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KLEINIAN GROUPS WITH DISCRETE LENGTH SPECTRUM

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Abstract

We characterize finitely generated torsion-free Kleinian groups whose real length spectrum (without multiplicities) is discrete.

Given a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}_+(\mathbb{H}^3)$, we define the *real length spectrum of* Γ *without multiplicities* to be the set of all real translation lengths of elements of Γ acting on $\mathbb{H}^3$

$$\mathfrak{L}(\Gamma) = \{l(\gamma) \mid \gamma \in \Gamma\} \subset \mathbb{R}.$$

Equivalently, $\mathfrak{L}(\Gamma) - \{0\}$ is the set of all lengths of closed geodesics in the quotient orbifold $\mathbb{H}^3/\Gamma$. Kim [**8**] has shown that if G is the fundamental group of a closed surface and $\Lambda \subset \mathbb{R}$ is discrete (that is, has no accumulation points in $\mathbb{R}$), then there exist only finitely many Kleinian groups, up to conjugacy, which are isomorphic to G and have real length spectrum Λ.

In this paper we characterize finitely generated, torsion-free Kleinian groups whose real length spectrum is discrete. In the process, we obtain a sharper version of the Covering Theorem which may be of independent interest. As a corollary of our main result and this new Covering Theorem, we see that geometrically infinite hyperbolic 3-manifolds have discrete length spectrum if and only if they arise as covers of finite volume hyperbolic orbifolds. Our work can thus be viewed as part of a family of results, including the Covering Theorem, which exhibit the very special nature of those geometrically infinite hyperbolic 3-manifolds with finitely generated fundamental group which cover finite volume hyperbolic orbifolds.

Main Theorem. *Suppose that* Γ *is a finitely generated, torsion-free Kleinian group. Then the real length spectrum* $\mathfrak{L}(\Gamma)$ *is discrete if and only if*

(1) Γ *is geometrically finite, or*
(2) *there exists a hyperbolic* 3*-manifold* $M = \mathbb{H}^3/G$ *which fibers over the circle and* Γ *is a fiber subgroup of* G*, or*
(3) *there exists a hyperbolic* 3*-manifold* $M = \mathbb{H}^3/G$ *which fibers over* $S^1/\langle z \mapsto \overline{z}\rangle$ *and* Γ *is a singular fiber subgroup of* G*.*

We recall that if a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ fibers over a 1-orbifold Q, then a fiber subgroup of G is the fundamental group of the pre-image of a regular point of Q, while a singular fiber subgroup is the fundamental group of the pre-image of a singular point of Q. (Notice that one only really obtains a conjugacy class of subgroups.)

Proof of Main Theorem. We refer the reader to [**3**] and [**4**] for details and terminology. Let $N = \mathbb{H}^3/\Gamma$, let $\epsilon > 0$ be less than the 3-dimensional Margulis constant, and let N^0_ϵ denote

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the complement of the ϵ-cuspidal thin part of N. By the Tameness Theorem (see Agol [**1**] or Calegari–Gabai [**2**]), N is topologically tame.

It is well known that a geometrically finite Kleinian group has discrete real length spectrum. We sketch a brief proof for the reader's convenience. If Γ is geometrically finite, then there exists a compact subset K of $N = \mathbb{H}^3/\Gamma$ such that any closed geodesic intersects K. (One may, for example, take K to be the intersection of the convex core of N with N_ϵ^0.) If $\mathfrak{L}(\Gamma)$ were indiscrete, then there would exist an infinite collection of distinct closed geodesics of length at most L (for some $L > 0$), all of which would intersect K. If this were the case, some sequence of closed geodesics in N would have to accumulate at a closed geodesic, which is impossible. Therefore, $\mathfrak{L}(\Gamma)$ is discrete.

Since a fiber subgroup, as described in (2), or a singular fiber subgroup, as described in (3), is a subgroup of a geometrically finite Kleinian group, they must also have discrete real length spectrum.

We now assume that Γ is a geometrically infinite Kleinian group with discrete real length spectrum and prove that Γ is either a fiber subgroup or a singular fiber subgroup of the fundamental group of a finite volume hyperbolic 3-manifold which fibers over a 1-orbifold. Let $U \cong \overline{S} \times [0,\infty) \subset N_\epsilon^0$ be a neighborhood of a geometrically infinite end of N_ϵ^0 and let $U_p \cong S \times [0,\infty) \subset N$ be its parabolic extension. By the main result of [**3**], there exists a sequence of useful simplicial hyperbolic surfaces $f_n : X_n \to U_p$, each properly homotopic in U_p to $S \times \{0\}$, which exits the end.

