FIBERED KNOTS AND SPHERICAL SPACE FORMS

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1. *Introduction*

A smooth $(n-1)$-*knot* is a smooth submanifold K of S^{n+1} diffeomorphic to S^{n-1}. If $S^{n-1} \times D^2$ is a tubular neighborhood of K, then $X = S^{n+1} - S^{n-1} \times \mathring{D}^2$ is called the *exterior* of K. Two $(n-1)$-knots are *equivalent* if there is a diffeomorphism of S^{n+1} throwing one onto the other. For $n \geqslant 3$, a given exterior corresponds to at most two inequivalent knots, K and K^* [**4**, **1**, **10**] (and see §3). Examples of inequivalent knots with diffeomorphic exteriors were given by Cappell and Shaneson [**2**] for $n = 4, 5$ (and $n = 3$ in TOP), using fibered knots with fiber a punctured n-torus. Gordon [**7**] gave examples for $n = 3$, using twist-spun knots. In view of [**16**], Gordon's result can be rephrased as follows.

THEOREM [**7**]. *Let $K = (S^4, S^2)$ be a fibered knot with closed fiber covered by $\mathbb{R}^3$, and odd order monodromy. Then the knots K and K^* are inequivalent.*

The theorem applies to all k-twist spun torus knots with k odd and greater than 1, except for a few which have closed fiber a spherical space form S^3/π, with $\pi = Q(8)$ or $SL(2,5)$. We extend Gordon's obstruction theoretic method to the remaining cases (Proposition 3.1). Furthermore, we show that these knots are the only non-trivial ones with closed fiber a spherical space form and odd order monodromy (see the table in §5). This gives the following.

THEOREM 1.1. *Let $K = (S^4, S^2)$ be a non-trivial fibered knot with closed fiber covered by S^3, and odd order monodromy. Then the knots K and K^* are inequivalent.*

In particular, this answers the question of Zeeman [**29**, p. 494, problem 2]. The 5-twist spun trefoil is not determined by its exterior. Actually, using different methods, the first author has generalized the above theorems to *any* non-trivial fibered 2-knot with odd order monodromy [**20**]. However, the methods employed here seem better suited to the spherical space form case.

Gordon's theorem is also valid in higher dimensions but, in order for it to produce examples, one first has to find knots of S^{n-2} in S^n, $n \geqslant 4$, with aspherical cyclic branched covers. Whether such knots exist is unknown. Another approach is to try to use Cappell's and Shaneson's method to construct examples with $n > 5$, but this requires finding certain elusive matrices. In the light of Theorem 1.1, one is tempted to consider higher dimensional knots with fiber a punctured space form. However, this is ruled out by the following.

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THEOREM 1.2. *Let $K = (S^{n+1}, S^{n-1})$ be a fibered knot with closed fiber a spherical space form S^n/π. If $n > 3$, then π is cyclic and K is determined by its exterior.*

This paper is organized as follows. In §2 we study based homotopy equivalences of $\Sigma \times S^1$, for Σ a spherical space form S^3/π. We identify the image of the evaluation map ev: $\mathscr{E}_0(\Sigma \times S^1) \to \mathrm{Aut}(\pi \times \mathbb{Z})$ and study in detail ker(ev) in the cases we need ($\pi = Q(8)$ and $SL(2,5)$), via obstruction theory. Section 3 starts with a review of equivalences of knots and twist spun knots. We then prove Theorem 1.1 for a twist spun torus knot, following the lines of [7]. In §4 we determine the possible finite groups π acting freely on S^n, $n \geqslant 3$, with the property that $(S^n/\pi)^\circ$ appears as the fiber of a knot in S^{n+1}. This leads to the proof of Theorem 1.2. In §5 we determine all the fibered knots in S^4 with fiber $(S^3/\pi)^\circ$, expanding previous work by Hillman [8] and Yoshikawa [28]. This, in turn, finishes the proof of Theorem 1.1.

2. *Homotopy equivalences of $S^3/\pi \times S^1$*

To describe the based self-homotopy equivalences of $S^3/\pi \times S^1$, we first recall some facts, due to Olum, concerning homotopy equivalences of spherical space forms.

Let $\mathscr{E}_0(\Sigma)$ denote the based homotopy classes of based self-homotopy equivalences of Σ. There is a natural (evaluation) map ev: $\mathscr{E}_0(\Sigma) \to \mathrm{Aut}(\pi_1(\Sigma, *))$ given by

$$\mathrm{ev}(f) = f_*: \pi_1(\Sigma, *) \longrightarrow \pi_1(\Sigma, *).$$

If $\Sigma = S^n/\pi$, where π is a finite group of order m acting freely on S^n, for n odd and greater than 1, then $H_n(\pi; \mathbb{Z}) \cong \mathbb{Z}_m$.

THEOREM [14, 17]. *The map* ev: $\mathscr{E}_0(S^n/\pi) \to \mathrm{Aut}\,\pi$ *is a monomorphism, with*

$$\mathrm{image(ev)} = \{\varphi \in \mathrm{Aut}\,\pi \mid \varphi_*: H_n(\pi) \longrightarrow H_n(\pi) \text{ is } \pm 1\}.$$

Here $\mathrm{ev}(f)_* = \pm 1$ *according to whether f is orientation preserving or reversing.*

Now let $n = 3$ and assume that π acts linearly on S^3, that is, S^3/π is a spherical space form. Then $\Sigma = S^3/\pi$ is a Seifert fibered space (see [15]). If Σ is not a lens space, it does not admit orientation reversing homotopy equivalences [13]. Let $Z(\pi)$ denote the center of π.

PROPOSITION 2.1. *The map* ev: $\mathscr{E}_0(S^3/\pi \times S^1) \to \mathrm{Aut}(\pi \times \mathbb{Z})$ *has image*

$$\{\alpha \in \mathrm{Aut}\,\pi \mid \alpha_*: H_3(\pi) \longrightarrow H_3(\pi) \text{ is } \pm 1\} \times (Z(\pi) \rtimes \mathbb{Z}_2).$$

Proof. Write $\pi_1(\Sigma \times S^1) = \pi \times \mathbb{Z}$, with the generator t of $\mathbb{Z}$ represented by $* \times S^1$, and let h be a principal orbit of the S^1-action on Σ, so that h represents the generator of $Z(\pi)$. If $\varphi \in \mathrm{Aut}(\pi \times \mathbb{Z})$, then $\varphi(t) = t^{\pm 1}h^i$. From this we easily see that

$$\mathrm{Aut}(\pi \times \mathbb{Z}) \cong \mathrm{Aut}\,\pi \times (Z(\pi) \rtimes \mathbb{Z}_2).$$

The generator of $\mathbb{Z}_2$ $(t \to t^{-1})$ is geometrically realized by a reflection of S^1. The generator of $Z(\pi)$ $(t \to th)$ is realized by

$$f: \Sigma \times S^1 \longrightarrow \Sigma \times S^1, \quad f(x, \theta) = (\theta \cdot x, \theta),$$

where $\theta \cdot x$ denotes the S^1-action on Σ. If $f \in \mathscr{E}_0(\Sigma \times S^1)$, then the component of ev(f) lying in $\mathrm{Aut}\,\pi$ is exactly ev(f_1), where $f_1 \in \mathscr{E}_0(\Sigma)$ is the composition

$$\Sigma \longrightarrow \Sigma \times S^1 \xrightarrow{\ f\ } \Sigma \times S^1 \longrightarrow \Sigma.$$

By taking the identity on the S^1 factor, we see that image(ev) is as claimed.

