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An Electronic Journal of Geography and Mathematics

25 YEARS, AND MORE, OF PUBLICATION!

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Articles

The word clouds (formed in Tagxedo, online) serve as a visual "abstract" of the adjacent article!

Symmetry Groups

S_3

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(**with figures by Sandra L. Arlinghaus**)

Tyger, tyger, burning bright,
In the forest of the night;
What immortal hand or eye,
Dare frame thy fearful symmetry?

William Blake

People have always been interested in symmetry. The fact that human bodies have bilateral symmetry makes us think the face we see in the mirror is a true reflection. The fact that this symmetry is not perfect makes actors and actressess demand that they be photographed from their 'good' sides.

Mathematicians measure symmetry using group theory. The set of transformations which leaves a geometric figure unchanged forms an object called a 'group'.

For example, an equilateral triangle can be rotated through 120 or 240 degrees, rotated through an axis between a vertex and its opposite side, or just left alone without altering its appearance. Thus there are six symmetries of an equilateral triangle, which form a symmetric group called S_3.

Graph theory studies connections among objects. Thus, to a graph theorist, an equilateral triangle has the same connections, and hence the same symmetries, as any other triangle. Nonetheless, it is easier to notice the symmetries if the geometric representation of a graph exhibits some of the symmetry.

There are two related but different problems which are of interest by themselves.

1) Given a graph, what are its symmetries?
2) Given a group of symmetries, what graph has a group of symmetries isomorphic to that group? In particular, how small a graph exists with that group of symmetries?

This article concentrates on the second part of the second problem, in particular for the groups of rotations of an equilateral triangle and of a square (cyclic groups of order 3 and 4, respectively).

Frucht has shown that any group has a graph with that graph as its group of symmetries (1949), and Arlinghaus has investigated how small these graphs can be for finite abelian groups (1977; 1985). Much notation is used to describe these graphs. But this article concentrates on 'nice' pictures of the two groups mentioned above (denoted Z_3 and Z_4, respectively).

For Z_3, one might start with an equilateral triangle, but that is known to have six symmetries, including the three extra rotations noted above. Eventually, one discovers that a graph with nine vertices is the smallest possible (Figure 1). It has a picture in which the rotations are visible.

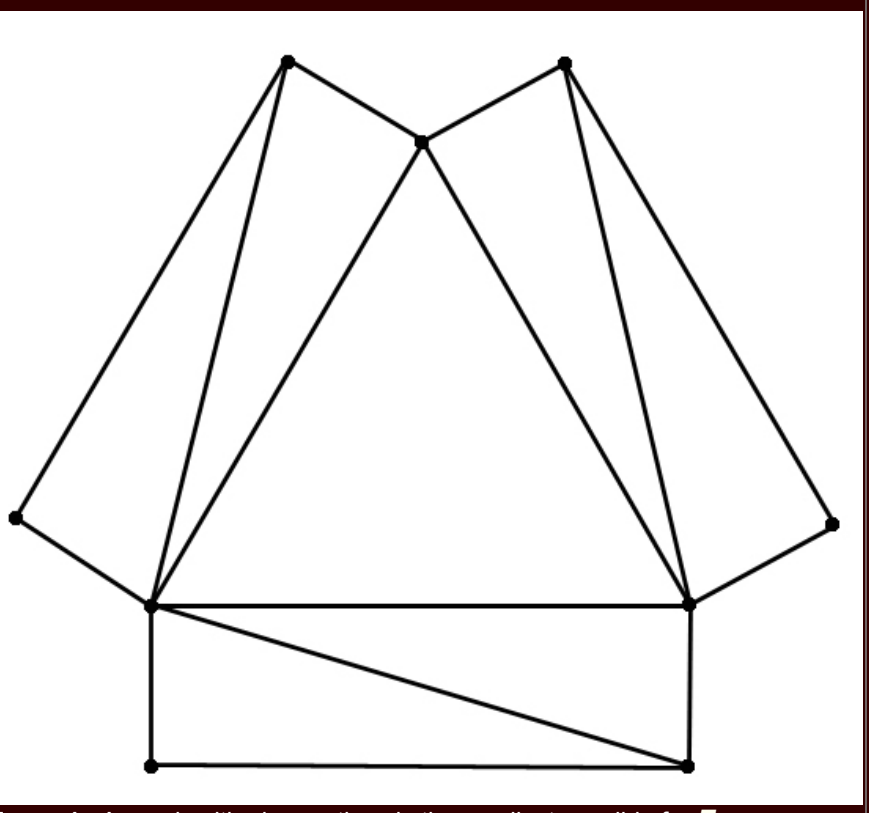

Figure 1. A graph with nine vertices is the smallest possible for Z_3.

