Economic MPC with a contractive constraint for nonlinear systems

Defeng He[1,2,*,†], Jing Sun[2] and Li Yu[1]

[1]*College of Information Engineering, Zhejiang University of Technology, Hangzhou 310023, China*
[2]*Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI 48109, USA*

SUMMARY

In this paper, we consider the stability issue of economic model predictive control (EMPC) for constrained nonlinear systems and propose a new contractive constraint formulation of nonlinear EMPC schemes. This formulation is one of Lyapunov-based approaches in which the contractive function chosen a priori can be used as a Lyapunov function. Some conditions are given to guarantee recursive feasibility and asymptotic stability of the EMPC. Moreover, we analyze the transient economic performance of the EMPC closed-loop system in some finite-time intervals. The proposed EMPC scheme is applied to a chemical reactor model to illustrate its utility and benefits. Copyright © 2016 John Wiley & Sons, Ltd.

1. INTRODUCTION

Economic model predictive control (EMPC) has received much attention because of its ability in integrating real-time process economic optimization and feedback control into an optimal control framework [1, 2]. Unlike traditional MPC, where target-tracking controllers are computed by minimizing positive definite cost functions [3], EMPC directly utilizes general economic functions as stage costs to design controllers. Therefore, EMPC can directly address the operational requirements and, hence, significantly improve system performance compared to the hierarchical control method in which an economically optimal operating point is computed by a real-time optimizer (RTO) in the upper layer and a target-tracking MPC in the lower layer is used to drive the system to the designed operating point [1, 4]. However, in general EMPC cannot guarantee stability of operating points using traditional MPC techniques because it minimizes a general (not necessarily convex or positive definite) cost function over a finite prediction horizon [4].

A special way to address stability of EMPC is to establish economic criterion-based Lyapunov functions by modifying economic cost functions. For example, in [5] the economic cost function was transformed to the so-called rotated cost, and then the monotonic decreasing property of the value function of the rotated cost is established by the assumption of strong duality and the terminal equality constraint. In [6], this terminal constraint was replaced by a terminal inequality constraint and a terminal penalty. Moreover, the assumption of strong duality was relaxed as a dissipativity condition with some elaborately chosen supply and storage functions, and the closed-loop stability of the optimal steady state was ensued by the dissipativity condition [4, 7, 8]. In [9] a generalized terminal equality constraint and a weighted terminal penalty were presented for both EMPC and target-tracking MPC, however with no stability analysis. The authors in [10, 11] presented a stabilizing EMPC without terminal constraints by imposing controllability conditions and using a

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*Correspondence to: Defeng He, College of Information Engineering, Zhejiang University of Technology, China.
†E-mail: hdfzj@zjut.edu.cn

sufficiently long prediction horizon. However, a long horizon makes the EMPC optimization problem a challenge to solve, thereby causing difficulties in real-time applications.

Another way to guarantee stability of EMPC is to enforce Lyapunov-based stability constraints to optimization problems of EMPC. For instance, [12] proposed a Lyapunov-based EMPC scheme which uses two different modes of operation such that the closed-loop system is ultimately bounded in a small region. In [13], the Lyapunov-based EMPC [12] was modified as a double-layer stabilizing EMPC structure, where three (economic) MPC controllers with different prediction horizons were considered, and stability of the closed-loop system was guaranteed by a controllability assumption. As one case of lexicographic multiobjective MPC [14], lately [15] proposed a Lyapunov-based EMPC scheme in a double-layer framework in which economic performance and stability were viewed as two conflicting objectives. To make a tradeoff for both objectives, a contractive constraint was designed using an upper-layer tracking-target MPC subject to a terminal equality constraint. Then this constraint imposed to the lower-layer economic MPC was employed to ensure stability of the EMPC, together with inherent robustness [16] of the upper-layer MPC. Besides solving two constrained optimization problems at each time, the global optimality of the upper-layer MPC is not necessarily guaranteed because of the non-convex nature of the optimization problem subject to nonlinear equality constraints [17]. In [18], we made use of control Lyapunov functions and the dual-mode approach to design a Lyapunov-based EMPC with a slight computational demand.

Motivated by the existing work in, e.g. [5, 11, 15], here we present a new stabilizing EMPC scheme with a contractive constraint for constrained nonlinear systems. Like other Lyapunov-based approaches, this scheme exploits the traditional terminal region and penalty function [3] to construct a contractive constraint. Sufficient conditions for guaranteed recursive feasibility and asymptotic stability are established. Moreover, the closed-loop transient performance evaluated over some finite-time intervals is analyzed under the same conditions. The proposed scheme has several special features.

1) It asymptotically stabilizes the optimal steady state without additional strong duality or dissipativity conditions w.r.t. the economic criterion; the conditions that are satisfied for linear systems with convex constraints and strictly convex costs, however, might not for other cases [5]. As a result, the proposed scheme can be used to optimized-based control of plants in general cases.

2) This scheme makes use of a terminal inequality constraint and the original economic criterion. The notion of terminal inequality constraints is widely used in traditional MPC [3] and was first extended to EMPC [6] because it can increase the size of the feasible set of initial conditions and decrease the differences between predicted and closed-loop trajectories. However, in these (economic) MPC problems, cost functions are often modified by adding penalty functions to establish stability of the optimal steady state. Note that the optimal trajectories minimizing the original cost function are generally different from those minimizing a modified function. Hence, MPC closed-loop trajectories corresponding to the original and modified cost functions are not expected to be the same.

3) The controller is computed by solving a single-layer optimization problem with a contractive constraint. Compared to the double-layer contractive EMPC [15], this scheme does not need the optimality of the contractive constraint, and its stability is established only using contraction of the constraint.

4) Recursive feasibility of this scheme always holds in the context of contractive constraints. It should be noted that to the best of our knowledge, the idea of imposing contractive constraints to MPC has been proposed for the first in [19, 20] for traditional MPC of continuous-time systems, and in [15] for economic MPC of discrete-time systems, but with different assumptions, Lyapunov functions and optimization problem frameworks.

These features of the proposed EMPC provide an alternative to stabilizing EMPC design, and will be illustrated by a comparison study of an example for a continuous stirred tank reactor (CSTR). The rest of this paper is as follows. In Section 2 we describe the problem to be addressed. In Section 3 we propose the new EMPC scheme and establish some conditions under which recursive feasibility and asymptotic stability are guaranteed. Furthermore, we analyze transient economic performance of the proposed EMPC scheme in the same section. In Section 4 the proposed EMPC scheme is applied to a chemical reaction process. We conclude the paper in Section 5.

2. PROBLEM SETUP

Notation

Let $R_{\geq 0}$ and $\mathbf{I}_{\geq 0}$ denote the sets of non-negative real and integer numbers, respectively. $\mathbf{I}_{a:b}$ is the set $\{i \in \mathbf{I}_{\geq 0}: a \leq i \leq b\}$ for some $a \in \mathbf{I}_{\geq 0}$ and $b \in \mathbf{I}_{\geq 0}$. Given two sets $S_1 \subseteq R^n$ and $S_2 \subseteq R^n$, define $S_1 \backslash S_2 = \{x \in R^n \mid x \in S_1 \ \& \ x \notin S_2\}$. A function $\alpha: R^n \to R_{\geq 0}$ is positive definite w.r.t. $x = s$ if it is continuous, $\alpha(s) = 0$, and $\alpha(x) > 0$ for all $x \neq s$. A continuous function $\alpha: R_{\geq 0} \to R_{\geq 0}$ is a class-K function if it is strictly increasing and $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$; it is a class-K_∞ function if it is a class-K function and $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\beta: R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is a class-KL function if $\beta(s, t)$ is a class-K function in s for each fixed $t \geq 0$; it is strictly decreasing in t for every fixed $s > 0$ and $\beta(s, t) \to 0$ as $t \to \infty$. For a given pair of functions α_1 and α_2, $\alpha_1 \circ \alpha_2(s)$ denotes the function $\alpha_1(\alpha_2(s))$ and $\alpha_1^k(s) = \alpha_1 \circ \alpha_1^{k-1}(s)$ for $k \in \mathbf{I}_{\geq 0}$, with $\alpha_1^0(s) = s$. The symbol ':=' denotes that the left-hand side of an equation is defined as the right-hand side. The converse applies to '=:'.