As the length spectrum of Γ is discrete, there is a uniform positive lower bound on the length of any closed geodesic in N. Therefore, there is a uniform positive lower bound on the length of any closed geodesic on X_n whose image under f_n is essential and not an accidental parabolic (that is, not homotopic into a cusp of N). On the other hand, [**3**, Lemma 7.1] guarantees that given any $A > 0$, there exists $n_A > 0$ so that for all $n \geqslant n_A$, any closed geodesic in X_n which is inessential or accidentally parabolic under f_n must have length at least A on X_n. Therefore, there is a uniform lower bound on the length of a closed geodesic on any X_n.

Then by [**4**], Lemma 7.1 there exists $C > 0$ and, for all n, a point x_n in the ϵ-thick part of X_n and a minimal generating set $\{a_1^n, \dots, a_m^n\}$ for $\pi_1(X_n, x_n)$ all of which are represented by curves (based at x_n) of length at most C; see [**4**] for definitions. By passing to a subsequence, we may assume that all the minimal generating sets are topologically equivalent. Therefore, there exist markings $g_n : S \to X_n$ so that if $s \in \pi_1(S)$, then $(g_n)_*(s)$ has a representative in X_n having length bounded independent of n (but not independent of s). Composing, we obtain a sequence of representations

$$\rho_n = (f_n \circ g_n)_* : \pi_1(S) \to \Gamma \subset \mathrm{PSL}_2(\mathbb{C}).$$

A result of Kim [**7**] shows that there exists a finite set $\{s_1, \dots, s_k\}$ of elements of $\pi_1(S)$ such that an irreducible representation of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{C})$ is determined up to conjugacy by the real translation lengths of their images. Since, for each i, each $(g_n)_*(s_i)$ has a representative of uniformly bounded length, each real translation length $l(\rho_n(s_i))$ is also uniformly bounded. Since $\mathfrak{L}(\Gamma)$ is discrete, we may pass to a subsequence, again called $\{\rho_n\}$, so that $l(\rho_n(s_i))$ is constant for all i. Therefore, all representations in this sequence are conjugate to ρ_1. Let $h_n \in \mathrm{PSL}_2(\mathbb{C})$ be the conjugating element, so that $\rho_n = h_n \rho_1 h_n^{-1}$.

We now observe that U_p is incompressible (that is, ρ_1 is injective). First note that, since all the ρ_n are conjugate, if $s \in \mathrm{Ker}(\rho_1)$ is non-trivial, then $s \in \mathrm{Ker}(\rho_n)$ for all n, and $(g_n)_*(s)$ has a representative of uniformly bounded length in X_n. However, since $f_n(g_n(s))$ is inessential, [**3**, Lemma 7.1] again guarantees that $\{l_{X_n}(g_n(s))\}$ converges to ∞, which is a contradiction.

Finally, consider the cover $\widetilde{N}_U$ of N corresponding to $\Gamma_U = \pi_1(U)$. The conjugating elements h_n descend to isometries $\widetilde{h}_n$ of $\widetilde{N}_U$. Let $\widetilde{f}_n : X_n \to \widetilde{N}_U$ denote the lift of f_n to $\widetilde{N}_U$ and let $\widetilde{x}_n = \widetilde{f}_n(x_n)$. Note that there exists $D > 0$ so that if $x \in \widetilde{N}_U$ and there exist representatives of $\{\widetilde{f}_1(a_1^1), \ldots, \widetilde{f}_1(a_m^1)\}$ based at x of length at most C, then $d(x, \widetilde{x}_1) \leqslant D$. It then follows that $d(\widetilde{x}_n, \widetilde{h}_n(\widetilde{x}_1)) \leqslant D$ for all n. Since $\{f_n(X_n)\}$ exits U, one sees that $\widetilde{h}_n(U) \cap U$ is non-empty for all large enough n, and that there are infinitely many distinct $\widetilde{h}_n$.

Let N_1 be the quotient of $\widetilde{N}_U$ by its full group of orientation-preserving isometries. We have established that the covering map $\pi_U : \widetilde{N}_U \to N_1$ is infinite-to-one on U.

We now recall the Covering Theorem (see [**9**, Theorem 9.2.2], [**4**] and Agol [**1**]).

