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MODULES WITH FINITE F-REPRESENTATION TYPE

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Abstract

Finitely generated modules with finite F-representation type over Noetherian (local) rings of prime characteristic p are studied. If a ring R has finite F-representation type or, more generally, if a faithful R-module has finite F-representation type, then tight closure commutes with localizations over R. F-contributors are also defined, and they are used as an effective way of characterizing tight closure. Then it is shown that $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ always exists under the assumption that $(R, \mathfrak{m})$ satisfies the Krull–Schmidt condition and M has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$, in which all the M_i are indecomposable R-modules that belong to distinct isomorphism classes and $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p]$.

0. *Introduction*

Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p. Let M be an R-module. Then, for any $e \geqslant 0$, we can derive an R-module structure on the set M with its scalar multiplication determined by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^eM$.

We say that M has *finite F-representation type* by finitely generated R-modules $M_1, M_2, \ldots, M_s$ if, for all $e \geqslant 0$, the R-modules ${}^eM$ are all isomorphic to finite direct sums of the R-modules $M_1, M_2, \ldots, M_s$. For each $i = 1, 2, \ldots, s$, we use $\#({}^eM, M_i)$ to denote the number of copies of M_i in the above direct sum decomposition of ${}^eM$. We say that M_i is an *F-contributor* if $\lim_{e\to\infty}(1/(ap^d)^e)\#({}^eM, M_i)$ is positive or, non-existent, or, equivalently, $\limsup_{e\to\infty}(1/(ap^d)^e)\#({}^eM, M_i) > 0$, where $d = \dim M$ and $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p] < \infty$.

Rings with finite F-representation type were first studied by Smith and van den Bergh in [**21**]. Discussion of the concept of F-contributors and the importance of R being an F-contributor can be found in recent work [**11**] by Huneke and Leuschke.

First we show that F-contributors exist and are Cohen–Macaulay.

Theorem A (see Lemma 2.1 and Lemma 2.2). *Suppose that $M \neq 0$ is a finitely generated R-module that has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$. Then at least one of the M_i is a non-zero F-contributor and every non-zero F-contributor is Cohen–Macaulay of dimension* $= \dim M$.

There is a closure operation, called 'tight closure', that is defined over rings of prime characteristic p [**7**]. Ever since the inception of the tight closure theory, the question of whether tight closure commutes with localizations has resisted resolution, although it has been proved to have a positive answer in special cases. The next result shows that finite F-representation type implies commutation

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of tight closure with localizations. It also demonstrates the importance of F-contributors in the computation of tight closures.

THEOREM B (see Theorem 2.3, Theorem 2.5 and Remark 2.6). *Suppose that R is a Noetherian ring of characteristic p.*

(i) *If there is a faithful R-module that has finite F-representation type (for example R has finite F-representation type), then tight closure commutes with localizations over R.*

(ii) *Assume that $(R, \mathfrak{m})$ is an analytically unramified, quasi-unmixed ring that has a completely stable test element (for example $(R, \mathfrak{m})$ is a complete domain) and that M is a faithful R-module with finite F-representation type by $M_1, M_2, \dots, M_s$, in which $M_1, M_2, \dots, M_r$ are all the F-contributors. Set $N = \bigoplus_{i=1}^{r} M_i$. Then $K_L^* = \ker(L \longrightarrow L/K \longrightarrow \operatorname{Hom}_R(N, L/K \bigotimes_R N))$ for any finitely generated R-modules $K \subseteq L$. (In particular, $I^* = (IN :_R N) = \operatorname{Ann}_R(N/IN)$ for any ideal I of R.) This also implies that tight closure commutes with localization.*

Under the assumption that $(R, \mathfrak{m})$ is a strongly F-regular local ring and satisfies the Krull–Schmidt condition, Smith and van den Bergh proved in [**21**] that if R has finite F-representation type by indecomposable modules $M_1, M_2, \dots, M_s$ that belong to distinct isomorphism classes, then $\lim_{e\to\infty}(\#({}^eR, M_i)/(ap^d)^e)$ always exists for every $i = 1, 2, \dots, s$.

We need to prove the existence of $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ in a more general situation.

THEOREM C (see Theorem 3.11). *Assume that $(R, \mathfrak{m})$ is a local ring that satisfies the Krull–Schmidt condition and that M has finite F-representation type by $\{M_1, M_2, \dots, M_s\}$, in which all the M_i are indecomposable R-modules belonging to distinct isomorphism classes. Then $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ exists and is rational for every i, where $a = [R/\mathfrak{m} : (R/\mathfrak{m})^p]$.*

In Section 1, we set up the notation carefully and review some known results. In Section 2, implications of the finite F-representation type condition and the importance of F-contributors are studied. In Section 3, we study the existence of $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$.

1. *Notation and known results*

All rings are assumed to be Noetherian and have prime characteristic p unless this is stated otherwise explicitly. For such a ring R, there is the Frobenius homomorphism $F : R \longrightarrow R$ defined by $r \longmapsto r^p$ for any $r \in R$. Therefore we have iterated the Frobenius homomorphism $F^e : R \longrightarrow R$ defined by $r \longmapsto r^{p^e}$ for any $r \in R$. Let M be an R-module. Then, for any $e \geqslant 0$, we can derive an R-module structure on M with its scalar multiplication determined by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. We denote the derived R-module by ${}^eM$. Notice ${}^0M = M$. It is straightforward to see that $\operatorname{Ass}_R(M) = \operatorname{Ann}_R({}^eM)$ and $\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_R({}^eM, {}^eN)$ for every $e \in \mathbb{N}$.

Let I be an ideal of R. Then for any $q = p^e$, we use $I^{[q]}$ to denote the ideal generated by $\{x^q \mid x \in I\}$. For any R-module M, it is easy to see that $R/I \bigotimes_R {}^eM \cong {}^eM/(I \cdot {}^eM) \cong {}^e(M/I^{[q]}M)$.

If ${}^1R$ is a finitely generated R-module (or equivalently ${}^eR$ is a finitely generated R-module for every $e \geqslant 0$), then we say that R is F-finite. If we denote by $k(P)$ the quotient field of R/P for $P \in \operatorname{Spec}(R)$, then by [**14**, Proposition 1.1, Proposition 2.3, Theorem 2.5], (also c.f. [**13**, Proposition 3.2]), we know that the F-finiteness of R implies that R has finite Krull dimension, that $[k(P) : k(P)^p] = [k(Q) : k(Q)^p]p^{\dim R_Q/PR_Q}$ for any $P, Q \in \operatorname{Spec}(R)$ such that $P \subseteq Q$, and that R is excellent.

In general, if ${}^1M$ is a finitely generated R-module, we say that M is F-finite. Notice that this implies that the ring $R/\operatorname{Ann}(M)$ is F-finite and therefore implies that ${}^eM$ is a finitely generated R-module for every $e \geqslant 0$.

Next we define finite F-representation type, which will be our main interest in the following sections. Some notation is needed. For an R-module M and an integer $n > 0$, we use nM to denote the direct sum of n copies of M while we agree that $0M = 0$. For non-negative integers $n_1, n_2, \ldots, n_s$ and R-modules $M_1, M_2, \ldots, M_s$, we use matrix multiplication $(n_1, n_2, \ldots, n_s)(M_1, M_2, \ldots, M_s)^{\mathrm{T}}$ to denote $n_1M_1 \oplus n_2M_2 \oplus \ldots \oplus n_sM_s = \bigoplus_{i=1}^s M_i^{\oplus n_i}$.

Rings with finite F-representation type were first studied by Smith and van den Bergh in [**21**].

Definition 1.1. Let R be a Noetherian ring of characteristic p and M a finitely generated R-module.

(i) We say that M *has finite F-representation type* by finitely generated R-modules $M_1, M_2, \ldots, M_s$ if, for every $e \geqslant 0$, the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, M_2, \ldots, M_s$, that is, there exist non-negative integers $n_{e1}, n_{e2}, \ldots, n_{es}$ such that

$$ {}^eM \cong (n_{e1}, n_{e2}, \ldots, n_{es})(M_1, M_2, \ldots, M_s)^{\mathrm{T}} = \bigoplus_{i=1}^{s} n_{ei}M_i. $$

(ii) We say that $M_1, M_2, \ldots, M_s$ *form a finite F-representation type system* if the R-modules ${}^1M_i$ are all isomorphic to finite direct sums of the R-modules $M_1, M_2, \ldots, M_s$, that is, there exist non-negative integers a_{ij} for $1 \leqslant i, j \leqslant s$ such that

$$ {}^1M_i \cong (a_{i1}, a_{i2}, \ldots, a_{is})(M_1, M_2, \ldots, M_s)^{\mathrm{T}} $$

for all $1 \leqslant i \leqslant s$.

(iii) We say that M *has finite F-representation type by a finite F-representation type system* $M_1, M_2, \ldots, M_s$ if the R-modules $M_1, M_2, \ldots, M_s$ form a finite F-representation type system and there exists an integer $e \geqslant 0$ such that the R-module ${}^eM$ is isomorphic to a finite direct sum of the R-modules $M_1, M_2, \ldots, M_s$, that is, there exist non-negative integers $n_{e1}, n_{e2}, \ldots, n_{es}$ such that

$$ {}^eM \cong (n_{e1}, n_{e2}, \ldots, n_{es})(M_1, M_2, \ldots, M_s)^{\mathrm{T}}. $$

Remark 1.2. We use the same notation as in Definition 1.1. Then the following hold.

(i) For the sake of convenience, we allow the M_i to be zero module or $M_i \cong M_j$ for some $i \neq j$.

(ii) If M has finite F-representation type, then M is F-finite.

(iii) Suppose that M has finite F-representation type by indecomposable R-modules $M_1, M_2, \ldots, M_s$ belonging to different isomorphism classes. If R satisfies the Krull–Schmidt condition and every M_i appears non-trivially in the direct sum decomposition of certain ${}^eM$, then M has finite F-representation type by the finite F-representation type system $M_1, M_2, \ldots, M_s$.

