

LIMIT GROUPS ARE CAT(0)

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ABSTRACT

We prove that every limit group acts geometrically on a CAT(0) space with the isolated flats property.

1. Introduction

A group is said to be a CAT(0) group if it acts geometrically, that is, properly discontinuously and cocompactly by isometries, on a CAT(0) space. One should think of a CAT(0) space as a geodesic metric space in which every geodesic triangle is at least as thin as its comparison triangle in the Euclidean plane. For basic facts about CAT(0) spaces and groups, a general reference is [4]. We will be interested mainly in geodesic spaces which are locally CAT(0) spaces (non-positively curved spaces), that is, every point has a neighborhood which is a CAT(0) space. All of our (locally) CAT(0) spaces will be proper. In this paper we show that the class of groups known as *limit groups* is CAT(0), thus answering a question circulated by Z. Sela.

Limit groups arise naturally in the study of equations over free groups. Historically they appear under different names: \exists -free groups [15], fully residually free groups [12, 13], and ω -residually free groups [18]. The term ‘limit group’ first appears in [18] and reflects the topological approach to the subject in which these groups arise in the limit of a sequence of homomorphisms $G \rightarrow F$ (F a free group) interpreted as a sequence of G -trees. Thus limit groups act isometrically on \mathbb{R} -trees (see also [16]) and the Rips machine can be brought to bear. For more information on limit groups, the reader is referred to the papers cited above and to the expositions [3], [5] and [14].

The quickest definition, in the incarnation of ω -residually free groups, is that a finitely generated group L is a limit group provided that for every finite subset $W \subset L$ there is a homomorphism $L \rightarrow F$ to a free group which is injective on W . A more constructive definition will be given in the Appendix (see Proposition A.2). For the convenience of the reader we summarize the basic properties of limit groups in the following proposition. Properties (1) and (3)–(5) follow easily from the definition, (2) and (6)–(9) can be found in the above papers (and can be proved quickly from the constructive definition), and (10) was proved independently by Dahmani [6] and Alibegović [1].

PROPOSITION 1.1. *Let L be a limit group. Then we have the following properties.*

- (1) The group L is torsion-free.
- (2) Every abelian subgroup of L is finitely generated.
- (3) Every non-trivial abelian subgroup is contained in a unique maximal abelian subgroup.
- (4) If two non-trivial commuting elements are conjugate then they are equal.
- (5) Roots are unique, that is, $x^n = y^n$ with $n \neq 0$ implies $x = y$.
- (6) The group L is finitely presented, and in fact has a finite Eilenberg–MacLane space.
- (7) The group L is coherent, that is, every finitely generated subgroup is finitely presented (and it is a limit group).
- (8) If L is not free then its cohomological dimension is $\max(2, n)$ where n is the maximal rank of an abelian subgroup.
- (9) There are finitely many conjugacy classes of maximal abelian subgroups of rank at least 2.
- (10) The group L is relatively hyperbolic with respect to the collection of maximal abelian subgroups of rank at least 2.

MAIN THEOREM. *Limit groups are CAT(0).*

Here is an outline of the argument, contained in Sections 2 and 3. An important class of limit groups consists of ω -residually free towers [18, Definition 6.1] (see also [13]). These are groups that have an explicit and inductive description in terms of gluing building blocks (graphs, surfaces, tori) in a certain way. We recall this definition in Section 3. It is easy to see that ω -residually free towers are CAT(0). We then make use of the fact [13, 19], whose proof due to Sela we outline in the Appendix, that every limit group occurs as a finitely generated subgroup L of an ω -residually free tower G . The associated CAT(0) space is then constructed as a ‘core’ of the covering space of G associated to L .

A closer analysis of the construction shows that the CAT(0) space we built satisfies the *isolated flats property* [9]. This analysis is contained in Section 4. After this paper was written, the results of Hruska and Kleiner [10] became available. In particular, it follows from [10, Theorem 1.2.1] and Proposition 1.1(10) that every CAT(0) space on which a limit group acts geometrically must have isolated flats.

2. Geometrically coherent locally CAT(0) complexes

Recall that a group is *coherent* if every finitely generated subgroup of it is finitely presented. We now introduce a CAT(0) version of coherence.

DEFINITION 2.1. Let X be a connected locally CAT(0) space. A subspace C of X is a *core* of X if it is compact, locally CAT(0) (with respect to the induced path metric), and inclusion $C \rightarrow X$ is a homotopy equivalence.

Note that if C is connected and locally CAT(0) then it is aspherical. Thus inclusion $C \rightarrow X$ is a homotopy equivalence if and only if it induces an isomorphism in π_1 .

Recall that a celebrated theorem of Scott [17] asserts that any connected 3-manifold M with finitely generated $\pi_1(M)$ contains a compact submanifold N such that $\pi_1(N) \rightarrow \pi_1(M)$ is an isomorphism, and in particular $\pi_1(M)$ is finitely presented. The compact submanifold N is known as a *Scott core* of M .

Scott's theorem is a motivation for our approach to proving that limit groups are CAT(0).

DEFINITION 2.2. A connected locally CAT(0) space Y is *geometrically coherent* if for every covering space $X \rightarrow Y$ with X connected and $\pi_1(X)$ finitely generated and every compact subset $K \subset X$ it follows that X contains a core $C \supset K$.

