STRONGLY ANNULAR TSUJI FUNCTIONS

C. L. BELNA, D. D. BONAR, F. W. CARROLL,
P. COLWELL AND G. PIRANIAN

1. *Annular functions*

A holomorphic function f in the unit disk D is *annular* if there exists a nested sequence $\{J_n\}$ of Jordan curves surrounding the origin such that each annulus $\{z : 1-\varepsilon < |z| < 1\}$ contains infinitely many of the curves J_n and such that the minimum of $|f(z)|$ on J_n tends to ∞ as $n \to \infty$. The function f is *strongly annular* if for $\{J_n\}$ we can choose a sequence of concentric circles.

2. *Tsuji functions and Julia points*

A holomorphic or meromorphic function f in D is a *Tsuji function* if in the interval $0 < r < 1$ the spherical length $L(r) = L(r, f)$ of the image of the circle $\gamma_r = \{z : |z| = r\}$ is a bounded function of r.

If f is meromorphic in D, if S is a line segment lying in D except for one endpoint, and if in each Stolz angle bisected by S the function f assumes every value in the extended plane, with at most two exceptions, then S is a *segment of Julia* for f. If each segment in D terminating at the point $e^{i\theta}$ is a segment of Julia for f, then $e^{i\theta}$ is a *Julia point* for f.

In [**4**], E. F. Collingwood and Piranian showed that the set of Julia points of a meromorphic Tsuji function may consist of the entire unit circle. They also showed that the holomorphic function

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)^2$$

is a Tsuji function with two segments of Julia, and they conjectured that a holomorphic Tsuji function can have at most finitely many segments of Julia. In [**6**], W. K. Hayman disproved this conjecture by constructing a holomorphic Tsuji function with a countably infinite set of Julia points. We shall extend Hayman's result by showing that the set of Julia points of a holomorphic Tsuji function may consist of the entire unit circle.

3. *Strongly annular functions*

The question whether a Tsuji function can be annular has circulated for several years (its first recorded discussion occurs in [**1**; Section 4]). We shall consider the problem in the context of strong annularity.

A natural way of constructing a strongly annular function is to choose a rapidly increasing sequence $\{k_n\}$ of positive integers and to define

$$f(z) = \sum k_n z^{k_n}.$$

Received 28 May, 1977; revised 24 April, 1978.

F. W. Carroll acknowledges support from the National Science Foundation.

[J. London Math. Soc. (2), 19 (1979), 79–92]

Because on the circle $|z| = 1-1/k_n$ the nth term dominates and its modulus is approximately k_n/e, the function f is strongly annular. Because f maps the circle $|z| = 1-1/k_n$ onto a curve that encircles the origin k_n times and at a distance approximately k_n/e, the spherical length $L(1-1/k_n, f)$ is approximately $k_n(2\pi e/k_n) = 2\pi e$. Nevertheless, f is not a Tsuji function; for on a certain circle of radius slightly greater than $1-1/k_n$, the nth and $(n+1)$st term in the series for f have equal modulus. The image on the Riemann sphere of that circle consists of many loops of sufficiently large diameter so that the length of the curve is large.

These considerations suggest that strong annularity and bounded spherical length might be incompatible. However, it has been known for some time that neither lacunarity nor unboundedness of the Taylor coefficients is essential for strong annularity. R. W. Howell [8] has shown that under the topology of locally uniform convergence, a residual subset of the space of functions $\Sigma\, \varepsilon_n z^n$ $(\varepsilon_n = \pm 1)$ consists of strongly annular functions. Also, Bonar, Carroll, and Piranian [2] have proved that if the sequence $\{k_n\}$ increases rapidly enough, then the function

$$f(z) = \sum_{n=1}^{\infty} \frac{{k_n}^{1/3}}{n^{1/2}} \left(\frac{z-1/2}{1-z/2}\right)^{k_n}$$

is strongly annular, although the sequence of its Taylor coefficients at the origin tends to 0.

Aware that strongly annular functions are more abundant than we had originally believed, we undertook an intensive search for a strongly annular Tsuji function. The example developed in Sections 4 to 13 resembles the function f given in the preceding paragraph; but there are two noticeable differences: In the new example, the ratio between the coefficient and the exponent in the nth term is much larger, and the sequence $\{k_n\}$ of exponents is subject to careful control.

In Section 14, we use a recent result of S. Dragosh [5] to obtain a theorem on the union of the sets of Julia points and points of spherical continuity of functions in Hayman's class T_2 (this class is substantially larger than the class of Tsuji functions).

4. *The main result*

THEOREM 1. *There exists a strongly annular Tsuji function whose set of Julia points is the unit circle.*

To prove the theorem, we shall construct a function

$$f(z) = \sum a_n z^{k_n} \tag{1}$$

such that, for each Möbius transformation

$$g_b(z) = \frac{z-b}{1-\bar{b}z} \qquad (0 < |b| < 1),$$

the composite function $f_b = f \circ g_b$ is a strongly annular Tsuji function. That each point $e^{i\theta}$ is a Julia point for f and for each f_b will be a trivial consequence of the construction.

