, "\gamma"] & D^{>0}\cap\mathcal{U}\backslash\mathcal{H} \arrow[u, "\cong"]
\end{tikzcd}$$

The top map sends $(z, [1])$ to $(z, x) = (\gamma z, \gamma x) = (\gamma z, [1])$. Hence, commutativity is clear. It remains to observe that the vertical arrows are isomorphisms. Without loss of generality, we observe this for the right vertical arrow. However, it is clear that

$$\Gamma_{\mathbb{C}} = \{\, z \in D^* \mid z(\mathcal{H}\times[1]) \cap (\mathcal{H}\times[1]) \neq \emptyset \,\}$$

considered as a subgroup of the automorphisms of $\mathcal{H}^{\pm}\times D_f^*/\mathcal{U}$. Hence, the vertical arrows are isomorphisms by Lemma 4.1.6.

■

The lemma below is used to write the action of T_q in terms of harmonic functions via the Čerednik-Drinfeld theorem.

Let $\mathcal{U}_1 \subset D_f^*$ be a compact open subgroup as above, and $x \in D_f^{*,p}$ an idele, identified as an element of D_f^* that is 1 at p. If $\mathcal{U}_1$ is small enough, then the diagram commutes:

$$\begin{tikzcd}
GL_2(\mathbb{Q}_p)\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{x\mathcal{U}_1x^{-1}} \arrow[r, "\cong"] \arrow[d, "(z,y)\mapsto(z,\mu(x)^{-1}y)"] & S_{x\mathcal{U}_1x^{-1}} \arrow[d, "x"] \\
GL_2(\mathbb{Q}_p)\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{\mathcal{U}_1} \arrow[r, "\cong"] & S_{\mathcal{U}_1}
\end{tikzcd}$$

The right vertical map is, on complex points, right multiplication by x. By the argument in [BC91, III.5.3.2] the same diagram commutes if we replace $\mathcal{U}_1$ by $\mathcal{U}$. Recall from §3.1.0.1 that,

$$\mu(x)^{-1} = \sigma(x)^{-1}\,\mathrm{Nr}(\sigma(x)) = x^{-1}\,\mathrm{Nr}(x) \qquad \forall x \in D_f^{*,p}$$

The idele $\mu(x) \in B_f^{*,p}$ is identified in B_f^* as an idele which is 1 at the p^{th} place.

Lemma 4.1.5. *Let $x \in D_f^{*,p}$ and recall that $\mu(\mathcal{U}^p) = U^p$. The double coset $U^p\mu(x)^{-1}B^*$ contains an element of the form $j(\gamma^{-1})$ for some $\gamma \in B_p^*$ with $\det\gamma$ a p-adic unit* [1]. *Fix such γ, and let Γ'_x be the group $\mu(x)U\mu(x)^{-1} \cap B^*$ be considered as a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ via $\phi_p : B_p^* \to \mathrm{GL}_2(\mathbb{Q}_p)$. The diagram commutes*

$$\begin{tikzcd}
GL_2(\mathbb{Q}_p)\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{\mu(x)U\mu(x)^{-1}} \arrow[r, "(y,z)\mapsto(y,\mu(x)^{-1}z)=(\gamma y,z)"] & GL_2(\mathbb{Q}_p)\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_U \\
\Gamma'_x\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \arrow[u, "\cong"] \arrow[r, "\gamma"] & \Gamma'\backslash \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \arrow[u, "\cong"'] \\
\widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \arrow[u] \arrow[r, "\gamma"] & \widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \arrow[u]
\end{tikzcd}$$

The top vertical arrows are induced by the natural inclusions of $\widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times [1]$ into $\widehat{\Omega} \hat{\otimes} \widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}} \times Z_{\mu(x)U\mu(x)^{-1}}$

[1]Recall that $j : B_p^* \to B_f^*$ identifies B_p^* with the ideles that are one away from p.

and $\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}\times Z_U$. *The bottom vertical arrows are also natural.*

Proof. By strong approximation and Lemma 2.3.1, the double coset $U^p\mu(x)^{-1}B^*$ contains an element $j(\gamma)^{-1}$ as above. The top square clearly commutes if it exists. To see this, it remains to check that the stated map is an isomorphism. This is clear for the corresponding functor of points. More specifically, first observe

Lemma 4.1.6. *Fix a group* G *and a subgroup* Γ'. *Let* X *be a set with a left* G *action,* $S\subset X$ *a subset, and*

$$\Gamma' = \{\,\gamma\in G \mid \gamma S\cap S\neq\emptyset\,\}$$

For each $x\in X$ *suppose that there is* $\gamma_x\in G$ *such that* $\gamma_x\cdot x\in S$. *Then, the inclusion* $S\to X$ *descends to a bijection* $\Gamma'\backslash S\xrightarrow{\phi}G\backslash X$.