Wise has studied this property for 2-complexes in the presence of sectional curvature conditions [21].

EXAMPLE 2.3. Any compact hyperbolic surface with totally geodesic boundary is geometrically coherent. Moreover, a core can always be chosen to be convex (see for example [11, Theorem 4.73, Exercise 4.75]).

EXAMPLE 2.4. A flat torus $T = \mathbb{R}^n/\Lambda$ is geometrically coherent. Indeed, any connected covering space admits a metric splitting $T' \times E$ where T' is a flat torus or a point and E is Euclidean space \mathbb{R}^k or a point. Thus $T' \times \{\text{point}\}$ is a (convex) core.

EXAMPLE 2.5. Without the curvature assumptions there are examples due to Wise [20] of finite 2-complexes X and covering spaces $\tilde{X} \rightarrow X$ such that $\pi_1(\tilde{X})$ is finitely presented and yet \tilde{X} does not contain a compact subset C with $\pi_1(C) \rightarrow \pi_1(\tilde{X})$ an isomorphism. We also point out that the product of two finite graphs with non-abelian fundamental groups is a two-dimensional locally CAT(0) complex which is not coherent, much less geometrically coherent. It appears to be unknown whether a locally CAT(0) complex which is coherent is necessarily geometrically coherent.

Many examples of geometrically coherent spaces can be constructed by repeatedly applying the following gluing theorem. The annulus $S^1 \times [0, 1]$ is endowed with a flat metric where S^1 has suitably chosen total length.

THEOREM 2.6. *Let M and N be geometrically coherent locally CAT(0) spaces. Let Y be the space obtained from the disjoint union of M , N and a finite collection of annuli $S_t^1 \times \{0\}$,*

$$M \sqcup \bigsqcup_t S_t^1 \times [0, 1] \sqcup N,$$

by gluing $S_t^1 \times \{0\}$ to a local geodesic in M by a local isometry and gluing $S_t^1 \times \{1\}$ to a local geodesic in N by a local isometry.

Then Y is geometrically coherent.

In the proof we shall need the following fact.

PROPOSITION 2.7 [4, 'Gluing with a tube', Proposition II.11.13]. *Let X and A be locally CAT(0) metric spaces. If A is compact and $\phi, \psi : A \rightarrow X$ are locally isometric immersions, then the quotient of $X \sqcup (A \times [0, 1])$ by the equivalence relation generated by $(a, 0) \sim \phi(a)$; $(a, 1) \sim \psi(a)$, for all $a \in A$, is locally CAT(0).*

Proof of Theorem 2.6. Let $p : X \rightarrow Y$ be a covering space with X connected and $\pi_1(X)$ finitely generated and let $K \subset X$ be a given compact set. Note that X is a graph of spaces with vertex spaces the components of $p^{-1}(M)$ and $p^{-1}(N)$ and with edge spaces the components of $p^{-1}(S_t^1 \times \{\frac{1}{2}\})$. Let Γ be the graph of groups associated to X , so that $\pi_1(X) = \pi_1(\Gamma)$. Thus Γ is a bipartite graph with two types of vertices which we will call M -type and N -type. All edge groups are either trivial or infinite cyclic. The graph Γ may be infinite, so consider an exhaustion of Γ by a chain of connected finite subgraphs. This induces an exhaustion of $\pi_1(X)$ by a chain of subgroups. Since $\pi_1(X)$ is finitely generated, this chain stabilizes, so there is a finite subgraph $\Gamma_0 \subset \Gamma$ with $\pi_1(\Gamma_0) = \pi_1(\Gamma)$. Let X_0 be the subspace of X corresponding to Γ_0 . Then X_0 may not contain the given compact set K , but K will certainly be contained in a finite subgraph of spaces of X , so we will simply enlarge Γ_0 and X_0 so as to contain K . Each edge of Γ_0 corresponds to an annulus or a strip $(\mathbb{R} \times [0, 1])$. One boundary component of each annulus or strip is identified with a circle or a line in an M -type vertex and the other is identified with a circle or a line in an N -type vertex. As a graph of groups, Γ_0 is finite with trivial or cyclic edge groups, and $\pi_1(\Gamma_0) = \pi_1(X)$ is finitely generated. It follows that all vertex groups are finitely generated so we are in position to apply our assumptions about the existence of cores in these vertex spaces.

We will now build a core C in X_0 (and this will be a core in X as well). The idea is to take the union of cores in vertex spaces, connecting annuli, plus rectangles in connecting strips. The cores and the rectangles have to be chosen with care so that the union contains K and so that the intersection between rectangles and neighboring cores is contractible.

Recall that X_0 contains finitely many vertex and edge spaces. Start by selecting a rectangle $R(S) = I(S) \times [0, 1]$ inside each strip $S = \mathbb{R} \times [0, 1] \subset X_0$ so that:

- (1) $R(S) \supset S \cap K$.

Now for each vertex space $V \subset X_0$ choose a core $C(V)$ such that:

- (2) $C(V) \supset V \cap K$;
- (3) $C(V) \supset V \cap R(S)$ for every strip $S \subset X_0$; and
- (4) $C(V) \supset V \cap A$ for every annulus $A \subset X_0$.

