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Exotic quasi-conformally homogeneous surfaces

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Abstract

We construct uniformly quasi-conformally homogeneous Riemann surfaces that are not quasi-conformal deformations of regular covers of closed orbifolds.

1. *Introduction*

Recall that a hyperbolic manifold M is K- *quasi-conformally homogeneous* if for all $x, y \in M$, there is a K-quasi-conformal map $f : M \to M$ with $f(x) = y$. It is said to be *uniformly quasi-conformally homogeneous* if it is K-quasi-conformally homogeneous for some K. We consider only complete and oriented hyperbolic manifolds.

In dimensions 3 and above, every uniformly quasi-conformally homogeneous hyperbolic manifold is isometric to the regular cover of a closed hyperbolic orbifold (see [**1**]). The situation is more complicated in dimension 2. It remains true that any hyperbolic surface that is a regular cover of a closed hyperbolic orbifold is uniformly quasi-conformally homogeneous. If S is a non-compact regular cover of a closed hyperbolic 2-orbifold, then any quasi-conformal deformation of S remains uniformly quasi-conformally homogeneous. However, typically a quasi-conformal deformation of S is not itself a regular cover of a closed hyperbolic 2-orbifold (see [**1**, Lemma 5.1].)

It is thus natural to ask if every uniformly quasi-conformally homogeneous hyperbolic surface is a quasi-conformal deformation of a regular cover of a closed hyperbolic orbifold. The goal of this note is to answer this question in the negative.

Theorem 1.1. *There are uniformly quasi-conformally homogeneous surfaces that are not quasi-conformal deformations of the regular cover of any closed hyperbolic* 2*-orbifold.*

In order to prove Theorem 1.1, we associate to every connected graph X with constant valence a hyperbolic surface S_X which is obtained by 'thickening' X. In particular, S_X is quasi-isometric to X. Each element $\varphi \in \mathrm{Aut}(X)$ gives rise to a quasi-conformal automorphism h_φ of S_X (with uniformly bounded dilatation). If $\mathrm{Aut}(X)$ acts transitively on the set of vertices of X, then the associated set of quasi-conformal automorphisms is coarsely transitive, that is, there exists D such that if $x, y \in S_X$, then there exists $\varphi \in \mathrm{Aut}(X)$ such that $d(h_\varphi(x), y) \leqslant D$. One may then use the work of Gehring and Palka [**5**] to show that S_X is uniformly quasi-conformally homogeneous.

We choose X to be a Diestel–Leader graph $\mathrm{DL}(m, n)$ with $m \neq n$. These graphs have transitive groups of automorphisms, but Eskin, Fisher and Whyte [**4**] recently showed that they are not quasi-isometric to the Cayley graph of any group. The proof is completed by

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the observation that any surface that is a quasi-conformal deformation of a regular cover of a closed orbifold is quasi-isometric to the Cayley graph of the deck transformation group.

On the other hand, it is easy to construct hyperbolic surfaces that are quasi-isometric to graphs with transitive automorphism group, that are not uniformly quasi-conformally homogeneous (see Section 5). So, one is left to wonder if there is a simple geometric characterization of uniformly quasi-conformally homogeneous surfaces.

2. *Turning graphs into surfaces*

For simplicity, let X be a connected countable graph such that every vertex has valence $d \geqslant 3$ and every edge has length 1. It will be convenient to assume that every edge of X has two distinct endpoints. In this section, we *thicken* X into a hyperbolic surface S_X quasi-isometric to X in such a way that whenever the group of automorphisms of X acts transitively on the set of vertices, then S_X is uniformly quasi-conformally homogeneous.