Covering Theorem. *Suppose that $\widehat{M}$ is a hyperbolic 3-manifold with finitely generated fundamental group and M is a hyperbolic 3-orbifold. If $p : \widehat{M} \to M$ is an orbifold cover which is infinite-to-one on a neighborhood U of a geometrically infinite end of $\widehat{M}_\epsilon^0$, then M has finite volume and has a finite manifold cover M' which fibers over the circle such that either*

(a) *$\widehat{M}$ is the cover of M' associated to the fiber, or*
(b) *$\widehat{M}_\epsilon^0$ is a twisted $\mathbb{R}$-bundle and $\widehat{M}$ is double covered by the manifold $\widehat{M}'$ which is the cover of M' associated to the fiber.*

The Covering Theorem immediately implies that Γ_U is the fiber group associated to a 3-manifold which fibers over the circle. If $\Gamma = \Gamma_U$, we are done. If not, we apply the Covering Theorem again to the map $p_U : \widetilde{N}_U \to N$ to conclude that p_U is finite-to-one. Hempel's Finite Index Theorem [**5**, Theorem 10.5] implies that N_ϵ^0 is a twisted $\mathbb{R}$-bundle and that the cover p_U is two-to-one.

The following addendum to the Covering Theorem allows us to conclude that Γ is a singular fiber subgroup, in the case that N_ϵ^0 is a twisted $\mathbb{R}$-bundle.

Proposition 1. *Suppose that $\widehat{M}$ is a geometrically infinite hyperbolic 3-manifold such that $\widehat{M}_\epsilon^0$ is a twisted $\mathbb{R}$-bundle over a compact surface, and let $p' : \widehat{M}' \to \widehat{M}$ be a two-fold cover such that $(\widehat{M}')_\epsilon^0$ is an untwisted $\mathbb{R}$-bundle.*

(1) *If there is a finite volume hyperbolic 3-manifold M'' that fibers over the circle with covering $q' : \widehat{M}' \to M''$ corresponding to the fiber, then $\widehat{M}$ covers a finite volume hyperbolic orbifold M.*
(2) *If $p : \widehat{M} \to M$ is a cover of a finite volume hyperbolic orbifold M, then M has a finite manifold cover $M' = \mathbb{H}^3/G'$ which fibers over the orbifold $S^1/\langle z \mapsto \overline{z}\rangle$ and $\widehat{M}$ is the cover associated to a singular fiber subgroup of G'.*

We now turn to the proof of Proposition 1 and refer the reader to Figure 1 for convenience. For the first statement, we note that the covers $p' : \widehat{M}' \to \widehat{M}$ and $q' : \widehat{M}' \to M''$ are regular covers with covering groups generated by an involution $a \in \mathrm{Isom}_+(\widehat{M}')$ and an infinite order isometry $b \in \mathrm{Isom}_+(\widehat{M}')$, respectively. Since $\mathrm{Isom}_+(\widehat{M}')$ acts properly discontinuously on $\widehat{M}'$ and $\langle b\rangle$ has quotient M'' of finite volume, the covering $\hat{p} : \widehat{M}' \to M = \widehat{M}'/\mathrm{Isom}_+(\widehat{M}')$ is onto the finite volume orbifold M. Since $\widehat{M}$ is the quotient of $\widehat{M}'$ by the subgroup $\langle a\rangle < \mathrm{Isom}_+(\widehat{M}')$, it follows that there is a covering $p : \widehat{M} \to M$ so that $\hat{p} = p \circ p'$, verifying the first statement.

For the second assertion, define $\hat{p} = p \circ p'$. The Covering Theorem, applied to $\hat{p} : \widehat{M}' \to M$, implies that M has a finite manifold cover M'' which fibers over the circle so that $\widehat{M}'$ is the cover associated to the fiber. Moreover, examining the proof of the Covering Theorem, we see that if $q : M'' \to M$ and $q' : \widehat{M}' \to M''$ are the associated covering maps, we may assume that $q \circ q' = \hat{p} = p \circ p'$.