Let C be the union of the following spaces:

- $C(V)$ for every vertex space $V \subset X_0$;
- A for every annulus $A \subset X_0$; and
- $R(S)$ for every strip $S \subset X_0$.

Then C is a compact space and it has the same fundamental group as X_0 . That C is locally CAT(0) follows from Proposition 2.7. □

REMARK 2.8. In the statement of Theorem 2.6 one can replace annuli $S^1 \times [0, 1]$ by products $T \times [0, 1]$ of flat tori T . The proof is similar.

3. ω -residually free towers

DEFINITION 3.1 [18, Definition 6.1]. A height 0 ω -residually free tower (ω -rft) is the wedge of finitely many circles, tori and closed hyperbolic surfaces excluding the surface of Euler characteristic -1 .

Assume that inductively the notion of a height $n - 1$ ω -rft has been defined. A height n ω -rft Y_n is obtained from a height $n - 1$ ω -rft Y_{n-1} by gluing a building block.

(A) (Abelian block) $Y_n = Y_{n-1} \sqcup S^1 \times [0, 1] \sqcup T^m / \sim$ where T^m is the m -torus, $S^1 \times \{1\}$ is identified with a coordinate circle in T^m and $S^1 \times \{0\}$ is identified with a non-trivial loop in Y_{n-1} that generates a maximal abelian subgroup in $\pi_1(Y_{n-1})$.

(Q) (Quadratic block) $Y_n = Y_{n-1} \sqcup \Sigma \sqcup_t S_t^1 \times [0, 1] / \sim$ where Σ is a connected compact hyperbolic surface with totally geodesic boundary and $\chi \leq -2$ or a punctured torus (see note below), $S_t^1 \times \{0\}$ is identified with a boundary component of Σ , and $S_t^1 \times \{1\}$ is identified with a homotopically non-trivial loop in Y_{n-1} . A further requirement is that there exists a retraction

$$r : Y_n \rightarrow Y_{n-1}$$

such that the restriction $r : \Sigma \rightarrow Y_{n-1}$ has non-abelian image in π_1 .

A group Γ is called an ω -rft if it is the fundamental group $\Gamma = \pi_1(Y)$ of a space which is an ω -rft.

NOTE. These are precisely the hyperbolic surfaces that support pseudo-Anosov homeomorphisms or equivalently two intersecting two-sided simple closed curves.

We took certain liberties stating this definition. In particular, Sela allows Y_n to be obtained from Y_{n-1} by attaching more than one building block and also allows wedging tori and closed surfaces. However, it is easy to see that both definitions define the same class of groups.

LEMMA 3.2. *Every ω -rft is geometrically coherent.*

Proof. Every ω -rft is given a locally CAT(0) metric by induction on height. When a surface is glued it is endowed with a hyperbolic metric with totally geodesic boundary. We use the fact that a hyperbolic structure can be given with prechosen lengths of boundary components [8, Exposé 3, Section II], so that these lengths match the lengths of circles to which they are glued. The statement now follows from Theorem 2.6 and induction on height. □

The importance of ω -residually free towers is that any limit group embeds in one. The main result of [13] is an essentially equivalent statement but expressed in a different language. We outline Sela’s construction in the Appendix.

THEOREM 3.3 [19, Definitions 1.11, 1.12]. *Every limit group is isomorphic to a finitely generated subgroup of an ω -residually free tower.*

This in turn immediately implies the result we are aiming for.

THEOREM 3.4. *Limit groups are CAT(0).*

4. Isolated flats

In this section we give a combination theorem for CAT(0) spaces with isolated flats which we will then use to show that limit groups act geometrically on such spaces. We recall the definition of the isolated flats property given by Hruska in [9]. A *flat* in a CAT(0) space X is an isometric embedding of a Euclidean space \mathbb{E}^k into X for some $k \geq 2$. A *half-flat* is an isometric embedding of $\mathbb{E}^{k-1} \times [0, \infty)$.

DEFINITION 4.1 [9, Definition 3.2]. A CAT(0) space X has the *isolated flats property* if it contains a family \mathcal{F} of flats such that the following are satisfied.

- (1) There is a constant B so that every flat in X is contained in the Hausdorff B -neighborhood of some flat $F \in \mathcal{F}$.
- (2) There exists $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ so that for every pair of distinct flats $F_1, F_2 \in \mathcal{F}$ and for every $k \geq 0$ the intersection of Hausdorff k -neighborhoods $N_k(F_1) \cap N_k(F_2)$ has diameter at most $\phi(k)$.

It is not hard to show that the family \mathcal{F} can be chosen so that it is invariant under all isometries of the space X .

Suppose that a group G acts geometrically on a metric space X . A k -flat F in X is *periodic* if there is a free abelian subgroup $A < G$ of rank k that acts on F by translations with quotient a k -torus. We will also say that a line (that is, a biinfinite geodesic) ℓ in X is periodic if there is an element $g \in G$ which acts on it as a non-trivial translation. If G acts geometrically on a CAT(0) space (X, \mathcal{F}, ϕ) with isolated flats, then every flat $F \in \mathcal{F}$ is periodic [9, Theorem 3.7].

