ON EXTENDING COMMUTATIVE SEMIGROUPS OF ISOMETRIES

R. G. DOUGLAS

Let V be an isometric operator defined on the Hilbert space $\mathscr{H}$, that is, $\|Vx\| = \|x\|$ for x in $\mathscr{H}$. From a result due to von Neumann [**5**] and Wold [**7**], it follows that there is a unitary operator W defined on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ that extends V. An analogous result was obtained by Cooper [**2**] for a continuous one parameter semi-group of isometries. Independently, Itô [**4**] and Brehmer [**1**] showed that every commutative semigroup of isometries on Hilbert space can be extended to a corresponding commutative semigroup of unitary operators on a larger Hilbert space. It is the purpose of this note to give a more direct and natural proof of this latter result which is valid for Banach spaces and to prove certain ancillary results concerning the commutant of the semigroup of isometries. The proof is based on the construction of the direct limit of Banach spaces. A precise statement of the result will be given after this construction has been carried out.

Let Σ be a commutative semigroup and $\mathscr{X}$ be a Banach space. An isometric representation of Σ on $\mathscr{X}$ is a map $\sigma \to V_\sigma$ such that V_σ is an isometry on $\mathscr{X}$ for each σ in Σ and $V_\sigma V_\tau = V_{\sigma\tau}$ for σ and τ in Σ.† Our object is to construct a Banach space $\mathscr{Y}$ containing $\mathscr{X}$ and a representation $\sigma \to W_\sigma$ consisting of *invertible* isometric operators on $\mathscr{Y}$ that extends $\sigma \to V_\sigma$. We begin by constructing $\mathscr{Y}$.

A commutative semigroup possesses a natural order making it into a directed set, namely $\sigma \geqslant \tau$ if $\sigma = \tau\gamma$ for some γ in Σ. Let $\mathscr{Y}_0$ denote the collection of functions f from Σ to $\mathscr{X}$ for which there exists σ_f in Σ such that $f(\sigma_f \gamma) = V_\gamma f(\sigma_f)$ for every γ in Σ. It is easily checked that $\mathscr{Y}_0$ is a linear space with respect to pointwise addition and scalar multiplication. Moreover, since $\|f(\tau)\| = \|V_\gamma f(\sigma_f)\| = \|f(\sigma_f)\|$ for $\tau \geqslant \sigma_f$ with $\tau = \sigma_f \gamma$, then $\lim_\sigma \|f(\sigma)\|$ exists for f in $\mathscr{Y}_0$ where the limit is taken over the directed set Σ. The function $\|f\| = \lim_\sigma \|f(\sigma)\|$ is a pre norm on $\mathscr{Y}_0$ and the collection $\mathscr{N} = \{f \in \mathscr{Y}_0 : \|f\| = 0\}$ is a subspace of $\mathscr{Y}_0$. Let $\mathscr{Y}_1$ denote the normed linear space $\mathscr{Y}_0/\mathscr{N}$ and $\mathscr{Y}$ the completion of $\mathscr{Y}_1$. For x in $\mathscr{X}$ define $f_x(\sigma) = V_\sigma x$. The map $x \to f_x + \mathscr{N}$ defines an isometric embedding of $\mathscr{X}$ into $\mathscr{Y}$ which we use to identify $\mathscr{X}$ as a subspace of $\mathscr{Y}$. Before continuing to the definition of the operators W_σ, let us observe that if $\mathscr{X}$ is assumed to be a Hilbert space, then an inner product can be defined on $\mathscr{Y}_0$ by $\lim_\sigma (f(\sigma), g(\sigma))$ so that $\mathscr{Y}$ is seen to be a Hilbert space also.

