POINT ARBORICITY CRITICAL GRAPHS EXIST

BÉLA BOLLOBÁS AND FRANK HARARY†

ABSTRACT

The point arboricity of a graph is the minimum number of colours assignable to the points so that no cycle is monochromatic. A graph is called k-critical if it is connected and the removal of any edge reduces the point arboricity from k to $k-1$. The existence of k-critical graphs of odd order only was established by Kronk and Mitchem. We construct here k-critical graphs of every possible even order.

Introduction

In general, we follow the notation and terminology of the book [**2**], but we use the terms *point* and *edge*. The *order* of a graph G is the number p of its points. The symbol $G \cup H$ will always refer to the union of disjoint graphs.

We write nG for the union of n disjoint copies of a connected graph G. Also G_1+G_2 is the join of G_1 and G_2, consisting of $G_1 \cup G_2$ and all edges joining (a point of) G_1 with G_2. As usual $\bar{H}$ denotes the complement of a graph H and $G-e$ is the subgraph of G resulting from the removal of the edge e. The addition of a *new edge* e joining two non-adjacent points of G results in the graph $G+e$.

The subject of the (edge) arboricity of a graph was launched by Nash-Williams [**4**]. He obtained an exact determination of the arboricity of a graph in terms of its subgraphs, see [**2**; p. 90].

A colouring of the points of a graph is called *acyclic* if no cycle has all its points the same colour. The *point arboricity* $\rho(G)$ of a graph G was introduced by Chartrand, Kronk and Wall [**1**] as the minimum number of colours in an acyclic colouring of the points of G. A graph G is *k-critical* with respect to point arboricity if G is *connected*, $\rho(G) = k$ and for each edge e, $\rho(G-e) = k-1$. Obviously the 2-critical graphs are the cycles, so henceforth we take $k \geqslant 3$.

Kronk and Mitchem [**3**] established that for each $k \geqslant 3$ and *odd* $p \geqslant 2k-1$, there exists a k-critical graph of order p. This was done by verifying that the graph $G = K_{2k-4}+C_{p-2k+4}$ has this property. They could not decide whether or not there exists a k-critical graph of even order but they showed that for $k \geqslant 3$ there is no k-critical graph of order $2k$ or of order $p \leqslant 2k-2$. Our object is to prove that there exist k-critical graphs of each even order greater than $2k$. This is done by constructing appropriate families of k-critical graphs.

Preliminary results

The following lemmas are needed in the proof of our main results. As Lemma 1 is an elementary observation, we state it without proof.

LEMMA 1. *If a connected graph G does not contain three independent points, then every induced subgraph of order 5 contains a cycle.*

Received 25 April 1974; revised 23 October 1974.

† Research supported in part by Grant 73–2302 from the Air Force Office of Scientific Research.

[J. LONDON MATH. SOC. (2), 12 (1975), 97–102]

LEMMA 2. *For each $h \geqslant 1$, there is a hamiltonian triangle-free graph H of order $4h+3$ such that the addition of any new edge e results in a graph $H+e$ which contains a spanning subgraph of the form $K_3 \cup 2hK_2$, the union of a triangle with $2h$ disjoint edges.*

Proof. If $h = 1$, it is easy to check that the required graph H may be obtained from C_7 by adding any three disjoint diagonals, which do not produce a triangle (Fig. 1).

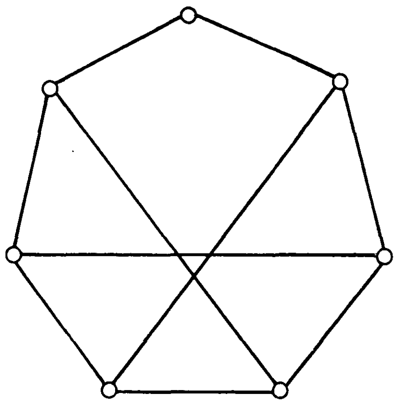

Fig. 1. A graph H for $h = 1$.

