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Secant varieties of $\mathbb{P}^2 \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$

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Abstract

We describe the defining ideal of the rth secant variety of $\mathbb{P}^2 \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$, for arbitrary n and $r \leqslant 5$. We also present the Schur module decomposition of the space of generators of each such ideal. Our main results are based on a more general construction for producing explicit matrix equations that vanish on secant varieties of products of projective spaces. This extends previous work of Strassen and Ottaviani.

1. *Introduction*

Let U, V, and W be complex vector spaces of dimension m, n and k, respectively, and let x be an element in the tensor product of their duals, $U^* \otimes V^* \otimes W^*$. The border rank of x is the minimal r such that the corresponding point $[x] \in \mathbb{P}(U^* \otimes V^* \otimes W^*)$ lies in the rth secant variety of the Segre variety of $\mathbb{P}(U^*) \times \mathbb{P}(V^*) \times \mathbb{P}(W^*)$. Similarly, for a symmetric tensor $x \in S^3U^*$ or a partially symmetric tensor $x \in U^* \otimes S^2V^*$, the symmetric border rank and the partially symmetric border rank are the smallest r such that $[x]$ is in the rth secant variety of the Veronese or the Segre–Veronese variety, respectively. Developing effective techniques for computing the border rank of tensors is an active area of research which spans classical algebraic geometry and representation theory [**12**–**16**, **20**].

In the partially symmetric case, the secant varieties of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$ are closely related to standard results about pencils of symmetric matrices. Moreover, the non-symmetric analog is $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{k-1}$ embedded by $\mathcal{O}(1,1,1)$, and the defining equations of all of its secant varieties are known by the work of Landsberg and Weyman [**16**, Theorem 1.1]. We record the partially symmetric analog in Proposition 5.1.

Our main result is Theorem 5.2, which focuses on the next case: secant varieties of $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$. We give two explicit matrices, and we prove that, when $r \leqslant 5$, their minors and Pfaffians, respectively, generate the defining ideal for these secant varieties. To illustrate, fix a basis $\{e_1, e_2, e_3\}$ of U^*. We may then express any point $x \in \mathbb{P}(U^* \otimes S^2V^*)$ as $x = e_1 \otimes A_1 + e_2 \otimes A_2 + e_3 \otimes A_3$ where each $A_i \in S^2V^*$ can be represented by an $n \times n$ symmetric matrix. With the ordered triplet of matrices (A_1, A_2, A_3) serving as coordinates on $\mathbb{P}(U^* \otimes S^2V^*)$, our main result is the following, which is a restatement of Theorem 5.2.

Theorem 1.1. *Let Y be the image of $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{3\binom{n+1}{2}-1} \cong \mathbb{P}(U^* \otimes S^2V^*)$ embedded by $\mathcal{O}(1,2)$. For any $r \leqslant 5$ the rth secant variety of Y is defined by the prime ideal generated by the $(r+1) \times (r+1)$ minors of the $n \times 3n$ block matrix*

$$\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \tag{1.1}$$

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and by the $(2r+2)\times(2r+2)$ *principal Pfaffians of the* $3n\times 3n$ *block matrix*

$$\begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}. \tag{1.2}$$

The matrices that appear in the statement of the above theorem are examples of what we call the 'exterior flattenings' of a 3-tensor (see § 2), and the construction of these matrices is motivated by the κ-invariant of a 3-tensor, as introduced in [7, § 1.1]. The minors of the exterior flattenings of a 3-tensor impose necessary equations on a wide array of secant varieties of Segre–Veronese embeddings of products of projective spaces. The minors of these exterior flattenings simultaneously generalize both the minors obtained from flattenings of a 3-tensor and the determinantal equations of Strassen [22, Lemma 4.4] and Ottaviani [20, Theorem 3.2].

Under the hypotheses of Theorem 1.1, the minors and Pfaffians of these exterior flattenings are insufficient to generate the ideal of the rth secant variety for $r\geqslant 7$. In other words, Theorem 1.1 would be false for $r\geqslant 7$, and we do not know if Theorem 1.1 holds when $r=6$; see Example 10 for more details. Note that by Abo and Brambilla [1, Corollary 1.4(ii)], these secant varieties have the expected dimension except when n is odd and $r=n+(n+1)/2$.

The proof of our main result uses a mix of representation theory and geometric techniques for studying determinantal varieties. We first introduce the relevant determinantal ideals and we use their equivariance properties to relate these ideals as the size of the tensor varies. Next, we apply this relation in order to understand the defining ideals of certain auxiliary varieties known as the subspace varieties $\mathrm{Sub}_{m',n'}$ (see Definition 6). We then prove our main result in the special case where $n=r$, by relating the secant variety with the variety of commuting symmetric $n\times n$ matrices. A similar idea has appeared in several instances previously [2, 7, 20, 22]. This step requires $r\leqslant 5$. Finally, we prove our main result by blending our results about subspace varieties with our knowledge about the case $n=r$.

Partially symmetric 3-tensors are closely related to the study of vector spaces of quadrics, which arise naturally in algebraic geometry. For instance, in the study of Hilbert schemes of points, border rank is connected to the smoothability of zero-dimensional schemes [4, 7]. As another example, [20, Proposition 6.3] relates the border rank of a partially symmetric tensor $x\in\mathbb{C}^3\otimes S^2(\mathbb{C}^n)$ with properties of the corresponding degree n determinantal curve in $\mathbb{P}^2$.

Questions about the border rank of partially symmetric tensors also arise in algebraic statistics [9, § 7]. For instance, the situation of the Theorem 5.2 corresponds to a mixture of random processes, each independently sampling from a distribution with three states and sampling twice from a distribution with n states. The border rank of the observed distribution corresponds to the number of processes in the mixture.

In signal processing, a partially symmetric tensor in $U^*\otimes S^2V^*$ can be constructed as the second derivative of the cumulant-generating function taken at m points; see [26]. The matrix equations in Theorem 5.2 can be used to study small border ranks of such tensors in the case $m=3$.

The defining ideal of the rth secant variety of $\mathbb{P}^2\times\mathbb{P}^{n-1}$ was previously known in the case when this secant variety is a hypersurface. This occurs when $n\geqslant 4$ is even, and $r=(3n-2)/2$, and this result follows from an analog to Strassen's argument [22, § 4], as shown by Ottaviani in the remark following [20, Theorem 4.1]. For historical interest, we note that the hypersurface case $n=4$ and $r=5$ dates to Toeplitz [23].

Theorem 5.2 thus provides a new family of examples where we can effectively compute the border rank of a partially symmetric tensor. Our main result also provides evidence for a partially symmetric analog of Comon's Conjecture, which posits that the symmetric rank of a tensor equals the rank [5, § 5], as discussed in Remark 8.

This paper is organized as follows. In § 2, we define a vector κ as an invariant of a 3-tensor. We use this κ-invariant to produce explicit matrix equations that vanish on the secant varieties of Segre–Veronese embeddings of projective spaces. To provide a more invariant perspective, and to connect with the previous literature [**12**, **13**, **16**, **17**], we also provide Schur module decompositions for our matrix equations. In § 3, we restrict to the case of the κ-invariant of a partially symmetric tensor. Here we also provide Schur module decompositions in the partially symmetric case. In § 4, we show that the κ_0 equations define subspace varieties. We prove our main result; Theorem 5.2 in § 5.

REMARK 1. Our results giving equations vanishing on the Segre and Segre–Veronese varieties (Propositions 2.1 and 3.1) hold in arbitrary characteristic. However, our proof of Theorem 5.2 does not extend to arbitrary characteristic because it relies on Lemmas 4.2 and 5.4 and [**3**, Theorem 3.1], all of which require characteristic 0.

2. *The κ-invariant of a 3-tensor*

From a tensor in $U^* \otimes V^* \otimes W^*$, we construct a series of linear maps, whose ranks we define to be the κ-invariants of the tensor. The κ-invariants give inequalities on the rank of the tensor, and thus, determinantal equations that vanish on the secant variety.

There is a natural map $U^* \otimes \bigwedge^j U^* \to \bigwedge^{j+1} U^*$ defined by sending $u \otimes u' \mapsto u \wedge u'$ for any $0 \leqslant j \leqslant m-1$. This induces an inclusion $U^* \subseteq \bigwedge^j U \otimes \bigwedge^{j+1} U^*$. By tensoring on both sides by $V^* \otimes W^*$, we get an inclusion $U^* \otimes V^* \otimes W^* \subseteq (V \otimes \bigwedge^j U^*)^* \otimes (W^* \otimes \bigwedge^{j+1} U^*)$. An element of the tensor product on the right-hand side may be interpreted as a linear homomorphism, meaning that, for any $x \in U^* \otimes V^* \otimes W^*$, we have a homomorphism

$$\psi_{j,x} \colon V \otimes \bigwedge^j U^* \to W^* \otimes \bigwedge^{j+1} U^*,$$

and $\psi_{j,x}$ depends linearly on x. We call $\psi_{j,x}$ an exterior flattening of x, as it generalizes the flattening of a tensor, as discussed below.

DEFINITION 1. Following [**7**, Definition 1.1], we define $\kappa_j(x)$ to be the rank of $\psi_{j,x}$, and we let $\kappa(x)$ denote the vector of κ-invariants $(\kappa_0(x), \dots, \kappa_{m-1}(x))$.

More concretely, by choosing bases for the vector spaces, we can represent $\psi_{j,x}$ as a matrix. If $e_1, \dots, e_m$ is a basis for U^*, then a basis for $\bigwedge^j U^*$ is given by the set of all $e_{i_1} \wedge \cdots \wedge e_{i_j}$ for $1 \leqslant i_1 < \cdots < i_j \leqslant m$, and analogously for $\bigwedge^{j+1} U^*$. For a fixed $u = \sum_{i=1}^m u_i e_i$ in U^*, the map $\bigwedge^j U^* \to \bigwedge^{j+1} U^*$ defined by $u' \mapsto u \wedge u'$ will send $e_{i_1} \wedge \cdots \wedge e_{i_j}$ to $\sum_i u_i e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_j}$. Thus, this map will be represented in the above bases by a matrix whose entries are either 0 or $\pm u_i$. The matrix for $\psi_{j,x}$ is the block matrix formed by replacing the scalar u_i with the matrix A_i, where A_i are the matrices such that $x = \sum_{i=1}^m e_i \otimes A_i$.