We say that a line ℓ in X is parallel to a flat or a line $F \subset X$ if ℓ is contained in a Hausdorff neighborhood of F . If a line is parallel to another line, then the two cobound a strip [4, Theorem II.2.13]. If a line ℓ is parallel to a flat F then F contains a line parallel to ℓ .

LEMMA 4.2. *Let (X, \mathcal{F}, ϕ) be a CAT(0) space with isolated flats on which a group G acts geometrically. Let ℓ be a periodic geodesic in X not parallel to any flat $F \in \mathcal{F}$. There exists a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ so that for every $k \geq 0$ and every $F \in \mathcal{F}$ the diameter of $N_k(\ell) \cap N_k(F)$ is no bigger than $\psi(k)$.*

Proof. We will assume without loss of generality that \mathcal{F} is G -invariant. Suppose that such a function does not exist. Then there exist $k \in \mathbb{N}$ and a sequence of flats $F_i \in \mathcal{F}$ so that $\text{diam}(N_k(\ell) \cap N_k(F_i)) > i$. Let $\ell_i = N_{2k}(F_i) \cap \ell$. Note that as $i \rightarrow \infty$ the length l_i of ℓ_i (possibly infinite) also tends to infinity. Fix a point $p \in \ell$. Since \mathcal{F} is preserved by G we may, if necessary, replace each F_i by a translate $g_i(F_i)$ for some element $g_i \in G$ that acts as a translation on ℓ so that for large i we have $p \in \ell_i$ and, moreover, the distance between p and the endpoints (if any) of ℓ_i goes to infinity. But then $N_{2k}(F_i) \cap N_{2k}(F_j)$ contains a large neighborhood in ℓ of p (for large i, j) which implies that $F_i = F_j$. Thus the sequence F_i eventually consists of a single flat and ℓ is parallel to it. □

We are interested in finite graphs of spaces that are locally CAT(0) and whose edge spaces are geodesic circles or segments. To reduce repetition we will call such a space X a *special CAT(0) graph of spaces*. Of course, the universal cover \tilde{X} of X has an induced decomposition as a tree of spaces whose edge spaces are lines or segments. In what follows we adopt the following terminology. Let $\pi : \tilde{X} \rightarrow T$ be the natural map to the tree that collapses each vertex space \tilde{X}_v to the associated vertex v and collapses products $\tilde{X}_e \times [0, 1]$ to edges of T . If \tilde{X}_v is a vertex space, we define the *extended vertex space* \tilde{X}_v^+ to be the inverse image of the star of v with respect to the barycentric subdivision of T . Thus \tilde{X}_v^+ is the union of \tilde{X}_v with products $\tilde{X}_e \times [0, \frac{1}{2}]$ (or $\tilde{X}_e \times [\frac{1}{2}, 1]$) for each $\tilde{X}_e \times [0, 1]$ attached to \tilde{X}_v .

An edge space \tilde{X}_e is identified with $\tilde{X}_e \times \{\frac{1}{2}\} \subset \tilde{X}_e \times [0, 1]$. Thus an edge space which is incident to a vertex space \tilde{X}_v is contained in the associated extended vertex space \tilde{X}_v^+ .

THEOREM 4.3. *Let X be a special CAT(0) graph of spaces whose vertex spaces come in two colors: G (for ‘good’) and B (for ‘bad’). We consider the induced coloring of the vertex spaces of \tilde{X} . We assume that:*

- (0) *each edge space \tilde{X}_e isometric to a line is incident to at least one G vertex;*
- (1) *distinct edge spaces in an extended G vertex space are not parallel to each other; and*
- (2) *an edge space in an extended G vertex space \tilde{X}_v^+ does not bound a half-flat in \tilde{X}_v^+ .*

If the universal cover of each vertex space X_v has the isolated flats property, then so does the universal cover \tilde{X} of X .

Note that (2) implies the following:

- (3) *an edge space in an extended G vertex space \tilde{X}_v^+ is not parallel to a flat in \tilde{X}_v^+ .*

In the proof of Theorem 4.3 we will need the following two lemmas.

LEMMA 4.4. *Let Z be a CAT(0) space, u, v two isometries of Z such that $\langle u, v \rangle \subset \text{Isom}(Z)$ is discrete, and A_u, A_v are axes of u, v respectively. If for some $k > 0$ the intersection $N_k(A_u) \cap N_k(A_v)$ has infinite diameter then some non-trivial powers of u and v coincide, and in particular A_u and A_v are parallel.*

Proof. Let $x_n \in A_u, y_n \in A_v$ be sequences of points going to infinity with $d(x_n, y_n) \leq 2k, n = 0, 1, 2, \dots$. Denote by $|u|, |v|$ the translation lengths of u, v respectively. For each n there are powers $u^{\alpha_n}, v^{\beta_n}$ of u, v taking x_0, y_0 to within $|u|, |v|$ of x_n, y_n . Thus $v^{-\beta_n} u^{\alpha_n}$ takes x_0 to within $2k + |u| + |v|$ of y_0 . By the discreteness assumption (along with the blanket assumption that Z is proper) there are only finitely many elements of $\langle u, v \rangle$ with this property and we conclude that

$$v^{-\beta_n} u^{\alpha_n} = v^{-\beta_m} u^{\alpha_m}$$

for many $\beta_m \neq \beta_n$ and the claim follows. □

LEMMA 4.5. *Under the hypotheses of Theorem 4.3 there is a function $\psi' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every extended G vertex space \tilde{X}_v^+ and any two distinct edge spaces $\tilde{X}_e, \tilde{X}_f \subset \tilde{X}_v^+$ the diameter of $N_k(\tilde{X}_e) \cap N_k(\tilde{X}_f)$ is no bigger than $\psi'(k)$.*