Proof. The defining property of Γ' implies that ϕ is injective: if $s,s'\in S$ and $\phi(s)=\phi(s')$ then $\exists\gamma\in\Gamma'$ such that $\gamma s=s'$. Hence, $s=s'$ as elements of $\Gamma'\backslash S$. The map is surjective by definition. ∎

Let $Y/\mathbb{Z}_p$ be a formal scheme and ϕ the top right vertical arrow and $g\in\mathrm{GL}_2(\mathbb{Q}_p)$. The induced map on Y-valued points is

$$\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}\times Z_U \xrightarrow{(x,y,z)\mapsto(gx,\widetilde{\mathrm{Fr}}^{-ord(\det g)}y,gz)} \widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}\times Z_U$$

In particular, $g\cdot(\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}\times[1](Y))\cap\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_{\mathrm{p}}^{\mathrm{unr}}}\times[1](Y)\neq\emptyset$ iff $g\in\Gamma'$. The induced action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on the functor of points is also a left action, and the map on Y-valued points is an isomorphism by the lemma above. Hence ϕ is an isomorphism.
The second square also commutes trivially. ∎

Lemma 4.1.7. *For each index* i, *let* $y_i\in B_f^*$ *an idele such that* $(y_i)_\ell=1$ *for* $\ell\neq q$, $\mathrm{Nr}(y_i)=j_q(q)$, *and*

$$Uj_q^B(\alpha_i)=Uy_i$$

Let $\omega' \in H^0(S_{\mathcal{U}} \times \mathbb{Q}_p, \Omega^{\otimes k/2})$, $x_i = y_i^{-1}$, and $\gamma_i \in B_p^*$ be defined by the property that $U^p j(\gamma_i^{-1}) B^* = U^p \mu(x_i)^{-1} B^* = U^p y_i j_q(1/q) B^*$. Then,

$$T_q \omega' = q^{k/2-1} \sum_i \alpha_i^*(\omega') = q^{k/2-1} \sum_i x_i^*(\omega') = q^{k/2-1} \sum_i \gamma_i^*(\omega')$$

Proof. If $f \in S_k(\Gamma_{\mathbb{C}})$ and $\omega_f \in \mathrm{H}^0(S_{\mathcal{U}}, \Omega^{\otimes k/2})$ is identified with $f(z)dz^{\otimes k/2}$ as in §4.1 then

$$\alpha_i^*(\omega_f) = f(\alpha_i z) d(\alpha_i z)^{\otimes k/2} = f(\alpha_i z) \det \alpha_i^{k/2} j(\alpha_i, z)^{-k} dz^{k/2} = f|_{\alpha_i} dz^{\otimes k/2} = \omega_{f|_{\alpha_i}}$$

This is the first equality. Since $\alpha_i x_i \in \mathcal{U}$, the maps

$$x_i, \alpha_i : S_{x_i \mathcal{U} x_i^{-1}} \to S_{\mathcal{U}}$$

are equal by Lemma4.1.4, this gives the second equality. The last equality holds by Lemma 4.1.5 ■

Recall that pullback induces a right action on the space of global differential forms, and the left action by an element $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ is defined by the equality

$$\gamma^*(\omega') = \gamma^{-1} \cdot \omega' \qquad \omega' \in \mathrm{H}^0(S_{\mathcal{U}} \times \mathbb{Q}_p, \Omega^{\otimes k/2})$$

Since

$$\mathrm{H}^0(\widehat{\Omega}, \omega^{k/2}) \to \mathcal{O}(k) \xrightarrow{I} \mathrm{C}_{\mathrm{har}}(k)$$

is equivariant for the *left* $\mathrm{GL}_2(\mathbb{Q}_p)$ action, the induced action on the space of harmonic functions is

$$T_q : \mathrm{C}_{\mathrm{har}}(k) \to \mathrm{C}_{\mathrm{har}}(k) \qquad f(e) \mapsto q^{k/2-1} \sum_i \gamma_i^{-1} f(\gamma_i e)$$

Corollary 4.1.8. *Recall that if $f \in \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$ the harmonic form associated to it by*

Theorem 3.3.2 is

$$\mathrm{JL}_f(z) = (-1)^{sign\, z} \tilde{D}(z) \cdot \phi \circ f \circ j \circ \tilde{D}(z) \cdot |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2}$$

Let $y_i = j_q^B(\xi_i)$*. We can take* $\gamma_i = \xi_i/q$*, which has determinant* $1/q$*. The* q^{th} *Hecke operator acts on* $\mathrm{JL}_f(z)$ *as*

$$T_q \,\mathrm{JL}_f(z) = q^{k/2-1} \sum_i \gamma_i^{-1}\, \mathrm{JL}_f(\gamma_i z) = \sum_i (-1)^{sign\, z} \tilde{D}(z) \cdot \phi \circ f \circ j \circ \tilde{D}(\gamma_i z) \cdot |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2}$$

Proof. The elements γ_i can be chosen as above because

$$U^p j_q(\xi_i/q) B^* = U^p (q\xi_i^{-1})^{(q)} (\xi_i/q)^{(q)} j_q(\xi_i/q) B^* = U^p j(q\xi_i^{-1}) B^*$$

Since $\det \gamma_i$ is a p-adic unit it commutes with $\tilde{D}$ and sign$(\gamma_i z) =$ sign(z) for all $z \in B_p^*$. ■

4.1.0.3 Definite quaternion algebras

The q^{th} Hecke operator of $\mathcal{L}_k(U, \mathbb{Q}_p)$ is $T_q^{def} = [U\eta_q U]$. Recall that

$$f|_g(z) = g_p f(zg) \qquad \forall g \in B_f^*, f \in \mathcal{L}_k(U, \mathbb{Q}_p)$$