For example, if $m = 4$, then $\psi_{j,x}$ are represented by the following matrices (in suitable coordinates):

$$\psi_{0,x} \colon V \otimes \bigwedge^0 U^* \xrightarrow{\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}} W^* \otimes \bigwedge^1 U^*,$$

$$\psi_{1,x} \colon V \otimes \bigwedge^1 U^* \xrightarrow{\begin{pmatrix} 0 & A_3 & -A_2 & 0 \\ -A_3 & 0 & A_1 & 0 \\ A_2 & -A_1 & 0 & 0 \\ A_4 & 0 & 0 & -A_1 \\ 0 & A_4 & 0 & -A_2 \\ 0 & 0 & A_4 & -A_3 \end{pmatrix}} W^* \otimes \bigwedge^2 U^*,$$

$$\psi_{2,x}\colon V\otimes \textstyle\bigwedge^2 U^* \xrightarrow{\left(\begin{smallmatrix} -A_4 & 0 & 0 & 0 & A_3 & -A_2 \\ 0 & -A_4 & 0 & -A_3 & 0 & A_1 \\ 0 & 0 & -A_4 & A_2 & -A_1 & 0 \\ A_1 & A_2 & A_3 & 0 & 0 & 0 \end{smallmatrix}\right)} W^*\otimes \textstyle\bigwedge^3 U^*,$$

$$\psi_{3,x}\colon V\otimes \textstyle\bigwedge^3 U^* \xrightarrow{(\,A_1\ A_2\ A_3\ A_4\,)} W^*\otimes \textstyle\bigwedge^4 U^*.$$

The entries of these matrices are linear forms on $\mathbb{P}(U^*\otimes V^*\otimes W^*)$, and the minors of these matrices $\psi_{j,x}$ are the 'explicit matrix equations' alluded to in § 1.

Note that, for $j=0$, the map $\psi_0\colon V\otimes \bigwedge^0 U^* \cong V \to W^*\otimes U^*$ is the homomorphism corresponding to x in the identification $U^*\otimes V^*\otimes W^* \cong \mathrm{Hom}(V, U^*\otimes W^*)$. In the literature, the matrix for ψ_0 is referred to as a 'flattening' of x by grouping W^* and U^*. Similarly, $\psi_{m-1,x}\colon V\otimes \bigwedge^{m-1}U^* \to W^*$ is a twist of the flattening formed by grouping U^* and V^*, because $\bigwedge^{m-1}U^* \cong U\otimes \det(U^*)$ as $\mathrm{GL}(U)$-modules.

If $[x]\in \mathbb{P}(U^*\otimes V^*\otimes W^*)$ is a tensor, then the rank of $[x]$ is the number r in a minimal expression $x = u_1\otimes v_1\otimes w_1 + \ldots + u_r\otimes v_r\otimes w_r$, where $u_i\in U$, $v_i\in V$ and $w_i\in W$ for $1\leqslant i\leqslant r$. The set of rank-1 tensors is closed and equals the Segre variety $\mathrm{Seg}(\mathbb{P}U^*\times\mathbb{P}V^*\times\mathbb{P}W^*)$. More generally, the Zariski closure of the set of tensors $[x]\in\mathbb{P}(U^*\otimes V^*\otimes W^*)$ having rank at most r is the rth secant variety of the Segre product, denoted by $\sigma_r(\mathrm{Seg}(\mathbb{P}U^*\times\mathbb{P}V^*\times\mathbb{P}W^*))$.

Definition 2. The border rank of $[x]\in\mathbb{P}(U^*\otimes V^*\otimes W^*)$ is the minimal r such that $[x]$ is in $\sigma_r(\mathrm{Seg}(\mathbb{P}U^*\times\mathbb{P}V^*\times\mathbb{P}W^*))$.

The following lemma generalizes the well-known fact that the ranks of the flattenings are bounded above by the tensor rank, and extends a result of Ottaviani [**20**, Theorem 3.2(i)].

Lemma 2.1. *If $x\in U^*\otimes V^*\otimes W^*$ has border rank at most r, then $\kappa_j(x)\leqslant r\binom{m-1}{j}$.*

Proof. Since κ_j is defined in terms of a matrix rank, an upper bound on κ_j is a closed condition on the set of tensors. It thus suffices to prove the statement with border rank replaced by rank. As observed above, $\psi_{j,x}$ depends linearly on x, so it is sufficient to assume that x is an indecomposable tensor, and then show that $\kappa_j(x)\leqslant\binom{m-1}{j}$. We can choose coordinates such that $x=e_1\otimes A$, and A is a matrix with only one non-zero entry. The non-zero rows of the matrix for $\psi_{j,x}$ will correspond to those basis elements $e_{i_1}\wedge\cdots\wedge e_{i_j}\in\bigwedge^j U^*$ such that $2\leqslant i_1<\cdots<i_j\leqslant m$, each of which is sent to a multiple of $e_1\wedge e_{i_1}\wedge\cdots\wedge e_{i_j}$. Since there are $\binom{m-1}{j}$ such basis elements, the rank of $\psi_{j,x}$ is equal to $\binom{m-1}{j}$. $\square$

The above lemma illustrates that the minors of the exterior flattenings provide equations that vanish on the secant variety of a Segre triple product. We write $S^\bullet(U\otimes V\otimes W)$ to denote the polynomial ring on the affine space $U^*\otimes V^*\otimes W^*$.

Definition 3. Let $c=(c_0,\ldots,c_{m-1})$ be a vector of positive integers. We define $I_{\kappa_i\leqslant c_i}$ to be the ideal generated by the $(c_i+1)\times(c_i+1)$-minors of $\psi_{j,x}$. Similarly, we use the notation $I_{\kappa\leqslant c}$ for the ideal generated by $I_{\kappa_i\leqslant c_i}$ for all $0\leqslant i\leqslant m-1$. Finally, we define $\Sigma_{\kappa_i\leqslant c_i}$ and $\Sigma_{\kappa\leqslant c}$ to be the subschemes of $\mathbb{P}(U^*\otimes V^*\otimes W^*)$ defined by the ideals $I_{\kappa_i\leqslant c_i}$ and $I_{\kappa\leqslant c}$, respectively.

The following proposition is an immediate consequence of Lemma 2.1.

PROPOSITION 2.2. *Fix $r \geqslant 1$. If c is the vector defined by $c_j = r\binom{m-1}{j}$ for $0 \leqslant j \leqslant m-1$, then*

$$\sigma_r(\mathrm{Seg}(\mathbb{P}U^* \times \mathbb{P}V^* \times \mathbb{P}W^*)) \subseteq \Sigma_{\kappa \leqslant c}.$$

REMARK 2. Fundamental in the construction of the exterior flattening $\psi_{j,x}$ was the inclusion of U^* into $\bigwedge^j U \otimes \bigwedge^{j+1} U^*$. More generally, any natural inclusion of U^* into the tensor product of two representations would yield an analog of $\psi_{j,x}$ as well as analogs of Lemma 2.1 and Proposition 2.2. For instance, from the inclusion $U^* \subseteq S_{(2,1)}(U) \otimes S_{(2,1,1)}(U^*)$, we may associate to a tensor x a homomorphism

$$V \otimes S_{(2,1)}(U^*) \to W^* \otimes S_{(2,1,1)}(U^*),$$

whose rank is at most five times the border rank of x. We restrict our attention to the κ-invariants because these seem to provide particularly useful inequalities in our cases of interest. However, an example of this generalized construction was introduced and applied in [**21**, Theorem 1.1] and has been further developed under the name of Young flattening in [**14**].

EXAMPLE 1. If $m = 2$, then as stated above, κ_0 and κ_1 are the ranks of the flattenings formed by grouping W^* with U^* and V^* with U^*, respectively. The ideal of the rth secant variety is $I_{\kappa \leqslant (r,r)}$ (see [**16**, Theorem 1.1]).

EXAMPLE 2. Let $m = 3$ and suppose that $n = k$ is odd. Denote by X the Segre product $\mathrm{Seg}(\mathbb{P}U^* \times \mathbb{P}V^* \times \mathbb{P}W^*)$ in $\mathbb{P}(U^* \otimes V^* \otimes W^*)$. Then $I_{\kappa_1 \leqslant 3n-1}$ is a principal ideal generated by the determinant of $\psi_{1,x}$, which defines the secant variety $\sigma_{(3n-1)/2}(X)$ (see [**22**, Lemma 4.4; see also **20**, Remark 3.3]).

EXAMPLE 3. The exterior flattening $\psi_{1,x}$ has also arisen in the study of totally symmetric tensors. For instance, in the case $n = 3$, the secant variety $\sigma_3(\nu_3(\mathbb{P}^2)) \subseteq \mathbb{P}^9$ is a hypersurface defined by the Aronhold invariant. Ottaviani has shown that this hypersurface is defined by any of the 8×8 Pfaffians of the matrix representing $\psi_{1,x}$ specialized to symmetric tensors [**21**, Theorem 1.2].

However, the ideals $I_{\kappa \leqslant c}$ do *not* equal the defining ideals of secant varieties even in relatively simple cases.

EXAMPLE 4. Let $n = m = k = 3$ and let Y be the image of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^{26}$ embedded by $\mathcal{O}(1,1,1)$. By Proposition 2.2, we know that $I_{\kappa \leqslant (3,6,3)}$ vanishes on $\sigma_3(Y)$, but we claim that it is not the defining ideal. Observe that the conditions $\kappa_0 \leqslant 3$ and $\kappa_2 \leqslant 3$ are trivial, and hence $I_{\kappa \leqslant (3,6,3)} = I_{\kappa_1 \leqslant 6}$. By definition, the ideal $I_{\kappa_1 \leqslant 6}$ is generated by the 7×7 minors of $\psi_{1,x}$. However, [**16**, Theorem 1.3] produces degree 4 equations that vanish on $\sigma_3(Y)$, and since $I_{\kappa_1 \leqslant 6}$ is generated in degree 7, we see that it does not equal the defining ideal of $\sigma_3(Y)$.

We now study our matrix equations from the perspective of representation theory, which connects them to previous work on secant varieties of Segre–Veronese varieties. The representation theory of our ideals $I_{\kappa_i \leqslant c_i}$ will also be necessary in the proof of Lemma 4.2.

Since the ideals $I_{\kappa_i \leqslant c_i}$ are invariant under the natural action of $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$, their generators can be decomposed as direct sums of irreducible representations of that group. Each polynomial representation of $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$ is of the form $S_\mu U \otimes S_\nu V \otimes$

$S_\omega W$, where $S_\mu U$, $S_\nu V$ and $S_\omega W$ are the Schur modules indexed by partitions μ, ν and ω with at most m, n and k parts, respectively, where if $\pi = (\pi_1, \dots, \pi_s)$ is a partition with $\pi_1 \geqslant \pi_2 \geqslant \cdots \geqslant \pi_s > 0$, then we say that s is the number of parts of π. For the summands of the degree d part of $I_{\kappa_i \leqslant c_i} \subset S^d(U \otimes V \otimes W)$, the partitions will always be partitions of d. For general background on Schur modules, see [**8**].

Lemma 2.3. *For each j and c_j, there is a* $\mathrm{GL}(U) \times \mathrm{GL}(V) \times \mathrm{GL}(W)$*-equivariant map*

$$\Phi_j\colon\ \bigwedge^{c_j+1}\left(V \otimes \bigwedge^{j} U^*\right) \otimes \bigwedge^{c_j+1}\left(W \otimes \bigwedge^{j+1} U\right) \to S^{c_j+1}\left(U \otimes V \otimes W\right),$$

whose image equals the vector space of generators of $I_{\kappa_j \leqslant c_j}$. In particular, every irreducible representation arising in the Schur module decomposition of the generators of $I_{\kappa_j \leqslant c_j}$ must be a submodule of both the source and target of Φ_j.