(iv) Suppose that M has finite F-representation type by the finite F-representation type system $M_1, M_2, \ldots, M_s$ as in Definition 1.1(iii), and let $A := (a_{ij})$ be the $n \times n$ matrix. Then

$$^{e+n}M \cong (n_{e1}, n_{e2}, \ldots, n_{es})A^n(M_1, M_2, \ldots, M_s)^{\mathrm{T}}$$

for all $n \geqslant 0$.

(v) If M has finite F-representation type or has finite F-representation type by a finite F-representation type system, then, for any multiplicatively closed set U in R, the localization $M_U = U^{-1}M$ also has finite F-representation type or has finite F-representation type by a finite F-representation type system. The same is true for the completions of M.

(vi) If R is F-finite and has finite Cohen–Macaulay representation type, then every finitely generated Cohen–Macaulay R-module M has finite F-representation type by the finite F-representation type system of all distinct indecomposable Cohen–Macaulay modules.

In general, if a finitely generated R-module M has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$, then the number of copies of M_i in decompositions of ${}^eM$ is not uniquely determined. However, we can fix a decomposition ${}^eM \cong (n_{e1}, n_{e2}, \ldots, n_{es})(M_1, M_2, \ldots, M_s)^{\mathrm{T}} = \bigoplus_{i=1}^{s} n_{ei}M_i$ of ${}^eM$ for each $e \geqslant 0$ in advance. Thus when we study an R-module M that has finite F-representation type, we agree on the fixed decompositions as above. To make our notation more transparent, we use $\#({}^eM, M_i)$ to denote n_{ei}, the number of copies of M_i in the pre-fixed decompositions of ${}^eM$. It is in this sense that the following notion of F-contributors is defined.

The concept of F-contributors and an explanation of its importance can be found in recent work [**11**] by Huneke and Leuschke. Here we give an explicit definition.

Definition 1.3. Let M be a finitely generated R-module that has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$ and let $P \in \operatorname{Spec}(R)$ be a prime ideal of R. Set $d(P) = \dim_{R_P}(M_P)$ and $a(P) = [k(P) : k(P)^p]$. We say that M_i, for some $1 \leqslant i \leqslant s$, is an F-contributor of M at P if $\limsup_{e\to\infty}(\#({}^eM, M_i)/(a(P)p^{d(P)})^e) > 0$, or, equivalently, $\lim_{e\to\infty}(\#({}^eM, M_i)/(a(P)p^{d(P)})^e)$ is either positive or non-existent.

Remark 1.4. Keep the notation of the above definition. Then the following hold.

(i) Our definition of *F-contributor* depends on the pre-fixed F-representation of ${}^eM$.

(ii) If $M_P \neq 0$ for some $P \in \operatorname{Spec}(R)$, then at least one of the M_i is an F-contributor at P. See Lemma 2.1.

(iii) Let $P, Q \in \operatorname{Spec}(R)$ be two prime ideals of R such that $a(P)p^{d(P)} = a(Q)p^{d(Q)}$. Then M has the same F-contributors at P and at Q. For this reason, when $a(P)p^{d(P)}$ is constant for all $P \in \operatorname{Spec}(R)$, we can simply say the

F-contributors of M. In particular, by [**14**], we know that $a(P)p^{d(P)}$ is constant for all $P \in \operatorname{Spec}(R)$ if $\operatorname{Spec}(R/\operatorname{Ann}(M))$ is connected and $R/\operatorname{Ann}(M)$ is locally equidimensional.

QUESTION 1.5. Does $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ always exist for every $i = 1, 2, \dots, s$?

There is a positive answer to Question 1.5 in [**21**] in the case when R is strongly F-regular. Recall that we say a reduced Noetherian ring R of characteristic p is strongly F-regular if, for any c in the complement of the union of all minimal primes of the ring R, the inclusion map $Rc^{1/p^e} \subset R^{1/p^e}$ splits for all $e \gg 0$ (or, equivalently, for some $e \geqslant 0$).

THEOREM 1.6 (see Smith and van den Bergh's results on finite F-representation type and growth [**21**]). *Let R be a strongly F-regular ring that satisfies the Krull–Schmidt condition. If R has finite F-representation type by indecomposable modules $M_1, M_2, \dots, M_s$ that belong to distinct isomorphism classes, then $\lim_{e\to\infty}(\#({}^eR, M_i)/(ap^d)^e)$ always exists for every $i = 1, 2, \dots, s$. Also $\lim_{e\to\infty}(\#({}^eR, M_i)/(ap^d)^e) > 0$ if M_i appears non-trivially as a direct summand of ${}^eR$ for some $e \geqslant 0$.*

DEFINITION 1.7 [**7**]. Let R be a Noetherian local ring of characteristic p and let L be an R-module. The tight closure of 0 in L, denoted by 0^*_L, is defined as follows. An element $x \in L$ is said to be in 0^*_L if there exists an element $c \in R^\circ$ such that $0 = x \otimes c \in L \bigotimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R. Given $K \subseteq L$, the tight closure of K in L, denoted by K^*_L, is then defined as the pre-image of $0^*_{L/K}$ under the natural map $L \longrightarrow L/K$.

If I is an ideal of R, then I^*_R is usually denoted by I^*. It is easy to see that an element $x \in R$ is in I^* if and only if there exists an element $c \in R^\circ$ such that $cx^{p^e} \in I^{[p^e]}$ for all $e \gg 0$.

An open question in the tight closure theory is that of whether tight closure commutes with localizations. Given R-modules $K \subseteq L$ and a multiplicatively closed set $U \subset R$, does $(U^{-1}K)^*_{U^{-1}L} = U^{-1}(K^*_L)$ always hold? It suffices to prove the case $K = 0$. We also mention that it is straightforward to show that $(U^{-1}K)^*_{U^{-1}L} \supseteq U^{-1}(K^*_L)$.

THEOREM 1.8 [**15**]. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic p and let $M \neq 0$ be a finitely generated R module with $\dim M = d$. Then the following hold.*

(i) *The limit (with $k = R/\mathfrak{m} = k(\mathfrak{m})$ and $a = [k : k^p]$)*

$$\lim_{e\to\infty} \frac{\lambda_R(M/I^{[p^e]}M)}{p^{de}} \qquad \left(= \lim_{e\to\infty} \frac{\lambda_R(R/I \bigotimes_R {}^eM)}{(ap^d)^e} \text{ if } a = [k : k^p] < \infty\right)$$

exists and is positive for every $\mathfrak{m}$-primary ideal I of R. The limit is called the Hilbert–Kunz multiplicity of M with respect to I.

(ii) *Hilbert–Kunz multiplicity is additive with respect to short exact sequence. Therefore we have the associativity formula.*

The existence of the Hilbert–Kunz multiplicity of M is generalized in [**17**].

Theorem 1.9 (Seibert's results [**17**, p. 278]). *Let $(R,\mathfrak{m})$ be an F-finite Noetherian local ring of characteristic p, let $k = R/\mathfrak{m}$ and let $a = [k : k^p]$. Suppose that j is an integer, that $\mathcal{C}$ is a family of finite R-modules with dimension at most j, and that g is a function from $\mathcal{C}$ to $\mathbb{Z}$, such that, for any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, the following hold.*

(a) *$M \in \mathcal{C}$ if and only if $M' \in \mathcal{C}$ and $M'' \in \mathcal{C}$.*
(b) *$g(M) \leqslant g(M') + g(M'')$, with equality if the sequence splits.*

Then we have the following conclusions.

(i) *If $M \in \mathcal{C}$, then ${}^eM \in \mathcal{C}$ for all $e \in \mathbb{N}$.*
(ii) *For each $M \in \mathcal{C}$, there is a real number $c(M)$ such that*

$$a^{-e}g({}^eM) = c(M)p^{je} + O(p^{(j-1)e}) \quad \text{for all } n \in \mathbb{N}.$$

Furthermore, $c(M)$ is an additive function of M on exact sequences.

(iii) *If g itself is additive on exact sequences, then, for any $M \in \mathcal{C}$, the function $a^{-e}g({}^eM)$ is a polynomial in p^e of the form*

$$a^{-e}g({}^eM) = b_0 + b_1p^e + b_2p^{2e} + \ldots + b_jp^{je},$$

with $b_k \in \mathbb{Q}$, for $k = 0, 1, 2, \ldots, j$.

Some examples of possible functions $g : \mathcal{C} \longrightarrow Z$ may be defined by $g(M) := \lambda_S(\mathrm{Tor}_i^S(L,M))$, $\lambda_S(\mathrm{Ext}_S^i(L,M))$ or $\lambda_S(\mathrm{Ext}_S^i(M,L))$ for any $i \geqslant 0$, any Noetherian local ring S of characteristic p such that $R \cong S/I$ for some ideal I of S, and any S-module L such that $\lambda_S(L) < \infty$.

Notation 1.10. Let $(R,\mathfrak{m})$ be an (F-finite) Noetherian local ring of prime characteristic p, and let L and M be finitely generated R-modules with $\lambda_R(L) < \infty$ and $\dim(M) = d$.

(i) We denote $e_{HK}(L,M) := \lim_{e\to\infty}(\lambda_R(L \bigotimes_R {}^eM)/(ap^d)^e)$, where $a = [k : k^p]$ with $k = R/\mathfrak{m}$.

(ii) In the case $L = R/I$ with I an $\mathfrak{m}$-primary ideal, we usually write $e_{HK}(L,M)$ as $e_{HK}(I,M)$, which is exactly the Hilbert–Kunz multiplicity of M with respect to I in Theorem 1.8.

(iii) Actually, the F-finite assumption can be avoided simply by considering the bimodule structure of ${}^eM$.