We now analyse the kernel of the evaluation map. Let $f \in \ker(\mathrm{ev})$. Since $\pi_2(\Sigma \times S^1) = 0$, we can homotop f to the identity on the 2-skeleton. The obstruction to homotoping f to the identity on $(\Sigma \times S^1)^{(3)}$ is the class of the difference cocycle

$$[d^3(f, \mathrm{id})] \in H^3(\Sigma \times S^1; \pi_3(\Sigma \times S^1)) \cong H^3(\Sigma; \mathbb{Z}) \oplus H^2(\Sigma; \mathbb{Z}).$$

The first component of the obstruction is detected on the top cell of Σ, and is precisely the obstruction for a homotopy $f_1 \simeq \mathrm{id}_\Sigma$. As f_1 is a homotopy equivalence, a well-known degree argument shows that this obstruction vanishes (see the proof of the theorem). Let $\theta_1(f)$ denote the second component of the obstruction. If $\theta_1(f) = 0$, we meet a final obstruction on the top cell of $\Sigma \times S^1$,

$$\theta_2(f) \in H^4(\Sigma \times S^1; \pi_4(\Sigma \times S^1)) \cong H^3(\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We shall not try to identify ker(ev) for an arbitrary group π, although this seems to be an interesting question in its own right. Instead, we analyse the two cases we need, namely SL(2, 5) and Q(8).

The case $\pi = \mathrm{SL}(2,5)$

The spherical space form $S^3/\mathrm{SL}(2,5) = \Sigma(2,3,5) = \Sigma$ is the binary icosahedral space, also known as the Poincaré sphere. Note that SL(2, 5) is perfect; in fact, $\Sigma(2,3,5)$ is the only known homology 3-sphere with non-trivial finite fundamental group. It is well known that $\mathrm{Aut}(\mathrm{SL}(2,5)) \cong \mathscr{S}_5$, the symmetric group on five symbols. The center of π is $\mathbb{Z}_2$, and the inner-automorphisms are $\mathrm{SL}(2,5)/\{\pm 1\} = \mathrm{PSL}(2,5) \cong \mathrm{A}_5$. The non-trivial outer-automorphism can be given as conjugation by

$$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \in \mathrm{GL}(2,5).$$

The effect of this outer-automorphism on $H_3(\pi) \cong \mathbb{Z}_{120}$ is multiplication by 49. The inner-automorphisms, of course, act by multiplication by 1 on $H_3(\pi)$. Thus, $\mathscr{E}_0(\Sigma(2,3,5)) \cong \mathrm{A}_5$ (see [**17**, §5] for proofs).

PROPOSITION 2.2. (1) *There is a short exact sequence*

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathscr{E}_0(\Sigma(2,3,5) \times S^1) \xrightarrow{\mathrm{ev}} \mathrm{A}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1.$$

(2) *Let* $f \in \mathscr{E}_0(\Sigma(2,3,5) \times S^1)$ *be such that* $\mathrm{ev}(f) \in \mathrm{A}_5$, *and let* $\tilde{f}$ *denote the lift of* f *to a homotopy equivalence of* $S^3 \times S^1$. *Then* $\tilde{f} \simeq \mathrm{id}_{S^3 \times S^1}$.

Proof. (1) By the remarks above and Proposition 2.1, image(ev) is as claimed. Let $f \in \ker(\mathrm{ev})$. We saw that $f \simeq \mathrm{id}$ on the 2-skeleton. In fact, any two homotopies of f to the identity on $(\Sigma \times S^1)^{(2)}$ are homotopic. To see this, observe that we can find a homotopy on $(\Sigma \times S^1)^{(1)} \times I$ since $\pi_2(\Sigma \times S^1) = 0$, and the obstruction to a homotopy on $(\Sigma \times S^1)^{(2)} \times I$ lies in

$$H^3(\Sigma \times S^1 \times I, \Sigma \times S^1 \times \partial I; \pi_3(\Sigma \times S^1)) \cong H^2(\Sigma \times S^1; \mathbb{Z}) \cong H^1(\Sigma; \mathbb{Z}) \oplus H^2(\Sigma; \mathbb{Z}) = 0,$$

since π is perfect.

The obstruction $\theta_1(f) \in H^2(\Sigma; \mathbb{Z})$ to homotoping f to the identity on $(\Sigma \times S^1)^{(3)}$ vanishes, again since π is perfect. The obstruction on the top cell of $\Sigma \times S^1$, namely $\theta_2(f)$, is of course realizable by a homotopy equivalence—modify the identity in a small ball by the generator of $\pi_4(\Sigma \times S^1)$—whose square is homotopic to the identity. Since this measures the obstruction to deforming f to the identity, up to modifications of the homotopy on $(\Sigma \times S^1)^{(3)}$, and since all homotopies are homotopic on $(\dot{\Sigma} \times S^1)^{(2)}$, it is well defined. Hence, $\ker(\mathrm{ev}) = \mathbb{Z}_2$, proving (1).

(2) Suppose that $f \in \ker(\mathrm{ev})$, with obstruction $\theta_2(f) \in H^4(\Sigma \times S^1; \mathbb{Z}_2)$. Let

$$p\colon S^3 \times S^1 \longrightarrow \Sigma^3 \times S^1$$

be the cover of $\Sigma \times S^1$ corresponding to $\pi_1(S^1)$, and let $\tilde{f}$ be a lift of f. Then it is easy to see that the obstruction to homotoping $\tilde{f}$ to the identity is given by

$$\theta_2(\tilde{f}) = p^*(\theta_2(f)) \in H^4(S^3 \times S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Since π has even order, $p^* = 0$, so $\tilde{f} \simeq \mathrm{id}_{S^3 \times S^1}$. In general, if $\mathrm{ev}(f) \subset \mathrm{A}_5$, we can write $f = f_0 \circ (f_1 \times \mathrm{id}_{S^1})$, where $f_0 \in \ker(\mathrm{ev})$ and $f_1 \in \mathscr{E}_0(\Sigma)$, $\mathrm{ev}(f_1) = \mathrm{ev}(f)$. As f_1 is orientation preserving, $\tilde{f}_1 \simeq \mathrm{id}_{S^3}$ and thus

$$\widetilde{f_1 \times \mathrm{id}_{S^1}} = \tilde{f}_1 \times \mathrm{id}_{S^1} \simeq \mathrm{id}_{S^3 \times S^1}.$$

This completes the proof.

