ON MATROID THEOREMS OF EDMONDS AND RADO

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Introduction

In this note I show how very general and powerful results about the union and intersection of matroids due to J. Edmonds **[19]** may be deduced from a matroid generalisation of Hall's theorem by R. Rado **[13]**.

Throughout, S, T, will denote finite sets, $|X|$ will denote the cardinality of the set X and $\{x_i : i \in I\}$ denotes the set whose distinct elements are the elements x_i.

A *matroid* $(S, \mathbf{M})$ is a finite set S together with a family $\mathbf{M}$ of subsets of S, called *independent* sets, which satisfies the following axioms

(1) $\emptyset \in \mathbf{M}$.

(2) If X is independent and $Y \subset X$, then Y is independent.

(3) If $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_{m+1}\}$ are members of $\mathbf{M}$ then there exists an element y_i of $Y - X$ such that $\{x_1, \ldots, x_m, y_i\} \in \mathbf{M}$.

It is easy to verify that these axioms are equivalent to many other sets of axioms given by Whitney **[18]** or Rado **[14]**. We often write $\mathbf{M}$ for the matroid $(S, \mathbf{M})$ and call $\mathbf{M}$ a matroid on S. The *rank* of a subset X of S is the cardinality of a maximal independent subset of X and is denoted by $r(X)$. The *rank of the matroid* is $r(S)$ and we often write this as $r(\mathbf{M})$. A *base* of $(S, \mathbf{M})$ is a maximal independent subset of S, and a well-known property of matroids is that if I is any independent set and B is any base, then there exists a subset Y of B such that $I \cup Y$ is also a base.

Associated with any matroid $(S, \mathbf{M})$ is a *dual matroid* $(S, \mathbf{M}^*)$ which is defined to have as its bases all sets of the form $S - B$, where B is a base of $\mathbf{M}$. Clearly the dual $\mathbf{M}^*$ is unique and it is not difficult to show that the rank functions r and r^* of a matroid and its dual are connected by

$$r^*(S-A) = |S| - |A| - r(S) + r(A) \tag{4}$$

for all subsets A of S.

We also point out that if $A \subset S$, then any matroid $\mathbf{M}$ on S induces a matroid $\mathbf{M} \times A$ on A in the natural way. $\mathbf{M} \times A$ consists of those subsets of A which are members of $\mathbf{M}$. It is called the *reduction* of $\mathbf{M}$ to A and clearly its rank function r_A is related to the rank function r of $\mathbf{M}$ by

$$r_A(B) = r(B) \tag{5}$$

for all subsets B of A.

Now if $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_k$ are matroids on S let $\mathbf{M}_1 \vee \mathbf{M}_2 \vee \ldots \vee \mathbf{M}_k$ denote the collection of subsets of S of the form $X_1 \cup X_2 \cup \ldots \cup X_k$ where $X_i \in \mathbf{M}_i$, $(1 \leqslant i \leqslant k)$.

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Edmonds [19] has proved that $\mathbf{M}_1 \vee \ldots \vee \mathbf{M}_k$ is a matroid on S. Also if r_i denotes the rank function of $\mathbf{M}_i$, and R_k denotes the rank function of $\mathbf{M}_1 \vee \ldots \vee \mathbf{M}_k$ it is easy to see that for any subset A of S,

$$R_k(S) \leqslant r_1(A)+r_2(A)+ \ldots +r_k(A)+|S-A|,$$

for let B be any base of $\mathbf{M}_1 \vee \ldots \vee \mathbf{M}_k$, then for any subset A of S

$$R_k(S) = |B| = |B \cap A|+|B \cap (S-A)| \leqslant r_1(A)+ \ldots +r_k(A)+|S-A|.$$

The union theorem of Edmonds is

THEOREM 1. *The rank of* $(S, \mathbf{M}_1 \vee \mathbf{M}_2 \vee \ldots \vee \mathbf{M}_k)$ *is given by*

$$R_k(S) = \min_{A \subset S} [r_1(A)+ \ldots +r_k(A)+|S-A|]. \tag{6}$$

COROLLARY. *The rank function* R_k *of* $(S, \mathbf{M}_1 \vee \ldots \vee \mathbf{M}_k)$ *is given by*

$$R_k(A) = \min_{B \subset A} [r_1(B)+ \ldots +r_k(B)+|A-B|], \tag{7}$$

for all subsets A *of* S.