For Z_4, again one starts with a square, but as before there are extra rotations. The smallest graph this time turns out to have 10 vertices. (A 12-vertex graph analogous to the 9-vertex graph for Z_3 does have group Z_4, but a smaller one exists.) Unfortunately, this graph is difficult to draw in the plane, and its symmetries are not easily visible if so drawn. Thus we exhibit this graph in other more pleasing settings. The graph has vertices 1, 2, 3, 4, 1', 2', 3', 4', 1", 2" and the symmetry group (isomorphic to Z_4) is $G = \{ (1), g, g^2, g^3\}$ where g = (1234)(1'2'3'4')(1"2") so g^2 = (13)(24)(1'3')(2'4')(1")(2") and g^3 = (1432)(1'4'3'2')(1"2"). Figure 2 shows an animation of one such arrangement; visual symmetry is not clear.

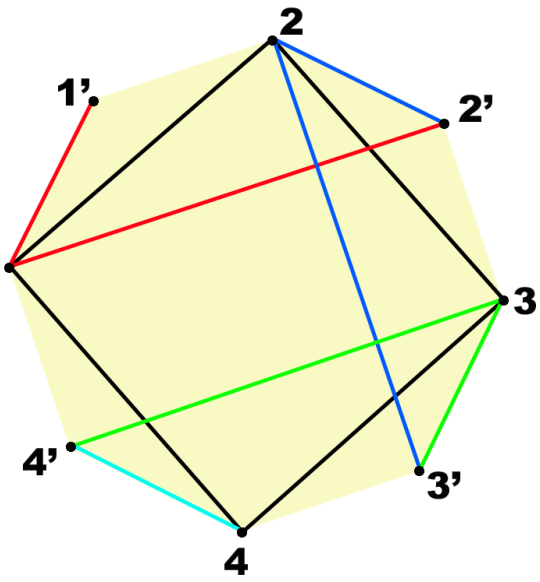

Figure 2. Natural minimal spatial arrangement for the cyclic group of order 4; notice the lack of visual symmetry portrayed in this correct representation of this symmetric group.

Figure 3 improves on Figure 2; it displays visual symmetry.

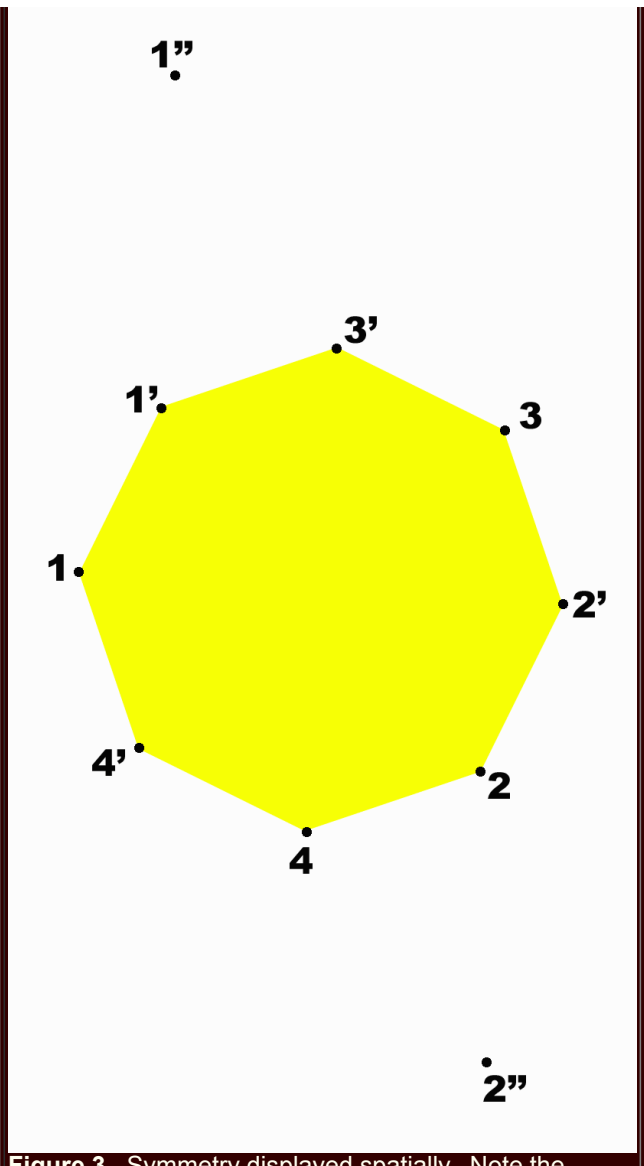

Figure 3. Symmetry displayed spatially. Note the position of 1, 2, 3, 4--it is similar in spatial style to the right-hand side of Figure 1.