Consider the discrete-time nonlinear systems of the form

$$x_{k+1} = f(x_k, u_k), \quad \forall k \in \mathbf{I}_{\geq 0} \tag{1}$$

where system state $x_k \in R^n$ and control input $u_k \in R^m$ at time k, and map $f: R^n \times R^m \to R^n$. It is assumed that the system has an equilibrium point (x_s, u_s) such that $x_s = f(x_s, u_s)$. The solution of the system for a given sequence of control inputs $\mathbf{u}$ and initial state x_0 is denoted as $x_k = \varphi(k; x_0, \mathbf{u})$ for $k \in \mathbf{I}_{\geq 0}$, where $x_0 = \varphi(0; x_0, \mathbf{u})$. The system is subject to the state and control constraints

$$x_k \in X, \quad u_k \in U, \quad \forall k \in \mathbf{I}_{\geq 0} \tag{2}$$

where $X \subset R^n$ and $U \subset R^m$ are compact sets, containing the equilibrium point in their interior. We assume that the states can be measured at each time $k \in \mathbf{I}_{\geq 0}$.

The economic criterion to be minimized is represented by the stage cost function $L_e: X \times U \to R$. The optimal operating point that stabilizes the system (1) is one of the steady-state points of (1), which is computed by

$$(x_s, u_s) = \arg\min_{(x,u)} \{L_e(x, u) \mid x = f(x, u), \quad x \in X, u \in U\}. \tag{3}$$

For simplicity, we assume that (x_s, u_s) is unique hereafter; otherwise, let (x_s, u_s) denote any of the steady-state points satisfying (3).

In target-tracking MPC, $L_e(x, u)$ is often chosen as a positive definite function w.r.t. (x_s, u_s), i.e. $L_e(x, u) \geq 0$ for all $(x, u) \in X \times U$ and $L_e(x, u) = 0$ if and only if $(x, u) = (x_s, u_s)$. In this case, the optimal operation often leads to closed-loop stability of x_s using the standard MPC stability designs [3]. In EMPC, however, $L_e(x, u)$ is chosen according to some economic criteria such as energy saving, production cost and yield, etc. These economic criteria have to be minimized or maximized in terms of profits and environmental concerns for plants. Hence, in EMPC $L_e(x, u)$ is not necessarily positive definite w.r.t. (x_s, u_s). Consequently, convergence and stability properties in the optimal economic operation are in general not guaranteed using the standard MPC stability designs because these designs depend on the positive definiteness of $L_e(x, u)$.

The control problem of this paper consists in computing an MPC law by minimizing the economic objective function evaluated over a prediction horizon $0 < N < \infty$,

$$\sum_{i=0}^{N-1} L_e(x_i, u_i) \tag{4}$$

subject to the constraints (1) and (2), which guarantees closed-loop asymptotic stability of the optimal operating point (x_s, u_s). Moreover, we will investigate the transient performance of the closed-loop system

$$J_T^{cl}(x) = \sum\nolimits_{k=0}^{T} L_e(x_k, \mu_N(x_k)) \qquad (5)$$

for some finite-time interval $T \in \mathbf{I}_{\geq 1}$, where $\mu_N(x)$ is the first element of the control sequence minimizing (4) for $x_0 = x$.

3. CONTRACTIVE EMPC SCHEME

Consider the system (1) and denote by $(x_{i|k}, u_{i|k})$ the state and control input at time $k + i$, predicted at time k for $i \in \mathbf{I}_{\geq 0}$. Let $\mathbf{u} = \{u_{0|k}, u_{1|k}, \ldots, u_{N-1|k}\}$ be a sequence of N predicted control inputs and $\mathbf{x} = \{x_{0|k}, x_{1|k}, \ldots, x_{N|k}\}$ be its corresponding predicted state trajectory according to the model (1). According to the economic criterion $L_e(x,u)$, we define the cost function to be minimized, J_N, as

$$J_N(x_k, \mathbf{u}) = \sum\nolimits_{i=0}^{N-1} L_e(x_{i|k}, u_{i|k}) \qquad (6)$$

where x_k is the state at the current time k and $x_{0|k} = x_k$.

Let $J_a: X \times U^N \to R_{\geq 0}$ be an auxiliary cost function and $\eta \in R_{\geq 0}$ be a scalar, which will be specified later on. We propose a new contractive EMPC formulation in which the following finite horizon optimal control problem is solved at each time $k \in \mathbf{I}_{\geq 0}$:

$$J_N^*(x_k) = \min_{\mathbf{u}} J_N(x_k, \mathbf{u}) \qquad (7a)$$

$$s.t. x_{i+1|k} = f(x_{i|k}, u_{i|k}), \quad \forall i \in \mathbf{I}_{0:N-1} \qquad (7b)$$

$$x_{i+1|k} \in X, \quad u_{i|k} \in U, \quad \forall i \in \mathbf{I}_{0:N-1} \qquad (7c)$$

$$x_{0|k} = x_k \in X \qquad (7d)$$

$$x_{N|k} \in X_f \qquad (7e)$$

$$J_a(x_k, \mathbf{u}) \leq \eta_k \qquad (7f)$$

where $J_N^*(x)$ is the optimal value function of (7) and terminal region X_f is a compact subset of X. The constraint (7d) is called the initial condition, and (7e) is the terminal constraint. Here the constraint (7f) will be designed as a contractive constraint used to establish closed-loop stability of (x_s, u_s). We denote the (possible local) optimal solution to the optimization problem as $\mathbf{u}^*(x_k) = \{u_{0|k}^*(x_k), \ldots, u_{N-1|k}^*(x_k)\}$ and then $J_N^*(x_k) = J_N(x_k, \mathbf{u}^*(x_k))$.

Consider the system (1) with the constraint (2). We define the set of admissible $(x, \mathbf{u})$ pairs as

$$Z_N = \left\{ (x, \mathbf{u}) | \varphi(k; x, \mathbf{u}) \in X, \quad u_k \in U, \quad \varphi(N; x, \mathbf{u}) \in X_f, \quad \forall k \in \mathbf{I}_{0:N-1} \right\} \qquad (8)$$

where x is an initial state at initial time $k = 0$ and $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$. Then the set of admissible initial states, X_N, is defined as the projection of Z_N onto X, i.e.

$$X_N = \left\{ x \in X | \exists \mathbf{u} \in R^{mN} \quad \text{such that} \quad (x, \mathbf{u}) \in Z_N \right\}. \qquad (9)$$

At time $k \in \mathbf{I}_{\geq 0}$, for the state $x_k \in X_N$ the set of admissible control sequences, U_N, is defined as

$$U_N(x_k, \eta_k) = \left\{ \mathbf{u} \in R^{mN} | \quad (x_k, \mathbf{u}) \in Z_N, \quad (7f) \right\}. \qquad (10)$$

Clearly, the optimal solution to (7) satisfies $\mathbf{u}^*(x_k) \in U_N(x_k, \eta_k)$.