To see how (7) follows from (6) it is sufficient to notice that for any subset A of S,

$$\big(A, (\mathbf{M}_1 \times A) \vee (\mathbf{M}_2 \times A) \vee \ldots \vee (\mathbf{M}_k \times A)\big) = \big(A, (\mathbf{M}_1 \vee \mathbf{M}_2 \vee \ldots \vee \mathbf{M}_k) \times A\big)$$

and since the rank of A in the matroid $(S, \mathbf{M})$ is just the rank of the matroid $(A, \mathbf{M} \times A)$, (7) follows.

Let S, T be finite sets and let $\sim$ be an incidence relation between the elements of S and the elements of T. If $s \in S$ and $t \in T$ and $s \sim t$ then we say that s and t are *incident*. For each $s \in S$, $\tilde{s} = (t \in T;\ s \sim t)$. If $X \subset S$, then $\tilde{X} = \bigcup_{s \in X} \tilde{s}$.

For notational convenience we let

$$S = \{s(i);\ 1 \leqslant i \leqslant m\} \quad \text{and} \quad T = \{t(j);\ 1 \leqslant j \leqslant n\}.$$

A *matching* between S and T with respect to the incidence relation $\sim$ is a pair of subsets (X, Y) where $X = \{s(i_1), \ldots, s(i_k)\}$ and $Y = \{t(j_1), \ldots, t(j_k)\}$ such that $i_p \neq i_q$ $(p \neq q)$, and $j_p \neq j_q$ $(p \neq q)$, and $s(\alpha) \neq s(\beta)$ $(\alpha \neq \beta)$, $t(\alpha) \neq t(\beta)$ $(\alpha \neq \beta)$ and such that for each λ, $1 \leqslant \lambda \leqslant k$, $s(i_\lambda) \sim t(j_\lambda)$. The common cardinality of X and Y is called the *cardinality of the matching* (X, Y). The theory of matchings is a much discussed topic in graph theory, see for example Ore [11].

If now $\mathbf{M}$, and $\mathbf{N}$ are matroids on S and T respectively we say that the pair (X, Y) is an *independent matching* between $(S, \mathbf{M})$ and $(T, \mathbf{N})$ with respect to the incidence relation $\sim$, if (X, Y) is a matching with respect to $\sim$ *and* X is independent in $(S, \mathbf{M})$ and Y is independent in $(T, \mathbf{N})$.

A "matching" form of Edmonds' intersection theorem as described by Brualdi [1] is

THEOREM 2. *If* r_1, r_2 *denote the rank functions of* $(S, \mathbf{M})$ *and* $(T, \mathbf{N})$ *respectively, then the maximum cardinality of an independent matching between* $(S, \mathbf{M})$ *and* $(T, \mathbf{N})$ *with respect to an incidence relation* $\sim$ *is equal to*

$$\min_{A \subset S} [r_2(\tilde{A})+r_1(S-A)].$$

Rado's Theorem. Let $\mathbf{A} = (A_i : 1 \leqslant i \leqslant n)$ be any collection of subsets of S. Let $\mathbf{M}$ be a matroid on S with rank function r. Then if $J \subset (1, \ldots, n)$ and $A(J)$ denotes $\bigcup (A_i : i \in J)$, we have the following result.

THEOREM 3. *The collection of subsets A has a transversal which is independent in $\mathbf{M}$ if and only if for all $J \subset (1, \ldots, n)$*

$$r(A(J)) \geqslant |J|.$$

A very simple proof of this is given in **[17]**, and also of the following "defect" version due to H. Perfect **[12]**.

THEOREM 3′. *If d is any non-negative integer $\leqslant n$, then there is a subcollection of $\mathbf{A}$ consisting of all but d of the A_i which has a transversal which is independent in $\mathbf{M}$ if and only if for all $J \subset (1, \ldots, n)$*

$$r(A(J)) \geqslant |J| - (n-d).$$

Brualdi **[1]** shows how Theorem 3 is deduced from Theorem 2.

