Bull. London Math. Soc. 34 (2002) 165–173
DOI: 10.1112/S0024609301008773

RESTRICTIONS OF ALGEBRAIC GROUP REPRESENTATIONS TO FINITE SUBGROUPS

HARM DERKSEN

Abstract

Suppose that H is a finite subgroup of a linear algebraic group, G. It was proved by Donkin that there exists a finite-dimensional rational representation of G whose restriction to H is free. This paper gives a short proof of this in characteristic 0. The author also studies more closely which representations of H can appear as a restriction of G.

1. *Introduction*

Suppose that G is an algebraic group over a field K, and that H is a finite subgroup of G. In this paper, we shall investigate which representations of H appear as a restriction of a representation of G. Throughout the paper, all representations are assumed to be finite-dimensional and rational. In [**2**], Donkin proves the following theorem.

Theorem 1.1 ([**2**]). *There exists a finite-dimensional rational representation V of G such that the restriction $V\,|_H$ of V to H is free; that is, $V\,|_H$ is isomorphic to W_{reg}^N, where N is a positive integer and W_{reg} is the regular representation of H.*

Donkin thereby answered a question raised in [**5**] by Kuzucuoğlu and Zalesskiĭ, who proved the theorem in the special case where K has positive characteristic and H is reduced. In this paper we give a short alternative proof of the results of Donkin in the case $\mathrm{char}(K) = 0$, using representation theory. In fact, we shall show the following theorem.

Theorem 1.2. *Suppose that the characteristic of the base field K is 0. There exists a GL_n-representation V_E of dimension $(E-1)^{n(n-1)/2}E^n$ such that $V_E\,|_H$ is free for every subgroup H of GL_n whose exponent divides E.*

Theorem 1.1 is an easy corollary of Theorem 1.2. Indeed, suppose that G is a linear algebraic group containing a finite subgroup H with exponent E. Now G is a Zariski closed subgroup of GL_n for some n. By Theorem 1.2, there exists a finite-dimensional rational representation V_E of GL_n, such that the restriction $V_E\,|_H$ of V_E to H is free. Clearly, the restriction $V_E\,|_G$ is a finite-dimensional rational representation of G whose restriction to H is free.

Suppose that V is a rational finite-dimensional representation of G with character τ. Let $V\,|_H$ be the restriction of H, and let $\chi = \tau\,|_H$ be the H-character of this

Received 3 March 2000; revised 10 February 2001.

2000 *Mathematics Subject Classification* 20C15 (primary), 20G05, 22E46 (secondary).

Supported by NSF, grant DMS-9970165. A first version was written with support of the Swiss National Science Foundation and the Freiwillige Akademische Gesellschaft Basel.

restriction. If $h, h' \in H$ are conjugate in G, then we must have $\chi(h) = \chi(h')$. This gives a necessary condition for an H-representation W to be liftable to a representation of G, but it is not sufficient, as we shall see later. However, we shall prove the following theorem.

THEOREM 1.3. *Suppose that G is a reductive group over a field K of characteristic 0, that $H \subset G$ is a finite subgroup, and that W is a finite-dimensional representation of H whose character χ satisfies $\chi(h) = \chi(h')$ for all pairs $h, h' \in H$ which are conjugate in G. Then there exists a rational finite-dimensional representation V of G such that $V|_H$ is isomorphic to $W^M \oplus W_{\mathrm{reg}}^N$ for some positive integers M and N.*

Define X_H^G to be the monoid of all restrictions $\{\tau|_H : \tau \text{ is a character of } G\}$. In the last section we shall prove the following theorem.

THEOREM 1.4. *If G is a connected reductive group over an algebraicially closed field K (of characteristic 0), and $H \subset G$ is a finite subgroup of G, then X_H^G is a finitely generated monoid.*

2. *Notation*

To avoid confusion, throughout the paper we shall stick to the following notation. We assume that G is a reductive group, that $\mathfrak{g}$ is its Lie algebra, that $e \in G$ is the identity element, and that $H \subset G$ is a finite subgroup. In general we shall denote representations of G by 'V', representations of H by 'W', characters of G by 'τ' and characters of H by 'χ'. The regular character of H is denoted by χ_{reg}. We choose a maximal torus $T \subset G$, and $\mathscr{W} = N_G(T)/T$ is the Weyl group, where $N_G(T)$ is the normalizer of T inside G. We shall write Φ for the set of roots, we choose simple roots $\alpha_1, \ldots, \alpha_r \in \Phi$, and Φ_+ will be the set of positive roots. If G is connected and semisimple, then we have the following additional notation. The weight lattice Λ is generated by fundamental weights $\lambda_1, \ldots, \lambda_r$, and $\Lambda_+ = \mathbb{N}\lambda_1 + \mathbb{N}\lambda_2 + \ldots + \mathbb{N}\lambda_r$ is the set of dominant weights. We shall write $\rho = \sum_{i=1}^r \lambda_i = \frac{1}{2}\sum_{\alpha \in \Phi_+} \alpha$. For $\lambda \in \Lambda_+$ we shall write V_λ for the $\mathfrak{g}$-module with heighest weight λ. We shall write τ_λ for the character of G on V_λ if the action of $\mathfrak{g}$ extends to an action of G.