5. *The coefficients and exponents*

The motivation for our choice of the coefficients a_n and the exponents k_n in (1) will become apparent in our later computations. At this stage, we merely plead the need for rapid growth of the sequence $\{k_n\}$ and for a carefully adjusted relation between $\{a_n\}$ and $\{k_n\}$. If the sequence $\{k_n\}$ grows too slowly, the annuli in which two consecutive terms of the series (1) have approximately equal modulus are not sufficiently separated, and unless we choose the coefficients a_n with proper care, the troublesome annuli are too broad for our purposes.

We set $k_1 = h_1 = 1$, and we defer until later the choice of the coefficient a_1. For $n = 2, 3, \ldots$, we define

$$h_n = a_{n-1}/h_{n-1}, \tag{2}$$

$$a_n = \exp h_n/h_{n-1}; \tag{3}$$

for each index n, we set $k_n = [h_n]$ (greatest-integer notation). A cursory estimate shows that if $a_1 \geqslant 7$, the sequences $\{a_n\}$ and $\{k_n\}$ grow rapidly; but we need a quantitative formulation of this.

LEMMA 1. *If* $p = 2, 3, \ldots$ *and* $a_1 \geqslant (p+6)!$, *then* $h_n > h_{n-1}^p$ *for* $n = 2, 3, \ldots$.

Proof. Since $h_1 = 1$ and $h_2 = a_1$, the inequality in the lemma is obviously satisfied for $n = 2$. Suppose it holds for some integer n. By condition (3), the inequality then implies that

$$a_n > \exp h_n^{1-1/p},$$

and from the power series expansion of the exponential function we deduce that

$$a_n > \frac{h_n^{(1-1/p)j}}{j!}$$

for each natural number j. From (2) it follows further that

$$h_{n+1} > \frac{1}{j!}\, h_n^{(1-1/p)j-1}.$$

For the special choice $j = p+6$, the exponent on the right is not less than $p+1$, and if in addition $h_n > j!$, it follows that $h_{n+1} > h_n^p$. In particular, if we choose $a_1 = j! = (p+6)!$, the required inequality holds for all indices n.

6. *One-term dominance*

On the circle $|z| = r_n = 1 - 1/h_n$, the value of $|z|^{k_n}$ is approximately $1/e$; therefore the modulus of the nth term in (1) is approximately a_n/e. We shall show that this is much larger than the sum of all earlier and all later terms.

By (2), $a_m = h_m h_{m+1}$, and by Lemma 1, we may assume that

$$\frac{a_m}{a_{m-1}} > \frac{h_m^{p+1}}{h_m h_{m-1}} = h_m^p/h_{m-1},$$

where p is some integer greater than 1. Therefore the sequence $\{a_m\}$ increases so rapidly that

$$a_1 + \ldots + a_{n-1} = o(a_n/e).$$

To estimate the later terms, we note that

$$a_m r_n^{k_m} = \exp\left(\frac{h_m}{h_{m-1}} + k_m \log\left(1 - \frac{1}{h_n}\right)\right) < \exp\left(\frac{h_m}{h_{m-1}} - \frac{h_m - 1}{h_n}\right).$$

If $m = n+1$, the last expression reduces to $\exp(1/h_n)$, and this is less than 3. If $m > n+1$, we use the inequality $h_m/h_{m-1} < h_m/3h_n$ and deduce that

$$a_m r_n^{k_m} < \exp(1 - h_m/2h_n).$$

Invoking Lemma 1 again, we see that

$$f(r_n e^{i\theta}) \sim a_n e^{ik_n\theta - 1}.$$

7. *The dirty annuli*

In Section 6, we saw that on the circle $|z| = r_n = 1 - 1/h_n$, the $(n+1)$st term of the series in (1) has modulus less than 3. We shall now examine the modulus of the same term on the slightly larger circle $|z| = r_n + 1/h_n^2$. By elementary computations,

$$a_{n+1}(1 - 1/h_n + 1/h_n^2)^{k_{n+1}} \geqslant a_{n+1}\left(1 - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right)\right)^{h_{n+1}}$$

$$= \exp\left\{\frac{h_{n+1}}{h_n} + h_{n+1} \log\left(1 - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right)\right)\right\}$$

$$= \exp\left\{h_{n+1}\left[\frac{1}{h_n} - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right) - \frac{1}{2h_n^2}\left(1 - \frac{1}{h_n}\right)^2 - \ldots\right]\right\}$$

$$\sim \exp h_{n+1}/2h_n^2.$$

By Lemma 1, we may assume that $h_{n+1} > 3h_n^3$, so that on the circle $|z| = r_n + 1/h_n^2$ the $(n+1)$st term in (1) has modulus greater than $\exp h_n$. By (3), this is much greater than a_n, and therefore the $(n+1)$st term dominates the series, on the circle

$$|z| = r_n + 1/h_n^2.$$

We call the annulus

$$r_n < |z| < r_n + 1/h_n^2$$

the nth *dirty annulus*, and we observe that each dirty annulus is narrow, compared with its distance from the unit circle. Because the $(n+1)$st term of (1) dominates the sum of all other terms, both on the outer boundary component of the nth dirty annulus and on the inner boundary component of the $(n+1)$st dirty annulus, and because the $(n+1)$st term has no zeros except at the origin, we can assert that

$$f(z) \sim a_{n+1} z^{k_{n+1}}$$

in the domain between the dirty annuli. Consequently, f is strongly annular.