Theorem 1.11 [**7**, Theorem 8.17]. *Let $(R,\mathfrak{m})$ be a local Noetherian ring, let M and $K \subseteq L$ be R-modules such that* $\dim(M) = \dim(R)$ *and* $\lambda(L) < \infty$, *and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals of R.*

(i) *If $K \subseteq 0_L^*$, then $e_{HK}(L,M) = e_{HK}(L/K,M)$. In particular, if $J \subseteq I^*$, then $e_{HK}(I,M) = e_{HK}(J,M)$.*

(ii) *Conversely, if R is an analytically unramified, quasi-unmixed ring with a completely stable test element (for example $(R,\mathfrak{m})$ is a complete domain), then $e_{HK}(L,R) = e_{HK}(L/K,R)$ implies that $K \subseteq 0_L^*$. In particular, $e_{HK}(I,R) = e_{HK}(J,R)$ implies that $J \subseteq I^*$.*

In [**7**, Theorem 8.17], more general results are proved.

2. F-contributors and tight closures

LEMMA 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of prime characteristic p, and let $M \neq 0$ be a finitely generated R-module that has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$. Set $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$ and $d = \dim(M)$. Then the sequence $\{\#({}^eM, M_i)/(ap^d)^e\}_{e=0}^{\infty}$ is bounded for every $i = 1, 2, \ldots, s$ such that $M_i \neq 0$ and at least one of the M_i is a non-zero F-contributor.*

Proof. Without loss of generality, assume that $M_i \neq 0$ for all $i = 1, 2, \ldots, s$. Then, by [**15**],

$$\lim_{e\to\infty} \frac{\lambda_R\big(M/\mathfrak{m}^{[q]}M\big)}{p^{de}} = \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(M_i/\mathfrak{m}M_i)$$

exists and is equal to $e_{HK}(\mathfrak{m}, M) > 0$. The existence of the limit and the fact that $\lambda_R(M_i/\mathfrak{m}M_i) > 0$ for all $i = 1, 2, \ldots, s$ prove the boundedness, while the fact that $e_{HK}(\mathfrak{m}, M) > 0$ proves the existence of at least one F-contributor. $\square$

LEMMA 2.2. *Let (R, m) be local, and let $M \neq 0$ be a finitely generated R-module that has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$. Set $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$ and $d = \dim(M)$. For any $i_0 = 1, 2, \ldots, s$, if $M_{i_0} \neq 0$ and $\liminf_{e\to\infty}(\#({}^eM, M_{i_0})/(ap^c)^e) > 0$, then $\operatorname{depth} M_{i_0} \geqslant c$. In particular, every non-zero F-contributor of M is Cohen–Macaulay of dimension $= \dim(M)$.*

Proof. Without loss of generality, we assume that M is a faithful R-module. Let $\underline{x} := x_1, x_2, \ldots, x_d$ be a system of parameters of R. Then $\underline{x}^q := x_1^q, x_2^q, \ldots, x_d^q$ is also a system of parameters of R for every $q = p^e$. Let $\mathrm{H}^j_R(\underline{x}^q, M)$ be the jth Koszul cohomology. Then we have $\lim_{e\to\infty}(\lambda_R(\mathrm{H}^j_R(\underline{x}^{p^e}, M))/p^{ce}) = 0$ for all $j = 0, 1, \ldots, c-1$ by a result which is implicit in [**16**] and explicitly stated in [**8**, Theorem 6.2]. On the other hand, we have

$$\begin{aligned}\lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^j_R(\underline{x}^{p^e}, M))}{p^{ce}} &= \lim_{e\to\infty} \frac{\lambda_R(\mathrm{H}^j_R(\underline{x}, {}^eM))}{(ap^c)^e} \\ &= \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^c)^e} \lambda_R(\mathrm{H}^j_R(\underline{x}, M_i)).\end{aligned}$$

Therefore $\lambda_R(\mathrm{H}^j_R(\underline{x}, M_{i_0})) = 0$ for all $j = 0, 1, \ldots, c-1$ by our assumption on M_{i_0}. Hence $\operatorname{depth} M_{i_0} \geqslant c$. In particular, every non-zero F-contributor of M is Cohen–Macaulay. $\square$

Next we study the localization problem under the assumption of finite F-representation type. One way to attack the question of whether tight closure commutes with localizations is to study, for a given $I \subset R$, the finiteness of $\bigcup_{e\geqslant 0} \mathrm{Ass}(R/I^{[p^e]})$ and the annihilators of $\mathrm{H}^0_{\mathfrak{m}}(R/I^{[p^e]})$ (see [**7**, **12**] and also [**10**, **22**] for results along this line), while another is to study the 'linear growth' property of the primary decompositions of $I^{[p^e]}$ in R (see [**20**] or [**19**]). Our next theorem shows that rings with finite F-representation type satisfy nice properties that one would want and consequently tight closure commutes with localizations whenever

R has finite F-representation type. The proof of Theorem 2.3(ii) below is similar to that of [**19**, Theorem 7.6(ii)] and that of [**2**, Theorem 3.7].

THEOREM 2.3. *Let R and S be Noetherian rings of prime characteristic p, and let M be a finitely generated R-module with finite F-representation type by $\{M_1, M_2, \dots, M_s\}$.*

(i) *For any finitely generated R-module L, the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass}(L \bigotimes_R {}^eM)$ is finite and there exists an integer $k \in \mathbb{N}$ such that* (a) *and* (b) *are satisfied.*

(a) *For every $e \in \mathbb{N}$, there exists a primary decomposition*

$$0 = Q_{e1} \cap Q_{e2} \cap \dots \cap Q_{es_e} \quad \text{of } 0 \text{ in } L \bigotimes_R {}^eM,$$

where $\operatorname{Ass}(L \otimes {}^eM) = \{P_{ej} \mid 1 \leqslant j \leqslant s_e\}$ and Q_{ej} are P_{ej}-primary components of $0 \subset L \bigotimes_R {}^eM$ satisfying $P_{ej}^k(L \bigotimes_R {}^eM) \subseteq Q_{ej}$ for all $1 \leqslant j \leqslant s_e$.

(b) *For all $J \subset R$ and for all $q = p^e$, we have*

$$J^k(0 :_{L\otimes_R {}^eM} J^\infty) = 0, \quad \text{that is, } J^k \operatorname{H}^0_J(L \bigotimes_R {}^eM) = 0.$$

(ii) *Consequently, tight closure commutes with localization if $\operatorname{Ann}_R(M) \subseteq \sqrt{(0)}$, the nilradical of R (for example M is faithful over R or $M = R$).*

(iii) *More generally, tight closure commutes with localizations over S provided that $S/\sqrt{(0)} \cong R/\sqrt{\operatorname{Ann}_R(M)}$ as rings.*

Proof. (i) For each $i = 1, 2, \dots, s$, write down a primary decomposition of 0 in $L \bigotimes_R M_i$ (ignore the M_i such that $L \bigotimes_R M_i = 0$) as follows.

$$0 = Q'_{i1} \cap Q'_{i2} \cap \dots \cap Q'_{it_i},$$

where Q'_{ij} is a P'_{ij}-primary component of $0 \subset L \bigotimes_R M_i$. Naturally we get an induced primary decomposition of $0 \subset L \bigotimes_R {}^eM$ for every e since ${}^eM$ is a direct sum of the M_i. Choose $k \in \mathbb{N}$ so that ${P'_{ij}}^k(L \bigotimes_R M_i) \subseteq Q'_{ij}$ for all $i = 1, 2, \dots, s$ and all $j = 1, 2, \dots, t_i$. Then (a) is evidently true. We also have $J^k(0 :_{L\otimes_R M_i} J^\infty) = 0$ for all i and all $J \subset R$. Thus $J^k(0 :_{L\otimes_R {}^eM} J^\infty) = 0$ for all $J \subset R$, $e \in \mathbb{N}$.

(ii) Let L be any finitely generated R-module, and let U be any multiplicatively closed subset of R. We need to show that $0^*_{U^{-1}L} \subseteq U^{-1}(0^*_L)$. We know that $\bigcup_{e\in\mathbb{N}} \operatorname{Ass}(L \bigotimes_R {}^eM)$ is finite by part (i), say $\bigcup_{e\in\mathbb{N}} \operatorname{Ass}(L \bigotimes_R {}^eM) = \{P_1, P_2, \dots, P_t\}$. Without loss of generality, we assume that, for some $1 \leqslant r \leqslant t$, $P_i \cap U = \varnothing$ and $P_j \cap U \neq \varnothing$ for all $1 \leqslant i \leqslant r$, $r+1 \leqslant j \leqslant t$. Then there exists $u \in U$ such that $u \in \bigcap_{j=r+1}^t P_j$. To prove $0^*_{U^{-1}L} \subseteq U^{-1}(0^*_L)$, it suffices to show that if $\frac{x}{1} \in 0^*_{U^{-1}L}$ with $x \in L$, then $x \in U^{-1}(0^*_L)$. The assumption that $\frac{x}{1} \in 0^*_{U^{-1}L}$ implies that there exist $c \in R^\circ$ and $u_e \in U$ such that $0 = u_e x \otimes c \in L \bigotimes_R {}^eR$ for all $e \gg 0$ (see [**2**, Lemma 3.3]). This implies that $0 = u_e x \otimes cm \in L \bigotimes_R {}^eM$ for all $m \in M$ and all $e \gg 0$ (since the R-linear map $R \longrightarrow M$ defined by $1 \longmapsto m \in M$ induces an R-linear map ${}^eR \longrightarrow {}^eM$). Since part (i)(a) holds for M, we adopt the notation there. In particular, for every $m \in M$ and $e \gg 0$,

$$u_e(x \otimes cm) = u_e x \otimes cm = 0 \in Q_{e1} \cap Q_{e2} \cap \dots \cap Q_{es_e} \subseteq L \bigotimes_R {}^eM,$$

as in (i)(a). Then, for each $e \gg 0$ and $1 \leqslant j \leqslant s_e$, we have $x \otimes cm \in Q_{ej}$ if $P_{ej} \cap U = \varnothing$ while $u^k x \otimes cm \in P_{ej}^k L \bigotimes_R {}^eM \subseteq Q_{ej}$ if $P_{ej} \cap U \neq \varnothing$. All in all, we have

$$u^k x \otimes cm \in \bigcap_{j=1}^{s_e} Q_{ej} = 0 \subseteq L \bigotimes_R {}^e M \quad \text{for all } e \gg 0 \text{ and all } m \in M.$$