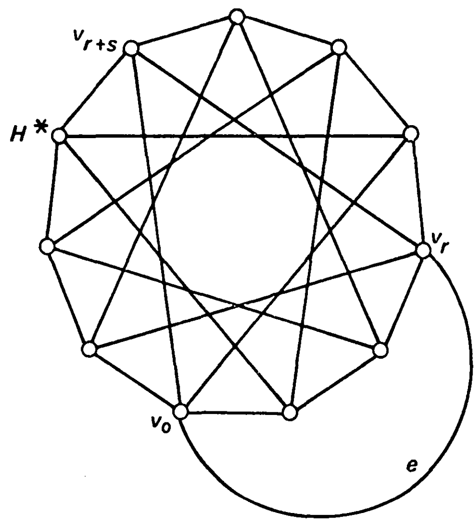

Fig. 2. A graph H^* for $h = 2$.

Suppose now that $h \geqslant 2$. In order to construct an appropriate graph H, start with the graph $C = C_{4h+3}$ and join every pair of points at distance $2h$ in C. Call these new edges the *chords* of C. The resulting graph has no triangles. Add edges arbitrarily until a maximal triangle-free graph H is obtained. Obviously H is hamiltonian and triangle-free, and has order $4h+3$.

We shall show that H satisfies the condition of Lemma 2. Let e be any edge of $\bar{H}$ and put $H^* = H+e$. Then H^* contains a triangle K (see Fig. 2). To prove the lemma we show that the subgraph $H^* - V(K)$ has a 1-factor, i.e.,

$$H^* - V(K) \supset 2h\,K_2.$$

Label the points of C by $v_0(=v_{4h+3}), v_1, \ldots, v_{4h+2}$, so that v_i is adjacent to v_{i+1}, $i = 0, \ldots, 4h+2$. Without loss of generality, let $V(K) = \{v_0, v_r, v_{r+s}\}$, where

$$0 < r \leqslant s \leqslant t = 4h+3-(r+s).$$

If r and s are odd then so is t. As $C - V(K)$ is the union of disjoint paths with $r-1$, $s-1$ and $t-1$ points, the alternating edges from these paths form the required 1-factor.

Thus we can suppose that exactly *one* of r, s and t is *odd* and *two* are even.

We now describe how to find a chord x independent of the edges in K such that $C - V(K) - V(x)$ is the union of disjoint paths of odd length. The edge x together with alternating edges from these paths will form the desired 1-factor.

If r is odd then choose the edge $v_{r+1}\,v_{r+1+2h}$ for the chord x. As s and t are even and $r+s < r+1+2h < 4h+3$, this chord has the required properties, since

$$r-1, r+s-(r+2) = s-2, r+1+2h-(r+s+1) = 2h-s$$

and

$$4h+3-(r+1+2h+1) = 2h+1-r$$

are all even.

If s is odd then we take x to be the chord $v_{r-1} v_{r+2h+2}$. As

$$r+2 < r+2h+2 < 4h+3,$$

the appropriate paths contain $r-2$, $s-1$, $2h+1-s$ and $2h-r$ points.

Finally, if t is odd then we can choose $x = v_{r+1} v_{r+2h+4}$. As

$$r+s < r+2h+3 < 4h+3,$$

$C-V(K)-V(x)$ is the disjoint union of four paths of even length, namely, of length $r-1$, $s-2$, $2h+2-s$ and $2h-1-r$.

The Existence Theorem

By the results of Kronk and Mitchem [3], the existence of a k-critical graph ($k \geqslant 3$) of order p has been established for all odd $p \geqslant 2k-1$ and the impossibility has been shown for $p - 2k$ and $p < 2k-1$. Our theorem asserts the existence for all remaining even $p \geqslant 2k+2$. The proof which we have been able to devise involves three separate propositions, whose combination yields the results: In Proposition 1, the existence of a k-critical graph of even order p is proved for $2k+2 \leqslant p \leqslant 3k-3$; Proposition 2 shows it for $3k-2 \leqslant p \leqslant 4k-4$; Proposition 3 handles the remaining $p \geqslant 4k-3$. The proof of each of these propositions has the same form, namely,

(a) an appropriate graph G is constructed,

(b) it is verified that $\rho(G) \geqslant k$,

(c) it is shown that for each edge e of G, $\rho(G-e) \leqslant k-1$.

It will be convenient in the proof of the theorem to have the following terminology. A colour class of a given colouring of the points of a graph containing just r points is called an *r-set of the colouring*.

THEOREM. *Let $k \geqslant 3$ and let p be an even number greater than $2k$. Then there exists a k-critical graph of order p.*

Proof. As mentioned above, the proof is divided into three stages.

