ESTIMATION OF COEFFICIENTS OF UNIVALENT FUNCTIONS BY A TAUBERIAN REMAINDER THEOREM

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Let $S$ denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

analytic and univalent in the unit disk $|z| < 1$. One of the most penetrating results on the coefficients $a_n$ is Hayman's theorem [6, 7] that

$$\lim_{n \to \infty} \frac{|a_n|}{n} = a \leq 1,$$  \hspace{1cm} (2)

for each $f \in S$, with equality only for the Koebe function

$$k(z) = z(1-z)^{-2} = \sum_{n=1}^\infty nz^n$$

or for one of its rotations. Hayman's proof begins with the elementary observation that each $f \in S$ has a direction $e^{i\theta_0}$ of maximal growth, in the sense that

$$\lim (1-r)^2 |f(re^{i\theta_0})| = \alpha \hspace{1cm} (3)$$

and the limit is 0 for every other direction. In particular, the direction $e^{i\theta_0}$ is unique if $\alpha > 0$ (see also Milin [12; p. 82]). The second step is the deduction of (2) from (3). Hayman's argument is relatively simple for $\alpha = 0$, but quite complicated for $\alpha > 0$.

Milin [11, 12] has recently simplified this latter step in the case $\alpha > 0$. His argument is essentially based upon the following result, which may be viewed as a new Tauberian theorem. Let

$$g(r) = \sum_{n=0}^\infty b_n r^n, \hspace{1cm} b_0 = 1,$$

be a power series with complex coefficients, convergent for $-1 < r < 1$. Let

$$s_n = \sum_{k=0}^n b_k, \hspace{1cm} \sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k,$$

and

$$\log g(r) = \sum_{n=1}^\infty c_n r^n.$$  \hspace{1cm} (4)

**Theorem A (Milin).** Suppose $|g(r)| \to \alpha$ as $r \to 1$, and

$$\sum_{n=1}^\infty n|c_n|^2 < \infty.$$
Then $|s_n| \to \alpha$ and $|\sigma_n| \to \alpha$ as $n \to \infty$.

This theorem is applied to the function
\[ g(r) = \frac{(1-r)^2}{r} f(r), \]
where $f$ is given by (1) and it is assumed, after a rotation, that $\theta_0 = 0$. Then a calculation gives
\[ s_n = a_{n+1} - a_n \quad \text{and} \quad \sigma_{n-1} = a_n/n. \]

The Tauberian condition $\sum n |c_n|^2 < \infty$ is a consequence of the following theorem of Bazilevich [2, 12], for $\alpha > 0$.

**Theorem B (Bazilevich).** Let $f \in S$, and let
\[ \log \left\{ \frac{f(z)}{z} \right\} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \]

Suppose $\alpha > 0$ and $\theta_0 = 0$. Then
\[ \sum_{n=1}^{\infty} n|\gamma_n - (1/n)|^2 \leq \frac{1}{2} \log 1/\alpha. \]

One curious feature of Milin's Tauberian theorem is the assumption that $|g(r)|$, rather than $g(r)$, has a limit. An early step in Milin's proof is to show that the partial sums $s_n$ are bounded. Hence, if $g(r) \to \alpha$, one can appeal to a classical Tauberian theorem [5; p. 154] to conclude that $\sigma_n \to \alpha$.

Under a much stronger hypothesis, Bazilevich [1, 2, 12] has estimated the rate of convergence of $|a_n/n|$ to $\alpha$. He assumes that $f \in S$ maps the unit disk onto the exterior of an analytic arc. Since the arc is analytic at $\infty$, a Schwarz reflection shows that the square-root transform of $f$ has the form
\[ \sqrt{(f(z^2))} = \frac{\sqrt{(\alpha)}z}{1-z^2} [1 + B_1(1-z^2) + B_2(1-z^2)^2 + \ldots] \]
near the pole at $z = 1$. In fact, an elementary argument [1] shows that $B_1$ is purely imaginary. Bazilevich concludes that
\[ |a_n| \leq \alpha n + C_1 \sqrt{(\log n)} + C_2, \]
where $C_1$ and $C_2$ depend only on $\alpha$, $B_1$, and $\text{Re } B_2$.

The purpose of the present paper is to show that under a relatively mild assumption on the behaviour of $f$ along its ray of maximal growth, one can obtain a somewhat weaker estimate on the rate of convergence of $a_n/n$. The result is as follows. Observe the implicit assumption, made without loss of generality, that $\theta_0 = 0$.