Now, the assumption that $\mathrm{Ann}_R(M) \subseteq \sqrt{(0)}$ implies that there is an R-linear map $h : M \longrightarrow R/\sqrt{(0)}$ such that

$$h(m_0) \in (R/\sqrt{(0)})^\circ$$

for some $m_0 \in M$. Applying h, we get

$$0 = u^k x \otimes ch(m_0) \in L \bigotimes_R {}^e(R/\sqrt{(0)})$$

for all $e \gg 0$. Notice that $h(m)$ can be lifted back to some $d \in R^\circ$ under the natural ring homomorphism $R \longrightarrow R/\sqrt{(0)}$. Also observe that, for any given $q_0 = p^{e_0}$, the Frobenius mapping $r \longmapsto r^{p^{e_0}}$ defines an R-linear map $F^{e_0} : {}^e R \longrightarrow {}^{e+e_0} R$ for all e. Choose q_0 large enough so that

$$\sqrt{(0)}^{[q_0]} = 0.$$

Then F^{e_0} factors through ${}^e(R/\sqrt{(0)})$, which means that there exists an R-linear map

$$G^{e_0} : {}^e(R/\sqrt{(0)}) \longrightarrow {}^{e+e_0} R$$

such that $G^{e_0}(h(m_0)) = d^{q_0} \in {}^{e+e_0}R$ for all e. Now apply G^{e_0} to the equation

$$0 = u^k x \otimes ch(m_0) \in L \bigotimes_R {}^e(R/\sqrt{(0)})$$

to get $0 = u^k x \otimes (cd)^{q_0} \in L \bigotimes_R {}^{e+e_0}R$ for all $e \gg 0$, which implies that $u^k x \in 0^*_L$ or, equivalently, $x \in U^{-1}(0^*_L)$.

(iii) This follows from part (ii) as, for a general ring T of characteristic p, tight closure commutes with localization over T if and only if it is true over $T/\sqrt{(0)}$. $\square$

Next we see the usefulness of F-contributors in the tight closure theory.

PROPOSITION 2.4. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of characteristic p, and let M be a finitely generated R-module with $\dim(M) = \dim(R)$. Assume that M has finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$ and that $\{M_1, M_2, \ldots, M_r\}$ is the set of all F-contributors for some $r \leqslant s$. Set $N = \bigoplus_{i=1}^r M_i$.*

(i) *For any finitely generated R-modules $K \subseteq L$, K^*_L is contained in the kernel of $L \longrightarrow L/K \longrightarrow \mathrm{Hom}_R(N, L/K \bigotimes_R N)$, the composition of the natural and the evaluation R-homomorphisms.*

(ii) *If, furthermore, R is analytically unramified and quasi-unmixed with a completely stable test element (for example $(R, \mathfrak{m})$ is a complete domain) and M is faithful over R, then $K^*_L = \ker(L \longrightarrow L/K \longrightarrow \mathrm{Hom}_R(N, L/K \bigotimes_R N))$.*

Proof. Without loss of generality, we assume that $K = 0$. Since $0^*_L \subseteq \bigcap_{n>0} (\mathfrak{m}^n L)^*_L$ and equality holds if there is a test element (by [**7**, Proposition 8.13(b)]) and

$$\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N)) = \bigcap_{n>0} \ker(L/\mathfrak{m}^n L \longrightarrow \mathrm{Hom}_R(N, L/m^n L \otimes N)),$$

we assume that $\lambda_R(L) < \infty$, still, without loss of generality. Let D be an arbitrary R-submodule of L and denote $L' := L/D$. Set $a = [k : k^p]$, and $d = \dim(R) = \dim(M)$. Then we have

$$\begin{aligned}
e_{HK}(L, M) - e_{HK}(L', M) &= \lim_{e\to\infty} \frac{\lambda_R(L \bigotimes_R {}^eM)}{(ap^d)^e} - \lim_{e\to\infty} \frac{\lambda_R(L' \bigotimes_R {}^eM)}{(ap^d)^e} \\
&= \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(L \bigotimes_R M_i) \\
&\quad - \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^d)^e} \lambda_R(L' \bigotimes_R M_i) \\
&= \lim_{e\to\infty} \sum_{i=1}^{s} \frac{\#({}^eM, M_i)}{(ap^d)^e} \big(\lambda_R(L \bigotimes_R M_i) - \lambda_R(L' \bigotimes_R M_i)\big) \\
&= \lim_{e\to\infty} \sum_{i=1}^{r} \frac{\#({}^eM, M_i)}{(ap^d)^e} \big(\lambda_R(L \bigotimes_R M_i) - \lambda_R(L' \bigotimes_R M_i)\big),
\end{aligned}$$

which implies that $e_{HK}(L, M) = e_{HK}(L', M) \iff \lambda_R(L \bigotimes_R M_i) = \lambda_R(L' \bigotimes_R M_i)$ for all $i = 1, 2, \ldots, r \iff \lambda_R(L \bigotimes_R N) = \lambda_R(L' \bigotimes_R N) \iff D \subseteq \{x \in L \,|\, 0 = x \otimes y \in L \bigotimes_R N,\ \forall\, y \in N\} = \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$.

(i) Since $e_{HK}(L, M) = e_{HK}(L/0^*_L, M)$ by Theorem 1.11, we have, by the above argument, $0^*_L \subseteq \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$.

(ii) Let $D' = \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$ and $L'' = L/D'$. Then, by the above argument again, $e_{HK}(L, M) = e_{HK}(L'', M)$. This implies that $e_{HK}(L, R/P) = e_{HK}(L'', R/P)$ for every $P \in \min(M) = \min(R)$ by the associativity formula, the fact that R is equidimensional, and the fact that, a priori, $e_{HK}(L, R/P) \geqslant e_{HK}(L'', R/P)$ for each minimal prime P. Hence $e_{HK}(L, R) = e_{HK}(L'', R)$, by the associativity formula again, which implies that $D' \subseteq 0^*_L$ by Theorem 1.11. Combined with the result in (i), this gives $0^*_L = \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$. □

The next theorem is a global version of Proposition 2.4. Notice that Theorem 2.5(iii) is just a special case of Theorem 2.3(ii), but is proved differently. Recall that persistence of tight closure holds if R is essentially of finite type over an excellent local ring or if $R/\sqrt{(0)}$ is F-finite by [**9**, Theorem 6.24].

THEOREM 2.5. *Let R be a Noetherian ring of characteristic p, and let M be a finitely generated R-module with finite F-representation type by $\{M_1, M_2, \ldots, M_s\}$. Consider the folowing conditions.*

(1) *$[k(\mathfrak{m}) : k(\mathfrak{m})^p]p^{\dim R_\mathfrak{m}}$ is constant for all maximal ideals $\mathfrak{m}$ of R. Under this condition, we set $N = \bigoplus_{i=1}^r M_i$ to be a direct sum of all the F-contributors* (*see Remark* 1.4(iii)).

(2) *Either* (a) *persistence of tight closure holds, or* (b) $\dim(M_\mathfrak{m}) = \dim(R_\mathfrak{m})$ *for all maximal ideals $\mathfrak{m}$ of R.*

(3) *M is faithful, R has a test element, and, for every maximal ideal $\mathfrak{m}$ of R, $R_\mathfrak{m}$ is analytically unramified and quasi-unmixed, and has a completely stable test element.*

Then the following hold.

(i) *Assume* (1) *and* (2). *Then* $K_L^* \subseteq \ker(L \to L/K \to \operatorname{Hom}_R(N, L/K \bigotimes_R N))$ *for any finitely generated R-modules* $K \subseteq L$.

(ii) *Assume* (1) *and* (3). *Then* $K_L^* = \ker(L \longrightarrow L/K \longrightarrow \operatorname{Hom}_R(N, L/K \bigotimes_R N))$ *for any finitely generated R-modules* $K \subseteq L$.

(iii) *Assume* (3). *Then tight closure commutes with localization over R, that is,* $(U^{-1}K)^*_{U^{-1}L} = U^{-1}(K_L^*)$ *for any finitely generated R-modules $K \subseteq L$ and for any multiplicatively closed set* $U \subset R$.

Proof. Without loss of generality, we assume that $K = 0$. Notice that condition (3) implies condition (2)(b).

(i) If condition (2)(a) is satisfied, then it is enough to prove the desired result over $R/\operatorname{Ann}(M)$ via the natural map $R \longrightarrow R/\operatorname{Ann}(M)$. However, notice that M is faithful over $R/\operatorname{Ann}(M)$; hence (2)(b) is satisfied. Therefore we assume (2)(b) without loss of generality. For every maximal ideal $\mathfrak{m}$ of R, we have $(0_L^*)_\mathfrak{m} \subseteq 0^*_{L_\mathfrak{m}}$. We then apply Proposition 2.4(i) to the local ring $R_\mathfrak{m}$ and get $0^*_{L_\mathfrak{m}} \subseteq (\ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N)))_\mathfrak{m}$. Hence $0_L^* \subseteq \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$.