3. *Examples*

EXAMPLE 3.1. Take $G = \mathrm{GL}_{2^s}$ (where $s \geqslant 1$) and $H = \{\mathrm{Id}, -\mathrm{Id}\} \subset G$. Let χ_0 be the trivial character, and let χ_1 be the signum character of H. Let V_λ be the irreducible representation corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$. The G-character of V_λ will be denoted by τ_λ. The restriction of the character $\tau_{(1)}$ of the representation $V = V_{(1)}$ to H is equal to $2^s\chi_1$. The restriction of the character $\tau_{(1,1,\ldots,1)}$ of $\bigwedge^{2^s} V$ to H is equal to χ_0. We shall show that X_H^G is generated by χ_0 and $2^s\chi_1$ as follows.

If $|\lambda| := \lambda_1 + \ldots + \lambda_r$ is even, then $\tau_\lambda|_H = \dim(V_\lambda)\chi_0$, and if $|\lambda|$ is odd, then $\tau_\lambda|_H = \dim(V_\lambda)\chi_1$. Suppose that $|\lambda|$ is odd, and let $\rho : \mathrm{GL}_{2^s} \to \mathrm{GL}(V_\lambda)$ be the group homomorphism corresponding to the action. Then $\det(\rho(g)) = \det(g)^l$ for some l. If we substitute $g = t \cdot \mathrm{Id}$, then we obtain $t^{\dim(V_\lambda)|\lambda|} = t^{l2^s}$. It follows that $\dim(V_\lambda)$ is divisible by 2^s.

The character $N\chi_{\text{reg}}$ is a restriction of a character of G if and only if N is divisible by 2^s.

EXAMPLE 3.2. Take $G = \text{GL}_6$ and let $H = \{\text{Id}, -\text{Id}\} \subset G$. Note that $\tau_{(1,1,1,1,1,1)} |_H = \chi_0$, $\tau_{(1)} |_H = 6\chi_1$, and $\tau_{(1,1,1)} |_H = 20\chi_1$. One can show that X_H^G is generated by χ_0, $6\chi_1$ and $20\chi_1$. Now $N\chi_{\text{reg}}$ is the restriction of a character of G if and only if $N = 6$, 12, 18, 20, 24, 26, 30, 32, or $N = 2M$ with $M \geqslant 18$.

EXAMPLE 3.3. Let $G = \text{SL}_2$, and take $H = \langle\sigma\rangle$, where

$$\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

and where ζ is a fifth root of unity. Let χ_i be the irreducible one-dimensional character defined by $\sigma \mapsto \zeta^i$ for $i = 0, 1, 2, 3, 4$. Denote the binary forms of degree d by V_d, and let τ_d be its character. Then we have

$$\begin{aligned} \tau_{5d} |_H &= (d+1)\chi_0 + d(\chi_1 + \chi_2 + \chi_3 + \chi_4), \\ \tau_{5d+1} |_H &= d(\chi_0 + \chi_2 + \chi_3) + (d+1)(\chi_1 + \chi_4), \\ \tau_{5d+2} |_H &= (d+1)(\chi_0 + \chi_2 + \chi_3) + d(\chi_1 + \chi_4), \\ \tau_{5d+3} |_H &= d\chi_0 + (d+1)(\chi_1 + \chi_2 + \chi_3 + \chi_4), \\ \tau_{5d+4} |_H &= (d+1)(\chi_0 + \chi_1 + \chi_2 + \chi_3 + \chi_4). \end{aligned}$$

The monoid X_H^G is generated by χ_0, $\chi_1 + \chi_4$, $\chi_0 + \chi_2 + \chi_3$ and $\chi_1 + \chi_2 + \chi_3 + \chi_4$.

Now σ and σ^4 are conjugate in G, σ^2 and σ^3 are conjugate in G, and Id, σ and σ^2 generate distinct conjugacy classes in G. Take $\chi = \chi_2 + \chi_3$, and note that $\chi(\sigma) = \chi(\sigma^4)$ and $\chi(\sigma^2) = \chi(\sigma^3)$. Now $(\tau_2 + \tau_3) |_H = \chi + \chi_{\text{reg}}$ where $\chi_{\text{reg}} = \chi_0 + \chi_1 + \chi_2 + \chi_3 + \chi_4$ is the regular character of H. However, there is no character τ of G such that $\tau |_H = M\chi$ with M positive.