8. *Strong annularity of the function f_b*

If $b = \beta e^{i\phi}$, then the function $g_b(z) = (z-b)/(1-\bar{b}z)$ attains its maximum and minimum moduli on the circle $|z| = r$ at the points $z = -re^{i\phi}$ and $z = re^{i\phi}$, and the two moduli are

$$\frac{r+\beta}{1+r\beta} = 1-(1-r)\frac{1-\beta}{1+r\beta}$$

and

$$\frac{|r-\beta|}{1-r\beta} = \left|1-(1-r)\frac{1+\beta}{1-r\beta}\right|.$$

If r is near 1, the coefficients of $1-r$ in the two expressions are approximately

$$(1-\beta)/(1+\beta) \qquad \text{and} \qquad (1+\beta)/(1-\beta),$$

respectively. In particular, the function g_b maps the circle $|z| = 1-1/h_n^2$ onto a circle lying between the nth and $(n+1)$st dirty annuli. By the argument in Section 7, each function f_b is strongly annular.

9. *The image of the circle $|z| = r$*

For each value b in D, the function

$$f_b(z) = \sum a_n \left(\frac{z-b}{1-\bar{b}z}\right)^{k_n}$$

maps each circle $\gamma_r = \{z : |z| = r\}$ onto a more or less convoluted curve Γ_r. We shall devote several sections to the task of showing that the spherical length $L(r, f_b)$ of Γ_r is a bounded function of r.

The analysis in Sections 6 and 7 shows that on the circle $|z| = r_n$ the nth term of the series in (1) is much larger than the sum of all the other terms, while on the circle $|z| = r_n + 1/h_n^2$ the $(n+1)$st term is much larger than the sum of all the other terms. Modifying the arguments slightly, we shall now establish similar one-term dominance for the power series of f'.

On the circle $|z| = r_n = 1-1/h_n$, the nth term in the differentiated series has the absolute value

$$a_n k_n r_n^{k_n-1} \sim a_n h_n/e = h_n^2 h_{n+1}/e$$

(the equality is a consequence of (2)). The sum of the preceding terms has absolute value less than

$$a_1 k_1 + a_2 k_2 + \ldots + a_{n-1} k_{n-1} \sim a_{n-1} h_{n-1} = h_{n-1}^2 h_n,$$

and clearly the nth term dominates the sum of the earlier terms. To estimate the later terms, we use the computations in Section 6 and obtain the bounds

$$a_{n+1} h_{n+1} r_n^{k_{n+1}} < 3h_{n+1} = o(h_n^2 h_{n+1})$$

and (for $m > n+1$)

$$h_m a_m r_n^{k_m} < \exp(1+\log h_m - h_m/2h_n)$$
$$< \exp(-\sqrt{h_m}).$$

We conclude that $f'(z) \sim a_n k_n z^{k_n - 1}$, on the circle $|z| = r_n$.

On the circle $|z| = r_n + 1/h_n^2$, the sum of the first n terms is less than

$$a_1 h_1 + \ldots + a_n h_n \sim a_n h_n = h_n^2 h_{n+1}.$$

For the $(n+1)$st term, the computations in Section 7 give

$$a_{n+1} h_{n+1} |z|^{h_{n+1}} \sim h_{n+1} \exp(h_{n+1}/2h_n^2) = (a_n/h_n) \exp(h_{n+1}/2h_n^2).$$

Because we may assume that $h_{n+1} > h_n^3$, this is much larger than the sum of the first n terms. Finally, for the terms with indices $m > n+1$, we use the estimate

$$a_m h_m \left[1 - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right)\right]^{h_m} = \exp\left\{\frac{h_m}{h_{m-1}} + \log h_m + h_m \log\left[1 - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right)\right]\right\}$$

$$< \exp\left\{h_m\left[\frac{1}{h_{m-1}} + \frac{\log h_m}{h_m} - \frac{1}{h_n}\left(1 - \frac{1}{h_n}\right)\right]\right\}$$

$$< \exp(-h_m/2h_n),$$

and we see that $f'(z) \sim a_{n+1} k_{n+1} z^{k_{n+1} - 1}$ on the circle $|z| = r_n + 1/h_n^2$. Because the estimate is also valid on the circle $|z| = r_{n+1}$, it holds throughout the domain between the nth and the $(n+1)$st dirty annuli. Using the notation $f\#$ for the spherical derivative of f, we deduce that in the domain between the dirty annuli,

$$f\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2} < |f'(z)|/|f(z)|^2 \sim k_{n+1}/(a_{n+1}|z|^{k_{n+1}})$$

$$< h_{n+1}\left(\exp\frac{-h_{n+1}}{h_n}\right)\left(1 - \frac{h_n - 1}{h_n^2}\right)^{-h_{n+1}}$$

$$= \exp\left\{\log h_{n+1} - \frac{h_{n+1}}{h_n} + h_{n+1}\left[\frac{h_n - 1}{h_n^2} + \frac{1}{2}\left(\frac{h_n - 1}{h_n^2}\right)^2 + \ldots\right]\right\}$$

$$\sim \exp\left\{\log h_{n+1} - \frac{h_{n+1}}{2h_n^2}\right\} = o(1).$$

In other words, $f\#(z) \to 0$ as z approaches the unit circle without entering any of the dirty annuli. This implies that if r lies near 1, then f_b maps γ_r onto a curve of small spherical length, except possibly for the portion of γ_r whose image under g_b falls into a dirty annulus.