Proof. Suppose the statement fails. Then there is $k > 0$ and a sequence of pairs of edge spaces in extended G vertex spaces whose k -neighborhoods have larger and larger intersections. Since there are only finitely many orbits of extended vertex spaces, and in each extended vertex space only finitely many orbits of edge spaces, we may assume that one of the edge spaces in these pairs is a fixed edge space \tilde{X}_e . Now we can translate the other edge spaces in these pairs by isometries stabilizing \tilde{X}_e (as in the proof of Lemma 4.2 except that flats are replaced by other edge spaces). Since a compact set intersects only finitely many edge spaces, we conclude that there is another edge space \tilde{X}_f so that $N_k(\tilde{X}_e) \cap N_k(\tilde{X}_f)$ has infinite diameter. Now we have a contradiction to Lemma 4.4. □

Proof of Theorem 4.3. Let $q : \tilde{X} \rightarrow X$ be the universal covering space. We need to find a family of flats \mathcal{F} in \tilde{X} and the function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ so that for any two flats $F_1, F_2 \in \mathcal{F}$ we have $\text{diam}(N_k(F_1) \cap N_k(F_2)) \leq \phi(k)$.

A fixed connected component of $q^{-1}(X_v)$ is the universal cover of X_v and, by our assumption, has a family of flats \mathcal{F}_v and a function ϕ_v as in Definition 4.1. The family of flats for any other connected component of $q^{-1}(X_v)$ is just a translate of \mathcal{F}_v by an appropriate element of $\pi_1(X)$. It is therefore immediate that the function we seek in Definition 4.1(2) for these families will be $k \mapsto \phi_v(2k) + 2k$ (a point x of \tilde{X} within k of both $F_1 \subset \tilde{X}_v$ and $F_2 \subset \tilde{X}_v$ is within k of a point in \tilde{X}_v which is within $2k$ of both F_1 and F_2 – this point can be taken to be the first point of \tilde{X}_v on the shortest geodesic from x to F_1).

Let $\mathcal{F} = \{gF : g \in \pi_1(X), F \in \mathcal{F}_v, \text{ for some vertex } v \in \Gamma\}$. We will show that this family satisfies the requirements from the definition of the isolated flats property for the space \tilde{X} .

Claim. Every flat in \tilde{X} is contained in some vertex space.

Indeed, suppose there is a flat $F \subset \tilde{X}$, necessarily two-dimensional, which is not contained in any vertex space. Consider the intersection of F with the collection of edge spaces in \tilde{X} . This intersection consists of parallel lines in F . Let ℓ be one of these lines. Thus ℓ is an edge space \tilde{X}_e and one of the two complementary regions in F adjacent to ℓ is a strip or a half-plane in an extended G vertex space, violating (1) or (2). The claim is proved.

Let F_1 and F_2 be two flats in \mathcal{F} contained in two distinct vertex spaces \tilde{X}_v and \tilde{X}_w respectively. There is an edge path $\tilde{X}_{e_1}, \dots, \tilde{X}_{e_n}$ in \tilde{X} between \tilde{X}_v and \tilde{X}_w . If \tilde{X}_{e_i} is a segment, for some i , then

$$\text{diam}(N_k(F_1) \cap N_k(F_2)) \leq \text{diam}(N_k(\tilde{X}_{e_i})) \leq \text{diam}(\tilde{X}_{e_i}) + 2k.$$

We now assume that all edge spaces in this edge path are lines. According to the assumption (0) there are two cases.

Case 1. One of \tilde{X}_v or \tilde{X}_w , say \tilde{X}_v , is a G vertex. Let $e = e_1$. By Lemma 4.2 (which applies by (3)), there is a function ψ_e so that the diameter of $Q = N_{2k}(F_1) \cap N_{2k}(\tilde{X}_e \times \{0\}) \cap \tilde{X}_v$ is at most $\psi_e(2k)$. Note that $N_k(F_1) \cap N_k(F_2)$ is contained in the k -neighborhood of Q (if $x \in N_k(F_1) \cap N_k(F_2)$ is on the \tilde{X}_v side of $\tilde{X}_e \times \{0\}$ then the first point of intersection with $\tilde{X}_e \times \{0\}$ of the shortest geodesic joining x to F_2 is in Q , and similarly for the other side), hence $\text{diam}(N_k(F_1) \cap N_k(F_2)) \leq \psi_e(2k) + 2k$.