Deduction of Theorem 1 *from Theorem* 3. Let $S = \{s_1, s_2, \ldots, s_n\}$. Let $\mathbf{M}_i$ $(1 \leqslant i \leqslant k)$ be matroids on S. For $1 \leqslant i \leqslant k$, let S_i be disjoint sets with

$$S_i = \{s_{1i}, s_{2i}, \ldots, s_{ni}\}.$$

Let $\mathbf{M}_i'$ be a matroid on S_i, isomorphic to $\mathbf{M}_i$ under the obvious mapping. Then since the S_i are disjoint, $\vee(\mathbf{M}_i' : 1 \leqslant i \leqslant k)$ is a matroid on $S' = \bigcup (S_i : 1 \leqslant i \leqslant k)$, with rank function ρ given in terms of the rank functions r_i of $\mathbf{M}_i$ by

$$\rho(X) = \sum_{i=1}^{k} r_i(X). \tag{8}$$

Let A_i $(1 \leqslant i \leqslant n)$ be subsets of S' defined by $A_j = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$. Then if $Y \subset S$, Y has rank $\geqslant u$ in $\vee(\mathbf{M}_i : 1 \leqslant i \leqslant k)$ if and only if the collection of subsets $(A_j : s_j \in Y)$ have a transversal which has rank $\geqslant u$ in $\vee(\mathbf{M}_i' : 1 \leqslant i \leqslant k)$. By Theorem 3′, necessary and sufficient conditions for this are that

$$\rho(A_{j_1} \cup \ldots, \cup A_{j_m}) \geqslant m - (|Y| - u) \tag{9}$$

for any $(j_1, \ldots, j_m)$ such that $(s_{j_1}, \ldots, s_{j_m}) \subset Y$. But for all p, $(1 \leqslant p \leqslant k)$,

$$r_p(A_{j_1} \cup A_{j_2} \cup \ldots \cup A_{j_m}) = r_p(s_{j_1}, \ldots, s_{j_m}).$$

Hence from (8), (9) reduces to the condition that for all $X \subset Y$,

$$\sum_{i=1}^{k} r_i(X) \geqslant |X| - |Y| + u$$

which completes the proof of Theorem 1.

Proof of Theorem 2 *from Theorem* 1. Let $(S, \mathbf{M})$ and $(T, \mathbf{N})$ be matroids and let $\sim$ be an incidence relation between S and T. Construct the bipartite graph G having vertex sets $S \cup T$ with $S \cap T = \emptyset$, and in which the edges of G join a pair of vertices $s \in S$, $t \in T$, if and only if $s \sim t$. Let E be the edge set of this graph.

For notational reasons we denote a typical member of E by $e(i, j)$, and this will signify that e is the edge joining $s(i)$ of S to $t(j)$ of T.

We let $\mathbf{M}'$ be the null set and those subsets

$$\{e(i_1, j_1), \ldots, e(i_k, j_k)\}$$

of E for which

(i) $s(i_1), s(i_2), \ldots, s(i_k)$ are *distinct* members of S,

(ii) The set $\{s(i_1), \ldots, s(i_k)\}$ is independent in $(S, \mathbf{M})$.

LEMMA 1. *$\mathbf{M}'$ is a matroid on E.*

Proof. If $A \in \mathbf{M}'$, then any subset of A is a member of $\mathbf{M}'$. Now let

$$W = \{e(i_1, j_1), \ldots, e(i_p, j_p)\}$$

$$W' = \{e(i_1', j_1'), \ldots, e(i_{p+1}', j_{p+1}')\}$$

be members of $\mathbf{M}'$. Since $\mathbf{M}$ is a matroid on S, there exists

$$s(i_k') \in \{s(i_1'), \ldots, s(i_{p+1}')\}, \quad \text{such that} \quad \{s(i_1), \ldots, s(i_p), s(i_k')\}$$

is an independent subset of $\mathbf{M}$ of cardinality $p+1$. Hence $W \cup \{e(i_k', j_k')\}$ is a member of $\mathbf{M}'$ of cardinality $p+1$, and thus $\mathbf{M}'$ is a matroid on E.