If $U\eta_q U = \coprod_i U y_i$ is a disjoint decomposition, then

$$T_q^{def} f = \sum_i f|_{y_i}$$

Let y_i be as in Lemma 4.1.3. There exists $u_i \in U_q$ such that $y_i j_q(u_i) = j_q(\bar{\xi}_i)$. Hence,

$$T_q^{def} f(z) = \sum_i f|_{y_i}(z) = \sum_i f(zy_i) = \sum_i f(zy_i j_q(u_i)) = \sum_i f(zj_q(\bar{\xi}_i))$$

Now, let $f \in \mathcal{L}_k(U, \mathbb{Q}_p)^{p-new}$. Observe that

$$\mathrm{JL}_{T_q^{def} f}(z) = \sum_i (-1)^{\operatorname{sign} z} \tilde{D}(z) \cdot \phi \circ f(j(\tilde{D}(z)) j_q(\bar{\xi}_i)) \cdot |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2}$$

Since

$$f(j(\tilde{D}(z)) j_q(\bar{\xi}_i)) = f(j_q(\bar{\xi}_i)(\bar{\xi}_i)^{(q)}(\bar{\xi}_i^{-1})^{(q)} j(\tilde{D}(z)))$$

Since $\bar{\xi}_i^{-1} = \xi/q$, the above expression is $f(j(\tilde{D}(\xi_i/q \cdot z)))$. Finally, we get

$$\mathrm{JL}_{T_q^{def} f}(z) = \sum_i (-1)^{\operatorname{sign} z} \tilde{D}(z) \cdot \phi \circ f \circ j \circ \tilde{D}(\xi_i/q \cdot z) \cdot |\mathrm{Nr}\, z|^{n/2} \det(z)^{n/2}$$

This proves the following

Theorem 4.1.9. *For each $q \nmid N$, the following diagram commutes*

$$\begin{tikzcd}
\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \arrow[r, "\mathrm{JL}"] \arrow[d, "T_q^{def}"'] & \mathrm{C}_{\mathrm{har}}(k) \arrow[d, "T_q"] \\
\mathcal{L}_k(U, \mathbb{Q}_p)^{p-new} \arrow[r, "\mathrm{JL}"] & \mathrm{C}_{\mathrm{har}}(k)
\end{tikzcd}$$

Remark 4.1.1. Notice how we defined T_q. It is an endomorphism of $\mathrm{H}^0(S_{\mathcal{U}}, \Omega^{\otimes k/2})$) which agrees with the classical Hecke operator. That endomorphism naturally induces an endomorphism of $\mathrm{C}_{\mathrm{har}}(k)$, which contains the target of our Explicit Jacquet-Langlands isomorphism from Theorem 3.3.2. The compatibility above in particular shows that the Hecke operators preserve $\mathrm{C}_{\mathrm{har}}(k)^{\Gamma}$, which is identified with global sections of an explicit twisted form of $S_{\mathcal{U}}$ defined in §3.1.0.3.

4.2 $\mathbb{Z}_p$-integral structure

E. de Shalit proved [dS89] that Drinfeld and Schneider's map [Sch84] is an isomorphism, and Teitelbaum [Tei93] gave an integral refinement. This is what we need to identify residues of integrally normalized modular forms.

Recall that we do not get p-adic uniformization of the Shimura curve $S/\mathbb{Z}_p$ from [Čer76], [Dri76], and [BC91]. Instead, we get p-adic uniformization of the explicit twisted form S' of S. The corresponding cocycle is defined by the Atkin-Lehner involution at p, and splits over an unramified quadratic extension [BC91, III.5.3.3] [JL85, Thm 4.3'].

Here, we show that the $\mathbb{Z}_p$ integral structure defined by the relative dualizing sheaf of $S'/\mathbb{Z}_p$ corresponds to a lattice

$$\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p} \subset \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$$

defined below.

4.2.1 Representations

In this section we recall various basic facts about the $\mathbb{Q}_p$ representations above.

Recall that for each n, r, $\mathbf{P}_{\mathrm{n}}^{\mathrm{r}}(\mathbb{Q}_{\mathrm{p}})$ and $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ are both the space of homogenous polynomials in x, y of degree n, as vector spaces. They are defined in §3.2.1 and Definition 2.1.1 respectively. Although equal as vector spaces, their left $\mathrm{GL}_2(\mathbb{Q}_p)$ actions are in general different and made explicit above. The total degree of our polynomials will remain fixed in a given context so we abuse notation to let P_i also denote the monomial $x^i y^{n-i}$. Consider the symmetric and non-degenerate bilinear form on this space of polynomials

$$\langle P_i, P_j\rangle = (-1)^i i!(n-i)!\delta_{i,n-j}$$

If $r = k-2$ then this inner product agrees with the inner product given in Definition 2.1.1 under the isomorphism ϕ from Lemma 3.2.1.

For the remainder of this section, the action of $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ is taken in $\mathbf{P}_{\mathrm{n}}^{\mathrm{r}}(\mathbb{Q}_{\mathrm{p}})$ and λ is the determinant γ (considered as an endomorphism of $\mathbb{Q}_p^2$). We also fix n, r with n even.