Case 2. One of the interior vertex spaces of the edge path is a G vertex. Say this is the vertex space \tilde{X}_u and the adjacent edge spaces are \tilde{X}_{e_i} and $\tilde{X}_{e_{i+1}}$, so that $\tilde{X}_{e_i} \subset \tilde{X}_u^+$ and $\tilde{X}_{e_{i+1}} \subset \tilde{X}_u^+$. By Lemma 4.5 we have that $Q = N_{2k}(\tilde{X}_{e_i}) \cap N_{2k}(\tilde{X}_{e_{i+1}}) \cap \tilde{X}_u^+$ has diameter at most $\psi'(2k)$. Now note that $N_k(F_1) \cap N_k(F_2)$ is contained in the k -neighborhood of Q , hence $\text{diam}(N_k(F_1) \cap N_k(F_2)) \leq \psi'(2k) + 2k$.

Hence, it suffices to define the function ϕ as

$$\phi(k) = \max\{\phi_v(2k) + 2k, \psi_e(2k) + 2k, \psi'(2k) + 2k, \text{diam}\tilde{X}_e + 2k\},$$

where the maximum is taken over all vertex spaces \tilde{X}_v , all inclusions of edge spaces which are lines into G vertex spaces, and all edge spaces \tilde{X}_e which are segments. \square

Now let us recall the setting. The group L is a limit group and it is embedded in an ω -rft $G = \pi_1(Y)$ of height $n > 0$. We view Y as a graph of spaces resulting from attaching the last stage; that is, either

- (1) $Y = M \sqcup \bigsqcup_t S_t^1 \times [0, 1] \sqcup N / \sim$ with N a hyperbolic surface, or
- (2) $Y = M \sqcup S^1 \times [0, 1] \sqcup N / \sim$ with N a torus, in which case the attaching circle in M generates a maximal abelian subgroup.

Let $p : X \rightarrow Y$ be the covering space with $\pi_1(X) = L$. According to Theorem 2.6, X contains a core $C \subset X$. The induced graph of spaces decomposition of C is a special graph of CAT(0) spaces, and there are two types of vertex spaces, the M -type and the N -type (these are in fact the cores of the covering spaces of M and N).

We will need several lemmas.

LEMMA 4.6. *Suppose a group G acts geometrically on a CAT(0) space with isolated flats (X, \mathcal{F}, ϕ) with \mathcal{F} G -invariant. Let g be an element of G that acts as a translation on a periodic geodesic $\ell \subset X$. If $\ell \parallel F$, for some $F \in \mathcal{F}$, then $g \in \text{Stab}(F)$ (and hence g does not generate a maximal abelian subgroup).*

Proof. Since ℓ is parallel to the flat F there is a neighborhood $N_k(F)$ that contains ℓ . It follows that $N_k(gF)$ contains $g\ell = \ell$, hence $N_k(F) \cap N_k(gF)$ contains ℓ . Since both F and gF belong to \mathcal{F} , Definition 4.1(2) of isolated flats implies $gF = F$, that is, $g \in \text{Stab}(F)$. □

LEMMA 4.7. *Suppose that N is a torus. No two distinct edge spaces incident to an M -type vertex space V in \tilde{C} cobound a strip in the associated extended vertex space V^+ .*

Proof. Suppose this is false and let ℓ_1, ℓ_2 be two distinct parallel lines that are edge spaces incident to V . Then ℓ_1, ℓ_2 are also edge spaces in \tilde{Y} contained in an extended vertex space $W^+ \supset V^+$. Since the graph of groups associated to Y has one edge and two vertices, there is $h \in \text{Stab}(W)$ with $h(\ell_1) = \ell_2$. Let $g_1 \in \pi_1(Y)$ be a primitive element with axis ℓ_1 . Thus $g_2 = hg_1h^{-1}$ is a primitive element with axis ℓ_2 . By Lemma 4.4 we conclude that $g_1^k = g_2^l$ for some k and l . That is to say,

$$g_1^k = hg_1^l h^{-1}.$$

This is an equation in $\text{Stab}(W)$, which is a limit group. It now follows that $k = l$ and $g_1h = hg_1$ (see Proposition 1.1(4), (5) – for the proof, if $g_1h \neq hg_1$ let $f : \text{Stab}(W) \rightarrow F$ be a homomorphism to a free group that does not kill $[g_1, h]$ and obtain a contradiction in F). However, h is not contained in the cyclic group generated by g_1 since it does not stabilize ℓ_1 . This contradicts the assumption that g_1 generates a maximal abelian group in $\text{Stab}(W)$. □

LEMMA 4.8. *Let $\ell \subset \tilde{C}$ be a periodic line in the preimage of the core in \tilde{Y} whose projection to Y is a loop that generates a maximal abelian subgroup of $\pi_1(Y)$. Then ℓ does not bound a half-flat in \tilde{C} .*

Proof. We argue by induction on the height of the tower.

Case 1. Suppose that ℓ transversely crosses an edge space. Of course, in this case N must be a torus or else it is clear that ℓ does not bound a half-flat. We will first argue that ℓ bounds at most two half-flats. Let S be the infinite strip associated with an edge space that ℓ crosses. The intersection $\ell \cap S$ is a segment and it separates S into two components, say S_1 and S_2 . Any half-flat bounded by ℓ must contain either S_1 or S_2 . If P, P' are two half-flats bounded by ℓ that both contain, say, S_1 , then $P \cap P'$ is a convex subset of P that contains $\ell \cup S_1$ and thus must equal P . This proves that $P = P'$ and that ℓ bounds at most two half-flats. Now suppose that ℓ bounds a half-flat P . Let g be an element that acts as a translation on ℓ . We now conclude that $g^2(P) = P$. In particular, $g^{2k}(S)$ are other strips associated to edge spaces and all are parallel to S . It follows that all edge spaces that intersect P are parallel to each other, and two consecutive edge spaces along ℓ that are incident to an M -type vertex space contradict Lemma 4.7.