As above, the covers $p' : \widehat{M}' \to \widehat{M}$ and $q' : \widehat{M}' \to M''$ are regular with covering group of order two, generated by $a \in \mathrm{Isom}_+(\widehat{M}')$, and infinite cyclic covering group, generated by $b \in \mathrm{Isom}_+(\widehat{M}')$, respectively. Since $\langle b\rangle$ has finite volume quotient M'' and $\mathrm{Isom}_+(\widehat{M}')$ acts properly

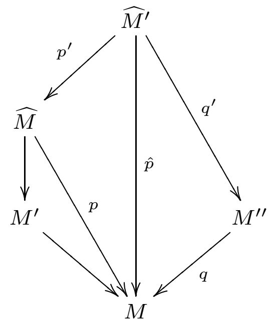

FIGURE 1.

discontinuously, it follows that $\langle b\rangle$ has finite index in $\mathrm{Isom}_+(\widehat{M}')$ and hence in $\langle a,b\rangle$. Therefore, there exists $k>0$ such that $\langle b^k\rangle$ is normal in $\langle a,b\rangle$.

Since $\langle b^k\rangle$ is normal in $\langle a,b\rangle$, we see that either $ab^ka=b^k$ or $ab^ka=b^{-k}$. We claim that it must be the case that $ab^ka=b^{-k}$. We may identify $(\widehat{M}')^0_\epsilon$ with $\overline{S}\times\mathbb{R}$ (for some compact connected surface $\overline{S}$) so that $T=\overline{S}\times\{0\}$ is preserved by a, that is $a(T)=T$. If $n>0$ is chosen large enough then $b^{nk}(T)$ does not intersect T. Since a interchanges the two components of $(N_U)^0_\epsilon - T$, we see immediately that $ab^{nk}a(T)$ and $b^{nk}(T)$ lie on opposite sides of T. Therefore, $ab^{nk}a\neq b^{nk}$, which implies that $ab^ka\neq b^k$. It follows that $ab^ka=b^{-k}$ and that $\langle a,b^k\rangle\cong\mathbb{Z}_2*\mathbb{Z}_2$ is generated by a and ab^k.

If n is even, then ab^{nk} is conjugate in $\langle a,b\rangle$ to a. Since a acts freely, so does ab^{nk}. Therefore, $\langle a,b^{2k}\rangle\cong\mathbb{Z}_2*\mathbb{Z}_2$ acts freely on M'. Let M' be the quotient of $\widehat{M}'$ by the group $\langle a,b^{2k}\rangle$. The main result of Hempel–Jaco [**6**] then implies that M' fibers over the orbifold $S^1/\langle z\mapsto\overline{z}\rangle$ and that $\widehat{M}'$ is the cover associated to a singular fiber subgroup. (Alternatively, one may assume that k is large enough that $b^k(T)$ does not intersect T and prove directly that the region between T and $b^k(T)$ is a fundamental region for the group $\langle a,b^{2k}\rangle$. One may then explicitly check that the quotient $\widehat{M}'/\langle a,b^{2k}\rangle$ fibers over the orbifold $S^1/\langle z\mapsto\overline{z}\rangle$ and that $\widehat{M}'$ is a cover associated to a singular fiber subgroup.) One may then check that $\hat{p}:\widehat{M}'\to M$ descends to a covering map $\hat{p}':M'\to M$. This completes the proof of Proposition 1, which in turn completes the proof of our Main Theorem.

If we combine Proposition 1 with our earlier statement of the Covering Theorem, we get the following slightly sharper version.

COVERING THEOREM (sharper version). *Suppose that $\widehat{M}$ is a hyperbolic 3-manifold with finitely generated fundamental group and M is a hyperbolic 3-orbifold. If $p:\widehat{M}\to M$ is an orbifold cover which is infinite-to-one on a neighborhood U of a geometrically infinite end of $\widehat{M}^0_\epsilon$, then M has finite volume and has a finite manifold cover $M'=\mathbb{H}^3/G'$ such that either*

(1) *M' fibers over the circle and $\widehat{M}$ is the cover associated to a fiber subgroup of G', or*
(2) *M' fibers over the orbifold $S^1/\langle z\mapsto\overline{z}\rangle$ and $\widehat{M}$ is the cover of M' associated to a singular fiber subgroup of G'.*

As a corollary of our Main Theorem and the new Covering Theorem we obtain a characterization of geometrically infinite hyperbolic 3-manifolds (with finitely generated fundamental group) which cover finite volume hyperbolic 3-orbifolds.

Corollary 1. *Let Γ be a finitely generated, torsion-free, geometrically infinite Kleinian group. Its real length spectrum $\mathfrak{L}(\Gamma)$ is discrete if and only if Γ is contained in a cofinite volume Kleinian group.*

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