Lemma 4.2.1. *Let $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $P, Q \in \mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$. Then,*

$$\langle \gamma P, \gamma Q \rangle = \lambda^{2r-n} \langle P, Q \rangle$$

By Lemma 3.2.1, $P_{k-2}(\mathbb{Q}_p) \cong P^{k-2}_{k-2}(\mathbb{Q}_p)$, hence in this setting the relation above becomes

$$\langle \gamma P, \gamma Q \rangle = \lambda^{k-2} \langle P, Q \rangle$$

Proof. Since the given bilinear form is $\mathrm{SL}_2(\mathbb{Q}_p)$ bilinear, we only need to check the determinant relation. Write $\gamma = \gamma' \left(\begin{smallmatrix} \lambda & \\ & 1 \end{smallmatrix} \right)$, and fix integers $0 \le i, j, \le n$. Since $\gamma' \in \mathrm{SL}_2(\mathbb{Q}_p)$,

$$\langle \gamma P_i, \gamma P_j \rangle = \langle \gamma' \left(\begin{smallmatrix} \lambda & \\ & 1 \end{smallmatrix} \right) P_i, \gamma' \left(\begin{smallmatrix} \lambda & \\ & 1 \end{smallmatrix} \right) P_j \rangle = \langle \left(\begin{smallmatrix} \lambda & \\ & 1 \end{smallmatrix} \right) P_i, \left(\begin{smallmatrix} \lambda & \\ & 1 \end{smallmatrix} \right) P_j \rangle = \langle P_i(\frac{1}{\lambda} \left(\begin{smallmatrix} 1 & \\ & \lambda \end{smallmatrix} \right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right))\lambda^r, P_j(\frac{1}{\lambda} \left(\begin{smallmatrix} 1 & \\ & \lambda \end{smallmatrix} \right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right))\lambda^r \rangle$$

$$= \lambda^{2r} \langle P_i \left(\begin{smallmatrix} x/\lambda \\ y \end{smallmatrix} \right), P_j \left(\begin{smallmatrix} x/\lambda \\ y \end{smallmatrix} \right) \rangle = \lambda^{2r-(i+j)} \langle P_i, P_j \rangle$$

$$= \lambda^{2r-(i+j)} (-1)^i i!(n-i)! \delta_{i,n-j} = \lambda^{2r-n} (-1)^i i!(n-i)! \delta_{i,n-j} = \lambda^{2r-n} \langle P_i, P_j \rangle$$

Which was the desired equality. Note that this implies Lemma 2.3.7. ■

The left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathrm{Hom}(\mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p)$ is given by $(\gamma f)(P) = f(\gamma^{-1} P)$

Proposition 4.2.2. *Let*

$$\tau : \mathrm{Hom}(\mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p) \to \mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}) \qquad f \mapsto \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i y^{n-i} f(x^{n-i} y^i)$$

This is an isomorphism of $\mathbb{Q}_p$ vector spaces with

$$\tau(\gamma \cdot f) = \gamma \cdot \tau(f) \cdot \lambda^{n-2r}$$

In particular, $\tau : \mathrm{Hom}(\mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p) \xrightarrow{\tau} \mathbf{P}^{3r-n}_{n}$ is an isomorphism of $\mathrm{GL}_2(\mathbb{Q}_p)$ representations, and if $r = n/2$ then $\mathbf{P}^{\mathrm{r}}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ is self-dual.

We need a few formulas to check the statement regarding the γ action.

Lemma 4.2.3. *For each* $0 \leq t \leq n, \gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ *let*

$$\gamma P_t = \sum_{j=0}^{n} a_{t,j}^{\gamma} P_j \qquad \textit{and} \qquad \mu_t = \frac{(-1)^t}{t!(n-t)!}$$

Then,

$$a_{t,s}^{\gamma} = \frac{\mu_s}{\mu_t} \lambda^{2r-n} a_{n-s,n-t}^{\gamma^{-1}}$$

Note that $\mu_t = \mu_{n-t}$.

Proof. Observe that

$$\langle \gamma P_t, P_{n-s} \rangle = \sum_{j=0}^{n} a_{t,j}^{\gamma} (-1)^j j!(n-j)! \delta_{j,s} = a_{t,s}^{\gamma} (-1)^s s!(n-s)!$$

Hence

$$a_{t,s}^{\gamma} = \frac{\langle \gamma P_t, P_{n-s} \rangle (-1)^s}{s!(n-s)!} = \langle \gamma P_t, P_{n-s} \rangle \mu_s$$

Further,

$$\langle \gamma P_t, P_{n-s} \rangle = \langle \gamma P_t, \gamma \gamma^{-1} P_{n-s} \rangle = \lambda^{2r-n} \langle P_t, \gamma^{-1} P_{n-s} \rangle = \lambda^{2r-n} \langle \gamma^{-1} P_{n-s}, P_t \rangle$$

By the formula above,

$$a_{n-s,n-t}^{\gamma^{-1}} \langle \gamma^{-1} P_{n-s}, P_t \rangle \mu_{n-t} = \langle \gamma^{-1} P_{n-s}, P_t \rangle \mu_t$$

and

$$\langle \gamma P_t, P_{n-s} \rangle = \lambda^{2r-n} \frac{a_{n-s,n-t}^{\gamma^{-1}}}{\mu_t}$$