The case $\pi = \mathrm{Q}(8)$

The spherical space form $\Sigma = \Sigma(2, 3, 3)$ is the quaternion space $S^3/\mathrm{Q}(8)$, where $S^3 = \{\text{unit quaternions}\}$ and $\mathrm{Q}(8) = \{\pm 1, \pm i, \pm j, \pm k\}$. The double cover corresponding to the cyclic subgroup generated by i is the lens space $L(4, 1)$. We have the following covers.

$$\begin{tikzcd} S^3 \arrow[r, "p_1"] \arrow[rr, bend right, "p"'] & L(4,1) \arrow[r, "p_2"] & \Sigma \end{tikzcd}$$

A homology computation gives the following.

$$\begin{array}{ccc} H^2(\Sigma; \mathbb{Z}) & \xrightarrow{p_2^*} & H^2(L(4,1); \mathbb{Z}) \\ \| & & \| \\ H_1(\mathrm{Q}(8); \mathbb{Z}) & \xrightarrow{\mathrm{tr}} & H_1(\mathbb{Z}_4; \mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \twoheadrightarrow & \mathbb{Z}_2 \subset \mathbb{Z}_4 \end{array}$$

The lens space $L(4, 1)$ admits an S^1-action $(\theta, x) \to \theta \cdot x$ with a circle e^1 of fixed points (see Figure 1). Pick the base point $\bar{*} = e^0$ on that circle. The map

$$\bar{g} \in \mathscr{E}_0(L(4,1) \times S^1), \quad \bar{g}(x, \theta) = (\theta \cdot x, x)$$

rotates e^2 around e^3 once. Hence $\theta_1(\bar{g}) = 1 \in H^2(L(4, 1); \pi_3(L(4, 1))) \cong \mathbb{Z}_4$. We shall pick base-points $* = p_2(\bar{*}) \in \Sigma$ and $\tilde{*} \in p_1^{-1}(\bar{*}) \in S^3$.

PROPOSITION 2.3. (1) *There is a short exact sequence*

$$1 \longrightarrow \ker(\mathrm{ev}) \longrightarrow \mathscr{E}_0(\Sigma(2,3,3) \times S^1) \xrightarrow{\mathrm{ev}} \mathscr{S}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1.$$

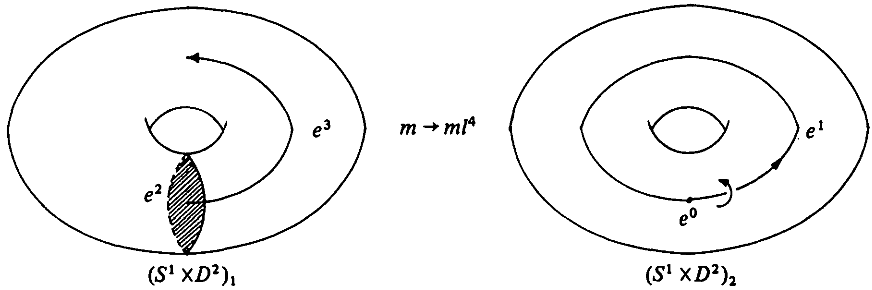

FIG. 1

(2) *Let* $f \in \mathscr{E}_0(\Sigma(2,3,3) \times S^1)$ *be such that* $\mathrm{ev}(f) \in \mathscr{S}_4$, *and let* $\tilde f$ *denote the lift of* f *to a homotopy equivalence of* $S^3 \times S^1$. *Then* $\tilde f \simeq \mathrm{id}_{S^3 \times S^1}$.

Proof. Every element of $\mathrm{Aut}(Q(8)) \cong \mathscr{S}_4$ induces the identity on $H_3(Q(8)) \cong \mathbb{Z}_8$ [**24**, Proposition 8.3]. Hence image(ev) is as claimed. To prove (2), we may assume, as in the proof of Proposition 2.2, that, $f \in \ker(\mathrm{ev})$. Pick lifts $\bar f$ and $\tilde f$ to $L(4,1) \times S^1$ and $S^3 \times S^1$, based at $\bar *$ and $\tilde *$ respectively. We have the following commutative diagram.

$$\begin{tikzcd}
S^3 \times S^1 \arrow[r, "\tilde f"] \arrow[d, "p_1 \times \mathrm{id}"] \arrow[dd, bend right=60, swap, "p \times \mathrm{id}"] & S^3 \times S^1 \arrow[d] \\
L(4,1) \times S^1 \arrow[r, "\bar f"] \arrow[d, "p_2 \times \mathrm{id}"] & L(4,1) \times S^1 \arrow[d] \\
\Sigma \times S^1 \arrow[r, "f"] & \Sigma \times S^1
\end{tikzcd}$$

As $\tilde f_* = 1$, we can homotop f to the identity on $(S^3 \times S^1)^{(3)}$. The remaining obstruction to homotoping $\tilde f$ to the identity is

$$\theta_2(\tilde f) \in H^3(S^3; \pi_4(S^3)) \cong \mathbb{Z}_2.$$

Case 1: $\theta_1(f) = 0$. In this case $\theta_2(\tilde f) = (p \times \mathrm{id})^*(\theta_2(f)) = 0$, since $\deg p = 8$. Hence $\tilde f \simeq \mathrm{id}$.

Case 2: $\theta_1(f) \neq 0$. Then $\theta_1(\bar f) = (p_2 \times \mathrm{id})^*(\theta_1(f)) = 0$ or $2 \in H^2(L(4,1); \mathbb{Z}) \cong \mathbb{Z}_4$.

(i) If $\theta_1(\bar f) = 0$, then $\theta_2(\tilde f) = (p_1 \times \mathrm{id})^*(\theta_2(\bar f)) = 0$, since $\deg p_1 = 4$. Hence $\tilde f \simeq \mathrm{id}$.

(ii) If $\theta_1(\bar f) = 2$, let $\bar h = \bar f \circ \bar g^{-2}$. Then, by additivity and naturality of the difference cocycle,

$$\theta_1(\bar h) = [d^3(\bar f \circ \bar g^{-2}, \bar f)] + [d^3(\bar f, \mathrm{id})] = \bar f_* \, \theta_1(\bar g^{-2}) + \theta_1(\bar f) = -2 + 2 = 0,$$

and thus, as in (i), $\tilde h \simeq \mathrm{id}$. Now $\tilde f = \tilde h \circ \tilde g^2 \simeq \tilde g^2$. As $\mathscr{E}_0(S^3 \times S^1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, it follows that $\tilde f \simeq \mathrm{id}$.