Figure 4 shows the final frame of the animation, as a static view of the symmetric spatial arrangement.

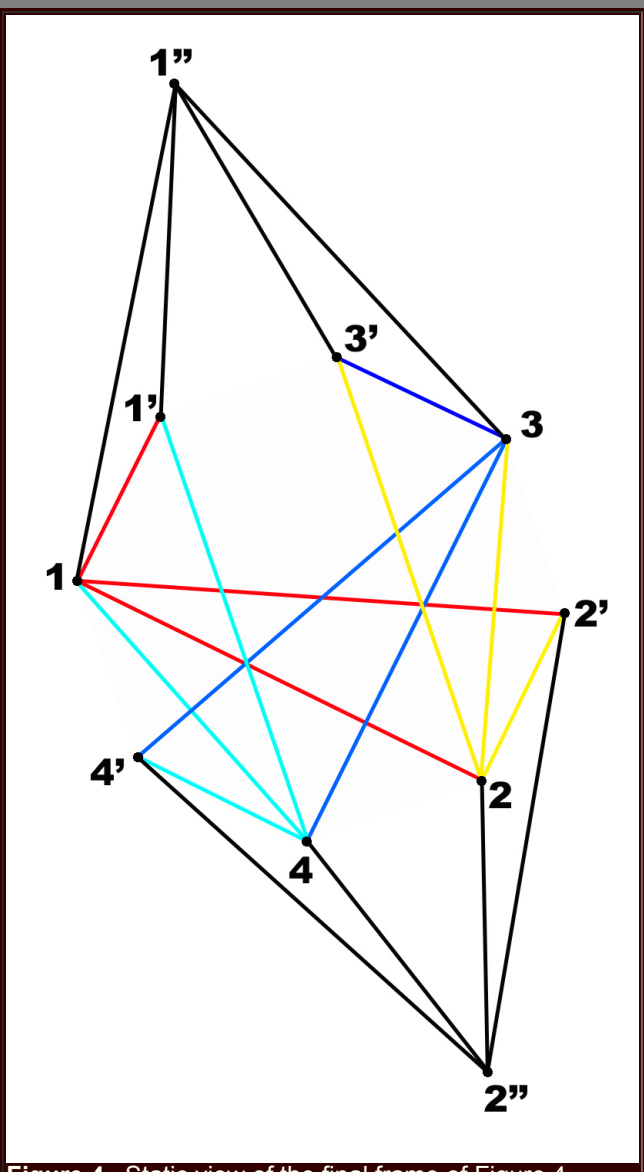

Figure 4. Static view of the final frame of Figure 4.

As may often be the case, geometric views that become complex can be improved, in terms of comprehension, with animation. Older texts might be made to come alive (Harary, 1969); more recent ones can be brightened (Arlinghaus, Arlinghaus, and Harary, 2002; Arlinghaus and Kerski (eBook version), 2013); most important, animation can do more than enhance existing research--as it opens better or new vistas, it can guide it!

References

Arlinghaus, Sandra L. and Kerski, Joseph. 2013. *Spatial Mathematics: Theory and Practice through Mapping*. Boca Raton: CRC Press.

Arlinghaus, Sandra L.; Arlinghaus, William C.; and Harary, Frank. 2002. *Graph Theory and Geography: An Interactive View*, eBook. New York: John Wiley & Sons.

Arlinghaus, William C. 1977. The Classification of Minimal Graphs with Given Abelian Automorphism Group. Ph.D. Dissertation, Department of Mathematics, Wayne State University.

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Frucht, Roberto. 1949. Graphs of degree three with a given abstract group. *Canadian Journal of Mathematics* 1 (4): 365-378.

Harary, Frank. 1969. *Graph Theory*. Reading, MA: Addison-Wesley.