PROPOSITION 1. *The theorem holds when*

$$2k+2 \leqslant p \leqslant 3k-3.$$

Proof of Proposition 1.

(a) Define h by $p = 2k+2h$. Let H be a graph of order $4h+3$ whose existence is guaranteed by Lemma 2. Then we construct the graph G of even order p as the join $G = \bar{H}+K_{p-(4h+3)}$.

(b) Let us show now that $\rho(G) \geqslant k$. Take an acyclic colouring of G with $\rho(G)$ colours. Denote by c_r the number of r-sets of the colouring. As $\rho(G)$ is the smallest number of colours in an acyclic colouring of G, at most one colour class has only one element, so $c_1 = 0$ or 1. As G does not have 3 independent points, Lemma 1 implies that every colour class has at most 4 points. Thus

$$\rho(G) = \sum_1^4 c_r$$

and

$$p = \sum_1^4 r\, c_r. \tag{1}$$

Furthermore, if a colour class does have 4 points they all belong to $\bar{H}$; if a colour class has 3 points then at least 2 of these points belong to $\bar{H}$. Consequently,

$$4c_4 + 2c_3 \leqslant 4h+3. \tag{2}$$

The relations (2) and (1) imply

$$\begin{aligned} k &= \tfrac{1}{4}(2p-4h) \leqslant \tfrac{1}{4}\{2p-(4c_4+2c_3-3)\} \\ &= \tfrac{1}{4}(4c_4+4c_3+4c_2+2c_1+3) \\ &= c_4+c_3+c_2+(2c_1+3)/4. \end{aligned}$$

As k is an integer and $c_1 = 0$ or 1,

$$k \leqslant \sum_1^4 c_r = \rho(G).$$

(c) Let e be an edge of G. We shall colour $G-e$ acyclically with $k-1$ colours. This will imply that $\rho(G) = k$ and G is k-critical.

Suppose first that e joins two points of $\bar{H}$. Then by Lemma 2, $H+e$ contains $K_3 \cup 2hK_2$ as a spanning subgraph. Colour the points of this K_3 and a point of $K_{p-(4h+3)}$ with the first colour. Use each of the next $2h$ colours to colour the points of a copy of K_2 (in $K_3 \cup 2hK_2 \subset H+e$) and a point of $K_{p-(4h+3)}$. Divide the remaining $p-(6h+4) = 2k-4h-4$ points into $(k-2h-2)$ 2-sets. This gives an acyclic colouring of G with $1+2h+(k-2h-2) = k-1$ colours.

Suppose now that e has an endpoint in $K_{p-(4h+3)}$. Using the facts that H is hamiltonian and $k \geqslant 2h+3$, one can easily find an acyclic colouring of $G-e$ which has $(2h+2)$ 3-sets and $\{p-3(2h+2)\}/2 = (k-2h-3)$ 2-sets. Thus $\rho(G-e) \leqslant k-1$, completing the proof of Proposition 1.

PROPOSITION 2. *The theorem holds when* $3k-2 \leqslant p \leqslant 4k-4$.

Proof of Proposition 2.

(a) Given integers $k \geqslant 3$ and p such that $3k-2 \leqslant p \leqslant 4k-4$, it is convenient to introduce an auxiliary parameter $n = p-3k$. Note then that $-2 \leqslant n \leqslant k-4$. Denote by L_{2m} the graph obtained from K_{2m} by removing the edges in an arbitrary 1-factor.

Construct the graph G by taking the following join:

$$G = 2K_{2n+5} + L_{p-4n-10}.$$

(b) Consider an acyclic colouring of G with $\rho(G)$ colours. Then by Lemma 1 every colour class contains at most 4 points. Even more, if 4 points are in a colour class, they are all in the $2K_{2n+5}$ part of G. Therefore there are at most $(n+2)$ 4-sets among the colour classes and the rest are 3-sets or smaller. Counting the number of points in the colour classes, we find that

$$\rho(G) \geqslant (n+2)+\frac{3k+n-4(n+2)}{3}$$

$$= (n+2)+(k-n-3+1) = k.$$

(c) Let e be an edge of G. Suppose first that e lies in the $2K_{2n+5}$. Then we can choose an acyclic colouring of G having a 5-set colour class in the $2K_{2n+5}$. Further, we can form $(n+1)$ 4-sets in the $2K_{2n+5}$ and the remaining points can be placed into 3-sets. Thus the number of colour classes in this acyclic colouring is just $k-1$.