Proof. Consider the map

$$\psi_{j,x}\colon V \otimes \bigwedge^{j} U^* \to W^* \otimes \bigwedge^{j+1} U^*.$$

After choosing bases of U, V and W, we may think of $\psi_{j,x}$ as a matrix of linear forms in $S^\bullet(U \otimes V \otimes W)$. Taking the $(c_j+1) \times (c_j+1)$-minors of $\psi_{j,x}$ then determines the map Φ_j. More concretely, our choice of bases for U, V and W determines a natural basis for the source of Φ_j consisting of indecomposable tensors; we define the map Φ_j by sending a basis element to the corresponding minor of the matrix $\psi_{j,x}$. Since the ideal $I_{\kappa_j \leqslant c_j}$ is defined as the ideal generated by the image of Φ_j, the lemma follows from Schur's Lemma. □

When $j = 0$, it is straightforward to compute the Schur module decomposition of $I_{\kappa_0 \leqslant c_0}$, as illustrated by the following example.

Example 5. The map

$$\psi_{0,x}\colon V \otimes \bigwedge^{0} U^* \longrightarrow W^* \otimes \bigwedge^{1} U^*$$

is a flattening of the tensor x by grouping U^* and W^*. As representations, the minors of $\psi_{0,x}$ decompose into irreducibles using the skew Cauchy formula [**8**, p. 80]

$$\bigwedge^{c_0+1} V \otimes \bigwedge^{c_0+1}(W \otimes U) = \bigwedge^{c_0+1} V \otimes \left(\bigoplus_{|\lambda| = c_0+1} S_\lambda W \otimes S_{\lambda'} U\right),$$

where λ ranges over all partitions of $c_0 + 1$ and λ' is the conjugate partition.

For instance, let $n = m = k = 3$ and consider the generators of $I_{\kappa_0 \leqslant 2}$. This is a vector space of cubic polynomials, and by Lemma 2.3, it must be the module

$$\bigwedge^{3} V \otimes ((S_3 W \otimes S_{1,1,1} U) \oplus (S_{2,1} W \otimes S_{2,1} U) \oplus (S_{1,1,1} \otimes S_3 U)).$$

After distributing, each irreducible module is the tensor product of three Schur functors applied to U, V and W, respectively, and we can thus drop the vector spaces and the tensor products from our notation, replacing $S_\mu U \otimes S_\nu V \otimes S_\omega W$ with $S_\mu S_\nu S_\omega$ without any ambiguity. Thus, we rewrite this module as

$$S_{1,1,1} S_{1,1,1} S_3 \oplus S_{2,1} S_{1,1,1} S_{2,1} \oplus S_3 S_{1,1,1} S_{1,1,1}.$$

The dimension of this space is $10 + 64 + 10 = 84$, which equals the number of maximal minors of the 3×9 matrix $\psi_{0,x}$.

When $j > 0$, the existence of the dual vector space U^* in the source of $\psi_{j,x}$ makes finding the Schur module decomposition of $I_{\kappa_i \leqslant c_i}$ more subtle. In Proposition 2.4, we provide an upper bound for the Schur module decomposition of $I_{\kappa_1 \leqslant c_1}$ in the case where $\dim U = 3$. To state the formula precisely, we first recall some notation.

For any vector space A, the Littlewood–Richardson formula is

$$S_\lambda A \otimes S_\mu A = \bigoplus_{|\pi|=|\lambda|+|\mu|} S_\pi A^{\oplus c^\pi_{\lambda,\mu}}, \tag{2.1}$$

where the multiplicities $c^\pi_{\lambda,\mu}$ are the Littlewood–Richardson numbers. For two vector spaces A and B, we use the outer plethysm formula

$$S_\pi(A \otimes B) = \bigoplus_{|\lambda|+|\mu|=|\pi|} (S_\lambda A \otimes S_\mu B)^{\oplus K_{\pi,\lambda,\mu}} \tag{2.2}$$

to define the Kronecker coefficients $K_{\pi,\lambda,\mu}$.

REMARK 3. In Propositions 2.4, 3.2 and 3.4, we use the fact that as $\mathrm{GL}(U)$ modules, $S_\pi U^* \otimes (\bigwedge^m U)^l \cong S_{l^m - \pi} U$, where l^m denotes the partition $(l, \dots, l)$. We caution that the entries in $l^m - \pi = (l - \pi_m, \dots, l - \pi_1)$ are reversed.

PROPOSITION 2.4. *Let $\dim(U) = 3$. For any Schur module $S_\pi U \otimes S_\lambda V \otimes S_\mu W$, let λ' and μ' denote the conjugate partitions of λ and μ, respectively, and let $(3)^{c_0+1} - \pi$ be the difference as in Remark 3. If $S_\pi U \otimes S_\lambda V \otimes S_\mu W$ occurs in the decomposition of $(I_{\kappa_1 \leqslant c_1})_{c_1+1}$ from Definition 3, then π, λ' and μ' have at most three parts, and the multiplicity of $S_\pi U \otimes S_\lambda V \otimes S_\mu W$ is at most the minimum of $c^{(3)^{c_0+1}-\pi}_{\lambda',\mu'}$ and $K_{\pi,\lambda,\mu}$.*

Computations with the software package LiE [**18**] suggest that the decomposition of $(I_{\kappa_1 \leqslant c_1})_{c_1+1}$ may equal the upper bound of Proposition 2.4, as is the case in Example 6.

Proof of Proposition 2.4. Using Lemma 2.3, the $(c_1 + 1) \times (c_1 + 1)$-minors of $\psi_{1,x}$ belong to the common submodules of the polynomials $S^{c_1+1}(U \otimes V \otimes W)$ and the domain of Φ_j, which we can rewrite using the Cauchy skew formula:

$$\left(\bigoplus_{|\lambda|=c_1+1} S_\lambda V \otimes S_{\lambda'} U^* \right) \otimes \left(\bigoplus_{|\mu|=c_1+1} S_\mu W \otimes S_{\mu'}(\textstyle\bigwedge^2 U) \right). \tag{2.3}$$

Here we note that λ' and μ' must have no more than three parts or else the summand is zero.

We focus on the U factor and compute

$$\begin{aligned} S_{\lambda'} U^* \otimes S_{\mu'}(\textstyle\bigwedge^2 U) &\cong S_{\lambda'} U^* \otimes S_{\mu'}(U^*) \otimes (\textstyle\bigwedge^3 U)^{c_0+1} \quad \text{because } \dim U = 3 \\ &\cong \bigoplus_{|\nu|=2(c_0+1)} (S_\nu U^*)^{\oplus c^\nu_{\lambda',\mu'}} \otimes (\textstyle\bigwedge^3 U)^{c_0+1} \quad \text{by (2.1)} \\ &\cong \bigoplus_{|\nu|=2(c_0+1)} (S_{(c_0+1)^3 - \nu} U)^{\oplus c^\nu_{\lambda',\mu'}} \quad \text{by Remark 3} \\ &\cong \bigoplus_{|\pi|=c_0+1} (S_\pi U)^{\oplus c^{(c_0+1)^3-\pi}_{\lambda',\mu'}} \quad \text{by taking } \pi = (c_0+1)^3 - \nu. \end{aligned}$$

Therefore, expression (2.3) becomes

$$\bigoplus_{|\lambda|=|\mu|=|\pi|=c_0+1} S_\pi U \otimes S_\lambda V \otimes S_\mu W^{\oplus c^{(c_0+1)^3-\pi}_{\lambda',\mu'}}.$$

Now we must decide which irreducible modules occur as a submodule of $S^{c_1+1}(U\otimes V\otimes W)$. For this, we decompose the space of polynomials using the Cauchy formula and the outer plethysm formula (2.2):

$$\begin{aligned} S^{c_1+1}(U\otimes V\otimes W) &\cong \bigoplus_{|\pi|=c_1+1} S_\pi U\otimes S_\pi(V\otimes W)\\ &\cong \bigoplus_{|\pi|=|\lambda|=|\mu|=c_0+1} (S_\pi U\otimes S_\lambda V\otimes S_\mu W)^{\oplus K_{\pi,\lambda,\mu}}. \end{aligned}$$

The proposition statement follows by Lemma 2.3. □

Example 6. Let $n=m=k=3$. Using LiE [18], we computed every decomposition of $(I_{\kappa_1\leqslant c_1})_{c_1+1}$ using Proposition 2.4. These decompositions appear below. To save space, we omit the notation of vector spaces and tensor products, as in Example 5. Further, we use the notation $\mathfrak{S}_s$ to indicate the direct sum over the (non-redundant) permutations of the subsequent s Schur modules.

$$\begin{aligned}
(I_{\kappa_1\leqslant 1})_2 &= (\mathfrak{S}_3\cdot S_{1,1}S_{1,1}S_2)\oplus S_2S_2S_2,\\
(I_{\kappa_1\leqslant 2})_3 &= (\mathfrak{S}_3\cdot S_{1,1,1}S_{2,1}S_{2,1})\oplus S_{2,1}S_{2,1}S_{2,1}\oplus(\mathfrak{S}_3\cdot S_{2,1}S_{2,1}S_3)\oplus S_3S_{1,1,1}S_{1,1,1},\\
(I_{\kappa_1\leqslant 3})_4 &= S_{2,2}S_{2,2}S_{2,2}\oplus(\mathfrak{S}_3\cdot S_{2,2}S_{2,1,1}S_{2,1,1})\oplus(\mathfrak{S}_3\cdot S_{2,2}S_{2,1,1}S_{3,1})\oplus(\mathfrak{S}_3\cdot S_{2,2}S_{3,1}S_{3,1})\\
&\quad\oplus S_{2,1,1}S_{2,1,1}S_{2,1,1}\oplus(\mathfrak{S}_3\cdot S_{2,1,1}S_{3,1}S_{3,1})\oplus S_{3,1}S_{2,1,1}S_{2,1,1}\oplus S_4S_{2,2}S_{2,2}\\
&\quad\oplus S_4S_{2,1,1}S_{2,1,1},\\
(I_{\kappa_1\leqslant 4})_5 &= S_{2,2,1}S_{2,2,1}S_{2,2,1}\oplus(\mathfrak{S}_3\cdot S_{2,2,1}S_{2,2,1}S_{3,1,1})\oplus(\mathfrak{S}_3\cdot S_{2,2,1}S_{3,2}S_{3,2})\\
&\quad\oplus(\mathfrak{S}_3\cdot S_{2,2,1}S_{3,2}S_{3,1,1})\\
&\quad\oplus(\mathfrak{S}_3\cdot S_{3,2}S_{3,2}S_{3,1,1})\oplus S_{3,2}S_{2,2,1}S_{2,2,1}\oplus S_{3,2}S_{3,1,1}S_{3,1,1}\oplus S_{3,1,1}(\mathfrak{S}_2\cdot S_{2,2,1}S_{3,1,1})\\
&\quad\oplus S_{3,1,1}S_{3,1,1}S_{3,1,1}\oplus S_{4,1}S_{2,2,1}S_{2,2,1}\oplus S_{4,1}(\mathfrak{S}_2\cdot S_{2,2,1}S_{3,2})\oplus S_{4,1}(\mathfrak{S}_2\cdot S_{2,2,1}S_{3,1,1})\\
&\quad\oplus S_5S_{2,2,1}S_{2,2,1},\\
(I_{\kappa_1\leqslant 5})_6 &= S_{2,2,2}S_{3,3}S_{3,3}\oplus(\mathfrak{S}_3\cdot S_{2,2,2}S_{3,2,1}S_{3,2,1})\oplus(\mathfrak{S}_3\cdot S_{3,3}S_{3,2,1}S_{3,2,1})\\
&\quad\oplus(S_{3,2,1}S_{3,2,1}S_{3,2,1})^{\oplus 2}\\
&\quad\oplus S_{4,2}S_{2,2,2}S_{2,2,2}\oplus S_{4,2}(\mathfrak{S}_2\cdot S_{2,2,2}S_{3,2,1})\oplus S_{4,2}S_{3,2,1}S_{3,2,1}\oplus S_{4,1,1}(\mathfrak{S}_2\cdot S_{2,2,2}S_{3,3})\\
&\quad\oplus S_{4,1,1}(\mathfrak{S}_2\cdot S_{2,2,2}S_{3,2,1})\oplus S_{4,1,1}S_{3,2,1}S_{3,2,1}\oplus S_{5,1}(\mathfrak{S}_2\cdot S_{2,2,2}S_{3,2,1})\\
&\quad\oplus S_6S_{2,2,2}S_{2,2,2},\\
(I_{\kappa_1\leqslant 6})_7 &= S_{3,3,1}S_{3,3,1}S_{3,3,1}\oplus(\mathfrak{S}_3\cdot S_{3,3,1}S_{3,2,2}S_{3,2,2})\oplus S_{3,2,2}S_{3,3,1}S_{3,3,1}\\
&\quad\oplus S_{4,2,1}(\mathfrak{S}_2\cdot S_{3,3,1}S_{3,2,2})\oplus S_{4,2,1}S_{3,2,2}S_{3,2,2}\oplus S_{5,1,1}S_{3,2,2}S_{3,2,2},\\
(I_{\kappa_1\leqslant 7})_8 &= S_{3,3,2}S_{3,3,2}S_{3,3,2}\oplus S_{4,2,2}S_{3,3,2}S_{3,3,2},\\
(I_{\kappa_1\leqslant 8})_9 &= S_{3,3,3}S_{3,3,3}S_{3,3,3}.
\end{aligned}$$

