LOCAL RESTRICTIONS FOR VARIOUS CLASSES OF DIRECTED GRAPHS

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The purpose of this article is to provide and establish criteria for determining whether a sequence of ordered pairs of integers is the sequence of degrees of the points of a directed graph possessing particular properties. After some necessary definitions and a preliminary result, such criteria are given for digraphs with different kinds of connectedness. In the final two sections we consider digraphs having the additional properties of being functional and traversable.

1. *Preliminaries*

A *directed graph*, or more briefly a *digraph* D, consists of a finite non-empty set $V = \{v_1, v_2, \ldots, v_p\}$ of *points* together with a given subset X of $V \times V$. The elements of X are called *lines*, and for any line $x = v_i v_j = (v_i, v_j)$, it is specified that $v_i \neq v_j$. Also, v_i is then said to be *adjacent to* v_j and v_j *adjacent from* v_i. The *outdegree* of a point v_i, denoted by $\operatorname{od} v_i = a_i$, is the number of points adjacent from v_i, and its *indegree*, $\operatorname{id} v_i = b_i$, is the number of points adjacent to it. *The degree* of point v_i is then the ordered pair (a_i, b_i), and the *degree sequence* $\sigma(D)$ of the digraph D is the sequence of degrees of its points, that is,

$$\sigma(D) = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big).$$

A sequence of points and lines of the form $u_1, u_1 u_2, u_2, u_2 u_3, \ldots, u_{n-1} u_n, u_n$, in which the n points are distinct, is called a *path* from u_1 to u_n. If there is a path from u to v in D, then v is said to be *reachable* from u. A *semipath* joining u and v is a sequence of points and lines which would become a path if the direction of some (possibly none) of the lines were reversed.

A digraph D is *strongly connected*, or *strong*, if every pair of points are mutually reachable; D is *weakly connected*, or *weak*, if every pair of points are joined by a semipath. A digraph is *disconnected* if it is not even weak.

A *subgraph* of D consists of subsets of the points and lines of D which themselves constitute a digraph. A *strong component* S of a digraph D is a maximal strongly connected subgraph; similarly a *weak component* W of D is a maximal weakly connected subgraph.

The *trivial digraph* consists of a single point, as does a *trivial strong component* of a digraph. A trivial weak component is called an *isolated point*.

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For any line x of D, let $D-x$ be the maximal subgraph not containing x. If u and v are distinct points of D and line $y = uv$ is not in D, call $D+y$ the union of D with line y.

Let σ be a sequence of p ordered pairs of non-negative integers. Fulkerson [2] and Ryser [6] have found the criteria of Theorem 0 for σ to be the degree sequence of some digraph. If there is such a digraph D, we say that σ is *graphical*, that σ belongs to D, and that D is a digraph of σ.

THEOREM 0. *Let σ be a sequence of ordered pairs of non-negative integers $\big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$, in which $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$, $0 \leqslant a_i \leqslant p-1$, and $0 \leqslant b_i \leqslant p-1$. This sequence is graphical if and only if the equation*

$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i, \tag{1}$$

and the following $p-1$ inequalities hold:

$$\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} \min\{k-1, b_i\} + \sum_{i=k+1}^{p} \min\{k, b_i\}; \quad 1 \leqslant k \leqslant p-1. \tag{2}$$

2. *Degree sequences and kinds of connectedness*

We will develop necessary and sufficient conditions for a graphical sequence σ to belong to disconnected, weak, and strong digraphs. The trivial digraph, whose degree sequence is $\big((0, 0)\big)$, is both weak and strong. Hence we only consider nontrivial digraphs.

THEOREM 1. *A graphical sequence $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ belongs to some disconnected digraph if and only if σ can be partitioned into two graphical subsequences.*

Both the necessity and sufficiency of this condition are immediate and their proofs are omitted. Instead we illustrate this with

$$\sigma = \big((2, 2), (2, 2), (2, 1), (1, 2), (1, 1), (1, 1)\big).$$

This sequence is graphical and partitions into the two graphical subsequences

$$\big((2, 2), (2, 1), (1, 2)\big) \text{ and } \big((2, 2), (1, 1), (1, 1)\big).$$

THEOREM 2. *A graphical sequence $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ belongs to some weak digraph if and only if it does not contain* $(0, 0)$ *and*

$$\sum_{i=1}^{p} a_i \geqslant p-1. \tag{3}$$

Proof. If σ is the degree sequence of a weak digraph D with p points, then σ does not contain $(0, 0)$ since D has no isolated points. Of course D has at least $p-1$ lines, as mentioned in [**1**]. Therefore, the necessity of these conditions is immediate.