EXAMPLE 3.4. Let $G = \text{GL}_{3^s}$, and take $H = \{\text{Id}, \zeta\,\text{Id}, \zeta^2\,\text{Id}\}$, where ζ is a primitive third root of unity. Let χ_i be the character $\zeta^i\,\text{Id} \mapsto \zeta^i$. Using similar arguments to those in Example 3.1, one can prove that whenever $\tau |_H = M\chi_1 + N\chi_{\text{reg}}$, it must be true that 3^s divides M and N. This shows that M cannot always be taken equal to 1 in Theorem 1.3.

4. *A lemma from representation theory*

At this beginning of this section we shall prove a lemma in representation theory, which will be applied to the proof of Theorem 1.2 at the end of this section.

Let $\mathfrak{g}$ be a semisimple Lie algebra. The character τ_λ has the following formal expression:

$$\sum_{\mu \in \Lambda} m_{\lambda,\mu} e^{\mu},$$

where $m_{\lambda,\mu}$ is the multiplicity of the weight μ in V_λ. A well-known formula of Weyl (see [**3**, 24.3]) states that for every $\lambda \in \Lambda_+$, we have

$$\left(\sum_{w \in \mathscr{W}} \text{sgn}(w) e^{w(\rho)}\right).\tau_\lambda = \sum_{w \in \mathscr{W}} \text{sgn}(w) e^{w(\rho+\lambda)}.$$

LEMMA 4.1. *If $\lambda + \rho + w(\mu) \in \Lambda_+$ for all $w \in \mathscr{W}$, then*

$$\Big(\sum_{w\in\mathscr{W}} e^{w(\mu)}\Big).\tau_\lambda$$

is a character of a representation of $\mathfrak{g}$.

Proof. We can write

$$\Big(\sum_{w\in\mathscr{W}} e^{w(\mu)}\Big).\tau_\lambda = \sum_v a_v \tau_v, \tag{4.1}$$

where v runs through a finite set of dominant weights, and where the a_v are integers. We shall prove that $a_v \geqslant 0$ for all v. We multiply (4.1) with

$$\sum_{w'\in\mathscr{W}} \operatorname{sgn}(w')e^{w'(\rho)},$$

and by using Weyl's formula we get

$$\Big(\sum_{w\in\mathscr{W}} e^{w(\mu)}\Big) \cdot \sum_{w'\in\mathscr{W}} \operatorname{sgn}(w')e^{w'(\rho+\lambda)} = \sum_v a_v \sum_{w'\in\mathscr{W}} \operatorname{sgn}(w')e^{w'(\rho+v)}. \tag{4.2}$$

If $w'(\rho+\lambda) + w(\mu) \in \rho + \Lambda_+$, then w' must be trivial because

$$(w')^{-1}(w'(\rho+\lambda) + w(\mu)) = \rho + \lambda + ((w')^{-1}w)(\mu) \in \Lambda_+.$$

Of we consider only the e^γ in (4.2) where γ lies in $\rho + \Lambda_+$, then we get

$$\sum_{\substack{w\in\mathscr{W} \\ \rho+\lambda+w(\mu)\in\rho+\Lambda_+}} e^{\rho+\lambda+w(\mu)} = \sum_v a_v e^{\rho+v}.$$

□

Proof of Theorem 1.2. Let $T \subset \mathrm{GL}_n$ be the set of diagonal matrices. Define $x_i \in X^\star(T)$ as the function which maps an element of T to its (i,i)-entry. The Weyl group $\mathscr{W}$ of GL_n is the symmetric group S_n which acts transitively on $x_1, x_2, \ldots, x_n$. Let us define u as

$$(1 + x_1 + x_1^2 + \ldots + x_1^{(E-1)})(1 + x_2 + x_2^2 + \ldots + x_2^{(E-1)})\ldots(1 + x_n + x_n^2 + \ldots + x_n^{(E-1)}).$$

Since u is $\mathscr{W}$-invariant, it can be seen as a class function on GL_n. Define a partition

$$p = ((E-2)(n-1), (E-2)(n-2), \ldots, (E-2), 0).$$

Let τ_p be the corresponding irreducible character of GL_n. We claim that the class function $\tau_p \cdot u$ is a character of a representation of GL_n. Let us put $x_i = e^{t_i}$. The formal character of u is

$$(1 + e^{t_1} + e^{2t_1} + \ldots + e^{(E-1)t_1})(1 + e^{t_2} + \ldots + e^{(E-1)t_2})\ldots(1 + e^{t_n} + e^{2t_n} + \ldots + e^{(E-1)t_n}).$$