In the following table, we record the dimension of each module of equations.

c_1	1	2	3	4	5	6	7	8
$\dim(I_{\kappa_1\leqslant c_1})_{c_1+1}$	378	2634	8910	12 420	7011	1296	81	1

Since these dimensions match the dimensions of the space of minors of $\psi_{1,x}$, as computed in Macaulay 2 (see [10]), all of the modules must be in the space of minors.

For $c_1=6$, we have $\dim(I_{\kappa_1\leqslant 6})_7 = 1296 = \binom{9}{7}^2$, and hence we see that all 7×7-minors of $\psi_{1,x}$ are linearly independent. By contrast, if $c_1=5$, then the fact that $\dim_{\mathbb{C}}(I_{\kappa_1\leqslant 5})_6 = 7011 < 7056 = \binom{9}{6}^2$ tells us that the 6×6 minors of $\psi_{1,x}$ are not all linearly independent. For example, the upper right and lower left 6×6 minors of

$$\psi_{1,x} = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}$$

are both equal to $\det(A_1) \cdot \det(A_3)$.

In the next section, we impose partial symmetry on our 3-tensors. We remark that we could impose other types of symmetry and this would lead to different investigations. For instance, we could restrict attention to 3-tensors in any of the following cases: $S^3(U^*)$, $U^* \otimes \bigwedge^2 V^*$, $\bigwedge^3 V^*$, or $S_{2,1}U^*$. In these cases, it would be straightforward to prove analogs of Lemma 2.1 and Proposition 2.2. However, if we hope to produce the ideal defining the appropriate secant varieties, then it is less obvious how to generalize Definition 3. It might be interesting to investigate the secant varieties of these other special types of 3-tensors.

3. *The κ-invariant for partially symmetric 3-tensors*

For the rest of the paper, we take $W = V$ and focus on partially symmetric 3-tensors $x \in U^* \otimes S^2V^* \subset U^* \otimes V^* \otimes V^*$. By this latter inclusion, we may extend the definition of $\kappa_j(x)$ to partially symmetric tensors. Not only does the κ-invariant provide a bound for the rank of x, but also for the partially symmetric rank, which is defined as the minimal r such that $x = \sum_{i=1}^r u_i \otimes v_i \otimes v_i$, for some $u_i \in U^*$ and $v_i \in V^*$. The set of rank-1 partially symmetric tensors is known as the Segre–Veronese variety of $\mathbb{P}U^* \times \mathbb{P}V^*$ embedded by $\mathcal{O}(1,2)$. Therefore, the Zariski closure of the set of partially symmetric tensors of rank at most r is the rth secant variety of the Segre–Veronese variety. We have the following analog of Definition 2.

Definition 4. The partially symmetric border rank of $[x] \in \mathbb{P}(U^* \otimes S^2V^*)$ is the minimal r such that $[x]$ is in $\sigma_r(\mathrm{Seg}(\mathbb{P}U^* \times v_2(\mathbb{P}V^*)))$.

Providing an analog of the equations from Definition 3 is a bit more subtle in the case of partially symmetric 3-tensors. In fact, it is necessary to refine the equations if we hope to produce ideals that are radical. To see this, consider the case where x is a partially symmetric $3 \times n \times n$ tensor. For such an x, the matrix representing $\psi_{1,x}$ has the form

$$\psi_{1,x} = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix},$$

where the A_i are symmetric $n \times n$-matrices. Since $\psi_{1,x}$ is a skew-symmetric matrix, all of the principal minors in $I_{\kappa_1 \leqslant c_1}$ are squares.

More generally, if $m = 4j+3$, then $\psi_{2j+1,x} \colon V \otimes \bigwedge^{2j+1} U^* \to V^* \otimes \bigwedge^{2j+2} U^*$ is represented by a skew-symmetric matrix in appropriate coordinates. Thus, the condition that $\psi_{2j+1,x}$ has rank at most an even integer c_{2j+1} is defined algebraically by the principal $(c_{2j+1}+2) \times (c_{2j+1}+2)$-Pfaffians of $\psi_{2j+1,x}$. These Pfaffians have degree $c_{2j+1}/2+1$, whereas the $(c_{2j+1}+1) \times (c_{2j+1}+1)$-minors have degree $c_{2j+1}+1$.

To encode this skew-symmetry into our matrix equations in the case of partially symmetric tensors, we introduce the following analog of Definition 3.

Definition 5. Let $I_{\kappa_j \leqslant c_j}$ in $S^\bullet(U \otimes S^2V)$ denote the ideal generated by the $(c_j+2) \times (c_j+2)$-Pfaffians of $\psi_{j,x}$, if $j = (m-1)/2$, j is an odd integer, and c_j is even. Otherwise,

$I_{\kappa_j\leqslant c_j}$ denotes the specialization of the ideal in Definition 3. As in Definition 3, for a vector c, $I_{\kappa\leqslant c}$ is defined to be the ideal generated by the $I_{\kappa_j\leqslant c_j}$ for all j, and $\Sigma_{\kappa_i\leqslant c_i}$ and $\Sigma_{\kappa\leqslant c}$ are the subschemes of $\mathbb{P}(U^*\otimes S^2V^*)$ defined by $I_{\kappa_i\leqslant c_i}$ and $I_{\kappa\leqslant c}$, respectively.

Note that, for partially symmetric tensors x, we have $\kappa_j(x)=\kappa_{m-1-j}(x)$ and likewise $I_{\kappa_j(x)\leqslant c_j}=I_{\kappa_{m-1-j}\leqslant c_j}$ for all j. With notation in Definition 5, we also obtain the following analog of Proposition 2.2.

Proposition 3.1. *Fix $r\geqslant 1$. Let X be the Segre–Veronese variety of $\mathbb{P}(U^*)\times\mathbb{P}(V^*)$ in $\mathbb{P}(U^*\otimes S^2V^*)$ and let c be the vector defined by $c_j=r\binom{m-1}{j}$ for $0\leqslant j\leqslant m-1$. Then $\sigma_r(X)\subseteq\Sigma_{\kappa\leqslant c}$.*

Remark 4. The rest of the paper concerns partially symmetric $3\times n\times n$ tensors, and we note that the equations given in Definition 5 would be insufficient to generate the ideal of the secant varieties for $m\geqslant 4$. For example, consider the case of partially symmetric $4\times n\times n$ tensors. If x is such a tensor, then the matrix representing $\psi_{2,x}$ has the form

$$\psi_{2,x}=\begin{pmatrix} -A_4 & 0 & 0 & \mathbf{0} & \mathbf{A_3} & -\mathbf{A_2}\\ 0 & -A_4 & 0 & -\mathbf{A_3} & \mathbf{0} & \mathbf{A_1}\\ 0 & 0 & -A_4 & \mathbf{A_2} & -\mathbf{A_1} & \mathbf{0}\\ A_1 & A_2 & A_3 & 0 & 0 & 0 \end{pmatrix},$$

where each A_i is an $n\times n$ symmetric matrix. If x has border rank at most r, then $\psi_{1,x}$ will have rank at most $3r$ by Proposition 3.1. However, the bold submatrix in the upper right will have rank at most $2r$. Moreover, since the **bold** submatrix is skew-symmetric, the condition that it has rank at most $2r$ is given by the vanishing of its $(2r+2)\times(2r+2)$-principal Pfaffians. Thus, the defining ideal of the rth secant variety must contain these Pfaffians, as well as three other sets of Pfaffians that arise by symmetry. Since the Pfaffians have degree $r+1$, they cannot be in the ideal of the $(3r+1)\times(3r+1)$-minors.

In effect, these Pfaffians amount to the generators of $I_{\kappa_1\leqslant 2r}$ applied to a $3\times n\times n$ subtensor. In the literature on tensors, this process for producing equations on larger tensors by applying known equations to all subtensors is known as inheritance. See [**13**, § 2.1] for a precise definition in the language of representation theory. The above analysis shows how the inheritance of κ-equations can produce new equations beyond the κ-equations themselves.

Proposition 3.2. *As a Schur module, we have the following decomposition of the generators $I_{\kappa_0\leqslant r}$ into irreducible representations of* $\mathrm{GL}(U)\times\mathrm{GL}(V)$:

$$(I_{\kappa_0\leqslant r})_{r+1}=\bigoplus_{|\pi|=r+1} S_\pi U\otimes S_{\pi'+1^{r+1}}V,$$

where π' is the conjugate partition to π, and $1^{r+1}=(1,\dots,1)$ is the partition with $r+1$ parts.

In the proof, we need the following observation.