If e is not in $2K_{2n+5}$, then there is an acyclic colouring of G with $(n+2)$ 4-sets of points in the $2K_{2n+5}$ and just one more 4-set in G. The remaining colour sets can be formed as 3-sets giving in all $k-1$ colour classes in $G-e$. Thus $\rho(G-e) \leqslant k-1$, as required.

PROPOSITION 3. *The theorem holds when* $p \geqslant 4k-4$.

Proof of Proposition 3.

(a) Let $p \geqslant 4k-4$ be even and let h_1, h_2 be natural numbers such that

$$4k-8+2h_1+2h_2 = p.$$

Let the graph G of order p be the join

$$(K_{2k-6}+C_{2h_1+1}) \cup (K_{2k-6}+C_{2h_2+1})+\bar{K}_2 = (H_1 \cup H_2)+\bar{K}_2. \tag{3}$$

Before proving that G is k-critical, we give another description of G.

Let $\tilde{G}_i = K_{2k-4}+C_{2h_i+1}$, $i = 1, 2$, and let $u_i, v_i \in V(K_{2k-4}) \subset V(\tilde{G}_i)$, $i = 1, 2$. Put $G_i = \tilde{G}_i - e_i$, $i = 1, 2$, where e_i is the edge $u_i v_i$ of $\tilde{G}_i$. Then G is obtained from $G_1 \cup G_2$ by identifying u_1 with u_2 and v_1 with v_2. Denote by $u, v \in V(G)$ the points obtained from u_i and v_i. Note that in (3), $V(\bar{K}_2) = \{u, v\}$.

(b) It is easily seen that $\rho(\tilde{G}_i) = k$ and $\rho(H_i) = k-1$. In fact these are the critical graphs of odd order constructed by Kronk and Mitchem [3].

Suppose G has an acyclic colouring with $k-1$ colours. Then, as $\rho(\tilde{G}_i) = k$, u and v must be of the same colour, say 1, and there exists a path P_i in $\tilde{G}_i - e_i \subset G$, connecting u and v, whose vertices all have colour 1. Thus $P_1 \cup P_2$ is a cycle whose vertices are all 1. This contradiction shows that $\rho(G) \geqslant k$.

(c) Let now e be an arbitrary edge of G. To complete the proof of the proposition we have to show that $\rho(G-e) \leqslant k-1$. Without loss of generality it suffices to discuss the following two cases.

(1) $e = uw$, where $w \in V(H_1)$. Colour with 1 the vertices u, v, w and another vertex z of H_2. Choose acyclic colourings of $H_1 - w$ and $H_2 - z$ with the same $k-2$ colours, different from 1. Thus we obtain an acyclic colouring of G with $k-1$ colours.

(2) e is an edge of H_1. Choose an acyclic colouring of the subgraph $\tilde{G}_2 - e_2$ of G with $k-1$ colours. In this colouring, u and v must have the same colour, say 1. As $\rho(H_1 - e_1) = k-2$, we can find an acyclic colouring of $H_1 - e$ with $k-2$ new colours. This gives an acyclic colouring of G with $k-1$ colours.

Thus the proof of the theorem is complete. On combining this theorem with the results of [3], we see that for $k \geqslant 3$, there exist k-critical graphs of order $p = 2k-1$ and $p \geqslant 2k+1$ only.

The above proof of the theorem involves the construction of three different families of graphs. This would be simplified by the discovery of a proof using only one construction.

References

1. G. Chartrand, H. V. Kronk, and C. E. Wall, "The point arboricity of a graph", *Israel J. Math.*, 6 (1968), 169–175.
2. F. Harary, *Graph theory* (Addison–Wesley, Reading, Mass., 1969).
3. H. V. Kronk and J. Mitchem, "Critical point-arboritic graphs", *J. London Math. Soc.* (2), 9 (1975), 459–466.
4. C. St. J. A. Nash-Williams, "Edge-disjoint spanning trees of finite graphs", *J. London Math. Soc.*, 36 (1961), 445–450.

Trinity College,
Cambridge University.

The University of Michigan and Wolfson College,
Oxford University.