This implies the formula above. ∎

Lemma 4.2.4. *For later reference, we record that*

$$\lambda^{n-2r} t!(n-t)!(-1)^t \delta_{t,n-s} = \langle \gamma^{-1} P_t, \gamma^{-1} P_s \rangle = \sum_{i=0}^{n} (-1)^i i!(n-i)! a_{t,i}^{\gamma^{-1}} a_{s,n-i}^{\gamma^{-1}}$$

Replacing t, s *with* $n-t$, $n-s$ *this is the equality*

$$\sum_{i=0}^{n} (-1)^i i!(n-i)! a_{n-t,i}^{\gamma^{-1}} a_{n-s,n-i}^{\gamma^{-1}} = \langle \gamma^{-1} P_{n-t}, \gamma^{-1} P_{n-s} \rangle = \lambda^{n-2r} t!(n-t)!(-1)^t \delta_{n-t,s}$$

Proof. Clear ■

Proof. (of Proposition 4.2.2)

Let $f_t \in \mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{r}}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p)$ be the linear function with $f_t(P_j) = \delta_{t,j}$. Note that

$$\gamma \cdot \tau(\gamma^{-1} f_t) = \gamma \sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i a_{n-i,t}^{\gamma}$$

and the coefficient of P_s in this sum is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_{n-i,t}^{\gamma} a_{i,s}^{\gamma}$$

Since

$$a_{n-i,t}^{\gamma} = \frac{\mu_t}{\mu_{n-i}} \lambda^{2r-n} a_{n-t,i}^{\gamma^{-1}} \qquad a_{i,s}^{\gamma} = \frac{\mu_s}{\mu_i} \lambda^{2r-n} a_{n-s,n-i}^{\gamma^{-1}}$$

The coefficient is

$$\lambda^{4r-2n} \mu_t \mu_s \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{\mu_i^2} a_{n-t,i}^{\gamma^{-1}} a_{n-s,n-i}^{\gamma^{-1}} = \lambda^{4r-2k} \mu_t \mu_s n! \sum_{i=0}^{n} (-1)^i i!(n-i)! a_{n-t,i}^{\gamma^{-1}} a_{n-s,n-i}^{\gamma^{-1}}$$

Combining the formulas above, the coefficient of P_s in the sum $\gamma\tau(\gamma^{-1} f_t)$ is $\lambda^{2r-n} \mu_s n! \delta_{t,n-s} = \lambda^{2r-n} \mu_s n! \delta_{n-t,s}$.

Finally, we calculate the coefficient of P_s in $\tau(f_t)$

$$\tau(f_t) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i f_t(P_{n-i}) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i \delta_{t,n-i}$$

Hence, the coefficient of P_s in $\tau(f_t)$ is

$$\frac{(-1)^s n!}{s!(n-s)!} \delta_{t,n-s} = n! \mu_s \delta_{t,n-s}$$

Comparing the coefficients of P_s in $\gamma\tau(\gamma^{-1} f_t)$ and $\tau(f_t)$ we have the desired formula. ■

4.2.2 Dualizing sheaf

In this section we calculate the integral structure on $\mathrm{C_{har}(k)}^{\Gamma}$ induced by the dualizing sheaf over $\mathbb{Z}_p$. Recall that the total degree of our polynomials is fixed at n, so we continue to abuse as above by letting P_i denote the monomial $x^i y^{n-i} \in \mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}})$.

If $f(z)$ is a rigid analytic function on the p-adic upper half plane Ω, then c_f and $I(f)$ are functions on $\overrightarrow{\mathcal{E}}(\mathcal{T})$ valued in $\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}})(\mathbb{Q}_p), \mathbb{Q}_p)$ and $\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}})(\mathbb{Q}_p)$ respectively. They are

$$c_f(e)(P_i) = \mathrm{Res}_e\, z^i f(z) dz$$

$$I(f)(e) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} P_i \,\mathrm{Res}_e\, z^{n-i} f(z) dz$$

Teitelbaum [Tei93] calculates the integral structure on harmonic forms via c_f. Proposition 4.2.2 gurantees that the diagram below commutes, and that the diagonal arrow τ^* is an isomorphism of left $\mathrm{GL}_2(\mathbb{Q}_p)$ modules.

$$\begin{tikzcd}
\Gamma(\Omega, \omega^{k/2}) \arrow[r, "f \mapsto c_f"] \arrow[d, "f \mapsto I(f)"'] & \mathrm{C_{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p)) \arrow[dl, "f \mapsto \tau \circ f"] \\
\mathrm{C_{har}}(\mathrm{k}) &
\end{tikzcd}$$