(ii) We have $0_L^* = \bigcap_\mathfrak{m} \bigcap_{n \geqslant 0} (\mathfrak{m}^n L)^*_L$ (by [**7**, Proposition 8.13(b)]), where $\mathfrak{m}$ runs over all maximal ideals of R. For each maximal ideal $\mathfrak{m}$ of R, let $\phi_\mathfrak{m}$ denote the natural ring homomorphism $R \longrightarrow R_\mathfrak{m}$. By [**7**, Proposition 8.9], we have

$$\begin{aligned}
(\mathfrak{m}^n L)^*_L &= \phi_\mathfrak{m}^{-1}\left(((\mathfrak{m}^n L)_\mathfrak{m})^*_{L_\mathfrak{m}}\right) \\
&= \phi_\mathfrak{m}^{-1}\left(\ker\left(L_\mathfrak{m} \longrightarrow \left(\frac{L}{\mathfrak{m}^n L}\right)_\mathfrak{m} \longrightarrow \operatorname{Hom}\left(N_\mathfrak{m}, \left(\frac{L}{\mathfrak{m}^n L}\right)_\mathfrak{m} \otimes N_\mathfrak{m}\right)\right)\right) \\
&= \phi_\mathfrak{m}^{-1}\left(\left(\ker\left(L \longrightarrow \frac{L}{\mathfrak{m}^n L} \longrightarrow \operatorname{Hom}\left(N, \frac{L}{\mathfrak{m}^n L} \otimes N\right)\right)\right)_\mathfrak{m}\right) \\
&= \ker\left(L \longrightarrow \frac{L}{\mathfrak{m}^n L} \longrightarrow \operatorname{Hom}\left(N, \frac{L}{\mathfrak{m}^n L} \otimes N\right)\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0_L^* &= \bigcap_\mathfrak{m} \bigcap_{n \geqslant 0} (\mathfrak{m}^n L)^*_L = \bigcap_\mathfrak{m} \bigcap_{n \geqslant 0} \ker\left(L \longrightarrow \frac{L}{\mathfrak{m}^n L} \longrightarrow \operatorname{Hom}\left(N, \frac{L}{\mathfrak{m}^n L} \otimes N\right)\right) \\
&= \ker(L \longrightarrow \operatorname{Hom}(N, L \otimes N)).
\end{aligned}$$

(iii) If $\operatorname{Spec}(R)$ is disconnected, that is $R = R_1 \times R_2$, then both R_1 and R_2 satisfy the conditions of the theorem. Also, to show that tight closure commutes with localization for R, it is enough to show the same results for both R_1 and R_2.

Hence we may assume that $\operatorname{Spec}(R)$ is connected so that $[k(P) : k(P)^p]p^{\dim R_P} = [k(Q) : k(Q)^p]p^{\dim R_Q}$ for any two prime ideals P and Q of R. Therefore condition (1) is satisfied by any localization of R and hence the result in part (i) applies.

To prove that tight closure commutes with localization, it is enough to show that, for any multiplicatively closed set $U \subset R$, $U^{-1}(0_L^*) = 0^*_{U^{-1}L}$. Applying the result in part (i) to $U^{-1}R$, we have $0^*_{U^{-1}L} \subseteq \ker(U^{-1}L \to \operatorname{Hom}(U^{-1}N, U^{-1}L \otimes U^{-1}N)) = U^{-1}(\ker(L \longrightarrow \operatorname{Hom}(N, L \otimes N)))$. However, we have $0_L^* = \ker(L \to \operatorname{Hom}(N, L \otimes N))$ by (ii) above. Hence $0^*_{U^{-1}L} \subseteq U^{-1}(0_L^*)$. We conclude that $U^{-1}(0_L^*) = 0^*_{U^{-1}L}$ as $U^{-1}(0_L^*) \subseteq 0^*_{U^{-1}L}$ is automatic. ☐

REMARK 2.6. We might be interested in the ideals cases of Theorem 2.3 and Theorem 2.5. It is straightforward to obtain the results by letting $L = R/I$.

(i) Theorem 2.3(i) states that the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass}(R/I \bigotimes_R {}^eM) = \bigcup_{e\in\mathbb{N}} \operatorname{Ass}(M/I^{[q]}M)$ is finite and $J^k \cdot \mathrm{H}^0_J({}^eM/I\cdot {}^eM) = 0$, for all $J \subset R$ and for all $q = p^e$, which implies that

$$J^{(k+\mu(J))q}(I^{[q]}M :_M J^\infty) \subseteq I^{[q]}M, \text{ that is, } J^{(k+\mu(J))q}\,\mathrm{H}^0_J\left(\frac{M}{I^{[q]}M}\right) = 0,$$

where $\mu(J)$ is the least number of generators of the ideal J.

(ii) Theorem 2.5(ii) simply states that $I^* = (IN :_R N) = \operatorname{Ann}_R(N/IN)$.

Remark 2.7. Let R be a Noetherian ring of characteristic p that has finite F-representation type by $\{M_1, M_2, \dots, M_s\}$. Say that $\{M_1, M_2, \dots, M_t\}$ is the set of all modules that appear in the decompositions of ${}^eR$ non-trivially for infinitely many e. Let $N' = \bigoplus_{i=1}^t M_i$. Then the Frobenius closure of 0 in an R-module L, denoted by 0^F_L, is determined by $0^F_L = \ker(L \longrightarrow \operatorname{Hom}_R(N', L \bigotimes_R N'))$. In particular, the Frobenius closure of an ideal I in R, denoted by I^F, is characterized by $I^F = (IN' :_R N')$. The proof is similar to that of Proposition 2.4 but more direct.

Discussion 2.8. Let R be as in Theorem 2.5(ii) and adopt the notation there. We furthermore assume that $\#({}^{e_0}R, M_i) > 0$ for some e_0 and for all $i = 1, 2, \dots, r$. Let $q_0 = p^{e_0}$. Then $N = \bigoplus_{i=1}^r M_i$ may be realized as a direct summand of R^{1/q_0} since ${}^eR \cong R^{1/p^e}$ as R-modules for every e. Say that $N = \bigoplus_{i=1}^r M_i$ is generated by $c_1^{1/q_0}, c_2^{1/q_0}, \dots, c_t^{1/q_0}$ as an R-submodule of R^{1/q_0}. Let $\tau_0 = (c_1, c_2, \dots, c_t)$ be the ideal of R generated by $c_1, c_2, \dots, c_t$. Then for any ideal I of R and an element $x \in R$, we have $x \in I^*$ if and only if $\tau_0 x^{q_0} \subseteq I^{[q_0]}$. Indeed, $x \in I^*$ if and only if $xN \subseteq IN$, that is, $x(c_1^{1/q_0}, c_2^{1/q_0}, \dots, c_t^{1/q_0}) \subseteq I(c_1^{1/q_0}, c_2^{1/q_0}, \dots, c_t^{1/q_0})$ if and only if $x(c_1^{1/q_0}, c_2^{1/q_0}, \dots, c_t^{1/q_0}) \subseteq IR^{1/q_0}$ if and only if $\tau_0 x^{q_0} \subseteq I^{[q_0]}$. Here the second 'if and only if' follows from the fact that N is a direct summand of R^{1/q_0}, while the third 'if and only if' follows by the taking of the q_0th Frobenius power or the q_0th root. Once again we deduce that tight closure commutes with localization in this case.

Remark 2.9. Of course we can talk about F-contributors for any F-finite R-module M without the assumption of finite F-representation type. If $a(P)p^{d(P)}$ is constant over $\operatorname{Spec}(R)$ and N is a non-zero F-contributor of M, then we always have the following.

(i) Suppose that $\dim M = \dim R$. Then for any finitely generated R-module L, we have $0^*_L \subseteq \ker(L \longrightarrow \operatorname{Hom}_R(N, L \bigotimes_R N))$.

(ii) N is necessarily a Cohen–Macaulay module if R is local. More generally, results similar to Lemma 2.2 can be proved.

3. *The sequence* $\{\#({}^eM, M_i)/(ap^d)^e\}_{e=0}^\infty$

In this section we study the growth of $\#({}^eM, M_i)$ as $e \to \infty$. We restrict ourselves to the case where $(R, \mathfrak{m})$ is local and $M \neq 0$ is a finitely generated R-module with finite F-representation type by a finite F-representation type system $M_1, M_2, \dots, M_s$. Without loss of generality, we may simply assume that $M \cong XY$ and ${}^eY \cong A^e Y$ for all $e \geqslant 0$, where $X = (n_1, n_2, \dots, n_s)$ is a $1\times s$ matrix, $A := (a_{ij})$ is an $s \times s$ matrix with non-negative integer entries, and $Y = (M_1, M_2, \dots, M_s)^{\mathrm{T}}$.

Consequently ${}^eM \cong XA^eY$ for all $e \geqslant 0$. For each $i = 1, 2, \ldots, s$, let $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$. Then we can easily see that $\#({}^eM, M_i) = XA^eE_i$. Then $\#({}^eM, M_i)/(ap^d)^e = XB^eE_i$, where $B = (1/ap^d)A$. We use E to denote the identity matrix of various sizes and use $Z = (z_1, z_2, \ldots, z_s)^T \in \mathbb{C}^s$ to denote an arbitrarily chosen and then fixed $s \times 1$ matrix with entries in $\mathbb{C}$. Similarly $X = (n_1, n_2, \ldots, n_s)$ is used to denote an arbitrarily chosen and then fixed vector. However, we may insist that the entries of X be non-negative integers in order to maintain the realization that $XB^eE_i = \#({}^eM, M_i)/(ap^d)^e$, where $M = \bigoplus_{i=1}^s n_iM_i$.

We also assume that $\{M_1, M_2, \ldots, M_r\}$ is the set of all F-contributors of $\bigoplus_{i=1}^s M_i$ so that, for any R-module $M \cong XY$, the set of F-contributors of M is contained in $\{M_1, M_2, \ldots, M_r\}$. We call $M_1, M_2, \ldots, M_r$ the general F-contributors of the finite F-representation type system $\{M_1, M_2, \ldots, M_s\}$. Also we set $Y' = (M_1, M_2, \ldots, M_r, 0, \ldots, 0)^T$, $a = [k(\mathfrak{m}) : k(\mathfrak{m})^p]$, and $d = \dim M$.

We will keep this notation throughout this section.