Lemma 3.3. *Suppose that π is a partition of d and that A is a vector space. If $S_\lambda A$ is a module occurring in the decomposition of $S_\pi(S^2A)$, then λ has at most d parts.*

Proof. Since π is a partition of d, we have an inclusion $S_\pi(S^2A) \subset (S^2A)^{\otimes d}$. The claim follows by induction and applying the Pieri formula to $(S^2A)^{\otimes d} = (S^2A)^{\otimes d-1} \otimes S^2A$. $\square$

Proof of Proposition 3.2. After choosing bases of U and V, we may view the map $\psi_{0,x}$ as a matrix of linear forms in $S^\bullet(U \otimes S^2V)$. By Lemma 2.3, $(I_{\kappa_0 \leqslant r})_{r+1}$ is the image of the $\mathrm{GL}(U) \times \mathrm{GL}(V)$-equivariant morphism

$$\Phi_0\colon \bigwedge^{r+1} V \otimes \bigwedge^{r+1}(U \otimes V) \to S^{r+1}(U \otimes S^2(V)),$$

which sends an indecomposable basis element in the source to the corresponding minor in the polynomial ring. The first step of our proof is to show that only those representations of the form $S_\pi U \otimes S_{\pi'+1^{r+1}} V$ appear in both the source and target of Φ_0. By Schur's Lemma, this will provide a necessary condition on the representations which can appear in the decomposition of $(I_{\kappa_0 \leqslant r})_{r+1}$. The second step of our proof is to show that each such representation actually arises; for this, we produce an explicit non-zero minor of $\psi_{0,x}$ that is in the image of Φ_0 restricted to $S_\pi U \otimes S_{\pi'+1^{r+1}} V$, so that Φ_0 restricted to $S_\pi U \otimes S_{\pi'+1^{r+1}} V$ is non-zero.

For the first step, suppose that $S_\pi U \otimes S_\lambda V$ is a module in $S^{r+1}(U \otimes S^2 V)$. If we apply the Cauchy decomposition formula to $S^d(U \otimes S^2V)$, and consider the resulting modules as $\mathrm{GL}(U)$-representations, then we must have $S_\pi U \otimes S_\lambda V$ contained in the summand $S_\pi U \otimes S_\pi(S^2V)$. In particular, we must have $S_\lambda V \subset S_\pi(S^2V)$. Therefore, by Lemma 3.3, λ has at most $r+1$ parts.

On the other hand, we can use the skew Cauchy formula to decompose

$$\bigwedge^{r+1} V \otimes \bigwedge^{r+1}(U \otimes V) = \bigwedge^{r+1} V \otimes \bigoplus_{|\pi|=r+1} S_\pi U \otimes S_{\pi'} V.$$

Applying the Pieri rule to $\bigwedge^{r+1} V \otimes S_{\pi'} V$, we see that all of the summands have more than $r+1$ parts except for $S_{\pi'+1^{r+1}} V$. Therefore, the decomposition of $I_{\kappa_0 \leqslant r}$ must consist only of the modules $S_\pi U \otimes S_{\pi'+1^{r+1}} V$, where π is a partition of $r+1$, and these modules must have multiplicity at most 1.

For the second step, fix π a partition of $r+1$. Suppose that $u_1, \dots, u_m$ is our ordered basis of U and $v_1, \dots, v_n$ is our ordered basis of V. Consider the indecomposable basis element

$$\begin{aligned} z_\pi = (v_1 \wedge \cdots \wedge v_{r+1}) \otimes \big(&(u_1 \otimes v_1) \wedge \cdots \wedge (u_1 \otimes v_{\pi_1}) \wedge (u_2 \otimes v_1) \wedge \cdots \wedge (u_2 \otimes v_{\pi_2}) \\ &\wedge \cdots \wedge (u_m \otimes v_1) \wedge \cdots \wedge (u_m \otimes v_{\pi_m}) \big) \end{aligned}$$

in $\bigwedge^{r+1} V \otimes \bigwedge^{r+1}(U \otimes V)$. We claim that z_π is in $S_\pi U \otimes S_{\pi'+1^{r+1}} V$, and, in fact is a non-zero highest weight vector in that representation. The vector z_π is non-zero because z_π is the tensor product of two tensors, each constructed as an exterior product of linearly independent tensors and hence non-zero. It is clear that z_π has weights π and $\pi' + 1^{r+1}$ in U and V, respectively, with respect to our chosen bases. Moreover, replacing v_i by v_j or u_i by u_j, with $j < i$ in either case, would result in a repeated term in the exterior product, and thus any raising operator would send z_π to zero, so z_π is a highest weight vector.

Let M_π be the submatrix of the block matrix $\psi_{0,x}^T = (A_1 \quad \cdots \quad A_m)$ defined by selecting the first $r+1$ rows and the first π_i columns of the ith block for each $i \leqslant n$. Then the map Φ_0 sends z_π to the determinant of M_π. For appropriate choices for A_i, we can make M_π equal the identity matrix, and therefore $\Phi(z_\pi) = \det(M_\pi)$ is non-zero. $\square$

In the case $\dim(U) = 3$, we similarly produce a formula for the decomposition of the modules generating $I_{\kappa_1 \leqslant 2r}$ in Proposition 3.4.

REMARK 5. Taken together, Propositions 3.2 and 5.1 provide a complete Schur module description of the generators of the ideal of any secant variety of the $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$. Similarly, Propositions 3.2 and 3.4, together with Theorem 5.2, provide a complete Schur module description of the generators of the ideal of the rth secant variety of $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$ for r at most 5.

PROPOSITION 3.4. *Suppose that* $\dim(U)$ *is* 3. *As a Schur module, we have the following decomposition of the generators* $I_{\kappa_1 \leqslant 2r}$ *into irreducible representations of* $\mathrm{GL}(U) \times \mathrm{GL}(V)$

$$(I_{\kappa_1 \leqslant 2r})_{r+1} = \bigoplus_{|\pi|=r+1} S_\pi U \otimes S_{(3)^{r+1}-\pi'} V,$$

where π' *is the conjugate partition to* π. *In order for the summand to be non-zero,* π *must have at most three parts, and* π_3 *must be at least* $r+1-n$ *(if the latter is positive).*

Proof. We consider the Pfaffians of a matrix representing the map $\psi_{1,x}\colon V \otimes U^* \to V^* \otimes \bigwedge^2 U^*$. In order to view $\psi_{1,x}$ as a skew-symmetric transformation, we identify $\bigwedge^2 U^*$ in the target with $U \otimes \bigwedge^3 U^*$. Then, we can view $\psi_{1,x}$ as a skew-symmetric form on $V \otimes U^*$, taking values in $\bigwedge^3 U^*$. Equivalently, a choice of a non-zero element in $\bigwedge^3 U$ gives a $\mathbb{C}$-valued skew-symmetric form.

The remainder of our proof essentially follows the same two steps as the proof of Proposition 3.2. The space of $(2r+2) \times (2r+2)$-Pfaffians of a skew-symmetric form on $V \otimes U^*$ is isomorphic to $\bigwedge^{2r+2}(V \otimes U^*)$. Therefore, similar to Lemma 2.3, $(I_{\kappa_1 \leqslant 2r})_{r+1}$ is the image of the map

$$\Phi_1\colon \ \bigwedge^{2r+2}(V \otimes U^*) \otimes (\bigwedge^3 U)^{r+1} \to S^{r+1}(U \otimes S^2(V)),$$

which sends an indecomposable basis element to the corresponding Pfaffian. Note that the power of $r+1$ in $(\bigwedge^3 U)^{r+1}$ is because the Pfaffian has degree $r+1$. First, we show that only modules of the form $S_\pi U \otimes S_{(3)^{r+1}-\pi'} V$ can arise as a representation in both the source and target of Φ_1. Second, we consider Φ_1 restricted to $S_\pi U \otimes S_{(3)^{r+1}-\pi'} V$ and we produce a Pfaffian in the image and explicitly show that it is non-zero. By Schur's Lemma, this will show that every such representation actually arises in the decomposition of $(I_{\kappa_1 \leqslant 2r})_{r+1}$.

For the first step, we use the skew Cauchy formula to decompose the source of Φ_1 as

$$\begin{aligned}\bigwedge^{2r+2}(U^* \otimes V) \otimes (\bigwedge^3 U)^{r+1} &= \bigoplus_{|\lambda|=2r+2} S_\lambda U^* \otimes S_{\lambda'} V \otimes (\bigwedge^3 U)^{r+1} \\ &= \bigoplus_{|\lambda|=2r+2} S_{(r+1)^3-\lambda} U \otimes S_{\lambda'} V,\end{aligned}$$

where we have used the duality formula from Remark 3 for the second equality. Every module in the source of Φ_1 is thus of the form $S_{(r+1)^3-\lambda} U \otimes S_{\lambda'} V$, where λ is a partition of $2r+2$. We make the substitution $\lambda = (r+1)^3 - \pi$ to arrive at the expression in the statement of the proposition.

For the second step, we explicitly produce a non-zero Pfaffian of $\psi_{1,x}$ in the image of Φ_1 restricted to $S_{\pi'} U \otimes S_{(r+1)^3-\pi} V$, and thus confirm that every module of the form $S_{\pi'} U \otimes S_{(r+1)^3-\pi} V$ (for appropriate π) occurs in the decomposition of $(I_{\kappa_1 \leqslant 2r})_{r+1}$. Suppose that u_1, u_2, u_3 is our ordered basis for U and $v_1, \ldots, v_n$ is our ordered basis for V. Let $\pi = (\pi_1, \pi_2, \pi_3)$ be a partition of $r+1$ with no more than three parts, and let $\lambda = (r+1)^3 - \pi = (r+1-\pi_3, r+1-\pi_2, r+1-\pi_1)$, as before. Consider the element

$$\begin{aligned}z_\pi = ((u_1^* \otimes v_1) \wedge \ldots \wedge (u_1^* \otimes v_{\lambda_3}) \wedge (u_2^* \otimes v_1) \wedge \ldots \wedge (u_2^* \otimes v_{\lambda_2}) \wedge (u_3^* \otimes v_1) \wedge \ldots \wedge (u_3^* \otimes v_{\lambda_1}))\\ \otimes (u_1 \wedge u_2 \wedge u_3)^{\otimes r+1}\end{aligned}$$

in $\bigwedge^{2r+2}(U^* \otimes V) \otimes (\bigwedge^3 U)^{r+1}$, where the u_i^* form the dual basis to the u_i. Note that z_π is non-zero since the vectors in each exterior product are linearly independent. Now, we will show that z_π is a highest weight vector in $S_\pi U \otimes S_{\lambda'} V$.

First, we claim that z_π has weight (π, λ'). By counting the occurrences of v_i in z_π, it is clear that the weight in the V-factor is λ'. For the U factor, we note that the weight of u_i^* is the negative of that of u_i, so that z_π has weight $(r+1-\lambda_3, r+1-\lambda_2, r+1-\lambda_1)$, which is equal to π.

Second, we must show that any raising operator will send z_π to zero, which will imply that z_π is a highest weight vector. In the V factor, sending v_i to v_j with $j < i$ would force a repeated vector in the exterior product. Likewise, a raising operator applied to the U factor would send u_i^* to u_j^* with $j > i$, which would, again, create a repeated factor in the exterior product.