4.2.2.1 Integral harmonic forms

Remark 4.2.1 (Comparison of notation). Unexplained notation in this remark is from [Tei93]. Teitelbaum identifies elements of B_p^* with vertices of the Bruhat-Tits tree diferently from us. Namely, he uses right cosets $\mathbb{Q}_p^* \mathcal{R}_p^* \backslash B_p^*$ while we use left coses. However,

$$B_p^*/\mathbb{Q}_p^* \mathcal{R}_p^* \xrightarrow{z \mapsto z^{-1}} \mathbb{Q}_p^* \mathcal{R}_p^* \backslash B_p^*$$

is an isomorphism of left $\mathrm{GL}_2(\mathbb{Q}_p)$ modules and using this, we translate his space of integral harmonic functions into the space $\mathrm{Z_{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_{\mathrm{p}}))$ below. The important points are that, in Teitelbaum's notation,

$$L_{\gamma v}(n) = \gamma L_v(n) \quad \forall \gamma \in B_p^*, v \in \mathcal{V}(\mathcal{T})$$
$$L_{\gamma e}(n) = \gamma L_e(N) \quad \forall e \in \overrightarrow{\mathcal{E}}(\mathcal{T})$$
$$L_{v(\gamma^{-1})}(n) = \gamma L(n)$$

Here, recall that v is the function associating an element $\gamma \in B_p^*$ with the associated coset $\mathbb{Q}_p^* \mathcal{R}_p^* \backslash B_p^*$. Finally, the edge $\epsilon = b(1) = [c(1), c(\beta)]$ that we define in Lemma 2.2.5, corresponds to the lattice

$$L(1-k) \cap \beta L(1-k)$$

(see [Tei93, Def. 6]). It is clear that $L(1-k) = \mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}})$. See §3.2.1.

Recall that the functions b and c from Lemma 2.2.5 associate elements of B_p^* with vertices and edges of the Bruhat-Tits tree and that $\beta = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ (2.2.1). Let $\epsilon = b(1)$ and

$$L_\epsilon = L_{b(1)} \cap L_{b(\beta)} \qquad L_{b(1)} = \mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_p) \qquad L_{b(\beta)} = L_{\beta b(1)} = \beta L_{b(1)}$$

The action of β on $\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_p)$ and $\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}})$ is recorded by the formulas

$$\beta(f_t) = p^{-n/2+t} f_t \qquad \beta P_t = p^{n/2-t} P_{n-t} \qquad \beta P_{n-t} = p^{t-n/2} P_t$$

Here, for $0 \leq t \leq n$, $f_t \in \mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_p)$ is defined by $f_t(P_i) = \delta_{i,t}$. Explicitly,

$$L_\epsilon = L_{b(1)} \cap \beta L_{b(1)} = \{p^{n/2-t} f_t |\, 0 \leq n/2 - t\} + \{f_t |\, n/2 - t < 0\}$$

The image of L_ϵ under τ is

$$\begin{aligned}
\tau(L_\epsilon) &= \mathbb{Z}_p \cdot \left\{ \binom{n}{t} p^{n/2-t} P_{n-t} \;\middle|\; 0 \leq n/2 - t \right\} + \mathbb{Z}_p \cdot \left\{ \binom{n}{t} P_{n-t} \;\middle|\; n/2 - t < 0 \right\} \\
&= \mathbb{Z}_p \cdot \left\{ \binom{n}{t} p^{t-n/2} P_t \;\middle|\; 0 \leq t - n/2 \right\} + \mathbb{Z}_p \cdot \left\{ \binom{n}{t} P_t \;\middle|\; t - n/2 < 0 \right\} \\
&\mathbb{Z}_p \cdot \left\{ \binom{n}{t} P_{n-t} p^{n/2-t} \;\middle|\; t \leq n/2 \right\} + \mathbb{Z}_p \cdot \left\{ \binom{n}{t} P_{n-t} \;\middle|\; t > n/2 \right\}
\end{aligned}$$

One checks directly that $\beta \cdot \tau(L_\epsilon) = \tau L_\epsilon$. If $p > n$, then the combinatorical expressions above are p-adic units. This lattice is also stable under the action of R_p^*. To see this, let $\mu \in R_p^*$. Then, by Lemma 2.2.1 $\mu\beta = \beta\mu'$ for some $\mu' \in R_p^*$. Hence, $\mu\tau(L_\epsilon) = \tau(\mu L_\epsilon) = \tau(\mu L_{b(1)} \cap \mu\beta L_{b(1)}) = \tau(\mu L_{b(1)} \cap \beta\mu' L_{b(1)}) = \tau L_\epsilon$.

Define

$$\mathrm{Z}_{\mathrm{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_{\mathrm{p}})) = \left\{ f \in \mathrm{C}_{\mathrm{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p)) \;\middle|\; f(z\epsilon) \in z L_\epsilon \, \forall z \in B_p^* \right\}$$

Recall from §3.2.2 that Teitelbaum defines an invertible sheaf ω on the formal p-adic upper half plane $\widehat{\Omega}$. It's global sections are identified with rigid analytic functions, and the Teitelbaum proves [Tei93, Thm. 17] that the restriction Res^0 is an isomorphism.

$$\begin{tikzcd}
\mathcal{O}(k) \arrow[rr, "f\mapsto c_f"] & & \mathrm{C_{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}}), \mathbb{Q}_p)) \\
\mathrm{H}^0(\widehat{\Omega}, \omega^{k/2}) \arrow[u] \arrow[r, "\mathrm{Res}^0"] & \mathrm{Z_{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_{\mathrm{p}})) \arrow[r] & \mathrm{C_{har}}(\mathrm{Hom}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Z}_{\mathrm{p}}), \mathbb{Z}_p)) \arrow[u]
\end{tikzcd}$$