We now construct the function $J_a(x, \mathbf{u})$ and scalar η. To this end, we introduce two auxiliary functions $L_a: X \times U \to R_{\geq 0}$ and $E_a: X \to R_{\geq 0}$, where L_a is positive definite w.r.t. (x_s, u_s) and E_a is positive definite w.r.t. x_s. We define $J_a(x, \mathbf{u})$ as follows:

$$J_a(x_k, \mathbf{u}) = E_a(x_{N|k}) + \sum_{i=0}^{N-1} L_a(x_{i|k}, u_{i|k}) \tag{11}$$

where $x_{i|k} = \varphi(k+i; x_k, \mathbf{u})$ for $i \in \mathbf{I}_{1:N-1}$ with $x_{0|k} = x_k$ and $\mathbf{u} \in U_N(x_k, \eta_k)$ corresponding to the state x_k at time $k \in \mathbf{I}_{\geq 0}$. Because $L_a(x,u)$ and $E_a(x)$ are positive definite w.r.t. (x_s, u_s) and x_s, we have $J_a(x_k, \mathbf{u}) \geq L_a(x_{0|k}, u_{0|k}) \geq 0$ with $\mathbf{u} \in U_N(x_k, \eta_k)$ for any $x_k \in X_N$. Substituting $\mathbf{u}^*(x_k)$ into (11), we have a value function of $J_a(x_k, \mathbf{u})$ as follows:

$$V(x_k) = E_a\left(x_{N|k}^*\right) + \sum_{i=0}^{N-1} L_a\left(x_{i|k}^*, u_{i|k}^*(x_k)\right) \tag{12}$$

where $x_{i|k}^* = \varphi(k+i; x_k, \mathbf{u}^*(x_k))$ for $i \in \mathbf{I}_{1:N-1}$. Note that $\mathbf{u}^*(x_k)$ is the optimal control for (7) and not for (11). Clearly, we have $V(x_k) \geq L_a(x_k, u_{0|k}^*(x_k)) \geq 0$ for all $x_k \in X_N$ because of the positive definiteness of $L_a(x,u)$ and $E_a(x)$. To obtain η, we need an assumption on the terminal region X_f and the functions L_a and E_a.

Assumption 1

There exists a compact terminal region $X_f \subset X$, containing x_s in its interior, and a continuous control law $\mu_f: X_f \rightarrow U$, with $\mu_f(x_s) = u_s$, such that

$$E_a\left(f\left(x, \mu_f(x)\right)\right) - E_a(x) \leq -L_a\left(x, \mu_f(x)\right), \quad \forall x \in X_f. \tag{13}$$

Assumption 1 implies that the region X_f is an invariant set of the system (1) in closed-loop with $\mu_f(x)$ provided that X_f is a sublevel set of $E_a(x)$. In the literature, many approaches have been proposed to design $\mu_f(x)$ as well as X_f and $E_a(x)$ satisfying this assumption; see, e.g. [21–24].Let $\mathbf{u}^*(x_{k-1})$ be the optimal solution to (7) at time $k-1$ and construct a sequence at time k as

$$\hat{\mathbf{u}}_k = \left\{ u_{1|k-1}^*(x_{k-1}), \cdots, u_{N-1|k-1}^*(x_{k-1}), \mu_f\left(x_{N|k-1}^*\right) \right\}. \tag{14}$$

Then we define η_k as

$$\eta_k = V(x_{k-1}) + \beta[J_a(x_k, \hat{\mathbf{u}}_k) - V(x_{k-1})] \tag{15}$$

with some $\beta \leq 1$. Because $V(x_k) \geq 0$ and $J_a(x_k, \mathbf{u}) \geq 0$ for any $x_k \in X_N$ and $\mathbf{u} \in U_N(x_k, \eta_k)$, it is straightforward to obtain that $\eta_k \geq 0$ for all $k \in \mathbf{I}_{>0}$. Note that although η_k depends on x_{k-1}, $V(x_k)$ is the function on x_k because of the receding horizon control principle.In MPC, the control input applied to the plant is the first action of $\mathbf{u}^*(x_k)$, which yields an implicit state feedback EMPC law $\mu_N(x)$ defined as

$$\mu_N(x_k) = u_{0|k}^*(x_k), \quad \forall k \in \mathbf{I}_{\geq 0}. \tag{16}$$

This controller gives rise to the closed-loop system

$$x_{k+1} = f(x_k, \mu_N(x_k)), \quad \forall k \in \mathbf{I}_{\geq 0}. \tag{17}$$

The procedure for implementing the EMPC controller (16) is summarized as Algorithm 1.

Algorithm 1 (Economic MPC with a contractive constraint)

1 (Initialization) Pick $N > 1$, $\beta \leq 1$ and $L_a(x,u)$, and compute (E_a, X_f, μ_f) to satisfy (13); set $\eta_0 := +\infty$ for a given $x_0 \in X_N$ and let $k=0$. Solve the optimization problem (7) and obtain its optimal solution $\mathbf{u}^*(x_0)$. Implement the control action $u_0 = \mu_N(x_0)$ to the system (1) and let $k=1$.
2 Compute $V(x_{k-1})$ and evaluate η_k by (14)–(15) with the state x_k at time k.

3 Solve the optimization problem (7) and obtain its optimal solution $\mathbf{u}^*(x_k)$.

4 Implement the control action $u_k = \mu_N(x_k)$ to the system (1).

5 Let $k = k + 1$ and go back to Step 2. ∎

Note that in Algorithm 1 the contractive constraint (7f) is inactive at initial time $k = 0$ because $\eta_0 := +\infty$. Hence, the set X_N defined as (9) is identified as the set of admissible initial states of the optimization problem (7).

Lemma 1

Consider Assumption 1 and the parameter $\beta \leq 1$. The optimization problem (7) is recursively feasible within the invariant set X_N.

Proof: Let $\mathbf{u}^*(x_k)$ be the optimal solution to (7) at time k, with $x_{0|k} = x_k \in X_N$. At the next time $k + 1$, we use (14) to choose a candidate sequence and a corresponding state sequence as follows:

$$\hat{\mathbf{u}}_{k+1} = \left\{ u^*_{1|k}(x_k), \cdots, u^*_{N-1|k}(x_k), \mu_f\left(x^*_{N|k}\right) \right\} \tag{18}$$

$$\hat{\mathbf{x}}_{k+1} = \left\{ x^*_{1|k}, \cdots, x^*_{N|k}, x_{N+1|k} \right\} \tag{19}$$

where $x^*_{i|k} = \varphi(k + i; x_k, \mathbf{u}^*(x_k)) \in X$ for $i \in \mathbf{I}_{1:N}$ and $x_{N+1|k} = f(x^*_{N|k}, \mu_f(x^*_{N|k}))$. Because of the terminal constraint (7e) and the invariance property of X_f, we have $x_{N+1|k} \in X_f$. Thus, the constraints in (7b)–(7e) are fulfilled.

Now we consider the contractive constraint (7f). Because this constraint is inactive at $k = 0$, we test it for $k \in \mathbf{I}_{\geq 1}$. In order to satisfy (7f), we substitute (18)–(19) to (11) and consider the following inequality:

$$J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) \leq \eta_{k+1} = V(x_k) + \beta[J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) - V(x_k)]. \tag{20}$$

Let $\alpha = 1 - \beta \geq 0$. Then (20) can be rewritten as

$$J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) \leq J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) + \alpha[V(x_k) - J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1})]. \tag{21}$$

From Assumption 1, it is known that

$$V(x_k) - J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) \geq L_a(x_k, \mu_N(x_k)) \geq 0. \tag{22}$$

Hence, the inequality (21) holds if $\alpha \geq 0$ and the equality is always true if $\alpha = 0$. Therefore, the inequality (20) holds for any $\beta \leq 1$. This implies that (18)-(19) satisfy (7f) and the candidate (18) satisfies the overall constraints in (7), which implies that this candidate is a feasible solution to (7) at time $k + 1$. Hence, $U_N(x_{k+1}, \eta_{k+1})$ is not empty and $x_{k+1} = f(x_k, \mu_N(x_k)) \in X_N$ for $\forall x_k \in X_N$. This establishes recursive feasibility of (7) with the invariant set X_N. ∎

Note that similar to the standard MPC [3, 21], a larger size of X_f generally leads to a larger size of X_N because of the property of the terminal region constraint.

Remark 1

It is possible that $V(x) > J_a(x, \hat{\mathbf{u}})$ because $\mathbf{u}^*(x)$ is not optimal for $J_a(x, \mathbf{u})$ but for $J_N(x, \mathbf{u})$. Moreover, the optimal path from $x_0 = x_s$ to $x_N = x_s$ is often different from $x_k \equiv x_s$ for all $k \in \mathbf{I}_{0:N}$ in the context of EMPC [4]. Consequently, the function $V(x)$ is not necessarily positive definite w.r.t. x_s as it is in target-tracking MPC. In other words, it is possible that $V(x_s) \neq 0$ unless some additional conditions are imposed. Note that $(x, u) = (x_s, u_s)$ if $V(x) = 0$ because of the positive definiteness of L_a and E_a w.r.t. (x_s, u_s) and x_s, respectively.