Therefore Question 1.5 can be restated as follows. Does $\lim_{e\to\infty} XB^eE_i$ exist for every $i = 1, 2, \ldots, s$? Or equivalently, does $\lim_{e\to\infty} XB^e$ exist? Or still equivalently, does $\lim_{e\to\infty} XB^eZ$ exist for every $Z \in \mathbb{C}^s$?

A slightly stronger question would be the following.

Question 3.1. Does the limit $\lim_{e\to\infty} XB^eE_i$ exist for every $X \in \mathbb{N}^s$ and every $i = 1, 2, \ldots, s$? Or equivalently, does $\lim_{e\to\infty} B^e$ exist? Or still equivalently, does $\lim_{e\to\infty} XB^eZ$ exist for every $X \in \mathbb{N}^s$ and every $Z \in \mathbb{C}^s$?

Example 3.2. Actually we should not expect a positive answer to the above question in general. There might be relations among $M_1, M_2, \ldots, M_s$ in terms of direct sums. Indeed, let $R = k$ be a field of characteristic $p = 2$ such that $[k : k^2] = 2$ and let $M = M_1 = M_2 = k$. Then M has finite F-representation type by a finite F-representation type system M_1, M_2 and we may pre-fix the direct sum decompositions of ${}^eM$ so that $X = (1, 0)$ and $A = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. However, it is easy to see that $\lim_{e\to\infty}(\#({}^eM, M_i)/2^e)$ do not exist for $i = 1, 2$. Or even more simply, let $R = k = M = M_1 = M_2$ where k is a perfect field and $X = (1, 0)$ so that $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By a result of Smith and van den Bergh, quoted as Theorem 1.6, the limit always exists and is always positive for $M = R$, where R is a strongly F-regular ring with finite F-representation type by finitely many indecomposable modules, which satisfies the Krull–Schmidt condition. Notice that in this case R does have finite F-representation type by a finite F-representation type system.

In this section, we first study the properties of the matrix B in the general situations of finite F-representation type by a finite F-representation type system. Then, in Theorem 3.11, we give a positive answer to Question 3.1 under the assumption that R satisfies the Krull–Schmidt condition and that $M_1, M_2, \ldots, M_s$ are all indecomposable, are non-zero and belong to different isomorphism classes.

Lemma 3.3. *All of the eigenvalues of B have absolute values of at most* 1.

Proof. This follows from Lemma 2.1. Suppose that, on the contrary, there exist a $\lambda \in \mathbb{C}$ and a complex vector $V = (v_1, v_2, \ldots, v_s)^{\mathrm{T}} \neq 0$ such that $|\lambda| > 1$ and $BV = \lambda V$. Then $B^eV = \lambda^eV$. By choosing a proper $X \in \mathbb{N}^s$ such that $XV \neq 0$, we have $|XB^eV| = |\lambda^eXV| = |\lambda|^e|XV| \to \infty$ as $e \to \infty$. However, by Lemma 2.1

applied to $M = X(M_1, M_2, \dots, M_s)^{\mathrm{T}}$, $|XB^eV| \leqslant \sum_{i=1}^{s} |v_i|(\#({}^eM, M_i)/(ap^d)^e)$ defines a bounded sequence, a contradiction. $\square$

Let $\lambda_1, \lambda_2, \dots, \lambda_l$ be the distinct eigenvalues of B such that $|\lambda_i| = 1$ for $i = 1, 2, \dots, k$ and $|\lambda_i| < 1$ for $i = k+1, k+2, \dots, l$. We can think of B as a $\mathbb{C}$-linear transformation of $\mathbb{C}^s$. Now, by the primary decomposition theorem (or Jordan canonical form theorem), we can write $\mathbb{C}^s$ as $\mathbb{C}^s = \bigoplus_{j=1}^{l} \mathcal{Z}_j$, where $\mathcal{Z}_j = \ker((\lambda_j E - B)^s) = \ker((\lambda_j E - B)^n)$ for sufficiently large n. Then every $Z \in \mathbb{C}^s$ can be written as $Z = \sum_{i=1}^{l} Z_i$, where $Z_i \in \mathcal{Z}_i$ for every $i = 1, 2, \dots, l$. In particular, $N_i^s Z_i = 0$ for every $i = 1, 2, \dots, l$, where B_i is the restriction of B to $\mathcal{Z}_i$ and $N_i := B_i - \lambda_i E$ for each $i = 1, 2, \dots, l$.

Then we have $XB^eZ = \sum_{i=1}^{l} XB^eZ_i$. For all $e \geqslant s$, we have $XB^eZ_i = X(\lambda_i E + N_i)^e Z_i = X(\sum_{j=0}^{s} \binom{e}{j} \lambda_i^{e-j} N_i^j) Z_i = \sum_{j=0}^{s} \binom{e}{j} \lambda_i^{e-j} X N_i^j Z_i$, which can be realized as $\lambda_i^e \sum_{j=1}^{s} c_{ij} \binom{e}{j} = \lambda_i^e P_i(e)$, where $c_{ij} = X((1/\lambda_i) N_i)^j Z_i$ and $P_i(e)$ is the value of the polynomial $P_i(W) = \sum_{j=1}^{s} c_{ij} \binom{W}{j} \in \mathbb{C}[W]$ at $W = e$ for each $1 \leqslant i \leqslant l$. (Here we assume that all the eigenvalues of B are non-zero. If 0 is an eigenvalue of B, we can treat the part corresponding to 0 separately to get a similar result.) Therefore we have $XB^eZ = \sum_{i=1}^{l} \lambda_i^e P_i(e)$.

Alternatively we can derive the above result in the following (essentially the same) way by means of matrices. By the primary decomposition theorem, there exists an invertible $s \times s$ matrix T with complex entries such that

$$T^{-1}BT = \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & B_l \end{pmatrix},$$

where, for each $i = 1, 2, \dots, l$, B_i is an $s_i \times s_i$ matrix such that $N'_i = B_i - \lambda_i E$ is nilpotent for each $i = 1, 2, \dots, l$. In particular, $(N'_i)^s = 0$.

Let $U = XT$ and $V = T^{-1}Z$. Corresponding to the partition of $T^{-1}BT$, we write $U = (U_1, U_2, \dots, U_l)$ and $V^T = (V_1^T, V_2^T, \dots, V_l^T)$ so that U_i and V_i^T are both $1 \times s_i$ complex matrices. Then we have $XB^eZ = \sum_{i=1}^{l} U_i B_i^e V_i$. For all $e \geqslant s$, we have $U_i B_i^e V_i = U_i(\lambda_i E + N'_i)^e V_i = U_i(\sum_{j=0}^{s} \binom{e}{j} \lambda_i^{e-j} (N'_i)^j) V_i = \sum_{j=0}^{s} \binom{e}{j} \lambda_i^{e-j} U_i (N'_i)^j V_i$, which can be realized as $\lambda_i^e \sum_{j=1}^{s} c_{ij} \binom{e}{j} = \lambda_i^e P_i(e)$, where $c_{ij} = U_i(\frac{1}{\lambda_i} N'_i)^j V_i$ and $P_i(e)$ is the value of the polynomial $P_i(W) = \sum_{j=1}^{s} c_{ij} \binom{W}{j} \in \mathbb{C}[W]$ at $W = e$ for each $1 \leqslant i \leqslant l$. (Here we assume that all the eigenvalues of B are non-zero. If 0 is an eigenvalue of B, then we can treat the part corresponding to 0 separately to get a similar result.) Therefore we have $XB^eZ = \sum_{i=1}^{l} \lambda_i^e P_i(e)$.

LEMMA 3.4. *Keep the notation as above. Then the following hold.*

(i) *The value* 1 *is an eigenvalue of* B.

(ii) $P_i(W) = c_{i0} = XZ_i$ *are constant polynomials for all* $i = 1, 2, \dots, k$.

(iii) *For some fixed* X *and* $Z = \sum_{i=1}^{l} Z_i$, *where* $Z_i \in \mathcal{Z}_i$ *for every* $i = 1, 2, \dots, l$, *we have* $\lim_{e\to\infty} XB^eZ$ *exists if and only if* $P_i(W) = c_{i0} = XZ_i = 0$ *for every* $i = 1, 2, \dots, k$ *such that* $\lambda_i \neq 1$.

The proof follows from a lemma in [**18**], either directly or indirectly. Also we need to use the fact that the set $\{\binom{W}{j} \mid j = 1, 2, \dots, s\}$, considered as a subset of the $\mathbb{C}$-vector space $\mathbb{C}[W]$, is linearly independent over $\mathbb{C}$. First we state the lemma.

LEMMA 3.5 [**18**, Lemma 2.3]. *We have $\gamma_1, \dots, \gamma_t \in \mathbb{C} \setminus \{0\}$ and $P_1(W), P_2(W), \dots, P_t(W) \in \mathbb{C}[W] \setminus \{0\}$ for some $t \in \mathbb{N}$. Assume that $\gamma_1, \gamma_2, \dots, \gamma_t$ are distinct. Set $f(e) := \sum_{i=1}^{t} \gamma_i^e P_i(e)$ for all $e \in \mathbb{N}$. Then we have the following.*

(i) *The following are equivalent.*
  (a) $\lim_{e\to\infty} f(e) = 0$.
  (b) $|\gamma_i| < 1$ *for all* $i = 1, 2, \dots, t$.

(ii) *For any $c \in \mathbb{C} \setminus \{0\}$, the following are equivalent.*
  (a) $\lim_{e\to\infty} f(e) = c$.
  (b) *There is an $i_0 \in \mathbb{N}$ with $1 \leqslant i_0 \leqslant t$ such that $\gamma_{i_0} = 1$, $P_{i_0} = c$ and $|\gamma_i| < 1$ for all $1 \leqslant i \leqslant t$ with $i \neq i_0$.*

Proof of Lemma 3.4. (i) This is basically proved in [**18**]. We include a proof for completeness.