We start by introducing some notation. Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of vertices and edges of the graph X. For each vertex $v \in \mathcal{V}$, let $\mathcal{E}_v$ be the set of edges of X that contain v. By assumption $\mathcal{E}_v$ has d elements for each v. For each v, choose a bijection

$$s_v : \mathcal{E}_v \longrightarrow \{1, \ldots, d\}.$$

Observe that if φ is an automorphism of X, then φ induces a bijection $(\varphi_*)_v : \mathcal{E}_v \to \mathcal{E}_{\varphi(v)}$ for all $v \in \mathcal{V}$. Consider the permutation

$$s_v^{\varphi} = s_{\varphi(v)} \circ (\varphi_*)_v \circ s_v^{-1} : \{1, \ldots, d\} \longrightarrow \{1, \ldots, d\}.$$

The building blocks of our construction will be copies of a fixed hyperbolic surface F that is homeomorphic to a sphere with d holes such that each boundary component of F is a geodesic of length 1. Label the components of ∂F by $\gamma_1, \ldots, \gamma_d$. For each i, choose a base point $p_i \in \gamma_i$ and observe that the choice of the base point together with the orientation of F determines uniquely a parameterization $\mathbb{S}^1 \to \gamma_i$ with constant velocity 1. We state the following observation as a lemma for future reference.

LEMMA 2.1. *For each $d \geqslant 3$, there exists $K_d > 1$ such that if $\sigma \in \mathfrak{S}_d$ is a permutation of the set $\{1, \ldots, d\}$ then, there is a K_d-quasi-conformal map $f_\sigma : F \to F$ which is an isometry when restricted to a neighborhood of ∂F and such that $f_\sigma(\gamma_i) = \gamma_{\sigma(i)}$ and $f_\sigma(p_i) = p_{\sigma(i)}$.*

Consider the hyperbolic surface $F \times \mathcal{V}$ and set $F_v = F \times \{v\}$. We construct S_X by gluing the components of $F \times \mathcal{V}$ together. The gluing maps are determined by the edges of X as follows. Given an edge $e \in \mathcal{E}$, let $v, v' \in \mathcal{V}$ be its two vertices, which we assume are always distinct. We identify the curves $\gamma_{s_v(e)} \times \{v\} \subset \partial F_v$ and $\gamma_{s_{v'}(e)} \times \{v'\} \subset \partial F_{v'}$. More precisely, let

$$g_e : \gamma_{s_v(e)} \times \{v\} \longrightarrow \gamma_{s_{v'}(e)} \times \{v'\}$$

be the unique orientation-reversing isometry which maps the marked point $(p_{s_v(e)}, v)$ to $(p_{s_{v'}(e)}, v')$. Let $\sim$ be the equivalence relation on $F \times \mathcal{V}$ generated by the maps g_e for all $e \in \mathcal{E}$. The equivalence classes of $\sim$ contain either one point in the interior of $F \times \mathcal{V}$ or two points in the boundary. In particular, the quotient space of $\sim$

$$S_X = F \times \mathcal{V} / \sim$$

is a surface. Moreover, since the gluing maps g_e are isometries, the hyperbolic metric on $F \times \mathcal{V}$ descends to a hyperbolic metric on S_X. By construction, this metric has injectivity radius bounded from above and below. In particular, if we choose ϵ_F to be a lower bound for the length of any homotopically non-trivial closed curve on F and δ_F to be a lower bound for the length of any properly embedded arc in F, which is not properly homotopic into the boundary of F, then $\epsilon_d = \min\{\epsilon_F/2, \delta_F\}$ is a lower bound for the injectivity radius of S_X.

Associated to every edge $e \in \mathcal{E}$, there is a simple closed geodesic c_e in S_X and c_e is disjoint from $c_{e'}$ for every pair of distinct edges $e, e' \in \mathcal{E}$. Let $\mathcal{C} = \{c_e | e \in \mathcal{E}\}$ be the collection of all such geodesics and note that $S_X \setminus \mathcal{C}$ is isometric to the interior of $F \times \mathcal{V}$.