Because the circle $g_b(\gamma_r)$ lies in the annulus with centre 0 and inner and outer radii

$$\frac{|r - \beta|}{1 - r\beta} \quad \text{and} \quad \frac{r + \beta}{1 - r\beta}$$

(notation of Section 8), and because $h_{n+1}/h_n \to \infty$ as $n \to \infty$, the circle $g_b(\gamma_r)$ meets at most one dirty annulus, if r is sufficiently near 1. Suppose that some points of $g_b(\gamma_r)$ lie in the nth dirty annulus. Then we can identify two maximal arcs γ_r^* and

γ_r^{**} such that $|g_b(re^{i\theta})|$ is monotone on each of the arcs and the image of each of the arcs lies in the dirty annulus. To be definite, we denote by γ_r^* the arc on which $|g_b(z)|$ is an increasing function of $\arg z$.

To facilitate the discussion of the spherical image of the arc γ_r^*, we denote by ψ the stereographic projection of the plane onto the Riemann sphere. If the arc $g_b(\gamma_r^*)$ begins at the inner boundary of the nth dirty annulus, then the behaviour of the function $\psi \circ f \circ g_b$ on the initial part of γ_r^* is determined almost entirely by the nth term in (1), and therefore the spherical image of the initial part of γ_r^* consists of smooth loops around the north pole, of approximate diameter $2e/a_n$. As $g_b(\gamma_r^*)$ penetrates deeper into the dirty annulus, the $(n+1)$st term in (1) affects the curve $\psi \circ f \circ g_b(\gamma_r^*)$ by generating shallow waves of high frequency. The amplitude of the waves increases gradually, and about at the image of the point on γ_r^* where

$$|(a_{n+1}\, g_b{}^{k_{n+1}})'| = |(a_n g_b{}^{k_n})'|,$$

the waves change to loops. The loops increase in diameter until

$$a_{n+1}|g_b{}^{k_{n+1}}| = a_n|g_b{}^{k_n}|;$$

then they begin to pass around the north pole, and as the term $a_n g_b{}^{k_n}$ fades into relative insignificance, the loops become progressively smaller. The proof that the length of the arc $\psi \circ f \circ g_b(\gamma_r^*)$ is a bounded function of r requires computations.

10. *A decomposition of* f_b

We write

$$f_b = \sum_{m=1}^{n} a_m g_b^{k_m} + a_{n+1} g_b^{k_{n+1}} + \sum_{m=n+2}^{\infty} a_m g_b^{k_m} = f_{b1} + f_{b2} + f_{b3}.$$

Because both f_{b3} and f'_{b3} are small in the nth dirty annulus, it follows from the triangle inequality that the effect of f_{b3} on the length of $\psi \circ f_b(\gamma_r^*)$ is negligible; therefore we shall restrict our attention to the length of the arc $\psi \circ (f_{b1} + f_{b2})(\gamma_r^*)$.

Although the first $n-1$ terms in the sum f_{b1} are much smaller than the nth term, we cannot ignore them; in regions where the sum $a_n g_b^{k_n} + a_{n+1} g_b^{k_{n+1}}$ is small, the first $n-1$ terms affect substantially the position of $f_b(\gamma_r^*)$, and therefore they may affect the length of $\psi \circ f_b(\gamma_r^*)$. But we note that

$$f'_{b2} = g'_b g_b^{-1} k_{n+1} f_{b2}$$

and

$$f'_{b1} = g'_b g_b^{-1} \sum_{1}^{n} a_m k_m g_b^{k_m} \sim g'_b g_b^{-1} k_n f_{b1},$$

so that

$$\frac{f'_{b2}}{f'_{b1}} \sim \frac{h_{n+1}}{h_n} \frac{f_{b2}}{f_{b1}}.$$

This implies that at each point on γ_r^* one of the two quantities $|f_{b2}/f_{b1}|$ and $|f'_{b2}/f'_{b1}|$ lies outside the interval $[1/2, 2]$.

It will be convenient to divide the arc γ_r^* into two arcs λ and μ (one of which may be empty). Recall that $|g_b(re^{i\theta})|$ is an increasing function of θ on γ_r^*.

If $|f_{b2}(z)| \geqslant |f_{b1}(z)|$ everywhere on γ_r^*, let $z_0 = re^{i\theta_0}$ denote the initial point of γ_r^*, and if $|f_{b2}(z)| \leqslant |f_{b1}(z)|$ everywhere on γ_r^*, let z_0 denote the terminal point of γ_r^*. If neither of the inequalities holds everywhere on γ_r^*, let z_0 denote the first point on γ_r^* where $|f_{b2}(z)| = |f_{b1}(z)|$. Let λ and μ denote the portions from the beginning of γ_r^* to z_0 and from z_0 to the end.

11. *The computations for* μ.

As we pointed our near the the end of Section 9, the image $\psi \circ f_b(\mu)$ consists of a sequence of progressively smaller loops around the north pole of the Riemann sphere. To estimate the combined length of these loops, we estimate first the position of the points $z_j = re^{i\theta_j}$ $(j = 1, 2, \ldots)$ on μ where

$$\arg f'_{b2}(z) = \arg f'_{b2}(z_0) + \pi j/3$$

for the first time. To a change $\Delta\theta$ in θ corresponds a change $(k_{n+1} + O(1))\Delta\theta$ in $\arg f'_{b2}(z)$, so that an increase $\pi/3$ in $\arg f'_{b2}(z)$ requires an increase greater than A_1/k_{n+1} in θ, where A_1 denotes a positive constant depending only on $|b|$.