For the sufficiency, suppose σ is a sequence which satisfies these conditions but belongs to no weak digraph. Let D be a digraph of σ with the minimum number of weak components. Then it can easily be shown that some weak component W has a line $x = uv$ such that $W-x$ is also weakly connected. Let $x' = u'v'$ be a line in any other weak component W'. Define two new lines $y_1 = uv'$ and $y_2 = u'v$. Then $E = D-x-x'+y_1+y_2$ is also a digraph of σ, and we now show that E has fewer weak components than D. Let w and w' be arbitrary points of W and W' respectively. In $W-x$ there are semipaths P and Q joining w with u and v respectively. In $W'-x'$ there is a semipath P' joining u' and w' or a semipath Q' joining v' and w'. If P' exists, then $P'+y_2+Q$ is a semipath joining w' and w; otherwise $P+y_1+Q'$ is such a semipath. Therefore any two points of $W \cup W'$ are joined by a semipath in E. Hence, E has fewer weak components than D, contrary to the supposition.

In order to develop a criterion for a sequence σ to belong to a strong digraph, we require the following preliminary observation.

LEMMA. *For a given graphical sequence σ, let D be a digraph of σ having the smallest number of strong components. Then either there are lines from every point of S to every point of S', or lines from every point of S' to every point of S.*

Proof. Suppose the lemma is false. Since S and S' are distinct strong components, there must by [5] be no lines in one direction, say from S' to S. Then by the assumption, there are points u in S and v' in S' such that there is no line from u to v'. Since S and S' are nontrivial, they contain lines $x = uv$ and $x' = u'v'$ respectively. Define the two new lines $y_1 = uv'$ and $y_2 = u'v$. Then of course $E = D-x-x'+y_1+y_2$ is also a digraph of σ. We will show that E has fewer strong components than D. Let w and w' be points of S and S', respectively. In $S-x$, there is a path P from w to u and a path Q from v to w; similarly, in $S'-x'$, there are paths P' and Q' from w' to u' and from v' to w'. Therefore

$$P+y_1+Q'+P'+y_2+Q$$

contains paths from w to w' and from w' to w. Hence, every pair of points are mutually reachable in E. Therefore E has fewer strong components than D, which is a contradiction, proving the lemma.

THEOREM 3. *Let $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ be a graphical sequence in which $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$. Then σ belongs to a strong digraph* *if and only if each* $a_i > 0$, *each* $b_j > 0$, *and for each* k: $1 \leqslant k \leqslant p-1$, *the following strict inequalities hold*:

$$\sum_{i=1}^{k} a_i < \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{p} \min\{k, b_i\}. \tag{4}$$

Proof. To prove the necessity, consider a strong digraph D whose points v_i are labelled so that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$. Suppose that for some integer k, equality holds in (4). Let $U = \{v_1, v_2, \ldots, v_k\}$ and let $W = \{v_{k+1}, \ldots, v_p\}$. Any point v_i in W can have at most $\min\{k, b_i\}$ lines to it from points of U, so there are at most $\sum_{k+1}^{p} \min\{k, b_i\}$ lines from points of U to points of W. Because there are $\sum_1^k a_i$ lines from points of U, there must thus be exactly $\sum_1^k b_i$ lines joining points of U. From this it follows that no point of W can reach any point of U, contradicting the assumption that D is strong.

The proof of the sufficiency of these conditions is considerably longer. We remark that in this proof, whenever a point is labelled with a subscript, *e.g.*, u_i or w_j, that label denotes that point throughout. On the other hand, the meaning of a point v without a subscript may vary as we prove various assertions. The same applies to lines x_i with subscripts and lines x and x' without. Lines y_i may vary during the proof, but these always denote lines not in the given digraph D.

Now let us assume that σ is a graphical sequence satisfying the stated conditions but not belonging to any strong digraph. Let D be a digraph of σ with the minimum number of strong components. There is a strong component S_0 such that no point of S_0 is adjacent from any point not in S_0 by [5]. Since by hypothesis no $b_i = 0$, S_0 must be nontrivial and hence by the lemma is unique. Let V_0 be the set of points of S_0, and say it has m points.