It follows that the graph X can be recovered from S_X as the dual graph to the multicurve $\mathcal{C}$. Moreover, there is a projection $\pi_X : S_X \to X$ that maps every component of $\mathcal{C}$ to the midpoint of its associated edge and maps every component of $S_X \setminus \mathcal{C}$ to its associated vertex. The map π_X is then a (K, C)-quasi-isometry, where $K = C = 2\mathrm{diam}(F)$. We recall that a map $g : Y \to Z$ between two metric spaces is a (K, C)-quasi-isometry if

$$\frac{1}{K} d_Y(x, y) - C \leqslant d_Z(g(x), g(y)) \leqslant K d_Y(x, y) + C$$

for all $x, y \in Y$ and if for every $z \in Z$ there exists $y \in Y$ such that $d_Z(g(y), z) \leqslant C$.

It also follows from the identification of X with the dual graph to $\mathcal{C}$ that every homeomorphism $f : S_X \to S_X$ that maps $\mathcal{C}$ to itself, meaning $f(\mathcal{C}) = \mathcal{C}$ and $f^{-1}(\mathcal{C}) = \mathcal{C}$, induces an automorphism of the graph X.

LEMMA 2.2. *Every automorphism of the graph X is induced by a K_d-quasi-conformal homeomorphism of S_X which preserves $\mathcal{C}$, where K_d is the constant provided by Lemma* 2.1.

Proof. Given an automorphism $\varphi : X \to X$, recall the definition of the permutation

$$s_v^{\varphi} : \{1, \dots, d\} \longrightarrow \{1, \dots, d\}$$

given above for each $v \in \mathcal{V}$. Let $f_v : F \to F$ be the K_d-quasi-conformal map associated by Lemma 2.1 to the permutation s_v^{φ} and define

$$H_{\varphi} : F \times \mathcal{V} \longrightarrow F \times \mathcal{V}, \quad H_{\varphi}(x, v) = (f_v(x), \varphi(v)).$$

Observe that H_{φ} is K_d-quasi-conformal. Moreover, if an edge $e \in \mathcal{E}$ contains v, then

$$H_{\varphi}(\gamma_{s_v(e)} \times \{v\}) = \gamma_{s_{\varphi(v)}(\varphi(e))} \times \varphi(v).$$

Also, by construction H_{φ} maps marked points to marked points. It follows that H_{φ} descends to a K_d-quasi-conformal homeomorphism

$$h_{\varphi} : S_X \longrightarrow S_X$$

with $h_{\varphi}(c_e) = c_{\varphi(e)}$ and $h_{\varphi}^{-1}(c_e) = c_{\varphi^{-1}(e)}$ for all $e \in \mathcal{E}$. In other words, h_{φ} induces φ. □

REMARK. It is not possible to construct the quasi-conformal automorphisms in Lemma 2.1 so that one obtains an action of Σ_d on F. Therefore, we do not in general obtain an action of $\mathrm{Aut}(X)$ on S_X.

We now combine Lemma 2.2 with a technique of Gehring and Palka [5] to show that if X is a graph with transitive automorphism group, then S_X is uniformly quasi-conformally homogeneous.

LEMMA 2.3. *Given $d \geqslant 3$, there exists $L_d > 1$ such that if X is a connected graph, such that every vertex has valence $d \geqslant 3$, every edge has length* 1 *and* $\mathrm{Aut}(X)$ *acts transitively on the vertices of X, then there is a L_d-quasi-conformally homogeneous hyperbolic surface S_X quasi-isometric to X.*

Proof. Let x and y be any two points on S_X. By Lemma 2.2 there exists a K_d-quasi-conformal automorphism $h : S_X \to S_X$ such that $h(x)$ and y both lie in (the image of) F_v for some vertex v of X. Therefore, $d(x, h(y)) \leqslant \mathrm{diam}(F)$.