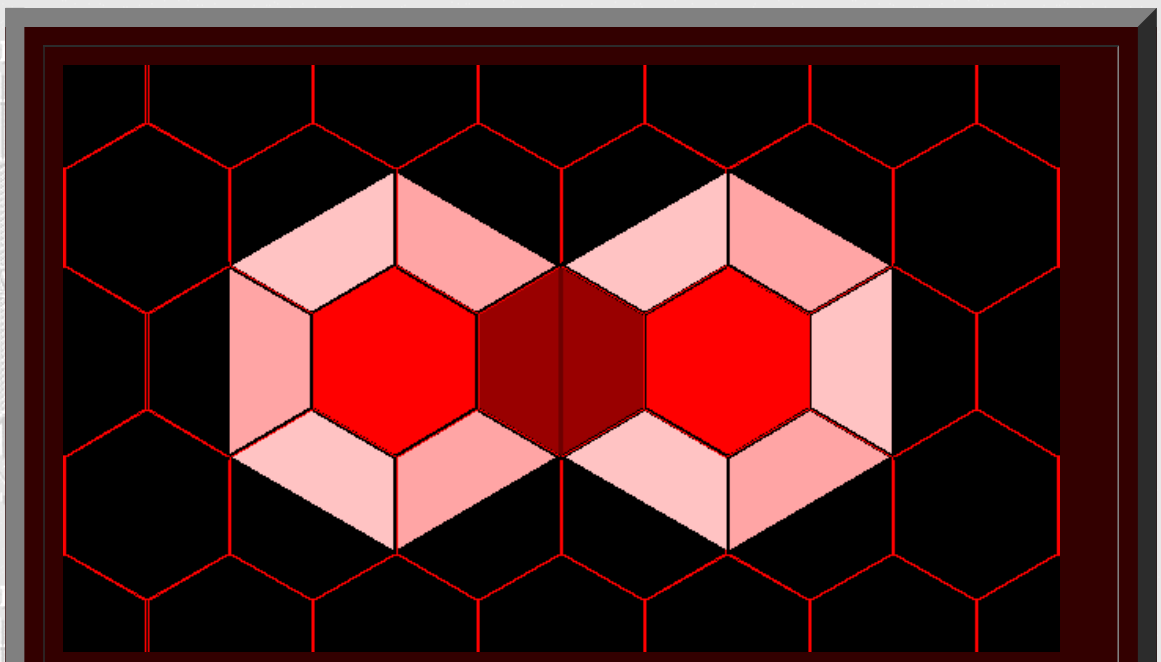
Wikipedia

- Graph Theory: https://en.wikipedia.org/wiki/Graph_theory
- Group Theory: https://en.wikipedia.org/wiki/Group_theory

Wolfram, Math World

- Isomorphic graphs: http://mathworld.wolfram.com/IsomorphicGraphs.html
- Graph automorphism: http://mathworld.wolfram.com/GraphAutomorphism.html

In the In



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RECENT NEWS...

1. *Quaestiones Geographicae*, Special Issue
2. Chene Street History Project.
3. ***Spatial Mathematics: Theory and Practice Through Mapping***. Sandra L. Arlinghaus and Joseph Kerski, (2013), CRC Press Linked video Published July 2013,
4. The work above is the first volume in a series of books to be published by CRC Press in its series "**Cartography, GIS, and Spatial Science: Theory and Practice**." If you have an idea for a book to include, or wish to participate in some other way, please contact the series Editor, Sandra L. Arlinghaus.

5. Virtual Cemetery with William E. Arlinghaus; an ongoing project that continues in development run in the virtual world in parallel with the trust-funded model of a real-world cemetery.

Solstice: An Electronic Journal of Geography and Mathematics,
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Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2001 (top 80 out of over 1000 entries worldwide)

One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2003 (Spatial Synthesis Sampler).

Solstice is listed in the Directory of Open Access Journals maintained by the University of Lund where it is maintained as a "searchable" journal.

Solstice is listed on the journals section of the website of the American Mathematical Society, http://www.ams.org/

Solstice is listed in *Geoscience e-Journals*

IMaGe is listed on the website of the Numerical Cartography Lab of The Ohio State University: http://ncl.sbs.ohio-state.edu/4_homes.html

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Congratulations to all *Solstice* contributors.

Remembering those who are gone now but who contributed in various ways to *Solstice* or to IMaGe projects, directly or indirectly, during the first 28 years of IMaGe:

Allen K. Philbrick Alma S. Lach Donald F. Lach | Frank Harary | William D. Drake | H. S. M. Coxeter | Saunders Mac Lane | Chauncy D. Harris | Norton S. Ginsburg | Sylvia L. Thrupp | Arthur L. Loeb | George Kish |

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