On γ_r^* (and therefore on μ) the modulus of $g_b(r\,e^{i\theta})$ is an increasing function of θ. The derivative of the modulus is 0 at θ_0, if $g_b(z_0)$ is the point of $g_b(\gamma_r)$ nearest the origin; but even in that worst of all possible cases,

$$|g_b(r\,e^{i\theta})| > |g_b(r\,e^{i\theta_0})|\,[1 + A_2(\theta - \theta_0)^2]$$

where A_2 is a positive constant proportional to the variation of $|g_b(z)|$ on the circle γ_r. Because this variation has the order of magnitude $1/k_n$, we can assert that

$$|g_b(z_j)| > |g_b(z_0)|(1 + A_3 j^2/(h_n h_{n+1}^2)), \tag{4}$$

where A_3 again depends only on $|b|$.

It follows that

$$\begin{aligned} |g_b(z_j)|^{k_{n+1}} &> |g_b(z_0)|^{k_{n+1}}(1 + A_3 j^2/(h_{n+1}^2 h_n))^{k_{n+1}} \\ &> |g_b(z_0)|^{k_{n+1}}(1 + A_3 j^2/(2h_{n+1} h_n)). \end{aligned}$$

It follows that

$$|g_b(z_j)|^{k_{n+1}} - |g_b(z_0)|^{k_{n+1}} > A_3 j^2 |g_b(z_0)|^{k_{n+1}}/(2h_{n+1}h_n).$$

From this we deduce that

$$\begin{aligned} |f_{b2}(z_j)| - |f_{b2}(z_0)| &> A_3 j^2 a_{n+1} |g_b(z_0)|^{k_{n+1}}/(2h_{n+1}h_n) \\ &= A_3 j^2 |f_{b2}(z_0)|/2h_{n+1}h_n \\ &\geqslant A_3 j^2 |f_{b1}(z_0)|/(2h_{n+1}h_n) \\ &\sim A_3 j^2 a_n/(2e\,h_{n+1}h_n). \end{aligned}$$

The last expression provides the motivation for the relation (2) in Section 5. By virtue of that relation, we can now assert that

$$|f_{b2}(z_j)| - |f_{b2}(z_0)| > A_3 j^2/6.$$

Because $|f_{b2}(z_j)| > |f_{b1}(z_0)|$ (see the end of Section 10), and because on the arc μ the response of $|f_{b1}(z)|$ to an increase in $|g_b(z)|$ is much smaller than the response of

$|f_{b2}(z)|$, we can assert that on the arc μ_j from z_j to z_{j+1} $(j = 1, 2, \ldots)$,

$$|f_b(z)| > A_3 j^2/7.$$

The referee believes that some readers will want to see analytical justification for the geometric argument in the preceding sentence. The authors thank the referee for supplying the following argument. We may assume that z lies on the arc μ_j from z_j to z_{j+1} $(j = 1, 2, \ldots)$. If we denote $g_b(z)-g_b(z_0)$ by δ and $|g_b(z)|-|g_b(z_0)|$ by η, then the argument at the beginning of this section shows that we have $\eta = O\big(j^2/(h_n h_{n+1}^2)\big)$. Moreover, it follows from (4) that

$$\eta > A_3 j^2/(2h_n h_{n+1}^2), \quad \text{while} \quad |\delta| < B_1 j/h_{n+1}.$$

Thus $|\delta| < B_2 \eta h_n h_{n+1}$. Let the function $\varepsilon(.)$ be defined by the formula

$$f_{b1}(z) = a_n\{g_b(z)\}^{k_n}[1+\varepsilon\{g_b(z)\}].$$

If K is the maximum of $|\varepsilon'(\zeta)|$ on the segment $[g_b(z_0), g_b(z)]$, we have the inequalities

$$\begin{aligned} \log|f_{b1}(z)|-\log|f_{b1}(z_0)| &< B_3\{h_n\eta+K|\delta|\} \\ &< B_4\eta\{h_n+Kh_n h_{n+1}\} \\ &< B_5\eta\left\{h_n+\frac{h_n a_{n-1} h_n h_{n+1}}{a_n}\right\} \\ &= B_5\eta\{h_n+h_n{}^2 h_{n-1}\} < 2B_5\eta h_n{}^2 h_{n-1}. \end{aligned}$$

Using Lemma 1, we deduce that

$$\log|f_{b1}(z)|-\log|f_{b1}(z_0)| = o(\eta h_{n+1}) = o\big(j^2/(h_n h_{n+1})\big).$$

On the other hand, we deduce from (4) that

$$\log|f_{b2}(z)|-\log|f_{b2}(z_0)| > \log\big(1+A_3 j^2/(2h_{n+1}h_n)\big).$$

We recall that $|f_{b1}(z_0)| = |f_{b2}(z_0)| \sim a_n/e$. Thus, when n is large,

$$\begin{aligned} \log|f_{b2}(z)|-\log|f_{b1}(z)| &> \log\big(1+A_3 j^2/(2h_{n+1}h_n)\big)-\log\Big(o\big(j^2/(h_{n+1}h_n)\big)\Big) \\ &> \log\big(1+A_3 j^2/(2.5h_{n+1}h_n)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} |f_b(z)| \geqslant |f_{b2}(z)|-|f_{b1}(z)|-o(1) &\geqslant |f_{b1}(z)|A_3 j^2/(2.5h_{n+1}h_n)-o(1) \\ &\sim a_n A_3 j^2/(2.5eh_{n+1}h_n)-o(1) > A_3 j^2/7. \end{aligned}$$

LEMMA 2. *If at each point of some differentiable arc C in the w-plane, the tangent line to C makes an angle at most $\pi/4$ with some fixed line S, and if the distance between C and the origin is δ, then the spherical length of C is less than* $6 \operatorname{arccot} \delta$.