**Theorem.** For some positive constants $B$ and $\delta$, and for some complex number $s \neq 0$, let $f \in S$ satisfy the inequality
\[ \left| \frac{(1-r)^2}{r} f(r) - s \right| \leq B(1-r)^{\delta}, \quad 0 < r < 1. \]
Then

\[ \left| \frac{a_n}{n} - s \right| \leq \frac{C}{\log n}, \quad n = 2, 3, \ldots, \]

where \( C \) depends only on \( |s|, B, \) and \( \delta \).

The proof depends upon a Tauberian remainder theorem, essentially due to Freud [3] and Korevaar [9], which we now state in the notation of Theorem A. A proof of Theorem C by the Karamata–Wielandt method is also implicit in Ganelius’ notes [4; pp. 3–6].

**Theorem C (Freud–Korevaar).** For some positive constants \( B \) and \( \delta \), suppose

\[ |g(r) - s| \leq B(1-r)^\delta, \quad 0 < r < 1, \]

and suppose \( |s_n| \leq M, \quad n = 1, 2, \ldots. \) Then

\[ |\sigma_n - s| \leq C/\log n, \quad n = 2, 3, \ldots, \]

where \( C \) depends only on \( B, M, \) and \( \delta \).

**Proof of Theorem.** Let \( \alpha = |s|, \) and define \( g \) by the relation (5). Since \( \sigma_{n-1} = a_n/n, \) we need only obtain a uniform bound for \( s_n = a_{n+1} - a_n \) in terms of \( \alpha \) and \( B. \) It is known, of course, that \( \left| a_{n+1} - a_n \right| \) has an absolute bound. Our hypothesis on \( f \) implies [12; p. 87] that \( |a_n| \to \infty \) and \( \arg a_n \to \arg s, \) but it seems difficult to estimate the rate of convergence in terms of \( \alpha \) and \( B \) alone. Accordingly, we follow Milin’s argument to estimate \( s_n. \)

Let \( c_n \) be defined by (4), and \( \gamma_n \) by (6). Then \( c_n = 2(\gamma_n - 1/n), \) and Theorem B gives

\[ \sum_{n=1}^{\infty} n|c_n|^2 \leq 2 \log 1/\alpha. \quad (7) \]

On the other hand,

\[ \sum_{n=0}^{\infty} s_n r^n = \frac{g(r)}{1-r} = \exp \left( \sum_{n=1}^{\infty} (c_n + (1/n)) r^n \right). \]

Hence, by an inequality of Lebedev and Milin [12; p. 51],

\[ |s_n|^2 \leq \exp \left\{ \sum_{k=1}^{n} k|c_k| + (1/k)^2 - \sum_{k=1}^{n} (1/k) \right\} \]

\[ \leq (1/\alpha^2) \exp \left\{ 2 \Re \sum_{k=1}^{n} c_k \right\}, \]

where (7) has been used. Furthermore, by hypothesis,

\[ \exp \left( \Re \sum_{k=1}^{n} c_k \right) = |g(r)| \exp \left( \Re \left( \sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k \right) \right) \]

\[ \leq (\alpha + B) \exp \left\{ \sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k \right\}. \]
But as in the proof of Fejér's Tauberian theorem [10; p. 65],

\[ \left| \sum_{k=1}^{n} c_k - \sum_{k=1}^{\infty} c_k r^k \right| \leq \left| \sum_{k=1}^{n} c_k (1 - r^k) \right| + \left| \sum_{k=n+1}^{\infty} c_k r^k \right| \]

\[ \leq \left\{ \sum_{k=1}^{n} |c_k|^2 \right\}^{1/2} \left\{ \sum_{k=1}^{n} (1/k) (1 - r^k)^2 \right\}^{1/2} + \frac{1}{n} \left\{ \sum_{k=n+1}^{\infty} k |c_k|^2 \right\}^{1/2} \left\{ \sum_{k=n+1}^{\infty} kr^{2k} \right\}^{1/2} \]

\[ \leq 2(2 \log (1/\alpha))^{1/2}, \]

with the choice \( r = 1 - (1/n) \). This establishes the required bound on \( s_n \), which completes the proof of the theorem.

One can make a similar quantitative statement when \( \alpha = 0 \). In this case one can proceed directly, without recourse to Tauberian remainder theorems. Since there is no longer a unique direction of maximal growth, the natural counterpart of our theorem simply estimates \( a_n \) in terms of the rate of growth of the maximum modulus of \( f \). See, for example, Hayman [8; p. 392].

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References

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