Assume there is a point w_1 not in V_0 which is not adjacent from $\min\{m, \mathrm{id}\, w_1\}$ points of V_0. Let u_0 and w_0 be points of V_0 such that u_0 is not adjacent to w_1 but is adjacent to w_0. Then by the lemma, w_1 must be a trivial strong component.

Assertion 1. The point w_1 is not adjacent from any point in any nontrivial strong component other than S_0. Suppose that this is not the case for some point v in another nontrivial strong component S. Let $x_0 = u_0 w_0$ and $x = v w_1$, and define two new lines $y_1 = u_0 w_1$ and $y_2 = v w_0$. Then $E = D - x_0 - x + y_1 + y_2$ is a digraph of σ with fewer strong components than D, because in E all points of S_0 and S are mutually reachable. This contradiction proves Assertion 1.

From this it follows that there is a point u_1 of a trivial strong component which is adjacent to w_1. Without loss of generality, we can take u_1 as adjacent from $\min\{m, \mathrm{id}\, u_1\}$ points of V_0. For otherwise the same argument as above would apply to u_1 as it did to w_1, and after a finite number of steps we would reach points having the properties of u_1 and w_1.

Assertion 2. The point u_1 is adjacent only from points of V_0. Suppose it is adjacent from some other point. Then it is adjacent from every point of V_0. Let $x_0 = u_0 w_0$, $x_1 = u_1 w_1$, $y_1 = u_0 w_1$, and $y_2 = u_1 w_0$. Then $E = D - x_0 - x_1 + y_1 + y_2$ has fewer strong components than D since all points of $V_0 \cup \{u_1\}$ are mutually reachable. Therefore, u_1 must be adjacent from $\mathrm{id}\, u_1$ points of V_0, and hence is adjacent only from those points.

Assertion 3. The point u_1 is adjacent to every point of every nontrivial strong component other than S_0. Suppose v is a point in another nontrivial strong component S and is not adjacent from u_1. Let $x_1 = u_1 w_1$, $x = u_0 v$, $y_1 = u_0 w_1$, and $y_2 = u_1 v$. Then $E = D - x_1 - x + y_1 + y_2$ has the same strong components as D. But S_0 and S are nontrivial strong components of E, neither of which has every point adjacent from every point of the other. Thus, E satisfies the hypothesis but not the conclusion of the lemma, which proves this assertion.

Let $V_1 = V_0 \cup \{u_1\}$. Assume there is a point w_2 not in V_1 which is not adjacent from $\min\{m+1, \mathrm{id}\, w_2\}$ points of V_1. By the lemma and Assertion 3, w_2 must be a trivial strong component.

Assertion 4. The point w_2 is not adjacent from any point in any nontrivial strong component other than S_0. Again we suppose this is not true. Let v be a point in a nontrivial strong component S and adjacent to w_2. Then by the same argument as that used in Assertion 1, it follows that w_2 is adjacent from every point of S_0. Hence it is not adjacent from u_1. Let $x_0 = u_0 w_0$, $x_1 = u_1 w_1$, $x = v w_2$, $y_1 = u_0 w_1$, $y_2 = u_1 w_2$, and $y_3 = v w_0$. Then $E = D - x_0 - x_1 - x + y_1 + y_2 + y_3$ has fewer strong components than D, since points of S_0 and S are now mutually reachable.

Therefore, there is a point u_2 of a trivial strong component which is adjacent to w_2, and without losing generality we can consider u_2 as being adjacent from $\min\{m+1, \mathrm{id}\, u_2\}$ points of V_1.

Assertion 5. The point u_2 is adjacent only from points of V_1. Suppose this is not the case. Then u_2 must be adjacent from every point of V_1. Also, by the argument used in proving Assertion 2, w_2 must be adjacent from every point of S_0. Hence, the line $u_1 w_2$ does not exist in D, and since u_2 is adjacent from every point of S_0, $u_2 \neq w_1$. Let $x_0 = u_0 w_0$, $x_1 = u_1 w_1$, $x_2 = u_2 w_2$, $y_1 = u_0 w_1$, $y_2 = u_1 w_2$, and $y_3 = u_2 w_0$. Then the digraph $E = D - x_0 - x_1 - x_2 + y_1 + y_2 + y_3$ has fewer strong components than D, which is impossible.

