# *CORRECTION:* ABELIAN VARIETIES DEFINED OVER THEIR FIELDS OF MODULI, I

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The proof of the theorem contains an error. Before giving a correct proof, we state two lemmas.

LEMMA 1. *Let $K/k$ be a cyclic Galois extension of degree $m$, let $\sigma$ generate* Gal $(K/k)$, *and let $(A, \mathscr{C}, \theta)$ be defined over $K$. Suppose that there exists an isomorphism $\lambda : (A, \mathscr{C}, \theta) \to (A^{\sigma}, \mathscr{C}^{\sigma}, \theta^{\sigma})$ over $K$ such that $v\lambda^{\sigma^{m-1}} \ldots \lambda^{\sigma}\ \lambda = 1$, where $v$ is the canonical isomorphism $(A^{\sigma^m}, \mathscr{C}^{\sigma^m}, \theta^{\sigma^m}) \to (A, \mathscr{C}, \theta)$. Then $(A, \mathscr{C}, \theta)$ has a model over $k$, which becomes isomorphic to $(A, \mathscr{C}, \theta)$ over $K$.*

*Proof.* This follows easily from [7], as is essentially explained on p. 371.

LEMMA 2. *Let $G$ be an abelian pro-finite group and let $\phi : G \to \mathbf{Q}/\mathbf{Z}$ be a continuous character of $G$ whose image has order $p$. Then either:*

(a) *there exist subgroups $G'$ and $H$ of $G$ such that $H$ is cyclic of order $p^m$ for some $m$, $\phi(G') = 0$, and $G = G' \times H$, or*

(b) *for any $m > 0$ there exists a continuous character $\phi_m$ of $G$ such that $p^m \phi_m = \phi$.*

*Proof.* If (b) is false for a given $m$, then there exists an element $\sigma \in G$, of order $p^r$ for some $r \leqslant m$, such that $\phi(\sigma) \neq 0$. (Consider the sequence dual to $0 \to \operatorname{Ker}(p^m) \to G \overset{p^m}{\to} G$). There exists an open subgroup $G_0$ of $G$ such that $\phi(G_0) = 0$ and $\sigma$ has order $p^r$ in $G/G_0$. Choose $H$ to be the subgroup of $G$ generated by $\sigma$, and then an easy application to $G/G_0$ of the theory of finite abelian groups shows the existence of $G'$ (note that $\phi(\sigma) \neq 0$ implies that $\sigma$ is not a $p$-th power in $G$).

We now prove the theorem. The proof is correct up to the statement (iv) (except that (i) should read: $F' \subset k_1 \subset F'_{ab}$). To remove a minor ambiguity in the proof of (iv), choose $\sigma$ to be an element of Gal $(F'_{ab}/k_2)$ whose image $\bar{\sigma}$ in Gal $(k_1/k_2)$ generates this last group. The error occurs in the statement that the canonical map $v : A^{\sigma^p} \to A$ acts on points by sending $a^{\sigma^p} \mapsto a$; it, of course, sends $a \mapsto a$.

The proof is correct, however, in the case that it is possible to choose $\sigma$ so that $\sigma^p = 1$ (in Gal $(F'_{ab}/k_2)$).

By applying Lemma 2 to $G = \operatorname{Gal}(F'_{ab}/k_2)$ and the map $G \to \operatorname{Gal}(k_1/k_2)$ one sees that only the following two cases have to be considered.

(a) It is possible to choose $\sigma$ so that $\sigma^{p^m} = 1$, for some $m$, and $G = G' \times H$ where $G'$ acts trivially on $k_1$ and $H$ is generated by $\sigma$.

(b) For any $m > 0$ there exists a field $K$, $F'_{ab} \supset K \supset k_1 \supset k_2$, such that $K/k_2$

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is a cyclic Galois extension of degree $p^m$.

In the first case, we let $K \subset F'_{ab}$ be the fixed field of $G'$. Then $(A, \mathscr{C}, \theta)$, regarded as being defined over $K$, has a model over $k_2$. Indeed, if $m = 1$, then this was observed above, but when $m > 1$ the same argument applies.

In the second case, let $\lambda : (A, \mathscr{C}, \theta) \to (A^{\bar{\sigma}}, \mathscr{C}^{\bar{\sigma}}, \theta^{\bar{\sigma}})$ be an isomorphism defined over $k_1$ and let $\nu\lambda^{\sigma} \ldots \lambda^{\sigma^{p-1}} \lambda = \alpha \in \mu(R)$.

If $\lambda$ is replaced by $\lambda\gamma$ for some $\gamma \in \mathrm{Aut}_{k_1}((A, \mathscr{C}, \theta))$ then $\alpha$ is replaced by $\alpha\gamma^p$. Thus, as $\mu(R)$ is finite, we may assume that $\alpha^{p^{m-1}} = 1$ for some $m$. Choose $K$, as in (b), to be of degree $p^m$ over $k_2$. Let $\sigma_m$ be a generator of Gal $(K/k_2)$ whose restriction to $k_1$ is $\bar{\sigma}$. Then

$$\lambda : (A, \mathscr{C}, \theta) \to (A^{\bar{\sigma}}, \mathscr{C}^{\bar{\sigma}}, \theta^{\bar{\sigma}}) = (A^{\sigma_m}, \mathscr{C}^{\sigma_m}, \theta^{\sigma_m})$$

is an isomorphism defined over $K$ and $\nu\lambda^{\sigma_m p^{m-1}}, \ldots \lambda^{\sigma_m} \lambda = \alpha^{p^{m-1}} = 1$, and so, by) Lemma 1, $(A, \mathscr{C}, \theta)$ has a model over $k_2$ which becomes isomorphic to $(A, \mathscr{C}, \theta$ over $K$.

The proof may now be completed as before.

*Addendum:* Professor Shimura has pointed out to me that the claim on lines 25 and 26 of p. 371, viz that $\mu(R)$ is a pure subgroup of $\Pi_R{}^*{}_l$, does not hold for all rings $R$. Thus this condition, which appears to be essential for the validity of the theorem, should be included in the hypotheses. It holds, for example, if $\mu(R)$ is a direct summand of $\mu(F)$.

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