Unlabeled arrows are the natural inclusions. See the proof of [Tei93, Lem. 16] for more information regarding the image of the left vertical map. Let τ be as in Proposition 4.2.2 and define

$$\mathrm{Z_{har}}(\mathrm{k}) = \left\{ f \in \mathrm{C_{har}}(\mathrm{k}) \;\middle|\; z^{-1}f(z\epsilon) \in \tau(L_\epsilon)\, \forall z \in B_p^* \right\}$$

Since $f \mapsto c_f$ is also Γ equivariant by Lemma 3.2.3 and §4.2.2, it descends to an isomorphism on Γ invariants. In terms of $\mathrm{Z_{har}}(\mathrm{k})^\Gamma$, Teitelbaum's result is then that the map

$$\mathrm{H}^0(\widehat{\Omega}, \omega^{k/2})^\Gamma \xrightarrow{\mathrm{Res}^0} \mathrm{Z_{har}}(\mathrm{k})^\Gamma$$

is an isomorphism of $\mathbb{Z}_p$ modules.

4.2.3 Definite quaternion algebras

In this section, we define a lattice $\mathcal{L}_k(U, \mathbb{Z}_p)^{p-new, W_p}$ and show that under Theorem 3.3.2 it corresponds to the integral structure defined above (if $p > n$). First, we characterize $\mathcal{L}_k(U, \mathbb{Z}_p)^{p-new, W_p}$ as the space of p-new forms with values in a lattice.

Definition 4.2.1. Let

$$\mathcal{L}_k(U, \mathbb{Z}_p)^{p-new, W_p} = \mathcal{L}_k(U, \mathbb{Z}_p)^{p-new} \cap W_p \mathcal{L}_k(U, \mathbb{Z}_p)^{p-new}$$

This is a finite index submodule of $\mathcal{L}_k(U, \mathbb{Z}_p)^{p-new}$ since W_p is injective and $p^M \cdot (\mathcal{L}_k(U, \mathbb{Z}_p)^{p-new} + W_p \mathcal{L}_k(U, \mathbb{Z}_p)^{p-new}) \subset W_p \mathcal{L}_k(U, \mathbb{Z}_p)^{p-new})$ for $M >> 0$. It is clear that $W_p \mathcal{L}_k(U, \mathbb{Z}_p)^{p-new, W_p} =$

$\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}$.

Lemma 4.2.5.

$$\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p} = \left\{ f \in \mathcal{L}_k(U,\mathbb{Q}_p)^{p-new} \;\middle|\; f(z) \in L \,\forall z \in B_f^* \right\}$$
$$= \left\{ f : B_f^* \to L \;\middle|\; \begin{matrix} {\scriptstyle f(zu)=u_p^{-1} f(z)\,\forall z \in B_f^*, u \in U} \\ f(bz) = f(z)\,\forall b \in B^* \;\; \langle f, g\rangle = 0 \,\forall\, g \in \mathcal{L}_k(U,\mathbb{Q}_p)^{p\text{-}old} \end{matrix} \right\}$$

where $L = \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}}) \cap \beta\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2}$.

Proof. First, let $M \subset \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})$ be the $\mathbb{Z}_p$ module generated by $\{f(z)|\, f \in \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}\,,\, z \in B_f^*\}$. If $\sum_i \lambda_i f_i(z_i) \in M$ then

$$\beta \left(\sum_i \lambda_i f_i(z_i) \right) p^{-n/2} = \sum_i \beta f_i(z_i j(\beta^{-1}) j(\beta)) p^{-n/2}) = \sum_i W_p f_i(z_i j(\beta^{-1})) \in M$$

Hence $\beta M p^{-n/2} \subset M$. Since $M \subset \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})$, and $\beta^2 M p^{-n} = M$, $\beta M p^{-n/2} \subset M \subset \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})$ implies $M \subset \beta\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2}$. Hence $M \subset L$.

If $f \in \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}$ then since $M \subset L$, f is an element of RHS. Conversely, if $f \in$ RHS, then clearly $f \in \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$, $W_p f \in \mathcal{L}_k(U,\mathbb{Q}_p)^{p-new}$, and for $z \in B_f^*$, $W_p f(z) = \beta f(z j(\beta)) p^{-n/2} \in \beta L p^{-n/2} \subset \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})$. Hence, $W_p f \in \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$ and $f \in W_p \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$. ∎

Lemma 4.2.6. *Let $M \subset \mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ a $\mathbb{Z}_p$ submodule stable under U_p, and*

$$\mathcal{M} = \left\{ f \in \mathcal{L}_k(U,\mathbb{Q}_p)^{p-new} | f(z) \in M \,\forall z \in B_f^* \right\}$$

Then, the image of $\mathcal{M}$ under the map JL *from Theorem 3.3.2 is*

$$\mathrm{C}_{\mathrm{har}}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}}))^{\Gamma}_{\mathcal{M}} = \left\{ \psi \in \mathrm{C}_{\mathrm{har}}(\mathbf{P}_{\mathrm{n}}^{\mathrm{n}/2}(\mathbb{Q}_{\mathrm{p}}))^{\Gamma} \;\middle|\; z^{-1}\psi(z) \in \phi(M)\,\forall z \in B_p^* \right\}$$

Here, ϕ is the isomorphism by Lemma 3.2.1.