Assertion 6. The point u_2 is adjacent to every point of every nontrivial strong component other than S_0. Once more we suppose the contrary and let v in S be some such point not adjacent from u_2. By the argument used in Assertion 3, w_2 is then adjacent from every point of S_0. Hence the line $u_1 w_2$ does not exist. Let $x = u_0 v$, $x_1 = u_1 w_1$, $x_2 = u_2 w_2$, $y_1 = u_0 w_1$, $y_2 = u_1 w_2$, and $y_3 = u_2 v$. Then in $E = D - x - x_1 - x_2 + y_1 + y_2 + y_3$, which has the same strong components as D, there is a point v in a nontrivial strong component which is not adjacent from every point of S_0. This is a contradiction to the lemma and proves the assertion.

Let $V_2 = V_1 \cup \{u_2\}$. If there is a point w_3 not in V_2 which is not adjacent from $\min\{m+2, \operatorname{id} w_3\}$ points of V_2, we repeat this process and get a point u_3, analogous to points u_1 and u_2. After a finite number of repetitions, we obtain a set of points $V_n = V_0 \cup \{u_1, u_2, \ldots, u_n\}$ such that (i) every point in every nontrivial strong component other than S_0 is adjacent from every point of V_n, (ii) every point v of every trivial strong component is adjacent from $\min\{m+n, \operatorname{id} v\}$ points of V_n, and (iii) no point of V_n is adjacent from any point not in V_n.

Let u be a point in V_n and w a point not in V_n. It follows that if w is adjacent to a point, then u is also adjacent to that point; hence, $\operatorname{od} u \geqslant \operatorname{od} w$. If u is in S_0, then it is adjacent to some other point of S_0, which is not adjacent from w. If u is not in S_0, it is adjacent to some point v not adjacent from every point of V_n and hence not adjacent from w. Therefore, $\operatorname{od} u > \operatorname{od} w$. Let $k = m+n$. Then

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{p} \min\{k, b_i\}.$$

This equality establishes the theorem.

3. *Degree sequences of functional digraphs*

A *functional digraph* is one in which every point has outdegree 1. Putting it another way, a functional digraph portrays an irreflexive function from a finite set V into itself. Clearly, in the degree sequence of a functional digraph, each $a_i = 1$. In the next theorem, we provide a criterion for a sequence of ordered pairs to belong to a functional digraph. We call such a sequence *functional.*

THEOREM 4. *Let* $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ *be a sequence of ordered pairs of integers in which each* $a_i = 1$ *and* $0 \leqslant b_i \leqslant p-1$. *This sequence belongs to some functional digraph if and only if*

$$\sum_{i=1}^{p} b_i = p.$$

Proof. The necessity of this equality follows since $\sum_1^p b_i = \sum_1^p a_i = p$.

For the sufficiency, we show that if σ is a sequence in which every $a_i = 1$, $b_i \leqslant p-1$, and $\sum_1^p b_i = p$, then σ is graphical. That it then belongs to a functional digraph is apparent.

It is obvious that (1) of Theorem 0 holds for σ. Also, at least two of the numbers b_i are non-zero. Let A_k, $1 \leqslant k \leqslant p-1$, denote the following sum:

$$A_k = \sum_{i=1}^{k} \min\{b_i, k-1\} + \sum_{i=k+1}^{p} \min\{b_i, k\}.$$

We consider several cases. If $k=1$, there is an $i>1$ for which $b_i \geqslant 1$, $A_1 \geqslant 1$. Now take $k>1$. If every $b_i \leqslant k-1$, then $A_k = \sum_1^p b_i = p > k$. The only remaining case is that in which some $b_j \geqslant k > 1$. This b_j then contributes at least $k-1$ to A_k and the other non-zero b_i contributes at least 1. Hence in this case as well, $A_k \geqslant k$. Therefore, for each k, $\sum_1^k a_i = k \leqslant A_k$, proving that (2) of Theorem 0 holds for σ. Hence σ is graphical, which serves to establish the theorem.

Strong and weak functional digraphs are characterised structurally in Harary [4]. Using these results, we find criteria for a functional sequence to belong to strong, weak, and disconnected digraphs.