Let $Z = (\lambda_R(M_1/\mathfrak{m}M_1), \lambda_R(M_2/\mathfrak{m}M_2), \dots, \lambda_R(M_s/\mathfrak{m}M_s))^{\mathrm{T}}$. Then

$$\lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e P_i(e) = \lim_{e\to\infty} \frac{\lambda_R({}^eM/\mathfrak{m}\cdot {}^eM)}{(ap^d)^e} = \lim_{e\to\infty} \frac{\lambda_R(M/\mathfrak{m}^{[p^e]}M)}{p^{de}} = e_{HK}(\mathfrak{m}, M)$$

and the fact that $e_{HK}(\mathfrak{m}, M) > 0$ implies that $\lambda_{i_0} = 1$ for some $1 \leqslant i_0 \leqslant l$ by Lemma 3.5(ii).

(ii) For each $i = 1, 2, \dots, l$, set $P_i'(W) = (P_i(W) - P_i(0)/W) \in \mathbb{C}[W]$. Since $\{XB^eZ = \sum_{i=1}^{l} \lambda_i^e P_i(e)\}_{e=0}^{\infty}$ is bounded, we have

$$0 = \lim_{e\to\infty} \frac{XB^eZ}{e} = \lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e \frac{P_i(e)}{e} = \lim_{e\to\infty} \sum_{i=1}^{l} \lambda_i^e P_i'(e),$$

which forces $P_i'(W) = 0$ for all $i = 1, 2, \dots, k$, which implies that $P_i(W) = c_{i0} = XZ_i$ are constant polynomials for all $i = 1, 2, \dots, k$.

(iii) This follows directly from part (ii) and Lemma 3.5(ii). □

LEMMA 3.6. *Keep the above notation. Then the following hold.*

(i) *$\mathcal{Z}_i = \ker(B - \lambda_i E) = \ker(N_i)$ is the eigenspace of λ_i (or, in matrix terms, $B_i = \lambda_i E$, that is, $N_i' = 0$) for all $i = 1, 2, \dots, k$.*

(ii) *Let $M = XY$ be a fixed R-module. Also we assume that $\lambda_k = 1$ without loss of generality. Then $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ exists for every $i = 1, 2, \dots, s$ if and only if $XZ = 0$ for every $Z \in \bigoplus_{j=1}^{k-1} \mathcal{Z}_j$.*

(iii) *We assume that $\lambda_k = 1$ without loss of generality. Let $Z = \sum_{i=1}^{l} Z_i$, where $Z_i \in \mathcal{Z}_i$ for every $i = 1, 2, \dots, l$. Then $\lim_{e\to\infty} XB^eZ$ exists for every X if and only if $Z_i = 0$ for $i = 1, 2, \dots, k-1$.*

(iv) *The limit $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e)$ exists for every module $M = XY$ and for every $i = 1, 2, \dots, s$ if and only if $k = 1$, that is, $\lambda_1 = 1$ is the only eigenvalue of B with absolute value equal to* 1.

Proof. (i) By the above lemma, we know that $\sum_{j=1}^{s} c_{ij}\binom{W}{j} = P_i(W) = c_i$. Since the set $\{\binom{W}{j} \mid j = 1, 2, \dots, r\}$, considered as a subset of the $\mathbb{C}$-vector space $\mathbb{C}[W]$, is linearly independent over $\mathbb{C}$, we have $c_{ij} = 0$ for all $j = 1, 2, \dots, s$. In particular, $c_{i1} = 0$. However, $c_{i1} = (1/\lambda_i)XN_iZ_i$. Therefore $XN_iZ_i = 0$. By running X over all possible choices and running Z over all vectors in $\mathbb{C}^s$ (actually it is enough to run Z over all vectors in $\mathcal{Z}_1$), we deduce that $N_1Z_1 = 0$ for all $Z_1 \in \mathcal{Z}_1$, which proves (i).

(ii) and (iii) immediately follow from the above lemma.

(iv) immediately follows from (ii) or (iii). Alternatively it can be proved directly. □

Discussion 3.7. For any $X \in \mathbb{N}^s$, let $\mathcal{V}_X$ be the set of all $s \times 1$ matrices $V \in \mathbb{C}^s$ with complex entries such that $\lim_{e\to\infty} XB^eV$ exists. It is easy to show that $\mathcal{V}_X$ is a B-subspace of $\mathbb{C}^s$ and that $\lim_{e\to\infty} XB^e$ exists if and only if $\mathcal{V}_X = \mathbb{C}^s$. By the definition of F-contributors, we know that $E_i \in \mathcal{V}_X$ for all $i = r+1, r+2, \ldots, s$ if $M_1, M_2, \ldots, M_r$ are all the F-contributors of $M = XY$.

Similarly, we define $\mathcal{V}$ to be the set of all $s \times 1$ matrices $V \in \mathbb{C}^s$ with complex entries such that $\lim_{e\to\infty} B^eV$ exists. It is easy to show that $\mathcal{V}$ is a B-subspace of $\mathcal{V}_X \subseteq \mathbb{C}^s$ for any $X \in \mathbb{N}^s$ and that $\lim_{e\to\infty} B^e$ exists if and only if $\mathcal{V} = \mathbb{C}^s$. By the definition of the general F-contributors, we know that $E_i \in \mathcal{V}$ for all $i = r+1, r+2, \ldots, s$ since $\{M_1, M_2, \ldots, M_r\}$ contains all the F-contributors of $M = XY$ for all possible X.

Let L be an R-module such that $\lambda_R(L) < \infty$ and $M \cong XY$ so that M has finite F-representation type by $M_1, M_2, \ldots, M_s$. By [**17**], we know that

$$\begin{aligned}\lim_{e\to\infty} XB^e\lambda_R(\mathrm{Hom}_R(Y', L)) &= \lim_{e\to\infty} XB^e\lambda_R(\mathrm{Hom}_R(Y, L)) \\ &= \lim_{e\to\infty} \frac{\lambda_R(\mathrm{Hom}_R({}^eM, L))}{(ap^d)^e}\end{aligned}$$

exists. Hence $\{\lambda_R(\mathrm{Hom}_R(Y', L)) \mid \lambda_R(L) < \infty\}$ and $\{\lambda_R(\mathrm{Hom}_R(Y, L)) \mid \lambda_R(L) < \infty\}$ are all contained in $\mathcal{V}$. Hence a sufficient condition for a positive answer to Question 3.1 would be that the $\{\lambda_R(\mathrm{Hom}_R(Y, L)) \mid \lambda_R(L) < \infty\}$ spans $\mathbb{Q}^s$ or that $\{\lambda_R(\mathrm{Hom}_R(Y', L)) \mid \lambda_R(L) < \infty\}$ spans $\mathbb{Q}^r$.

In the remaining part of this section we assume that the R-modules $M_1, M_2, \ldots, M_r$ satisfy the following unique condition.

$$\sum_{i=1}^r n_iM_i \cong \sum_{i=1}^r m_iM_i \quad \text{if and only if } m_i = n_i \text{ for all } 1 \leqslant i \leqslant r. \tag{3.1}$$

This condition is satisfied if, for example, R satisfies the Krull–Schmidt condition and $M_1, M_2, \ldots, M_r$ are all indecomposable, are non-zero and belong to different isomorphism classes. Indeed, under the uniqueness condition (3.1), we can show that $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e) = \lim_{e\to\infty} XBE_i$ exists for every $i = 1, 2, \ldots, s$ and every $X \in \mathbb{N}^s$. Its proof uses the following theorems of Guralnick [**6**] and Auslander [**3**]. We only quote a special version of each of the theorems. See the original papers for their general versions and proofs.

Theorem 3.8 [**6**, Corollary 1]. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, not necessarily of characteristic p, and let M and N be finite R-modules. If $M/\mathfrak{m}^nM \cong N/\mathfrak{m}^nN$ for a sufficiently large $n \in \mathbb{N}$, then $M \cong N$.*

The next theorem of Auslander can be found in [**3**, **4**]. A simple and direct proof of the result is provided by Bongartz in [**5**].

THEOREM 3.9 [**3**, **4**, **5**]. *Let R be a Artinian ring, not necessarily of characteristic p, and let M and N be finite R-modules. Then $M \cong N$ if and only if $\lambda_R(\mathrm{Hom}_R(M,L)) = \lambda_R(\mathrm{Hom}_R(N,L))$ for all finite R-modules L, which is also equivalent to $\lambda_R(M \otimes L) = \lambda_R(N \otimes L)$ for all finite R-modules L.*

Actually it is the following corollary of the above two theorems that is used in the proof of Theorem 3.11.

COROLLARY 3.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, not necessarily of characteristic p, and let M and N be finite R-modules. Then $M \cong N$ if and only if $\lambda_R(\mathrm{Hom}_R(M,L)) = \lambda_R(\mathrm{Hom}_R(N,L))$ for all finite R-modules L such that $\lambda_R(L) < \infty$ if and only if $\lambda_R(M \otimes L) = \lambda_R(N \otimes L)$ for all finite R-modules L such that $\lambda_R(L) < \infty$.*

Proof. For any $n \in \mathbb{N}$ and for any finitely generated $R/\mathfrak{m}^n$-module L, we have $\lambda_R(\mathrm{Hom}_R(M,L)) = \lambda_R(\mathrm{Hom}_R(N,L))$ by assumption. This is the same as saying that $\lambda_{R/\mathfrak{m}^n}(\mathrm{Hom}_{R/\mathfrak{m}^n}(M/\mathfrak{m}^n M, L)) = \lambda_{R/\mathfrak{m}^n}(\mathrm{Hom}_{R/\mathfrak{m}^n}(N/\mathfrak{m}^n N, L))$ for any finitely generated $R/\mathfrak{m}^n$-module L. Hence, by Theorem 3.9, $M/\mathfrak{m}^n \cong N/\mathfrak{m}^n$ as $R/\mathfrak{m}^n$-modules (and as R-modules) for any $n \in \mathbb{N}$. Then Theorem 3.8 gives the desired result that $M \cong N$ as R-modules. $\square$

THEOREM 3.11. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of characteristic p, and let M be a finitely generated R-module with finite F-representation type by a finite F-representation type system $\{M_1, M_2, \ldots, M_s\}$, of which $M_1, M_2, \ldots, M_r$ are the general F-contributors which satisfy the uniqueness condition* (3.1). *Then $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e) = \lim_{e\to\infty} XB^eE_i$ exists and is rational for every $i = 1, 2, \ldots, s$ and every $X \in \mathbb{N}^s$, where $M \cong XY$, or, equivalently, the matrix B has exactly one eigenvalue, that is,* 1, *with absolute value equal to* 1.