Proof. After a rotation, we may assume that S is the real line. Let C_1, C_2, C_3, and C_4 denote the portions of C lying in the half-plane $u \leqslant -\delta/\sqrt{2}$, in the strip $-\delta/\sqrt{2} \leqslant u \leqslant 0$, in the strip $0 \leqslant u \leqslant \delta/\sqrt{2}$, and in the half-plane $u \geqslant \delta/\sqrt{2}$, respectively.

If C_4 is not empty, let t denote arc length on C_4, measured from left to right. The inequality $du/dt \geqslant 1/\sqrt{2}$, $(w = u+iv)$, implies that the spherical length of C_4 is at most

$$\int_0^\infty \frac{dt}{1+u^2} \leqslant \int_0^\infty \frac{dt}{1+(\delta+t)^2/2}$$

$$= \sqrt{2}\arctan\frac{\delta+t}{\sqrt{2}}\Bigg|_0^\infty = \sqrt{2}\arctan\frac{\sqrt{2}}{\delta}.$$

Because on the positive half-line the arctangent is a concave function,

$$\arctan \alpha x < \alpha \arctan x$$

where $\alpha > 1$, and therefore the spherical length of C_4 is less than

$$2\arctan 1/\delta = 2\operatorname{arccot}\delta.$$

If C_3 is not empty, let s denote arc length on C_3. Because $0 \leqslant s \leqslant \delta$ on C_3, the spherical length of C_3 is not greater than

$$\int_0^\delta \frac{ds}{1+\delta^2} = \frac{\delta}{1+\delta^2} = \frac{1}{\delta}\,\frac{1}{1+1/\delta^2} \leqslant \int_0^{1/\delta} \frac{dx}{1+x^2} = \operatorname{arccot}\delta.$$

Analogous computations yield similar bounds for the spherical length of C_1 and C_2, and Lemma 2 follows immediately.

Returning to the task of estimating the spherical length of $f_b(\mu)$, we recall that on each of the arcs $\mu_j (j = 0, 1, \ldots)$ the variation of $\arg f'_{b2}(z)$ is $\pi/3$. Because $f'_{b1}(z)$ and $f'_{b3}(z)$ are much smaller than $f'_{b2}(z)$, the values of $\arg f'_b(z)$ lie in an interval whose length is only slightly greater than $\pi/3$. Moreover, on each arc μ_j the argument of dw/dz is restricted to a short interval, and therefore we can apply Lemma 2 and deduce that for all values r sufficiently near 1, the spherical length of $f_b(\mu)$ is less than

$$6\sum_0^\infty \operatorname{arccot} A_3 j^2/7.$$

Because of the constant A_3, the value of this bound depends on $|b|$; but it is independent of r, and therefore the spherical length of $f_b(\mu)$ is a bounded function of r.

12. *The computations for the arc λ.*

The arc $\psi \circ f_b(\mu)$ studied in Section 11 winds around the Riemann sphere's north pole in progressively tighter loops. The arc $\psi \circ f_b(\lambda)$, which will occupy us now, consists largely of loops about a centre that moves slowly at a distance approximately e/a_n from the north pole.

First we dispatch the portion of λ where

$$a_{n+1}h_{n+1}|g_b|^{h_{n+1}} \leqslant 2a_n h_n |g_b|^{h_n}.$$

Here $|f'_{b2}| < 3|f'_{b1}|$ and f_{b2}/f_{b1} is small. Consequently,

$$f_b\# < 5|f'_{b1}/f_{b1}{}^2| \sim 5h_n/|f_{b1}| \sim 5eh_n/a_n = 5e/h_{n+1}.$$

Therefore f_b maps this portion of λ onto an arc of small spherical length.

Next we deal with the portion where

$$2a_n h_n |g_b|^{h_n} \leqslant a_{n+1} h_{n+1} |g_b|^{h_{n+1}} \leqslant a_n h_{n+1} |g_b|^{h_n}/2.$$

The second inequality implies that $|f_b| > |f_{b1}|/3$, and therefore, by virtue of the first inequality,

$$f\#_b < \frac{2|f'_{b2}|}{|f_{b1}/3|^2} \sim \frac{18h_{n+1}|f_{b2}|}{|f_{b1}|^2} < 18\, h_{n+1}/|f_{b1}| \sim 18\, eh_{n+1}/a_n = 18\, e/h_n.$$

Again, we conclude that f_b maps the portion of λ onto an arc of small spherical length.