Assumption 2

The functions f and L_e are continuous on the compact sets $X \times U$, respectively, and there exist some class-K_∞ functions α_f, α_l, and γ_i, $i = 1,\ldots,4$ such that $\|f(x,u) - f(z,v)\| \leq \alpha_f(\|(x,u) - (z,v)\|)$, $\|L_e(x,u) - L_e(z,v)\| \leq \alpha_l(\|(x,u) - (z,v)\|)$, $L_a(x,u) \leq \gamma_1(\|x - x_s\|) + \gamma_2(\|u - u_s\|)$, and $\gamma_3(\|x - x_s\|) \leq E_a(x) \leq \gamma_4(\|x - x_s\|)$, $\forall (x,u)$, $(z,v) \in X \times U$ for some vector norm $\|\cdot\|$.

Because L_a and E_a are positive definite w.r.t. (x_s, u_s) and x_s, respectively, the class-K_∞ functions γ_i, $i = 1,\ldots,4$ exist in the finite dimensional case with X and U, and with continuity of L_a and E_a.

Assumption 3

The optimal solution to (7), $\mathbf{u}^*(x)$, satisfies that $u_{i|k}^*(x_s) = u_s$ for $i \in \mathbf{I}_{0:N-1}$. Moreover, there exists a class-K_∞ function α_u such that $\|u_{i|k}^*(x) - u_{i|k}^*(x_s)\| \leq \alpha_u(\|x - x_s\|)$ for any $x \in X_f$ and $i \in \mathbf{I}_{0:N-1}$.

Assumption 3 holds if $\mathbf{u}^*(x)$ is continuous on X_f. As a candidate solution to (7), the sequence $\mathbf{u}(x)$ with $u_{i|k}(x) = u_f(x)$ for $i \in \mathbf{I}_{0:N-1}$ satisfies Assumption 3. Combining Assumptions 1–3, we have the following stability result.

Theorem 1

Suppose that Assumptions 1–3 hold and the parameter $0 < \beta \leq 1$. If (7) is initially feasible, then x_s is an asymptotically stable equilibrium point of the closed-loop system (17) with the region of attraction X_N.

Proof: By applying Lemma 1 and the assumptions recursively, it is obtained that (7) is feasible for $x_k \in X_N$ at time $k \in \mathbf{I}_{\geq 0}$. Thus, X_N is an invariant region of the closed-loop system.

Let $\mathbf{u}^*(x_k)$ and $\mathbf{u}^*(x_{k+1})$ be the optimal solutions to (7) at time k and $k+1$, respectively. Consider a candidate Lyapunov function $V(x)$ given as (12). For all $x_k \in X_N$, the constraint (7f) and the definition of $V(x)$ lead to

$$V(x_{k+1}) - V(x_k) \leq \beta[J_a(x_{k+1}, \hat{\mathbf{u}}_{k+1}) - V(x_k)] \leq -\beta L_a(x_k, \mu_N(x_k)) \tag{23}$$

where $\hat{\mathbf{u}}_{k+1}$ is given by (18). Because of the positive definiteness of $L_a(x,u)$ w.r.t. (x_s, u_s), $\beta L_a(x,u) \geq \rho_1(\|x - x_s\|)$ for all $(x,u) \in X \times U$, where ρ_1 is a class-K_∞ function. Hence, $V(x)$ is a strictly monotone decreasing function along the trajectories of (17) and $V(x_k) \leq V(x_0)$ for all $k \in \mathbf{I}_{\geq 0}$. Moreover, by (23) we have $V(x) \geq \rho_1(\|x - x_s\|)$ because $V(x) \geq 0$ for all $x \in X_N$.

In order to obtain the upper bound of $V(x)$, we separately consider the cases of $x \in X \backslash X_f$ and $x \in X_f$. Let $L_{\max} = \max\{L(x,u) \mid (x,u) \in X \times U\}$ and $E_{\max} = \max\{E(x) \mid x \in X\}$. Then for the state $x \in X \backslash X_f$, it holds that $V(x) \leq E_{\max} + N L_{\max} =: V_{\max}$. Considering any state $x \in X_f$ at time k, i.e. $x_k = x$, it is obtained from Assumptions 2 and 3 that

$$
\begin{aligned}
V(x) &\leq \gamma_4\left(\left\|x_{N|k}^* - x_s\right\|\right) + \sum_{i=0}^{N-1}\left[\gamma_1\left(\left\|x_{i|k}^* - x_s\right\|\right) + \gamma_2\left(\left\|u_{i|k}^*(x) - u_s\right\|\right)\right] \\
&= \gamma_4\left(\left\|f\left(x_{N-1|k}^*, u_{N-1|k}^*\right) - f(x_s, u_s)\right\|\right) + \gamma_1\left(\left\|x_{0|k} - x_s\right\|\right) \\
&\quad + \sum_{i=1}^{N-1}\gamma_1\left(\left\|f\left(x_{i|k}^*, u_{i|k}^*\right) - f(x_s, u_s)\right\|\right) + \sum_{i=0}^{N-1}\gamma_2\left(\left\|u_{i|k}^*(x) - u_{i|k}^*(x_s)\right\|\right) \\
&\leq \gamma_4 \circ \alpha_N(\|x - x_s\|) + N\gamma_2 \circ \alpha_u(\|x - x_s\|) + \sum_{i=0}^{N-1}\gamma_1 \circ \alpha_i(\|x - x_s\|) \\
&=: \rho(\|x - x_s\|).
\end{aligned}
$$

where $\alpha_0(r) = r$ and $\alpha_i(r) = \alpha_f \circ \alpha_{i-1}(r) + \alpha_f \circ \alpha_u(r) \in K_\infty$ for $i \in \mathbf{I}_{1:N}$. Clearly, ρ is a class-K_∞ function. If $\rho(\|x - x_s\|) \geq V_{\max}$ for $x \in X \backslash X_f$, we get $V(x) \leq \rho(\|x - x_s\|)$ for all $x \in X$. Otherwise, we multiply $\rho(\|x - x_s\|)$ by a constant C such that $C\rho(\|x - x_s\|) \geq V_{\max}$ for $x \in X \backslash X_f$. This operation yields $V(x) \leq \rho_2(\|x - x_s\|)$ for

all $x \in X$, where $\rho_2(r) := \max\{1, C\}\rho(r)$. Hence, we have $\rho_1(\|x - x_s\|) \leq V(x) \leq \rho_2(\|x - x_s\|)$ for all $x \in X_N$, with class-$\mathcal{K}_\infty$ functions ρ_1 and ρ_2. Combining the inequality (23), $V(x)$ is a Lyapunov function of the closed-loop system (17) and x_s is an asymptotically stable equilibrium point of (17) within X_N. Because X_N is invariant for (17), it is a region of attraction of the closed-loop system. ∎

Remark 2

From the proof of Lemma 1 and Theorem 1, one can see that the contractive property of (7f) is crucial in guaranteeing stability of the optimal steady state of the closed-loop system. Hence, the resulting EMPC (here referred to as *Contractive EMPC*) scheme can be viewed as a variant of contractive MPC [19, 20, 25], but with several different features. In traditional contractive MPC [3], a Lyapunov function is chosen a priori as a positive definite function $M(x)$, e.g. $M(x) = x^T P x$ with $P > 0$, to ensure that $M(x(k;x,\mathbf{u})) \leq M(x)$ with $k = 1,2,\ldots,N$. Then a variable horizon (N^*) optimal control problem is solved online, where X_f depends on the current state x and $M(x)$. In the original version of contractive MPC, the whole control sequence $\mathbf{u}^*$ is applied to plants in an open-loop fashion and the procedure is repeated at every time interval N^*. Under assumptions of feasibility, exponential stability of the origin is ensured by the contractive constraint. Unlike the traditional contractive MPC, here the Lyapunov function is defined as $V(x) = J_a(x,\mathbf{u}^*(x))$ and it is not necessarily positive definite w.r.t. x_s because $\mathbf{u}^*(x)$ is not optimal for $J_a(x,\mathbf{u})$ but for $J_N(x,\mathbf{u})$. Moreover, the optimization (7) is a fixed horizon problem where X_f is chosen to satisfy Assumption 1, i.e. X_f is not related to the current state x and $V(x)$, and recursive feasibility is always guaranteed. The controller obtained here follows the standard receding horizon principle, in the sense that only the first control element of the solution to (7) is applied to (1) at each time. In addition, here the contractive constraint (7f) does not lead to exponential stability of the closed-loop system (17).