Now p corresponds to the weight $(E-2)(n-1)t_1 + (E-2)(n-2)t_2 + \ldots + (E-2)t_{n-1}$ and $\rho = \frac{1}{2}((n-1)t_1 + (n-3)t_2 + \ldots + (1-n)t_n)$. A weight $a_1t_1 + a_2t_2 + \ldots + a_nt_n$ is dominant if and only if $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$. A weight μ appearing in u is of the form $a_1t_1 + a_2t_2 + \ldots + a_nt_n$, with $a_1, a_2, \ldots, a_n \in \{0, 1, \ldots, E-1\}$, and it is easy to check that $p + \rho + \mu$ is dominant. Define $\deg(e^{a_1t_1+\ldots+a_nt_n}) = a_1 + a_2 + \ldots a_n$, and write $u = \sum_{i=0}^{n(E-1)} u_i$, where u_i is the homogeneous part of degree i. We restrict to $\mathfrak{sl}_n$, the Lie algebra of SL_n, and we apply Lemma 4.1. So $\tau_p \cdot u_i$ is a character of a representation V_i of $\mathfrak{sl}_n$, and the Lie algebra action extends to an action of the simply connected group SL_n. For $\lambda \in K \setminus \{0\}$ we have $\tau_p \cdot u_i(\lambda \mathrm{Id}) = \tau_p \cdot u_i(\mathrm{Id})\lambda^i = \dim(V_i)\lambda^i$.

Now we can extend the action of SL_n on V_i to GL_n by defining $\lambda\mathrm{Id} \cdot v = \lambda^i v$ for all $v \in V$, and we see that $\tau_p \cdot u_i$ is the GL_n-character of V_i. So $\tau_p \cdot u$ is a character of the GL_n module $V = \bigoplus_i V_i$ of dimension $\tau_p(\mathrm{Id})u(\mathrm{Id})$. Clearly, $u(\mathrm{Id}) = E^n$, and by Weyl's formula we get

$$\tau_p(\mathrm{Id}) = \prod_{\alpha\in\Phi_+} \frac{\langle p+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} = \prod_{\alpha\in\Phi_+} \frac{\langle (E-2)\rho+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle} = (E-1)^{n(n-1)/2}.$$

The function u vanishes on $H \setminus \{\mathrm{Id}\}$, and so does $u \cdot \tau_p$. It follows that $V|_H$ is a free H-module. □

5. *The monoid* $\mathscr{C}$

Let d be the order of H, and write $H = \{e, h_1, h_2, \ldots, h_{d-1}\}$. Let X_G be the set of characters of all finite-dimensional representations of G over $\overline{K}$, the algebraic closure of K. This monoid is generated by infinitely many irreducible characters. Characters of H have values in $\mathbb{C}$ (or to be precise in $\overline{\mathbb{Q}} \subset \mathbb{C}$, the algebraic closure of $\mathbb{Q}$). We define a map $X_G \to \mathbb{C}^{d-1}$ by

$$\pi(\tau) := \big(\tau(h_1), \tau(h_2), \ldots, \tau(h_{d-1})\big).$$

Let $\mathscr{C}$ be the image of π.

Lemma 5.1. *The monoid $\mathscr{C}$ is in fact a $\mathbb{Z}$-module.*

Proof. It is enough to show that $-\pi(\tau) \in \mathscr{C}$ for all $\tau \in X_G$. By Theorem 1.1, there exists a representation V of G such that $V|_H$ is a free H-module. Let τ_0 be the character of V. We have $\pi(\tau_0) = 0$. Let $\overline{\tau_0}$ be the character of $V^\star$, the dual space. Then $\tau_0\overline{\tau_0} = 1 + \tau_1$, where 1 is the trivial character and τ_1 is a character. We have $\tau_0\overline{\tau_0}\tau = \tau + \tau\tau_1$ and $\pi(\tau) + \pi(\tau\tau_1) = \pi(\tau + \tau\tau_1) = \pi(\tau_0\overline{\tau_0}\tau) = 0$, so $-\pi(\tau) = \pi(\tau\tau_1) \in \mathscr{C}$. □

For $h \in H$, we write $[h]_G$ to denote the conjugacy class of h in G.

Lemma 5.2. *The rank of $\mathscr{C}$ as a $\mathbb{Z}$-module is equal to r, where r is the cardinality of the set of conjugacy classes $\{[h_1]_G, [h_2]_G, \ldots, [h_{d-1}]_G\}$.*