This leaves us with the portion of λ where

$$a_{n+1}|g_b|^{h_{n+1}} \geqslant a_n |g_b|^{h_n}/2. \tag{5}$$

We recall that, by virtue of the definition of λ,

$$a_{n+1}|g_b|^{h_{n+1}} \leqslant |f_{b1}|,$$

so that now $|f_{b2}|$ lies between $|f_{b1}|$ and approximately $|f_{b1}|/2$. We use the remainder of this section to show that f_b maps the portion of λ determined by (5) onto a curve whose spherical length has an upper bound independent of r.

We assign to $\arg\{f_{b2}(z_0)/f_{b1}(z_0)\}$ its lowest possible positive value; for $j = 0, 1, \ldots$ we define the points $t_j = r\, e^{i\tau_j}$ on λ by the condition

$$\arg\{f_{b2}(t_j)/f_{b1}(t_j)\} = -(2j+1)\pi,$$

and we denote by $\lambda_j (j = 0, 1, \ldots)$ the portion of λ between t_j and t_{j+1} (except that the last of the arcs λ_j is the arc from the corresponding point t_j to the point where

$$|f_{b2}| = a_n |g_b|^{h_n}/2).$$

Because $\arg f'_{b2}$ is monotone on λ and has values restricted to the interval $[-3\pi, 2\pi]$ on the arc from z_0 to t_1, and because the variation of $\arg f'_{b1}$ on that arc is small, we can divide the arc from z_0 to t_1 into eleven subarcs on each of which $\arg f'_b$ is restricted to an interval of length less than $\pi/2$. To each of the eleven subarcs we apply Lemma 2 of Section 11.

On each of the arcs $\lambda_j (j = 1, 2, \ldots)$, the ratio f'_b/f'_{b2} is nearly 1, and therefore the direction of the tangent line to the curve $f_b(\lambda_j)$ differs only slightly from the direction of the tangent line at the corresponding point of the curve defined by the equation

$$w(z) = a_{n+1}(g_b(z))^{k_{n+1}}.$$

In other words, the curve $f_b(\lambda)$ consists of smooth loops. Our next task is to prove that the distance between the loop $f_b(\lambda_j)$ and the origin is fairly large.

Reasoning as in the first two paragraphs of Section 11, we see that if $\tau_j \geqslant \theta \geqslant \tau_{j+1}$, then

$$|g_b(r\, e^{i\theta})| < |g_b(t_0)|(1 - A_4 j^2/h_n h_{n+1}^2),$$

in other words, that

$$|g_b(r\, e^{i\theta})|^{k_{n+1}} \leqslant |g_b(t_0)|^{k_{n+1}}(1 - A_4 j^2/a_n h_{n+1})^{k_{n+1}}.$$

As we remarked after (5), $|f_{b2}|$ lies between $|f_{b1}|$ and approximately $|f_{b1}|/2$, so that the second factor on the right is larger than 2/5, say. Now if $0 < u < 1$ and $x > 1$, and if

$$F(u) = (1-u)^x > 2/5,$$

it follows from the mean-value theorem that

$$F(u) - F(0) = -ux(1-u_0)^{x-1}$$

for some u_0 in the interval $(0, u)$. It follows that

$$1-(1-u)^x = ux(1-u_0)^{x-1} > ux(1-u)^{x-1} > ux(1-u)^x > 2ux/5,$$

and therefore

$$(1-u)^x < 1-2ux/5.$$

Setting $u = A_4 j^2/a_n h_{n+1}$ and $x = k_{n+1}$, we deduce that

$$(1-A_4 j^2/a_n h_{n+1})^{k_{n+1}} < 1-2A_4 j^2 k_{n+1}/5a_n k_{n+1} < 1-\frac{A_4 j^2}{3a_n}$$

and hence that

$$|g_b(t_0)|^{k_{n+1}} - |g_b(t_j)|^{k_{n+1}} \geqslant |g_b(t_0)|^{k_{n+1}} A_4 j^2/(3a_n).$$

Because $f_{b2} = a_{n+1}\, g_b^{k_{n+1}}$, we can write this in the form

$$|f_{b2}(t_0)| - |f_{b2}(t_j)| > |f_{b2}(t_0)| A_4 j^2/(3a_n),$$

and by (5), the right hand member is greater than

$$a_n |g_b(t_0)|^{h_n} A_4 j^2/(6a_n) \sim A_4 j^2/(6e).$$

Because on λ_j the derivative of $|f_{b2}|$ with respect to arc length is much larger than the derivative of $|f_{b1}|$, and because the spherical variation of f_{b1} on λ_j is much smaller than $1/a_n$, the length of $\psi \circ f_b(\lambda_j)$ is not greater than twice the length of the spherical image of the circle with centre $f_{b1}(t_j)$ and radius $|f_{b2}(t_j)|$. A diameter of that circle is a radial segment whose endpoints have modulus

$$|f_{b1}(t_j)| - |f_{b2}(t_j)| \quad \text{and} \quad |f_{b1}(t_j)| + |f_{b2}(t_j)|.$$

The chordal distance between two points w_1 and w_2 is

$$|w_1 - w_2|/\surd(1+|w_1|^2) \qquad \surd(1+|w_2|^2);$$

suppressing the variable t_j, we can therefore write the spherical length of our circle in the form

$$2\pi |f_{b2}|/\surd(1+(|f_{b1}|-|f_{b2}|)^2) \qquad \surd(1+(|f_{b1}|+|f_{b2}|)^2).$$

Obviously, this is less than

$$2\pi/(|f_{b1}(t_j)| - |f_{b2}(t_j)|).$$

Because $|f_{b1}(t_j)| - |f_{b2}(t_j)| \sim |f_{b1}(t_0)| - |f_{b2}(t_j)|$ and $|f_{b1}(t_0)| \geqslant |f_{b2}(t_0)|$, it follows from an argument like that in the preceding section that the spherical length of $f_b(\lambda_j)$ is less than A_5/j^2. This concludes the proof that f_b is a Tsuji function.