Remark 3

In the context of EMPC, the idea of imposing a contractive property on the closed-loop behavior is also presented in [15]. In that work, the author regarded stability and economics as two conflicting objectives and proposed a double-layer lexicographic optimization formulation, i.e.

$$1^{st} \text{ layer}: \begin{cases} \mathbf{u}_1^*(x_k) = \arg\min_{\mathbf{u}}\left\{J_{tr}(x_k, \mathbf{u}) = \sum_{i=0}^{N-1} L_a\left(x_{i|k}, , u_{i|k}\right)\right\} \\ s.t. \quad (7b) - (7d), \quad x_{N|k} = x_s \end{cases} \tag{24}$$

$$2^{nd} \text{ layer}: \begin{cases} \mathbf{u}_2^*(x_k) = \arg\min_{\mathbf{u}} J_N(x_k, \mathbf{u}) \\ s.t. \quad (7b) - (7d), \quad x_{N|k} = x_s \\ \quad\quad J_{tr}(x_k, \mathbf{u}) \leq \xi_k := J_{tr}^*(x_k) + a\left[V_{tr}(x_{k-1}) - J_{tr}^*(x_k)\right] \end{cases} \tag{25}$$

where $J_{tr}(x,\mathbf{u})$ is the target-tracking function, $J_N(x,\mathbf{u})$ is the economic function given as (6), $J_{tr}^*(x) = J_{tr}(x,\mathbf{u}_1^*(x))$, $V_{tr}(x) = J_{tr}(x,\mathbf{u}_2^*(x))$, and some $0 \leq a < 1$. The last constraint in (25), i.e. $J_{tr}(x,\mathbf{u}) \leq \xi$, is a contractive constraint, which, together with inherent robustness of target-tracking MPC, ensures that $V_{tr}(x)$ is a Lyapunov function of the system (1) in closed-loop with this EMPC law specified as the first element of $\mathbf{u}_2^*(x)$. Because of the global optimality of $\mathbf{u}_1^*(x)$ and the positive definiteness of $J_{tr}^*(x)$ w.r.t. x_s, $V_{tr}(x)$ is positive definite w.r.t. x_s. Note that this formulation needs to online solve two non-convex, nonlinear optimization problems, which significantly increases the computational load of implementing MPC [14]. In particular, non-convexity may lead to non-global solution in both layers and therefore, no guarantee of feasibility for (25) because of discontinuous Pareto fronts of non-convex multiobjective optimization (15). Finally, the terminal equality will reduce the size of feasible set of initial conditions (6).

Note that the contractive constraint (7f) can be replaced with some conditions, e.g. [26]

$$J_a(x_k, \mathbf{u}) - V(x_{k-1}) \leq -\sigma L_a(x_{k-1}, \mu_N(x_{k-1})) \qquad (26)$$

for some $\sigma \in R_{\geq 0}$. In this case, recursive feasibility and stability also hold for $\sigma \in (0, 1]$ from the proof of Lemma 1 and Theorem 1. Additionally, because recursive feasibility of (7), which is essential in establishing stability, is ensured for $\beta \in (0,1]$ but not for $\beta > 1$, stability of (17) cannot be established for the general case of $\beta > 1$. But on a case by case it is possible to find conditions for $\beta > 1$ to establish stability of (17).

Let the balls B_r be defined as $B_r = \{x \in X: \|x - x_s\| \leq r\}$ for some norm $\|\cdot\|$ and $r > 0$, and the level sets of $V(x)$ as $S_c = \{x \in X: V(x) \leq c\}$ for some $c > 0$. Because the system (17) asymptotically converges to (x_s, u_s) while remaining bounded for all $k \in \mathbf{I}_{\geq 0}$, for a small $c > 0$ there exists a finite time interval $T_c \in \mathbf{I}_{\geq 1}$ such that for all $k \geq T_c$, $\varphi(k; x_0, \mu_N) \in S_c$ with $x_0 \in X_N$. Then from Theorem 1 we have the following corollary on $V(x)$ and the terminal predicted state $x_{N|k}$.

Corollary 1

Consider a small number $c > 0$ such that $S_c \subset X_f$. Under the assumptions in Theorem 1, the closed-loop states of (17) enter S_c in such way that $V(x)$ decreases exponentially. Moreover, if the state $x_k \in X_N \backslash S_c$, then its optimal predicted state $x_{N|k}^*$ satisfies that

$$\|x_{N|k}^* - x_s\| \leq \theta_1(\|x_0 - x_s\|, k) \qquad (27)$$

where θ_1 is a class-KL function.

Proof: Let $\hat{S}$ be the closure of set $X_N \backslash S_c$. We compute the minimal value of $L_a(x,u)/J_a(x,\mathbf{u})$ for $\forall(x, \mathbf{u}) \in (\hat{S}, U^N)$ and denote it as $\bar{a}$, where u is the first element of sequence $\mathbf{u}$. Note that $\bar{a}$ exists because of Assumptions 1–3 and the positive definiteness of $L_a(x,u)$ and $E_a(x,u)$. Clearly, we have $0 < \bar{a} < 1$. By the inequality (23), we have $V(x_{k+1}) - V(x_k) \leq -\bar{a}\beta V(x_k)$. This implies that $V(x_k) \leq (1 - \bar{a}\beta)^k V(x_0)$. Because $e^{(s-1)} - s \geq 0 \Leftrightarrow s^k \leq e^{(s-1)k}$ for $\forall s \in [0,1)$ and $k \in \mathbf{I}_{\geq 0}$, we have

$$V(x_k) \leq V(x_0)e^{-(\bar{a}\beta)k}, \quad \forall x_k \in X_N \backslash S_c. \qquad (28)$$

Because S_c is the sublevel set of $V(x)$ and the system [17] is asymptotically stable, $V(x)$ decreases exponentially until $V(x) \leq c$, i.e. $x \in S_c$.

By Assumption 2 and the definition of $V(x)$, it is derived that $\gamma_3(\|x_{N|k}^* - x_s\|) \leq E_a(x_{N|k}^*) \leq V(x_k)$ for $\forall x_k \in X_N$. Consider any state $x_k \in X_N \backslash S_c$. Then combining (28), we have

$$\|x_{N|k}^* - x_s\| \leq \gamma_3^{-1}\left(\rho_2(\|x_0 - x_s\|)e^{-(\bar{a}\beta)k}\right) =: \theta_1(\|x_0 - x_s\|, k) \qquad (29)$$

for $\forall x_k \in X_N \backslash S_c$ of (17). Because the functions $\gamma_3, \rho_2 \in K_\infty$, θ_1 is a class-KL function and thus (27) holds. ∎

Remark 4

It is observed from (28) that the larger the value of β, the faster the decaying of $V(x)$. This implies that a lager value of β speeds up the closed-loop system (17) to approach to the optimal steady-state point. In this sense, β is named as the *contractive factor* reflecting the contractive property of the closed-loop behavior.

Now we analyze economic performance of the proposed EMPC scheme. Without loss of generality, here we assume that $L_e(x,u) > 0$ for all $x \in X$ and $u \in U$. Given an initial state $x_0 \in X_N \backslash S_c$, from Corollary 1 we can find a finite time interval $T > 0$ such that $\varphi(k; x_0, \mu_N(\cdot)) \in X_N \backslash S_c$ for $k \in \mathbf{I}_{1:T}$ and $\varphi(k; x_0, \mu_N(\cdot)) \in S_c$ for $k \in \mathbf{I}_{\geq T+1}$. Moreover, we consider any admissible control sequence $\mathbf{u}$ and its associated trajectory $\varphi(k; x_0, \mathbf{u})$ which satisfies $\|\varphi(T; x_0, \mathbf{u}) - x_s\| \leq \|\varphi(T; x_0, \mu_N(\cdot)) - x_s\| \leq r$ for some $r > 0$. Then we have the following results on the transient performance (5) evaluated over the finite time interval T.