Case 2. Suppose that ℓ is contained in a vertex space V . If V is an N -type vertex space then either N is a surface, and then ℓ cannot bound a half-flat, or N is a torus, and then the image loop does not generate a maximal abelian subgroup. So we may assume that V is an M -type vertex space. By induction, ℓ does not bound any half-flats in V . It remains to rule out half-flats P with $\partial P = \ell$ and P intersecting some edge spaces. These intersections must be lines parallel to ℓ . Let ℓ_1, ℓ_2, \dots be the (finite or infinite) sequence of lines of intersection between P and the edge spaces ordered according to distance from ℓ . Thus the strip S between ℓ and ℓ_1 is contained in V and the strip between ℓ_1 and ℓ_2 (or the half-flat in case there is no ℓ_2) is contained in the adjacent N -type vertex space W . Thus N is a torus. Let g_1 be a primitive element that translates along ℓ_1 . As in the proof of Lemma 4.4 we see that $g^k \in \langle g_1 \rangle$. It follows that g^k , and therefore g (see note below), belong to the non-cyclic abelian subgroup $\text{Stab}(W) \cong \pi_1(N)$, a contradiction. \square

NOTE. This is based on the property of limit groups that if a, b, c are non-trivial elements and $[a, b] = [b, c] = 1$ then $[a, c] = 1$; see Proposition 1.1(3).

THEOREM 4.9. *Limit groups act geometrically on CAT(0) spaces with the isolated flats property.*

Proof. Let L be a limit group and $L \subset G$ an embedding of L in an ω -rft G . Let n be the height of G . If $n = 0$ then L is the free product of free groups, free abelian groups, and hyperbolic surface groups, and the fact that such groups have isolated flats is straightforward from the definition. We will now assume $n > 0$ and that the theorem holds for limit groups that can be embedded in ω -residually free towers of height less than n .

Let $p : X \rightarrow Y, C \subset X$ be as in our setting in this section. We will verify the conditions of Theorem 4.3 for the graph of spaces decomposition of C inherited from X . If N is a hyperbolic surface then the N -type vertices cannot contain flats, half-flats or strips, so declare that N -type vertices are G vertices and M -type vertices are B vertices. Now assume that N is a torus. Then declare that M -type vertices are G vertices and N -type vertices are B vertices. That M -type vertices satisfy Theorem 4.3(1) and (2) follows from Lemmas 4.7 and 4.8. \square

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Appendix. Sketch of proof of Theorem 3.3

In this Appendix we outline a proof of Theorem 3.3 due to Sela. The reader is assumed to have some familiarity with limit groups, for example as in the expository papers [3], [5] and [14] as well as with the language of JSJ decompositions.

First, there is a basic fact about ω -residually free towers easily proved by induction on height.

LEMMA A.1. *Let Y be an ω -rft and A a non-cyclic abelian subgroup of $\pi_1(Y)$. Then A is conjugate into the fundamental group of a torus added at some stage in the construction of Y .*

It then follows that in the definition of an ω -rft we could add another building block

(T) (Torus block) $Y_n = Y_{n-1} \sqcup T^k \times [0, 1] \sqcup T^l / \sim$ where T^k and T^l are k - and l -tori, $k < l$, $T^k \times \{1\}$ is identified with a coordinate torus in T^l , and $T^k \times \{0\}$ is glued to Y_{n-1} by a map that is π_1 -injective and its image is a maximal abelian subgroup.

If $k = 1$ this is exactly building block (A) and if $k > 1$ then it follows from Lemma A.1 that the gluing map to Y_{n-1} can be taken to be an isomorphism to a previously added torus. Thus we could modify this earlier step and glue in the l -torus instead of the k -torus and the modified ω -rft Y'_{n-1} would be homotopy equivalent to Y_n .

Let L be a given limit group. We want to show that L embeds in some ω -rft.

An important fact about limit groups is that any sequence of epimorphisms $L_1 \rightarrow L_2 \rightarrow \dots$ eventually consists of isomorphisms. (For the proof, note that the sequence of algebraic varieties $\text{Hom}(L_1, SL_2(\mathbb{R})) \supset \text{Hom}(L_2, SL_2(\mathbb{R})) \supset \dots$ eventually consists of equalities.) We can thus use induction and assume that all proper limit group quotients of L embed in ω -residually free towers.

If $L = L_1 * L_2$ is a free product and $L_i \subset \Gamma_i$ ($i = 1, 2$) are embeddings in ω -residually free towers then $L = L_1 * L_2 \subset \Gamma_1 * \Gamma_2$ and the free product of ω -residually free towers is an ω -rft. Thus we can assume that L is freely indecomposable.

If L is abelian it is already an ω -rft. Otherwise, one carefully constructs a proper quotient $q : L \rightarrow L'$ with certain properties. This amounts to finding a 'strict quotient' (see [18, Proposition 5.9, Definition 5.10]). We state this result as it appears in [3, Definition 1.14, Theorem 1.25].