Proof. If $[h_i]_G = [h_j]_G$, then $\tau(h_i) = \tau(h_j)$ for all $\chi \in X(G)$, so it is clear that the rank of $\mathscr{C}$ is $\leqslant r$. Without loss of generality, we may assume that $[h_1]_G, [h_2]_G, \ldots, [h_r]_G$ are all different. The elements $h_1, \ldots, h_r$ are semisimple in G, so the conjugacy classes $[h_i]_G$ are Zariski-closed subsets of G (see [**1**, 9.2], or [**4**, I.3]). Let $\mathscr{O}(G)$ be the coordinate ring G over the algebraic closure $\overline{K}$, and we let G act on itself by conjugation. There exists an invariant $f \in \mathscr{O}(G)^G$ (a class function) such that $f(h_1), f(h_2), \ldots, f(h_r)$ are all different (see [**6**, Corollary 1.2]). We can write $f = \sum_{i=1}^{s} a_i\tau_i$ with $a_i \in \overline{K}$ and τ_i a character for $i = 1, \ldots, s$. For a generic choice of positive integers $b_1, \ldots, b_s$ we see that $\tau := \sum_{i=1}^{s} b_i\tau_i$ is a character such that $\tau(h_1), \tau(h_2), \ldots, \tau(h_r)$ are all different. Now $\pi(1), \pi(\tau), \pi(\tau^2), \ldots, \pi(\tau^{r-1})$ are linearly independent, because these vectors form the matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \tau(h_1) & \tau(h_2) & \ldots & \tau(h_{d-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \tau^{r-1}(h_1) & \tau^{r-1}(h_2) & \ldots & \tau^{r-1}(h_{d-1}) \end{pmatrix},$$

and the first $r \times r$ minor is a Vandermonde determinant, whose value is

$$\prod_{1 \leqslant i < j \leqslant r} (\tau(h_i) - \tau(h_j)),$$

which is nonzero. □

Proof of Theorem 1.3. Let $\mathscr{D} \subset \mathbb{C}^{d-1}$ be the set of all

$$\big(\chi(h_1), \chi(h_2), \ldots, \chi(h_{d-1})\big),$$

where χ is a character of H with the property that $\chi(h_i) = \chi(h_j)$ for all $h_i, h_j \in H$ which are conjugate in G. Let $\mathscr{D}'$ be the $\mathbb{Z}$-module generated by $\mathscr{D}$. It is easy to see that $\mathscr{D}'$ has rank less than or equal to r, and clearly $\mathscr{C} \subseteq \mathscr{D} \subseteq \mathscr{D}'$, so $\mathscr{C} \subseteq \mathscr{D}'$ is a submodule of finite index (and in fact it follows that $\mathscr{D} = \mathscr{D}'$). Suppose that W is a finite-dimensional H-module, and that χ is its character. For some M we have

$$M\big(\chi(h_1), \ldots, \chi(h_{d-1})\big) = \pi(\tau) \in \mathscr{C}.$$

So $\tau\,|_H - M\chi$ is a class function on H vanishing on $h_1, h_2, \ldots, h_{d-1}$, so it must be a multiple of the regular character χ_{reg} of H, say $\tau\,|_H - M\chi = N\chi_{\mathrm{reg}}$ with $N \in \mathbb{Z}$. Without loss of generality, we may assume that $N > 0$. (We can replace τ by $\tau + l\tau'$ where l is a positive integer and τ' is a character such that $\tau'\,|_H = A\chi_{\mathrm{reg}}$ for some positive integer A as in Theorem 1.1.) Let V be the representation of G (defined over $\overline{K}$) corresponding to τ. Then $V\,|_H \cong W^M \oplus W_{\mathrm{reg}}^N$. There exists a finite algebraic extension L of K such that the representation V is defined over L. Let V' be the $G(L)$ module V seen as a $G(K)$-module. Then $V'\,|_H = W^{Ms} \oplus W_{\mathrm{reg}}^{Ns}$, where s is the degree of the field extension $[L : K]$. □

6. *The finitely generated monoid of restricted characters*

LEMMA 6.1. *Suppose that G is connected, that $\mathfrak{g}$ is simple, and that $h \in G$ is of finite order and not in the center of G. Then for every $\varepsilon > 0$ there exists a positive integer N such that for every dominant weight λ with $\langle \rho, \lambda \rangle > N$ we have*

$$\left| \frac{\tau_\lambda(h)}{\tau_\lambda(e)} \right| < \varepsilon.$$

Proof. We choose a maximal torus T of G containing h. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} K x_\alpha$$

be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{h}$ is the Lie algebra of T. Because h is not in the center of G, there exists an x_α such that $hx_\alpha = \zeta x_\alpha$ with $\zeta \neq 1$. We shall show that there exists a constant $M > 0$ such that

$$|\tau_\lambda(h)| \leqslant M \dim V_\lambda^{x_\alpha}$$

for all dominant weights λ, where $V_\lambda^{x_\alpha}$ is the kernel of x_α acting on V_λ. The elements x_α and $x_{-\alpha}$ generate a sub-Lie algebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$. Having obtained an $\mathfrak{sl}_2$-module, we have a decomposition

$$V_\lambda = \bigoplus_{i=1}^{r} R_i$$

where R_i is h-stable, and irreducible as an $\mathfrak{sl}_2$-module for all i.