Proposition 1

Consider a small number $c > 0$ such that $S_c \subset X_f$. Under the assumptions in Theorem 1, the transient performance (5) evaluated over the finite time interval T satisfies that

$$J_T^{cl}(x_0) \leq J_T(x_0, \mathbf{u}) + (N - T)\alpha_L \circ \rho_2^{-1}(c) + NL_e(x_s, u_s) + \sum_{k=0}^{T-1} \theta_2(||x_0 - x_s||, k). \tag{30}$$

where α_L is a class-K_∞ function and θ_2 is a class-KL function.

Proof: Let the sequences $\mathbf{u}^*(x_k)$ and $\mathbf{u}^*(x_{k+1})$ be the optimal solutions to (7) at time k and $k+1$, respectively, where $x_{k+1} = f(x_k, \mu_N(x_k))$. Considering the sequence (18), we compute the optimal value functions of J_N as follows:

$$J_N^*(x_k) = L_e(x_k, \mu_N(x_k)) + \sum_{i=1}^{N-1} L_e\left(x_{i|k}^*, u_{i|k}^*\right)$$

$$J_N^*(x_{k+1}) \leq J_N(x_{k+1}, \hat{\mathbf{u}}_{k+1}) = L_e\left(x_{N|k}^*, \mu_f\left(x_{N|k}^*\right)\right) + \sum_{i=0}^{N-2} L_e\left(x_{i+1|k}^*, u_{i+1|k}^*\right)$$

which lead to

$$J_N^*(x_{k+1}) - J_N^*(x_k) \leq L_e\left(x_{N|k}^*, \mu_f\left(x_{N|k}^*\right)\right) - L_e(x_k, \mu_N(x_k)). \tag{31}$$

Adding (31) from $k = 0$ to $k = T - 1$, it is obtained that

$$J_T^{cl}(x_0) \leq J_N^*(x_0) - J_N^*(x_T) + \sum_{k=0}^{T-1} L_e\left(x_{N|k}^*, \mu_f\left(x_{N|k}^*\right)\right) \leq J_N^*(x_0) \tag{32}$$
$$+ \sum_{k=0}^{T-1} L_e\left(x_{N|k}^*, \mu_f\left(x_{N|k}^*\right)\right).$$

By Assumption 2, for $x \in X_f$ we have $L_e(x, \mu_f(x)) - L_e(x_s, u_s) \leq \alpha_l(||x - x_s||) + \alpha_l \circ \alpha_{\mu f}(||x - x_s||) =: \alpha_L(||x - x_s||)$ with the class-K_∞ function α_L. Because of (29), we have that

$$J_T^{cl}(x_0) \leq J_N^*(x_0) + TL_e(x_s, u_s) + \sum_{k=0}^{T-1} \theta_2(||x_0 - x_s||, k) \tag{33}$$

where $\theta_2(||x_0 - x_s||, k) = \alpha_L \circ \gamma_3^{-1}(\rho_2(||x_0 - x_s||)e^{-\bar{a}\beta k})$ and it is a class-KL function.

From the dynamic programming principle, $J_N^*(x)$ satisfies that

$$J_N^*(x_0) \leq \inf_{\mathbf{u} \in U_N(x_0)} \left\{ J_T(x_0, \mathbf{u}) + J_{N-T}^*(\varphi(T; x_0, \mathbf{u})) \right\}.$$

Let $r = \rho_2^{-1}(c)$. For $T \in \mathbf{I}_{1:N-1}$ and $\mathbf{u}$ steering (1) to the ball B_r within T time steps, we obtain that

$$J_T(x_0, \mathbf{u}) = J_T(x_0, \mathbf{u}) + J_{N-T}^*(\varphi(T; x_0, \mathbf{u})) - J_{N-T}^*(\varphi(T; x_0, \mathbf{u})) \tag{34}$$
$$\geq J_N^*(x_0) - (N - T)[\alpha_L(r) + L_e(x_s, , u_s)].$$

Note that for $T \geq N$, the non-negativity of $L_e(x, u)$ implies that the inequality $J_T(x, \mathbf{u}) \geq J_N^*(x)$ holds for all admissible $\mathbf{u} \in U_T(x)$, implying (34) again. Substituting (34) to (33) leads to

$$J_T^{cl}(x_0) \leq J_T(x_0, \mathbf{u}) + (N - T)\alpha_L(r) + NL_e(x_s, u_s) + \sum_{k=0}^{T-1} \theta_2(||x_0 - x_s||, k)$$

for all admissible $\mathbf{u} \in U_T(x)$ steering (1) to B_r within T time steps, which is equal to (30). ∎

It is remarked that from Proposition 1 the transient performance given here does not admit transient optimality in the sense of (11) because the error term will not vanish as $N \to \infty$ and $||\varphi(T; x_0, \mu_N(\cdot)) - x_s|| \to 0$. Nevertheless, the closed-loop system (17) admits the asymptotic time average performance.

Proposition 2

Under the assumptions in Theorem 1, the closed-loop system (17) admits the asymptotic time average performance, i.e.

$$\lim_{K\to\infty} \inf J_K^{cl}(x)/K \le L_e(x_s, u_s) \tag{35}$$

with the region of attraction X_N.

   Proof: It is straightforward to test that the suboptimal controller defined by the sequence (18) can asymptotically stabilize the system [1] to x_s with X_N. Then following the proof of Theorem 7 in [13], one can show the inequality (35) and hence, the proof of this claim is omitted. ■

   Note that because the asymptotic average performance may not be a singleton in the context of economic NMPC, a lower bound on the asymptotic average performance under the proposed EMPC scheme is considered in (36) from [4, 8, 13]. In addition, the average transient performance over a finite time window is not guaranteed to be better than $L_e(x_s, u_s)$ and it may take any value. Recently [11] established optimal transient performance estimates for the EMPC without terminal constraint.

   Because of the constraint (7f), the contractive factor β will affect the predicted optimal performance $J_N^*(x)$. For clarity, let $x = x_k$, $J_N^*(x, \beta) := J_N^*(x)$, $\eta(\beta) := \eta$ and $U_N(x, \beta) := U_N(x, \eta)$ for a selected $0 < \beta \le 1$. From Theorem 1, the feasible set $U_N(x, \beta)$ is always non-empty for any $x \in X_N$.

Proposition 3

Under the assumptions in Theorem 1, the predicted optimal performance satisfies that $J_N^*(x, \beta_1) \le J_N^*(x, \beta_2)$ for any $0 < \beta_1 < \beta_2 \le 1$ and $x \in X_N$.

   Proof: The proof consists of two parts. We first prove that $U_N(x, \beta_2) \subseteq U_N(x, \beta_1)$ for $0 < \beta_1 < \beta_2 \le 1$ and any $x \in X_N$, and then prove the proposition by contradiction under the assumptions.

   Part 1. From the proof of Lemma 1, it is known that $\eta(\beta) > 0$ for any $0 < \beta \le 1$. For two values $0 < \beta_1 < \beta_2 \le 1$, then we have $\eta(\beta_2) < \eta(\beta_1)$ because of the inequality (22). Consider the sets $U_N(x, \beta_1)$ and $U_N(x, \beta_2)$, and pick any $\mathbf{u}(\beta_2) \in U_N(x, \beta_2)$. We obtain that $J_a(x, \mathbf{u}(\beta_2)) \le \eta(\beta_2) < \eta(\beta_1)$, which leads to $\mathbf{u}(\beta_2) \in U_N(x, \beta_1)$. Because of arbitrariness of $\mathbf{u}(\beta_2) \in U_N(x, \beta_2)$, it is obtained that $U_N(x, \beta_2) \subseteq U_N(x, \beta_1)$ for $0 < \beta_1 < \beta_2 \le 1$ and any $x \in X_N$.