Finally, we check that $\Phi_1(z_\pi) \neq 0$. Let M_π be the principal submatrix obtained from $\psi_{1,x}$ by selecting the rows and columns with indices $\{1, \ldots, \lambda_3, n+1, \ldots, n+\lambda_2, 2n+1, \ldots, 2n+\lambda_1\}$. Then $\Phi_1(z_\pi)$ equals the Pfaffian of M_π. To check that the Pfaffian of M_π is non-zero, it suffices to produce a specialization of M_π which has full rank. Note that if B_i is the appropriate submatrix from the upper-left corner of A_i, then M_π has the following shape:

$$M_\pi = \begin{array}{c} \\ \lambda_3 \\ \lambda_2 \\ \lambda_1 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \lambda_3 & \lambda_2 & \lambda_1 \end{array} \\ \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3^t & 0 & B_1 \\ B_2^t & -B_1^t & 0 \end{pmatrix} \end{array}.$$

We have $\lambda_1 = \pi_1 + \pi_2$, $\lambda_2 = \pi_1 + \pi_3$, $\lambda_3 = \pi_2 + \pi_3$. If we specialize the A_i such that the B_i are as follows

$$B_1 = \begin{array}{c} \\ \pi_1 \\ \pi_3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \pi_2 & \pi_1 \end{array} \\ \begin{pmatrix} 0 & \mathrm{Id}_{\pi_1} \\ 0 & 0 \end{pmatrix} \end{array}, \quad B_2 = \begin{array}{c} \\ \pi_3 \\ \pi_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \pi_2 & \pi_1 \end{array} \\ \begin{pmatrix} 0 & 0 \\ -\mathrm{Id}_{\pi_2} & 0 \end{pmatrix} \end{array}, \quad \text{and} \quad B_3 = \begin{array}{c} \\ \pi_3 \\ \pi_2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \pi_1 & \pi_3 \end{array} \\ \begin{pmatrix} 0 & \mathrm{Id}_{\pi_3} \\ 0 & 0 \end{pmatrix} \end{array},$$

then the specialization of M_π has full rank, since it is the standard block skew-symmetric matrix. $\square$

EXAMPLE 7. Consider the case $n = 4$ and $c_1 = 10$. Since $\psi_{1,x}$ is a skew-symmetric 12×12 matrix, we expect the ideal $I_{\kappa_1 \leqslant 10}$ to be a principal ideal, generated by a polynomial in $S^6(U \otimes S^2(V))$. Applying Proposition 3.4, we must have a sum over partitions π of 6 such that $(3)^6 - \pi'$ has at most four parts. This forces π' to equal $(3,3)$, and thus $\pi = (2,2,2)$. The generators of $I_{\kappa_1 \leqslant 10}$ are therefore equal to the 1-dimensional representation $S_{2,2,2}(U) \otimes S_{3,3,3,3}(V)$, corresponding to the Pfaffian of $\psi_{1,x}$.

EXAMPLE 8. Consider the case $n = 4$ and $c = (3,6,3)$, which we revisit in Example 9. Propositions 3.2 and 3.4 give us the decompositions

$$(I_{\kappa_0 \leqslant 3})_4 = S_{2,2}S_{3,3,1,1} \oplus \mathbf{S_{2,1,1}S_{4,2,1,1}} \oplus S_{3,1}S_{3,2,2,1} \oplus S_4 S_{2,2,2,2},$$
$$(I_{\kappa_1 \leqslant 6})_4 = S_{2,2}S_{3,3,1,1} \oplus \mathbf{S_{2,1,1}S_{3,3,2}} \oplus S_{3,1}S_{3,2,2,1} \oplus S_4 S_{2,2,2,2}.$$

Both modules are 495-dimensional and consist of quartic polynomials. The ideal $I_{\kappa \leqslant (3,6,3)}$, which equals $I_{\kappa_6 \leqslant 6} + I_{\kappa_0 \leqslant 3}$ by definition, is generated by the 630-dimensional space of quartics obtained by taking the sum of the above decompositions. Note that, due to the highlighted modules in the above decompositions, neither $I_{\kappa_0 \leqslant 3}$ nor $I_{\kappa_1 \leqslant 6}$ contains the other. In particular, the 4×4-minor formed by taking columns 1, 2, 5 and 9 from the flattening $\psi_{0,x}$, which is the transpose of (1.1), is not in the ideal of Pfaffians. On the other hand, the Pfaffian formed by taking the rows and columns of (1.2) with indices 1, 2, 5, 6, 7, 9, 10 and 11 is not contained in the ideal of the minors.

Note that the formulas in Propositions 3.2 and 3.4 are multiplicity free, unlike, for example, the ideal generators computed in Example 6.

4. *Subspace varieties of partially symmetric tensors*

We next give a geometric interpretation for the varieties $\Sigma_{\kappa_0\leqslant r}$. These are the partially symmetric analogs of the subspace varieties defined in [**16**, Definition 1]; Proposition 4.1 forms an analog to [**16**, Theorem 3.1].

Definition 6. The subspace variety $\mathrm{Sub}_{m',n'}$ is the variety of tensors $x \in (U^* \otimes S^2V^*)$ such that there exist vector spaces $\tilde{U}^* \subset U^*$ and $\tilde{V}^* \subset V^*$ of dimensions m' and n', respectively, with $x \in (\tilde{U}^* \otimes S^2\tilde{V}^*) \subset (U^* \otimes S^2V^*)$.

Remark 6. The variety $\mathrm{Sub}_{m',n'}$ has a nice desingularization, analogous to that used to prove the results in [**16**, § 3]. Consider the product of Grassmannians $\mathrm{Gr}(m', U^*) \times \mathrm{Gr}(n', V^*)$, and let E be the total space of the vector bundle $\mathcal{R}_U \otimes S^2\mathcal{R}_V$, where $\mathcal{R}_U$ and $\mathcal{R}_V$ are the tautological subbundles over $\mathrm{Gr}(m', U^*)$ and $\mathrm{Gr}(n', V^*)$, respectively. Then there is a natural map $\pi\colon E \to \mathrm{Sub}_{m',n'}$, which desingularizes $\mathrm{Sub}_{m',n'}$. Moreover, one can verify that Weyman's geometric technique can be applied in this situation [**25**, § 5]. In fact, a straightforward adaptation of the argument in [**16**, Theorem 3.1] implies that $\mathrm{Sub}_{m',n'}$ is normal with rational singularities.

We next directly calculate the generators of the ideal of the subspace variety when $m' = m$, which is the case we need.

Proposition 4.1. *The defining ideal of* $\mathrm{Sub}_{m,n'}$ *equals* $I_{\kappa_0\leqslant n'}$.

The following lemma plays a crucial technical role in the proof of both Proposition 4.1 and Theorem 5.2, as it provides a criterion for determining the reducedness of some of the ideals that we are studying.

Lemma 4.2. *Let Z be a $\mathrm{GL}(U) \times \mathrm{GL}(V)$-invariant reduced subscheme of the desingularization E from Remark* 6. *Suppose that I is an invariant ideal in $S^\bullet(U \otimes S^2V)$, which contains the ideal of $\mathrm{Sub}_{m',n'}$, and whose pullback to E defines Z. Then I is the ideal of $\pi(Z)$. In particular, I is a radical ideal.*

Proof. Let $J \subseteq S^\bullet(U \otimes S^2V)$ be the defining ideal of $\pi(Z)$. Recall that $q\colon E \to \mathrm{Gr}(m', U^* \times \mathrm{Gr}(n', V^*)$ is the total space of a vector bundle, as defined in Remark 6. Our setup is the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & E \arrow[d, "\pi"] & \\ \pi(Z) \arrow[r, hook] & \mathrm{Sub}_{m',n'} \arrow[r, hook] & (U^* \otimes S^2V^*). \end{tikzcd}$$

The hypothesis that the pullback of I defines Z in E guarantees $I \subseteq J$. We thus need to show the reverse inclusion.

A point $P \in \mathrm{Gr}(m', U^*) \times \mathrm{Gr}(n', V^*)$ corresponds to vector subspaces $\tilde{U}^* \subset U^*$ and $\tilde{V}^* \subset V^*$, and this induces a surjection of rings $\phi_P\colon S^\bullet(U \otimes S^2V) \to S^\bullet(\tilde{U} \otimes S^2\tilde{V})$. The fiber $q^{-1}(P) \subseteq E$ is isomorphic to the affine space $(\tilde{U}^* \otimes S^2\tilde{V}^*)$, and we define $\tilde{Z}_P := Z \cap q^{-1}(P)$.

We claim that a polynomial $f \in S^\bullet(U \otimes S^2V)$ belongs to J if and only if $\phi_P(f)$ vanishes on $\pi(\tilde{Z}_P)$ for every choice of P. The 'only if' direction of the claim is straightforward. For the 'if' direction, we first note that, since Z is assumed to be reduced, the condition that $f \in J$ is equivalent to the condition that the pullback of f vanishes on every point $y \in \pi(Z)$. This is in turn equivalent to the condition f vanishes on each point of Z, which is implied by the hypothesis that $\phi_P(f)$ vanishes on $\pi(\tilde{Z}_P)$ for each P.

In fact, since $\mathrm{GL}(U) \times \mathrm{GL}(V)$ acts transitively on $\mathrm{Gr}(m', U^*) \times \mathrm{Gr}(n', V^*)$, we conclude that f vanishes on $\pi(Z)$ if and only if $\phi_P(g \cdot f)$ vanishes on $\pi(\tilde{Z})$ for *any* fixed choice of P and all $g \in \mathrm{GL}(U) \times \mathrm{GL}(V)$. Therefore, J is the sum of all irreducible Schur submodules M of $S^\bullet(U \otimes S^2V)$ such that $\phi_P(M)$ vanishes on $\tilde{Z}_P$. For the rest of the proof we fix $\tilde{U}^*$ and $\tilde{V}^*$, and denote the induced map ϕ_P by ϕ and $\tilde{Z}_P$ by $\tilde{Z}$.

To show that $J \subseteq I$, let M be an irreducible Schur submodule of the ideal J; we want to show that M is contained in our given ideal I. If M is isomorphic to $S_\mu U \otimes S_\nu V$, then the construction of Schur modules implies that $\phi(M)$ is isomorphic as a $\mathrm{GL}(\tilde{U}) \times \mathrm{GL}(\tilde{V})$-representation to $S_\mu \tilde{U} \otimes S_\nu \tilde{V}$, which is either trivial or an irreducible representation. We know that $\phi(M)$ vanishes on $\tilde{Z}$ and thus, since I pulls back to the defining ideal of Z, it follows that $\phi(M)$ is contained in $\phi(I)$. There is thus an irreducible Schur submodule $N \subset I$ such that $\phi(N) = \phi(M)$, and hence N is isomorphic to $S_\mu U \otimes S_\nu V$. If N equals M, then we are done. Otherwise, ϕ sends the submodule $N + M$, spanned by two copies of $S_\mu U \otimes S_\nu V$, to the submodule $\phi(M)$, which is a single copy of $S_\mu \tilde{U} \otimes S_\nu \tilde{V}$. Thus, some subrepresentation L of $N + M$ is sent to zero by ϕ. Since L is a representation in the kernel of ϕ, L belongs to the ideal of $\mathrm{Sub}_{m',n'}$, which is contained in I by assumption. It follows that I contains the span of N and L, and hence I contains M. We conclude that $I = J$, as desired. □

Proof of Proposition 4.1. First, we prove the claim set-theoretically. The $(n'+1) \times (n'+1)$ minors of $\psi_{0,x}$ vanish if and only if the map has rank at most n'. By linear algebra, this is equivalent to the existence of a change of basis in which $\psi_{0,x}$ uses only the first n' rows, which is the definition of $\mathrm{Sub}_{m,n'}$.