For each i, $R_i^{x_\alpha}$ is one-dimensional and spanned by a heighest weight vector v_i. A basis of R_i is given by

$$v_i, x_{-\alpha}v_i, x_{-\alpha}^2 v_i, \ldots, x_{-\alpha}^{d_i} v_i$$

for some nonnegative integer d_i. Now $hv_i = \gamma_i v_i$ for some γ_i, and we get

$$h(x_{-\alpha}^j v_i) = \zeta^{-j} x_{-\alpha}^j h v_i = \zeta^{-j}\gamma_i(x_{-\alpha}^j v_i).$$

Let $\tau_i(h)$ be the trace of h on R_i. Then we have

$$|\tau_i(h)| = |\gamma_i(1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{-d})| = \left|\gamma_i \frac{1-\zeta^{-d-1}}{1-\zeta^{-1}}\right| \leqslant \frac{2}{|\zeta - 1|}.$$

For an integer $M \geqslant 2/|\zeta - 1|$ we have

$$|\tau_\lambda(h)| \leqslant Mr = M \dim V_\lambda^{x_\alpha}.$$

We shall now show that $\dim V_\lambda^{x_\alpha}/\dim V_\lambda \to 0$ if $\langle \rho, \lambda\rangle \to \infty$. For a pair of dominant weights λ and μ, there exists a natural G-equivariant multiplication $V_\lambda \times V_\mu \to V_{\lambda+\mu}$ constructed as follows. Let $U \subset G$ be the maximal unipotent subgroup. Let $\mathcal{O}(G)$ be the coordinate ring, and let $\mathcal{O}(G/U)$ be the subring of U-invariant functions, where U acts on G by right-multiplication. It is known that, as a left-module, $\mathcal{O}(G/U)$ has the decomposition

$$\mathcal{O}(G/U) = \bigoplus_\lambda V_\lambda,$$

where λ runs through all the dominant weights. Now $\mathcal{O}(G/U)$ is a graded ring, graded by the monoid of dominant weights. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, and let $\lambda_1, \ldots, \lambda_l$ be the fundamental weights. For the moment we shall fix an i with $1 \leqslant i \leqslant l$. The element x_α acts non-trivially on V_{λ_i}, because otherwise G would act trivially on V_{α_i} by the simplicity of $\mathfrak{g}$. Choose an element $p \in V_{\lambda_i}$ such that $q := x_\alpha p \neq 0$ and $x_\alpha^2 p = 0$. For all nonnegative integers j and k with $j \leqslant k$, we have $p^{k-j}q^j V_\lambda \subset V_{\lambda+k\lambda_i}$. So we have an inclusion $p^k V_\lambda^{x_\alpha} + p^{k-1}qV_\lambda^{x_\alpha} + \ldots + q^k V_\lambda^{x_\alpha} \subset V_{\lambda+k\lambda_i}$. In fact, the sums are direct because $p \in \mathcal{O}(G/U)$ is transcendental over $\mathcal{O}(G/U)^{x_\alpha}$. ($x_\alpha$ acts as a derivation on $\mathcal{O}(G/U)$, so the kernel of x_α is algebraically closed within $\mathcal{O}(G/U)$.) Note that multiplication with p or q is injective, since $\mathcal{O}(G/U)$ is a domain. So we get $\dim V_{\lambda+k\lambda_1} \geqslant (k+1)\dim V_\lambda^{x_\alpha}$. By Weyl's formula, we get

$$\frac{\dim V_\lambda^{x_\alpha}}{\dim V_\lambda} \leqslant \frac{\dim V_{\lambda+k\lambda_i}}{(k+1)\dim V_\lambda} = \frac{1}{k+1}\prod_{\alpha\in\Phi_+}\frac{\langle \rho+\lambda+k\lambda_i, \alpha\rangle}{\langle \rho+\lambda, \alpha\rangle}$$

$$= \frac{1}{k+1}\prod_{\alpha\in\Phi_+}\left(1 + \frac{k\langle\lambda_i, \alpha\rangle}{\langle\rho+\lambda, \alpha\rangle}\right) \leqslant \frac{1}{k+1}\prod_{\alpha\in\Phi_+}\left(1+\frac{k\langle\lambda_i,\alpha\rangle}{\langle\lambda,\alpha_i\rangle}\right)$$

because $\langle\lambda_i, \alpha\rangle = 0$ or $\langle\rho+\lambda, \alpha\rangle \geqslant \langle\lambda, \alpha_i\rangle$ for all α. There is a constant $M_i(k)$ such that

$$\frac{\dim V_\lambda^{x_\alpha}}{\dim V_\lambda} \leqslant \frac{1}{k}$$

for all λ with $\langle\lambda, \alpha_i\rangle > M_i(k)$. Write $\rho = \sum_{i=1}^l k_i\alpha_i$. Take

$$N = k_1 M_1(kM) + k_2 M_2(kM) + \ldots + k_l M_l(kM).$$

If $\langle\rho, \lambda\rangle > N$, then $\langle\lambda, \alpha_i\rangle > M_i(kM)$ for some i, and

$$\left|\frac{\tau_\lambda(h)}{\tau_\lambda(e)}\right| \leqslant M\frac{\dim V_\lambda^{x_\alpha}}{\dim V_\lambda} \leqslant \frac{M}{Mk} \leqslant \frac{1}{k}. \qquad \square$$