Let $\epsilon_d > 0$ be a lower bound for the injectivity radius of S_X. (Note that ϵ_d depends only on d and the choice of surface F above.) Lemma 2.6 in [1] (which is derived from [5, Lemma 3.2]) implies that there exists a K'_d-quasi-conformal map $\psi : S_X \to S_X$ such that $\psi(h(x)) = y$ where

$$K'_d = (e^{\epsilon_d/2} + 1)^{4\mathrm{diam}(F)/\epsilon_d + 2}.$$

Then $\psi \circ h$ is a $K_d K'_d$-quasi-conformal map taking x to y. Therefore S_X is L_d-quasi-conformally homogeneous, where $L_d = K_d K'_d$. □

3. *Diestel–Leader graphs*

Diestel and Leader [3] constructed a family of graphs whose automorphism groups act transitively on their vertices and conjectured that these graphs are not quasi-isometric to the Cayley graph of any finitely generated group. Eskin, Fisher and Whyte [4] recently established this conjecture. In this section, we give a brief description of the Diestel–Leader graphs (see Diestel and Leader [3] or Woess [8] for more detailed descriptions).

Given $m, n \geqslant 2$, consider two trees T_m and T_n of valence $m+1$ and $n+1$, respectively, and such that every edge has length 1. Choose points $\theta_m \in \partial_\infty T_m$ and $\theta_n \in \partial_\infty T_n$ in the corresponding Gromov boundaries and vertices $0_m \in T_m$ and $0_n \in T_n$. Finally, consider $\mathbb{R}$ as a graph with vertices of valence 2 at every integer $k \in \mathbb{Z}$. Observe that the Busemann function

$$\beta_m : T_m \longrightarrow \mathbb{R}$$

centered at θ_m and normalized at 0_m is a simplicial map between both graphs. Note that, for any two vertices $v, w \in T_m$, there exists an automorphism φ of T_m such that $\varphi(v) = w$ and

$$\beta_m(\varphi(x)) - \beta_m(x) = \beta_m(w) - \beta_m(v)$$

for all $x \in T_m$. Clearly, the same is true for the corresponding Busemann function

$$\beta_n : T_n \longrightarrow \mathbb{R}.$$

We orient the tree T_m or T_n in such a way that every positively oriented edge points toward θ_m or θ_n, respectively.

Let $T_m \times T_n$ be the product of the two trees T_m and T_n in the category of graphs. In other words, the set of vertices of $T_m \times T_n$ is the product of the set of vertices of T_m and T_n, and an edge in $T_m \times T_n$ with vertices (v, v') and (w, w') is a pair (e, e'), where e is an edge in T_m with vertices v and w, and e' is an edge in T_n with vertices v' and w'. See [6] for a more precise description of the product.

The automorphism groups of the two oriented trees T_m and T_n act transitively on the set of vertices and every pair $(\varphi, \psi) \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ of automorphisms induces an automorphism of $T_m \times T_n$. It follows that $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ acts transitively on the set of vertices of $T_m \times T_n$.

Consider the simplicial map

$$f : T_m \times T_n \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \beta_m(x) - \beta_n(y).$$

The pre-image $\mathrm{DL}(m, n) = f^{-1}(0)$ of 0 is a connected graph and it is clear from the discussion above that the subgroup of $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ which preserves $f^{-1}(0)$ acts transitively on the vertices of $\mathrm{DL}(m, n)$. The following result of Eskin, Fisher and Whyte [4] is the key fact needed to prove our main theorem.

THEOREM 3.1 (Eskin, Fisher, Whyte). *If $m \neq n$, then* $\mathrm{DL}(m, n)$ *is not quasi-isometric to the Cayley graph of any finitely generated group.*

4. *The proof of the main theorem*

We are now ready to give the proof of our main theorem. We first observe that a quasi-conformal deformation of a regular cover of a closed orbifold is quasi-isometric to the Cayley graph of a finitely generated group.