   Part 2. By contradiction, it is assumed that $J_N^*(x, \beta_1) > J_N^*(x, \beta_2)$ for any $0 < \beta_1 < \beta_2 \le 1$ and $x \in X_N$. Let $\mathbf{u}^*(\beta_i) \in U_N(x, \beta_i)$ be the optimal solution corresponding with $J_N^*(x, \beta_i)$ for $i = 1, 2$. From Part 1, we have $\mathbf{u}^*(\beta_2) \in U_N(x, \beta_1)$, which implies that there exists a solution $\mathbf{u}^*(\beta_2)$ such that $J_N^*(x, \beta_2) < J_N^*(x, \beta_1)$ in $U_N(x, \beta_1)$. This contradicts the optimality of $J_N^*(x, \beta_1)$ in $U_N(x, \beta_1)$. Hence, this proposition holds. ■

Proposition 3 shows that the predicted optimal performance $J_N^*(x)$ will be degraded when increasing the value of β, which, on the other hand, speeds up the decaying of $V(x)$ and hence improves stability of the closed-loop system. In this sense, the loss of optimality of $J_N^*(x)$ can be regarded as a price that one has to pay for ensuring feasibility and stability of the EMPC controller. This mechanism is the so-called *Stability Price* in [15].

Remark 5

It should be emphasized that $J_N^*(x)$ is a measure of open-loop performance and generally $J_N^*(x) \ne J_N^{cl}(x)$ because of the receding horizon nature of MPC. Hence, Proposition 3 and Stability Price may not hold for $J_N^{cl}(x)$. Given that two different values of β will generate different closed-loop trajectories, even if they start from the same initial state x_0, it is a great challenge to theoretically establish analogues of both Proposition 3 and Stability Price for $J_N^{cl}(x)$. Nevertheless, in practice Stability Price can provide guidance to tune the contractive factor β and this will be illustrated by a numerical example.

4. NUMERICAL EXAMPLE

To illustrate the performance of the proposed EMPC scheme, we consider the isothermal chemical reactor system [5]

$$\frac{dc_a}{dt} = \frac{Q_{in}}{V_R}(c_{ain} - c_a) - \sigma c_a^2$$

$$\frac{dc_b}{dt} = \frac{Q_{in}}{V_R}(c_{bin} - c_b) + \sigma c_a^2 \tag{36}$$

where c_a and c_b are the molar concentrations of species 'a' and 'b', respectively, c_{ain} and c_{bin} are the feed concentrations of 'a' and 'b' and Q_{in} is the flow through the reactor. The volume of the reactor $V_R = 10$ L and the rate constant $\sigma = 1.2$ L/(mol min). In this study, the concentrations c_a and c_b are the states x_1 and x_2, respectively, and the flow Q_{in} is the control u. The constraints are imposed on the state and control variables by the form of $x_i \in [0, 1]$ for $i = 1, 2$ and $u \in [0, 15]$. The process economics are defined as the price of product b and a separation cost

$$L_e(x, u) = 0.5u - 2x_2u. \tag{37}$$

The optimal steady-state point for this cost is computed as $(x_s, u_s) = (0.5785, 0.4215, 9.5258)$. In order to design the contractive constraint (7f), we define the following positive definite function

$$L_a(x, u) = ||x - x_s||_2^2 + ||u - u_s||_2^2. \tag{38}$$

Let the system (A, B) be the linearized model of (36) at (x_s, u_s). From the LQ optimal control approach, solving the Riccati equation $PA + A^T P - PBB^T P + I = 0$ leads to a positive definite matrix solution $P = \begin{bmatrix} 0.3448 & 0.2213 \\ 0.2213 & 0.5248 \end{bmatrix}$. Then we construct the triplet as $E_a(x) = (x - x_s)^T P(x - x_s)$, $X_f = \{x \in R^2 : E_a(x) \leq 0.0613\}$ and $\mu_f(x) = [-0.0052\ 0.0128](x - x_s) + u_s$, which satisfies Assumption 1 of Section 3. Assumption 2 is also satisfied given the function forms of f and L_e. Because it is hard to directly validate Assumption 3 during numerical optimization, here we imposed the condition $||u_{i|k}(x) - u_s|| \leq \sigma||x - x_s||$ with $\sigma = 2000$ for $i \in I_{0:N-1}$ and $x \in X_f$ on the optimization operation to fulfill Assumption 3.

The system (36) is discretized with a sampling time 0.1 min and the Euler's first-order approximation is employed for all derivatives. Let the prediction horizon steps $N = 4$ and the simulation time steps be 70. The solution at time k was used as an initial guess for solving the optimization problems at $k + 1$. Moreover, all optimization problems had been solved by the *fmincon* function with the SQP algorithm in MATLAB V7.1 on the computer of MS WINDOWS 7.0 Enterprise and an Intel® Core i5 CPU with 2.3 GHz and 4 GB RAM.

We pick two initial states (I) (0.4, 0.1) and (II) (0.6, 0.6). Figures 1, 2 separately show the time profiles of $V(x)$ and the state trajectories for different values of the contractive factor β, where the left plot is associated with the initial state (I) and the right the initial state (II). The profiles depicted
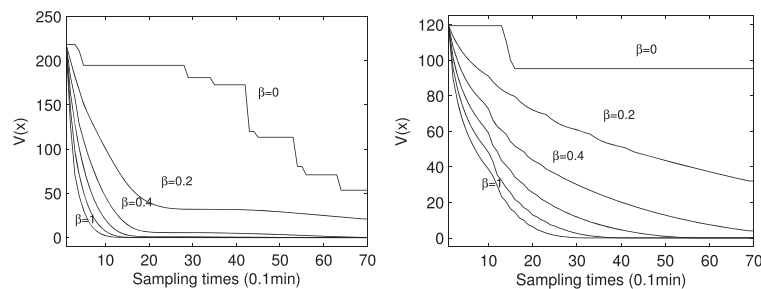

Figure 1. Time profiles of $V(x)$ for different values of β. Left plot: initial state (I); right plot: initial state (II).
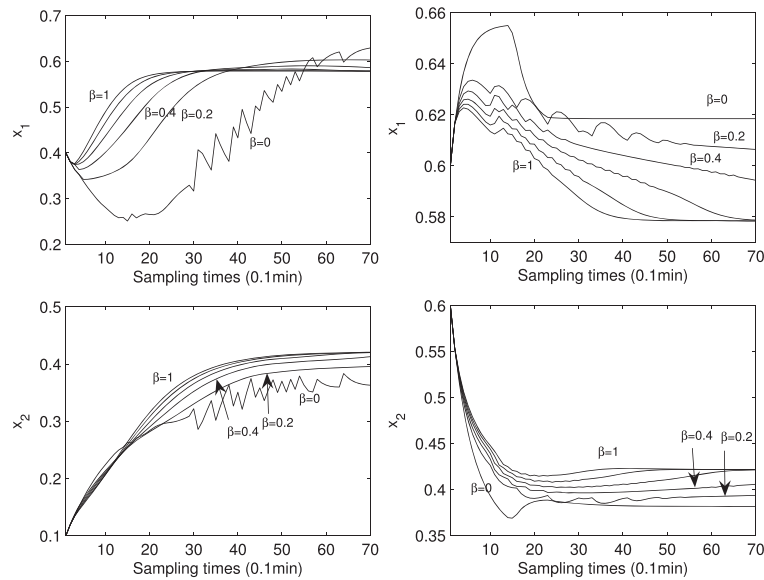
Figure 2. Closed-loop state trajectories for different values of β. Left plot: initial state (I); right plot: initial state (II).

in both figures are for β in the interval [0, 1] with increments of 0.2. It can be seen from both figures that (i) for all $0 < \beta \leq 1$, the value functions are strictly decreasing and thus stability of the closed-loop system is established; (ii) for larger values of β, the value functions decay more quickly, which implies that the closed-loop system approaches to the steady-state point (x_s, u_s) more quickly. It should be emphasized that the condition of $0 < \beta \leq 1$ is sufficient but not necessary to guarantee stability of the closed-loop system, which can be illustrated by the profile of $V(x)$ corresponding to $\beta = 0$ in the left subfigure.