Proof. We first arbitrarily choose and then fix an $X \in \mathbb{N}^s$ and set $M \cong XY$. By Discussion 3.7, it suffices to show that the set of vectors $\{\lambda_R(\mathrm{Hom}_R(Y', L)) = (\lambda_R(\mathrm{Hom}_R(M_1, L)), \lambda_R(\mathrm{Hom}_R(M_2, L)), \ldots, \lambda_R(\mathrm{Hom}_R(M_r, L))) \in \mathbb{Q}^r \mid \lambda(L) < \infty\}$ spans $\mathbb{Q}^r$. Suppose that this is not the case. Then there are integers $c_1, c_2, \ldots, c_r$, not all zero, such that $(c_1, c_2, \ldots, c_r)\lambda_R(\mathrm{Hom}_R(Y', L)) = 0$, that is,

$$c_1\lambda_R(\mathrm{Hom}_R(M_1, L)) + c_2\lambda_R(\mathrm{Hom}_R(M_2, L)) + \ldots + c_r\lambda_R(\mathrm{Hom}_R(M_r, L)) = 0$$

for all R-modules L such that $\lambda_R(L) < \infty$. Without loss of generality, we may assume that $c_i \geqslant 0$ for $i = 1, 2, \ldots, t$ and $c_j = -b_j < 0$ for $j = t+1, t+2, \ldots, r$. Let $N' = \bigoplus_{i=1}^t c_iM_i$ and $N'' = \bigoplus_{j=t+1}^r b_jM_j$. Then $(c_1, c_2, \ldots, c_r)\lambda_R(\mathrm{Hom}_R(Y', L)) = 0$ means that $\lambda_R(\mathrm{Hom}_R(N', L)) = \lambda_R(\mathrm{Hom}_R(N'', L))$ for all R-modules L such that $\lambda_R(L) < \infty$, which implies that $N' \cong N''$ from Corollary 3.10. However, this is impossible as $M_1, M_2, \ldots, M_r$ satisfy the uniqueness condition (3.1).

It remains to show that $\lim_{e\to\infty}(\#({}^eM, M_i)/(ap^d)^e) = \lim_{e\to\infty} XBE_i$ is rational for every $i = 1, 2, \ldots, s$ and every $X \in \mathbb{N}^s$. This follows directly from a lemma of Seibert [**18**, Lemma 2.4]. We include a proof for completeness. Indeed, since we know that the only unimodular eigenvalue of B is 1 and the zero space of $B - E$ is the same as the zero space of $(B - E)^n$ for all $n \in \mathbb{N}$, there exists an invertible matrix $T \in M_{s\times s}(\mathbb{Q})$ such that

$$T^{-1}BT = \begin{pmatrix} E_{s_1\times s_1} & 0 \\ 0 & B_{s_2\times s_2}\end{pmatrix},$$

where $E_{s_1\times s_1}$ is the $s_i \times s_i$ identity matrix and $B_{s_2\times s_2}$ is an $s_2 \times s_2$ matrix with all its eigenvalues having absolute values strictly less than 1. In particular, $\lim_{n\to\infty} B^n_{s_2\times s_2} = 0$.

Write $XT^{-1} = (X', X'')$ and $TE_i = (E'_i, E''_i)^T$, where X', X'', E'_i and E''_i are $1 \times s_1$, $1 \times s_2$, $s_1 \times 1$ and $s_2 \times 1$ matrices respectively, with rational entries. Then

$$\lim_{n\to\infty} XB^nE_i = \lim_{n\to\infty}(X'E'_i + X''B^n_{s_2\times s_2}E''_i) = X'E'_i,$$

which is rational. □

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of characteristic p (not necessarily satisfying the Krull–Schmidt condition), and let M be a finitely generated R-module with finite F-representation type. If we use $\#({}^eM, R)$ to denote the maximal number of copies of R appearing as a direct summand of ${}^eM$, then $\lim_{e\to\infty}(\#({}^eM, R)/(ap^d)^e)$ exists.*

Proof. We may assume that R is complete since $\#({}^eM, R) = \#({}^e\widehat{M}, \widehat{R})$. Then the existence of the limit follows immediately from Theorem 3.11 as complete rings satisfy the Krull–Schmidt condition. □

Remark 3.13. The limit $\lim_{e\to\infty}(\#({}^eR, R)/(ap^d)^e)$ was studied in [**11**] by Huneke and Leuschke and is called the F-signature of R there.

Question 3.14. Now let us return to the general situation at the beginning of the section, that is, we do not assume that R satisfies the Krull–Schmidt condition or that $M_1, M_2, \dots, M_s$ are all indecomposable and belong to distinct isomorphism classes. Let $P(W) \in \mathbb{Q}[W]$ be the characteristic polynomial of B. Suppose that $\lambda \in \mathbb{C}$ is a root of $P(W)$ and $|\lambda| = 1$. Then is λ an nth root of 1?

Does Theorem 3.11 help with anything in this direction, as we can complete the ring R without loss of generality? If the answer to the above question is positive, then we can show that the sequence $\{\#({}^eM, M_i)/(ap^d)^e\}_{e=0}^{\infty}$ is 'periodically convergent', that is, there exists an integer $k > 0$ such that, for every $i = 1, 2, \dots, k$, $\lim_{n\to\infty}(\#({}^{nk+i}M, M_i)/(ap^d)^{nk+i})$ exists.

4. *About* $\bigcap_L \mathrm{Ann}_R(\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N)))$

Let us return to the situation of Proposition 2.4(ii) and Theorem 2.5(ii) and keep the notation. Both results claim that $K^*_L = \ker(L \longrightarrow L/K \longrightarrow \mathrm{Hom}_R(N, L/K \otimes N))$ for any finitely generated R-modules $K \subseteq L$, in which N is the direct sum of all F-contributors. Thus the test ideal of R is

$$\tau = \bigcap_{K\subseteq L}(K :_R (\ker(L \longrightarrow L/K \longrightarrow \mathrm{Hom}_R(N, L/K \otimes N)))),$$

where $K \subseteq L$ run over all finitely generated R-modules. As $K_L^*/K = 0_{L/K}^*$, we may always assume that $K = 0$ to get $\tau = \bigcap_L \mathrm{Ann}_R(\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N)))$, and it is easy to see that $\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N))$ consists of $x \in L$ such that $x \otimes N$ is zero in $L \bigotimes_R N$. In the case of R being approximately Gorenstein, the test ideal can be simplified as $\tau = \bigcap_{I \subset R}(I :_R I^*) = \bigcap_{I \subset R}(I :_R (IN :_R N))$. Our next definition is inspired by this observation.

DEFINITION 4.1. Let R be a Noetherian ring, not necessarily of characteristic p. For any R-module N, we define $\tau(N) = \bigcap_L \mathrm{Ann}_R(\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N)))$, with L running over all finitely generated R-modules.

LEMMA 4.2. *Let R be a Noetherian ring, not necessarily of characteristic p, let N be a finitely generated R-module, and let U be a multiplicatively closed subset of R. Then $\tau(N) \cap U \neq \varnothing$ if and only if there exists $n \in \mathbb{N}$ such that $nN_U = N_U \oplus \ldots \oplus N_U$ has a direct summand isomorphic to R_U ($n = 1$ if $R_U = U^{-1}R$ is local).*

Proof. First we assume that nN_U has a direct summand isomorphic to R_U for some positive integer n. Since $\tau(nN) = \tau(N)$, we may assume that $n = 1$. Therefore there exists an element $c \in U$ such that R_c is a homomorphic image of N_c. That is the same as saying that there is an R-homomorphism $f : N \longrightarrow R$ such that $c^i \in f(N)$ for some i. We may as well assume that $i = 1$. Then, for any finitely generated R-module L and for any $x \in \ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N))$, we have $x \otimes N = 0$ in $L \bigotimes_R N$. Applying $1_L \otimes f$ to $L \bigotimes_R N$, we get $cx = 0 \in L \cong L \otimes R$, which in turn implies that $c \in \mathrm{Ann}_R(\ker(L \longrightarrow \mathrm{Hom}_R(N, L \otimes N)))$. Hence $c \in \tau(N)$, which gives $\tau(N) \cap U \neq \varnothing$, the desired result.

For the converse implication, we assume that $\tau(N) \cap U \neq \varnothing$. By relabeling R_U and N_U with R and N respectively, we may simply assume that $\tau(N) = R$ and prove that nN has a direct summand isomorphic to R for some $n \in \mathbb{N}$. Say that N is generated by $x_1, x_2, \ldots, x_n$. Define an R-linear map $\phi : R \longrightarrow nN$ by $r \longmapsto (rx_1, rx_2, \ldots, rx_n)$. The assumption that $\tau(N) = R$ says exactly that the induced map $1_L \otimes \phi : L \bigotimes_R R \longrightarrow L \bigotimes_R nN$ is injective for any finitely generated (and hence any) R-module L, that is, ϕ is pure. Since nN is Noetherian, we find that $\phi : R \longrightarrow nN$ is a split injection and hence nN has a direct summand isomorphic to R. $\square$

REMARK 4.3. Let us again return to Proposition 2.4(ii) and Theorem 2.5(ii), with M being a finite F-representation type faithful R-module. Then R is weakly F-regular if and only if $\tau(N) = R$ if and only if R is an F-contributor of M (by Lemma 4.2) if and only if R is strongly F-regular (by a recent result in [**1**]).

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