3. *Twist-spun torus knots*

We first review some notation and definitions. Let $K = (S^{n+1}, S^{n-1})$ be a knot with exterior $X = S^{n+1} - S^{n-1} \times \mathring{D}^2$. Given a gluing diffeomorphism

$$\gamma\colon S^{n-1} \times S^1 \longrightarrow S^{n-1} \times S^1,$$

the pair $K_\gamma = (S^{n-1} \times D_2 \bigcup_\gamma X, S^{n-1} \times 0)$ depends only on the pseudo-isotopy class of γ. The group of pseudo-isotopy classes of $S^{n-1}\times S^1$ is the same as the group of self-homotopy equivalences, and is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ [**4**, **1**, **10**]. The first two factors correspond to orientation-reversals of S^{n-1} and S^1 respectively, and the third to the 'twist' τ, defined by

$$\tau(x,\theta) = (\rho(\theta)\cdot x,\theta),$$

where $\rho(\theta)$ rotates S^{n-1} about its polar S^{n-3} through angle θ. Since the generators of the first two factors extend to diffeomorphisms of $(S^{n-1}\times D^2, S^{n-1}\times 0)$, we see that K_γ is equivalent to either $K = K_{\mathrm{id}}$ or $K^* = K_\tau$. Further, K and K^* are equivalent if, and only if, there is a diffeomorphism of X which restricts on ∂X to $\varepsilon\tau$, where ε belongs to the first two factors of $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.

If K is a knot of S^1 in S^3, and k is a positive integer, the *k-twist spin* of K is a knot $K^k = (S^4, S^2)$, with exterior $X^k = \mathring{M}_k\times_\sigma S^1$. Here M_k is the k-fold cyclic branched cover of (S^3, K), and $\mathring{M}_k = M_k$ minus an invariant ball. The monodromy σ is the canonical branched covering transformation ($2\pi/k$ rotation in a normal disk to the branch set) [**29**]. Gordon [**7**] showed that $S^2\times D^2\bigcup_\tau X^k\cong S^4$ and proved that, if $\tilde{M}_k\cong\mathbb{R}^3$ and k is odd, then $K^k\not\cong K^{k*}$. More generally, if $(s,k)=1$, and $K^{k,s}$ denotes the s-fold cyclic branched cover of the k-twist spin of K, Pao [**16**] showed that $K^{k,s}$ is a knot in S^4, and $S^2\times D^2\bigcup_\tau X^{k,s}\cong S^4$. Gordon's proof also gives $K^{k,s}\not\cong K^{k,s*}$, if $\tilde{M}_k\cong\mathbb{R}^3$ and k is odd.

Let $K_{p,q}$ be the (p,q)-torus knot, with $p,q>1$ and coprime. Its k-fold cyclic branched cover $M^k_{p,q}$ is well known to be diffeomorphic to the Brieskorn manifold

$$\Sigma(p,q,k) = \{(Z_0,Z_1,Z_2)\in\mathbb{C}^3\cap S^5\,|\,Z_0^p+Z_1^q+Z_2^k=0\}.$$

If $1/p+1/q+1/r\leqslant 1$, $\widetilde{\Sigma(p,q,k)}\cong\mathbb{R}^3$; otherwise, $\widetilde{\Sigma(p,q,k)}\cong S^3$. Gordon's theorem [**7**] shows that $K^{k,s}_{p,q}$ is not determined by its exterior if k is odd and greater than 1, $(s,k)=1$, and $(\{k,s\},\{p,q\})$ is not $(\{3,1\},\{2,3\})$, $(\{3,1\},\{2,5\})$, $(\{5,1\},\{2,3\})$ or $(\{5,2\},\{2,3\})$. The knot $K^3_{2,3}$ has fiber

$$\Sigma(2,3,3)^\circ = (S^3/Q(8))^\circ.$$

The knots $K^3_{2,5}$, $K^5_{2,3}$ (and also $K^2_{3,5}$) have fiber $\Sigma(2,3,5)^\circ\cong(S^3/\mathrm{SL}(2,5))^\circ$. Although they have the same fundamental group $\mathrm{SL}(2,5)\times\mathbb{Z}$, they have different π_2 (as $\mathbb{Z}\pi_1$-modules) [**6**]. The knots $K^5_{2,3}$ and $K^{5,2}_{2,3}$ have isomorphic π_1 and π_2 (as $\mathbb{Z}\pi_1$-modules), but their exteriors are not homotopy equivalent (rel ∂), although it is conceivable that they are homotopy equivalent [**18**].

PROPOSITION 3.1. *Let K be one of the knots $K^3_{2,3}$, $K^3_{2,5}$, $K^5_{2,3}$ or $K^{5,2}_{2,3}$. Then $K\not\cong K^*$.*

Proof. The knot K has exterior $X=\mathring{\Sigma}\times_\sigma S^1$, where Σ is $\Sigma(2,3,3)$ (or $\Sigma(2,3,5)$) and σ has odd order k ($=3$ or 5). If $K\cong K^*$, then there is a diffeomorphism of X restricting to $\varepsilon\tau\colon\partial X\to\partial X$. Now pass to the k-fold cyclic cover $\mathring{\Sigma}\times S^1$, obtaining a diffeomorphism

$$f_0\colon\mathring{\Sigma}\times S^1\longrightarrow\mathring{\Sigma}\times S^1,$$

restricting on $\partial(\mathring{\Sigma}\times S^1)$ to $\varepsilon\tau^k=\varepsilon\tau$ (since k is odd). Extend f_0 to a diffeomorphism f of $\Sigma\times S^1=\mathring{\Sigma}\times S^1\cup B^3\times S^1$ by taking $f|_{B^3\times S^1}$ to be the cone on $\varepsilon\tau$. By reversing the orientation of S^1 if necessary, we have that $f\in\mathscr{E}_0(\Sigma\times S^1)$, $\mathrm{ev}(f)\in\mathscr{S}_4$ (or A_5). Letting $\tilde{f}$ be a lift of f to $S^3\times S^1$, Proposition 2.3 (or 2.2) gives $\tilde{f}\simeq\mathrm{id}_{S^3\times S^1}$. Let B^3_+ be one lift of B^3 to S^3. Then $\tilde{f}|_{B^3_+\times S^3}$ is the cone on $\varepsilon\tau$. Writing $S^3=B^3_+\cup B^3_-$, it is easy to define a (radial) homotopy from $\tilde{f}$ to $\varepsilon'\tau'$, where ε', τ' correspond to ε, τ in one higher dimension. This gives $\varepsilon'\tau' \simeq \mathrm{id}_{S^3\times S^1}$; a contradiction.

REMARK. For any knot K in S^3, we have $K^2 \cong K^{2*}$ (Litherland [7, footnote, p. 595], Montesinos [12], Plotnick [19]).