COROLLARY 4a. *Let* $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ *be a functional sequence.*

1. *σ belongs to some strong digraph if and only if each* $b_i = 1$.
2. *σ belongs to some weak digraph which is not strong if and only if some* $b_i = 0$.
3. *σ belongs to some disconnected digraph if and only if at least four* b_i *are non-zero.*

4. *Degree sequences of traversable digraphs*

An *isograph* is a digraph in which each point has equal outdegree and indegree. A weak isograph possesses the following interesting property: it has a sequence $u_1, u_1u_2, u_2, u_2u_3, \ldots$ in which points and lines alternate which begins at any point and proceeds in such a way that no line is repeated, that traverses each line exactly once and returns to the starting point. Such digraphs are called *traversable*, and weak isographs are the only traversable digraphs by a well-known result of Euler.

Our next result provides a criterion for a sequence of ordered pairs to belong to some isograph. We remark that every weak isograph is necessarily strong, see [5]. Therefore, using Theorem 2, a graphical sequence of ordered pairs of equal integers belongs to some strong digraph if and only if it contains no (0, 0).

THEOREM 5. *Let* $\sigma = \big((a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)\big)$ *be a sequence of ordered pairs of integers in which* $0 < a_i = b_i \leqslant p-1$ *and* $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$. *Then σ belongs to some traversable digraph if and only if the following* $p-1$ *inequalities hold:*

$$\sum_{i=1}^{k} a_i \leqslant k(k-1) + \sum_{i=k+1}^{p} \min\{k, a_i\}; \quad 1 \leqslant k \leqslant p-1. \tag{5}$$

Proof. If σ is a sequence belonging to a traversable digraph, the inequalities (2) of Theorem 0, and therefore the inequalities (5), hold.

For the converse, we suppose that σ satisfies the above conditions, but that for some integer k,

$$\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} \min\{k-1, b_i\} + \sum_{i=k+1}^{p} \min\{k, b_i\}.$$

It is obvious that $k \neq 1$. If $a_k \geqslant k-1$, then

$$\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} \min\{k-1, a_i\} + \sum_{i=k+1}^{p} \min\{k, b_i\} = k(k-1) + \sum_{i=k+1}^{p} \min\{k, b_i\}.$$

Because this contradicts the hypothesis that (5) holds for each k, it follows that $a_k < k-1$. Clearly $a_1 > k-1$, for otherwise,

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \min\{k-1, a_i\}, \text{ contradicting the supposition.}$$

Let j be the greatest integer such that $a_j \geqslant k-1$. Then

$$\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} \min\{a_i, k-1\} + \sum_{i=k+1}^{p} \min\{a_i, k\} = j(k-1) + \sum_{i=j+1}^{k} a_i + \sum_{i=k+1}^{p} a_i.$$

Therefore,

$$\sum_{i=1}^{j} a_i > j(k-1) + \sum_{i=k+1}^{p} a_i.$$

By hypothesis,

$$\sum_{i=1}^{j} a_i \leqslant j(j-1) + \sum_{i=j+1}^{p} \min\{a_i, j\} \leqslant j(j-1) + \sum_{i=j+1}^{k} \min\{a_i, j\} + \sum_{i=k+1}^{p} a_i.$$

Combining these two inequalities, we have

$$j(k-1) + \sum_{i=k+1}^{p} a_i < \sum_{i=1}^{j} a_i < \sum_{i=1}^{p} a_i \leqslant j(j-1) + \sum_{i=j+1}^{k} \min\{a_i, j\} + \sum_{i=k+1}^{p} a_i.$$

Therefore,

$$j(k-j) < \sum_{i=j+1}^{k} \min\{a_i, j) \leqslant j(k-j).$$

This contradiction establishes the theorem since it follows from our earlier remarks that being graphical, σ must belong to a traversable digraph.

It is interesting to note the similarity of (5) to the criteria of Erdős and Gallai [3] for a sequence of positive integers $\rho = (a_1, a_2, \ldots, a_p)$: $p-1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_p$ to be the sequence of degrees of some graph (undirected). The only requirement in addition to (5) is that $\sum_1^p a_i$ be even. From this it follows, of course, that the sequence σ of Theorem 5 belongs to a symmetric digraph if and only if $\sum_1^p a_i$ is even.

5. *Summary*

In this note we have provided criteria for determining whether a sequence of ordered pairs of integers belongs to digraphs of various classes, *viz.* strong, weak, disconnected, functional, and traversable digraphs.

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