13. *The Julia points of f_b*

On both boundary components of the nth dirty annulus, the modulus of $f(z)$ is at least $a_n/3$. The same is true on the radii (in the annulus) on which $(k_{n+1}-k_n)\arg z$ is an even multiple of π. On the radii on which $(k_{n+1}-k_n)\arg z$ is an odd multiple of π, there exists a point in the nth dirty annulus ($n = 1, 2, \ldots$) where the sum of the nth and $(n+1)$st terms of (1) is 0, and at this point $|f(z)| < 2a_{n-1}$. By Rouché's theorem, $f(z)$ assumes equally often (and therefore at least once) all values of modulus less than $a_n/3$, in each sector of the nth dirty annulus determined by a relation of the form

$$2\pi \frac{m-1}{k_{n+1}-k_n} < \arg z < 2\pi \frac{m}{k_{n+1}-k_n} \quad (m = 0, 1, \ldots).$$

Clearly, the spherical image of each sector covers the Riemann sphere, except for a small neighbourhood of the north pole. Because the diameter of the sector is small compared with its distance from the unit circle, it follows that each point $e^{i\theta}$ is a Julia point of f.

If each point $e^{i\theta}$ is a Julia point of f, then clearly each point $e^{i\theta}$ is a Julia point of f_b. This concludes the proof of Theorem 1.

14. *The Julia points of functions in Hayman's class T_2*

We conclude with a proof that for every annular Tsuji function, the set of Julia points is large in both the topological and the measure-theoretic sense.

E. F. Collingwood [3] proved that *if f is a meromorphic Tsuji function in D, then almost every point on the unit circle C is either a Fatou point or a Julia point for f.* Because the Fatou points of an annular function constitute a set of measure 0, the set of Julia points of an annular Tsuji function must have measure 2π. That it is also a residual subset of C follows immediately from our second theorem. The theorem deals with a class of functions more general than the class of Tsuji functions.

Following Hayman, we say that a meromorphic function f in D belongs to the class T_2 provided there exists a sequence $\{J_n\}$ of Jordan curves whose interiors expand to D (in the sense of Section 1) and whose images $f(J_n)$ have bounded spherical length. We denote by $J(f)$ the set of Julia points of f, and we say that a point of C belongs to the set $X(f)$ provided it is the midpoint of some arc on C to which f has a continuous extension (in the spherical metric). With this notation, our result takes the following form.

THEOREM 2. *If $f \in T_2$, then $X(f) \cup J(f)$ is residual on C.*

Proof. Let $N(f)$ denote the set of points on C where f is normal, that is, the set of points $e^{i\theta}$ to which there corresponds a pair of positive numbers r and M such that

$$f^{\#}(z) < M/(1-|z|)$$

whenever $z \in D$ and $|z-e^{i\theta}| < r$. Because f is meromorphic, it follows from results of S. Dragosh [5; Theorems 8 and 9] that the set $N(f) \cup J(f)$ is residual on C. Furthermore, Hayman [7; Theorem 3] has proved that $N(f) = X(f)$, for functions f in T_2; this proves Theorem 2.

Theorem 2 contradicts the statement at the end of page 199 of Collingwood's paper [3]. Collingwood stated that even if a Tsuji function has no Fatou points, the set of its Julia points may still be of first category. The proposed proof is based on an example furnished by Piranian in an informal communication. Piranian regrets that his construction involves an error: the first clause in the final sentence (middle of page 200) in [3] is incorrect.

References

1. D. D. Bonar, F. W. Carroll and P. Colwell, " Category results for Tsuji functions ", *Canad. J. Math.*, 29 (1977), 552–558.
2. D. D. Bonar, F. W. Carroll and G. Piranian, " Strongly annular functions with small Taylor coefficients ", *Math. Z.*, 156 (1977), 85–91.
3. E. F. Collingwood, " A boundary theorem for Tsuji functions ", *Nagoya Math. J.*, 29 (1967), 197–200.
4. E. F. Collingwood and G. Piranian, " Tsuji functions with segments of Julia ", *Math. Z.*, 84 (1964), 246–253.
5. S. Dragosh, " The spherical derivative of meromorphic functions ", *J. Reine Angew. Math.*, 252 (1972), 51–67.
6. W. K. Hayman, " Regular Tsuji functions with infinitely many Julia points ", *Nagoya Math. J.*, 29 (1967), 185–196.
7. W. K. Hayman, " The boundary behaviour of Tsuji functions", *Michigan Math. J.*, 15 (1968), 1–26.
8. R. W. Howell, "Annular functions in probability ", *Proc. Amer. Math. Soc.*, 52 (1975), 217–221.

Pennsylvania State University, University Park, Pennsylvania 16802.

Denison University, Granville, Ohio 43023.

Ohio State University, Columbus, Ohio 43210.

Iowa State University, Ames, Iowa 50011.

University of Michigan, Ann Arbor, Michigan 48109.