4. *Proof of Theorem* 1.2

We wish to determine the fibered knots (S^{n+1}, S^{n-1}), n odd and $n \geqslant 3$, with closed fiber a spherical space form S^n/π. The exterior of such a knot is $(S^n/\pi)^\circ \times_\sigma S^1$. The monodromy σ induces an automorphism of π, call it also σ, with the following properties:

$$\pi/\langle g^{-1}\sigma(g)\rangle = \{1\} \text{ ('}\sigma\text{ kills the group')}, \tag{1}$$

$$\sigma_* - 1 \colon H_i(\pi) \longrightarrow H_i(\pi) \text{ is an isomorphism, } 0 < i < n, \tag{2_i}$$

$$\sigma_* \colon H_n(\pi) \longrightarrow H_n(\pi) \text{ is } +1. \tag{3}$$

Here $\langle\ \rangle$ denotes normal closure. Condition (2_1) is just the abelianization of condition (1). Condition (2) comes from the Wang sequence and condition (3) follows from Olum's theorem (see §2) and the fact that σ is orientation preserving. Notice that, if N is a normal subgroup with $\sigma(N) = N$, the induced automorphism $\bar{\sigma} \in \mathrm{Aut}(\pi/N)$ also satisfies (1).

PROPOSITION 4.1. *Let π be a finite group acting freely on S^n, n odd and $n \geqslant 3$, and $\sigma \in \mathrm{Aut}\,\pi$ satisfying conditions* (1) *and* (2). *Then $\pi = G \times \mathbb{Z}_m$, where $G = \{1\}$,* Q(8), *$T(k)$ or* SL(2, 5), *m is odd, and $(|G|, m) = 1$. Moreover, if π is not cyclic, $n = 3$.*

Proof. Finite groups acting freely on spheres are completely classified in six types (see [27, pp. 179, 195–198; 25]). Condition (2_1) permits us to delete from the list any groups with abelianization cyclic of even order. We are left with types I, II, III, V. We now decide what are the admissible groups in each type.

Type I.

$$\pi = \left(A, B \,\middle|\, \begin{matrix} A^{m_1} = B^{m_2} = 1 & m_1 \geqslant 1, m_2 \geqslant 1, (m_2(r-1), m_1) = 1 \\ BAB^{-1} = A^r & r^{m_2} \equiv 1 \quad (m_1) \end{matrix} \right).$$

The commutator subgroup is $\mathbb{Z}_{m_1} = (A | A^{m_1} = 1)$, and

$$H_1(\pi) = \pi/\mathbb{Z}_{m_1} \cong \mathbb{Z}_{m_2} = (B | B^{m_2} = 1).$$

The automorphism σ induces automorphisms $\sigma|_{\mathbb{Z}_{m_1}} = \cdot i$ and $\bar{\sigma} = \cdot k \in \mathrm{Aut}\,\mathbb{Z}_{m_2}$. By condition (2_1), $\bar{\sigma} - 1 = \cdot(k-1) \in \mathrm{Aut}\,\mathbb{Z}_{m_2}$. Hence σ has the form

$$\sigma \colon \begin{cases} A \longrightarrow A^i & (i, m_1) = 1, \\ B \longrightarrow B^k A^j & (k, m_2) = (k-1, m_2) = 1. \end{cases}$$

The condition $\sigma(BAB^{-1}) = \sigma(A^r)$ imposes $ir^k \equiv ir\ (m_1)$, whence $r \equiv 1\ (m_1)$. Therefore $\pi = \mathbb{Z}_{m_1 m_2}$.

Type II.

$$\pi = \left(A, B, R \;\middle|\; \begin{array}{ll} \text{As in I; also} & l_1^2 \equiv r^{l_2-1} \equiv 1\ (m_1) \\ R^2 = B^{m_2/2} & m_2 = 2^u v, u \geqslant 2, v \text{ odd} \\ RAR^{-1} = A^{l_1}, RBR^{-1} = B^{l_2} & l_2 = 2^u\mu - 1 \\ & l_2^2 \equiv 1\ (m_2) \end{array} \right).$$

The conditions $(m_2(r-1), m_1) = 1$, $r^{m_2} \equiv 1\ (m_1)$ imply that m_1 is odd, with $(m_1, m_2) = 1$. This, together with the above relations holding in π, easily show the subgroup $\mathbb{Z}_{m_1}$ generated by A to be characteristic. Let

$$G = \pi/\mathbb{Z}_{m_1} = (B, R \mid B^{m_2} = 1, R^2 = B^{m_2/2}, RBR^{-1} = B^{l_2}).$$

It is the semidirect product $\mathbb{Z}_{m_2/4} \rtimes Q(8)$, with $\mathbb{Z}_{m_2/4}$ generated by B^4 and Q(8) by $B^{m_2/4}$ and R. The elements of G are of the form $R^\varepsilon B^k$, $\varepsilon = 0$ or 1. If $v = 1$ and $u = 2$, $G \cong Q(8)$. Otherwise, the induced automorphism $\bar{\sigma} \in \operatorname{Aut} G$ takes B to B^k or RB^k. If $\bar{\sigma}(B) = B^k$, $\bar{\sigma}$ induces $\tilde{\sigma} \in \operatorname{Aut} Q(8)$, $\tilde{\sigma}(B) = B^k$, which does not kill Q(8). Hence $\bar{\sigma}(B) = RB^k$. From $\bar{\sigma}(B^4) = B^{2k(l_2+1)} = (B^4)^{2^{u-1}k\mu}$, we see that $(2^{u-1}k\mu, 2^{u-2}v) = 1$, which gives v odd, $u = 2$ and $(\mu, v) = 1$. From $(l_2+1, v) = 1$ and $l_2^2 \equiv 1\ (4v)$, we get $l_2 - 1 \equiv 0\ (v)$, which, together with $l_2 \equiv -1\ (4)$, implies that $l_2 = 2v+1$. This shows that $RB^4R^{-1} = B^4$ and thus $G \cong Q(8) \times \mathbb{Z}_v$.

The automorphism $\bar{\sigma}$ has the form

$$\bar{\sigma}: \begin{cases} B^4 \longrightarrow B^{4s}, & (s, v) = (s-1, v) = 1, \\ B^v \longrightarrow RB^v, & \\ R \longrightarrow B^v. & \end{cases}$$

(When $v = 1$, we could also have $B \to R$, $R \to RB^v$; we then continue in the same manner.) Hence σ has the form

$$\sigma: \begin{cases} A \longrightarrow A^{i_1}, & (i_1, m_1) = 1, \\ B^4 \longrightarrow B^{4s}A^{i_2}, & \\ B^v \longrightarrow RB^vA^{i_3}, & \\ R \longrightarrow B^vA^{i_4}. & \end{cases}$$

Since σ is a homomorphism and $B^vAB^{-v} = A^{r^v}$, $B^4AB^{-4} = A^{r^4}$, $RAR^{-1} = A^{l_1}$, we get $i_1 l_1 r^v \equiv i_1 r^v$, $i_1 r^{4s} \equiv i_1 r^4$, $i_1 r^v \equiv i_1 l_1\ (m_1)$. This, together with $(i_1, m_1) = 1$, $l_1^2 \equiv 1\ (m_1)$ and $(4(s-1), v) = 1$, yields $r \equiv l_1 \equiv 1\ (m_1)$. Therefore $\pi = Q(8) \times \mathbb{Z}_{m_1 v}$.