PROPOSITION A.2. *The class of limit groups coincides with the following hierarchy of groups (Constructible Limit Groups).*

Level 0 of the hierarchy consists of finitely generated free groups, finitely generated free abelian groups, and fundamental groups of closed surfaces that support pseudo-Anosov homeomorphisms.

*A group L belongs to level at most $n + 1$ if and only if either it has a free product decomposition $L = L_1 * L_2$ with L_1 and L_2 of level at most n or it has a homomorphism $\rho : L \rightarrow L'$ with L' of level at most n and it has a generalized abelian decomposition such that the following hold.*

- (i) *The homomorphism ρ is injective on the peripheral subgroup of each abelian vertex group.*
- (ii) *The homomorphism ρ is injective on each edge group E and at least one of the images of E in a vertex group of the one-edged splitting induced by E is a maximal abelian subgroup.*
- (iii) *The image of each QH -vertex group is a non-abelian subgroup of L' .*
- (iv) *For every rigid vertex group V , ρ is injective on the ‘envelope’ \tilde{V} of V , defined by first replacing each abelian vertex with the peripheral subgroup and then letting \tilde{V} be the subgroup of the resulting group generated by V and by the centralizers of incident edge groups.*

REMARK A.3. Sela’s proof that every limit group belongs to this hierarchy goes like this. We may assume that L is freely indecomposable. Consider the abelian JSJ decomposition of L . Let $f_i : L \rightarrow F$ be a sequence of homomorphisms to a free group so that each $1 \neq x \in L$ is killed by only finitely many f_i . Let g_i be obtained from f_i by conjugating and composing with the elements of the modular group of the JSJ decomposition so that the sum of the lengths of the images of fixed generators of L is as small as possible. A subsequence of the g_i converges to an action of L on an \mathbb{R} -tree and $q : L \rightarrow L'$ divides out the kernel of the action. One then checks the properties stated in Proposition A.2.

So now suppose that we have a freely indecomposable, non-abelian limit group L . Let $q : L \rightarrow L'$ be as above and let $i : L' \hookrightarrow \Gamma'$ be an embedding in an ω -rft Γ' . The idea is to construct an ω -rft Γ by attaching things to Γ' so that L embeds in Γ . This is explained in [19, Definition 1.12]. Rather than repeat the definition we illustrate the construction on examples that cover the cases when the JSJ decomposition has one edge. The reader can easily extrapolate the general case (or refer to [19, Definition 1.12]).

EXAMPLE A.4. Suppose that $L = A *_E B$ with A, B rigid. We have the composition $\nu = iq : L \rightarrow \Gamma'$ which embeds A and B and their envelopes. Let U be the maximal abelian subgroup of Γ' that contains $\nu(E)$. Form Γ as

$$\Gamma = \Gamma' *_U (U \times \mathbb{Z}).$$

This corresponds to step (T) of the construction of an ω -rft. Let t be a generator of \mathbb{Z} . Now define

$$j : L \rightarrow \Gamma$$

by $j|_A = \nu|_A$ and $j|_B = (\nu|_B)^t$ (that is, $j(b) = t\nu(b)t^{-1}$ for $b \in B$).

It is interesting to note that a special case of this construction was discovered by Baumslag [2] in 1962!

EXAMPLE A.5. Let $L = A *_E$ with A rigid. We again have $\nu : L \rightarrow \Gamma'$ and form Γ similarly

$$\Gamma = \Gamma' *_U (U \times \mathbb{Z}),$$

where U is the maximal abelian subgroup of Γ' that contains $\nu(E)$. Then define $j : L \rightarrow \Gamma$ by $j|_A = \nu|_A$ and j sends the ‘stable letter’ s to $t\nu(s)$ where t is a generator of \mathbb{Z} .

EXAMPLE A.6. Let $L = A *_E B$ with B rigid and A abelian. Then perform step (T) to the ω -rft Γ' by gluing an abelian group to the maximal abelian subgroup containing the image of E , and send A there.

EXAMPLE A.7. Let $L = A *_E B$ where B is rigid and A is a QH-vertex group and E corresponds to the boundary of the surface. Let $\nu : L \rightarrow \Gamma'$ be the composition $L \rightarrow L' \rightarrow \Gamma'$ as above and let $E' = \nu(E)$. Now apply step (Q) and form

$$\Gamma = \Gamma' *_E A',$$

where (A', E') is a copy of (A, E) . Define the embedding $j : L \rightarrow \Gamma$ by $j|_B = \nu|_B$ and $j : A \rightarrow A'$ is the identifying isomorphism.

REMARK A.8. We outlined a construction of an embedding $j : L \rightarrow \Gamma$ of a given limit group L into an ω -rft Γ . This embedding depends on the choice of a sequence of strict quotients defined on the free factors of L and the free factors of subsequent quotients ending in free groups. Such a sequence is called a *strict resolution* and $j : L \rightarrow \Gamma$ is the associated *completion*. Every limit group has a *canonical* finite collection of resolutions and each resolution gives rise to a completion $L \rightarrow \Gamma$ (which may not be an embedding if the resolution is not strict). Thus every limit group has a canonical finite collection of completions. The study of completions is the main topic of [19].

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