Proof. Recall that the harmonic functions are determined by their values on norm one elements by Lemma 2.4.5. If $z \in B_p^{*,1}$ and $f \in \mathcal{M}$ then

$$\mathrm{JL}_f(z) = z \cdot \phi \circ f \circ j(z)$$

Conversely, if $\mathrm{JL}_f(z) \in z\phi(M)$ for all $z \in B_p^{*,1}$ then $f \circ j(z) \in M$ for all $z \in B_p^{*,1}$. Since M is stable under U_p, $f \in \mathcal{M}$ by strong approximation (2.1). ■

Consider the monomials $P_i = x^i y^{n-i}$ as elements of the space $\mathbf{P}_{\mathrm{n}}(\mathbb{Q}_{\mathrm{p}})$ of homogenous polynomials of degree n. Then,

$$\beta P_i p^{-n/2} = P_{n-i} p^{i-n/2} \qquad \beta P_{n-i} p^{-n/2} = P_i p^{n/2-i} \qquad 0 \leq i \leq n$$

$$L = \mathbb{Z}_p \cdot \left\{ P_i p^{n/2-i} \;\middle|\; i \leq n/2 \right\} + \mathbb{Z}_p \cdot \{ P_i \mid i > n/2 \}$$

Recall from Lemma 3.2.1 that $\phi(P_i) = P_i(y, -x) = (-1)^i P_{n-i}$.

$$\phi(L) = \mathbb{Z}_p \cdot \left\{ P_{n-i} p^{n/2-i} \;\middle|\; i \leq n/2 \right\} + \mathbb{Z}_p \cdot \{ P_{n-i} \mid i > n/2 \}$$

Finally, it is clear that L (or equivalently $\phi(L)$, where ϕ is as in Lemma 3.2.1) is stable under the action of U_p. To see this, let $\mu \in U_p$. Then, $\mu\beta = \beta\mu'$ for some $\mu' \in U_p$ by Lemma 2.2.1. Hence, $\mu L = \mu(\mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}}) \cap \beta\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2}) = \mu\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}}) \cap \mu\beta\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2} = \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}}) \cap \beta\mu'\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2} = \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}}) \cap \beta\, \mathbf{P}_{\mathrm{n}}(\mathbb{Z}_{\mathrm{p}})\, p^{-n/2}$.
It is clear that

Lemma 4.2.7. *For all primes p,*

$$\tau(L_\epsilon) \subseteq \phi(L)$$

is a finite index inclusion of $\mathbb{Z}_p$ lattices. If $p > n = k - 2$, then this inclusion is an equality.

Then, Lemma 4.2.6 implies

Theorem 4.2.8. *Let $\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p} \subset \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new}$ the lattice in the space of p-new forms on B above (Definition 4.2.1), and $\mathrm{Z}_{\textit{har}}(\mathrm{k})^{\Gamma} \subset \mathrm{C}_{\textit{har}}(\mathrm{k})^{\Gamma}$ from §4.2.2.1 the space of integral harmonic forms. Then,*

$$\begin{array}{ccc} \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p} & \xrightarrow{\mathrm{JL}} & \mathrm{C}_{\textit{har}}(\mathrm{k})^{\Gamma} \\ & & \uparrow \\ & & \mathrm{Z}_{\textit{har}}(\mathrm{k})^{\Gamma} \end{array}$$

the explicit Jacquet-Langlands isomorphism induces a finite index inclusion of $\mathbb{Z}_p$ lattices

$$\mathrm{Z}_{\textit{har}}(\mathrm{k})^{\Gamma} \subseteq \mathrm{JL}\left(\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}\right)$$

If $p > k-2$ then this inclusion is an equality. In particular, if $k = 2$ then $\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}$ and $\mathrm{Z}_{\textit{har}}(\mathrm{k})^{\Gamma}$ are identified for all p.

Theorem 4.2.9. *If $p > 3$, then then by Lemma 3.2.2, $\mathrm{Z}_{\textit{har}}(\mathrm{k})^{\Gamma} \subset \mathrm{C}_{\textit{har}}(\mathrm{k})^{\Gamma}$ is identified with the lattice $H^0(S',\omega_{S'/\mathbb{Z}_p}^{\otimes k/2})$ of forms defined by the relative dualizing sheaf $\omega_{S'/\mathbb{Z}_p}$ of $S'/\mathbb{Z}_p$. In this case, the theorem above takes the form*

$$\begin{array}{ccc} \mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p} & \xrightarrow{\mathrm{JL}} & \mathrm{C}_{\textit{har}}(\mathrm{k})^{\Gamma} \\ & & \uparrow \\ & & H^0(S',\omega_{S'/\mathbb{Z}_p}^{\otimes k/2}) \end{array}$$

the explicit Jacquet-Langlands isomorphism induces a finite index inclusion of $\mathbb{Z}_p$ lattices

$$H^0(S',\omega_{S'/\mathbb{Z}_p}^{\otimes k/2}) \subseteq \mathrm{JL}\left(\mathcal{L}_k(U,\mathbb{Z}_p)^{p-new,W_p}\right)$$

If in addition $p > k-2$ then this inclusion is an equality.

Proof. This follows from Theorem 4.2.8. ∎

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