Type III.

$$\pi = \left(A, B, P, Q \;\middle|\; \begin{array}{ll} \text{As in I; also} & \\ P^2 = Q^2 = (PQ)^2 & m_2 \equiv 1\ (2) \\ AP = PA, AQ = QA & m_2 \equiv 0\ (3) \\ BPB^{-1} = Q, BQB^{-1} = PQ & \end{array} \right).$$

The subgroup $Q(8) = (P, Q \mid P^2 = Q^2 = (PQ)^2)$ is characteristic, and $\pi/Q(8) \cong \mathbb{Z}_{m_1} \rtimes \mathbb{Z}_{m_2}$ is of type I; hence $r = 1$. Write $m_2 = 3^k m$, $(m, 3) = 1$. As

$B^{3^k}PB^{-3^k} = P$, $B^{3^k}QB^{-3^k} = Q$, the subgroup $\mathbb{Z}_{mm_1}$ generated by $B^{3^k}A$ is central. If we let $T(k) = \mathrm{Q}(8) \rtimes \mathbb{Z}_{3^k}$ be the subgroup generated by P, Q and B^m, then $\pi = T(k) \times \mathbb{Z}_{mm_1}$.

Type V. $\pi = \mathrm{SL}(2,p) \times K$, p prime and at least 5, K of type I, $(|K|, |\mathrm{SL}(2,p)|) = 1$.

The group K has to be cyclic, by the discussion for type I. Since π acts freely on S^n, its cohomology period divides $n+1$. The l-Sylow subgroups of $\mathrm{SL}(2,p)$ are $\mathrm{Q}(2^k)$, for $l = 2$, and $\mathbb{Z}_l$, for $l \neq 2$, and

$$H^*(\mathrm{SL}(2,p)) = H^{4*}(\mathrm{Q}(2^k)) \oplus H^{(p-1)*}(\mathbb{Z}_p) \oplus \bigoplus_{l \neq p} H^{4*}(\mathbb{Z}_l),$$

where $H^{4*}(\mathrm{Q}(2^k)) = \mathbb{Z}_{2^k}$ [9]. This shows that $\mathrm{SL}(2,p)$ has cohomology period $p-1$ if $p \equiv 1$ (4) and $2(p-1)$ if $p \equiv 3$ (4). Therefore $(p-1) \mid n+1$ if $p \equiv 1$ (4) and $2(p-1) \mid n+1$ if $p \equiv 3$ (4). As $H_3(\pi) = H^4(\pi)$ is cyclic of even order, condition (2_3) forces $p \leqslant 5$. Hence $\pi = \mathrm{SL}(2,5) \times K$.

We have shown that $\pi = G \times \mathbb{Z}_m$, where $G = \{1\}$, $\mathrm{Q}(8)$, $T(k)$ or $\mathrm{SL}(2,5)$, and $(|G|, m) = 1$. By condition (2_1), m is odd. In particular, π is a 3-manifold group (see [15] or [27, p. 224]), and so $H_3(\pi) \cong \mathbb{Z}_{|\pi|}$. If π is not cyclic, π has even order, and condition (2_3) forces $n = 3$.

In order to finish the proof of Theorem 1.2, we analyse the knots with closed fiber a spherical space form $S^n/\mathbb{Z}_m$, $n = 2d-1 > 3$. These manifolds are generalized lens spaces $L(m; q_1, \ldots, q_d)$. Since lens spaces admit circle actions with codimension 2 fixed point set, they are 'spun' in the sense of [4, §17]. Therefore knots with fiber a punctured lens space are determined by their exterior.

REMARK. If a punctured lens space $L(m; q_1, \ldots, q_d)^\circ$ is the fiber of a fibered knot in S^{2d}, with monodromy σ, then $\sigma = \cdot r \in \operatorname{Aut} \mathbb{Z}_m$ satisfies conditions (1) to (3), namely, $(r,m) = 1$, $(r^i - 1, m) = 1$ for $1 \leqslant i < d$, $(r^d, m) = 1$. Ruberman [21] shows that this implies that $L(m; q_1, \ldots, q_d) = L(m; 1, r, \ldots, r^{d-1})$ if $d = 2l+1$ and $L(m; q_1, \ldots, q_d) = L(m; 1, r, \ldots, r^{l-1}, b, br, \ldots, br^{l-1})$, with $(b,m) = 1$, if $d = 2l$. Conversely, given $\sigma \in \operatorname{Aut} \mathbb{Z}_m$ satisfying the above conditions, one can find a fibered knot in S^{2d} with fiber a punctured lens space and monodromy σ [5, 21].

The conclusion that $\pi = G \times \mathbb{Z}_m$, with G and m as in Proposition 4.1, was reached by Hillman [8] for knots in S^4, under the (weaker?) assumption that the knot have finite commutator subgroup. These groups can be realized as the commutator subgroups of knots in S^4 by 2-twist spinning, except in the case when $\pi = \mathrm{Q}(8) \times \mathbb{Z}_m$, $m \neq 1$, when one only gets a knot in a homotopy 4-sphere [28]. In the next section we turn to the more delicate question of realizing all the possible pairs (π, σ).

5. *Realizing monodromies*

Let (S^4, S^2) be a fibered knot with closed fiber a spherical space form S^3/π. The monodromy σ satisfies conditions (1) and (3) of §4 (condition (2_1) follows from (1) and (2_2) from $H_2(\pi) = 0$). Given an orientation-preserving diffeomorphism σ of S^3/π satisfying (1), we can assume that σ fixes a small D^3; the manifold

$$\Sigma^4 = (S^3/\pi)^\circ \times_\sigma S^1 \cup S^2 \times D^2$$

is a homotopy 4-sphere (homeomorphic to S^4 by Freedman's work [3]). The pair (Σ^4, S^2) is a knot with fiber $(S^3/\pi)^\circ$ and monodromy σ. If σ is periodic, with $(S^3/\pi)/\sigma = S^3$, then σ will necessarily have a circle of fixed points, and we are just constructing a twist spun knot (or a branched cover) in S^4 (see [16]). By [26], a periodic diffeomorphism σ of S^3/π with a one-dimensional fixed point set is geometric, that is, preserves the Seifert structure. If σ reverses the orientation of the fibers, then it is well known that σ expresses S^3/π as the 2-fold cyclic branched cover of S^3 along a pretzel knot [11]. Otherwise, S^3/π is a cyclic branched cover of S^3 along a torus knot. All such possibilities will occur below.