LEMMA 6.2. *Suppose that $\mathcal{M} \subset \mathbb{N}^n$ is a submonoid. Let $C = \mathbb{R}_{\geqslant 0}\mathcal{M}$ be the cone in $\mathbb{R}^n$ spanned by $\mathcal{M}$. If C has finitely many extremal rays, then $\mathcal{M}$ is finitely generated.*

Proof. We can choose $m_1, \dots, m_r \in \mathcal{M}$ which span the cone C. If $m \in \mathcal{M}$, then there exist real numbers $a_1, a_2, \dots, a_r \geqslant 0$ such that $m = \sum_{i=1}^r a_i m_i$. Assume that the cardinality of the set $S = \{i \in \{1, \dots, r\} \mid a_i \neq 0\}$ is minimal. It is easy to see that the set $\{m_i \mid i \in S\}$ is linearly independent. It follows that $a_1, \dots, a_r \in \mathbb{Q}$. Choose an N such that $Na_i \in \mathbb{N}$ for all i. It follows that $Nm \in \mathcal{M}$. Let $\mathcal{M}'$ be the monoid generated by $m_1, \dots, m_r$. Let $\mathbb{C}[\mathcal{M}]$ and $\mathbb{C}[\mathcal{M}']$ be the algebras on the monoids $\mathcal{M}$ and $\mathcal{M}'$. The algebra $\mathbb{C}[\mathcal{M}]$ is integral over $\mathbb{C}[\mathcal{M}']$. The quotient field of $\mathbb{C}[\mathcal{M}]$ is a finite extension of the quotient field of $\mathbb{C}[\mathcal{M}']$ because $\mathbb{Z}\mathcal{M}' \subset \mathbb{Z}\mathcal{M}$ has finite index. We conclude that $\mathbb{C}[\mathcal{M}]$ is a finite module over $\mathbb{C}[\mathcal{M}']$, and therefore $\mathcal{M}$ must be finitely generated. □

Proof of Theorem 1.4. The monoid X_H of characters of H is isomorphic to $\mathbb{N}^n$, where n is the number of irreducible representations of H. The set X_H^G is a submonoid of X_H. Let $C \subset \mathbb{R}^n$ be the cone spanned by X_H^G. By Lemma 6.2 we have to show that C has finitely many extremal rays. By Theorem 1.1, χ_{reg} lies in C, and moreover, by Theorem 1.2, χ_{reg} lies in the *relative interior*: inside the vector space $\mathbb{R}C$, χ_{reg} lies in the interior of C.

Step 1. First we shall deal with the case where G is connected and simple. Because χ_{reg} lies in the relative interior of C, we can choose finitely many $f_1, f_2, \dots, f_l \in C$ such that χ_{reg} lies in the interior of the cone spanned by $f_1, \dots, f_l$ (inside the topological space C). There exists an $\varepsilon > 0$ such that every $f \in C$ with

$$\max_{h \in H} \left| f(h) - \frac{\chi_{\text{reg}}(h)}{|H|} \right| < \varepsilon$$

lies in the cone spanned by $f_1, \dots, f_l$. The cone C is spanned by all $\tau_\lambda \mid_H$. By Lemma 6.1 there are only finitely many λ such that

$$\max_{h \in H} \left| \frac{\tau_\lambda(h)}{\tau_\lambda(e)} - \frac{\chi_{\text{reg}}(h)}{|H|} \right| \geqslant \varepsilon.$$

This proves that C is spanned by $f_1, \dots, f_l$ and $\tau_\lambda \mid_H$, for $\lambda \in I$, where I is a finite set of dominant weights. By Lemma 6.2, X_H^G is finitely generated.

Step 2. We assume that G is connected, and that $\mathfrak{g}$ is simple. The group G has a finite center $Z(G)$. For an irreducible character ξ of $Z(G)$ we define $X_H^G(\xi) \subset X_H$ as the monoid of all $\tau \mid_H$ where τ is a character of G satisfying $\tau \mid_{Z(G)} = \tau(e)\xi$. We also define $C(\xi)$ as the cone generated by $X_H^G(\xi)$. We shall prove that $C(\xi)$ has finitely many extremal rays. Assume that $X_H^G(\xi) \neq \{0\}$. Choose a non-zero character τ of G such that $\tau \mid_{Z(G)} = \tau(e)\xi$, and let $\bar{\tau}$ be the dual character in $X_H^G(\xi^{-1})$. We can choose τ in such a way that for every $h \in H$ we have $\tau(h) = 0$ if and only if $\chi(h) = 0$ for all $\chi \in X_H^G(\xi)$. We define maps $u : C(\xi) \to C(1)$ and $v : C(1) \to C(\xi)$ by

$$u(f) = f\bar{\tau} \mid_H \quad (\text{where } f \in C(\xi)) \qquad \text{and} \qquad v(f) = f\tau \mid_H \quad (\text{where } f \in C(1)).$$