We define a transient time window [0, T] of (36) such that the closed-loop state $\varphi(T; x_0, \mu_N(\cdot))$ enters the range of $\pm 5\%$ the steady state x_s. Denote by $\mathcal{J}_T^{cl}(I)$ and $\mathcal{J}_T^{cl}(II)$ the transient economic performances obtained by applying Algorithm 1, starting from the initial states (I) and (II), respectively. Table I presents these values for the different values of β. As can be seen, the transient performance obtained by the proposed scheme is a decreasing function on β. That is, the transient performance will be improved by reducing the value of β. This observation may be explained to some extent by the Stability Price of the predicted performance because the constraint (7f) is relaxed by reducing β and hence, the predicted performance can be improved. However, it is seen from Figure 2 that the convergence speed of the closed-loop system gets slower as β becomes smaller. In this sense, we can select the value of β to make a tradeoff between the economic performance and stability of the closed-loop system.

Table II tabulates the transient economic performances and computational times by applying Algorithm 1 for different lengths of the prediction horizon N under the fixed $\beta = 0.8$. It is observed that the obtained transient performance cannot be improved by only lengthening or reducing the prediction horizon N. In other words, the obtained transient performance is not a monotone function on N. However, the computational times for calculating control actions, over all sampling times of all simulations, are reduced when we select a shorter N.

Table I. The transient economic performances for different values of β under $N = 4$.

β	0.2	0.4	0.6	0.8	1.0
$\mathcal{J}_T^{cl}(I)$	-209.8552	-60.0956	-45.4511	-38.4738	-36.7232
$\mathcal{J}_T^{cl}(II)$	-417.1054	-205.7893	-114.7035	-104.2869	-93.4900

Table II. The transient economic performances and computational times for different values of N under $\beta = 0.8$.

N	4	5	10	15	20
$\mathcal{J}_T^l(\mathrm{I})$	−38.4738	−42.0817	−40.3958	−38.3085	−32.0736
$\mathcal{J}_T^l(\mathrm{II})$	−104.2869	−85.6629	−81.8655	−99.9914	−107.4492
T_{max} (ms)	446.1	554.4	735.4	957.7	2209
T_{ava} (ms)	29.5	31.2	51.9	86.3	170.1

Table III. The transient economic performances and computational times for three controllers.

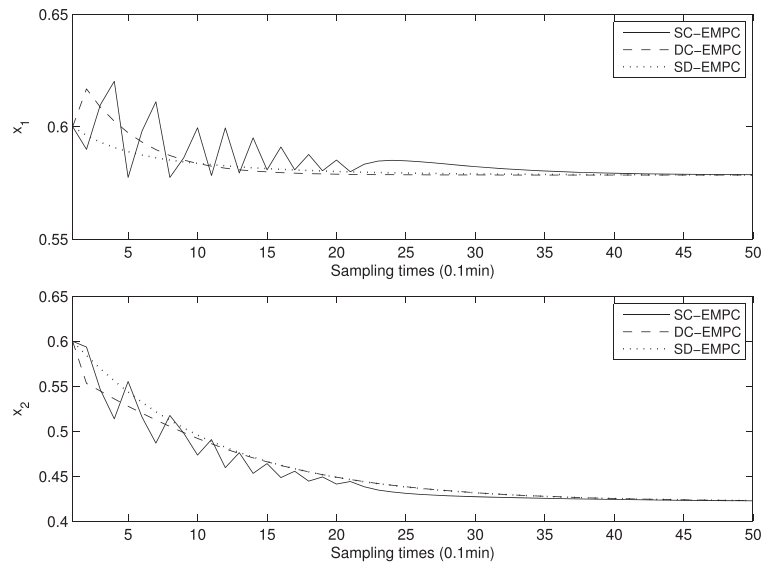|	SC-EMPC (15,0.8)	DC-EMPC (15,0.2)	SD-EMPC (15,--)	SC-EMPC (4,0.6)
$\mathcal{J}_T^l(\mathrm{II})$	−99.9914	−109.5727	−109.6129	−114.7035
T_{max} (ms)	957.7	1254.9	498.8	508.2
T_{ava} (ms)	86.3	186.1	40.6	32.1

Figure 3. Time evolutions of the closed-loop states starting from the initial state II, associated with three controllers.
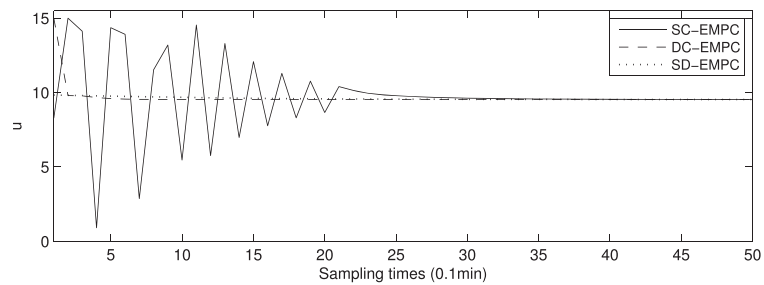

Figure 4. Time profiles of the inputs applied by three controllers, associated with the initial state II.

In what follows, we consider the initial state (II) and assess the three controllers applied by the single-layer contractive EMPC proposed here (SC-EMPC), the double-layer contractive EMPC in [15] (DC-EMPC) and the single-layer dissipative EMPC in [6] (SD-EMPC). Note that the system (37) is not dissipative w.r.t. the economic criterion (38). Hence the following regularized function followed from [6] is used for SD-EMPC:

$$L_e(x, u) = 0.5u - 2x_2u + 0.1(u - u_s)^2 \qquad (39)$$

with a penalty function $V_f(x) = E_a(x) + [-4.0541 - 9.7533](x - x_s)$ and the terminal region X_f. Moreover, in order to guarantee initial feasibility of DC-EMPC, we select the prediction horizon $N = 15$ for all three controllers. It is noted that initial feasibility of DC-EMPC is ensured only if $N \geq 10$ but initial feasibility of SC-EMPC and SD-EMPC is ensured for $N = 4$ because of the terminal inequality constraint.

Table III gives the transient economic performances and computational times obtained by separately applying the three controllers, respectively, where $\beta = 0.8$ for SC-EMPC and $a = 0.2$ for DC-EMPC. From the first to third column of Table III, it is known that the transient economic performance of DC-EMPC is almost equal to that of SD-EMPC, and both DC-EMPC and SD-EMPC have better transient performances than SC-EMPC. This implies that the transient performance of SC-EMPC admits no transient optimality in the sense of [11]. However, we have known from Tables I and II that the transient economic performance of SC-EMPC has no monotonicity w.r.t. N but it is a decreasing function on β. Furthermore, it is observed from the three columns that there is significant difference in the computational time among three controllers. We can select a shorter N and a smaller β, e.g. $(N,\beta) = (4,0.6)$, to improve the transient economic performance of SC-EMPC as well as reducing its computational times, as shown in the last column of this table.

Figures 3 and 4 show the time evolutions of the closed-loop states and control inputs by separately applying the three controllers, where $(N,\beta) = (4,0.6)$ for SC-EMPC. As expected, the three closed-loop systems are asymptotically stable at x_s in the presence of the state and control constraints, but they approach the economic setpoint in different ways. In particular, comparing the solid lines to the dashed and dotted lines in Figures 3 and 4, it is that both DC-EMPC and SD-EMPC produce smoother closed-loop state responses and control signals than SC-EMPC. Note that one can adjust β to mitigate the oscillatory behavior resulted from SC-EMPC (see Figure 2).

5. CONCLUSIONS

This paper proposed a contractive EMPC scheme with guaranteed asymptotic stability for constrained nonlinear systems. By imposing a special contractive constraint, based on the terminal region and terminal penalty, into the EMPC optimization problem, we derived the sufficient conditions for guaranteeing recursive feasibility and stability of EMPC. Moreover, we analyzed the transient economic performance and established the relation of the predicted performance to the contractive factor. The example of an isothermal chemical reactor demonstrated the effectiveness of the EMPC scheme proposed here.

D.F. He and L. Yu are with the College of Information Engineering, Zhejiang University of Technology, Hangzhou, Zhejiang 310023, P.R. China. D.F. He is also with the Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI 48109, USA (e-mail: hdfzj@zjut.edu.cn; lyu@zjut.edu.cn).
J. Sun is with the Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI 48109, USA (e-mail: Jingsun@umich.edu).

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