PROPOSITION 5.1. *Given a pair* (π, σ), *with* π *a finite group acting linearly on* S^3 *and* $\sigma \in \operatorname{Aut}\pi$ *satisfying conditions* (1) *and* (3), *there is a fibered knot* (Σ^4, S^2), Σ^4 *homeomorphic to* S^4, *with fiber* $(S^3/\pi)^\circ$ *and monodromy* σ.

Proof. Suppose that $\sigma \in \operatorname{Aut}\pi$ can be realized as the monodromy of a knot with exterior $(S^3/\pi)^\circ \times_\sigma S^1$. Then so can $\mu\sigma^{\pm 1}\mu^{-1}$, for $\mu \in \operatorname{Inn}\pi$, and the resulting knot exteriors are diffeomorphic. Consequently, we shall only consider σ up to this equivalence relation. If $\varphi = \sigma \circ \mu_g$, $g \in \pi$, is another automorphism killing the group, then (π, φ) can be realized by a fibered knot (Σ^4, S^2) with exterior $S^3/\pi \times_\sigma S^1$-{neighborhood of tg}, where $t = * \times S^1$.

Now let (π, σ) be a pair as in the hypothesis. By Proposition 4.1 (or [8]), $\pi = G \times \mathbb{Z}_m$, $G = \{1\}$, Q(8), $T(k)$ or SL(2,5), m odd and $(|G|, m) = 1$. We write $\sigma = (\sigma|_G, \sigma|_{\mathbb{Z}_m})$. If $\sigma|_{\mathbb{Z}_m} = \cdot r$, then $(r-1, m) = 1$, by condition (1), and $r^2 \equiv 1\ (m)$, by condition (3). Hence $\sigma|_{\mathbb{Z}_m} = -1$. We now analyse each type.

Type I. $\pi = \mathbb{Z}_m = (A \mid A^m = 1, m \text{ odd})$, $\sigma(A) = A^{-1}$. This is realized by 2-twist spinning a certain 2-bridge knot [23].

Type II. $\pi = \mathrm{Q}(8) \times \mathbb{Z}_m = (B, R \mid B^2 = R^2 = (BR)^2) \times (A \mid A^m = 1,\ m \text{ odd})$. Any automorphism of Q(8) which kills it is equivalent to ψ: $B \to R$, $R \to RB$, of order 3. The pair $(\mathrm{Q}(8), \psi)$ can be realized by 3-twist spinning the trefoil knot. The automorphism $(\psi, -1) \in \operatorname{Aut}(\mathrm{Q}(8) \times \mathbb{Z}_m)$, $m \neq 1$, is realizable by a diffeomorphism of S^3/π [22], and thus by a knot in a Σ^4.

The manifold S^3/π is Seifert fibered over S^2, with invariants $(b; (2,1), (2,1), (2,1))$, where $m = |2b+3|$ (see [15]). The group π has a Seifert presentation

$$\pi = (q_1, q_2, q_3, h \mid q_1^2 h = q_2^2 h = q_3^2 h = q_1 q_2 q_3 h^{-b} = [q_i, h] = 1),$$

where $q_1 = B^{-1}RA^{-1}$, $q_2 = BA^{-1}$, $q_3 = RA^{-1}$, $h = B^2A^2$. The automorphism $(\psi, -1)$ translates to $q_1 \to q_3^{-1}q_2^{-1}q_3 h^m$, $q_2 \to q_3^{-1}h^m$, $q_3 \to q_1^{-1}$, $h \to h^{-1}$. Notice that if $m = 1$ $(b = -1, \pi = \mathrm{Q}(8))$, then $h^2 = 1$, and $(\psi, -1) = \psi$ can be simplified to $q_1 \to q_3^{-1}q_2 q_3$, $q_2 \to q_3$, $q_3 \to q_3^{-1}q_1 q_3$, $h \to h$. Arrange the three (2, 1) fibers to lie equally spaced on the equator, and put the $(1, -1)$ fiber at the north pole. Then ψ is geometrically realized by a map of period three given in the complement of these fibers by a $2\pi/3$ polar rotation of S^2 and a $2\pi/3$ rotation of the S^1 fibers. One checks that this extends to a map of period three on $S^3/\mathrm{Q}(8)$, the (2, 1) fibers produce one (2, 1) fiber in the quotient, and the (regular) fiber over the south pole produces a (3, 1) fiber. The map is branched along the $(1, -1)$ fiber, which descends to a regular fiber, and we have the 3-fold cyclic branched cover of S^3 along the trefoil knot. When $m \neq 1$, $(\psi, -1)$ has order 6, but cannot be realized by a periodic diffeomorphism.

Type III.

$$\pi = T(k)\times\mathbb{Z}_m = \left(P,Q,B \,\middle|\, \begin{matrix} P^2=Q^2=(PQ)^2, B^{3^k}=1 \\ BPB^{-1}=Q, BQB^{-1}=PQ \end{matrix}\right)\times(A\,|\,A^m=1,(m,6)=1).$$

By conditions (1) and (3), the automorphism $(\sigma|_{T(k)})_*$ of

$$H_1(T(k))\cong\mathbb{Z}_{3^k}=(B\,|\,B^{3^k}=1)$$

is multiplication by -1. Using the presentation of $\mathrm{Aut}(T(k))$ given in **[8]**, we deduce that $\sigma|_{T(k)}$ belongs to the subgroup of $\mathrm{Aut}(T(k))$ generated by μ_P, μ_Q, μ_B and $\rho\colon P\to Q^{-1}, Q\to P^{-1}, B\to B^{-1}$. It is now easy to see that $\sigma|_{T(k)}$ is conjugate (by inner automorphisms) to either the involution ρ or the order 4 automorphism

$$\varphi=\rho\mu_{PB}\colon P\longrightarrow P,\ Q\longrightarrow PQ,\ B\longrightarrow B^{-1}P.$$

(The possible automorphism φ was overlooked in **[8]**.)

The pairs $(T(1),\rho)$ and $(T(1),\varphi)$ can be realized by 2-twist spinning the $(3,4)$ torus knot, respectively 4-twist spinning the trefoil knot. The pairs $(T(k),\rho)$ and $(T(k)\times\mathbb{Z}_m,(\rho,-1))$ can be realized by 2-twist spinning certain pretzel knots **[11, 28]**. As $\varphi=\rho\mu_{PB}$ and $(\varphi,-1)=(\rho,-1)\circ\mu_{PB}$, the pairs $(T(k),\varphi)$ and $(T(k)\times\mathbb{Z}_m,(\varphi,-1))$ can be realized by knots in a Σ^4.