The maps u and $v \circ u$ are injective on $C(\xi)$, because of our choice of τ. Note that $X_H^G(1)$ can be identified with $X_{H/(H \cap Z(G))}^{G/Z(G)}$. Let χ_{reg} be the regular character of $H/(H \cap Z(G))$. We see that $u(v(\chi_{\text{reg}})) = \tau(e)^2 \chi_{\text{reg}}$, so χ_{reg} is in the image of u.

Moreover, χ_{reg} lies in the relative interior of $u(v(C(1))$, because χ_{reg} lies in the relative interior of $C(1)$. We have

$$\dim u(v(C(1)) = \dim v(C(1)) \geqslant \dim v(u(C(\xi)) = \dim C(\xi) \geqslant \dim u(C(\xi)).$$

It follows that χ_{reg} is in the relative interior of $u(C(\xi))$. If $\langle \rho, \lambda \rangle \to \infty$ (with $\tau_\lambda |_{Z(G)} = \tau_\lambda(e)\xi$), then by Lemma 6.1,

$$\frac{\tau_\lambda \tau}{\tau_\lambda(e)\tau(e)} \to \frac{\chi_{\text{reg}}}{|H/(H \cap Z(G))|}.$$

With similar arguments to those in Step 1, we see that $u(C(\xi))$ has finitely many extremal rays; therefore $C(\xi)$ has finitely many extremal rays. It follows that $X_H^G(\xi)$ is finitely generated by Lemma 6.2. Now X_H^G is finitely generated, because it is generated by all $X_H^G(\xi)$, where ξ is a character of $Z(G)$.

Step 3. Suppose now that $G = G_1 \times G_2 \times \ldots \times G_m$ where, for each i, G_i is connected with a simple Lie algebra, or G_i is a one-dimensional torus. The irreducible representations of G are exactly all $V_1 \otimes V_2 \otimes \ldots \otimes V_m$, where V_i is an irreducible representation of G_i for all i. Let $p_i : G \to G_i$ be the projection onto G_i. By restricting to H, we get a homomorphism $p_i |_H : H \to G_i$. We know that $X_{\pi_i(H)}^{G_i}$ is finitely generated (if G_i is connected with a simple Lie algebra, then this was done in Step 2; if G_i is a torus, then this is very easy), and therefore $X_H^{G_i}$ is finitely generated. Let $S_i \subset X_H^{G_i}$ be a finite set of generators. Now X_H^G is generated by all $\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_m$ with $\chi_i \in S_i$ for all i. We conclude that X_H^G is finitely generated.

Step 4. Now G is connected and reductive. It is known that G is a quotient of some $\tilde{G}$ with $\tilde{G}$ as in Step 3, and some finite central group Z. Let $\tilde{H}$ be the inverse image of H of the map $\tilde{G} \to \tilde{G}/Z \cong G$. Now X_H^G is the monoid of all characters $\chi \in X_{\tilde{H}}^{\tilde{G}}$ with $\chi(z) = \chi(e)$ for all $z \in Z \cap \tilde{H}$. Let L be the $\mathbb{Z}$-module of all $\chi \in \mathbb{Z}X_H^G$ satisfying $\chi(z) = \chi(e)$ for all $z \in Z$. Then X_H^G is the intersection of a finitely generated monoid $X_{\tilde{H}}^{\tilde{G}}$ with a $\mathbb{Z}$-module L, and therefore X_H^G is finitely generated. □

References

1. A. Borel, *Linear algebraic groups*, 2nd edn, Grad. Texts in Math. 126 (Springer, New York/Berlin, 1991).
2. S. Donkin, 'On free modules for finite subgroups of algebraic groups', *J. London Math. Soc.* (2) 55 (1997) 287–296.
3. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Grad. Texts in Math. 9 (Springer, New York/Berlin, 1972).
4. H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects Math. (Vieweg, Braunschweig, 1984).
5. M. Kuzucuoğlu and A. E. Zalesskiĭ, 'Hall universal group as a direct limit of algebraic groups', *J. Algebra* 192 (1997) 55–60.
6. D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, 3rd edn, Ergeb. Math. Grenzgeb. 34 (Springer, Berlin, 1994).

Department of Mathematics
University of Michigan
East Hall
525 East University
Ann Arbor, MI 48109-1109
USA

hderksen@umich.edu