Let $G$ be a finite group and $H$ a subgroup with the property that any two elements in $H$ which are conjugate in $G$ must be conjugate in $H$. Frobenius [2] proved in 1901 that if $H$ is an Abelian Hall subgroup of $G$ (i.e. $|H|$ and $|G:H|$ are coprime), then $H$ has a normal complement (i.e. there is a normal subgroup $N$ of $G$ with $HN = G$ and $N \cap H = 1$). In fact, this conclusion is valid if “Abelian” is replaced by “nilpotent” (e.g. see Sah [4]).

There is a stronger hypothesis on the subgroup $H$ than the conjugacy hypothesis. Suppose that every complex irreducible character of $H$ extends to an irreducible character of $G$. Then since the irreducible characters of $H$ are a basis for the space of class functions on $H$, these characters must separate classes. Because the irreducible characters of $H$ extend to irreducible characters of $G$ (which are class functions of $G$), distinct $H$-conjugacy classes must belong to distinct $G$-conjugacy classes. Consequently, the character extension property on $H$ implies the conjugacy property.

Sah [4] has also proved that if $H$ is a solvable Hall subgroup and if $H$ has the character extension property, then $H$ has a normal complement. The purpose of this note is to prove a normal complementation theorem along these lines for solvable groups, retaining the character extension property on $H$ while replacing the Hall condition on $H$ by a condition which controls the imbedding of $H$ in $G$.

The notation and concepts are standard. Recall that a finite solvable group contains a nilpotent self-normalizing subgroup called a Carter subgroup and that any two Carter subgroups are conjugate [1]. If $G$ is a finite solvable group, the unique smallest normal subgroup $N$ of $G$ such that $G/N$ is nilpotent is called the nilpotent residual of $G$. From these facts and definitions it is easy to see that if $C$ is a Carter subgroup of $G$ then $NC = G$, and if $L \triangleleft G$ then $LC/L$ and $NL/L$ are a Carter subgroup and the nilpotent residual of $G/L$ respectively. $Z(H)$ denotes the centre of a group $H$. The kernel of a character $\zeta$ on $H$ is the kernel of any representation affording $\zeta$.

**Theorem** Let $G$ be a finite solvable group, $C$ a Carter subgroup and $N$ the nilpotent residual. If every complex irreducible character of $C$ extends to an irreducible character of $G$, then $N$ is a normal complement to $C$.

**Proof.** It suffices to prove that $N \cap C = 1$. If the theorem is false let $G$ be a counter-example of the smallest order. If $\zeta$ is an irreducible character of $C$ let $\zeta^1$ denote some fixed character of $G$ such that $\zeta^1|C = \zeta$. Let $K = \cap \ker \zeta^1$ where the intersection is over all irreducible characters $\zeta$ of $C$. Then $K \triangleleft G$ and $K \cap C = 1$ since $K \cap C = \cap (\ker \zeta^1 \cap C) = \cap \ker \zeta$.

Suppose $K \neq 1$. Then $G/K$ has smaller order than $G$, and $CK/K$ is a Carter subgroup of $G$. Moreover, every character $CK/K$ extends to a character of $G/K$.

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To see this let $\chi$ be an irreducible character of $CK/K$. The isomorphism $C \cong C/(C \cap K) \cong CK/K$ gives that the function $\zeta$ defined by $\zeta(c) = \chi(cK)$ is an irreducible character of $C$. The extension $\xi^1$ of $\zeta$ to $G$ has $K$ in its kernel by the definition of $K$. Thus $\chi^1$ defined by $\chi^1(gK) = \xi^1(g)$ is an irreducible character of $G/K$ extending $\chi$. Minimality of the counter-example $G$ implies that $NK \cap CK = K$. Thus $N \cap C \leq K$ and so $N \cap C = N \cap C \cap K = 1$ contrary to $G$ being a counter-example.

Thus $K = 1$. Let $T_\zeta$ be a matrix representation of $G$ affording the character $\zeta^1$, and let $T$ be the representation of $G$ which is the direct sum of the $T_\zeta$'s as $\zeta$ ranges over all irreducible characters of $C$. Suppose $x \in Z(G)$. Then by Schur's lemma $T_\zeta(x)$ is a scalar matrix and so $T_\zeta(x)$ commutes with $T_\zeta(g)$ for all $g \in G$. Thus $T(x)$ commutes with $T(g)$ for all $g \in G$. But the kernel of $T$, $K$, is 1 and so $T$ is a monomorphism. Hence $x$ commutes with all $g \in G$, and $Z(C) \leq Z(G)$. But $Z(G) \leq Z(C)$ also, since $C$ is self-normalizing. It now follows that $Z(G)$ is not trivial since $C$ is nilpotent.

Let $A$ be a minimal normal subgroup of $G$ in $Z(G) \leq C$. Then as before we can see that $G/A$ satisfies the hypotheses of the theorem. By the choice of $G$ we have $NA \cap CA = A$. But this gives $N \cap C \leq A$ for any minimal normal subgroup of $G$ which is contained in $Z(G)$. Thus $Z(G)$ contains a unique minimal normal subgroup of $G$, and as a result it is a $p$-group for some prime $p$. Thus $Z(C)$ is a $p$-group, and since $C$ is nilpotent $C$ is a $p$-group.

Now $C$ is a self-normalizing $p$-subgroup of $G$, whereupon $C$ is a $p$-Sylow subgroup of $G$. Since $C$ has the character extension property, whenever $x$ and $y$ in $C$ are $G$-conjugate they are also $C$-conjugate. But then in this case a standard argument with the transfer shows (or see [3; p. 432, 4.9]) that $G$ has a normal $p$-complement $L$. Since $C \cong G/L$ is nilpotent it follows that $N \leq L$. Thus $N \cap C \leq L \cap C = 1$, and $G$ is no counter-example. This contradiction concludes the proof.

Note that the converse of the theorem is trivial.

The same proof proves an apparently more general result. Recall that a subgroup $H$ of a finite group $G$ is called abnormal if $g \in \langle H, H^g \rangle$ for all $g \in G$. (See [1] for basic properties of abnormal subgroups.) The following is true: if $G$ is a finite group with a nilpotent, abnormal subgroup $H$ and if $H$ has the character extension property, then $H$ has a normal complement. However, we have no example which shows this to be more general than the theorem.

Note that the example of $G$ cyclic of order 4 and $C$ the subgroup of index 2 shows that some assumption on $C$ beyond the character-extension assumption is necessary to achieve the conclusion. The example of $SL_2(3)$ where a Carter subgroup $C$ is cyclic of order 6 and the nilpotent residual $N$ is the quaternion subgroup and where $N \cap C$ is the centre of $SL_2(3)$ shows that the character extension property of $C \leq G$ is really stronger than the conjugacy property of $C \leq G$. Moreover it shows that the theorem is false with the conjugacy hypothesis on $C \leq G$ replacing the character extension hypothesis.

References

A CHARACTER-THEORETIC COMPLEMENTATION THEOREM FOR CARTER SUBGROUPS


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