We first define W_σ from $\mathscr{Y}_0$ to $\mathscr{Y}_0$ such that $(W_\sigma f)(\tau) = V_\sigma f(\tau)$ for f in $\mathscr{Y}_0$ and τ in Σ. Since $(W_\sigma f)(\sigma_f \gamma) = V_\sigma f(\sigma_f \gamma) = V_\sigma V_\gamma f(\sigma_f) = V_\gamma V_\sigma f(\sigma_f) = V_\gamma (W_\sigma f)(\sigma_f)$

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† Multiplicative notation will be used for the binary operation on Σ.

it follows that $W_\sigma f$ is in $\mathscr{Y}_0$ and $\|W_\sigma f\| = \|f\|$. Thus W_σ defines an isometry on $\mathscr{Y}_1$ which we also denote by W_σ. Further, since

$$(W_\sigma f_x)(\tau) = V_\sigma f_x(\tau) = V_\sigma V_\tau x = V_\tau V_\sigma x = f_{V_\sigma x}(\tau)$$

it follows that W_σ extends V_σ. Moreover, the identity

$$(W_{\sigma\tau} f)(\gamma) = V_{\sigma\tau} f(\gamma) = V_\sigma V_\tau f(\gamma) = (W_\sigma W_\tau f)(\gamma) = V_\sigma V_\tau f(\gamma) = V_\tau V_\sigma f(\gamma)$$
$$= (W_\tau W_\sigma f)(\gamma)$$

shows that $W_{\sigma\tau} = W_\sigma W_\tau = W_\tau W_\sigma$. Thus the unique extensions of the operators to $\mathscr{Y}$ yield an isometric representation $\sigma \to W_\sigma$ of Σ on $\mathscr{Y}$ that extends $\sigma \to V_\sigma$. It remains only to establish that each W_σ is invertible and for this it is sufficient to show that $W_\sigma \mathscr{Y}_1 = \mathscr{Y}_1$ for σ in Σ. For f in $\mathscr{Y}_0$, we define $g(\sigma \sigma_f \gamma) = f(\sigma_f \gamma)$ for γ in Σ and $g(\tau) = 0$ for τ in Σ, $\tau \not\geqslant \sigma \sigma_f$. Then g is in $\mathscr{Y}_0$ and $\|W_\sigma g - f\| = 0$. Thus $W_\sigma(\{g + \mathscr{N}\}) = \{f + \mathscr{N}\}$ so that $W_\sigma \mathscr{Y}_1 = \mathscr{Y}_1$ and W_σ is seen to be invertible.

The extension $\sigma \to W_\sigma$ just constructed is minimal and the minimal extension is uniquely determined. The first statement follows from the fact that

$$\bigcup_{\sigma \in \Sigma} W_\sigma^{-1} \mathscr{X} = \mathscr{Y}_1$$

is dense in $\mathscr{Y}$. Since any extension of $\sigma \to V_\sigma$ is uniquely determined on this subspace, the uniqueness of a minimal extension follows.

We summarize the preceding in the following:

THEOREM 1. *Let Σ be a commutative semigroup, $\mathscr{X}$ be a Banach space, and $\sigma \to V_\sigma$ be an isometric representation of Σ on $\mathscr{X}$. Then there exists a unique representation $\sigma \to W_\sigma$ of Σ on a Banach space $\mathscr{Y}$ containing $\mathscr{X}$ that extends $\sigma \to V_\sigma$, consists of invertible isometric operators and is minimal. Moreover, $\mathscr{Y}$ is a Hilbert space if $\mathscr{X}$ is.*

We now want to consider the relation which exists between the commutants $\mathscr{A}$ and $\mathscr{B}$ of the representations $\sigma \to V_\sigma$ and $\sigma \to W_\sigma$, respectively. If we set $\mathscr{B}_{\mathscr{X}} = \{B \mid \mathscr{X} : B \in \mathscr{B},\ B\mathscr{X} \subset \mathscr{X}\}$, then it is easy to see that $\mathscr{B}_{\mathscr{X}} \subset \mathscr{A}$ and

$$\|B\| \geqslant \|B \mid \mathscr{X}\|.$$

That the preceding inclusion and inequality are actually equalities is the content of:

THEOREM 2. *Let Σ be a commutative semigroup, $\sigma \to V_\sigma$ be an isometric representation of Σ on a Banach space $\mathscr{X}$, and $\sigma \to W_\sigma$ be the minimal extension to invertible isometric operators on a Banach space $\mathscr{Y}$ containing $\mathscr{X}$. If $\mathscr{A}$ and $\mathscr{B}$ denote the commutants of $\sigma \to V_\sigma$ and $\sigma \to W_\sigma$, respectively, then $\mathscr{A} = \mathscr{B}_{\mathscr{X}}$. Moreover, each A in $\mathscr{A}$ has a unique extension to a B in $\mathscr{B}$, $\|A\| = \|B\|$ and the relative bounded strong [weak (in case $\mathscr{X}$ is a Hilbert space)] operator topology on $\mathscr{B}_{\mathscr{X}}$ coincides with the bounded strong [weak] operator topology on $\mathscr{A}$.*

Proof. The remark preceding the theorem shows that $\mathscr{A} \subset \mathscr{B}_{\mathscr{X}}$ and $\|B\| \geqslant \|B \mid \mathscr{X}\|$. Suppose A is in $\mathscr{A}$ and define B on $\mathscr{Y}_0$ such that $(Bf)(\sigma) = Af(\sigma)$

for σ in Σ. Since $\|Bf\| = 0$ if $\|f\| = 0$, then B defines an operator on $\mathscr{Y}$ that commutes with all the W_σ and $B \mid \mathscr{X} = A$. Thus B is in $\mathscr{B}$, A belongs to $\mathscr{B}_{\mathscr{X}}$ and $\mathscr{A} = \mathscr{B}_{\mathscr{X}}$.

Let B be in $\mathscr{B}_{\mathscr{X}}$. For x in $\mathscr{X}$ and σ in Σ we have

$$\|BW_\sigma^{-1}x\| = \|W_\sigma^{-1}Bx\| = \|Bx\| \leqslant \|B \mid \mathscr{X}\| \, \|x\|$$

so that $\|B\| = \|B \mid \mathscr{X}\|$. Thus each A in $\mathscr{A}$ has a unique extension to an operator B in $\mathscr{B}$. Now suppose $\{B_\alpha\}$ is a uniformly bounded net of operators in $\mathscr{B}$ such that $\{B_\alpha \mid \mathscr{X}\}$ is Cauchy in the strong operator topology on $\mathscr{A}$. Then $\{B_\alpha \mid \mathscr{X}\}$ converges strongly to A in $\mathscr{A}$ with extension B in $\mathscr{B}$. For x in $\mathscr{X}$ and σ in Σ we have $\lim_\alpha \|(B - B_k) W_\sigma^{-1} x\| = \lim_\alpha \|(B - B_k)x\| = 0$. Since $\bigcup_{\sigma \in \Sigma} W_\sigma^{-1} \mathscr{X}$ is dense in $\mathscr{Y}$ and the $\{B_\alpha\}$ are uniformly bounded, it follows that the net $\{B_\alpha\}$ converges strongly to B. Thus the relative bounded strong operator topology on $\mathscr{B}_{\mathscr{X}}$ coincides with the bounded strong operator topology on $\mathscr{A}$. A similar argument establishes the corresponding fact for the weak operator topology when $\mathscr{X}$ is a Hilbert space.

The identification $\mathscr{A} = \mathscr{B}_{\mathscr{X}}$ is well-known for the case of the unilateral shift and was recently extended to the case of an arbitrary isometry in [3]. The relation between topologies is established for the case of the unilateral shift in [6]. The following corollary appears in [4] in the context of Hilbert space.

COROLLARY. *Let Σ be a topological semigroup and $\sigma \to V_\sigma$ be an isometric representation of Σ on $\mathscr{X}$ which is continuous in the uniform, strong or weak operator topologies. Then the minimal extension $\sigma \to W_\sigma$ of $\sigma \to V_\sigma$ to a representation consisting of invertible isometric operators is continuous in the corresponding topology.*

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University of Michigan,
Ann Arbor,
Michigan 48104.