Second, we show that $I_{\kappa_0 \leqslant n'}$ is radical in the case when $n' = n-1$. Note that $\mathrm{Sub}_{m,n-1}$ has dimension $m\binom{n}{2} + n$, and thus $\mathrm{Sub}_{m,n-1}$ and $\Sigma_{\kappa_0 \leqslant n-1}$ have codimension $mn - n + 1$. This is the same as the codimension of the maximal minors of a generic $n \times mn$ matrix, so $I_{\kappa_0 \leqslant n-1}$ is Cohen–Macaulay by Eisenbud [**6**, Theorem 18.18], and it suffices to show that the affine cone over $\Sigma_{\kappa_0 \leqslant n-1}$ is reduced at some point. Consider a neighborhood of the point $u_1 \otimes v_1^2 + \ldots + u_1 \otimes v_{n-1}^2$. In coordinates around this point, $I_{\kappa_0 \leqslant n-1}$ consists of the maximal minors of the $n \times mn$ matrix:

$$\begin{bmatrix} 1 + x_{1,1,1} & \cdots & x_{1,1,n-1} & x_{1,1,n} & x_{2,1,1} & \cdots & x_{2,1,n} & \cdots & x_{m,1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots \\ x_{1,1,n-1} & \cdots & 1 + x_{1,n-1,n-1} & x_{1,n-1,n} & \vdots & & \vdots & \cdots & \vdots \\ x_{1,1,n} & \cdots & x_{1,n-1,n} & x_{1,n,n} & x_{2,1,n} & \cdots & x_{2,n,n} & \cdots & x_{m,n,n} \end{bmatrix}.$$

The $mn - n + 1$ minors that use the first $n - 1$ columns form part of a regular sequence, and thus the affine cone over $\Sigma_{\kappa_0 \leqslant n-1}$ is reduced in a neighborhood of this point. Since $I_{\kappa_0 \leqslant n-1}$ is a Cohen–Macaulay ideal, it follows that $\Sigma_{\kappa_0 \leqslant n-1}$ is everywhere reduced.

Third, we show that $I_{\kappa_0 \leqslant n'}$ defines $\mathrm{Sub}_{m,n'}$ for arbitrary n'. By reverse induction on n', we assume that $I_{\kappa_0 \leqslant n'}$ equals the ideal of $\mathrm{Sub}_{m,n'}$, and we seek to show equality for $n' - 1$. We will apply Lemma 4.2, where E is the vector bundle over $\mathrm{Gr}(m, U^*) \times \mathrm{Gr}(n', V^*) = \mathrm{Gr}(n', V^*)$ desingularizing $\mathrm{Sub}_{m,n'}$ as in Remark 6. Note that, by cofactor expansion, $I_{\kappa_0 \leqslant n'-1}$ contains

$I_{\kappa_0 \leqslant n'}$, which is the ideal of $\mathrm{Sub}_{m,n'}$ by the inductive hypothesis. We describe Z, which is defined by the pullback of $I_{\kappa_0 \leqslant n'-1}$, on a local trivialization $(U^* \otimes S^2\tilde{V}^*) \times Y$ of the vector bundle E, where $\tilde{V}$ is n'-dimensional and Y is an open subset of $\mathrm{Gr}(n', V^*)$. The pullbacks of the $(n'-1) \times (n'-1)$ minors of $\psi_{0,x}$ do not involve the base Y, and are the $\kappa_0 \leqslant n'-1$ equations applied to $U^* \otimes S^2\tilde{V}^*$. These are maximal minors of the matrix $\psi_{0,x}$ for $U^* \otimes S^2\tilde{V}^*$, and hence they define a reduced subscheme of $U^* \otimes S^2\tilde{V}^*$ by the previous paragraph. In the local trivialization, their scheme is the product of this reduced scheme with Y, so the preimage of $\Sigma_{\kappa_0 \leqslant n'}$ in E is reduced. We may thus apply Lemma 4.2 and conclude that $I_{\kappa_0 \leqslant n'-1}$ is reduced. □

Remark 7. When $m' < m$, the ideal of $\mathrm{Sub}_{m',n'}$ is similarly generated by the sum of $I_{\kappa_0 \leqslant n'}$ and the irreducible modules in $\bigwedge^{m'+1} U \otimes \bigwedge^{m'+1}(S^2V)$. A decomposition of the latter space, in somewhat different notation, can be found at [**19**, p. 47].

5. *Secant varieties of* $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ *embedded by* $\mathcal{O}(1,2)$

In this section, we prove the main result of our paper, which is to show that the equations given in Definition 3 generate the defining ideal of the rth secant variety of $\mathbb{P}^2 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$ when $r \leqslant 5$.

We first consider a simpler case: the secant varieties of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$. All such secant varieties are defined by κ-equations, which, in this case, are simply the minors of flattenings. The analogous statement for non-symmetric matrices appears as Theorem 1.1 in [**16**]. However, we know of no proof in the literature for the case of partially symmetric tensors, so we provide one below.

Definition 7. For a variety $X \subseteq \mathbb{P}^N$ we denote the affine cone of X in $\mathbb{A}^{N+1}$ by $\hat{X}$.

Proposition 5.1. *Suppose $m = 2$, and let $Y \subseteq \mathbb{P}(U^* \otimes S^2V^*)$ be the image of $\mathbb{P}(U^*) \times \mathbb{P}(V^*)$ under the embedding by $\mathcal{O}(1,2)$. For any $r > 1$ and any n, the secant variety $\sigma_r(Y)$ is defined ideal-theoretically by the ideal $I_{\kappa_0 \leqslant r}$.*

Proof. We have $\widehat{\sigma_r}(Y) \subseteq \hat{\Sigma}_{\kappa_0 \leqslant r} = \mathrm{Sub}_{2,r}$, where the inclusion follows from Proposition 3.1 and the equality follows from Proposition 4.1. Since $\mathrm{Sub}_{2,r}$ is integral, it suffices to prove that $\widehat{\sigma_r}(Y)$ and $\mathrm{Sub}_{2,r}$ have the same dimension. By Abo and Brambilla [**1**, Corollary 1.4(i)], the former has the expected dimension $rn + r$. From the definition of $\mathrm{Sub}_{2,r}$, we can compute its dimension to be $r(n-r) + 2\binom{r+1}{2} = rn + r$. □

For the remainder of this section, we restrict to the case when $\dim U^* = 3$, which is the next partially symmetric case. We let $\dim V^* = n$, and we consider partially symmetric tensors $x \in U^* \otimes S^2V^*$. We fix $\mathbb{P}^N := \mathbb{P}(U^* \otimes S^2(V^*))$ and we let $X \subset \mathbb{P}^N$ denote the embedding of $\mathbb{P}(U^*) \times \mathbb{P}(V^*)$ by $\mathcal{O}(1,2)$. Let $S := S^\bullet(U \otimes S^2(V))$ be the homogeneous coordinate ring of $\mathbb{P}^N$, which contains the ideals $I_{\kappa_j \leqslant c_j}$ and $I_{\kappa \leqslant c}$ as in Definition 5.

Theorem 5.2. *For $r \leqslant 5$ the defining ideal of the variety $\sigma_r(X)$ is $I_{\kappa \leqslant (r,2r,r)}$.*

Our method of proof is as follows. When n equals r, we relate the ideal $I_{\kappa_1 \leqslant 2r}$ to the ideal of commuting symmetric matrices. This is a variant of an idea that has appeared in several instances previously [**2**, **7**, **20**, **22**]. This relation only holds away from a certain closed subvariety of $\mathbb{P}^N$, and in order to extend to all of $\mathbb{P}^N$, we need a bound on the dimension of this variety. Such a bound is given in [**7**, § 5], and only holds for $r \leqslant 5$. Finally, we reduce the general case to the case of $n = r$, using Lemma 4.2.

Before the proof, we examine the secant varieties of $\mathbb{P}^2 \times \mathbb{P}^3$ in more detail.

Example 9. Let $X \subseteq \mathbb{P}^{29}$ be the image of $\mathbb{P}^2 \times \mathbb{P}^3$ embedded by $\mathcal{O}(1,2)$. The defining ideal of $\sigma_5(X)$ was previously known. The secant variety $\sigma_5(X)$ is deficient, and is in fact a hypersurface in $\mathbb{P}^{29}$. This hypersurface is defined by the Pfaffian of $\psi_{1,x}$ (see [**20**, Theorem 4.1]).

In the non-symmetric case, [**16**, Theorem 1.1] illustrates that the defining ideal for the second secant variety is generated by the 3×3 minors of the various flattenings. This suggests that a similar result holds in the partially symmetric case, although we know of no explicit reference for such a result. Nevertheless, in the situation of this example, a direct computation with [**10**] confirms that the defining ideal of $\sigma_2(X)$ is indeed generated by the 3×3 minors of the flattening $\psi_{0,x}$ and by the 3×3 minors of the other flattening of x, that is, by considering x in $\mathrm{Hom}(U, S^2V^*)$. Theorem 5.2 provides an alternate description, illustrating that the 3×3 minors of $\psi_{0,x}$ and the 6×6 principal Pfaffians of $\psi_{1,x}$ also generate the ideal of $\sigma_2(X)$.

As far we are aware, the defining ideals for $\sigma_3(X)$ and $\sigma_4(X)$ were not previously known. In the case of $\sigma_4(X)$, the defining ideal is given by $I_{\kappa \leqslant (4,8,4)}$. Since the ideals $I_{\kappa_0 \leqslant 4}$ and $I_{\kappa_2 \leqslant 4}$ are trivial, this equals the ideal $I_{\kappa_1 \leqslant 8}$. Thus, $\sigma_4(X)$ is defined by the 10×10 principal Pfaffians of $\psi_{1,x}$.

The case of $\sigma_3(X)$ is perhaps the most interesting, since this case requires minors from both $\psi_{0,x}$ and $\psi_{1,x}$ (and, unlike the case of $\sigma_2(X)$, the Pfaffians from $\psi_{1,x}$ do not arise from an alternative flattening). Here, $\sigma_3(X)$ is defined by the maximal minors of $\psi_{0,x}$, as well as the 8×8 principal Pfaffians of $\psi_{1,x}$. By Example 8, we see that neither $I_{\kappa_0 \leqslant 3}$ nor $I_{\kappa_1 \leqslant 6}$ is sufficient to generate the ideal of $\sigma_3(X)$.

In fact, neither $I_{\kappa_0 \leqslant 3}$ nor $I_{\kappa_1 \leqslant 6}$ is sufficient to define $\sigma_3(X)$ even set-theoretically. For $I_{\kappa_0 \leqslant 3}$, this follows from the fact that a generic element $y \in \Sigma_{\kappa_0 \leqslant 3}$ has $\kappa_1(y) = 8$. On the other hand, one may check that if

$$x := \sum_{i=1}^{3} u_i \otimes (v_1 \otimes v_{i+1} + v_{i+1} \otimes v_1) \in U^* \otimes S^2V^*,$$

then $\kappa(x) = (4, 6, 4)$, and hence $[x]$ belongs to $\Sigma_{\kappa_1 \leqslant 6}$, but not to $\sigma_3(X)$.