The manifold $S^3/T(k)\times\mathbb{Z}_m$ is Seifert fibered over S^2, with invariants $(b;(2,1),(3,1),(3,1))$, where $m=|6b+7|$, if $k=1$, and $(b;(2,1),(3,1),(3,2))$, where $3^km=|6b+9|$, if $k\neq1$. When $k=1$,

$$\pi=(q_1,q_2,q_3,h\,|\,q_1^2h=q_2^3h=q_3^3h=q_1q_2q_3h^{-b}=[q_i,h]=1),$$

where $q_1=PA^{3b+3}$, $q_2=P^{-1}B^{-1}A^{2b+2}$, $q_3=P^2B^{-1}A^{2b+2}$, $h=P^2A$ (or $h=P^2$ if $m=1$). The automorphism $(\varphi,-1)$ translates to $q_1\to q_1h^{6b+8}$, $q_2\to q_3h^{8b+10}$, $q_3\to q_1^{-1}q_2q_1h^{8b+10}$, $h\to h^{-1}$. If $m=1$ ($b=-1$, $\pi=T(1)$), $h^2=1$ and $(\varphi,-1)=\varphi$ simplifies to $q_1\to q_1$, $q_2\to q_3$, $q_3\to q_1^{-1}q_2q_1$, $h\to h$. Arrange the two $(3,1)$ fibers to lie on the equator of S^2, put the $(2,1)$ fiber at the north pole and the $(1,-1)$ fiber at the south pole. Then φ is geometrically realizable by a map of period four which rotates S^2 by π, and rotates the S^1 fibers by $\frac{1}{2}\pi$. The two $(3,1)$ fibers produce one $(3,1)$ fiber in the quotient, the $(1,-1)$ fiber produces a $(2,1)$ fiber, and the map is branched along the $(2,1)$ fiber, which descends to a regular fiber. Thus we have the 4-fold cyclic branched cover of S^3 along the trefoil knot. If $m\neq1$, $(\varphi,-1)$ has order four, but cannot be realized by a periodic map.

When $k\neq1$,

$$\pi=(q_1,q_2,q_3,h\,|\,q_1^2h=q_2^3h=q_3^3h^2=q_1q_2q_3h^{-b}=[q_i,h]=1),$$

where $q_1=P^{-1}h^{b+1}$, $q_2=P^{-1}B^{-1}A^i$, $q_3=B^{-2}A^{-i-1}$, $h=P^2B^3A$ (or $h=P^2B^3$ if $m=1$), where, if $m=3j\pm1$, then $i=\pm j$. The automorphism $(\varphi,-1)$ translates to $q_1\to q_1h^{-2b-2}$, $q_2\to q_3h$, $q_3\to q_1^{-1}q_2q_1h$, $h\to h^{-1}$, but cannot be realized by a periodic map.

Type V. $\pi=\mathrm{SL}(2,5)\times\mathbb{Z}_m$, $(m,30)=1$. First consider the case when $m=1$. Then

$$\pi=\mathrm{SL}(2,5)=(q_1,q_2,q_3\,|\,q_1^2=q_2^3=q_3^5=q_1q_2q_3),$$

the fundamental group of the Seifert manifold $\Sigma(2,3,5)=(-1;(2,1),(3,1),(5,1))$. Given $\sigma\in\mathrm{Aut}\,\pi$ with $\sigma_*=1\colon H_3(\pi)\to H_3(\pi)$, the result in **[17]** mentioned in §2 shows

σ to be inner. As the identity does not kill the group, we are left with four conjugacy classes (in Inn(SL(2, 5)) $\cong A_5$), represented by conjugation by q_1, q_2, q_3 and q_3^2 (q_3 and q_3^2 are conjugate in Aut(SL(2, 5)) $\cong \mathscr{S}_5$, but not in A_5). These automorphisms are realized by $K^2_{3,5}$, $K^3_{2,5}$, $K^5_{2,3}$ and $K^{5,2}_{2,3}$ **[18]**.

For $\pi = \mathrm{SL}(2,5) \times \mathbb{Z}_m$, the automorphism σ is of the form $(\mu_g, -1)$, where g is q_1, q_2, q_3 or q_3^2. The involution $(\mu_{q_1}, -1)$ can be realized by 2-twist spinning a certain pretzel knot **[11, 28]**. The automorphisms $(\mu_g, -1) = (\mu_{q_1}, -1) \circ \mu_{q_1^{-1}g}$ ($g = q_2$, q_3 or q_3^2) can be realized by knots in a Σ^4. One can translate these automorphisms in terms of the Seifert presentation of π corresponding to $S^3/\pi = (b; (2,1), (3,1), (5,1))$, where $m = |30b+31|$. As before, the diffeomorphisms realizing these automorphisms (of order 6, 10 and 10) are not periodic.

The information gained in proving Propositions 5.1 and 3.1 is summarized in Table 1.

TABLE 1

π	σ	order of σ	realized by	is $K \cong K^*$?
$\mathbb{Z}_m$	-1	2	K^2, K = 2-bridge knot	yes
Q(8)	ψ	3	$K^3_{2,3}$	no
Q(8) $\times \mathbb{Z}_m$	$(\psi, -1)$	6	knot in Σ^4	?
$T(1)$	ρ	2	$K^2_{3,4}$	yes
	φ	4	$K^4_{2,3}$	?
$T(k)$	ρ	2	K^2, K = pretzel knot	yes
	φ	4	knot in Σ^4	?
$T(k) \times \mathbb{Z}_m$	$(\rho, -1)$	2	K^2, K = pretzel knot	yes
	$(\varphi, -1)$	4	knot in Σ^4	?
SL(2, 5)	μ_{q_1}	2	$K^2_{3,5}$	yes
	μ_{q_2}	3	$K^3_{2,5}$	no
	μ_{q_3}	5	$K^5_{2,3}$	no
	$\mu_{q_3^2}$	5	$K^{5,2}_{2,3}$	no
SL(2, 5) $\times \mathbb{Z}_m$	$(\mu_{q_1}, -1)$	2	K^2, K = pretzel knot	yes
	$(\mu_{q_2}, -1)$	6	knot in Σ^4	?
	$(\mu_{q_3}, -1)$	10	knot in Σ^4	?
	$(\mu_{q_3^2}, -1)$	10	knot in Σ^4	?

We can complete now the proof of Theorem 1.1. Let K be a non-trivial fibered 2-knot, with fiber $(S^3/\pi)^\circ$ and odd order monodromy. From Table 1, we see that π = Q(8) or SL(2, 5). If π = Q(8), the π acts linearly on S^3 **[22]**, and thus Proposition 2.3 (2) holds. If π = SL(2, 5), it is not known whether π acts linearly, but the proof of Proposition 2.2 (2) does not require that. Now the proof of Proposition 3.1 shows that $K \not\cong K^*$.

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