We let $\mathbf{N}'$ be the matroid induced on E by $\mathbf{N}$ in the analogous way, and now we can state the obvious lemma

LEMMA 2. *$(S, \mathbf{M})$ and $(T, \mathbf{N})$ have an independent matching of cardinality k with respect to $\sim$ if and only if the matroids $(E, \mathbf{M}')$ and $(E, \mathbf{N}')$ have a common independent set of cardinality k.*

We now use Theorem 1 and duality in essentially the same way as Edmonds to prove

THEOREM 4. *Two matroids $(S, \mathbf{M}_1)$ and $(S_2, \mathbf{M}_2)$ with rank functions r_1 and r_2 have a common independent set of cardinality k if and only if for all subsets $A \subset S$,*

$$r_1(A) + r_2(S-A) \geqslant k.$$

Proof. If a subset I is independent in $\mathbf{M}$, and $\mathbf{M}_2$ then $S-I$ contains a base of $\mathbf{M}_2^*$, and thus the rank of the matroid $\mathbf{M}_1 \vee \mathbf{M}_2^*$ is not less than $|I| + r_2^*(S)$.

Conversely if $r[\mathbf{M}_1 \vee \mathbf{M}_2^*] \geqslant k + r_2^*(S)$, then since any base B^* of $\mathbf{M}_2^*$ is independent in $\mathbf{M}_1 \vee \mathbf{M}_2^*$, there must exist a subset I of $S - B^*$ which is independent in $\mathbf{M}_1$ and has cardinality not less than k.

Thus $\mathbf{M}_1$ and $\mathbf{M}_2$ have a common independent set of cardinality k if and only if

$$r(\mathbf{M}_1 \vee \mathbf{M}_2^*) \geqslant k + r_2^*(S).$$

Using Theorem 1 this implies that for any $A \subset S$,

$$r_1(A) + r_2^*(A) + |S-A| \geqslant k + r_2^*(S).$$

Using (4), this reduces to

$$r_1(A) + r_2(S-A) \geqslant k.$$

Now combining Lemma 2 and Theorem 4 we see that $(S, \mathbf{M})$ and $(T, \mathbf{N})$ have an independent matching of cardinality k with respect to $\sim$ if and only if

$$r_1'(A) + r_2'(E-A) \geqslant k$$

for all subsets A of E, where r_1', r_2' are the rank functions of $\mathbf{M}'$ and $\mathbf{N}'$ respectively.

By the definition of $\mathbf{M}'$ and $\mathbf{N}'$ on E, this is clearly equivalent to

$$\min_{S_0 \subset S} [r_2(\tilde{S}_0)+r_1(S-S_0)] \geqslant k,$$

and thus Theorem 2 follows.

Conclusion

By using the theorems here together with suitably chosen matroids one gets easy proofs of many apparently unrelated combinatorial results. For example, taking $\mathbf{M}_i = \mathbf{M}$ for all i, we see that the necessary and sufficient conditions for a matroid $\mathbf{M}$ to have k disjoint bases is that $V(\mathbf{M}_i : 1 \leqslant i \leqslant k)$ has a basis of cardinality $kr(S)$, which is so if and only if $\forall A \subset S$,

$$kr(A)+|S-A| \geqslant kr(S). \tag{10}$$

Similarly $\mathbf{M}$ is such that S is the union of as few as k independent sets if and only if $\forall A \subset S$.

$$kr(A) \geqslant |A|. \tag{11}$$

These results were originally proved for matroids by Edmonds [**2**] and [**3**]. By applying (11) when $\mathbf{M}$ is the natural matroid induced on a vector space by linear independence we get the theorem of Horn [**6**]. By applying (10) and (11) to the cycle matroid of a graph G we deduce the necessary and sufficient conditions (a) for a graph to have k edge disjoint spanning forests and (b) for a graph to be the union of k subforests, thus obtaining graph theorems of Tutte [**15**] and Nash-Williams [**9**], [**10**]. By choosing $\mathbf{M}_i$ to be the matroid $\mathbf{M}$ truncated at r_i, we get necessary and sufficient conditions for a matroid to have disjoint independent sets of g prescribed cardinalities r_i. Applying this to the special case when $\mathbf{M}$ is a transversal matroid we thus get the result of P. J. Higgins [**5**] who gives conditions for a family Q of sets to have k mutually disjoint partial transversals of prescribed sizes $n_1, n_2, \ldots, n_k$. Many other covering and packing theorems of this nature proved by Edmonds and Fulkerson [**4**] follow by a similar argument.

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