Remark 8. Let $x \in U^* \otimes S^2V^*$ and let $r \leqslant 5$. Theorem 5.2 implies that the border rank of x, considered as an element of $U^* \otimes V^* \otimes V^*$, equals the partially symmetric border rank of x. This is because the ideal $I_{\kappa \leqslant (r,2r,r)}$ is (up to radical) the restriction to $\mathbb{P}(U^* \otimes S^2V^*)$ of an ideal on $\mathbb{P}(U^* \otimes V^* \otimes V^*)$ which vanishes on the rth secant variety of $\mathbb{P}(U^*) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ (see Proposition 2.2 and Definition 5). This can thus be viewed as evidence for a partially symmetric analog of Comon's Conjecture [**5**, § 5].

Definition 8. If we write $x = e_1 \otimes A_1 + e_2 \otimes A_2 + e_3 \otimes A_3$ for e_1, e_2, e_3 a basis of U and the A_i symmetric matrices, then $\det(t_1A_1 + t_2A_2 + t_3A_3)$ is a polynomial in t_1, t_2 and t_3. We define $P \subset \mathbb{A}^{N+1}$ to be the subset of those x such that this polynomial vanishes identically.

Remark 9. Note that $\mathbb{A}^{N+1} - P$ is exactly the $\mathrm{GL}(U^*) \times \mathrm{GL}(V^*)$-orbit of the set $\{e_1 \otimes \mathrm{Id} + e_2 \otimes B + e_3 \otimes C \mid B, C \in S^2V^*\}$. This set was also studied by Wall [**24**] who described it as the unstable locus for the $\mathrm{SL}(U) \times \mathrm{SL}(V)$-action.

LEMMA 5.3. *Let $n = r$. Then $\hat{\Sigma}_{\kappa \leqslant (r,2r,r)} - P$ is an irreducible locus of codimension at least $\binom{r}{2}$ on $\mathbb{A}^{N+1} - P$.*

In fact, the codimension is exactly $\binom{r}{2}$, as will be shown in the proof of Lemma 5.5. See [20, Theorem 7.2] for a related result in a more general context.

Proof. Since $n = r$, and $\kappa_0 = \kappa_2$ are always at most n, we have that $\Sigma_{\kappa_1 \leqslant r} = \Sigma_{\kappa \leqslant (r,2r,r)}$. For convenience, we denote this scheme by Σ, and we seek to show that $\hat{\Sigma} - P$ is irreducible and of codimension $\binom{r}{2}$.

We let $\mathcal{W} \subseteq \mathbb{A}^{N+1}$ be the set $\{e_1 \otimes \mathrm{Id} + e_2 \otimes B + e_3 \otimes C \mid B, C \in S^2V^*\}$ as in Remark 9, and we identify points in $\mathcal{W}$ with pairs of symmetric matrices (B, C). Let $Z \subseteq \mathcal{W}$ be the subscheme defined by the equations $[B, C] = 0$. By Brennan, Pinto and Vasconcelos [3, Theorem 3.1], Z, known as the variety of commuting symmetric matrices, is an integral subscheme of codimension $\binom{r}{2}$ in $\mathcal{W}$.

We claim that $\hat{\Sigma} - P$ is irreducible. To see this, we note the following equivalence of matrices under elementary row and column operations:

$$\begin{bmatrix} 0 & \mathrm{Id} & -B \\ -\mathrm{Id} & 0 & C \\ B & -C & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & BC - CB \end{bmatrix}.$$

Therefore, the scheme-theoretic intersection of $\hat{\Sigma}$ with $\mathcal{W}$ is exactly Z, the variety of commuting symmetric matrices. By Remark 9 and the fact that κ_1 is $\mathrm{GL}(U^*) \times \mathrm{GL}(V^*)$-invariant, we see that $\hat{\Sigma} - P$ is exactly the $\mathrm{GL}(U^*) \times \mathrm{GL}(V^*)$ orbit of the irreducible variety Z, and therefore irreducible.

Finally, since $Z = \mathcal{W} \cap \hat{\Sigma}$, the codimension of $\hat{\Sigma} - P$ in $\mathbb{A}^{N+1}$ is at least the codimension of Z in $\mathcal{W}$, which is $\binom{r}{2}$. □

The following result is contained in [7, Proof of Corollary 5.6].

LEMMA 5.4. *If $n \leqslant 5$, then the codimension of P in $\mathbb{A}^{N+1}$ is strictly greater than $\binom{n}{2}$.*

LEMMA 5.5. *Let $n = r \leqslant 5$. Then $\sigma_r(X)$ is defined scheme-theoretically by $I_{\kappa_1 \leqslant 2r} = I_{\kappa \leqslant (r,2r,r)}$. Moreover, the ring $S/I_{\kappa_1 \leqslant 2r}$ is Gorenstein, that is, $\sigma_r(X)$ is arithmetically Gorenstein.*

Proof. The ideal of the principal $(2r+2) \times (2r+2)$-Pfaffians of a generic skew-symmetric matrix is a Gorenstein ideal of codimension $\binom{r}{2}$ (see [11, Theorem 17]). Our ideal $I_{\kappa_1 \leqslant 2r}$ is a linear specialization of this ideal, and by Lemmas 5.3 and 5.4, it must be irreducible and have the same codimension. Therefore, the linear specialization is defined by a regular sequence, so $\Sigma_{\kappa_1 \leqslant 2r}$ is also arithmetically Gorenstein and irreducible.

Hence, $\hat{\Sigma}_{\kappa_1 \leqslant 2r}$ is either reduced or everywhere non-reduced. As in the proof of Lemma 5.3, let $\mathcal{W} \subseteq \mathbb{A}^{N+1}$ be the linear space defined by $A_1 = \mathrm{Id}$, and consider the scheme-theoretic intersection $\hat{\Sigma}_{\kappa_1 \leqslant 2r} \cap \mathcal{W}$. Again, the codimension of $\hat{\Sigma}_{(r,2r,r)} \cap \Lambda$ in $\mathcal{W}$ is $\binom{r}{2}$, so the generators of the ideal of Λ form a regular sequence on the local ring of any point of $\hat{\Sigma}_{(r,2r,r)}$ contained in $\mathcal{W}$. The intersection is isomorphic to the variety of commuting symmetric matrices from the proof of Lemma 5.3, which is reduced. This implies that $\hat{\Sigma}_{\kappa\leqslant(r,2r,r)}$ is reduced as well, and thus that $\Sigma_{\kappa\leqslant(r,2r,r)}$ is reduced. $\square$

Proof of Theorem 5.2. Lemma 5.5 proves the theorem in the case when $n = r$, and so we just need to extend this result to the cases when $n \neq r$. We let $N' = 3\binom{r}{2} - 1$, so that $\mathbb{P}^{N'}$ is the projective space of partially symmetric $3 \times r \times r$ tensors. We write $X' \subset \mathbb{P}^{N'}$ for the image of $\mathbb{P}^2 \times \mathbb{P}^{r-1}$ embedded by $\mathcal{O}(1,2)$.

First, suppose that $n < r$. We pick an inclusion of V^* into $\mathbb{C}^r$, and also a projection from $\mathbb{C}^r$ back to V^*. These define an inclusion $\mathbb{P}^N \to \mathbb{P}^{N'}$ and a rational map $\pi\colon \mathbb{P}^{N'} \to \mathbb{P}^N$, respectively. Because the projection is linear, it commutes with taking secant varieties, so $\sigma_r(X) = \pi(\sigma_r(X'))$. Applying Lemma 5.5, we get the first equality of

$$\pi(\sigma_r(X')) = \pi(\Sigma_{\kappa_1\leqslant 2r}) \supset \pi(\Sigma_{\kappa_1\leqslant 2r} \cap \mathbb{P}^N) = \Sigma_{\kappa_1\leqslant 2r} \cap \mathbb{P}^N \supset \sigma_r(X) = \pi(\sigma_r(X')).$$

Note that the middle equality follows from the fact that π is the identity on $\mathbb{P}^N$. We conclude that $\sigma_r(X)$ is defined by $I_{\kappa_1\leqslant 2r}$, which is the statement of the theorem, since the conditions on κ_0 and κ_2 are trivial when $n < r$.

Second, we want to prove the theorem when $n > r$, for which we use Lemma 4.2. We consider the subspace variety $\mathrm{Sub}_{3,r} \subset \mathbb{A}^{N+1}$ and its desingularization $\pi\colon E \to \mathrm{Sub}_{3,r}$. By Proposition 4.1, $\mathrm{Sub}_{3,r}$ is the affine cone over $\Sigma_{\kappa_0\leqslant r}$, which contains $\hat{\Sigma}_{\kappa\leqslant(r,2r,r)}$. We set $Z := \pi^{-1}(\hat{\Sigma}_{\kappa\leqslant(r,2r,r)})$. Note that, along any fiber $U^* \otimes S^2\tilde{V}^*$ of $q\colon E \to \mathrm{Gr}(r, V^*)$, we have that $Z \cap (U^* \otimes S^2\tilde{V}^*)$ is defined by the $\kappa_1 \leqslant 2r$ equations applied to $U^* \otimes S^2\tilde{V}^*$. It follows that $Z \subseteq E$ is defined by the pullback of $I_{\kappa\leqslant(r,2r,r)}$. Since $\tilde{V}^*$ is r-dimensional, Lemma 5.5 implies that $Z \cap (U \otimes S^2\tilde{V}^*)$ is the cone over the rth secant variety of $\mathbb{P}(U^*) \times \mathbb{P}(\tilde{V}^*)$ in $U^* \otimes S^2\tilde{V}^*$. In particular, Z is reduced. We thus have the inclusions

$$\pi(Z) \subset \widehat{\sigma_r}(X) \subset \hat{\Sigma}_{\kappa\leqslant(r,2r,r)} = \pi(Z).$$

The first inclusion is clear, the second is by Proposition 3.1, and the equality follows from Lemma 4.2. Therefore, these schemes must be equal, which is the desired statement. $\square$

We conclude by observing that Theorem 5.2 is false for $r = 7$. (We do not know whether or not it holds for $r = 6$.)

EXAMPLE 10. Set $n = \dim V^* = 6$, in which case $\Sigma_{\kappa\leqslant(7,14,7)} = \Sigma_{\kappa_1\leqslant 14}$. Let X be the Segre–Veronese variety of $\mathbb{P}^2 \times \mathbb{P}^5$ embedded by $\mathcal{O}(1,2)$ in $\mathbb{P}^{62}$. We use a simple dimension count to show that the secant $\sigma_7(X)$ is properly contained in $\Sigma_{\kappa_1\leqslant 14}$.

The secant variety $\sigma_7(X)$ is not defective [**1**, Corollary 1.4(ii)], so it has the expected dimension, namely, $\dim \sigma_7(X) = 7 \cdot \dim X + 6 = 55$. On the other hand, since $I_{\kappa_1\leqslant 14}$ is a Pfaffian ideal, its codimension is at most $\binom{4}{2}$. We thus have

$$\dim \Sigma_{\kappa_1\leqslant 14} \geqslant \dim \mathbb{P}^{62} - \binom{4}{2} = 62 - 6 = 56.$$

Since $56 > 55$, it follows that $\sigma_7(X) \subsetneq \Sigma_{\kappa_1\leqslant 14}$.

Note that $\dim V^* = 6$ is the smallest dimension such that the seventh secant variety is properly contained within $\mathbb{P}^N$.

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