LEMMA 4.1. *Suppose that a surface Σ is a quasi-conformal deformation of a surface S which normally covers a closed orbifold $\mathcal{O}$, then Σ is quasi-isometric to the Cayley graph of the* (*finitely generated*) *group of deck transformations of the covering map $S \to \mathcal{O}$.*

Proof. Since any K-quasi-conformal map is a $(K, K\log 4)$-quasi-isometry (see [**7**, Theorem 11.2]), Σ is quasi-isometric to S. Let G be the necessarily finitely generated group of deck transformations of the covering $S \to \mathcal{O}$. Since G acts on S co-compactly and discretely, the Svarc–Milnor lemma (for example, see [**2**, Proposition 8.19]) implies that S is quasi-isometric to the Cayley graph of G. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $X = \mathrm{DL}(2, 3)$ be the $(2, 3)$-Diestel–Leader graph and let S_X be the Riemann surface associated to X in the previous section. Since $\mathrm{Aut}(X)$ acts transitively on the vertices of X, it follows from Lemma 2.3 that S_X is uniformly quasi-conformally homogeneous. Suppose, for the sake of contradiction, that S_X is a quasi-conformal deformation of a Riemann surface S which is a regular cover $S \to \mathcal{O}$ of a compact orbifold $\mathcal{O}$. By Lemma 4.1, the surface S_X is quasi-isometric to the Cayley graph of a finitely generated group. Since S_X is quasi-isometric to X, the same is true for $X = \mathrm{DL}(2, 3)$. This contradicts Eskin, Fisher and Whyte's Theorem 3.1. □

5. *Surfaces quasi-isometric to Cayley graphs need not be uniformly quasi-conformally homogeneous*

It is easy to check that every hyperbolic surface S is quasi-isometric to a graph X with unit-length edges and bounded valence. Any quasi-conformal automorphism of S induces a quasi-isometry of X (which is only coarsely well defined), and the quasi-isometry constants may be uniformly bounded by the dilatation of the quasi-conformal map. One may then readily show that if S is uniformly quasi-conformally homogeneous, then S is quasi-isometric to a graph X such that there exists $C, L > 0$ such that the set of (L, C)-quasi-isometries of X acts transitively on X.

One might hope this construction, which is a sort of quasi-inverse to the construction in Section 2, could be used to construct a characterization of uniformly quasi-conformally homogeneous surfaces. However, uniform quasi-conformal homogeneity is not a quasi-isometry invariant. For example, if we let X be the 'ladder' graph made by joining equal integer points on two copies of the real line, then S_X is quasi-isometric to the real line as is any finite area hyperbolic surface S homeomorphic to a twice-punctured torus. The thickened ladder S_X is uniformly quasi-conformally homogeneous, by Lemma 2.3, but S is not, as it has no lower bound on its injectivity radius (see [**1**, Theorem 1.1]).

One may further construct hyperbolic surfaces with bounded geometry (that is, having upper and lower bounds on their injectivity radius) that are quasi-isometric to graphs with transitive automorphism group that are not uniformly quasi-conformally homogeneous.

EXAMPLE 5.1. *A bounded geometry surface S' that is quasi-isometric to the Cayley graph of the free group F_2 on 2-generators, but is not uniformly quasi-conformally homogeneous.*

Construction of Example 5.1. Let T be the infinite 4-valent tree and let S_T be the uniformly quasi-conformally homogeneous surface constructed by Lemma 2.3. One may form a new surface S' by removing a disk D from S_T and replacing it by a surface F that is homeomorphic to a torus with a disk removed. We place a hyperbolic structure on S' such that there is an isometry from $S_T - U$ to $S' - V$, where U is a bounded neighborhood of D and V is a bounded neighborhood of F. One may further assume that the boundary ∂F of F is totally geodesic in the resulting hyperbolic structure. It follows that S' is also quasi-isometric to T, which is the Cayley graph of F_2.

Every non-separating closed geodesic on S' must intersect F. One may then readily check, using the fact that a K-quasi-conformal automorphism is a $(K, K\log 4)$-quasi-isometry, that given a non-separating closed geodesic α in F and any $K > 1$, there exists R_K such that if $g : S \to S'$ is K-quasi-conformal, then $g(\alpha)$ lies in the neighborhood of radius R_K about F. It immediately follows that S' cannot be uniformly quasi-conformally homogeneous.

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